Faster 3-Colouring Algorithm for Graphs of Diameter 3

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Abstract

We show that given an n-vertex graph G of diameter 3 we can decide if G is 3-colourable in time $2^{O(n^{2/3-\varepsilon})}$ for any $\varepsilon < 1/33$. This improves on the previous best algorithm of $2^{O((n\log n)^{2/3})}$ from Dębski, Piecyk and Rzążewski [Faster 3-coloring of small-diameter graphs, *ESA 2021*].

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1 Introduction

In the past two decades, much attention has been given to when central NP-hard algorithmic problems on graphs can be solved faster when restrictions are placed on the input graph. For example, after a sequence of works [2, 7, 10, 11, 12, 13, 14, 22], the complexity of k-COLOURING has been determined on H-free graphs for all connected H, with the remaining open case(s) being $k = 3$ and $H = P_t$ a path on $t \geq 8$ vertices.

For 3-COLOURING, a natural algorithmic approach (which is also the one used for the first subexponential time algorithm for P_t-free graphs) is to "branch" for well-chosen vertices on the colour that is assigned to them. This can be formalized by first generalizing 3-COLOURING to LIST-3-COLOURING, where lists $L(v) \subseteq \{1, 2, 3\}$ are given for each vertex v of the input graph and we need to decide whether the graph admits a proper colouring that assigns each vertex a colour from its list. Often problems are restricted to hereditary classes (that is, those closed under taking induced subgraphs) which is convenient since "branching" on a colour keeps the graph within the class. This makes a recursive argument possible.

An example of a natural restriction which does not lead to a hereditary graph class, is bounding the diameter of the input graph. This setting has been explored for many problems, such as problems related to various colouring variants [4, 21], feedback vertex sets [1], partitioning [5], and dominating sets [3]. For 3-colouring specifically, various authors have considered the setting of bounding the diameter, sometimes in combination with other

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constraints [6, 8, 16, 17, 15, 18]. Many problems remain NP-hard when adding a restriction on the diameter: for example, there is an easy reduction from 3-COLOURING to 4-COLOURING graphs of diameter 2 (by adding a universal vertex to the input graph). In particular, assuming the Exponential Time Hypothesis (ETH), there is no subexponential-time algorithm for k-colouring diameter-2 graphs for any $k \geq 4$. Since 2-COLOURING is polynomial-time solvable via reduction to 2-SAT [9], this reduction does not work for $k = 3$. In fact, 3-colouring can be solved in subexponential time on diameter-2 graphs and diameter-3 graphs, though not on diameter-4 graphs assuming ETH [8, 18].

In this paper, we study 3-COLOURING on graphs of diameter 3. Mertzios and Spirakis [18] proved that under ETH, this problem cannot be solved in time $2^{o(\sqrt{n})}$ where n denotes the number of vertices of the input graph, whereas Dębski, Piecyk and Rzążewski [8] provided an algorithm running in time $2^{O((n \log n)^{2/3})}$. This leaves the question of whether one of these two bounds may be close to tight. We show the upper bound is not tight, providing a slightly faster algorithm.

▶ **Theorem 1.** *LIST-3-COLOURING can be solved in time $2^{O(n^{2/3-\varepsilon})}$ for any fixed $\varepsilon < 1/33$ on the class of n-vertex graphs of diameter 3.*

The restriction of $\varepsilon < 1/33$ follows from a probabilistic argument used for proving the main structural result forming the basis of our algorithm.

The complexity of (LIST-)3-COLOURING diameter-2 graphs also remains an interesting open problem and it is possible that this problem is solvable in polynomial time.

Mertzios and Spirakis [18] provided the first subexponential-time algorithm for LIST-3-COLOURING on diameter-2 graphs with running time $2^{O(\sqrt{n \log n})}$. In the particular case of diameter-2 graphs, if we colour a vertex v and all its neighbours, then this immediately leaves a 2-LIST-COLOURING instance (which can be solved in polynomial time). This immediately provides an algorithm running in time $3^{\delta(G)} n^{O(1)}$ where $\delta(G)$ denotes the minimum degree of the graph G. On the other hand, high degree vertices are also useful, since "branching" on their colour may remove many colours from the lists of their neighbours. We can then track the progress of our algorithm through a potential function such as $\sum_{v \in V(G)} |L(v)|$ and roughly speaking the recursive calculations will work out because at most one "branch" has a small improvement whereas all other branches have an improvement of at least $\Delta(G)/2$ where $\Delta(G)$ denotes the maximum degree of graph G. Instead of such a branching approach, Mertzios and Spirakis [18] find a small dominating set when $\delta(G)$ is large to obtain their running time.

Similarly, for a diameter-3 graph we may aim for a running time of $2^{O((n \log n)^{2/3})}$ as follows. When the maximum degree is at most $(n \log n)^{1/3}$, then the second neighbourhood of any vertex is of size at most $(n \log n)^{2/3}$. Guessing the colours of the vertices in $v \cup N(v) \cup N(N(v))$ of a vertex v in a diameter-3 graph leaves a 2-LIST-COLOURING instance again which is solvable in polynomial time. By "branching" on the possible colours for a vertex of highest degree, removing vertices with a single colour in their list and updating colours throughout, we can hope to reduce to such a situation.

The only small issue with this reasoning is that the graph classes of diameter-2 and diameter-3 graphs are not hereditary, and so we cannot simply "ignore" vertices whose colour has already been assigned while keeping our diameter assumption. Nevertheless, with some additional structural reasoning, the overall approach of branching on the colours of vertices of maximum degree until there is a vertex of small degree can be made to work. Dębski, Piecyk and Rzążewski [8] provide such an algorithm that solves LIST-3-COLOURING on n-vertex graphs of diameter 3 in time $2^{O((n \log n)^{2/3})}$.

Our algorithm builds on this strategy, and is similar in approach to the algorithm Dębski, Piecyk and Rzążewski [8] introduce for the diameter-2 setting. To extend their approach, we use new structural insights as well as introducing additional branching rules to handle the issue of increasing diameter when removing coloured vertices.

2 Preliminaries

All graphs we consider in this paper are simple and finite. We use $[k]$ to denote the set $\{1,\dots,k\}$ for positive integers k and $\log n$ for the natural (e-based) logarithm.

A problem instance for LIST-3-COLOURING consists of a graph G and a list assignment $L : V(G) \to 2^{[3]}$ (that is, $L(v) \subseteq \{1,2,3\}$ for all $v \in V(G)$). The decision problem is then to determine whether there exists a proper colouring $f : V(G) \to [3]$ of G such that $f(v) \in L(v)$ for all $v \in V(G)$. A problem instance for 2-LIST-COLOURING consists of a graph G and a list assignment L such that $|L(v)| \leq 2$ for all $v \in V(G)$.

We say two instances (G,L) and (G',L') are *equivalent* if (G,L) is a YES-instance if and only if (G',L') is.

Given a graph G and a list assignment $L : V(G) \to 2^{[3]}$ and $i \in [3]$, let

$$L_i = \{v \in V(G) \text{ with } |L(v)| = i\}.$$

For a positive integer k and a vertex $v \in V(G)$, let $N^{(k)}(v)$ be the set of all vertices at distance exactly k from v, and let $N^{(\leq k)}(v)$ be the set of all vertices at distance at most k from v, excluding v itself. Let $N^{(\leq k)}[v]$ be the closed neighbourhood equivalent, that is $N^{(\leq k)}[v] := N^{(\leq k)}(v) \cup \{v\}$. Additionally, let $N^{(k)}_{L_i}(v)$ and $N^{(\leq k)}_{L_i}(v)$ for $i \in [3]$ be the sets $N^{(k)}(v) \cap L_i$ and $N^{(\leq k)}(v) \cap L_i$ respectively. If $k = 1$, we omit the superscript and simply use $N_{L_i}(v)$. Analogously we define $N^{(\leq k)}_{L_i}[v]$ and $N_{L_i}[v]$ for the closed neighbourhood equivalents. Finally, for a set $S \subseteq V(G)$, we define $N^{(\leq k)}(S)$ to be the set of all vertices in $V(G) \setminus S$ at distance at most k from S, and $N^{(\leq k)}[S] := N^{(\leq k)}(S) \cup S$.

We say that a probabilistic event E happens *with high probability* with respect to a variable ℓ if $\lim_{\ell\to\infty} \mathbb{P}[E] = 1$. In this paper, the relevant variable will always be the number of vertices in the graph.

We also use the following Chernoff bounds (see [19] for a proof).

▶ **Lemma 2** (Chernoff bound). *Let $X \sim Bin(n,p)$ be the sum of n independent random variables taking value 0 with probability $1-p$ and value 1 with probability p. Then for $\delta \in [0,1]$,*

$$\mathbb{P}(|X - np| \geq \delta np) \leq \exp(-\delta^2 np/3).$$

For $\delta > 0$,

$$\mathbb{P}(X \geq (1+\delta)np) \leq \exp(-\delta^2 np/(2+\delta)).$$

In particular, we use multiple times that if $\mathbb{E}[X] = np \geq 3\log n$ for n sufficiently large, then

$$\mathbb{P}(X \geq 2\mathbb{E}[X]) \leq \mathbb{P}(|X - np| \geq np) \leq \exp(-np/3) \leq \exp(-\log n) = n^{-1} \to 0$$

as $n \to \infty$. One time, we will require a slightly stronger version of the Chernoff bound above:

▶ **Lemma 3** (Chernoff bound – upper bound variant). *Let $X \sim Bin(n,p)$ be the sum of n independent random variables taking value 0 with probability $1-p$ and value 1 with probability p. Then for $\delta > 0$ and $b \geq np$,*

$$\mathbb{P}(X \geq (1+\delta)b) \leq \exp(-\delta^2 b/(2+\delta)).$$

The above lemma follows from the observation that the standard proof of Chernoff tail bounds based on the moment generating function (see [19]) goes through with an upper bound on the expected value of X rather than the expected value itself as well.

3 Reduction and branching rules

We first consider a number of instance features we may exploit to reduce the complexity of the problem instance. Note that if no vertex contains three colours in its list, the problem may be reduced to 2-SAT. Hence, we have the following theorem.

▶ **Theorem 4** (Edwards [9]). *Let G be a graph and let $L : V(G) \to 2^{[3]}$ be a list assignment such that $|L(v)| \leq 2$ for all $v \in V(G)$. Then it is decidable in polynomial time whether G admits a proper L-colouring.*

As a first step in simplifying our problem instances, we have the following reduction rules:

R1 If there exists a vertex $v \in V(G)$ with $L(v) = \emptyset$, the instance is infeasible. Return NO.

R2 If $L_3 = \emptyset$, the instance may be solved in polynomial time using Theorem 4.

R3 If there exist adjacent vertices $u, v \in V(G)$ with $L(u) = \{c\}$ for some $c \in [3]$ and $c \in L(v)$, remove c from $L(v)$.

R4 If $|L_3| < c$ for some constant c, exhaustively guess the colours assigned to the vertices in L_3, and solve the rest of the instance using rule R2.

The constant c used in R4 will be determined later.

Note that none of these rules affects the feasibility of the instance, and each may be checked and executed in polynomial time. Moreover, for R2-R4, the function $\sum_{v\in V(G)} |L(v)|$ strictly decreases by applying the reduction rule. Hence, each of these reduction rules is applied only a polynomial number of times for a given instance.

In addition to these reduction rules, we introduce a number of branching rules. In contrast to the reduction rules, these branching rules do not directly simplify the problem instance. Rather, the branching rules each generate a number of new instances, each of which is simpler than the original instance. Solving these simpler instances in turn allows us to solve the original instance, enabling a recursive algorithm. Before introducing the first branching rules, we need the following lemma.

▶ **Lemma 5.** *Let $u \in V(G)$. Then there exists a colour $c \in L(u)$ such that if we assign c to u and apply reduction rule R3 repeatedly, we obtain an instance (G, L') with $|L_3'| \leq |L_3| - |N_{L_3}(N_{L_2}(N_{L_2}(u)))|/3$.*

Proof. Let $x, y \in L_2$ and $v \in L_3$ such that $u - x - y - v$ is a path of length 3 in G. Let $c \in L(x) \cap L(y)$ and let c' be the other colour in $L(x)$ (that is, $L(x) = \{c, c'\}$). If we assign the colour c' to u, applying reduction rule R3 will result in vertex x being assigned colour c. Hence, applying R3 again results in $L(y)$ losing a colour, determining the colour of y as well. As a result, $L(v)$ will lose the colour assigned to y and hence v will be removed from L_3. In particular, for each $v \in N_{L_3}(N_{L_2}(N_{L_2}(u)))$, there is a colour c that can be assigned to v such that v is removed from L_3 and so the statement follows from the pigeonhole principle. ◀

Let $\mu := |L_3|$. We have the following branching rules:

B1 If there exists a vertex $v \in L_2 \cup L_3$ with $|N_{L_3}(v)| \geq \mu^{1/3+\varepsilon}$, branch on the colour assigned to v.

B2 If there exists a vertex $v \in L_3$ that has via neighbours in L_2 at least $\mu^{1/3+\varepsilon}$ second neighbours in L_3, that is,

$$|N_{L_3}(N_{L_2}(v))| \geq \mu^{1/3+\varepsilon},$$

then branch on the colour assigned to v.

B3 If there exists a vertex $v \in L_3$ with

$$|N_{L_3}(N_{L_2}(N_{L_2}(v)))| \geq \mu^{1/3+\varepsilon},$$

using Lemma 5 there exists a colour $c \in [3]$ such that if we assign c to v, the lists of at least $\mu^{1/3+\varepsilon}/3$ vertices in $N_{L_3}(N_{L_2}(N_{L_2}(v)))$ decrease in size after repeatedly applying R3. We branch on this colour for v: either we assign v the colour c, or we remove c from the list $L(v)$.

B4 If there exist two distinct vertices $u, v \in L_3$ with

$$|N_{L_3}(N(u) \cap N(v))| \geq \mu^{1/3+\varepsilon},$$

we branch on the colours of u and v. If u and v are adjacent, we create six branches, one for each of the possible colour assignments for u and v in which they are assigned different colours. If u and v are non-adjacent, we additionally consider a branch where we merge u and v into a single vertex but do not fix any colours.

We omit the proof of the correctness of these branching rules.

We say an instance (G, L) is *reduced* if none of the above reduction or branching rules can be applied to reduce the complexity of the instance. Note that other than merging two vertices in branching rule B4, none of the branching rules remove any of the vertices. When a vertex v is in L_1, it has effectively been coloured and no longer plays a role for the remainder of a reduced instance. We do not delete it to maintain diameter at most 3.

Branching rules B2 and B3 specifically are designed to establish and maintain the property that $G[L_3]$ is close to having diameter 3 in the following sense:

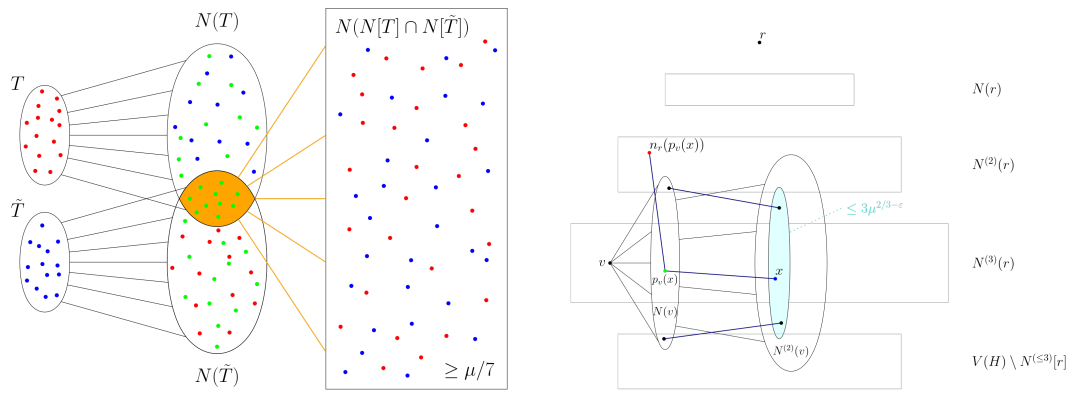
▶ **Lemma 6.** *Let (G, L) be a reduced instance with G a diameter-3 graph. Then for all vertices $u \in L_3$, it holds that $|N^{(\leq 3)}_{G[L_3]}(u)| \geq \mu - 4\mu^{2/3+2\varepsilon}$.*

Proof. Let $u \in L_3$. For each $v \in L_3$, there is a path between u and v in G of length at most 3 since G has diameter 3. We prove the following stronger statement: there are at most $4\mu^{2/3+\varepsilon}$ vertices $v \in L_3$ such that there is a $u - v$ path of length at most 3 containing vertices in $V(G) \setminus L_3$.

By the reduction rules, no vertex in L_1 is adjacent to a vertex in L_3, and so a $u - v$ path of length 3 has no vertices in L_1. Therefore, we consider the paths of length at most 3 containing vertices in L_2, which must have one of the following forms:

1. $u - x - v$ where $x \in L_2$,
2. $u - x - y - v$ where $x, y \in L_2$,
3. $u - x - y - v$ where $x \in L_2$ and $y \in L_3$, or
4. $u - x - y - v$ where $x \in L_3$ and $y \in L_2$.

Because B2 is not applicable to the reduced instance, we find that $|N_{L_3}(N_{L_2}(u))| < \mu^{1/3+\varepsilon}$ and in particular there are at most $\mu^{1/3+\varepsilon}$ vertices $v \in L_3$ reachable from u via paths of type 1. Similarly, because B3 is not applicable, there are at most $\mu^{1/3+\varepsilon}$ vertices $v \in L_3$ reachable via paths of type 2. Because B1 and B2 are not applicable, we moreover find that there are at most $\mu^{2/3+2\varepsilon}$ vertices $v \in L_3$ reachable via paths of type 3, and at most $\mu^{2/3+2\varepsilon}$



(a) Outcome 2. **(b)** Outcome 3.

Figure 1 Illustrations of the second and third outcomes in Lemma 7 respectively.

vertices $v \in L_3$ reachable via paths of type 4. In total, there are at most $4\mu^{2/3+2\varepsilon}$ vertices $v \in L_3$ reachable from u via paths of length at most 3 that are not fully contained in $G[L_3]$, as desired. ◀

In the next section, we will describe one more branching rule, which is only applied when the instance is reduced.

4 Main lemma and the last branching rule

To introduce the final branching rule, we first require the following structural result that we will apply to $H = G[L_3]$. See Figure 1 for an illustration of the second and third outcomes of the lemma below.

▶ **Lemma 7.** *Let $\varepsilon < 1/33$. There exists an $\ell > 0$ such that the following holds for all $\mu \geq \ell$. Let H be a μ-vertex graph such that for each vertex $v \in V(H)$,*

- $|N(v)| \leq \mu^{1/3+\varepsilon}$ *and* $|N^{(2)}(v)| \leq \mu^{2/3+2\varepsilon}$, *and*
- $|N^{(\leq 3)}(v)| \geq \mu - 4\mu^{2/3+2\varepsilon}$,

and for all pairs of distinct vertices $u, v \in V(H)$,

- $|N(N(u) \cap N(v))| \leq \mu^{1/3+\varepsilon}$.

Let $r \in V(H)$. For each vertex $a \in N^{(3)}(r)$, fix a neighbour $n_r(a) \in N^{(2)}(r)$. Additionally, for each $v \in V(H)$ and each $x \in N^{(2)}(v)$, fix a vertex $p_v(x) \in N(v) \cap N(x)$. Then one of the following statements must hold.

1. *H is not 3-colourable.*
2. *H admits a 3-colouring $\phi : V(H) \to [3]$ and there exist disjoint sets $T, \widetilde{T} \subseteq V(H)$ of size at most $2\mu^{2/3-\varepsilon}$ such that*
$$|N(N[T] \cap N[\widetilde{T}])| \geq \mu/7,$$
both T and $\widetilde{T}$ are monochromatic under ϕ, and the vertices in T receive a colour different than the one the vertices in $\widetilde{T}$ receive.
3. *H admits a 3-colouring $\phi : V(H) \to [3]$ and there exists a vertex $v \in V(H)$ such that*
$$|\{x \in N^{(2)}(v) : p_v(x) \notin N^{(3)}(r) \text{ or } \phi(n_r(p_v(x))) \neq \phi(x)\}| \leq 3\mu^{2/3-\varepsilon}.$$

Moreover, if H admits a 3-colouring $\phi : V(H) \to [3]$, then either outcome 2 or 3 must hold for this specific 3-colouring ϕ.

Proof. We may assume that H is 3-colourable, as otherwise outcome 1 of the lemma holds. Let $\phi : V(H) \to [3]$ be a proper 3-colouring of G. It remains to show that either outcome 2 or 3 holds for ϕ.

As in the assumptions of the lemma, let $r \in V(H)$ (arbitrarily). We randomly pick two sets of vertices S and $\widetilde{S}$ as follows.

- Each vertex in $V(H)$ is included in S independently with probability $\mu^{-(1/3+\varepsilon)}$.
- Each vertex in $N^{(2)}(r)$ is included in $\widetilde{S}$ independently with probability $\mu^{-3\varepsilon}$.

We show that with high probability, either there are (monochromatic) subsets T and $\widetilde{T}$ of S and $\widetilde{S}$ respectively such that the second outcome in the lemma applies, or the third outcome in the lemma applies. Since $|V(H)| = \mu$ and $|N^{(2)}(r)| \leq \mu^{2/3+2\varepsilon}$, the following claim follows from Chernoff bounds (see Lemma 2).

▷ Claim 8. With high probability, $|S| \leq 2\mu^{2/3-\varepsilon}$ and $|\widetilde{S}| \leq 2\mu^{2/3-\varepsilon}$.

Let $R := V(H) \setminus N^{(3)}(r)$. Let $V' \subseteq N^{(3)}(r)$ be the set of all vertices $v \in N^{(3)}(r)$ such that $|R \cap N(v)| \leq \mu^{1/3-2\varepsilon}$. We show that most of the vertices in H belong to V'.

▷ Claim 9. For μ sufficiently large, $|V'| \geq \mu - 7\mu^{2/3+5\varepsilon}$.

Proof. We first observe that by the first three assumptions in the lemma statement,

$$|N^{(\leq 2)}[r]| \leq \mu^{1/3+\varepsilon} + \mu^{2/3+2\varepsilon} + 1 \text{ and } |V(H) \setminus N^{(\leq 3)}(r)| \leq 4\mu^{2/3+2\varepsilon}.$$

This implies that $|R| = |V(H) \setminus N^{(3)}(r)| \leq 6\mu^{2/3+2\varepsilon} \leq \mu^{2/3+5\varepsilon}$ for μ sufficiently large.

Let $B := N^{(3)}(r) \setminus V'$ be the set of all vertices $v \in N^{(3)}(r)$ such that $|R \cap N(v)| > \mu^{1/3-2\varepsilon}$. We show that $|B| \leq 6\mu^{2/3+5\varepsilon}$. Let $E' \subseteq E(H)$ be the set of all edges with one endpoint in B and one endpoint in R. By the definition of B, every vertex $v \in B$ is adjacent to more than $\mu^{1/3-2\varepsilon}$ vertices in R. Therefore, $|E'| > \mu^{1/3-2\varepsilon} \cdot |B|$. Since each vertex in H has degree at most $\mu^{1/3+\varepsilon}$, we find that

$$\mu^{1/3-2\varepsilon} \cdot |B| < |E'| \leq |R|\mu^{1/3+\varepsilon} \leq 6\mu^{2/3+2\varepsilon} \cdot \mu^{1/3+\varepsilon}.$$

This shows that $|B| \leq 6\mu^{2/3+5\varepsilon}$. Since $|V'| = |V(H)| - |R| - |B|$, the claim follows. ◁

It is not quite enough to show that $V' \subseteq N(N[S] \cap N[\widetilde{S}])$ for the second outcome of the lemma: ideally, the vertices in S and $\tilde{S}$ that are used to achieve this feat also have different colours. For $i \in [3]$, let $S_i = S \cap \phi^{-1}\{i\}$ and $\widetilde{S}_i = \widetilde{S} \cap \phi^{-1}\{i\}$. We will show that with high probability, either the third outcome of the lemma holds, or for all $v \in V'$,

$$v \in \bigcup_{i \neq j} N(N(S_i) \cap N(\widetilde{S}_j)). \tag{1}$$

The second outcome then follows in the latter case from the previous claims and the pigeonhole principle.

As in the assumptions of the lemma, for each vertex $a \in N^{(3)}(r)$, we fix a neighbour $n_r(a) \in N^{(2)}(r)$ and for all vertices $v \in V(H)$ and $x \in N^{(2)}(v)$, we fix a vertex $p_v(x) \in N(v) \cap N(x)$. We call $p_v(x)$ the *favourite parent* of x with respect to v.

For a vertex $v \in V'$, we say that a vertex $x \in N^{(2)}(v)$ is *v-deducing* if $p_v(x) \in N^{(3)}(r)$ and $\phi(x) \neq \phi(n_r(p_v(x)))$. Note that if moreover $x \in S$ and $n_r(p_v(x)) \in \widetilde{S}$, then $p_v(x) \in N(S_i) \cap N(\widetilde{S}_j)$ for $i = \phi(x)$ and $j = \phi(n_r(p_v(x)))$ with $i \neq j$. Hence v satisfies (1).

Recall that $n_r(a)$ is only defined for $a \in N^{(3)}(r)$. For each vertex $v \in V'$ and each vertex $a \in N(v)$, we define the *bucket*

$$B_a^{(v)} := \{x \in N^{(2)}(v) \,:\, p_v(x) = a\} \subseteq N(a).$$

That is, the bucket $B_a^{(v)}$ is the set of all vertices in the second neighbourhood of v which have a as their favourite parent with respect to v. For a vertex $v \in V'$, we say that a vertex $a \in N(v)$ is *v-fruitful* if $B_a^{(v)}$ contains at least $\mu^{1/3-2\varepsilon}$ v-deducing vertices. Note that this implies that $a \in N^{(3)}(r)$ by definition of v-deducing. Similarly, we call the corresponding bucket $B_a(v)$ itself v-fruitful as well.

▷ Claim 10. If there exists a vertex $v \in V'$ with at most $\mu^{1/3-2\varepsilon}$ v-fruitful buckets, then vertex v satisfies the third outcome of the lemma.

Proof. Suppose that $v \in V'$ has at most $\mu^{1/3-2\varepsilon}$ vertices $a \in N(v) \cap N^{(3)}(r)$ that are v-fruitful. We need to show that $|\{x \in N^{(2)}(v) : p_v(x) \notin N^{(3)}(r) \text{ or } \phi(n_r(p_v(x))) \neq \phi(x)\}| \leq 3\mu^{2/3-\varepsilon}$.

We first show that for each $v \in V'$, we have $|\{x \in N^{(2)}(v) : p_v(x) \notin N^{(3)}(r)\}| \leq \mu^{2/3-\varepsilon}$. Since $v \in V'$, by definition $|N(v) \setminus N^{(3)}(r)| \leq \mu^{1/3-2\varepsilon}$. For each vertex $y \in N(v) \setminus N^{(3)}(r)$, there are at most $|N(y)| \leq \mu^{1/3+\varepsilon}$ vertices x with $p_v(x) = y$. Hence there are at most $\mu^{1/3-2\varepsilon}\mu^{1/3+\varepsilon} = \mu^{2/3-\varepsilon}$ vertices x with $p_v(x) \notin N^{(3)}(r)$.

It now suffices to show that $|\{x \in N^{(2)}(v) : x \text{ is } v\text{-deducing}\}| \leq 2\mu^{2/3-\varepsilon}$. Since each vertex in H has degree at most $\mu^{1/3+\varepsilon}$,

- there are at most $\mu^{1/3+\varepsilon}$ buckets $B_a^{(v)}$ with $a \in N(v)$, and
- $|B_a^{(v)}| \leq \mu^{1/3+\varepsilon}$ (since $B_a^{(v)} \subseteq N(a)$).

Our assumption on v is that at most $\mu^{1/3-2\varepsilon}$ buckets are v-fruitful. The v-fruitful buckets together thus contain at most $\mu^{1/3-2\varepsilon} \cdot \mu^{1/3+\varepsilon} = \mu^{2/3-\varepsilon}$ v-deducing vertices. Moreover, each bucket that is not v-fruitful contains at most $\mu^{1/3-2\varepsilon}$ v-deducing vertices, and so the buckets that are not v-fruitful also contain at most $\mu^{1/3-2\varepsilon} \cdot \mu^{1/3+\varepsilon} = \mu^{2/3-\varepsilon}$ v-deducing vertices. Hence $N^{(2)}(v)$ contains at most $2\mu^{2/3-\varepsilon}$ v-deducing vertices, as desired. ◁

For each vertex $v \in V'$, let $A_v \subseteq N(v) \cap N^{(3)}(r)$ be the set of v-fruitful vertices a for which $B_a^{(v)} \cap S$ contains at least one v-deducing vertex x. In particular, A_v is a random variable that depends on S. We will later analyse the probability that $n_r(p_v(x)) = n_r(a) \in \widetilde{S}$.

▷ Claim 11. With high probability, for all vertices $v \in V'$, $|A_v| \geq \mu^{1/3-5\varepsilon}/(4\log\mu)$.

Proof. We first show that, with high probability, each bucket $B_a^{(v)}$ contains at most $2\log\mu$ vertices from S for each $a \in N(v)$. Recall that $n := |B_a^{(v)}| \leq |N(a)| \leq \mu^{1/3+\varepsilon}$. Each element of $B_a^{(v)}$ is included in S independently with probability $p := \mu^{-(1/3+\varepsilon)}$. Let $X := |B_a^{(v)} \cap S|$. Then $\mathbb{E}[X] = np \leq 1$. So by the variant of Chernoff bounds with an upper bound on the expected value (Lemma 3 with $\delta = 2\log\mu - 1$ and $b = 1$), and as for μ sufficiently large we have that $\delta^2/(2+\delta) \geq 0.9(\delta+1)$,

$$\mathbb{P}(|B_a^{(v)} \cap S| > 2\log\mu) = \mathbb{P}(|B_a^{(v)} \cap S| > 1+\delta) \leq \exp(-0.9(\delta+1)) = \mu^{-1.8}.$$

There are at most $|N(v)| \leq \mu^{1/3+\varepsilon}$ buckets. So by the union bound, the probability that a bucket $B_a^{(v)}$ (for some $v \in V'$ and $a \in N(v)$) contains more than $2\log\mu$ vertices from S is at most $\mu \cdot \mu^{1/3+\varepsilon} \cdot \mu^{-1.8}$, which tends to 0 as $\mu \to \infty$.

By Claim 10, we may assume that for all vertices $v \in V'$, there are at least $\mu^{1/3-2\varepsilon}$ v-fruitful vertices $a \in N(v) \cap N^{(3)}(r)$, meaning that that the bucket $B_a^{(v)}$ contains at least $\mu^{1/3-2\varepsilon}$ v-deducing vertices. Since these buckets are by definition disjoint, there exist at least $\mu^{2/3-4\varepsilon}$ v-deducing vertices $x \in N^{(2)}(v)$ such that $p_v(x)$ is v-fruitful. Let X_v denote the number of such vertices x that are part of S. Each such x is part of S independently with probability $\mu^{-(1/3+\varepsilon)}$ and so $\mathbb{E}[X_v] \geq \mu^{1/3-5\varepsilon}$. Hence, by Chernoff bounds (the first inequality of Lemma 2 with $\delta = 1/2$), $\mathbb{P}(X_v \leq \mu^{1/3-5\varepsilon}/2) \leq \mu^{-2}$ for μ sufficiently large, using that $1/3 - 5\varepsilon > 0$. So with high probability, for each $v \in V'$, the pigeonhole principle implies that there are at least $\frac{1}{2}\mu^{1/3-5\varepsilon}/(2\log\mu)$ different buckets that contain v-deducing vertices in S. Hence, with high probability, $|A_v| \geq \mu^{1/3-5\varepsilon}/(4\log\mu)$. ◁

If for a vertex $v \in V'$ one of the vertices in $\{n_r(a) : a \in A_v\}$ is included in $\widetilde{S}$, then v satisfies (1) as desired. We therefore next show that the set $\{n_r(a) : a \in A_v\}$ is sufficiently large.

▷ Claim 12. With high probability, for all vertices $v \in V'$, we have $|\{n_r(a) : a \in A_v\}| \geq 4\mu^{3\varepsilon} \log \mu$.

Proof. We show that $|\{n_r(a) : a \in A_v\}| \geq 20\mu^{3\varepsilon} \log \mu$ for any vertex v with $|A_v| \geq \mu^{1/3-5\varepsilon}/(4 \log \mu)$. The claim then follows from Claim 11.

Suppose towards a contradiction that there is a vertex v with $|A_v| \geq \mu^{1/3-5\varepsilon}/(4 \log \mu)$ yet $|\{n_r(a) : a \in A_v\}| < 4\mu^{3\varepsilon} \log \mu$. For $y \in N^{(2)}(r)$, let $A_v(y) := \{a \in A_v : n_r(a) = y\}$. By the pigeonhole principle, there exist a vertex $y \in N^{(2)}(r)$ such that $|A_v(y)| > \mu^{1/3-5\varepsilon}/(\mu^{3\varepsilon}(4 \log \mu)^2) = \mu^{1/3-8\varepsilon}/(4 \log \mu)^2$.

By definition of A_v, we have that each $a \in A_v(y)$ is v-fruitful, and hence $|B_a^{(v)}| \geq \mu^{1/3-2\varepsilon}$ for each $a \in A_v(y)$. Since all buckets are disjoint,

$$\begin{aligned}
|N(A_v(y))| &\geq \left| \bigcup_{a \in A_v(y)} B_a^{(v)} \right| \\
&= \sum_{a \in A_v(y)} \left| B_a^{(v)} \right| \\
&> \mu^{1/3-2\varepsilon} \cdot \mu^{1/3-8\varepsilon}/(4 \log \mu)^2 \\
&= \mu^{2/3-10\varepsilon}/(4 \log \mu)^2.
\end{aligned}$$

Each vertex in $A_v(y)$ is a neighbour of both y and v and so

$$|N(N(y) \cap N(v))| \geq |N(A_v(y))| > \mu^{2/3-10\varepsilon}/(4 \log \mu)^2.$$

Note that

$$2/3 - 10\varepsilon > 1/3 + \varepsilon \iff 1/3 > 11\varepsilon \iff \varepsilon < 1/33.$$

For μ sufficiently large, as $\varepsilon < 1/33$, we found above that $|N(N(y) \cap N(v))| > \mu^{1/3+\varepsilon}$. However, by the fourth assumption in the lemma statement, $|N(N(y) \cap N(v))| \leq \mu^{1/3+\varepsilon}$. This gives a contradiction. ◁

Let $v \in V'$. Then by the previous claim we may assume that $|\{n_r(a) : a \in A_v\}| \geq 20\mu^{3\varepsilon} \log \mu$. Since each vertex in $N^{(2)}(r)$ is included in $\widetilde{S}$ independently with probability $\mu^{-3\varepsilon}$, by Chernoff bounds (Lemma 2 with $X = |\widetilde{S} \cap \{n_r(a) : a \in A_v\}|$, $np \geq 20 \log \mu$ and $\delta = 1/2$), for μ sufficiently large

$$\mathbb{P}(|\widetilde{S} \cap \{n_r(a) : a \in A_v\}| < 10 \log \mu) \leq \exp(-20 \log \mu/12) \leq \mu^{-1.1}.$$

In particular, by a union bound, with high probability for each $v \in V'$ we find that $\widetilde{S} \cap \{n_r(a) : a \in A_v\} \neq \emptyset$.

We next spell out the final pigeonhole principle application. For each vertex $v \in V'$, we assign a pair of colours $(i, j) \in [3]^2$ with $i \neq j$ as follows. Let $a \in A_v$ with $n_r(a) \in \widetilde{S}$ and $x \in B_a^{(v)} \cap S$ be a v-deducing vertex. Let $i = \phi(x)$ and $j = \phi(n_r(a)) = \phi(n_r(p_v(x)))$. By the definition of v-deducing, $i \neq j$.

By the pigeonhole principle, there exists a pair (i, j) which is assigned to v for at least $\frac{1}{6}|V'|$ vertices $v \in V'$. Let

$$\begin{aligned}
T &:= \{v \in S : \phi(v) = i\}, \\
\widetilde{T} &:= \{v \in \widetilde{S} : \phi(v) = j\}.
\end{aligned}$$

By Claim 8, as $T \subseteq S$ and $\widetilde{T} \subseteq \widetilde{S}$, we have that $|T|, |\widetilde{T}| \leq 2\mu^{2/3-\varepsilon}$. By definition, T and $\widetilde{T}$ are monochromatic under ϕ for different colours and

$$|N(N[T] \cap N[\widetilde{T}])| \geq \frac{1}{6}|V'| \geq \frac{1}{6}(\mu - 9\mu^{2/3+5\epsilon}) \geq \frac{1}{7}\mu$$

by Claim 9 for μ sufficiently large. This means the second outcome of the lemma holds. We define ℓ such that all "for all μ sufficiently large" statements hold. Note that ℓ only depends on our choice of ε. ◀

The constant ℓ given in the statement of Lemma 7 is the value we will use for constant c in reduction rule R4.

Based on Lemma 7, we introduce one final branching rule. This branching rule is only applied if none of the previous reduction and branching rules are applicable. Namely, if the instance is reduced, $H = G[L_3]$ satisfies the conditions of Lemma 7 (we omit the proof).

B5 If (G, L) is reduced, we create a number of new instances.
As in the statement of Lemma 7, fix $r \in V(H)$, neighbours $n_r(a) \in N^{(2)}_{L_3}(r)$ for vertices $a \in N^{(3)}_{L_3}(r)$, and vertices $p_v(x) \in N_{L_3}(v) \cap N_{L_3}(x)$ for $v \in L_3$ and $x \in N^{(2)}_{L_3}(v)$.
First, for each pair of disjoint subsets $T, \widetilde{T} \subseteq L_3$ such that each has size at most $2\mu^{2/3-\varepsilon}$ and $|N_{L_3}(N_{L_3}[T] \cap N_{L_3}[\widetilde{T}])| \geq \mu/7$, we create six new branches: one for each pair of colours $(i, j) \in [3]^2$ such that $i \neq j$. In each of the six respective branches, assign each vertex in T the colour i, and each vertex in $\widetilde{T}$ the colour j. These branches correspond to the second outcome of Lemma 7.
Second, we create branches corresponding to the third outcome of Lemma 7. For each vertex $v \in N^{(3)}_{L_3}(r)$, we create a branch for each possible valid combination of a 3-colouring of $N_{L_3}[v]$, a 3-colouring of $\{n_r(a) : a \in N_{L_3}(v)\}$, a set $S \subseteq N^{(2)}_{L_3}(v)$ of size at most $3\mu^{2/3-\varepsilon}$ such that $\{x \in N^{(2)}_{L_3}(v) : p_v(x) \notin L_3 \setminus N^{(3)}(r)\} \subseteq S$, and each 3-colouring of S.
In these branches, colour the vertices of

$$N_{L_3}[v] \cup \{n_r(a) : a \in N_{L_3}(v)\} \cup S$$

according to the colourings defining the branch. This gives a partial colouring ϕ. We extend ϕ to the remaining vertices $x \in N^{(2)}_{L_3}(v) \setminus S$ via $\phi(x) := \phi(n_r(p_v(x)))$.

We omit the proof of the correctness of this final branching rule.

While branching rule B5 creates many new branches every time it is applied, its main purpose is to reduce the size of L_3. It indeed does so significantly on every application, as shown in the following lemma.

▶ **Lemma 13.** *Let (G, L) be a reduced instance and let $\mu = |L_3|$. Then, in the new instances created by applying branching rule B5, after exhaustively applying the reduction rules, the number of vertices of list size 3 is*
1. *at most $(6/7) \cdot \mu$ in the branches corresponding to the second outcome of Lemma 7, and*
2. *at most $4\mu^{2/3+2\varepsilon}$ in the branches corresponding to the third outcome of Lemma 7.*

Proof. For the branches corresponding to the second outcome of Lemma 7, since T and $\widetilde{T}$ are assigned different colours, the colour of each vertex in $N_{L_3}[T] \cap N_{L_3}[\widetilde{T}]$ is determined after applying the reduction rules exhaustively. Hence, each vertex in $N_{L_3}(N_{L_3}[T] \cap N_{L_3}[\widetilde{T}])$ loses a colour from its list. By the definition of branching rule B5 this set has size at least $\mu/7$, and thus the created new instances each have at least $\mu/7$ fewer vertices of list size 3, as desired.

Next we consider the branches corresponding to the third outcome of Lemma 7. In each of these branches, for some vertex $v \in N^{(3)}_{L_3}$ we colour all of $N^{(\leq 2)}_{L_3}[v]$. Thus, after applying the reduction rules exhaustively, none of the vertices in $N^{(\leq 3)}_{L_3}[v]$ still have list size 3. Hence, using Lemma 6, the resulting instance has at most $4\mu^{2/3+2\varepsilon}$ vertices of list size 3, as desired. ◀

5 Running time analysis

Note that branching rule B5 always applies if the instance is in a reduced state. Since an instance is by definition reduced if and only if none of the other branching and reduction rules may be applied, the branching algorithm is guaranteed to terminate. Hence, by the correctness of the reduction and branching rules, the algorithm correctly determines whether the input graph is 3-colourable and constructs a valid 3-colouring if it exists. The fifth branching rule depends on the choice of ε. We show that for each $\delta \in (0, 1/33)$, there exists an $\varepsilon \in (0, 1/33)$ such that the branching algorithm using this ε will run in time $2^{O(n^{2/3-\delta})}$. We briefly sketch the runtime analysis here. We create a recurrence relation for each of the branching rules capturing the number of new instances it creates. For each of the branching rules B1-B4, at most one instance is created with one fewer vertex of list-size 3, and a constant number (say k) of instances where the number of vertices in L_3 drops by at least $\mu^{1/3+\varepsilon}$. If we look at the number of leaves in the corresponding tree (that is, the total number of instances created following rules B1-B4), we can describe each leaf by the path from the root. This will be a sequence with symbols in $\{0, 1, \dots, k\}$, say, of length at most n in this case (an upper bound for μ), and at most $n^{2/3-\varepsilon}$ of the entries are nonzero ("the big steps," whereas the label 0 is used for branches along which the size decreases by 1). This gives $\binom{n}{n^{2/3-\varepsilon}} \cdot k^{n^{2/3-\varepsilon}}$ as upper bound. Moreover, whereas branching rule B5 generates many more new instances, each of them drops the number of vertices in L_3 by either a constant factor or an exponential factor, which sufficiently counteracts the exponential explosion in the number of instances.

6 Conclusion

We made progress on one of the two "notorious" [20] open cases of the complexity of k-colouring on graphs of bounded diameter, by providing a faster algorithm for 3-colouring diameter-3 graphs. Assuming the Exponential Time Hypothesis (ETH), 3-colouring cannot be solved in time $2^{o(n)}$ on diameter-4 graphs with n vertices [18], nor can k-colouring be solved in time $2^{o(n)}$ on diameter-2 graphs for $k \geq 4$. The most intriguing open problem is whether diameter-2 graphs can be 3-coloured in polynomial time. We therefore explicitly highlight the following problem.

▶ **Problem 14.** *Can 3-COLOURING be solved on diameter-2 graphs in quasi-polynomial time?*

We provided, for any $\varepsilon < 1/33$, an algorithm for 3-COLOURING on diameter-3 graphs that runs in time $2^{O(n^{2/3-\varepsilon})}$. The limiting factor on 1/33 in our proof occurs in the proof of Claim 12. Before our work, the exponent of 2/3 could have been a natural guess for the optimal exponent, with the ETH-based lower bound on the exponent of 1/2 by Mertzios and Spirakis [18] being another natural option.

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