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UNIVERSAL AUTOHOMEOMORPHISMS OF $\mathbb{N}^*$

KLAAS PIETER HART AND JAN VAN MILL

(Communicated by Vera Fischer)

To the memory of Cor Baayen, who taught us many things

ABSTRACT. We study the existence of universal autohomeomorphisms of $\mathbb{N}^*$. We prove that the Continuum Hypothesis (CH) implies there is such an autohomeomorphism and show that there are none in any model where all autohomeomorphisms of $\mathbb{N}^*$ are trivial.

INTRODUCTION

This paper is concerned with universal autohomeomorphisms on $\mathbb{N}^*$, the Čech-Stone remainder of $\mathbb{N}$.

In very general terms we say that an autohomeomorphism h on a space X is *universal* for a class of pairs (Y, g) , where Y is a space and g is an autohomeomorphism of Y , if for every such pair there is an embedding $e : Y \rightarrow X$ such that $h \circ e = e \circ g$, that is, h extends the copy of g on $e[Y]$.

In [1, Section 3.4] one finds a general way of finding universal autohomeomorphisms. If X is homeomorphic to X^ω then the shift mapping $\sigma : X^\mathbb{Z} \rightarrow X^\mathbb{Z}$ defines a universal autohomeomorphism for the class of all pairs (Y, g) , where Y is a subspace of X . One embeds Y into $X^\mathbb{Z}$ by mapping each $y \in Y$ to the sequence $\langle g^n(y) : n \in \mathbb{Z} \rangle$.

Thus, the Hilbert cube carries an autohomeomorphism that is universal for all autohomeomorphisms of separable metrizable spaces and the Cantor set carries one for all autohomeomorphisms of zero-dimensional separable metrizable spaces. Likewise the Tychonoff cube $[0, 1]^\kappa$ carries an autohomeomorphism that is universal for all autohomeomorphisms of completely regular spaces of weight at most κ , and the Cantor cube 2^κ has a universal autohomeomorphism for all zero-dimensional such spaces.

Our goal is to have an autohomeomorphism h on $\mathbb{N}^*$ that is universal for all autohomeomorphisms of all *closed* subspaces of $\mathbb{N}^*$. The first result of this paper is that there is no trivial universal autohomeomorphism of $\mathbb{N}^*$, and hence no universal autohomeomorphism at all in any model where all autohomeomorphisms of $\mathbb{N}^*$ are trivial. On the other hand, the Continuum Hypothesis implies that there is a universal autohomeomorphism of $\mathbb{N}^*$. The proof of this will have to be different from the results mentioned above because $\mathbb{N}^*$ is definitely not homeomorphic to its power $(\mathbb{N}^*)^\omega$; it will use group actions and a homeomorphism extension theorem.

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We should mention the dual notion of universality where one requires the existence of a surjection $s : X \rightarrow Y$ such that $g \circ s = s \circ h$. For the space $\mathbb{N}^*$ this was investigated thoroughly in [2] for general group actions.

1. SOME PRELIMINARIES

Our notation is standard. For background information on $\mathbb{N}^*$ we refer to [5].

We denote by Aut the autohomeomorphism group of $\mathbb{N}^*$. We call a member h of Aut *trivial* if there are cofinite subsets A and B of $\mathbb{N}$ and a bijection $b : A \rightarrow B$ such that h is the restriction of βb to $\mathbb{N}^*$.

In both sections we shall use the G_δ -topology on a given space (X, τ) ; this is the topology τ_δ on X generated by the family of all G_δ -subsets in the given space. It is well-known that $w(X, \tau) \leq w(X, \tau)^\aleph_0$; we shall need this estimate in Section 3.

2. WHAT IF ALL AUTOHOMEOMORPHISMS ARE TRIVIAL?

To begin we observe that fixed-point sets of trivial autohomeomorphism of $\mathbb{N}^*$ are clopen. Therefore, to show that no trivial autohomeomorphism is universal it would suffice to construct a compact space that can be embedded into $\mathbb{N}^*$ and that has an autohomeomorphism whose fixed-point set is not clopen.

The example. We let L be the ordinal $\omega_1 + 1$ endowed with its G_δ -topology. Thus all points other than ω_1 are isolated and the neighbourhoods of ω_1 are exactly the co-countable sets that contain it. Then L is a P -space of weight $\aleph_1$ and hence, by the methods in [4, Section 2], its Čech-Stone compactification βL can be embedded into $\mathbb{N}^*$.

We define $f : L \rightarrow L$ such that ω_1 is the only fixed point of βf . We put

$$\begin{aligned} f(\omega_1) &= \omega_1, \\ f(2 \cdot \alpha) &= 2 \cdot \alpha + 1, \text{ and} \\ f(2 \cdot \alpha + 1) &= 2 \cdot \alpha. \end{aligned}$$

This defines a continuous involution on L .

If $p \in \beta L \setminus L$ then $p \in \text{cl } \alpha$ for some $\alpha < \omega_1$ and then either $E = \{2 \cdot \beta : \beta < \alpha\}$ or $O = \{2 \cdot \beta + 1 : \beta < \alpha\}$ belongs to the ultrafilter p . But $f[E] \cap E = \emptyset = f[O] \cap O$, hence $\beta f(p) \neq p$.

Since ω_1 is not an isolated point of βL , no matter how this space is embedded into $\mathbb{N}^*$ there is no trivial autohomeomorphism of $\mathbb{N}^*$ that would extend βf .

3. THE CONTINUUM HYPOTHESIS

Under the Continuum Hypothesis the space $\mathbb{N}^*$ is generally very well-behaved and one would expect it to have a universal autohomeomorphism as well. We shall prove that this is indeed the case. We need some well-known facts about closed subspaces of $\mathbb{N}^*$.

First we have Theorem 1.4.4 from [5] which characterizes the closed subspaces of $\mathbb{N}^*$ under CH: they are the compact zero-dimensional F -spaces of weight $\mathfrak{c}$, and, in addition: every closed subset of $\mathbb{N}^*$ can be re-embedded as a nowhere dense closed P -set.

Second we have the homeomorphism extension theorem from [3]: CH implies that every homeomorphism between nowhere dense closed P -sets of $\mathbb{N}^*$ can be extended to an autohomeomorphism of $\mathbb{N}^*$.

Step 1. We consider the natural action of $\mathbf{Aut}$ on $\mathbb{N}^*$; the map $\sigma : \mathbf{Aut} \times \mathbb{N}^* \rightarrow \mathbb{N}^*$ given by $\sigma(f, p) = f(p)$. This action is continuous when $\mathbf{Aut}$ carries the compact-open topology τ and hence also when $\mathbf{Aut}$ carries the G_δ -modification τ_δ of τ . For the rest of the construction we consider the topology τ_δ .

Using this action we define an autohomeomorphism $h : \mathbf{Aut} \times \mathbb{N}^* \rightarrow \mathbf{Aut} \times \mathbb{N}^*$ by $h(f, p) = (f, f(p))$. The map h is continuous because its two coordinates are and it is a homeomorphism because its inverse $(f, p) \mapsto (f, f^{-1}(p))$ is continuous as well.

Now if X is a closed subset of $\mathbb{N}^*$ and $g : X \rightarrow X$ is an autohomeomorphism then we can re-embed X as a nowhere dense closed P -set and we can then find an $f \in \mathbf{Aut}$ such that $f \upharpoonright X = g$. We transfer this embedded copy of X to $\{f\} \times \mathbb{N}^*$ in $\mathbf{Aut} \times \mathbb{N}^*$; for this copy of X we then have $h \upharpoonright X = g$. It follows that h satisfies the universality condition.

Step 2. We embed $\mathbf{Aut} \times \mathbb{N}^*$ into $\mathbb{N}^*$ in such a way that there is an autohomeomorphism H of $\mathbb{N}^*$ such that $H \upharpoonright (\mathbf{Aut} \times \mathbb{N}^*) = h$. Then H is the desired universal autohomeomorphism of $\mathbb{N}^*$.

To this end we list a few properties of this product.

Weight. The weight of the product is equal to $\mathfrak{c}$, as both factors have weight $\mathfrak{c}$. For $\mathbb{N}^*$ this is clear and for $\mathbf{Aut}$ this follows because the topology τ has weight $\mathfrak{c}$ and one obtains a base for τ_δ by taking the intersections of all countable subfamilies of a base for τ .

Zero-dimensional and F . The product is a zero-dimensional F -space as the product of the P -space $\mathbf{Aut}$ and the compact zero-dimensional F -space $\mathbb{N}^*$, see [6, Theorem 6.1].

Strongly zero-dimensional. The product $\mathbf{Aut} \times \mathbb{N}^*$ is not compact, but we shall construct a compactification of it that is also a zero-dimensional F -space of weight $\mathfrak{c}$.

For this we need to prove that $\mathbf{Aut} \times \mathbb{N}^*$ is actually strongly zero-dimensional. We prove more: the product is ultraparacompact, meaning that every open cover has a *pairwise disjoint* open refinement.

Let $\mathcal{U}$ be an open cover of the product consisting of basic clopen rectangles.

For each $f \in \mathbf{Aut}$ there is a finite subfamily $\mathcal{U}_f$ of $\mathcal{U}$ that covers $\{f\} \times \mathbb{N}^*$, say $\mathcal{U}_f = \{C_i \times D_i : i < k_f\}$. Let $C_f = \bigcap_{i < k_f} C_i$ and $D_{f,i} = D_i \setminus \bigcup_{j < i} D_j$ for $i < k_f$. Then $\mathcal{C}_f = \{C_f \times D_{f,i} : i < k_f\}$ is a disjoint family of clopen rectangles that covers $\{f\} \times \mathbb{N}^*$ and refines $\mathcal{U}$.

Because $\mathbf{Aut}$ has weight $\mathfrak{c}$, and we assume CH, there is a sequence $\langle f_\alpha : \alpha \in \omega_1 \rangle$ in $\mathbf{Aut}$ such that $\{C_{f_\alpha} : \alpha \in \omega_1\}$ covers $\mathbf{Aut}$. Next we let $V_\alpha = C_{f_\alpha} \setminus \bigcup_{\beta < \alpha} C_{f_\beta}$ for all α . Because $\mathbf{Aut}$ is a P -space the family $\{V_\alpha : \alpha \in \omega_1\}$ is a disjoint open cover of $\mathbf{Aut}$.

The family $\{V_\alpha \times D_{f_\alpha,i} : i < k_{f_\alpha}, \alpha \in \omega_1\}$ then is a disjoint open refinement of $\mathcal{U}$.

A compactification. To complete Step 2 we construct a compactification of $\mathbf{Aut} \times \mathbb{N}^*$ that is a zero-dimensional F -space of weight $\mathfrak{c}$ and that has an autohomeomorphism that extends h . The Čech-Stone compactification would be the obvious candidate, were it not for the fact that its weight is equal to $2^{\mathfrak{c}}$. More precisely, using some continuous onto function from $(\mathbf{Aut}, \tau)$ onto $[0, 1]$ one obtains a clopen partition of $(\mathbf{Aut}, \tau_\delta)$ of cardinality $\mathfrak{c}$. This shows that $\beta(\mathbf{Aut} \times \mathbb{N}^*)$ admits a continuous surjection onto the space $\beta\mathfrak{c}$ (where $\mathfrak{c}$ carries the discrete topology).

To create the desired compactification we build, either by transfinite recursion or by an application of the Löwenheim-Skolem theorem, a subalgebra $\mathbb{B}$ of the algebra of clopen subsets of $\mathbf{Aut} \times \mathbb{N}^*$ that is closed under h and h^{-1} , of cardinality $\mathfrak{c}$, and that has the property that for every pair of countable subsets A and B of $\mathbb{B}$ such that $a \cap b = \emptyset$ whenever $a \in A$ and $b \in B$ there is a $c \in \mathbb{B}$ such that $a \subseteq c$ and $c \cap b = \emptyset$ for all $a \in A$ and $b \in B$. The latter condition can be fulfilled because $\mathbf{Aut} \times \mathbb{N}^*$ is an F -space — $\bigcup A$ and $\bigcup B$ have disjoint closures — and strongly zero-dimensional — the closures can be separated using a clopen set.

The Stone space $\mathrm{St}(\mathbb{B})$ of $\mathbb{B}$ is then a compactification of $\mathbf{Aut} \times \mathbb{N}^*$ that is a compact zero-dimensional F -space of weight $\mathfrak{c}$, with an autohomeomorphism $\bar{h}$ that extends h . We embed $\mathrm{St}(\mathbb{B})$ into $\mathbb{N}^*$ as a nowhere dense P -set and extend $\bar{h}$ to an autohomeomorphism H of $\mathbb{N}^*$.

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