Measuring Adjoint-invariance of Neighborhoods in Solvable Lie Groups

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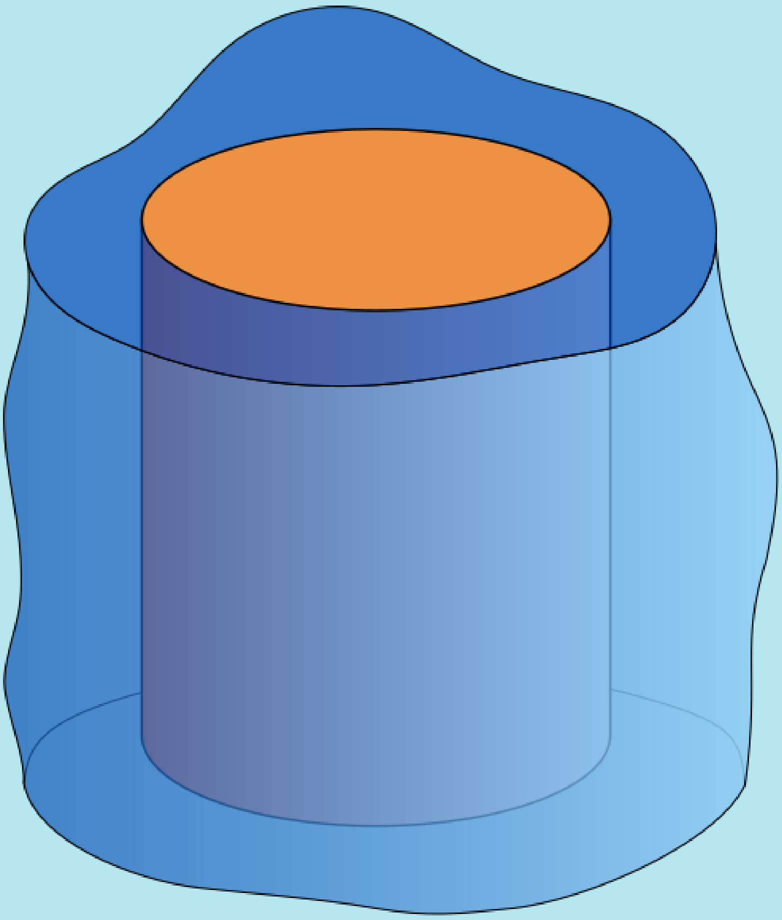

TU Delft

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by

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Abstract

In this thesis, we derive a lower bound on a quantity appearing in a Fourier multiplier inequality on solvable Lie groups. In [2], a classical result by de Leeuw about the restriction of Fourier multipliers on $\mathbb{R}^n$ to a discrete subgroup is extended to a noncommutative setting. It is shown that a compactly supported p-multiplier m on a locally compact group G has the following relation to its restriction to a discrete subgroup Γ:

$$c(\mathrm{supp}(m|_\Gamma))\left\|T_{m|_\Gamma}\right\|_p \leq \|T_m\|_p.$$

Here $c(U) = \inf\left\{\sqrt{\delta_F} \mid F \subseteq U \text{finite}\right\}$, where δ_F is a quantity that determines to what extent small neighborhoods of the identity in G are left invariant by conjugation by elements of F. In this thesis, we estimate δ_F for connected solvable Lie groups.

Our main result is theorem 9, which states that for a connected solvable Lie group G with Lie algebra $\mathfrak{g}$, if $\lambda_1, \ldots, \lambda_n \colon \mathfrak{g}_\mathbb{C} \to \mathbb{C}$ are the generalized weights of the complexification $\mathfrak{g}_\mathbb{C}$, there exist unique homomorphisms $\chi_1, \ldots, \chi_n \colon G \to \mathbb{R}_{>0}$ such that $\chi_i = \mathrm{d}\lambda_i$, and

$$\delta_F \geq \prod_{i=1}^{n} \inf_{g \in F} \chi_i(g).$$

Preface

In this thesis, I present the results of the research I have done over the period from January to September 2023. Having spent most of my education learning about the mathematics that has been built up over the last couple of centuries, I was excited to start this project to hopefully contribute something new to the field. During the process of writing this thesis, I gained valuable insights about doing mathematical research, both on an academic and personal level.

I am grateful to my supervisor, Bas Janssens, for his guidance, and invaluable input throughout the project. His expertise and mentorship have been instrumental in shaping this thesis into its final form. I would also like to thank Jan van Neerven and David de Laat for participating in the thesis committee.

Furthermore, I want to express my appreciation of my friends and family. Their support and encouragement have helped me through the challenges of I encountered while writing this thesis. In particular, I would like to express my gratitude to my friend Rik, who, as a fellow mathematics student, was frequently helpful in providing another perspective.

This work would not have been possible without the support and guidance of those in my academic and personal life. Through this thesis, I hope to have contributed some insights that are useful to those researching this topic further.

Benjamin Oudejans
Delft, September 2023

Contents

Introduction

In this thesis, we will be looking at a generalization of de Leeuw's theorem on Fourier multipliers to a noncommutative setting. We can generalize the Fourier transform on $\mathbb{R}^n$ to any locally compact abelian topological group G by means of Pontryagin duality, which will be explained in more detail in chapter 1. We denote the Fourier transform of $f\colon G \to \mathbb{R}$ by $\hat{f}$. Then we call a bounded continuous function $m\colon G \to \mathbb{R}$ a p-multiplier if the Fourier multiplier operator T_m, defined by $\widehat{T_m f} = m\hat{f}$ for all f in some set dense in $L^p(\widehat{G})$, extends to a bounded operator on $L^p(G)$. $\|T_m\|_p$ denotes the norm of this operator on $L^p(\widehat{G})$.

De Leeuw's theorem states for a discrete subgroup H of $\mathbb{R}^n$, if m is a continuous p-multiplier on $\mathbb{R}^n$, the restriction $m|_H$ is a p-multiplier on $\widehat{H}$ [11] [3], and $||T_m|_H|| \leq ||T_m||$.

In [2], an analogue of this theorem is proven for locally compact unimodular groups G, which we now allow to be non-abelian. It is possible to define Fourier multipliers in this more general setting as well, which will be described in chapter 1. We set Γ to be a discrete subgroup of G, that is, Γ is discrete in the topology of G. In [2], theorem A states that for a compactly supported continuous p-multiplier m,

$$c(\operatorname{supp}(m|_\Gamma))\left\|T_{m|_\Gamma}\right\|_p \leq \|T_m\|_p.$$

For any $U \subseteq G$, $c(U)$ is a number between 0 and 1 that quantifies how much the adjoint action of elements in U affects small neighborhoods of the identity in G.

For $c(U) = 1$ we have the strongest bound, which corresponds to a situation where the adjoint action of U almost leaves the small neighborhoods invariant. In general, this will not be the case, but for many types of Lie groups and multipliers, it is still possible to find a lower bound. In this thesis, we will investigate this for solvable Lie groups.

To define $c(U)$, we will need several intermediate quantities. For a neighborhood $V \subseteq G$ of the identity and a compact set $F \subseteq G$ we define

$$\delta_F(V) = \frac{\mu\left(\bigcap\limits_{s\in F} \operatorname{Ad}_s(V)\right)}{\mu(V)}$$

where μ is the Haar measure of G.

Then for a neighborhood basis $\mathcal{V}$ of the identity, we can define

$$\delta_F(\mathcal{V}) = \liminf_{V\in\mathcal{V}} \delta_F(V),$$

and

$$\delta_F = \sup\{\delta_F(\mathcal{V}) \mid \mathcal{V} \text{ symmetric neighborhood basis}\}.$$

Finally,

$$c(U) = \inf\left\{\sqrt{\delta_F} \mid F \subseteq U \text{ finite}\right\}.$$

We will look at the case where G is a Lie group. By the Levi decomposition, any Lie algebra is a semidirect product of a semisimple Lie algebra and a solvable one. For the semisimple case, theorem B in [2] gives a lower bound on δ_{B_ρ} for certain balls B_ρ in G. The nilpotent case is also solved as a special case of amenable groups in [3]. We give an alternative geometrical proof for the nilpotent case, and generalize this to a new result: a lower bound on δ_F for solvable Lie groups.

In chapter 1, we give an overview of the noncommutative analysis that we need to understand Fourier multipliers in this context. Chapter 2 covers some Lie group and algebra theory that we need. In chapter 3, we will show some properties of δ_F for Lie groups. Chapter 4 treats the known result for semisimple Lie groups. In chapter 5, we show how to compute δ_F for the Heisenberg group. In chapter 6, we prove that $\delta_F = 1$ for any compact F in a nilpotent Lie group. Lastly, in chapters 7 and 8, we will present our new result on δ_F for split-solvable Lie groups and solvable Lie groups respectively.

1

Noncommutative Fourier Analysis

The main focus of this thesis is estimating δ_F, which does not involve any noncommutative analysis. However, we are interested in δ_F because of its use in the noncommutative de Leeuw theorem. Therefore it is relevant to understand Fourier multipliers in the noncommutative case and what this result says about them. This chapter serves to give a brief overview of the theory of noncommutative Fourier analysis. This chapter is mostly a summary of chapter 2 in [2]. Some background information about von Neumann algebras is needed, for which we refer to [12].

1.1. Locally Compact Abelian Groups

Before we describe the noncommutative case, let us review the abelian case. If G is an abelian locally compact topological group, there is a natural way to generalize the Fourier transform on $\mathbb{R}^n$ to G. To define it we need two ingredients. The first is the Pontryagin dual of G. A character on G is a continuous homomorphism from G to the circle group $\mathbb{T}$. The set of all characters is called the Pontryagin dual and is denoted $\widehat{G}$. This is again a topological group, with pointwise multiplication and topology generated by uniform convergence on compact sets. The second ingredient is the Haar measure: every locally compact group has a regular measure μ which is left-invariant, meaning that $\mu(gV) = \mu(V)$ for any $g \in G$ and measurable $V \subseteq G$, and this measure is unique up to a scaling by a constant. Then for any $f \in L^1(G,\mu)$ and $\chi \in \widehat{G}$ we can define $\hat{f}(\chi) = \int_G f(x)\overline{\chi(x)}\,\mathrm{d}\mu(x)$. For locally compact abelian groups, Pontryagin's duality theorem states that

$$\widehat{\widehat{G}} \cong G,$$

with the isomorphism being the evaluation map $\mathrm{ev} : G \to \widehat{\widehat{G}}$: $\mathrm{ev}_x(\chi) = \chi(x)$. This allows us to construct an inverse Fourier transform as well. Indeed, for $f \in L^1(\widehat{G},\hat{\mu})$ we can set $\check{f}(x) = \int_{\widehat{G}} f(\chi)\mathrm{ev}_x(\chi)\,\mathrm{d}\hat{\mu}(\chi) = \int_{\widehat{G}} f(\chi)\chi(x)\,\mathrm{d}\hat{\mu}(\chi)$ for $x \in G$ [6].

For a function $m : \widehat{G} \to \widehat{G}$, we can define the Fourier multiplier T_m, by $\widehat{T_m f} = m\hat{f}$ for f in some function space such that the right hand side is well defined. If the operator extends to a bounded operator $L^p(G) \to L^p(G)$, we call m a p-multiplier. Then we define $\|m\|_p = \|T_m : L^p(G) \to L^p(G)\|$.

De Leeuw's theorem states that for a discrete subgroup H of $\mathbb{R}^n$, if m is a continuous p-multiplier on $\mathbb{R}^n$, the restriction $m|_H$ is again a p-multiplier on $\widehat{H}$, and $\|m\|_p = \|m|_H\|_p$ [11] [3].

1.2. Noncommutative Groups

In the noncommutative setting, the definitions become more complicated, and Pontryagin duality fails in general. A more detailed treatment can be found in [2].

We will now consider G to be a locally compact group, not necessarily abelian. The left regular representation $\lambda : G \to B(L^2(G))$ is defined as $(\lambda(s)g)(t) = g(s^{-1}t)$ for $g \in L^2(G)$, $s,t \in G$. Then for $f \in L^1(G)$ we can define the operator $\lambda(f) = \int_G f(s)\lambda(s)\,\mathrm{d}\mu(s)$ as a Bochner integral. This serves as the noncommutative equivalent of the Fourier transform. In the abelian case, the Fourier transform is given by an integral over the characters of G, weighted by the input function f. In this setting, the

characters are replaced by the translation operators $\lambda(s)$. Note that $\lambda(f)$ is an operator on $L^2(G)$, as opposed to the abelian case where the Fourier transform returns functions $\widehat{G} \to \mathbb{C}$.
The space $\mathcal{L}(G) = \overline{\text{span}}\left\{\lambda(f) \mid f \in L^1(G)\right\}$ will act as the noncommutative equivalent of $\widehat{G}$, where $\overline{\text{span}}$ is the strong closure of the span. For $1 \leq p < \infty$ there is a natural norm $||\cdot||_p$ on this space, which allows us to define $L_p(\widehat{G})$ as the completion of $\left\{x \in \mathcal{L}(G) \mid ||x||_p < \infty\right\}$. To define this norm we note that some elements in $\mathcal{L}(G)$ are convolution operators, meaning that for $x \in \mathcal{L}(G)$, there exists an $f \in L^2(G)$ such that $x\xi = f * \xi$ for all compactly supported continuous $\xi \colon G \to \mathbb{R}$. If so, this f is uniquely defined. We will denote such a convolution operator x_f. Then we define

$$\varphi_G(x^*x) = \begin{cases} ||f||_2 & x = x_f \text{ for some } f \in L^2(G) \\ \infty & \text{otherwise} \end{cases}.$$

If G is unimodular, then $\varphi_G(x^*x) = \varphi_G(xx^*)$ for all $x \in \mathcal{L}(G)$. Then we define our p-norm as $||x||_p = \varphi_G(|x|^p)^{1/p}$.
We can call a bounded continuous function $m : G \to \mathbb{C}$ a p-multiplier if there exists a bounded linear operator operator $T_m : L^p(\widehat{G}) \to L^p(\widehat{G})$ such that $T_m(\lambda(f)) = \lambda(mf)$ for f in a dense set where $\lambda(mf)$ is always well-defined. For operators on $L^p(\widehat{G})$ we will denote the operator norm by $||\cdot||_{L^p(\widehat{G})}$.
If G has the property that its Haar measure is also right-invariant, it is called *unimodular*. Furthermore, we call a subgroup $\Gamma \subset G$ *discrete* if it is discrete in the topology of G. Then we have the following result on p-multipliers on G:

Theorem 1 (Theorem A [2]). *If m is a compactly supported p-multiplier on a unimodular group G and $\Gamma \subset G$ is a discrete subgroup, then*

$$c(\text{supp}(m|_\Gamma))\left\|T_{m|_\Gamma}\right\|_{L^p(\widehat{\Gamma})} \leq \|T_m\|_{L^p(\widehat{G})}.$$

Here $c(U)$ is as defined in the introduction. Since $c(U)$ is defined as an infimum of $\sqrt{\delta_F}$ over certain compact sets F, we are interested in calculating or finding lower bounds on δ_F. In chapter 3, we will investigate properties of δ_F for Lie groups, and in further chapters use that to calculate δ_F in nilpotent Lie groups and show a lower bound for solvable Lie groups in terms of F.

2

Lie Groups

In this chapter, we briefly cover some of the basic theory of Lie groups, and establish some definitions and results that we will need in the rest of this thesis.

Definition 1. A Lie group is a smooth manifold G with a group structure $(G, \cdot)$, where both multiplication $\cdot\colon G \times G \to G$ and inversion $\cdot^{-1} : G \to G$ are smooth maps.

We will only consider real Lie groups in this thesis, and all Lie algebras will be real unless stated otherwise. A closely related structure is the Lie algebra:

Definition 2. A Lie algebra is a vector space $\mathfrak{g}$ with a bilinear operation $[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ that satisfies the following conditions:

1. $[x, x] = 0$ for all $x \in \mathfrak{g}$
2. $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$ for all $x, y, z \in \mathfrak{g}$.

There are several important classes of Lie algebras. One is the class of *nilpotent* Lie algebras. For a Lie algebra $\mathfrak{g}$ we define its *lower central series* by $\mathfrak{g}^0 = \mathfrak{g}$, and $\mathfrak{g}^{k+1} = [\mathfrak{g}, \mathfrak{g}^k]$. If this series terminates, i.e. $\mathfrak{g}^k = \{0\}$ for some k, then $\mathfrak{g}$ is called nilpotent. In chapter 5, we will present an example of a nilpotent Lie algebra: the Heisenberg algebra. In particular, *abelian* Lie algebras, those that satisfy $[\mathfrak{g}, \mathfrak{g}] = \{0\}$, are nilpotent.
There is also the broader class of *solvable* Lie algebras. For this class, we define the derived series by $\mathfrak{g}^{(0)} = \mathfrak{g}$ and $\mathfrak{g}^{(k+1)} = [\mathfrak{g}^{(k)}, \mathfrak{g}^{(k)}]$. If $\mathfrak{g}^{(k)} = \{0\}$ for some k, $\mathfrak{g}$ is called solvable. Since $\mathfrak{g}^{(k)} \subseteq \mathfrak{g}^k$ for all k, we see that solvability implies nilpotency.

Another important class of Lie algebras are the *semisimple* ones. An ideal $\mathfrak{a}$ of a Lie algebra $\mathfrak{g}$ is a linear subspace such that $[\mathfrak{g}, \mathfrak{a}] \subseteq \mathfrak{a}$. An ideal is itself again a Lie algebra, hence it makes sense to consider solvable ideals. A semisimple Lie algebra is one that has no nontrivial solvable ideals. A *reductive* Lie algebra is one where all solvable ideals are abelian.
Solvable and semisimple Lie algebras are particularly important classes, since any Lie algebra $\mathfrak{g}$ over a field of characteristic 0 can be decomposed as a semidirect product $\mathfrak{g} = \mathfrak{s} \ltimes \mathfrak{r}$ where $\mathfrak{s}$ is semisimple and $\mathfrak{r}$ is solvable. This is known as the Levi decomposition.

2.1. Relation between Lie groups and algebras

A Lie group has an associated Lie algebra, which is the space of all smooth left-invariant vector fields: vector fields X such that $\mathrm{d}L_g(X_h) = X_{gh}$ for all $g, h \in G$, where L_g denotes left-multiplication by g. The set of all left-invariant vector fields forms a real vector space. If we view tangent vectors of G as derivations, then for any smooth vector fields X, Y we can define a new vector field $[X, Y]$ by $[X, Y](f) = X(Y(f)) - Y(X(f))$ for all smooth $f\colon G \to \mathbb{R}$. If X and Y are left-invariant, then so is $[X, Y]$, and it is easily verified that the bracket $[\cdot, \cdot]$ satisfies the properties in definition 2. Hence the set of left-invariant vector fields forms a Lie algebra, denoted $\mathrm{Lie}(G)$.
Left-invariant vector fields are uniquely determined by their vector at the identity e, since all other

vectors follow from $X_g = \mathrm{d}L_g(X_e)$. Therefore, it is common to view $\mathrm{Lie}(G)$ as T_eG, the tangent space at the identity.
One important function is the exponential map $\exp\colon \mathfrak{g} \to G$. For any $x \in \mathfrak{g}$, there is a unique *one-parameter group*, that is a homomorphism $\varphi\colon \mathbb{R} \to G$, such that $\varphi'(0) = x$. Then $\exp(x)$ is defined as $\varphi(1)$. This map gives us a way to relate the Lie algebra to the Lie group, although not in a one-to-one way in general. It is always a local diffeomorphism though, meaning that around any point we can find a neighborhood such that $\exp$ restricts to an injective map.

2.1.1. Homomorphisms

If G and H are Lie groups with Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$ respectively, we call a smooth group homomorphism $\Phi : G \to H$ a *Lie group homomorphism*. These are closely related to Lie algebra homomorphisms, which are linear maps $\phi : \mathfrak{g} \to \mathfrak{h}$ such that $\phi([x, y]) = [\phi(x), \phi(y)]$ for all $x, y \in \mathfrak{g}$.
Any homomorphism of Lie groups induces a Lie algebra homomorphism between the associated Lie algebras: if $\Phi\colon G \to H$ is a Lie group homomorphism, then $\mathrm{d}\Phi_e : \mathfrak{g} \to \mathfrak{h}$ is a Lie algebra homomorphism. Furthermore, $\Phi \circ \exp = \exp \circ \, \mathrm{d}\Phi_e$ [9]. Since we always consider differentials at e, we will leave out the subscript e in future chapters.

Conversely, we would hope that for every Lie algebra homomorphism ϕ, there exists a unique Lie group homomorphism such that $\mathrm{d}\Phi_e = \phi$. However, this is not quite true. Suppose Φ is a nontrivial homomorphism from the circle group $\mathbb{T} \cong \mathbb{R}/\mathbb{Z}$ to the additive group $\mathbb{R}$. Then by continuity we can find a nonzero rational number p/q with $p \in \mathbb{Z}$ and $q \in \mathbb{N}$ such that $\Phi(p/q) \neq 0$. But $0 = \Phi(0) = \Phi(p) = \Phi(q \cdot p/q) = q \cdot \Phi(p/q) \neq 0$. Hence there are no nontrivial homomorphisms. However, both $\mathbb{T}$ and $\mathbb{R}$ have $\mathbb{R}$ as their Lie algebra, and any linear function $\mathbb{R} \to \mathbb{R}$ is a Lie algebra . Thus the correspondence between homomorphisms breaks down, as there are no Lie group homomorpisms corresponding to the nontrivial Lie algebra homomorphisms.

The Lie group homomorphisms are constrained by the fact that they have to invariant modulo $\mathbb{Z}$. Essentially, there are different non-homotopic paths from 0 to any point in $\mathbb{T}$, as it is possible to loop around multiple times. But since the Lie algebra only captures the behavior of $\mathbb{T}$ around 0, this restriction does not apply to the Lie algebra. This discrepancy disappears when we demand our Lie group to be simply connected, as any two paths between two points are homotopic in such a space. Indeed, the following result holds:

Theorem 2. *[13, Thm 7.13] Let G, H be Lie groups with Lie algebras $\mathfrak{g}, \mathfrak{h}$ respectively. Suppose G is simply connected. Then for any homomorphism $\phi\colon \mathfrak{g} \to \mathfrak{h}$, there exists a unique homomorphism $\Phi\colon G \to H$ such that $\mathrm{d}\Phi_e = \phi$*

However, in this thesis we are mostly concerned with connected, but not necessarily simply connected Lie groups. In this case existence of a Lie group homomorphism that integrates a Lie algebra homomorphism is not guaranteed, but if it exists, it is unique:

Theorem 3. *[13, Prop 7.8] Let G, H be Lie groups with Lie algebras $\mathfrak{g}, \mathfrak{h}$ with G connected. Then if $\Phi, \Psi\colon G \to H$ are homomorphism such that $\mathrm{d}\Phi_e = \mathrm{d}\Psi_e$, then $\Phi = \Psi$*

2.1.2. The adjoint representation

A Lie group acts on its Lie algebra with the adjoint representation, which is the differential of conjugation in the Lie group. For any $g \in G$, we can define the conjugation map $C_g\colon G \to G$ by $C_g(h) = ghg^{-1}$. We then define $\mathrm{Ad}_g = \mathrm{d}C_g$. Since C_g is a diffeomorphism, $\mathrm{Ad}_g \in GL(\mathfrak{g})$. In fact, $\mathrm{Ad}\colon G \to GL(\mathfrak{g})$ is a representation of G.

Lie groups also have an adjoint representation, denoted $\mathrm{ad}\colon \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$, defined by $\mathrm{ad}_x(y) = [x, y]$ for $x, y \in \mathfrak{g}$. It turns out that $\mathrm{ad} = \mathrm{d}\,\mathrm{Ad}$, so for any $x \in \mathfrak{g}$, $\mathrm{Ad}_{\exp(x)} = \exp(\mathrm{ad}_x)$. This is a useful relation that we will frequently use to investigate the relation between the structure of a Lie algebra and the adjoint action of the Lie group on it.

Conjugation is sometimes also referred to as the adjoint action, and C_g can be denoted Ad_g as well. Since this Ad_g takes elements of a Lie group as its argument, it can be distinguished from the adjoint representation of G on $\mathfrak{g}$. In further chapters, we will use this notation to be consistent with the previous literature about δ_F.

2.2. Haar and Lebesgue measures

As discussed in chapter 1, every locally compact group has a Haar measure. In the special case of Lie groups, the Haar measure is generated by a volume form: for any Lie group G we can find a left-invariant volume form Vol_G which is unique up to a constant multiple [10, Prop 16.10]. We find that the Haar measure μ on G corresponds to integration against this form, since for $g \in G$ and $V \subseteq G$,

$$\mu(gV) = \int_G 1_{gV} \mathrm{Vol}_G = \int L^*_{g^{-1}}(1_V \mathrm{Vol}_G) = \int 1_V \mathrm{Vol}_G = \mu(V).$$

The Haar measure and left-invariant volume form are only defined up to a constant.

Let $\mathfrak{g}$ be the Lie algebra of G. Since $\mathfrak{g}$ is a real vector space, we can measure subsets with the Lebesgue measure Λ, which corresponds to a constant volume form $\mathrm{Vol}_\mathfrak{g}$. This is again only defined up to a constant, but we will always choose the volume forms so that $(\exp^* \mathrm{Vol}_G)_0 = (\mathrm{Vol}_\mathfrak{g})_0$.

Proposition 1. *A Lie group G is unimodular if and only if $|\det(\mathrm{Ad}_g)| = 1$ for all $g \in G$*

Since we will be looking at real connected Lie groups, this means that $\det(\mathrm{Ad}_g) = 1$, since $\det(\mathrm{Ad}_e) = 1$, so by continuity, it must be 1 on the entire connected component.

If G is connected, we can also define unimodularity in terms of its Lie algebra $\mathfrak{g}$. Indeed, for a linear operator A we have $\det(\exp(A)) = e^{\mathrm{Tr}(A)}$ [1], so for $A = \mathrm{ad}_x$ for $x \in \mathfrak{g}$, we obtain

$$e^{\mathrm{Tr}(\mathrm{ad}_x)} = \det(\exp(\mathrm{ad}_x)) = \det\left(\mathrm{Ad}_{\exp(x)}\right).$$

Since $\Delta \colon g \mapsto \det(\mathrm{Ad}_g)$ is a homomorphism, the previous relation shows that $d\Delta(x) = \mathrm{Tr}(\mathrm{ad}_x)$. If G is connected, then as a consequence of theorem 3, Δ is trivial if and only if $d\Delta$ is, hence $\mathrm{Tr}(\mathrm{ad}_x) = 0\ \forall x \in \mathfrak{g}$ if and only if G is unimodular.

Definition 3. A Lie group is called nilpotent/solvable if it is connected and its Lie algebra is nilpotent/solvable.

Note that we require the Lie group to be connected for this definition. For other kinds of Lie group, in particular reductive Lie groups, the definition typically does not require this. To prevent any ambiguity, we state the connectedness assumption explicitly in our main results.

It turns out that many kinds of Lie groups are unimodular. Compact Lie groups are unimodular since $\Delta(G)$ is a compact subgroup of $\mathbb{R}_{>0}$, hence it must equal $\{1\}$. Nilpotent Lie groups are always unimodular, since for $x \in \mathfrak{g}$, ad_x is nilpotent and thus only has 0 as its eigenvalues. Therefore $\mathrm{Tr}(\mathrm{ad}_x) = 0\ \forall x \in \mathfrak{g}$.

Connected semisimple Lie groups are also unimodular. $d\Delta(x) = \mathrm{Tr}(\mathrm{ad}_x)$ is a Lie algebra homomorphism from $\mathfrak{g}$ to the abelian Lie algebra $\mathbb{R}$. Therefore $[\mathfrak{g}, \mathfrak{g}] \subseteq \ker(\Delta)$, but $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$, so $d\Delta = 0$.
It is easily verified that direct products of connected unimodular groups are again unimodular, in particular connected reductive Lie groups are unimodular. Solvable Lie groups are, however, not unimodular in general. This will be treated in more detail in chapter 7.

2.3. Connected Lie groups

While a Lie algebra does not contain all the information about the Lie group it comes from, a lot can be said about the Lie group if we know that it is connected. The following proposition gives one such result.

Proposition 2. *If G is a connected Lie group with Lie algebra $\mathfrak{g}$, then $\exp(\mathfrak{g})$ generates G.*

Proof. Proposition A4.25 in [7] states that if G is a topological group with subgroup H, then H is open if and only if it contains a nonempty open subset of G, and in that case it contains the identity component G_0 of G. The exponential map is a local diffeomorphism and thus an open map [10, Prop 4.6c]. Hence $\exp(\mathfrak{g})$ is open, thus the group it generates contains G_0, which equals G by connectedness. □

For solvable Lie groups we can make an even stronger statement:

Proposition 3. *[5] If G is a connected solvable Lie group with Lie algebra $\mathfrak{g}$, then $\exp(\mathfrak{g})$ is dense in G.*

[1] This is easily verified for diagonalizable matrices, the general case follows by density

3

Properties of δ_F for Lie groups

In this chapter, we will show some of the basic properties of δ_F.

Since δ_F indicates how well the adjoint action preserves neighborhoods of the identity, we would expect that $\delta_F \leq 1$. If $e \in F$, this is immediately true since $\mathrm{Ad}_e = I$, hence $\bigcap_{g\in F} \mathrm{Ad}(V) \subseteq V$. However, this assumption will not be necessary. For unimodular groups, we also have $\delta_F \leq 1$. Indeed, since the Haar measure is then also right-invariant, we have $\mu(\mathrm{Ad}_g(V)) = \mu(gVg^{-1}) = \mu(V)$ for all $g \in G$ and measurable $V \subseteq G$. Hence for $F \subseteq G$ nonempty, $\bigcap_{g\in F} \mathrm{Ad}_g(V)$ is contained in $\mathrm{Ad}_s(V)$ for any $s \in F$, hence its measure cannot be more than $\mu(\mathrm{Ad}_s(V)) = \mu(V)$. In other words, $\delta_F(V) \leq 1$. As a result, the same holds for δ_F.

The following is also useful to note:

Proposition 4. *If $F \subseteq F'$, then $\delta_F \geq \delta_{F'}$*

Proof. For any $V \subseteq G$, $\bigcap\limits_{g\in F'} \mathrm{Ad}_g(V) \subseteq \bigcap\limits_{g\in F} \mathrm{Ad}_g(V)$, hence $\delta_F(V) \geq \delta_{F'}(V)$. Therefore also $\delta_F \geq \delta_{F'}$. □

3.1. Defining δ_F in terms of the Lie algebra

The adjoint action and Haar measure of a Lie group can be difficult to work with, making it challenging to compute δ_F. Therefore, we will work in the Lie algebra instead, and use the Lebesgue measure Λ to define equivalent δ quantities. We will show that under the exponential correspondence between G and its Lie algebra, the Lebesgue measure approximates the Haar measure around the identity, so the results from the Lie algebra can be transferred to the Lie group.

Let $\mathfrak{g}$ be the Lie algebra of G. For $V \subseteq \mathfrak{g}$ and $F \subseteq G$, we define

$$\delta_F^0(V) = \frac{\Lambda\left(\bigcap\limits_{s\in F} \mathrm{Ad}_s(V)\right)}{\Lambda(V)}.$$

For a neighborhood basis $\mathcal{V}$ of 0 in $\mathfrak{g}$, we set

$$\delta_F^0(\mathcal{V}) = \liminf_{V\in\mathcal{V}} \delta_F^0(V),$$

and

$$\delta_F^0 = \sup\{\delta_F^0(\mathcal{V}) \mid \mathcal{V} \text{ symmetric neighborhood basis}\}.$$

We will show that δ_F and δ_F^0 are the same. First we will need a lemma to establish the relationship between the Haar measure of G and the Lebesgue measure of $\mathfrak{g}$. Note that we defined the normalization of these measures such that $\exp^* \mathrm{Vol}_G = \mathrm{Vol}_{\mathfrak{g}}$.

Lemma 1. *Let G be a locally compact Lie group with Haar measure μ, and let $\mathfrak{g}$ be its Lie algebra with Lebesgue measure Λ. Then for all $\varepsilon > 0$, there exists an open neighborhood U of 0 in $\mathfrak{g}$ such that for all measurable $V \subseteq U$, $|\Lambda(V) - \mu(\exp(V))| < \varepsilon\Lambda(V)$.*

Proof. Since $\exp^* \mathrm{Vol}_G$ and $\mathrm{Vol}_{\mathfrak{g}}$ are both non-vanishing top forms on $\mathfrak{g}$, there exists a smooth function $f\colon \mathfrak{g} \to \mathbb{R}$ such that $\mathrm{Vol}_G = f\,\mathrm{Vol}_{\mathfrak{g}}$. Because the forms agree at 0, we have $f(0) = 1$. Furthermore, by continuity we can find a neighborhood W of 0 such that $|f(x)-1| < \varepsilon$ for all $x \in W$. Furthermore, since $\exp : \mathfrak{g} \to G$ is a local diffeomorphism, we can find a neighborhood U of 0 such that $\exp : U \to \exp(U)$ is a diffeomorphism. Without loss of generality, we can assume $U \subseteq W$.

Note that for $V \subseteq U$

$$\begin{aligned}\mu(\exp(V)) &= \int\limits_{\exp(U)} 1_{\exp(V)} \mathrm{Vol}_G \\ &= \int\limits_U \exp^*\left(1_{\exp(V)} \mathrm{Vol}_G\right) \\ &= \int\limits_U 1_V \exp^* \mathrm{Vol}_G \\ &= \int\limits_U 1_V f\,\mathrm{Vol}_{\mathfrak{g}}.\end{aligned}$$

Hence

$$\begin{aligned}|\mu(\exp(V)) - \Lambda(V)| &= \left|\int\limits_U 1_V(f-1)\,\mathrm{Vol}_{\mathfrak{g}}\right| \\ &\leq \int\limits_U 1_V|f-1|\,\mathrm{Vol}_{\mathfrak{g}} \\ &\leq \varepsilon \int\limits_U 1_V \,\mathrm{Vol}_{\mathfrak{g}} \\ &= \varepsilon\Lambda(V).\end{aligned}$$

□

In [2], a variant of this lemma is shown for G semisimple. An explicit formula for f is given, which is shown to be adjoint-invariant. Hence the estimate can be extended to subsets of $\mathrm{Ad}_G(U)$. This is a necessary step in the semisimple case, but we will not need it.
We will now use the previous lemma to show that δ_F and δ_F^0 are equal.

Proposition 5. *Let G be a locally compact Lie group and let $F \subseteq G$ compact. Then $\delta_F = \delta_F^0$*

Proof. First we will show that for a neighborhood basis $\mathcal{V}$ around 0 in $\mathfrak{g}$, $\delta_F^0(\mathcal{V}) = \delta_F(\exp(\mathcal{V}))$

Indeed, let $\varepsilon > 0$. Then by lemma 1, we can find a neighborhood U of 0 such that for all measurable $V \subset U$, $|\mu(\exp(V)) - \Lambda(V)| < \varepsilon\Lambda(V)$
Without loss of generality, we can take U to be small enough such that $\exp|_U$ is a diffeomorphism onto its image and such that $\Lambda(U) \leq 1$.
Since F is compact, $M := \sup_{g\in F} \|\mathrm{Ad}_g\|$ is finite. Therefore we can find an index N_ε such that for all $i \geq N_\varepsilon$, $M \cdot V_i \subseteq U$. Now let $i \geq N_\varepsilon$. Then $\mathrm{Ad}_g(V_i) \subseteq M \cdot V_i \subset U$ for all $g \in F$. Since $\exp$ is injective on U, it follows that

$$\bigcap_{g\in F} \mathrm{Ad}_g(\exp(V_i)) = \bigcap_{g\in F} \exp\big(\mathrm{Ad}_g(V_i)\big) = \exp\left(\bigcap_{g\in F} \mathrm{Ad}_g(V_i)\right).$$

[1] Since both V_i and $\bigcap_{g\in F} \mathrm{Ad}_g(\exp(V_i)$ are subsets of U, we can apply lemma 1 to both. For V_i, we obtain $|\mu(\exp(V_i)) - \Lambda(V_i)| < \varepsilon\Lambda(V_i) \leq \varepsilon\Lambda(U) \leq \varepsilon$, hence

$$(1-\varepsilon)\Lambda(V_i) \leq \mu(\exp(V_i)) \leq (1+\varepsilon)\Lambda(V_i).$$

[1]If $f\colon A \to B$ is injective and $(A_g)_{g\in I}$ are sets in A, then $f\left(\bigcap_{g\in I} A_i\right) = \bigcap_{g\in I} f(A_g)$ This does not work in general if f is only injective on $\bigcap_{g\in I} A_g$, hence the need to take V_i small enough that $\mathrm{Ad}_g(V_i)$ are all in the injectivity radius of $\exp$.

For $\bigcap_{g\in F} \mathrm{Ad}_g(\exp(V_i))$, we find

$$\begin{aligned}\left|\mu\left(\bigcap_{g\in F}\mathrm{Ad}_g(\exp(V_i))\right)-\Lambda\left(\bigcap_{g\in F}\mathrm{Ad}_g(V_i)\right)\right| &= \left|\mu\left(\bigcap_{g\in F}\mathrm{Ad}_g(\exp(V_i))\right)-\Lambda\left(\bigcap_{g\in F}\mathrm{Ad}_g(V_i)\right)\right| \\ &< \varepsilon\Lambda\left(\bigcap_{g\in F}\mathrm{Ad}_g(V_i)\right) \le \varepsilon\Lambda(U)\le\varepsilon,\end{aligned}$$

so

$$(1-\varepsilon)\Lambda\left(\bigcap_{g\in F}\mathrm{Ad}_g(V_i)\right) \le \mu\left(\bigcap_{g\in F}\mathrm{Ad}_g(\exp(V_i))\right) \le (1+\varepsilon)\Lambda\left(\bigcap_{g\in F}\mathrm{Ad}_g(V_i)\right).$$

Now we let $A_V = \Lambda\left(\bigcap_{g\in F}\mathrm{Ad}_g(V)\right)$, $B_V = \mu\left(\bigcap_{g\in F}\mathrm{Ad}_g(\exp(V))\right)$, $C_V = \Lambda(V)$, and $D_V = \mu(\exp(V))$. The above conditions then become $(1-\varepsilon)C_V \le D_V \le (1+\varepsilon)C_V$ and $(1-\varepsilon)A_V \le B_V \le (1+\varepsilon)A_V$. Then

$$\begin{aligned}|\delta_F^0(\mathcal{V})-\delta_F(\exp(\mathcal{V}))| &= \left|\liminf_{V\in\mathcal{V}}\frac{A_V}{C_V}-\liminf_{V\in\mathcal{V}}\frac{B_V}{D_V}\right| \\ &= \liminf_{V\in\mathcal{V}}\left|\frac{A_V}{C_V}-\frac{B_V}{D_V}\right|\end{aligned}$$

Hence if we let $\varepsilon>0$ and let $V=V_i$ for some $i\ge N_\varepsilon$, we have

$$\frac{A_V}{C_V}-\frac{B_V}{D_V}\le\frac{A_V}{C_V}-\frac{(1-\varepsilon)A_V}{(1+\varepsilon)C_V}=\left(1-\frac{1-\varepsilon}{1+\varepsilon}\right)\frac{A_V}{C_V}=\frac{2\varepsilon}{1+\varepsilon}\frac{A_V}{C_V}\le\frac{2\varepsilon}{1-\varepsilon}\frac{A_V}{C_V}$$

and

$$\frac{A_V}{C_V}-\frac{B_V}{D_V}\ge\frac{A_V}{C_V}-\frac{(1+\varepsilon)A_V}{(1-\varepsilon)C_V}=\left(1-\frac{1+\varepsilon}{1-\varepsilon}\right)\frac{A_V}{C_V}=\frac{-2\varepsilon}{1-\varepsilon}\frac{A_V}{C_V},$$

hence $\left|\frac{A_V}{C_V}-\frac{B_V}{D_V}\right|\le\frac{2\varepsilon}{1-\varepsilon}\frac{A_V}{C_V}$.

Since the liminf does not depend on V_i for $i\le N_\varepsilon$, it follows that

$$|\delta_F^0(\mathcal{V})-\delta_F(\exp(\mathcal{V}))|=\liminf_{V\in\mathcal{V}}\left|\frac{A_V}{C_V}-\frac{B_V}{D_V}\right|\le\liminf_{V\in\mathcal{V}}\frac{2\varepsilon}{1-\varepsilon}\frac{A_V}{C_V}=\frac{2\varepsilon}{1-\varepsilon}\delta_F^0(\mathcal{V}).$$

This holds for any $\varepsilon>0$, hence we must have $\delta_F^0(\mathcal{V})=\delta_F(\exp(\mathcal{V}))$.

We now know $\delta_F^0(\mathcal{V}) = \delta_F(\exp(\mathcal{V}))$ for any neighborhood basis $\mathcal{V}$ around 0. Note that if $\mathcal{V}$ is a symmetric neighborhood basis in $\mathfrak{g}$, then $\exp(\mathcal{V})$ is a symmetric neighborhood basis in G. Conversely, if $\mathcal{W}$ is a symmetric neighborhood basis in G, then $\mathcal{V}=\exp^{-1}(\mathcal{W})$ is a symmetric neighborhood basis in $\mathfrak{g}$. Eventually, $V\in\mathcal{V}$ become small enough to be in the injectivity radius of $\exp$, meaning that $\exp(V_i) = W_i$ for i large enough. Hence $\delta_F^0(\mathcal{V}) = \delta_F(\exp(\mathcal{V})) = \delta_F(\mathcal{W})$. So any value that can be attained by $\delta_F(\mathcal{W})$, can also be attained by a $\mathcal{W}$ that is the exponential of a symmetric neighborhood basis from the Lie algebra. Hence $\delta_F(\exp(\mathcal{V}')) = \delta_F(\mathcal{W}') = \delta_F(\mathcal{W})$. Hence

$$\begin{aligned}\delta_F &= \sup\{\delta_F(\mathcal{W}) \mid \mathcal{W}\text{ symmetric neighborhood basis in } G\} \\ &= \sup\{\delta_F(\exp(\mathcal{V})) \mid \mathcal{V}\text{ symmetric neighborhood basis in } \mathfrak{g}\} \\ &= \sup\{\delta_F^0(\mathcal{V}) \mid \mathcal{V}\text{ symmetric neighborhood basis in } \mathfrak{g}\} \\ &= \delta_F^0\end{aligned}$$

□

4

Reductive Lie Groups

Before we investigate any nilpotent or solvable Lie groups, we briefly cover the known result on δ_F for reductive Lie groups. For reductive Lie groups, a lower bound on δ_{B_ρ} is shown in [2], where B_ρ are certain balls in the Lie group G. Any reductive Lie algebra has an invariant bilinear form $B\colon \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$. Invariance means that $B([x,y],z) = B(x,[y,z])$ for all $x, y, z \in \mathfrak{g}$. Furthermore, there exists an involution $\theta\colon \mathfrak{g} \to \mathfrak{g}$ called the Cartan involution that makes $B_\theta(x,y) = -B(x,\theta(y))$ into an inner product. This induces a norm on $\mathfrak{g}$, which in turn induces the operator norm $\mathfrak{gl}(\mathfrak{g})$. Then we can define $B_\rho = \left\{g \in G \mid \left\|\mathrm{Ad}_g\right\| \leq \rho\right\}$ for $\rho > 1$.

A Lie group G acts on points in its Lie algebra $\mathfrak{g}$ with the adjoint representation, and this is a group action. Hence any point $x \in \mathfrak{g}$ generates an orbit

$$\mathcal{O}_x = \left\{\mathrm{Ad}_g(x) \mid g \in G\right\}.$$

An element $x \in \mathfrak{g}$ is called nilpotent if ad_x is nilpotent as a linear operator. If $\mathfrak{g}$ is reductive and x is nilpotent, then all elements of $\mathcal{O}_x$ are nilpotent, hence this is called a *nilpotent orbit*. There is a unique nilpotent orbit of highest dimension, whose closure is the *nilpotent cone*, which is the set of all nilpotent elements.[4]

Theorem B in [2] states that

$$\delta_{B_\rho} \geq \rho^{-d/2},$$

where d is the dimension of the highest dimensional nilpotent orbit. Nilpotent orbits can be made into symplectic manifolds by endowing them with the KKS-form[4]. This means that nilpotent orbits always have even dimension, so that the exponent in the lower bound is always an integer.

5

Heisenberg group

Now that we have shown some results about δ_F, we can show an example of how to compute it, and what conclusion we can draw with theorem 1.

5.1. Preliminaries

Definition 4 (Heisenberg group)**.** The Heisenberg group H is defined as

$$\left\{ \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \mid a, b, c \in \mathbb{R} \right\}$$

With matrix multiplication as the group operation.

There are several different definitions of the Heisenberg group in the literature, though this one is convenient for our purposes as it is a matrix Lie group.

It is not difficult to see that H is a closed subgroup of $GL(3, \mathbb{R})$, and is therefore a matrix Lie group.

Definition 5. The Heisenberg algebra $\mathfrak{h}$ is the Lie algebra

$$\left\{ \begin{pmatrix} 0 & x & z \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix} \mid x, y, z \in \mathbb{R} \right\}$$

with the standard commutator

We will often identify $\mathfrak{h}$ with $\mathbb{R}^3$ using the standard basis

$$X = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \; Y = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \; Z = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

so (x, y, z) will be shorthand for $xX + yY + zZ$.

The commutation relations of this basis are $[X, Y] = Z$, $[X, Z] = 0$, $[Y, Z] = 0$. Thus $[\mathfrak{h}, \mathfrak{h}] = \operatorname{span}\{Z\}$ and $[\mathfrak{h}, [\mathfrak{h}, \mathfrak{h}]] = \{0\}$, hence $\mathfrak{h}$ is nilpotent.

If we consider a path of the form $\gamma(t) = \begin{pmatrix} 1 & at & ct \\ 0 & 1 & bt \\ 0 & 0 & 1 \end{pmatrix}$ through H, we see that $\gamma'(0) = \begin{pmatrix} 0 & a & c \\ 0 & 0 & b \\ 0 & 0 & 0 \end{pmatrix}$. Furthermore, if γ is any path in H, we see that $\gamma'(0) \in \mathfrak{h}$ since the only non-constant matrix entries are strictly upper diagonal. Therefore, $\mathfrak{h}$ is the Lie algebra of H.

5.2. Adjoint action and exponential

Since H is a matrix Lie group, it is easy to compute the adjoint action of H on $\mathfrak{h}$.
If $g = \left(\begin{smallmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{smallmatrix}\right) \in H$, we have $g^{-1} = \left(\begin{smallmatrix} 1 & -a & ab-c \\ 0 & 1 & -b \\ 0 & 0 & 1 \end{smallmatrix}\right)$. Therefore, for $A = \left(\begin{smallmatrix} 0 & x & z \\ 0 & 0 & y \\ 0 & 0 & 0 \end{smallmatrix}\right) \in \mathfrak{h}$, we have

$$\mathrm{Ad}_g(A) = gAg^{-1} = \begin{pmatrix} 0 & x & z + ay - bx \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix}$$

To make notation more compact, we will denote $H(a, b, c) = \left(\begin{smallmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{smallmatrix}\right)$. Then $\mathrm{Ad}_{H(a,b,c)}((x, y, z)) = (x, y, z + ay - bx)$.

For matrix Lie groups, the exponential map is given by the infinite series formula $\exp(A) = \sum\limits_{k=0}^{n} \frac{1}{k!} A^k$, hence

$$\exp((x, y, z)) = \exp \begin{pmatrix} 0 & x & z \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix} = I + \begin{pmatrix} 0 & x & z \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 0 & 0 & xy \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = (x, y, z + \frac{1}{2} xy).$$

5.3. Computing $c(H)$

In this section, we show that $c(H) = 1$.

In order to do this, we will show that $\delta_F^0(\mathcal{V}) = 1$ for a particular choice of $\mathcal{V}$, where F will range over a family of compact sets that cover H. Our neighborhood basis will consist of cylinders: define $C_{r,h} := \{(x, y, z) \in \mathfrak{h} \mid x^2 + y^2 \le r^2, |z| \le h\}$, as depicted in figure 5.1(a). Then we define the neighborhood basis $\mathcal{C} = \{C_{h^2,h} \mid h > 0\}$.
For F, we will consider the sets $F_\rho := \{H(a, b, c) \mid a^2 + b^2 \le \rho^2, |c| < \rho\}$ on $C_{h^2,h}$. We note that the bound on c is only there to ensure compactness, it has no effect on any computations.

Note that the adjoint action of H only affects the z coordinate of a point in $\mathfrak{h}$. Since we chose F_ρ conveniently, we can easily bound how much a point $A = (x, y, z) \in \mathfrak{h}$ is shifted by the adjoint action of any $s = H(a, b, c) \in F_\rho$. Indeed, since $\mathrm{Ad}_s(A) = (x, y, z + ay - bx)$, we see that the first two coordinates are unaffected, while the last coordinate is shifted by $ay - bx$. By the reverse triangle inequality, we see that $|z + ay - bx| \ge |z| - |ay - bx|$. Hence, if we can bound $|ay - bx|$ in terms of ρ, x and y as a, b range over F_ρ, this tells us how much by how much the third coordinate will shrink due to the adjoint action.

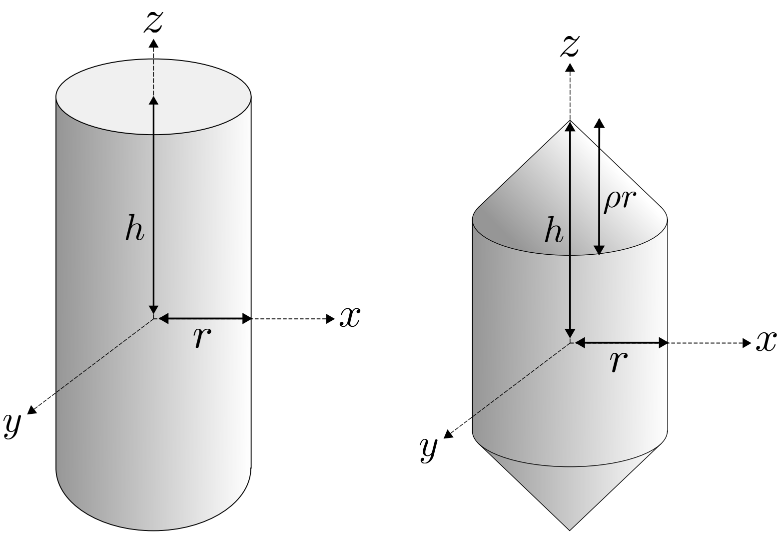
We find that

$$\begin{aligned} |ay - bx| &= \left| \begin{pmatrix} a \\ -b \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} \right| \\ &\le \left\| \begin{pmatrix} a \\ -b \end{pmatrix} \right\| \cdot \left\| \begin{pmatrix} x \\ y \end{pmatrix} \right\| \text{ by Cauchy-Schwarz} \\ &= \sqrt{a^2 + b^2} \cdot \sqrt{x^2 + y^2} \\ &\le \rho \sqrt{x^2 + y^2} \text{ by definition of } F_\rho \end{aligned}$$

Thus $-\rho\sqrt{x^2 + y^2} \le ay - bx \le \rho\sqrt{x^2 + y^2}$, and for $\begin{pmatrix} a \\ -b \end{pmatrix} = \pm \frac{\rho}{\sqrt{x^2+y^2}} \begin{pmatrix} x \\ y \end{pmatrix}$ the minimum and maximum are attained. Since we look at the intersection of $\mathrm{Ad}_s(C_{r,h})$ over all $s \in F_\rho$, the maximal shift in z tells us how much of the cylinder is removed: Consider a vertical line segment in the cylinder, so x and y are fixed and z varies from $-h$ to h. By the previous computation, the adjoint action of $s \in F_\rho$ shifts the line segment up or down by at most $\rho\sqrt{x^2 + y^2}$. Taking the intersection over all $s \in F_\rho$, we therefore see that pieces of length $\rho\sqrt{x^2 + y^2}$ are removed from the line segment, leaving the segment where $-h + \rho\sqrt{x^2 + y^2} \le z \le z - \rho\sqrt{x^2 + y^2}$.

Doing this for all the vertical line segments that make up $C_{r,h}$, i.e. for all x and y such that $x^2 + y^2 \le r^2$, we see that

$$\bigcap_{s\in F_\rho} \mathrm{Ad}_s(C_{r,h}) = \left\{(x,y,z) \mid x^2+y^2 \le r^2, |z| \le h - \rho\sqrt{x^2+y^2}\right\}. \tag{5.1}$$



(a) $C_{r,h}$ (b) $\bigcap_{s\in F_\rho} \mathrm{Ad}_s(C_{r,h})$

Figure 5.1: $C_{r,h}$ before and after taking the intersection of adjoints

which is displayed in figure 5.1(b), for the case where h is at least ρr. If instead $h < \rho r$, then $h - \rho\sqrt{x^2+y^2}$ can become negative, which means that the cones on the top and bottom meet, and the cylindrical section vanishes. The radius of the circle where the cones meet will also be smaller than r, which means a lot of volume is lost. For that reason, we are interested in tall cylinders, where only bits from the top and bottom are removed, leaving most of the volume. Therefore, the neighborhood basis $\mathcal{C}$ was chosen to contain the cylinders $C_{h^2,h}$, which become relatively taller as h goes to 0, i.e. $h/r \to \infty$ as $h \to 0$.

Now we can compute $\delta^0_{F_\rho}(C_{r,h})$. The denominator $\Lambda(C_{r,h})$ is simply the volume of the cylinder, which is $2\pi r^2 h$. For the denominator $\Lambda\left(\bigcap_{s\in F_\rho} \mathrm{Ad}_s(C_{r,h})\right)$, we will compute the volume of the part of above the $z = 0$ plane, which we will call V_u. By symmetry, the total volume is $2V_u$. The shape consists of a cone of height ρr and a cylinder of height $h - \rho r$, both with radius r. Hence we find $V_u = \frac{1}{3}\pi r^2(\rho r) + \pi r^2(h-\rho r) = \pi r^2(h - \frac{2}{3}\rho r)$. Thus $\delta^0_{F_\rho}(C_{r,h}) = \frac{2V_u}{\Lambda(C_{r,h})} = \frac{2\pi r^2(h-\frac{2}{3}\rho r)}{2\pi r^2 h} = 1 - \frac{2\rho r}{3h}$. In particular, $\delta^0_{F_\rho}(C_{h^2,h}) = 1 - \frac{2}{3}\rho h$.

Therefore we find that

$$\delta^0_{F_\rho}(\mathcal{C}) = \liminf_{C\in\mathcal{C}} \frac{\lambda\left(\bigcap_{s\in F_\rho} \mathrm{Ad}_s(C)\right)}{\lambda(C)} = \liminf_{h\to 0}\left(1 - \frac{2}{3}\rho h\right) = 1.$$

Note that $\delta^0_{F_\rho}(\mathcal{V})$ is at most 1 for any neighborhood basis $\mathcal{V}$ and $\mathcal{C}$ is a symmetric neighborhood basis for which $\delta^0_{F_\rho}(\mathcal{C}) = 1$. Hence $\delta^0_{F_p}$, being the supremum over all symmetric neighborhood bases of $\delta^0_{F_\rho}(\mathcal{V})$, must equal 1.

Now if $F \subseteq G$ is finite, there must exist a $\rho > 0$ such that $F \subseteq F_\rho$, hence $\delta^0_F \ge \delta^0_{F_\rho} = 1$. Therefore $\delta^0_F = 1$, so by proposition 5, this implies that $\delta_F = 1$. It then also follows that $c(U) = \inf\left\{\sqrt{\delta_F} \mid F \subseteq U \text{ finite}\right\} = 1$.

5.4. De Leeuw's theorem

The subset $H(\mathbb{Z}) = \{H(a,b,c) \mid a,b,c \in \mathbb{Z}\}$ of H is a subgroup: Since $H(a,b,c) \cdot H(a',b',c') = H(a + a', b{+}b', c{+}c'{+}ab')$, the result has integer entries again, and the same goes for the inverse $H(a,b,c)^{-1} = H(-a,-b,ab{-}c)$. Clearly, $H(\mathbb{Z})$ is a discrete subgroup of H. Since we determined that $c(H) = 1$, we find that for any compactly supported continuous p-multiplier $H \to \mathbb{C}$, $\left\|T_m|_{H(\mathbb{Z})}\right\|_p \leq ||T_m||$. The support of m does not matter in this case, since $c(H) = 1$, so $c(F) = 1$ for any compact $F \subseteq H$.

6

Nilpotent Lie Groups

Now that we have shown that $c(H) = 1$ for the Heisenberg group, we will generalize this to nilpotent all Lie groups. Although it may not be obvious at first, a crucial step in computing δ_F^0 for the Heisenberg group is that the Z coordinate is affected by the adjoint action by an amount proportional to the X and Y coordinates, but the X and Y coordinates themselves were unaffected. This allowed us to choose neighborhoods such that the X and Y coordinates shrank much faster than the Z coordinate, so that the effect of the adjoint action becomes arbitrarily small in the limit. If the X and Y coordinates were affected by an amount depending on the Z coordinate, this method would not have worked, since we already chose our neighborhoods so that Z can be large compared to X and Y, so we cannot make the adjoint action disappear in the limit.

In the proof of the general nilpotent case, we show that it is always possible to choose neighborhoods that shrink in such a way that the adjoint action disappears in the limit. We can relate the adjoint action of G on $\mathfrak{g}$ to the adjoint representation ad on $\mathfrak{g}$, and make use of the following result:

Theorem 4. *[8, Cor 3.3] If $\mathfrak{g}$ is a nilpotent Lie algebra, there exists a basis of $\mathfrak{g}$ such that ad_x is strictly upper triangular for all $x \in \mathfrak{g}$.*

Hence, if $\mathfrak{g}$ is nilpotent, we can construct a flag of subspaces such that applying ad_x for any $x \in \mathfrak{g}$ to an element of a subspace results in an element in a smaller subspace. Because of this, the adjoint action can only move a point along vectors in a smaller subspace. Because of proposition 5, in order to show that $c(G) = 1$ it will be sufficient to show that $\delta_F^0(\mathcal{V}) = 1$ for a particular choice of a neighborhood basis $\mathcal{V}$ and F ranging over a family of sets that cover G. By choosing a neighborhood basis which shrinks faster in the smaller subspaces, we can make the distance that any point is shifted by the adjoint action arbitrarily small, meaning the neighborhood is almost adjoint-invariant.

Theorem 5. *Let G be a connected nilpotent Lie group. Then $c(G) = 1$*

Proof. By theorem 4, there exists a basis $v_1, \dots, v_n$ of $\mathfrak{g}$ such that ad_x is strictly upper triangular for all $x \in \mathfrak{g}$. Then

$$\mathrm{Ad}_{\exp(x)} = \exp(\mathrm{ad}_x) = \exp \begin{pmatrix} 0 & & * \\ & \ddots & \\ 0 & & \ddots \;\; 0 \end{pmatrix} = \begin{pmatrix} 1 & & * \\ & \ddots & \\ 0 & & \ddots \;\; 1 \end{pmatrix}$$

Since G is connected, $\exp(\mathfrak{g})$ generates G by proposition 2. Hence we can write any $g \in G$ as a product as exponentials, and since Ad is a homomorphism, Ad_g is a product of upper-triangular matrices with 1's on the diagonal. Thus Ad_g itself is upper-triangular with 1's on the diagonal. We will write $g_{i,j}$ for the i, j element of the matrix Ad_g in the $\{v_1, \dots, v_n\}$ basis.

Let $x = \sum_{i=1}^n x_i v_i \in \mathfrak{g}$, and let $F \in G$ be compact. Since the matrix entries of Ad_g are continuous in G, there is a bound M on all matrix components of $\mathrm{Ad}_{g^{-1}}$ uniform over $g \in F$.

We will choose the neighborhood basis $\mathcal{C} = \{C_h \mid h > 0\}$ where $C_h = \{(x_1, \dots, x_n) \in \mathfrak{g} \mid |x_k| < h^k \ \forall k = 1, \dots, n\}$. We claim that

$$\widehat{C_h} \subseteq \bigcap_{g \in G} \mathrm{Ad}_g(C_h),$$

where $\widehat{C_h} = \{(x_1, \dots, x_n) \in \mathfrak{g} \mid |x_k| < h^k - nMh^{k+1}\}$. In order to do so, we will show that $\mathrm{Ad}_{g^{-1}}(\widehat{C_h}) \subseteq C_h$ for all $g \in F$. Since Ad_g and $\mathrm{Ad}_{g^{-1}}$ are inverse functions, it then follows that $\widehat{C_h} \subseteq \mathrm{Ad}_g(C_h)$ for all $g \in F$, so the claim follows.

Let $h < 1$, $x = \sum_{i=n}^{n} x_i v_i \in \widehat{C_h}$ and $g \in F$. Note that $|x_k| \leq h^k - nMh^{k+1} \leq h^k \leq h^l$ for $l \leq k$.

Then $\left(\mathrm{Ad}_{g^{-1}}(x)\right)_l = x_l + \sum\limits_{i=l+1}^{n} x_i (g^{-1})_{l,i}$, so

$$\begin{aligned}
|\left(\mathrm{Ad}_{g^{-1}}(x)\right)_l| &= |x_l + \sum_{k=l+1}^{n} (g^{-1})_{l,k}|x_k \\
&\leq |x_l| + \sum_{k=l+1}^{n} |x_k| \cdot |(g^{-1})_{l,k}| \\
&\leq h^l - nMh^{l+1} + nMh^{l+1} \\
&= h^l
\end{aligned}$$

This shows that $\mathrm{Ad}_{g^{-1}}(\widehat{C_h}) \subseteq C_h$.

Now note that $\Lambda\left(\bigcap\limits_{g \in F} \mathrm{Ad}_g(C_h)\right) \geq \Lambda(\widehat{C_h}) = 2^n \prod\limits_{m=1}^{n} (h^m - nMh^{m+1}) = 2^n h^N + \mathcal{O}(h^{N+1})$ where $N = 1 + \dots + n$. [1] Furthermore, $\Lambda(C_h) = 2^n h^N$, hence

$$\delta_F^0(\mathcal{C}) = \liminf_{C \in \mathcal{C}} \frac{\Lambda\left(\bigcap_{g \in F} \mathrm{Ad}_g(C)\right)}{\Lambda(C)} = \liminf_{h \to 0} \frac{h^N + \mathcal{O}(h^{N+1})}{h^N} = 1.$$

Taking the supremum over all symmetric neighborhood bases then gives $\delta_F^0 = 1$ since 1 is the maximal possible value. Then it follows from proposition 5 that $\delta_F = 1$.
This holds for all compact $F \subseteq G$, so certainly all finite F, hence $c(G) = 1$.

□

[1] For convenience we normalize the Lebesgue measure such that the unit cube in the $\{v_1, \dots, v_n\}$ basis has volume 1. The choice of normalization has no effect on the value of δ_F^0.

7

Split-solvable Lie Groups

In the nilpotent case, we were able to show that δ_F is always 1, using the fact that the adjoint representation can be made strictly upper diagonal. For solvable Lie groups, we can't do this in general, but in this chapter we will study a class of Lie algebras called split-solvable, where we can make the adjoint representation upper diagonal. The diagonal elements will determine the lower bound we obtain for δ_F. After showing an example of estimating δ_F for a unimodular split-solvable Lie group, we prove a general lower bound.

7.1. Example: Sol_3

We will first look at an example of a small solvable Lie group. We define Sol_3 to be $\mathbb{R}^3$ with multiplication $(a,b,c)\cdot(a',b',c') = (e^c a' + a, e^{-c}b' + b, c')$. Note that this is a semidirect product $\mathbb{R}^2 \rtimes \mathbb{R}$. It is easily verified that the inverse of (a,b,c) is $(-ae^{-c}, -be^{c}, -c)$.Then its Lie algebra $\mathfrak{sol}_3$ is again $\mathbb{R}^3$, where we will choose X, Y, Z to be the canonical basis of $\mathbb{R}^3$. We can compute the adjoint with $\mathrm{Ad}_g(x) = \frac{\partial}{\partial t}\mid_{t=0} g\exp(tx)g^{-1}$ and the Lie bracket using $\mathrm{ad}_x = \frac{\partial}{\partial t}\mid_{t=0} \mathrm{Ad}_{\exp(tx)}$. Then we find that Lie bracket satisfies by $[X,Y]=0$, $[Z,X]=X$, and $[Z,Y]=-Y$. Z commutes with everything. Furthermore, we have $\mathrm{Ad}_{(a,b,c)}X = e^c X$, $\mathrm{Ad}_{(a,b,c)}Y = e^{-c}Y$ and $\mathrm{Ad}_{(a,b,c)}Z = -aX + bY + Z$. As a matrix,

$$\mathrm{Ad}_{(a,b,c)} = \begin{pmatrix} e^c & 0 & -a \\ 0 & e^{-c} & b \\ 0 & 0 & 1 \end{pmatrix}.$$

Hence $\det\left(\mathrm{Ad}_{(a,b,c)}\right) = e^c \cdot e^{-c} \cdot 1 = 1$ for any $(a,b,c) \in \mathrm{Sol}_3$, thus Sol_3 is unimodular.
$\mathfrak{sol}_3$ is indeed solvable, since $[\mathfrak{sol}_3, \mathfrak{sol}_3] = \mathrm{span}\{X,Y\}$. Then taking the commutator of $\mathrm{span}\{X,Y\}$ with itself gives $\{0\}$, so the derived series terminates. However, it is not nilpotent, as $[\mathfrak{sol}_3, \mathrm{span}\{X,Y\}] = \mathrm{span}\{X,Y\}$, hence the lower central series does not terminate.

We will compute $\delta_{F_{\rho,\varepsilon}}$, where $F_{\rho,\varepsilon} = \{(a,b,c) \in \mathrm{Sol}_3 \mid a^2+b^2 \le \rho^2,\ |c| \le \varepsilon\}$. This is similar to the set F_ρ we used in the Heisenberg group, however, now we have a bound on $|c|$, on which our lower bound on $\delta_{F_{\rho,\varepsilon}}$ will depend.

Just like in the Heisenberg example, we see that there is a certain symmetry between X, Y, while Z behaves differently. Therefore, it makes sense to again choose a neighborhood basis of cylinders. However, in the Heisenberg algebra, the adjoint left the first two coordinates unchanged, while in this example, the last coordinate is invariant under the adjoint. As a result, we will now let the height of the cylinder go to 0 faster than the radius. Concretely, we set $C_{h,r} = \{xX + yY + zZ \mid x^2+y^2 \le r^2,\ |z| \le h\}$ and define the neighborhood basis $\mathcal{C} = \{C_{r,r^2} \mid r > 0\}$.

We will now compute a lower bound on $\delta_{F_{\rho,\varepsilon}}(C_{r,h})$ for some fixed ρ, ε, r and h. Let $(a,b,c) \in F_{\rho,\varepsilon}$ and we will choose $xX+yY+zZ \in C_{r,h}$ on the circular boundary, i.e. $x^2+y^2=r^2$. We see that $\mathrm{Ad}_{(a,b,c)}(xX+yY+zZ) = (e^c x - az)X + (e^{-c}y + bz)Y + zZ$, and since the last coordinate is unaffected, we will only look at the first two. By the reverse triangle inequality,

$$\begin{aligned}\left\|\begin{pmatrix} e^c x - az \\ e^{-c}y + bz\end{pmatrix}\right\| &\geq \left|\left\|\begin{pmatrix} e^c x \\ e^{-c}y\end{pmatrix}\right\| - \left\|\begin{pmatrix} az \\ -bz\end{pmatrix}\right\|\right| \\ &= \left\|\begin{pmatrix} e^c x \\ e^{-c}y\end{pmatrix}\right\| - |z|\left\|\begin{pmatrix} a \\ -b\end{pmatrix}\right\| \quad (*) \\ &\geq e^{-|c|}\left\|\begin{pmatrix} x \\ y\end{pmatrix}\right\| - |z|\rho \\ &\geq e^{-\varepsilon}r - \rho h\end{aligned}$$

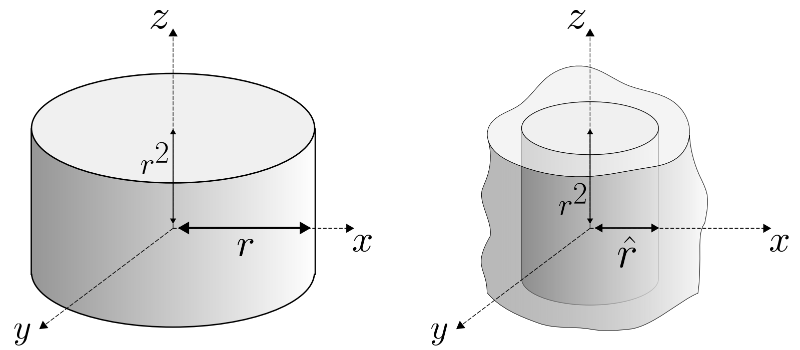
Note that at $(*)$, we can remove the absolute value since we consider the neighborhood basis C_{r,r^2}, so we have $|z| \leq h = r^2 \ll r$ for r small.

Since Ad_g is a homeomorphism, it maps connected sets to connected sets. Hence the cylinder $C_{e^{-\varepsilon}r-\rho h,h}$ is contained in $\mathrm{Ad}_g(C_{r,h})$ for all $g \in F_{\rho,\varepsilon}$, and thus in $\bigcap_{g\in F_{\rho,\varepsilon}} \mathrm{Ad}_g(C_{r,h})$, as shown in fig. 7.1. Now using the formula $\Lambda(C_{r,h}) = \pi h r^2$ for the volume of a cylinder, we see $\Lambda\left(\bigcap_{g\in F_{\rho,\varepsilon}} \mathrm{Ad}_g(C_{r,h})\right) \geq \Lambda(C_{e^{-\varepsilon}r-\rho h,h}) = \pi r^2(e^{-\varepsilon}r - \rho h)^2$. Thus

$$\delta^0_{F_{\rho,\varepsilon}}(C_{r,h}) = \frac{\Lambda\left(\bigcap_{g\in F_{\rho,\varepsilon}} \mathrm{Ad}_g(C_{r,h})\right)}{\Lambda(C_{r,h})} \geq \frac{\pi(e^{-\varepsilon}r - \rho h)^2 \cdot h}{\pi r^2 \cdot h} = \left(e^{-\varepsilon} - \frac{\rho h}{r}\right)^2.$$

In particular, $\delta_{F_{\rho,\varepsilon}}(C_{r,r^2}) \geq (e^{-\varepsilon} - \rho r)^2$. Then

$$\delta^0_{F_{\rho,\varepsilon}} \geq \delta^0_{F_{\rho,\varepsilon}}(\mathcal{C}) = \liminf_{C\in\mathcal{C}} \delta^0_{F_{\rho,\varepsilon}}(C) = \lim_{r\to 0} \delta^0_{F_{\rho,\varepsilon}}(C_{r,r^2}) = \lim_{r\to 0}(e^{-\varepsilon} - \rho r)^2 = e^{-2\varepsilon}.$$



(a) C_{r,r^2}

(b) The outer shape shows what $\bigcap_{s\in F_{\rho,\varepsilon}} \mathrm{Ad}_s(C_{r,r^2})$ could look like. Inside it is the cylinder of radius $\hat{r} := e^{-\varepsilon}r - r^2\rho$

Figure 7.1: C_{r,r^2} before and after taking the intersection of adjoints.

Therefore, we see that in the solvable case, we do not find $\delta_{F_{\rho,\varepsilon}} = 1$. The crucial difference with the nilpotent case is that the diagonal of Ad_g is not all 1s anymore. In the nilpotent case, we showed that the elements above the diagonal have no effect on δ_F: for $x \in g$ and $g \in G$ we found that the l-th component of $\mathrm{Ad}_g(x)$ equals

$$\left(\mathrm{Ad}_g(x)\right)_l = x_l + \sum_{i=l+1}^{n} g_{l,i}x_i$$

where $g_{l,i}$ are off-diagonal elements of Ad_g. By choosing our neighborhood basis such that x_i shrinks faster than x_l for $i > l$, all but the x_l term vanishes in the limit. However, in the solvable case, we also find a factor in front of x_l, which remains unaffected as we go to smaller neighborhoods, and hence our

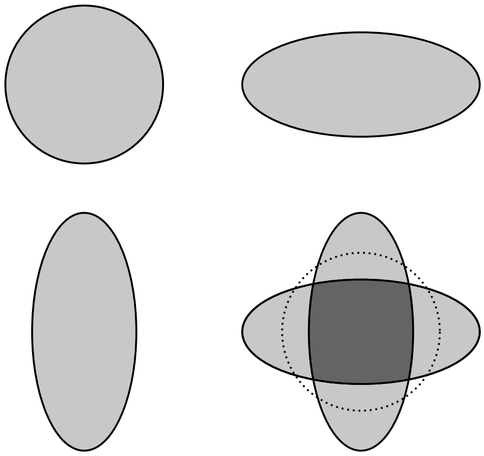


Figure 7.2: An example of adjoint action on a circle. Stretching in two different directions preserves area, but the intersection (shaded dark) is clearly smaller than the original circle (indicated by the dotted line).

neighborhoods are not almost adjoint-invariant.

However, since G is unimodular, $\det(\mathrm{Ad}_g) = 1$ for any particular $g \in G$, thus we find that $\mathrm{Ad}_g(V)$ has the same volume as V for any $g \in G$. We only lose volume in the intersection $\bigcap_{g \in F} \mathrm{Ad}_g(V)$. For instance in the case of Sol_3, the adjoint action of the element $(0, 0, 1)$ stretches a set V by e in the X direction while shrinking it by a factor e^{-1} in the Y direction. The element $(0, 0, -1)$ does the reverse. Separately they preserve the volume, but taking the intersection, the stretched out parts are lost, since they stretch in different directions, while the shrinking factor cuts in both directions. This is illustrated in fig. 7.2. This suggests that the minimum values of the diagonal elements over all $g \in F$ will determine δ_F. We will show that this is indeed the case in theorem 7 after introducing the necessary theory in section 7.2.

7.2. Structure of Solvable Lie groups

To compute δ_F for solvable Lie groups, we first need to understand their structure and what the adjoint action looks like. Since the adjoint is a representation, we will present some representation theory of solvable Lie algebras. Perhaps the most important result is the following:

Theorem 6 (Lie's theorem [9]). *Let $\mathfrak{g}$ be a solvable Lie algebra over a field $\mathbb{K} \subseteq \mathbb{C}$ of characteristic 0 and V be a nonzero $\mathbb{K}$-vector space. If $\pi : \mathfrak{g} \to \mathfrak{gl}(V)$ is a representation and the eigenvalues of $\pi(x)$ lie in $\mathbb{K}$ for all $x \in \mathfrak{g}$, then there exists a common eigenvector $v \in V$ for all $\pi(x)$, $x \in \mathfrak{g}$.*

Using this theorem, we can construct a full flag $V = V_0 \supset \ldots \supset V_n = \{0\}$ of invariant subspaces in V. For $\dim(V) = 1$ this is immediate. If $\dim(V) = n + 1$, we can proceed by induction: we assume that for n-dimensional representations, we can find an invariant flag. We can use Lie's theorem to find a simultaneous eigenvector $v \in V$. Then $W = V/\operatorname{span}\{v\}$ is an n-dimensional vector space, and since $\operatorname{span} v$ is an invariant subspace, we can define the quotient representation $\pi'(x)(w + \operatorname{span}\{v\}) = \pi(x)w + \operatorname{span}\{v\}$ which still satisfies the condition of Lie's theorem. Hence we can find a full flag $W = W_0 \supset \ldots \supset W_n = \{0\}$ which is invariant under π'. Then we can set $V_i = p^{-1}(W_i)$ for $i = 0, \ldots, n$ and $V_{n+1} = \{0\}$, where $p : V \to W$ is the canonical projection. This is an invariant flag in V, since if $x \in \mathfrak{g}$ and $u \in V_j$, then $u + \operatorname{span}\{v\} = p(u) \in W_j$ so $\pi(x)(v) + \operatorname{span}\{v\} = \pi'(x)(u + \operatorname{span}\{v\}) \subseteq W_j$, hence $\pi(x)(v) \in p^{-1}(W_j) = V_j$.

Now let $i \in \{1, \ldots, n\}$. Then under the quotient representation $\pi' : \mathfrak{g} \to \mathfrak{gl}(V/V_{i-1})$, V_i/V_{i-1} is an invariant subspace of V/V_{i-1} of dimension 1. Hence for each $x \in \mathfrak{g}$, $\pi'(x)$ acts as a scalar on V_i/V_{i-1}. As a result, we can define the map $\lambda_i : \mathfrak{g} \to \mathbb{C}$ which sends each x to that scalar: $\pi(x)v = \lambda_i(x)v$ for all $v \in V_i/V_{i-1}$. These λ_i are examples of *weights*. In general, any $\mathbb{K}$-linear function $\mathfrak{g} \to \mathbb{C}$ such that the *weight space*$\{v \in V \mid \pi(x)v = \lambda(x)v\}$ is nontrivial is called a weight of π. Since π is a homomorphism, it is immediate that weights are also homomorphisms.

The weight λ_i is only a weight of π', not of π: for $x \in \mathfrak{g}$ and $v \in V_i$ we have $\lambda_i(x)(v + V_{i-1}) = \pi'(x)(v + V_{i-1}) = \pi(x)v + V_{i-1}$, hence $\pi(x)v = \lambda_i(x)v + w$ for some $w \in V_{i-1}$, i.e. $\pi(x)$ acts as a scalar modulo V_{i-1}. A $\mathbb{K}$-linear function $\lambda\colon \mathfrak{g} \to \mathbb{C}$ will be called a *generalized weight* if there exists an invariant subspace W of V such that λ is a weight of the quotient representation on V/W. It turns out that $\lambda_1, \dots, \lambda_n$ are all the generalized weights of V [1].

This tells us a lot about what such a representation looks like: if we pick a basis $v_1, \dots, v_n$ such that $v_i \in V_i \quad V_{i-1}$, which we will call an *adapted basis*, then we see that for any $x \in \mathfrak{g}$, $\pi(x)v_i = \lambda_i(x)v_i + w$ for some $w \in V_{i-1} = \operatorname{span}\{v_1, \dots, v_{i-1}\}$. In other words, $\pi(x)$ is upper triangular with $\lambda_1(x), \dots, \lambda_n(x)$ on the diagonal. Hence $\lambda_1(x), \dots, \lambda_n(x)$ are the eigenvalues of $\pi(x)$.

This theorem is particularly powerful for complex Lie algebras, since $\mathbb{C}$ is algebraically closed, so we can always find a basis to make any representation upper-triangular. However, we work with real Lie algebras here, hence representations cannot be made upper-triangular in general, as they may have complex eigenvalues. We will first restrict our attention to the case where $\mathfrak{g}$ is a real Lie algebra where the eigenvalues of the adjoint representation are all real. Such a Lie algebra is called *split-solvable* or *completely solvable*.

If $\mathfrak{g}$ is split-solvable, we can apply Lie's theorem on the adjoint representation, hence we have weights $\lambda_1, \dots, \lambda_n\colon \mathfrak{g} \to \mathbb{R}$ such that for all $x \in \mathfrak{g}$,

$$\operatorname{ad}_x = \begin{pmatrix} \lambda_1(x) & & * \\ & \ddots & \\ 0 & & \lambda_n(x) \end{pmatrix}$$

in the adapted basis. On the Lie group level, this gives us

$$\operatorname{Ad}_{\exp(x)} = \exp(\operatorname{ad}_x) = \begin{pmatrix} e^{\lambda_1(x)} & & * \\ & \ddots & \\ 0 & & e^{\lambda_n(x)} \end{pmatrix} \tag{7.1}$$

Now since $\exp(\mathfrak{g})$ generates G by proposition 2, we see that for $g \in G$, Ad_g is a product of such matrices, and is therefore again upper-triangular. Therefore, the diagonal entries (and thus eigenvalues) of Ad_g are homomorphisms in g. We denote the i'th diagonal entry of Ad_g by $\chi_i(g)$, and we will call these the *weights* of G. From eq. (7.1), it is evident that $\chi_i(\exp(x)) = e^{\lambda_i(x)}$, and thus $d\chi_i = \lambda_i$. By theorem 3, this relation uniquely determines χ_i for each i.

7.3. Lower bound on δ_F for split-solvable Lie groups

Theorem 7. *Let G be a connected split-solvable Lie group with generalized weights $\lambda_1, \dots, \lambda_n$. Then there exist unique homomorphisms $\chi_1, \dots, \chi_n\colon G \to \mathbb{R}_{>0}$ such that $\mathrm{d}\chi_i = \lambda_i$ for each i, and for any compact set $F \subseteq G$,*

$$\delta_F \geq \prod_{k=1}^{n} \inf_{g \in F} \chi_k(g).$$

Proof. For the upper triangular elements, we will use the notation $g_{i,j}$ for the (i, j) matrix element of Ad_g in the $v_1, \dots, v_n$ basis. As a matrix,

$$\operatorname{Ad}_g = \begin{pmatrix} \chi_1(g) & g_{1,2} & \cdots & g_{1,n} \\ & \chi_2(g) & \ddots & \vdots \\ & & \ddots & g_{n-1,n} \\ & & & \chi_n(g) \end{pmatrix} \tag{7.2}$$

Let $C_h = \{x \in \mathfrak{g} \mid |x_k| \leq h^k \ \forall k = 1, \dots, n\}$. We consider the symmetric neighborhood basis $\mathcal{C} = \{C_h \mid h > 0\}$.

We define $M = \sup_{g \in F} \| \operatorname{Ad}_{g^{-1}} \|$, which is finite since $g \mapsto \| \operatorname{Ad}_{g^{-1}} \|$ is continuous and F is compact. Note in particular that for any $g \in F$, $|(g^{-1})_{i,j}| \leq M$ for all i, j. Furthermore, define $\mu_i := \inf_{g \in F} \chi_k(g)$ and

let $\mu_{\max} = \max\{\mu_1, \dots, \mu_n\}$.
We define $\widehat{C_h} = \{x \in \mathfrak{g} \mid |x_k| \leq \mu_k \left(h^k - n\mu_{\max} M h^{k+1}\right) \;\; \forall k = 1, \dots, n\}$, which is nonempty for h small enough. We will show that for $g \in F$, $\mathrm{Ad}_{g^{-1}}(\widehat{C_h}) \subseteq C_h$. It follows that $\widehat{C_h} \subseteq \mathrm{Ad}_g(C_h)$ for all $g \in F$, i.e. $\widehat{C_h} \subseteq \bigcap_{g \in F} \mathrm{Ad}_g(C_h)$. Then a simple computation of the volumes of C_h and $\widehat{C_h}$ allows us find the lower bound on δ_F.

Let $x \in \widehat{C_h}$. If we apply $\mathrm{Ad}_{g^{-1}}$ to x, we see from equation 7.2 that the resulting v_l component is $(\mathrm{Ad}_{g^{-1}}(x))_l = \chi_l(g^{-1})x_l + \sum\limits_{k=l+1}^{n} (g^{-1})_{l,k} x_k = \chi_l(g)^{-1}x_l + \sum\limits_{k=l+1}^{n} (g^{-1})_{l,k} x_k$.

Hence for $g \in F$ and $h < 1$

$$\begin{aligned}
|(\mathrm{Ad}_{g^{-1}}(x))_l| &\leq |\chi_l(g)|^{-1}|x_l| + \sum_{k=l+1}^{n} \left|(g^{-1})_{l,k}\right| \cdot |x_k| \quad (*) \\
&\leq \mu_l^{-1}\left(\mu_l\left(h^l - n\mu_{\max} M h^{l+1}\right)\right) + nM\mu_{\max} h^{l+1} \\
&= h^l
\end{aligned}$$

Where the summation at $(*)$ can be estimated by noting that for $k \geq l+1$,

$$\begin{aligned}
|x_k| &\leq \mu_k(h^k - n\mu_{\max} M h^{k+1}) \\
&\leq \mu_k h^k \leq \mu_{\max} h^k \\
&\leq \mu_{\max} h^{l+1} \text{ when } h < 1.
\end{aligned}$$

Since there are always fewer than n terms in the sum, we find the upper bound $nM\mu_{\max} h^{l+1}$.
Looking at the bounds on the coordinates of $\mathrm{Ad}_{g^{-1}}(x)$, we see that $\mathrm{Ad}_{g^{-1}}(x) \in C_h$, thus indeed $\mathrm{Ad}_{g^{-1}}(\widehat{C_h}) \subseteq C_h$.

To compute the volumes of C_h and $\widehat{C_h}$, we note that both sets are hyperrectangles where the side length along the v_k axis is twice the upper bound on $|x_k|$. Hence for h small enough,

$$\Lambda\left(\widehat{C_h}\right) = \prod_{k=1}^{n} 2\mu_k\left(h^k - n\mu_{\max} M h^{k+1}\right) = 2^n \left(\prod_{k=1}^{n} \mu_k\right) h^N + \mathcal{O}(h^{N+1})$$

where $N = 1 + \dots + n$.
The volume of C_h is simply $\Lambda(C_h) = \prod\limits_{k=1}^{n} 2h^k = 2^n h^N$. This gives

$$\delta_F^0(C_h) = \frac{2^n \left(\prod\limits_{k=1}^{n} \mu_k\right) h^N + \mathcal{O}(h^{N+1})}{2^n h^N} = \prod_{k=1}^{n} \mu_k + \mathcal{O}(h).$$

Then $\delta_F^0(\mathcal{C}) = \liminf\limits_{C \in \mathcal{C}} \delta_F^0(C) = \lim\limits_{h \to 0} \left(\prod\limits_{k=1}^{n} \mu_k + \mathcal{O}(h)\right) = \prod\limits_{k=1}^{n} \mu_k$, so $\delta_F^0 \geq \prod\limits_{k=1}^{n} \mu_k$. Then by proposition 5, $\delta_F \geq \prod\limits_{k=1}^{n} \mu_k = \prod\limits_{k=1}^{n} \inf\limits_{g \in F} \chi_k(g)$ □

Note that for unimodular groups, the given lower bound of δ_F is at most 1:

$$\prod_{k=1}^{n} \inf_{g \in F} \chi_k(g) \leq \inf_{g \in F} \prod_{k=1}^{n} \chi_k(g) = \inf_{g \in F} \det\left(\mathrm{Ad}_g\right) = 1.$$

This is to be expected, since for unimodular groups we have $\delta_F \leq 1$.

7.4. Lower Bound in Terms of the Lie Algebra

In many cases, it may be difficult to compute the weights of G, but it is easier to work with the adjoint representation of the Lie algebra. For solvable Lie groups, $\exp(\mathfrak{g})$ is dense in G [5]. This can be used to evaluate or estimate $\inf\limits_{g \in F} \chi_i(g)$ in terms of λ_i. For example, one can use the following statement:

Proposition 6. *Let G be a solvable Lie group. Suppose $\phi\colon G \to \mathbb{R}_{>0}$ is a homomorphism and $F \subseteq G$ is nonempty, compact and has dense interior. Then*

$$\inf_{g \in F} \phi(g) = \inf_{x \in \exp^{-1}(F)} e^{d\phi(x)}.$$

Proof. Let $\mathfrak{g} = \mathrm{Lie}(G)$. For solvable Lie groups, $\exp(\mathfrak{g})$ is dense in G. Now $\mathrm{int}(F)$ is open, and nonempty by density, hence $\exp(\mathfrak{g}) \cap \mathrm{int}(F)$ is dense in $\mathrm{int}(F)$, and thus also in F.

Therefore

$$\inf_{x \in \exp^{-1}(F)} e^{d\phi(x)} = \inf_{x \in \exp^{-1}(F)} \phi(\exp(x)) = \inf_{g \in F \cap \exp(\mathfrak{g})} \phi(g),$$

since $\{\exp(x) | x \in \exp^{-1}(F)\} = F \cap \exp(\mathfrak{g})$.
Then by density of $F \cap \exp(\mathfrak{g})$ in F and continuity of ϕ, the result follows. $\square$

Note that any nonempty F is compact and has dense interior if and only if it is the closure of an open relatively compact set.
Using this proposition, if F is nice enough, the lower bound on δ_F can be computed in the Lie algebra, without needing to compute the weights of G. This is particularly useful when defining F in terms of the Lie algebra.

8

Solvable Lie groups

In this chapter, we will extend our result for split-solvable Lie groups to all solvable Lie groups. We work out an example to show how our previous method fails and how it can be amended to work in this more general case. Then we explain how to use complexification in combination with Lie's theorem in order to prove a lower bound for all solvable Lie groups.

8.1. Example: the diamond Lie algebra

The diamond Lie algebra $\mathfrak{d}$ is an example of a solvable Lie algebra that is not split-solvable. It is generated by 4 elements X, Y, Z, A, where the bracket is defined by $[A, X] = Y$, $[A, Y] = -X$, $[X, Y] = Z$, and Z is a central element. Then $\mathfrak{d}^{(1)} = [\mathfrak{d}, \mathfrak{d}] = \operatorname{span}\{X, Y, Z\}$ which is the Heisenberg algebra $\mathfrak{h}$, which is nilpotent, so $\mathfrak{d}$ is solvable. To see that it is not split-solvable, we observe that in the ordered basis (X, Y, Z, A),

$$\operatorname{ad}_A = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

which has eigenvalues 0 and $\pm i$.
Furthermore, it is easily seen that ad_x has no nonzero diagonal entries for any $x \in \mathfrak{d}$, hence $\mathfrak{d}$ is unimodular.

Note that $\mathfrak{d} = \mathfrak{h} \rtimes \operatorname{span}\{A\} \cong \mathfrak{h} \rtimes \mathbb{R}$, hence $\mathfrak{d}$ is the Lie algebra of $D = H \rtimes \mathbb{R}$ for some semidirect product structure, where H is the Heisenberg group and $\mathbb{R}$ is the additive Lie group. Using the notation of chapter 5 for elements of H, we can construct any element of $\mathfrak{d}$ as $D(a, b, c, u) := (H(a, b, c), u)$ for $a, b, c, u \in \mathbb{R}$. For convenience, we will write $H(a, b, c)$ and u for the natural embedding in D, so $D(a, b, c, u) = H(a, b, c) \cdot u$ as a product in D.

To compute the adjoint action of D on $\mathfrak{d}$, we can split it up as $\operatorname{Ad}_{D(a,b,c,u)} = \operatorname{Ad}_{H(a,b,c)} \operatorname{Ad}_u$. For the additive Lie group $\mathbb{R}$, the exponential map $\operatorname{span}\{A\} \to \mathbb{R}$ is simply the identity, in the sense that $\exp(uA) = u$. Note that if $\varphi \colon \mathbb{R} \to H$ is a one-parameter group, then through the natural embedding it is also a one-parameter group in D. Hence it follows that for $x \in \mathfrak{h}$, the exponential maps of H and D agree. The same holds for the exponential map of $\mathbb{R}$. Hence there is no ambiguity when using the exponential of an element in $\mathfrak{h}$ or $\mathbb{R}$, as it coincides with the exponential on $\mathfrak{d}$.

Hence

$$\operatorname{Ad}_u = \operatorname{Ad}_{\exp(uA)} = \exp(\operatorname{ad}_{uA}) = \exp \begin{pmatrix} 0 & -u & 0 & 0 \\ u & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} \cos(u) & -\sin(u) & 0 & 0 \\ \sin(u) & \cos(u) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

We also need to know how $\mathrm{Ad}_{H(a,b,c)}$ acts on A. Since $\exp(aX) = H(a,0,0)$, we can compute

$$\begin{aligned}\mathrm{Ad}_{H(a,0,0)} A = \mathrm{Ad}_{\exp(aX)} A = \exp(a\cdot \mathrm{ad}_X)A &= \exp\begin{pmatrix}0&0&0&0\\0&0&0&-a\\0&a&0&0\\0&0&0&0\end{pmatrix}\cdot\begin{pmatrix}0\\0\\0\\1\end{pmatrix}\\ &= \begin{pmatrix}1&0&0&0\\0&1&0&-a\\0&a&1&-a^2/2\\0&0&0&1\end{pmatrix}\begin{pmatrix}0\\0\\0\\1\end{pmatrix} = \begin{pmatrix}0\\-a\\-a^2/2\\1\end{pmatrix} = -aY - \frac{1}{2}a^2Z + A,\end{aligned}$$

where the exponential of the matrix can be evaluated with the power series. Similarly $\mathrm{Ad}_{H(0,b,0)} A = bX - \frac{1}{2}b^2Z + A$ and $\mathrm{Ad}_{H(0,b,0)} Z = Z$, and $\mathrm{Ad}_{H(0,0,c)} = I$ since it is a central element.
Thus, all combined this gives

$$\begin{aligned}
\mathrm{Ad}_{D(a,b,c,u)} X &= \mathrm{Ad}_{H(a,b,c)} \mathrm{Ad}_u X = \mathrm{Ad}_{H(a,b,c)}(\cos(u)X + \sin(u)Y)\\
&= \cos(u)X + \sin(u)Y + (a\sin(u) - b\cos(u))Z\\
\mathrm{Ad}_{D(a,b,c,u)} Y &= \mathrm{Ad}_{H(a,b,c)} \mathrm{Ad}_u X\\
&= \mathrm{Ad}_{H(a,b,c)}(-\sin(u)X + \cos(u)Y) = -\sin(u)X + \cos(u)Y + (a\cos(u) + b\sin(u))Z\\
\mathrm{Ad}_{D(a,b,c,u)} Z &= \mathrm{Ad}_{H(a,b,c)} Z = Z\\
\mathrm{Ad}_{D(a,b,c,u)} A &= \mathrm{Ad}_{H(a,b,c)} \mathrm{Ad}_u A = \mathrm{Ad}_{H(a,b,c)} A = \mathrm{Ad}_{H(0,b,0)} \mathrm{Ad}_{H(a,0,0)} \mathrm{Ad}_{H(0,0,c)} A\\
&= \mathrm{Ad}_{H(0,b,0)} \left(-aY - \frac{1}{2}a^2Z + A\right)\\
&= -aY - \frac{1}{2}a^2Z + bX - \frac{1}{2}b^2Z + A = bX - aY - \frac{1}{2}(a^2+b^2)Z + A
\end{aligned}$$

If we order our basis as (Z, X, Y, A), the adjoint has the following matrix form:

$$\mathrm{Ad}_{D(a,b,c,u)} = \begin{pmatrix}1 & a\sin(u) - b\cos(u) & a\cos(u) + b\sin(u) & -\frac{1}{2}(a^2+b^2)\\ 0 & \cos(u) & -\sin(u) & b\\ 0 & \sin(u) & \cos(u) & -a\\ 0&0&0&1\end{pmatrix}. \tag{8.1}$$

This is not quite upper triangular, but if we consider the rotation matrix in the middle as one block, we do see that it has a block-diagonal structure. We will use the notation $R(\theta) = \left(\begin{smallmatrix}\cos(\theta) & -\sin(\theta)\\ \sin(\theta) & \cos(\theta)\end{smallmatrix}\right)$ for the rotation matrix on $\mathbb{R}^2$ around θ radians clockwise.

We consider the adjoint action of $F_{\rho,\varepsilon} := \{D(a,b,c,u) \mid a^2+b^2 \le \rho^2, |c| \le \varepsilon, |u| \le \varepsilon\}$ on the neighborhoods $C_{h,r,\mu} = \{xX + yY + zZ + tA \mid x^2+y^2 \le r^2, |z| \le t, |t| \le \mu\}$.

We let $D(a,b,c,u) \in F_{\rho,\varepsilon}$, and take a point $p = xX + yY + zZ + tA$ on the boundary of $C_{r,\mu}$, so $x^2 + y^2 = r^2$, $|t| = \mu$ and $|z| = h$, and apply $\mathrm{Ad}_{D(a,b,c,u)}$ to it. First looking at the X and Y components of $\mathrm{Ad}_{D(a,b,c,u)}(p)$

$$\begin{aligned}
\left\|\begin{pmatrix}\left(\mathrm{Ad}_{D(a,b,c,u)}(p)\right)_X\\ \left(\mathrm{Ad}_{D(a,b,c,u)}(p)\right)_Y\end{pmatrix}\right\| &= \left\|\begin{pmatrix}\cos(u)x - \sin(u)y + bt\\ \sin(u)x + \cos(u)y - at\end{pmatrix}\right\|\\
&= \left\|R(u)\begin{pmatrix}x\\y\end{pmatrix} - t\begin{pmatrix}-b\\a\end{pmatrix}\right\|\\
&\ge \left\|R(u)\begin{pmatrix}x\\y\end{pmatrix}\right\| - |t|\cdot\left\|\begin{pmatrix}-b\\a\end{pmatrix}\right\|\\
&\ge \sqrt{x^2+y^2} - |t|\sqrt{a^2+b^2} \text{ if } t \ll r\\
&\ge r - \rho\mu
\end{aligned}$$

The A component is invariant under the adjoint, $\left(\mathrm{Ad}_{D(a,b,c,u)}(p)\right)_A = t$. For the Z component, we have

$$\begin{aligned}
\left|\left(\mathrm{Ad}_{D(a,b,c,u)}(p)\right)_Z\right| &= \left|(a\sin(u) - b\cos(u))x + (a\cos(u) + b\sin(u))y - \frac{1}{2}(a^2+b^2)t + z\right| \\
&\geq \left||z| - |(a\sin(u) - b\cos(u))x + (a\cos(u) + b\sin(u))y - \frac{1}{2}(a^2+b^2)t|\right| \\
&\geq h - \left|\begin{pmatrix}-b\\a\end{pmatrix}\cdot R(u)\begin{pmatrix}x\\y\end{pmatrix}\right| - \frac{1}{2}(a^2+b^2)|t| \textit{ if } x,y,t \ll h \\
&\geq h - \left\|\begin{pmatrix}-b\\a\end{pmatrix}\right\|\cdot\left\|\begin{pmatrix}x\\y\end{pmatrix}\right\| - \rho^2 h \\
&\geq h - \rho r - \frac{1}{2}\rho^2\mu
\end{aligned}$$

Hence $\mathrm{Ad}_{D(a,b,c,u)}$ maps any point on the boundary of $C_{r,h,\mu}$ to a point outside the interior of $\widehat{C}_{r,h,\mu} := C_{\hat{r},\hat{h},\mu}$ with $\hat{r} = r - \rho h$ and $\hat{h} = h - \rho h - \frac{1}{2}\rho^2\mu$. Since $\mathrm{Ad}_{D(a,b,c,u)}$ is a homeomorphism, it maps boundaries to boundaries and interiors to interiors, and hence the image of $C_{r,h,\mu}$ contains $\widehat{C}_{r,h,\mu}$. This holds for any $D(a,b,c,u) \in F_{\rho,\varepsilon}$, hence $\widehat{C}_{r,h,\mu} \subseteq \bigcap_{g\in F_{\rho,\varepsilon}} \mathrm{Ad}_g(C_{r,h,\mu})$.

It follows that

$$\begin{aligned}
\delta^0_{F_{\rho,\varepsilon}}(C_{h,r,\mu}) &= \frac{\Lambda\left(\bigcap_{g\in F_{\rho,\varepsilon}} \mathrm{Ad}_g(C_{r,h,\mu})\right)}{\Lambda(C_{r,h,\mu})} \geq \frac{\Lambda\left(\widehat{C}_{r,h,\mu}\right)}{\Lambda(C_{r,h,\mu})} \\
&= \frac{\pi(r-\rho\mu)^2\cdot 2(h-\rho r - \frac{1}{2}\rho^2\mu)\cdot 2\mu}{\pi r^2\cdot 2h\cdot 2\mu} = \left(1 - \frac{\rho\mu}{r}\right)^2\left(1 - \frac{\rho r}{h} - \frac{\rho^2\mu}{h}\right)
\end{aligned}$$

In our computations, we needed $x, y, t \ll h$ and $t \ll r$, so we will accordingly choose our neighborhoods such that $\mu = h^4$ and $r = h^2$, hence we define $\mathcal{C} = \left\{C_{h^2,h,h^4} \mid h > 0\right\}$. Then

$$\delta^0_{F_{\rho,\varepsilon}}(\mathcal{C}) = \liminf_{C\in\mathcal{C}} \delta^0_{F_{\rho,\varepsilon}} = \lim_{h\to 0}\left(1-\rho h^2\right)^2\left(1 - \rho h - \rho^2 h^3\right) = 1.$$

Hence $\delta_{F_{\rho,\varepsilon}} = 1$.

Apparently, using mostly the same technique as in the split-solvable case, it is possible to work out δ_F for a solvable Lie group which is not split-solvable. The main difference is that in this case, the X and Y variables had to be treated together, as the adjoint action caused a rotation in the $X-Y$ plane. In the split-solvable case, we chose our neighborhood basis so that it shrunk at different rates along each of the basis vectors. However, now the neighborhood basis needs to shrink at the same rate in the X and Y directions, so that the cross-section of the neighborhoods is always circular. Then the rotating action of the adjoint keeps everything in this circle (up to the action of the A coordinate which disappears in the limit as we go to smaller neighborhoods). If instead the neighborhood basis shrinks as h^k in the X direction and h^{k+1} in the Y direction for some k, then the adjoint action would rotate the neighborhood so that the image of the neighborhood only shrinks as h^k in the Y direction, which would break our computation of δ_F. This is a direct result of the one nonzero entry below the diagonal in eq. (8.1).

In the rest of the chapter, we show that these pairs of coordinates in which the adjoint action acts as a rotation occur more generally in solvable Lie algebras, and are essentially the only thing that makes the situation more complicated than the split-solvable case. In the general case, the rotation can also be paired by a shrinking factor determined by the generalized weights, as in the split-solvable case, which will determine the lower bound on δ_F that is presented in theorem 9.

8.2. Complexification

In the example, we used a similar method as in the previous chapter to compute δ_F for a solvable Lie algebra which is not split-solvable. We would like to extend our result from the previous chapter to the general solvable case, however, Lie's theorem is not applicable to all real solvable Lie algebras. It is always possible to embed the real Lie algebra in a complex Lie algebra, called its *complexification*. For any real vector space V, we can define the complexification $V_{\mathbb{C}} = V \otimes_{\mathbb{R}} \mathbb{C}$. It is conventional to suggestively denote $v \otimes a$ by av. We can define conjugation on $V_{\mathbb{C}}$ by setting $\overline{v \otimes a} = v \otimes \overline{a}$ and extending linearly. If $v \in V_{\mathbb{C}}$ satisfies $\overline{v} = v$, we will call v a real vector. We can view V to be the subset $V \otimes 1$ in $V_{\mathbb{C}}$, so that the set of real vectors in $V_{\mathbb{C}}$ is V. Any $v \in V$ can be uniquely decomposed as $v = u + iw$ with u and w real vectors, in fact $\mathrm{Re}(v) := (v + \overline{v})/2$ and $\mathrm{Im}(v) := (v - \overline{v})/2i$ are real vectors and $v = \mathrm{Re}(v) + i\,\mathrm{Im}(v)$. This is often a more practical way of working with a vector in V_C in terms of vectors in V.

A real Lie algebra $\mathfrak{g}$ can also be complexified: as a vector space, the complexification of $\mathfrak{g}$ is $\mathfrak{g}_{\mathbb{C}}$, and we define its Lie bracket by setting $[v \otimes a, w \otimes b] = [v, w] \otimes ab$, which extends uniquely to a bilinear map $\mathfrak{g}_{\mathbb{C}} \times \mathfrak{g}_{\mathbb{C}} \to \mathfrak{g}_{\mathbb{C}}$.

If $\mathfrak{g}$ is solvable, then so is $\mathfrak{g}_{\mathbb{C}}$. This follows from the fact that $[\mathfrak{g}_{\mathbb{C}}, \mathfrak{g}_{\mathbb{C}}] = [\mathfrak{g}, \mathfrak{g}]_{\mathbb{C}}$, since $[\mathfrak{g}_{\mathbb{C}}, \mathfrak{g}_{\mathbb{C}}] = \mathrm{span}\{[v \otimes a, w \otimes b] \mid v, w \in \mathfrak{g},\ a, b \in \mathbb{C}\} = \mathrm{span}\{[v, w] \otimes ab \mid v, w \in \mathfrak{g},\ a, b \in \mathbb{C}\} = [\mathfrak{g}, \mathfrak{g}]_{\mathbb{C}}$. Then $\mathfrak{g}_{\mathbb{C}}^n = (\mathfrak{g}^n)_{\mathbb{C}}$, and if $\mathfrak{g}$ is solvable there exists an m such that $\mathfrak{g}^m = \{0\}$, so $\mathfrak{g}_{\mathbb{C}}^m = \{0\}$ too.

Therefore, if we have a real Lie algebra $\mathfrak{g}$ with a representation $\pi\colon \mathfrak{g} \to \mathfrak{gl}(V)$ of a real vector space V, we can complexify $\mathfrak{g}$ and extend π to a complex representation $\mathfrak{g}_{\mathbb{C}} \to \mathfrak{gl}(V_{\mathbb{C}})$. Note that any $T \in \mathfrak{gl}(V)$ extends naturally to a complex linear operator on $V_{\mathbb{C}}$ by setting $T(v + iw) = Tv + iTw$ for $v, w \in V$, and hence π extends naturally to $\mathfrak{g}_{\mathbb{C}}$ by setting $\pi(x + iy) = \pi(x) + i\pi(y)$. Then we can find a basis of $\mathfrak{g}_{\mathbb{C}}$ that makes $\pi(x)$ upper triangular for all $x \in \mathfrak{g}_{\mathbb{C}}$. However, if not all eigenvalues of the representation are real, this basis will not consist of real vectors.

8.3. Adapting Lie's theorem to real Lie algebras

Even though $\mathbb{R}$ is not algebraically closed, the roots of real polynomials are either real or come in conjugate pairs. As a result, we can adapt Lie's theorem somewhat to obtain a flag of invariant subspaces where some of the subspaces are 2 dimensions larger than the previous one.

Effectively, Lie's theorem in the complex case says that any representation has a one-dimensional (and thus irreducible) invariant subspace. In the real case, it is still always possible to find an irreducible invariant subspace, though it might be 2-dimensional.

Proposition 7. *Let $\mathfrak{g}$ be a real solvable lie algebra, and $\pi\colon \mathfrak{g} \to \mathfrak{gl}(V)$ be an irreducuble representation on a nontrivial real vector space V. Then V is one of the following:*

(i) $V = \mathbb{R}v$ *for some* $0 \neq v \in V$ *and* π *has a real-valued weight*

$$\pi(x)v = \lambda(x)v \ \forall x \in \mathfrak{g}$$

(ii) $V = \mathrm{span}\{v_1, v_2\}$ *for* $v_1, v_2 \in V$ *independent, and* π *has a has two weights* λ *and* $\overline{\lambda}$ *which are each other's conjugates, such that*

$$\pi(x)(v_1 + iv_2) = \lambda(x)(v_1 + iv_2)$$

$$\pi(x)(v_1 - iv_2) = \overline{\lambda(x)}(v_1 - iv_2)$$

for all $x \in \mathfrak{g}$

Any finite-dimensional representation has an irreducible subrepresentation, since we can keep extracting invariant subspaces from reducible subrepresentations until we encounter an irreducible one, which we are guaranteed to find since one-dimensional representations are irreducible.

Hence, for any representation π of a real solvable Lie algebra, we can always find an invariant subspace on which π acts as either case (1) or (2).

Similarly to Lie's theorem, we can use an induction argument to obtain a flag of invariant subspaces.

Theorem 8 (Prop 1.3.7 [1]). *Let $\mathfrak{g}$ be a real solvable Lie algebra with a representation $\pi : \mathfrak{g} \to \mathfrak{gl}(V)$ for a nontrivial real vector space V. Then there exists a basis $\{v_1, \dots, v_n\}$ of $U = V_{\mathbb{C}}$ such that*

(a) *For all* $i = 1, \dots, n$, $U_i = \mathrm{span}\{v_1, \dots, v_i\}$ *is an invariant subspace, and we have a corresponding generalized weight* λ_i *which is a weight on* U/U_{i-1}.

(b) If $U_i \neq \overline{U_i}$, then $v_{i+1} = \overline{v_i}$ and $\lambda_{i+1} = \overline{\lambda_i}$

(c) If $U_i = \overline{U_i}$ and $U_{i-1} = \overline{U_{i-1}}$, then $v_i \in V$ and λ_i is real-valued.

This means that we have some indices corresponding to a real-valued generalized weight and a real basis vector, and some consecutive pairs of indices corresponding to conjugate pairs of generalized weights and basis vectors. We will call such $i, i+1$ *paired indices*. An index that is not in any such pair will be called *unpaired*.

Using this theorem, we can define an adjusted basis $\{v_1', \dots, v_n'\}$ where if $i, i+1$ are paired indices, we set $v_i' = \frac{1}{2}(v_i + v_{i+1})$ and $v_{i+1}' = \frac{1}{2i}(v_i - v_{i+1})$. Otherwise, we set $v_i' = v_i$. Therefore, all v_i are real vectors. We will call such a basis an $\mathbb{R}$-*adapted basis*. Let $i, i+1$ be paired indices and let π' be the quotient representation on U/U_{i-1}. Then in the basis $\{v_i + U_{i-1}, v_{i+1} + U_{i-1}\}$, we have

$$\pi'|_{U_{i+1}/U_i}(x) = \begin{pmatrix} \lambda_i(x) & 0 \\ 0 & \overline{\lambda_i(x)} \end{pmatrix}.$$

To convert this to the new basis, we conjugate by the transition matrix from $\{v_i, v_{i+1}\}$ to $\{v_i', v_{i+1}'\}$ to obtain

$$\begin{aligned} \pi'|_{U_{i+1}/U_i}(x) &= \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2i} & -\frac{1}{2i} \end{pmatrix} \begin{pmatrix} \lambda_i(x) & 0 \\ 0 & \overline{\lambda_i(x)} \end{pmatrix} \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2i} & -\frac{1}{2i} \end{pmatrix}^{-1} \\ &= \begin{pmatrix} \frac{1}{2}(\lambda_i(x) + \overline{\lambda_i(x)}) & -\frac{1}{2i}(\lambda_i(x) - \overline{\lambda_i(x)}) \\ \frac{1}{2i}(\lambda_i(x) - \overline{\lambda_i(x)}) & \frac{1}{2}(\lambda_i(x) + \overline{\lambda_i(x)}) \end{pmatrix} = \begin{pmatrix} \operatorname{Re}\lambda_i(x) & -\operatorname{Im}\lambda_i(x) \\ \operatorname{Im}\lambda_i(x) & \operatorname{Re}\lambda_i(x) \end{pmatrix} \end{aligned}$$

Therefore, for paired indices $i, i+1$,

$$\begin{aligned} \pi'(x)(av_i' + bv_{i+1}' + U_{i-1}) &= \begin{pmatrix} \operatorname{Re}\lambda_i(x) & -\operatorname{Im}\lambda_i(x) \\ \operatorname{Im}\lambda_i(x) & \operatorname{Re}\lambda_i(x) \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} + U_{i-1} \\ &= (a \operatorname{Re}\lambda_i(x) - b \operatorname{Im}\lambda_i(x))v_i' + (a \operatorname{Im}\lambda_i(x) + b \operatorname{Re}\lambda(x))v_{i+1}' + U_{i-1} \end{aligned}$$

And for an unpaired index i we still have $\pi'(x)(v_i + U_{i-1}) = \lambda_i(x)v_i + U_{i-1}$.

Hence in the $\{v_1', \dots, v_n'\}$ basis, $\pi(x)$ is matrix containing singular entries and 2 by 2 blocks on the diagonal, with all entries below the block-diagonal being 0. So, for example, if 1 and 6 are unpaired indices, and 2,3 and 4,5 are paired indices, we have

$$\pi(x) = \begin{pmatrix} \lambda_1(x) & & & & & \\ & \operatorname{Re}\lambda_2(x) & -\operatorname{Im}\lambda_2(x) & & * & \\ & \operatorname{Im}\lambda_2(x) & \operatorname{Re}\lambda_2(x) & & & \\ & & & \operatorname{Re}\lambda_4(x) & -\operatorname{Im}\lambda_4(x) & \\ & 0 & & \operatorname{Im}\lambda_4(x) & \operatorname{Re}\lambda_4(x) & \\ & & & & & \lambda_6(x) \end{pmatrix} \tag{8.2}$$

Furthermore, when restricted to $V = \operatorname{span}_{\mathbb{R}}\{v_1', \dots, v_n'\}$, π is the original real representation on $\mathfrak{g}$, hence in this basis $\pi(x)$ is a real matrix for all x.

We will again apply this to the adjoint representation of $\mathfrak{g}$, and use this to derive the structure of the adjoint representation on G using the equality $\operatorname{Ad}_{\exp(x)} = \exp(\operatorname{ad}_x)$. We use the following proposition to compute $\operatorname{Ad}_{\exp(x)}$:

Proposition 8. *Let A, B be block-upper-triangular matrices with matching diagonal structure, i.e.*

$$A = \begin{pmatrix} A_1 & & * \\ & \ddots & \\ 0 & & A_k \end{pmatrix}, \quad B = \begin{pmatrix} B_1 & & * \\ & \ddots & \\ 0 & & B_k \end{pmatrix}$$

where A_i and B_i are square blocks of the same size for each i. Then

(a)

$$AB = \begin{pmatrix} A_1B_1 & & * \\ & \ddots & \\ 0 & & \ddots & \\ & & & A_kB_k \end{pmatrix}$$

(b)

$$\exp(A) = \begin{pmatrix} \exp(A_1) & & * \\ & \ddots & \\ 0 & & \ddots & \\ & & & \exp(A_k) \end{pmatrix}$$

Now using the fact that

$$\exp\begin{pmatrix} a & -b \\ b & a \end{pmatrix} = e^a \begin{pmatrix} \cos(b) & -\sin(b) \\ \sin(b) & \cos(b) \end{pmatrix} = e^a R(b)$$

for all $a, b \in \mathbb{R}$, we have

$$\exp\begin{pmatrix} \operatorname{Re}\lambda & -\operatorname{Im}\lambda \\ \operatorname{Im}\lambda & \operatorname{Re}\lambda \end{pmatrix} = e^{\operatorname{Re}\lambda}\begin{pmatrix} \operatorname{Im}\cos(\lambda) & -\sin(\operatorname{Im}\lambda) \\ \sin(\operatorname{Im}\lambda) & \cos(\operatorname{Im}\lambda) \end{pmatrix} = e^{\operatorname{Re}\lambda} R(\operatorname{Im}\lambda) \tag{8.3}$$

for any $\lambda \in \mathbb{C}$. Hence we know what the block-diagonal of $\mathrm{Ad}_{\exp(x)}$ looks like. For example, the exponential of the matrix in eq. (8.2) is

$$\begin{pmatrix} e^{\lambda_1(x)} & & & & & * \\ & e^{\operatorname{Re}\lambda_2(x)}\cos(\operatorname{Im}\lambda_2(x)) & -e^{\operatorname{Re}\lambda_2(x)}\sin(\operatorname{Im}\lambda_2(x)) & & & \\ & e^{\operatorname{Re}\lambda_2(x)}\sin(\operatorname{Im}\lambda_2(x)) & e^{\operatorname{Re}\lambda_2(x)}\cos(\operatorname{Im}\lambda_2(x)) & & & \\ & & & e^{\operatorname{Re}\lambda_4(x)}\cos(\operatorname{Im}\lambda_4(x)) & -e^{\operatorname{Re}\lambda_4(x)}\sin(\operatorname{Im}\lambda_4(x)) & \\ & 0 & & e^{\operatorname{Re}\lambda_4(x)}\sin(\operatorname{Im}\lambda_4(x)) & e^{\operatorname{Re}\lambda_4(x)}\cos(\operatorname{Im}\lambda_4(x)) & \\ & & & & & e^{\lambda_6(x)} \end{pmatrix}$$

Now we can use proposition 2 and proposition 8 to deduce that for any $g \in G$, Ad_g again has the same block sizes on the diagonal. The 1×1 blocks corresponding to unpaired indices i will again be denoted $\chi_i(g)$, and we still have $d\chi_i = \lambda_i$. For paired indices $i, i+1$, we denote the 2×2 block by $T_i(g)$. Since $T_i(g)$ is a product of matrices of the same form as in eq. (8.3), we see that $T_i(g)$ is a real scalar times an orthogonal matrix, and if we choose the scalar positive, they are uniquely determined, and will be denoted $\chi_i(g)$ and $O_i(g)$ respectively. It follows from proposition 8(a) that they are both homomorphisms. Also note that $||T_i(x)|| = |\chi_i(g)| \cdot ||O_i(x)|| = \chi_i(g)$.
Since $T_i(\exp(x)) = \chi_i(\exp(x))O_i(\exp(x)) = e^{\operatorname{Re}\lambda_i(x)}R(\operatorname{Im}\lambda_i(x))$, we see that $\chi_i(\exp(x)) = e^{\operatorname{Re}\lambda_i(x)}$, hence $d\chi_i = \operatorname{Re}\lambda_i$ (Note that the same holds for unpaired indices, since then λ_i is real-valued). For convenience, we will set $\chi_{i+1} = \chi_i$, and again call $\chi_1, \dots, \chi_n$ the weights of G.

Theorem 9. *Let G be a connected solvable Lie group of dimension n with generalized weights $\lambda_1, \dots, \lambda_n$. Then there exist unique homomorphisms $\chi_1, \dots, \chi_n$ such that $\mathrm{d}\chi_i = \operatorname{Re}\lambda_i$. Then for any compact $F \subseteq G$,*

$$\delta_F \geq \prod_{i=1}^{n} \inf_{g \in F} \chi_i(g).$$

Proof. The existence of $\chi_1, \dots, \chi_n$ is shown in the preceding text. Uniqueness follows from theorem 3.

We will fix an $\mathbb{R}$-adapted basis and for any $x \in \mathfrak{g}$, let x_i be the component of the i-th vector in the basis.

Define $p_1 = 1$ and $p_{i+1} = p_i$ if $i, i+1$ are paired indices and $p_{i+1} = p_i + 1$ otherwise. Then we will define our neighborhood basis as $\mathcal{C} = \{C_h \mid h > 0\}$ with

$$C_h = \left\{ x \in \mathfrak{g} \mid \begin{array}{ll} \sqrt{x_i^2 + x_{i+1}^2} \leq h^{p_i} & i, i+1 \text{ paired indices} \\ |x_i| \leq h^{p_i} & i \text{ unpaired} \end{array} \right\}.$$

We again define $g_{i,j}$ to be the entries of Ad_g in the $\mathbb{R}$-adapted basis, and set $M = \sup_{g\in F} || \mathrm{Ad}_{g^{-1}} ||$, noting that for any $g \in F$, $|(g^{-1})_{i,j}| \leq M$ for all i, j. Furthermore, let $\mu_i = \inf_{g\in F} \chi_k(g)$ and $\mu_{\max} = \max\{\mu_1, \dots, \mu_n\}$.
Moreover, we define

$$\widehat{C_h} = \left\{ x \in \mathfrak{g} \mid \begin{array}{ll} \sqrt{x_i^2 + x_{i+1}^2} \leq \mu_i \left(h^{p_i} - 2n\mu_{\max} M h^{p_i+1}\right) & i, i+1 \text{ paired indices} \\ |x_i| \leq \mu_i \left(h^{p_i} - 2n\mu_{\max} M h^{p_i+1}\right) & i \text{ unpaired} \end{array} \right\}.$$

Let $x \in \widehat{C_h}$. We will show upper bounds on the components of $\mathrm{Ad}_{g^{-1}}(x)$ to show that it is in C_h.
First, consider an unpaired index l. Then

$$(\mathrm{Ad}_{g^{-1}}(x))_l = \chi_l(g)^{-1} x_l + \sum_{i=l+1}^{n} (g^{-1})_{l,i} x_i$$

We first note that if i is an unpaired index, then $|x_i| \leq \mu_i \left(h^{p_i} - 2n\mu_{\max} M h^{p_i+1}\right)$, and if i, j are paired indices, then $|x_i| \leq \sqrt{x_i^2 + x_j^2} \leq \mu_i \left(h^{p_i} - 2n\mu_{\max} M h^{p_i+1}\right)$ (noting that $p_i = p_j$) by definition of $\widehat{C_h}$, so we find the same upper bound for both paired and unpaired indices. Hence

$$\begin{aligned} |x_i| &\leq \mu_i \left(h^{p_i} - 2n\mu_{\max} M h^{p_i+1}\right) \\ &\leq \mu_i h^{p_i} \leq \mu_{\max} h^{p_i} \\ &\leq \mu_{\max} h^{p_l+1} \text{ when } h < 1. \end{aligned} \tag{8.4}$$

Noting that $p_i \geq p_l + 1$ as $i > l$ and l is unpaired.
This allows us to bound $\sum_{i=l+1}^{n} (g^{-1})_{l,i} x_i$ for $i > l$: Since there are always fewer than n terms in the sum, we find the upper bound

$$\left| \sum_{i=l+1}^{n} (g^{-1})_{l,i} x_i \right| \leq nM\mu_{\max} h^{p_l+1} \leq 2nM\mu_{\max} h^{p_l+1} \tag{8.5}$$

Hence for $g \in F$ and $h < 1$

$$\begin{aligned} |(\mathrm{Ad}_{g^{-1}}(x))_l| &\leq |\chi_l(g)|^{-1} |x_l| + \sum_{i=l+1}^{n} \left|(g^{-1})_{l,i}\right| \cdot |x_i| \\ &\leq \mu_l^{-1} \left(\mu_l \left(h^{p_l} - 2n\mu_{\max} M h^{p_l+1}\right)\right) + 2nM\mu_{\max} h^{p_l+1} \\ &= h^{p_l} \end{aligned}$$

For paired indices $l, l+1$, we have

$$\begin{pmatrix} \left(\mathrm{Ad}_{g^{-1}}(x)\right)_l \\ \left(\mathrm{Ad}_{g^{-1}}(x)\right)_{l+1} \end{pmatrix} = e^{-\operatorname{Re}\lambda_l(x)} R(-\operatorname{Im}\lambda_l(x)) \begin{pmatrix} x_l \\ x_{l+1} \end{pmatrix} + \sum_{i=l+2}^{n} \begin{pmatrix} (g^{-1})_{l,i} x_i \\ (g^{-1})_{l+1,i} x_i \end{pmatrix}.$$

We can again bound the coordinate $|x_i|$, now for $i > l+1$. With the same computation as in eq. (8.4), we see that $|x_i| \leq \mu_{\max} h^{p_l+1}$. (Note that in the paired case we need $i > l+1$ rather than $i > l$, since $p_i \geq p_l + 1$ will only hold for $i > l+1$ now). Hence we can bound

$$\sum_{i=l+2}^{n} \left(|(g^{-1})_{l,i}| + |(g^{-1})_{l+1,i}|\right) \cdot |x_i| \leq 2nM\mu_{\max} h^{p_l+1},$$

since there are no more than n terms, and $\left(|(g^{-1})_{l,i}| + |(g^{-1})_{l+1,i}|\right) \leq 2M$.

Hence

$$\begin{aligned}\left\|\begin{pmatrix}\left(\mathrm{Ad}_{g^{-1}}(x)\right)_l\\ \left(\mathrm{Ad}_{g^{-1}}(x)\right)_{l+1}\end{pmatrix}\right\| &\leq \chi_l(g)^{-1}\left\|O_i(g)^{-1}\begin{pmatrix}x_l\\ x_{l+1}\end{pmatrix}\right\| + \left\|\sum_{i=l+2}^{n}\begin{pmatrix}(g^{-1})_{l,i}x_i\\ (g^{-1})_{l+1,i}x_i\end{pmatrix}\right\|\\ &\leq \chi_l(g)^{-1}\left\|\begin{pmatrix}x_l\\ x_{l+1}\end{pmatrix}\right\| + \sum_{i=l+2}^{n}\left(|(g^{-1})_{l,i}| + |(g^{-1})_{l+1,i}|\right)\cdot|x_i|\\ &\leq \mu_l^{-1}\left(\mu_l\left(h^{p_i} - 2n\mu_{\max}Mh^{p_l+1}\right)\right) + 2n\mu_{\max}Mh^{p_{l+2}}\\ &\leq h^{p_l} - 2n\mu_{\max}Mh^{p_l+1} + 2n\mu_{\max}Mh^{p_l+1} \quad \text{since } p_{l+2}\leq p_l+1\\ &= h^{p_l}\end{aligned}$$

This shows that for $g \in F$ and $x \in \widehat{C_h}$, $\mathrm{Ad}_{g^{-1}}(x) \in C_h$, and thus $\widehat{C_h} \subseteq \bigcap_{g\in F}\mathrm{Ad}_g(C_h)$.

C_h and $\widehat{C_h}$ are higher dimensional cylinder-like shapes. For a 3-dimensional cylinder, the volume is the area of the circular base multiplied by the height. A similar formula holds true for the volume of C_h.
We can write C_h as a cartesian product

$$C_h = \left(\prod_{i\text{unpaired}}[-h^{p_i}, h^{p_i}]\right) \times \left(\prod_{i,i+1\text{paired}} D_{h^{p_i}}\right),$$

where D_r denotes a 2-dimensional disk of radius r. Hence if we let Λ_i be the i-dimensional Lebesgue measure,

$$\begin{aligned}\Lambda(C_h) &= \left(\prod_{i\text{unpaired}}\Lambda_1([-h^{p_i}, h^{p_i}])\right)\cdot\left(\prod_{i,i+1\text{paired}}\Lambda_2(D_{h^{p_i}})\right)\\ &= \left(\prod_{i\text{unpaired}} 2h^{p_i}\right)\cdot\left(\prod_{i,i+1\text{paired}}\pi\left(h^{p_i}\right)^2\right)\\ &= 2^r\pi^s h^P,\end{aligned}$$

where r is the amount of unpaired indices, s is the amount of index pairs and $P = \sum_{i=1}^n p_i$. Note that the exponent of h is P since for paired indices $i, i+1$, $p_i = p_{i+1}$, so $\left(h^{p_i}\right)^2 = h^{p_i}h^{p_{i+1}}$. Hence, the product contains h^{p_i} once for every $i = 1, \ldots, n$.
With a similar computation, we find that

$$\begin{aligned}\Lambda\left(\widehat{C_h}\right) &= \left(\prod_{i\text{unpaired}} 2\mu_i\left(h^{p_i} - 2n\mu_{\max}Mh^{p_i+1}\right)\right)\cdot\left(\prod_{i,i+1\text{paired}}\pi\left(\mu_i\left(h^{p_i} - 2n\mu_{\max}Mh^{p_i+1}\right)\right)^2\right)\\ &= 2^r\pi^s h^P\prod_{i=1}^n\mu_i + \mathcal{O}(h^{P+1}),\end{aligned}$$

noting that $\mu_i = \mu_{i+1}$ if i and $i+1$ are paired.
Hence

$$\delta_F^0(C_h) = \frac{2^r\pi^s h^P\prod_{k=1}^n\mu_k + \mathcal{O}(h^{P+1})}{2^r\pi^s h^P} = \prod_{k=1}^n\mu_k + \mathcal{O}(h).$$

It follows that $\delta_F^0(\mathcal{C}) = \prod_{k=1}^n\mu_k$. Hence $\delta_F = \delta_F^0 \geq \prod_{k=1}^n\mu_k$.

□

9

Outlook

In this thesis, we have shown a lower bound on δ_F for solvable Lie groups, meaning that we now have lower bounds for both components of the Levi decomposition of a Lie algebra. This is a step towards the goal of finding a lower bound on δ_F for all Lie groups. However, since the semisimple part acts on the solvable ideal in the Levi decomposition, it cannot simply be concluded that the product of the solvable and semisimple bounds gives a bound for the general case.

During the writing of this thesis, it was discovered by my supervisor that the presented methods can be amended to show that $c(G) = 1$ for all connected solvable G. Furthermore, δ_F can be written as a product of a semisimple contribution and a solvable contribution. The semisimple contribution can be estimated as in chapter 4, but the solvable contribution concerns the adjoint action of the whole Lie group on the radical ideal of the Lie algebra, rather than the action of a solvable Lie group on its Lie algebra as we considered in this thesis. Now knowing that $\delta_F = 1$ for solvable Lie groups, only the action of the semisimple subgroup on the radical ideal will determine the solvable contribution. An article presenting these findings is currently in the works. Future research could investigate how the solvable contribution could be estimated, potentially by first investigating the action of $\mathfrak{sl}_2(\mathbb{R})$ on solvable ideals, and then exploiting the structure theory of semisimple Lie algebras to obtain a general result.

Bibliography

[1] Didier Arnal and Bradley Currey. *Representations of solvable Lie groups*. New Mathematical Monographs. Cambridge University Press, 2020.

[2] Martijn Caspers, Bas Janssens, Amudhan Krishnaswamy-Usha, and Lukas Miaskiwskyi. *Local and multilinear noncommutative de Leeuw theorems*. 2023. arXiv: `2201.10400 [math.OA]`.

[3] Martijn Caspers, Javier Parcet, Mathilde Perrin, and Éric Ricard. "Noncommutative de Leeuw Theorems". In: *Forum of Mathematics, Sigma* 3 (2015).

[4] David H Collingwood and William M McGovern. *Nilpotent Orbits In Semisimple Lie Algebra*. en. CRC Press, 1993.

[5] D. Ž. Đoković and Nguyêñ Q. Thâńg. "Lie groups with dense exponential image". In: *Mathematische Zeitschrift* 225.1 (1997), pp. 35–47.

[6] Gerald B. Folland. *A Course in Abstract Harmonic Analysis*. Chapman and Hall/CRC, 2016.

[7] Karl H Hofmann and Sidney A Morris. *The structure of compact groups*. en. De Gruyter Studies in Mathematics. De Gruyter, 1998.

[8] J E Humphreys. *Introduction to Lie algebras and representation theory*. en. Graduate texts in mathematics. Springer, 1973.

[9] Anthony W Knapp. *Lie Groups Beyond an Introduction*. 2nd ed. Progress in Mathematics. Birkhauser Boston, 2002.

[10] John M Lee. *Lie Groups*. Graduate texts in mathematics. Springer New York, 2013.

[11] Karel de Leeuw. "On Lp Multipliers". In: *Annals of Mathematics* 81.2 (1965), pp. 364–379. ISSN: 0003486X.

[12] Gerald J Murphy. *C*-algebras and operator theory*. en. Academic Press, 1990.

[13] Luiz A B San Martin. *Lie Groups*. Springer International Publishing, 2021.