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Valuation of Mortgage Offer Options

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Li Sun
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VALUATION OF MORTGAGE OFFER OPTIONS

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Li SUN

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Applied Mathematics

Responsible Supervisor:	Prof. dr. ir. C. W. Oosterlee,	TU Delft
Daily Supervisor:	Prof. dr. ir. C. W. Oosterlee,	TU Delft
	drs. M. Keijzers,	ABN AMRO Bank N.V.



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PREFACE

This thesis consists of two parts. The first part is based on my work from November 01 2015 to April 30 2016 in ABN AMRO Bank where I got the opportunity to take on this project. The second part is based on my work from May 01, 2016 to July 2016 in TU Delft where I went further with this project with the application of some advanced numerical techniques. The defence date of this thesis is on August 31, 2016.

The information of the mortgages mentioned in this thesis is gathered from ABN AMRO Hypotheken Groep BV (AAHG). AAHG, one of the subsidiaries of ABN AMRO Bank, is responsible for the mortgage activities of ABN AMRO, providing mortgage products through various channels and under different brands, such as ABN AMRO(AAB) label, Florius label, MoneYou label. Among these mortgage labels, 40% of AAHG's portfolio share is consists of AAB label, which makes AAB label the main brand in AAHG. In the scope of this thesis, we focus on the mortgage (offers) under AAB label only.

The mortgage offer process considered in this thesis is the mortgage offer process before March 21 2016. On March 21 2016, the Mortgage Credit Directive (MCD) was implemented in Dutch legislation, requiring banks to meet the new rules. As a consequence, AAHG has adapted its mortgage offer process accordingly. The changed offer process has not been considered in this thesis due to the insufficient historical data. For the information of readers, MCD and the changed mortgage offer process is described in [Appendix B](#).

Li Sun

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I

FIRST PART

1

INTRODUCTION

A mortgage offer is a contract listing commitments between a prospective mortgage borrower and a mortgage lender. After the lender's approval on the borrower's creditworthiness, the borrower has an option, but not an obligation, to close this approved mortgage offer and pass the mortgage contract at the notary. Such option is called a mortgage offer option. Only if a mortgage offer is closed, the terms in the mortgage offer come into force. Mortgage offer options can be utilized by a rational borrower, which is beneficial to the borrowers' position and adverse to the lenders' position. To offset the adverse position, lenders can charge borrowers a reasonable value for the mortgage offer option, which poses the topic of this thesis.

Before introducing the topic of this thesis, we introduce some background knowledge related to mortgage offers. In Section 1.1, the basic knowledge about mortgages is introduced. The mortgage offer process, as a part of a mortgage application process, is introduced in Section 1.2. Afterwards, the value of mortgage offer options is explained, which leads to the research question of this thesis in Section 1.3.

1.1. MORTGAGE INTRODUCTION

"In general, a mortgage is a loan that is secured by underlying assets that can be repossessed in the event of default." quoted from [1]. In this thesis, we restrict our scope of mortgages to the loans which are required for buying residential real estate (RRE) with the RRE as collateral. Borrowers are obliged to make interest payments and repayments for their loans as stated in the mortgage contracts. If default happens during the loan term¹, the lender has the right to possess and sell the collateral compensating for a relevant loss.

A mortgage process is defined as the process where a mortgage borrower needs to go

¹The loan term of a mortgage is the period over which the loan runs [2].

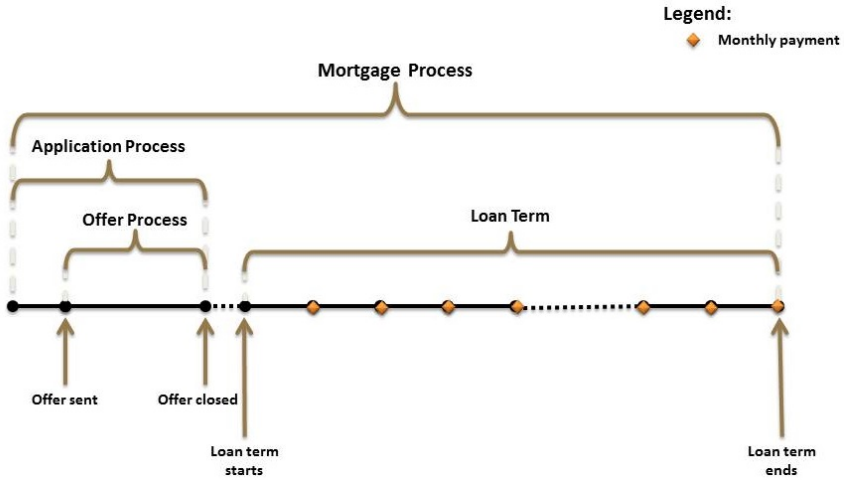


Figure 1.1: mortgage process

through from a mortgage application to the end of the loan term², which is illustrated in Figure 1.1. Before having a mortgage, a borrower applies for a mortgage from a mortgage lender. After the lender's processing of the application, a mortgage offer is sent to that borrower, which is the start point of a mortgage offer process. The mortgage offer is a contract which states loan commitments of the lender and the borrower. Generally, the commitments can not be changed after the offer is sent, but it is only when the offer is closed that the commitments need to be fulfilled. In Figure 1.1, the offer process ends when the offer is closed, after which the borrower officially starts a mortgage contract. Under ABN AMRO (AAB) label, the loan term starts on the first day of the next month when the first payment is made by the bank for that loan [2]³. The date on which the client starts to pay interest for the loan is the date on which the bank transfers the loan funds, which is called the loan starting date. To facilitate the valuation in this thesis, we make Assumption 1 regarding the loan starting date.

Assumption 1. *We assume the bank transfers the loan funds for a closed mortgage offer on the first day of the next month when the mortgage offer is closed.*

Therefore, the loan starting date is the same as the starting date of the loan term, which is the first day of the next month when the mortgage offer is closed. During the loan term, the borrower is obliged to pay a prescribed monthly payment to the lender until the loan term ends. At the end of the loan term, the outstanding principal of that loan will be paid if no default happens.

In a mortgage offer, the lender's commitment in the offer option is regarding the mortgage rate which is combined with the outstanding principal each month resulting

²Mortgage prepayment and default during the loan term are not considered in this thesis.

³If a mortgage offer is closed before the bank makes the first payment for that loan, the loan term will start on the first day of the month when the first payment is made [2].

in the borrower's monthly interest payment for that loan. The mortgage rate is priced based on the following mortgage contract variables: mortgage type, mortgage offer type, amortization scheme, Loan to Market Value (LtV), mortgage insurance or not. The explanation of those variables is as follows.

Mortgage Type

Under the AAB label, basically there are two types of mortgages, variable-rate mortgages and fixed-rate mortgages.

variable-rate mortgage:

A variable-rate mortgage is a mortgage with the interest rate adjusted by the lender each month according to the money market developments.

fixed-rate mortgage:

A fixed-rate mortgage is a mortgage with a fixed interest rate for a certain agreed period. This agreed period is known as the fixed-rate period (at least one year). Generally, the longer the fixed-rate period is chosen, the higher the corresponding fixed rate will be. At the end of the fixed-rate period, the borrower either pays off the outstanding principal or resets a mortgage rate for the outstanding principal.

Mortgage Offer Type

There are two available types of mortgage offers under the AAB label for fixed-rate mortgages, i.e., the *budget mortgage offer* and the *regular mortgage offer*.

budget mortgage offer:

A *budget mortgage offer* is a mortgage offer with a *budget option* which gives borrowers the right to lock a fixed mortgage rate for the applied mortgage. If a *budget option* holder closes the offer, the interest rate for this holder to pay is the locked fixed mortgage rate, regardless of a mortgage rate change during the offer period.

regular mortgage offer:

A *regular mortgage offer* is a mortgage offer with a *regular option* (also called *lock-or-lower option*) which gives a borrower the right to have the lower mortgage rate of the initially locked mortgage rate and the mortgage rate on the offer closing day. As a cost of this more beneficial option compared to a *budget option*, under the AAB label, a *regular option* holder needs to pay extra 10 basis points on the settled mortgage rate if the offer finally is closed, compared to the mortgage rate a budget holder pays.

Amortization Scheme

After closing the offers, borrowers are obliged to pay a monthly interest payment for their loan, meanwhile repaying their loan according to an agreed repayment scheme. Such a monthly payment scheme is called an amortization scheme. Specifically, an amortization scheme, or called redemption scheme, can be described by an amortization schedule which states the scheduled principal repayment, the interest payment, and the remaining principal in each month during the loan term. There are three basic amortization schemes under the AAB label, namely the *bullet amortization scheme*, the *linear amortization scheme*, and the *annuity amortization scheme*⁴. Figure 1.2, Figure 1.3 and Figure 1.4 demonstrate monthly payments under different amortization schemes.

Linear mortgage:

For a linear mortgage, a fixed repayment is made every month in the loan term. And the monthly payment will decrease steadily as the monthly interest rate payment does.

⁴see <https://www.abnamro.nl/en/personal/mortgages/mortgage-types.html>

Annuity mortgage (also called level-paying mortgage):

For an annuity mortgage, the monthly payment remains the same amount. Among the monthly payment, the interest payment part decreases while the repayment part increases during the loan term.

Bullet mortgage (also called interest-only mortgage):

For a bullet mortgage, the monthly payment is the interest payment part only until the end of the loan term. The outstanding principal is to be fully repaid at the end of the loan term.



Figure 1.2: linear amortization scheme



Figure 1.3: annuity amortization scheme

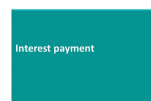


Figure 1.4: bullet amortization scheme

Loan to Market Value

Loan to market value (or loan-to-value (LtV)) of a mortgage is the ratio of the outstanding principal to the market value of the collateral. In the Netherlands, the max LtV is 102% in 2016 (103% in 2015) as regulated by the Mortgage Code of Conduct⁵. LtV affects the risk class which a mortgage falls into. For a mortgage, the higher its LtV is, the riskier class it is in, and the higher the mortgage rate generally is. On the other hand, a mortgage insurance for this mortgage is an alternative to improve its risk class without changing its LtV.

Mortgage Insurance

Mortgage insurance (also known as mortgage guarantee or home-loan insurance) is an insurance which is to compensate mortgage lenders for the losses caused by mortgage payment defaults. For instance, in the Netherlands, mortgages with the National Mortgage Guarantee Scheme (in Dutch: Nationale Hypotheek Garantie (NHG)) are evaluated limited financial risk to lenders, since if a mortgage with NHG defaults, NHG is obliged to pay 90% of losses to the lender⁶.

The combination of these price setting variables gives a specific mortgage product along with its corresponding mortgage rate, which determines the mortgage payment cash flows as long as the mortgage is closed⁷.

1.2. INTRODUCTION TO MORTGAGE OFFER PROCESS

The information in this section is based on the features of AAB label mortgages and therefore we refer to the mortgage lender as the bank. Before officially having a mortgage from a mortgage lender, a borrower usually makes a mortgage application, which is the starting point of an application process (see Figure 1.5). As Figure 1.5 demonstrates,

⁵See <http://www.dnb.nl/en/news/news-and-archive/dnbulletin-2015/dnb322357.jsp#>.

⁶The introduction of a loss-sharing mechanism was announced on 31 October 2013. The loss-sharing mechanism states that lenders take 10% of the losses if a mortgage with NHG defaults, unlike losses totally paid by NHG before this announcement. See https://www.fitchratings.com/gws/en/fitchwire/fitchwirearticle/Dutch-Mortgage-Guarantee?pr_id=807733

⁷The uncertainty of prepayments and default during a mortgage term is not considered in this scope.

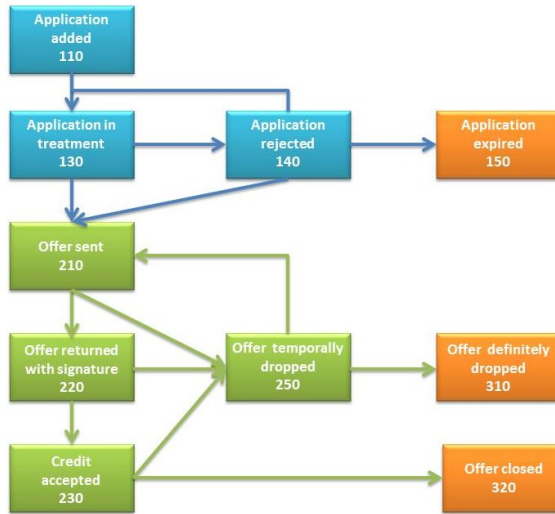


Figure 1.5: mortgage application process (the numbers represent the status codes in the system)

after processing the borrower's application (state 130), the bank sends the borrower a mortgage offer (state 210) consistent with the spot market as well as the borrower's preferred mortgage product. Under the AAB label, a borrower has maximum 2 weeks to consider and return the offer with signature, otherwise the offer expires. After the signed offer is returned (state 220), the bank starts to process a creditworthiness assessment for this returned offer within 21 days. If the borrower meets the credit requirements for the mortgage in the returned offer, the bank will approve the credit (state 230). Otherwise the bank rejects the credit (state 250), which means the returned offer is invalid and the borrower falls out of the right to close this rejected offer. Only after the bank accepts the credit (state 230), can the borrower close the offer within the offer period. Under the AAB label, the initial offer period, for both *budget offers* and *regular offers*, is 3 months. *Budget offers* cannot be extended beyond 3 months and therefore the offers must be closed within three months since the offer sent date. A regular offer's validity can be extended by 6 months to a total of 9 months. There is no extension fee. However, if a regular offer is not closed after an extension is requested, a cancellation fee of 1% of the loan principal is applicable. After closing the mortgage offer (state 320), the borrower gets the loan from the bank to buy the residential real estate. Figure 1.5 diagrammatically describes the above process. As long as an application reaches the state colored orange in Figure 1.5, this application officially ends, which means this application is not active anymore. So we could call these orange colored states (namely, state 150, state 310, state 320) the *application end states*.

Within an application process, an offer process starts when a mortgage offer is sent out (state 210). At this moment, the commitments made by the bank are specified in this sent offer, which means the bank is obliged to fulfill these commitments as long as

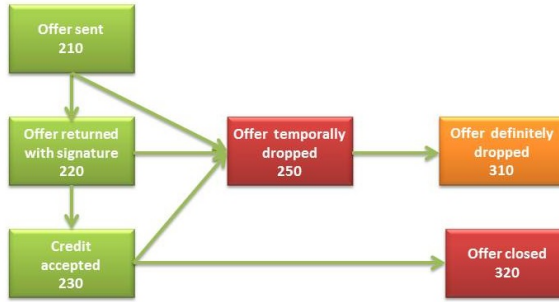


Figure 1.6: offer process

the borrower meets the requirements stated in the offer. However, the borrower is not obliged to finally close this offer. The end of an offer process means either this sent offer is dropped (state 250) or comes into force (state 320). We call state 250 and state 320 the *offer end states*, which are colored red in Figure 1.6. As a sent offer reaches either *offer end state*, the outcome of this offer process is known.

Offer processes can be differentiated by the type of rights given to borrowers during the offer period, because borrowers' rights imply the commitments of a lender. The potential loss to fulfill the commitments is a risk to the bank. As introduced in Section 1.1, only the offer type is related to a borrower's right in the offer process. So we differentiate a *budget offer* process from a *regular offer* process.

1.3. RESEARCH QUESTION

In [3], a mortgage default can be viewed as an option on collateral possession. Analogously, a *budget option* can be seen as an option on the possession of a forward loan with a fixed interest rate. A mortgage offer is close to a fixed-interest-rate forward loan contract to the borrower, which is committed by the lender. But the major differences are that the loan starting time in a mortgage offer is not fixed due to the offer option. The main research question of this thesis is:

- *How to value offer options?*

The price of a mortgage offer option should be the expected loss to the lender caused by the offer option. No entirely satisfactory terminology has ever been created, but a clear distinction exists between two directions in pricing offer options. One is based on the endogenous termination modelling in an offer process, and the other one is based on the exogenous termination modelling [3].

Endogenous Termination:

Offer options are exercised as a result of the borrower's minimizing the market value of the loan. Such exercise is regarded endogenous, or "optimal" termination in an offer process, because the market value of the loan can be viewed as the market cost to

the borrower by closing the mortgage offer and a rational borrower always tries to minimize the cost. In the absence of credit risk, this financial termination is independent of the borrower's individual characteristics, depends only on the term structure of interest rates. In the direction of the endogenous termination modelling in offer option valuation, contingent-claim models are feasible candidates in generating such endogenous termination so that the offer option price can be achieved.

Exogenous Termination:

Termination of a mortgage offer occurs for extraneous reasons, while not necessarily minimizing the objective market cost of the mortgage. The motivation of such termination can arise from personal circumstances of the borrower, such as the borrower's preference or belief about expected return on the underlying. In the direction of the exogenous termination modelling, the knowledge of human behaviour in an offer period is required. By predicting human behaviour in a coming offer period of a newly sent mortgage offer, the lender buys corresponding hedging instruments at the offer sending moment. The expected loss of this hedging portfolio consisting of the offer and the hedging instruments at the offer sending moment is a reasonable option price for this newly sent offer.

In this thesis, both proposed directions in pricing offer options are going to be performed.

2

EXOGENOUS TERMINATION MODELLING

This chapter is going to present the exogenous termination modelling for valuation of mortgage offer options. In Section 2.1, a hedge strategy for fixed-rate mortgage offers is proposed. Based on the hedge strategy, we get the quantity called *hit ratio* to be estimated for the mortgage offer process. An existing exogenous model called transition matrix method (TMM) for the *hit ratio* prediction is studied and implemented in Section 2.2. It motivates the improved transition matrix method in Section 2.3. In Section 2.4, some correlation tests for the *hit ratio* are performed. In Section 2.5, some other models regarding the *hit ratio* prediction are presented and analyzed, which concludes this chapter.

Some terms in the subsequent calculations are explained below.

Definition 2 (Discount Factor). *The discount factor $D(t, T)$ is to discount the cash flow occurring at time T to the value at time t . If the discount rate is continuous compounding and constant during the period from time t to time T , denoted as r ,*

$$D(t, T) = e^{r(t-T)};$$

if the discount rate is yearly compounding and constant during the period from time t to time T ,

$$D(t, T) = (1 + r)^{t-T},$$

where $(t - T)$ is in year fraction; if the discount rate is continuous compounding and floating during the period from time t to time T , denoted as $r(s)$ for the discounted rate at time s ($t \leq s \leq T$),

$$D(t, T) = \exp \left\{ - \int_t^T r(s) ds \right\}.$$

Definition 3 (Present Value (or Market Value)). *The present value of a future cash flow is defined as the current worth of that cash flow in the market. At time t , given that the rate of return for a Y -amount cash flow occurring at time T ($t < T$) is r , the present value (or the market value) of that cash flow is given by*

$$PV = Y \cdot D(t, T),$$

where $D(t, T)$ is the discount factor at the discount rate r .

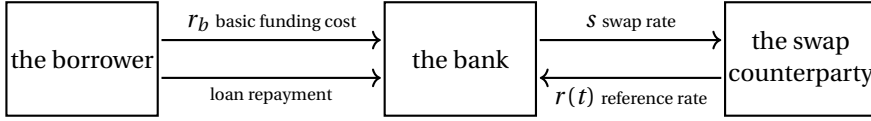
2.1. HEDGE STRATEGY

For a fixed-rate mortgage, the borrower is committed to a fixed mortgage rate m for a fixed rate period. The fixed mortgage rate m generally consists of two parts, the fixed rate of the basic funding cost r_b and the other cost spread ($m - r_b$). (Hereafter, we assume the other cost spread is constant during the offer period.) However, the market rate of the basic funding cost, i.e., the discount rate for the mortgage payments, which is indexed to a reference rate, such as the Euro Interbank Offered Rate (Euribor), is floating over time. The varying reference rate gives rise to the varying market value of the loan with the interest rate r_b during the fixed rate period. An upward movement of the reference rate during the fixed rate period can lead to the market value of the loan with the interest rate r_b less than the outstanding principal, which poses interest rate risk to the bank.

On the other hand, the market value of a loan with the reference rate as the interest rate is equal to the outstanding loan principal during the fixed rate period. For hedging the interest rate risk of the fixed-rate loan, the bank can construct a hedging portfolio of the loan to exchanged the fixed rate r_b charged from the borrower with the reference rate in the market. By such exchange, the market value of the hedging portfolio is equal to the outstanding loan principal during the fixed rate period.

Suppose a *budget offer* with the committed mortgage rate m_0 will be closed on a deterministic date. At the offer sending moment, the bank can hedge the interest rate risk by entering into a forward interest rate swap (FIRS) in which the bank receives the reference rate and pays a fixed rate (the swap rate) on a specified notional amount during the fixed rate period, as illustrated in Figure 2.1. Without considering mortgage prepayment and default, the outstanding loan amount during the fixed rate period is deterministic at the offer sending moment because of the known amortization scheme in the offer, which is the specified notional amount of the FIRS. By charging the swap rate as the fixed rate of the basic funding cost in m_0 , i.e., $r_b = s$ in Figure 2.1, the present value of the hedge portfolio of this budget offer is equal to the outstanding loan principal during the fixed rate period. Hence, the interest rate risk of the loan on the *budget offer* is totally hedged by the bank.

However, the above hedge strategy works based on two assumptions. One is the *budget offer* will be closed, and the other one is the offer closing date is deterministic. Although these two assumptions do not hold in reality, the number to be predicted for the hedge strategy can be derived. That is the so-called *hit ratio*.

Figure 2.1: hedge portfolio of a closed *budget* offer during the fixed-rate period

Definition 4 (Hit Ratio). *The hit ratio $h(t)$ of a mortgage offer is defined as the probability for the offer to be closed at time t . If t is a week, $h(t)$ is called weekly hit ratio. If t is a month, $h(t)$ is called monthly hit ratio.*

The *hit ratio* prediction is crucial for the hedge strategy and its hedge effectiveness. This is explained separately for the *budget offer* case and for the *regular offer* case.

In the *budget offer* case, without loss of generality, we suppose at the beginning of month T_0 , i.e., at time t_0 , the bank sends a batch of *budget offers* of the same mortgage contract variables. In the offers, the committed mortgage rate is m_0 (yearly compounding) for a T -year fixed rate period. The total mortgage principal of these offers is M . In the 3-month *budget offer* period, the monthly closed offer principal in month T_i ($i = 0, 1, 2$) is predicted as

$$M^{(p)}(T_i) = h^{(p)}(T_i) \cdot M,$$

where $h^{(p)}(T_i)$ (shorthand notation $h_i^{(p)}$) is the predicted month T_i *hit ratio*. In order to hedge the interest rate risk of the offers which are predicted to be closed in month T_i , at the offer sending moment t_0 the bank enters into a FIRS in which the bank pays the swap rate $s_i(t_0)$ and receives the reference rate on the outstanding principal of the loans during the T -year fixed rate period. The FIRS will be settled at the beginning of month T_{i+1} . The portfolio of the *budget offers* and the hedging instrument (the FIRS) is the hedging portfolio of the *budget offers*.

We take the reference rate as the risk-free rate in the risk-neutral measure $(\Omega, \mathcal{F}, Q)$ with filtration $\{\mathcal{F}_t\}_{t \geq t_0}$. Under the risk-neutral measure, the fixed rate of the basic funding cost in m_0 , denoted as $r_b(t_0)$, is the value satisfying the following equation.

$$\begin{aligned} \mathbb{E}^Q \left[\sum_{i=0}^2 \sum_{j=1}^n M^{(p)}(T_i, T_{i+j}) (r_b(t_0) - s_i(t_0)) \Delta t \cdot D(t_0, T_{i+j+1}) \middle| \mathcal{F}_{t_0} \right] &= 0, \\ r_b &= \frac{\mathbb{E}^Q \left[\sum_{i=0}^2 \sum_{j=1}^n M^{(p)}(T_i, T_{i+j}) s_i(t_0) \Delta t \cdot D(t_0, T_{i+j+1}) \middle| \mathcal{F}_{t_0} \right]}{\mathbb{E}^Q \left[\sum_{i=0}^2 \sum_{j=1}^n M^{(p)}(T_i, T_{i+j}) \Delta t \cdot D(t_0, T_{i+j+1}) \middle| \mathcal{F}_{t_0} \right]}, \end{aligned} \quad (2.1)$$

where the terms are explained as follows.

$M^{(p)}(T_i, T_{i+j})$ ($i = 0, 1, 2$; $j = 1, 2, \dots, n$): the outstanding principal in month T_{i+j} of the offers which are predicted to be closed in month T_i ;

$D(t_0, T_{i+j+1})$: the discount factor with the reference rate as the discount rate to discount the payment occurring at the beginning of month T_{i+j+1} to the value at the offer sending moment t_0 ;

n : the number of the scheduled payment times stated in the offer;

Δt : the time interval (in year fraction) between adjacent mortgage payments, i.e., $\Delta t = \frac{T}{n}$. Generally, $\Delta t = \frac{1}{12}$, meaning monthly mortgage payments in the offers.

Equation (2.1) implies that if the sent offers are closed exactly as predicted, the fixed rate of the basic funding cost settled by Equation (2.1) results in the value of the hedging portfolio equal to the outstanding loan amount at the offer sending moment.

At the end of month T_i ($i = 0, 1, 2$), it turns out that the realized closed amount in month T_i is $M(T_i)$. If $M(T_i) < M^{(p)}(T_i)$, it means that the *hit ratio* $h_i^{(p)}$ is under estimation, and the hedged amount by the FIRS is not enough so that an additional swap need to be entered for hedging the interest rate risk of the $(M^{(p)}(T_i) - M(T_i))^+$ -amount fixed-rate loan. The expected loss of the hedging portfolio to the bank is the market value of the hedging portfolio minus the outstanding loan principal plus the cost of the hedging instruments, i.e., the present value of the fixed rate payments according to the settled swap minus the present value of the basic funding cost payments by the borrowers plus the cost of the hedging instruments. Since there is no cost of entering the FIRS, the expected loss of the hedging portfolio to the bank at the beginning of month T_3 is calculated as

$$\begin{aligned}
 L_b = & \underbrace{+\mathbb{E}^Q \left[\sum_{i=0}^2 \sum_{j=1}^n (M(T_i, T_{i+j}) - M^{(p)}(T_i, T_{i+j}))^+ s_i(T_{i+1}) \Delta t \cdot D(T_3, T_{i+j+1}) \middle| \mathcal{F}_{T_3} \right]}_{\text{present value of the cash outflows of the fixed-rate payments by the additionally entered swaps}} \\
 & + \underbrace{\mathbb{E}^Q \left[\sum_{i=0}^2 \sum_{j=1}^n M^{(p)}(T_i, T_{i+j}) s_i(t_0) \Delta t \cdot D(T_3, T_{i+j+1}) \middle| \mathcal{F}_{T_3} \right]}_{\text{present value of the cash outflows of the fixed-rate payments by the initially entered FIRS}} \\
 & - \underbrace{\mathbb{E}^Q \left[\sum_{i=0}^2 \sum_{j=1}^n M(T_i, T_{i+j}) r_b(t_0) \Delta t \cdot D(T_3, T_{i+j+1}) \middle| \mathcal{F}_{T_3} \right]}_{\text{present value of the cash inflows of the basic funding cost payments by the borrowers}},
 \end{aligned} \tag{2.2}$$

where $M^{(p)}(T_i, T_{i+j})$ ($i = 0, 1, 2; j = 1, 2, \dots, n$), n , and Δt are explained in Equation (2.1), $M(T_i, T_{i+j})$ is the outstanding principal in month T_{i+j} of the offers which are indeed closed in month T_i , $s_i(T_{i+1})$ is the swap rate at the beginning of month T_{i+1} for hedging the interest rate risk of the *budget offers* closed in month T_i , $D(T_3, T_{i+j+1})$ is the discount factor with the reference rate as the discount rate to discount the payment occurring at the beginning of month T_{i+j+1} to the value at the beginning of month T_3 .

Without considering mortgage prepayments and defaults in the fixed rate period, by the same amortization scheme in the offers, the ratios of the realized monthly outstanding principals to the predicted monthly outstanding principals are the same. Hence, we

rewrite Equation (2.2) as

$$\begin{aligned}
 L_b(T_3) &= \mathbb{E}^Q \left[\sum_{j=1}^n \sum_{i=0}^2 \left(\left(\frac{M(T_i, T_{i+j})}{M^{(p)}(T_i, T_{i+j})} - 1 \right)^+ s_i(T_{i+1}) + s_i(t_0) \right. \right. \\
 &\quad \left. \left. - \frac{M(T_i, T_{i+j})}{M^{(p)}(T_i, T_{i+j})} r_b(t_0) \right) M^{(p)}(T_i, T_{i+j}) \Delta t \cdot D(T_3, T_{i+j+1}) \middle| \mathcal{F}_{T_3} \right] \\
 &= \mathbb{E}^Q \left[\sum_{j=1}^n \sum_{i=0}^2 \left(\left(\frac{h(T_i)}{h^{(p)}(T_i)} - 1 \right)^+ s_i(T_{i+1}) + s_i(t_0) \right. \right. \\
 &\quad \left. \left. - \frac{h(T_i)}{h^{(p)}(T_i)} r_b(t_0) \right) M^{(p)}(T_i, T_{i+j}) \Delta t \cdot D(T_3, T_{i+j+1}) \middle| \mathcal{F}_{T_3} \right],
 \end{aligned} \tag{2.3}$$

where $h(T_i)$ ($i = 0, 1, 2$) (shorthand notation h_i) is the realized month T_i *hit ratio*.

The expected loss of the hedging portfolio at the offer sending moment t_0 can be compensated by charging the *budget option* holders a reasonable *budget option* price. That is, the price of one-unit-principal *budget option* at t_0 is given by

$$\mathbb{E}^Q \left[-\frac{L_b(T_3)}{M} \cdot D(t_0, T_3) \middle| \mathcal{F}_{t_0} \right]. \tag{2.4}$$

In view of Equation (2.3), to obtain the expectation (2.4), we need to know a function of state factors to represent the variable $h(T_i)$ ($i = 0, 1, 2$) accurately, i.e., knowing the mechanics how the value $h(T_i)$ ($i = 0, 1, 2$) changes in the market environment. Additionally, the *budget option* price can be minimized by choosing an optimal $h^{(p)}(T_i)$ ($i = 0, 1, 2$) in the hedge strategy after knowing the function of $h(T_i)$ ($i = 0, 1, 2$).

When we turn to *regular mortgage offers*, the mortgage rate commitment on a *regular offer* not only gives a borrower the right to lock in a mortgage rate, but also the right to have a lower mortgage rate of the initially locked one and the one on the offer closing date. By such commitment, the mortgage interest payments generated by a closed *regular offer* can differ from the ones by the closed *budget offer*, which also implies the hedge strategy of the *regular offer* is not the same as the one of the *budget offer*. Without loss of generality, suppose at the beginning of month T_0 , i.e., at time t_0 , the bank sends a batch of *regular offers* of the same mortgage contract variables. The initially locked mortgage rate is m_0 for a T -year fixed rate period in the offers. The fixed rate of the basic funding cost in m_0 is $r_b(t_0)$. The total mortgage principal of these offers is M . For the 9-month *regular offer* period, the monthly closed offer principal in month T_i ($i = 0, 1, \dots, 8$) is predicted by $M^{(p)}(T_i) = h_i^{(p)} M$, where $h_i^{(p)}$ is the predicted month T_i *hit ratio*. If the regular offers are closed at time t with the mortgage rate $m_t > m_0$ on the offer closing date, the borrowers pay the mortgage rate m_0 , which means that in order to hedge the interest rate risk of these closed regular offers, the bank should exchange the fixed rate of the basic funding cost in m_0 with the reference rate during the fixed rate period. In this situation, a FIRS at the offer sending moment t_0 should be entered by the bank to hedge the interest rate risk of these closed regular offers, which is the same hedge strategy as the one in the *budget offer* case. However, if the regular offers are closed at time t with the mortgage

rate $m_t \leq m_0$ on the offer closing date, the borrowers pay the mortgage rate m_t , which means that in order to hedge the interest rate risk of these closed regular offers, the bank should exchange the fixed rate of the basic funding cost in m_t , denoted as $r_b(t)$, with the reference rate by entering a FIRS on the offer closing date. The floating mortgage rate on the offer closing day implies that the hedge strategy for the regular offers should be able to hedge the interest rate risk of both the loan with the interest rate $r_b(t_0)$ and the loan with the interest rate $r_b(t)$ during the fixed rate period. Hence, in addition to entering a FIRS at the offer sending moment t_0 , the bank also buys a swaption on the notional amount equal to the FIRS'. The swaption is a European put option with the swap rate of the FIRS as the strike price, and matures on the settlement date of the FIRS, so that it helps the bank to dump the unused part of the entered FIRS without loss.

The detailed operation can be seen as follows:

In order to hedge the interest rate risk of the *regular offers* closed in month T_i ($i = 0, 1, \dots, 8$), at the offer sending moment t_0 the bank enters into a FIRS in which the bank pays a fixed rate (or the swap rate) $s_i(t_0)$ and receives the reference rate on the outstanding principal during the T -year fixed rate period. The FIRS settles at the beginning of month T_{i+1} . At the offer sending moment, the bank also buys a swaption on the notional amount equal to the FIRS's. The swaption is a European put option with the swap rate $s_i(t_0)$ as the strike rate, which is to hedge the risk of selling the unused FIRS at the settlement. At the settlement of the FIRS, i.e., at the beginning of month T_{i+1} , if the swap rate is $s_i(T_{i+1})$, the payoff of the bought swaption is given by

$$\mathbb{E}^Q \left[\sum_{j=1}^n M^{(p)}(T_i, T_{i+j}) \left(s_i(t_0) - s_i(T_{i+1}) \right)^+ \Delta t \cdot D(T_{i+1}, T_{i+j+1}) \middle| \mathcal{F}_{T_{i+1}} \right],$$

where $M^{(p)}(T_i, T_{i+j})$ ($i = 0, 1, \dots, 8; j = 1, 2, \dots, n$), $D(T_{i+1}, T_{i+j+1})$, n and Δt are explained in Equation (2.2). On the offer closing day $t \in [T_i, T_{i+1}]$ ¹, if the swap rate $s_i(t)$ is not greater than the one of the initially entered FIRS, i.e., $s_i(t) \leq s_i(t_0)$, the bank enters another FIRS with the swap rate $s_i(t)$ to exchange with the reference rate on the outstanding principal of the closed offers during the fixed rate period. The settlement date of this newly entered FIRS is the beginning of month T_{i+1} . If the swap rate $s_i(t)$ is greater than the one of the initially entered FIRS, i.e., $s_i(t) > s_i(t_0)$, the initially entered FIRS is used to hedge the interest rate risk of the closed offers. At the end of the offer period, the unused part of the initially entered FIRS will be sold without any loss by holding the swaption. By such operation, the bank can hedge the interest rate risk of the regular offers.

Now let us check the loss of the hedging portfolio of the *regular offers*. Suppose in month T_0 , among these *regular offers*, there are some closed at time $t \in [T_0, T_1]$ with the fixed rate of the basic funding cost

$$\min(r_b(t_0), r_b(t)).$$

¹ $[T_i, T_{i+1})$ here represents a discrete set containing the days within month T_i .

The total principal of the offers closed at t is denoted as $M(t)$. At the beginning of month T_1 , the loss of the hedging portfolio for the *regular offers* closed in month T_0 can be obtained as follows.

In the case of $\sum_{t \in [T_0, T_1]} M(t) \cdot \mathbb{1}_{\{s_0(t) > s_0(t_0)\}} > M^{(p)}(T_0)$ ²:

$\sum_{t \in [T_0, T_1]} M(t) \cdot \mathbb{1}_{\{s_0(t) > s_0(t_0)\}} > M^{(p)}(T_0)$ means that the predicted *hit ratio* $h_0^{(p)}$ is underestimated and the notional amount in the initially entered FIRS is insufficient for hedging the interest rate risk of the closed offers in month T_0 . So at the beginning of month T_1 , the bank needs to enter an interest rate swap (IRS) on the notional amount

$$\left(\sum_{t \in [T_0, T_1]} M(t) \cdot \mathbb{1}_{\{s_0(t) > s_0(t_0)\}} - M^{(p)}(T_0) \right)^+.$$

The entered IRS has the same payment feature as the initially entered FIRS, except for the swap rate. The swap rate of the IRS is denoted as $s_0(T_1)$. All the bought month T_1 matured swaption are left without the need to hedge the loss of dumping the unused FIRS. At the beginning of month T_1 , the expected loss of the hedging portfolio is

$$\begin{aligned} L_r(T_1) = & \underbrace{+\mathbb{E}^Q \left[\sum_{j=1}^n \left(\sum_{t \in [T_0, T_1]} M(t, T_j) \mathbb{1}_{\{s_0(t) > s_0(t_0)\}} - M^{(p)}(T_0, T_j) \right) s_0(T_1) \Delta t \cdot D(T_1, T_{j+1}) \right]_{\mathcal{F}_{T_1}}}_{\text{present value of the cash outflows of the fixed-rate payments by the swap entered at the end of month } T_0} \\ & + \underbrace{\mathbb{E}^Q \left[\sum_{j=1}^n \sum_{t \in [T_0, T_1]} M(t, T_j) \mathbb{1}_{\{s_0(t) \leq s_0(t_0)\}} s_0(t) \Delta t \cdot D(T_1, T_{j+1}) \right]_{\mathcal{F}_{T_1}}}_{\text{present value of the fixed-rate payments by the additional FIRSs entered during month } T_0} \\ & + \underbrace{\mathbb{E}^Q \left[\sum_{j=1}^n M^{(p)}(T_0, T_j) s_0(t_0) \Delta t \cdot D(T_1, T_{j+1}) \right]_{\mathcal{F}_{T_1}}}_{\text{present value of the cash outflows of the fixed-rate payments by the intially entered FIRS}} \\ & - \underbrace{\mathbb{E}^Q \left[\sum_{j=1}^n \sum_{t \in [T_0, T_1]} \left(M(t, T_j) \cdot \mathbb{1}_{\{r_b(t) > r_b(t_0)\}} r_b(t_0) + M(t, T_j) \cdot \mathbb{1}_{\{r_b(t_0) \geq r_b(t)\}} r_b(t) \right) \Delta t \cdot D(T_1, T_{j+1}) \right]_{\mathcal{F}_{T_1}}}_{\text{present value of the cash inflows of the basic funding cost payments by the borrowers}} \\ & - \underbrace{\mathbb{E}^Q \left[\sum_{j=1}^n \min \left\{ \sum_{t \in [T_0, T_1]} M(t, T_j) \cdot \mathbb{1}_{\{s_0(t) > s_0(t_0)\}}, M^{(p)}(T_0, T_j) \right\} \left(s_0(t_0) - s_0(T_1) \right)^+ \Delta t \cdot D(T_1, T_{j+1}) \right]_{\mathcal{F}_{T_1}}}_{\text{payoff of the month } T_0 \text{ matured swaption}} \end{aligned} \quad (2.5)$$

plus the cost of the bought swaption.

In the case of $\sum_{t \in [T_0, T_1]} M(t) \cdot \mathbb{1}_{\{s_0(t) > s_0(t_0)\}} = M^{(p)}(T_0)$:

$\sum_{t \in [T_0, T_1]} M(t) \cdot \mathbb{1}_{\{s_0(t) > s_0(t_0)\}} = M^{(p)}(T_0)$ means that the hedged amount in the FIRS is exactly the used amount and no extra IRS is needed at the beginning of month T_1 . And all the bought month T_1 matured swaption are left without the need to hedge the risk of

² $\mathbb{1}_{\{\mathcal{A}\}}$ is an indicator function which equals 1 when event $\mathcal{A}$ is true, otherwise equal to zero.

dumping the FIRS. At the beginning of month T_1 , the expected loss of the hedging portfolio is given by Equation (2.5) plus the cost of the bought swaption.

2

In the case of $\sum_{t \in [T_0, T_1]} M(t) \cdot \mathbb{1}_{\{s_0(t) > s_0(t_0)\}} < M^{(p)}(T_0)$:

$\sum_{t \in [T_0, T_1]} M(t) \cdot \mathbb{1}_{\{s_0(t) > s_0(t_0)\}} < M^{(p)}(T_0)$ means that the *regular offers* is over hedged by the initially entered FIRS. Hedging the offer principal $(M^{(p)}(T_0) - \sum_{t \in [T_0, T_1]} M(t) \cdot \mathbb{1}_{\{s_0(t) > s_0(t_0)\}})^+$ in the initially entered FIRS is redundant. Only $\min\{\sum_{t \in [T_0, T_1]} M(t) \cdot \mathbb{1}_{\{s_0(t) > s_0(t_0)\}}, M^{(p)}(T_0)\}$ notional amount of the initially entered FIRS is used at the beginning of month T_1 , while the unused part of the FIRS is hedged by the month T_1 matured swaption. At the beginning of month T_1 , the expected loss of the hedging portfolio is given by Equation (2.5) plus the cost of the bought swaption.

At the end of the offer period, i.e., the beginning of month T_9 , the expected loss of the hedging portfolio is given by

$$\begin{aligned}
 L_r(T_9) = \mathbb{E}^Q \left[\sum_{i=0}^8 \sum_{j=1}^n \left(\left(\sum_{t \in [T_i, T_{i+1}]} M(t, T_{i+j}) \mathbb{1}_{\{s_i(t) > s_i(t_0)\}} - M^{(p)}(T_i, T_{i+j}) \right)^+ s_i(T_{i+1}) \right. \right. \\
 + M^{(p)}(T_i, T_{i+j}) s_i(t_0) + \sum_{t \in [T_i, T_{i+1}]} M(t, T_{i+j}) \mathbb{1}_{\{s_i(t) \leq s_i(t_0)\}} s_i(t) \\
 - \sum_{t \in [T_i, T_{i+1}]} M(t, T_{i+j}) \cdot \mathbb{1}_{\{r_b(t) > r_b(t_0)\}} r_b(t_0) - \sum_{t \in [T_i, T_{i+1}]} M(t, T_{i+j}) \cdot \mathbb{1}_{\{r_b(t_0) \geq r_b(t)\}} r_b(t) \\
 \left. \left. - \min \left\{ \sum_{t \in [T_i, T_{i+1}]} M(t, T_{i+j}) \cdot \mathbb{1}_{\{s_i(t) > s_i(t_0)\}}, M^{(p)}(T_i, T_{i+j}) \right\} \left(s_i(t_0) - s_i(T_{i+1}) \right)^+ \right) \right. \\
 \left. \cdot \Delta t \cdot D(T_9, T_{i+j+1}) \right] \Big| \mathcal{F}_{T_9} \Big] \quad (2.6)
 \end{aligned}$$

plus the cost of the bought swaption.

Under the risk-neutral measure $(\Omega, \mathcal{F}, Q)$ with filtration $\{\mathcal{F}_t\}_{t \geq t_0}$, a reasonable price of one-unit-principal *regular option* at t_0 is given by

$$\mathbb{E}^Q \left[-\frac{L_r(T_9)}{M} \cdot D(t_0, T_9) \Big| \mathcal{F}_{t_0} \right]. \quad (2.7)$$

To obtain the expectation (2.7), we need know a function of state factors to represent the variable $h(T_i)$ ($i = 0, 1, \dots, 8$) of the *regular offers* accurately, i.e., the mechanics how the value $h(T_i)$ changes in the market environment. Additionally, the *regular option* price can be minimized by choosing an optimal $h^{(p)}(T_i)$ ($i = 0, 1, \dots, 8$) in the hedge strategy after knowing the function of $h(T_i)$ ($i = 0, 1, \dots, 8$).

Therefore, we conclude that the *hit ratio* is crucial in the hedge strategy for mortgage offer options and its dynamics determines the hedge effectiveness and the offer option price.

2.2. TRANSITION MATRIX METHOD (TMM)

In this section, a *hit ratio* prediction model called transition matrix method is introduced. In order to check its predictive performance, we perform some replication tests of TMM based on the data in Appendix A. From the replication results, we can not only deduce some drawbacks of TMM, but also propose a way to improve TMM, which motivates the improved transition matrix method (ITMM) in Section 2.3

2.2.1. MODEL DESCRIPTION

The transition matrix method (TMM) is used to predict monthly *hit ratios* of newly sent offers in one month [4]. It is based on a six-state transition matrix. The six states are the possible offer states in an offer process (see Figure 1.6), namely state 210, state 220, state 230, state 250, state 310, and state 320. Based on the state transition information of the active offers in month t , the month t transition matrix $P(t)$ is defined as

$$P(t) = \begin{pmatrix} P_{210,210}(t) & P_{210,220}(t) & P_{210,230}(t) & P_{210,250}(t) & P_{210,310}(t) & P_{210,320}(t) \\ 0 & P_{220,220}(t) & P_{220,230}(t) & P_{220,250}(t) & P_{220,310}(t) & P_{220,320}(t) \\ 0 & 0 & P_{230,230}(t) & P_{230,250}(t) & P_{230,310}(t) & P_{230,320}(t) \\ 0 & 0 & 0 & P_{250,250}(t) & P_{250,310}(t) & P_{250,320}(t) \\ 0 & 0 & 0 & 0 & P_{310,310}(t) & P_{310,320}(t) \\ 0 & 0 & 0 & 0 & 0 & P_{320,320}(t) \end{pmatrix} \quad (2.8)$$

where $0 \leq P_{i,j}(t) \leq 1$ ($i \leq j$, $i, j \in \mathcal{J} := \{210, 220, 230, 250, 310, 320\}$.) is the transition probability that the offers in i -state at the beginning of month t transit to state j at the end of month t . In fact,

$$P_{250,320}(t) = 0, \quad P_{310,310}(t) = 1, \quad P_{310,320}(t) = 0, \quad P_{320,320}(t) = 1,$$

because offers in state 250 or state 310 are invalid and can not transit to state 320, vice versa. The transition probabilities are measured in the principal ratio, i.e.,

$$P_{i,j}(t) = \frac{M_{i,j}(t)}{\sum_{l \in \mathcal{J}} M_{i,l}(t)}, \quad i \leq j, \quad i, j \in \mathcal{J}, \quad (2.9)$$

where $M_{i,j}(t)$ is the total principal amount of the offers which stay in state i at the beginning of month t and transit to state j at the end of month t . The i -state ratio of the active offers at the beginning of month t , denoted as $S_i(t)$, is defined as the percentage (in offer principal) of the offers in state i at the beginning of month t , given by

$$S_i(t) := \frac{\sum_{l \in \mathcal{J}}^{l \geq i} M_{i,l}(t)}{\sum_{i \in \mathcal{J}} \sum_{l \in \mathcal{J}}^{l \geq i} M_{i,l}(t)}. \quad (2.10)$$

We define the state ratio of the offers at the beginning of month t as a vector containing the i -state ratio ($i \in \mathcal{J}$), given by

$$\mathbf{S}(t) := \begin{pmatrix} S_{210}(t) \\ S_{220}(t) \\ S_{230}(t) \\ S_{250}(t) \\ S_{310}(t) \\ S_{320}(t) \end{pmatrix}.$$

Without the inflow of newly sent offers in month t , it holds that

$$\begin{pmatrix} S_{210}(t) \\ S_{220}(t) \\ S_{230}(t) \\ S_{250}(t) \\ S_{310}(t) \\ S_{320}(t) \end{pmatrix}^{\text{tr}} P(t) = \begin{pmatrix} S_{210}(t+1) \\ S_{220}(t+1) \\ S_{230}(t+1) \\ S_{250}(t+1) \\ S_{310}(t+1) \\ S_{320}(t+1) \end{pmatrix}^{\text{tr}}. \quad (2.11)$$

where tr means the transpose of the vector. Without the inflow of newly sent offers in month t , the month t *hit ratio* $h(t)$ of the offers which are active in the offer process at the beginning of month t , can be obtained by

$$h(t) = S_{320}(t+1) - S_{320}(t).$$

It should be mentioned that the transition matrix calculation differentiates the offer information in mortgage labels and mortgage offer types [4]. Differentiation over mortgage labels and mortgage offer types is necessary, since different kinds of rights may be given to borrowers by different mortgage labels or different mortgage offer options, which impacts borrowers' decisions making in the offer process.

According to [4], J. Karelse and I. Ent have shown that based on historical data, the 3-month transition matrix of the offers sent in month t is roughly similar to $\bar{P}^3(t)$, where $\bar{P}(t) = \frac{1}{9} \sum_{q=t-9}^{q=t-1} P(q)$, and hence presumed that the Markov property holds in the transition process. Under this assumption, $\bar{P}^n(t)$ can be used to obtain an n -month ahead forecast of the state ratio $S(t+n)$ of the offers sent in month t .

To predict the monthly *hit ratios* of the offers sent in month t , there are two possible approaches of TMM. The differences are the input of the initial state ratio.

FIRST APPROACH IN TMM

For the offers sent in month t , first we get the predicted transition matrix $\bar{P}(t)$ by averaging the last nine one-month transition matrices, i.e.,

$$\bar{P}(t) = \frac{1}{9} \sum_{q=t-9}^{q=t-1} P(q).$$

Second, we use $\bar{P}(t)$ to predict the monthly *hit ratios* of the offers sent in month t . The predicted month $(t+1)$ *hit ratio* is given by

$$h(t+1) = \left(\mathbf{S}^{\text{tr}}(t+1) \cdot \bar{P}(t) \right)_{320} - S_{320}(t+1),$$

where $\mathbf{S}(t+1)$ is the state ratio vector at the beginning of month $(t+1)$ for the offers sent in month t . The predicted *hit ratio* of month $(t+i)$, $i \in \mathbb{N}^+$, is given by

$$h(t+i) = \left(\mathbf{S}^{\text{tr}}(t+1) \cdot \bar{P}^i(t) - \mathbf{S}^{\text{tr}}(t+1) \cdot \bar{P}^{i-1}(t) \right)_{320},$$

until the maximum offer period is included. The predicted *hit ratios* obtained under the above approach are called the transition matrix method (TMM) predicted *hit ratios*.

SECOND APPROACH IN TMM

One problem in the first approach is that the month t *hit ratio* of the offer sent in month t , i.e., $S_{320}(t+1)$, can not be predicted due to the use of $\mathbf{S}(t+1)$ in the calculation. To tackle this problem, we make Assumption 5.

Assumption 5. *For the offers sent in month t , we assume the offer state of those offers at the beginning of month t is state 210.*

By Assumption 5, we obtain the state ratio $\mathbf{S}(t)$ of the offers sent in month t as follows.

$$\mathbf{S}(t) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad (2.12)$$

And then $\mathbf{S}^{\text{tr}}(t) \cdot \bar{P}(t)$ gives the predicted resulting state ratio at the end of month t . Hence, combined with Assumption 5, the second approach in TMM gives the monthly *hit ratios* as follows.

$$\begin{aligned} h(t) &= \left(\mathbf{S}^{\text{tr}}(t) \cdot \bar{P}(t) \right)_{320} = \bar{P}_{210,320}(t) \\ h(t+i) &= \left(\mathbf{S}^{\text{tr}}(t) \cdot \bar{P}^{i+1}(t) - \mathbf{S}^{\text{tr}}(t) \cdot \bar{P}^i(t) \right)_{320}, \quad i \in \mathbb{N}^+, \end{aligned}$$

until the final month in the offer period is included. The predicted *hit ratios* obtained under this approach are called the alternative transition matrix method (ATMM) predicted *hit ratios*.

2.2.2. REPLICATION RESULTS

In this section, the data in the AST data sheet (see Appendix A) are going to be used in the calculation of realized historical monthly *hit ratios* as well as the predicted monthly *hit ratios* by TMM. Due to the limitation of the collected data, the transition probability here is measured in offer number ratio instead of offer principal ratio, which means that Equation (2.9) is replaced with Equation (2.13) in the following replication tests of the transition matrix method.

$$P_{i,j}(t) = \frac{N_{i,j}(t)}{\sum_{l \in \mathcal{J}} N_{i,l}(t)}, \quad i \leq j, \quad i, j \in \mathcal{J}, \quad (2.13)$$

where $N_{i,j}(t)$ is the number of offers which are in state i at the beginning of month t and transit to state j at the end of month t . And the calculation of the i -state ($i \in \mathcal{J}$) ratio in Equation (2.10) is replaced with

$$S_i(t) = \frac{\sum_{l \in \mathcal{J}}^{l \geq i} N_{i,l}(t)}{\sum_{i \in \mathcal{J}} \sum_{l \in \mathcal{J}}^{l \geq i} N_{i,l}(t)}. \quad (2.14)$$

The goodness of fit of TMM may indicate its predictive performance. The applied statistic tools to measure the goodness of fit of TMM in this section are the **R-squared** and the **Root Mean Squared Error (RMSE)**. Suppose $\{x_i\}_{1 \leq i \leq n}$ are realized data points which are predicted as $\{\hat{x}_i\}_{1 \leq i \leq n}$ respectively, by the prediction model. The sum of squares total (SST) and the sum of squares errors (SSE) are given by

$$SST = \sum_{i=1}^n (x_i - \bar{x})^2, \quad SSE = \sum_{i=1}^n (x_i - \hat{x}_i)^2,$$

where

$$\bar{x} = \frac{\sum_{i=1}^n x_i}{n}.$$

The SST measures how far the realized data points are from their mean, and the SSE measures how far the realized data are from the predicted values. So,

$$\mathbf{R-squared} = \frac{SST - SSE}{SST}, \quad \mathbf{RMSE} = \sqrt{\frac{SSE}{n}}.$$

The **R-squared** is a relative measure of fit, measuring the proportional prediction improvement by the prediction model, compared to the mean model, while the **RMSE** is an absolute measure of fit. One of the results is presented in Table 2.1.

Monthly <i>hit ratio</i> Prediction	TMM – R-squared	TMM – RMSE	ATMM – R-squared	ATMM – RMSE
in month T_0			-0.07141	0.25559
in month T_1	0.42534	0.046992	0.20892	0.055135
in month T_2	0.23995	0.064938	0.063747	0.072073
in month T_3	-0.0694	0.060646	-0.17284	0.063511

Table 2.1: the **R-squared** and the **RMSE** of (A)TMM in the monthly *hit ratio* prediction for *budget offers*

From Table 2.1, the **R-squared** and the **RMSE** both suggest that the TMM prediction preforms better than the ATMM prediction. The main reason is that the TMM prediction has the input $\mathbf{S}(t+1)$ — the state ratio result of the first month transition. The **R-squared** measures the predictive performance of the model, compared to the mean model. Normally, the value of **R-squared** ranges from zero to one, where one indicates that the model explains all the movements of the response data around its mean, while zero indicates that the model explains none of the movements of the response data around its mean. Negative R-squared values indicate that in view of the sum of prediction errors, the predictive performance of the model even falls behind the mean model's. From the results in Table 2.1, neither TMM nor ATMM can explain more than 50% of the movements of the response data around its mean, and the **RMSE** results are relatively large, compared to the realized monthly *hit ratios* in Figure 2.3. Hence, in terms of the **R-squared** and **RMSE** measurements, TMM and ATMM can not be accepted in the *hit ratio* prediction.

In addition, statistical hypothesis testing can also be applied to measure the goodness of fit of the prediction model. Before starting a hypothesis test, we do the following preparation:

Hypothesis Test A:

A standard criterion to judge a prediction model is the extent to which the predicted values resemble the observed data. So if the predicted values are exactly the observed data, the model is a perfect prediction model. Testing whether the predicted values are the same as the realized data is equivalent to testing whether the prediction error equals zero, where the prediction error is equal to the observed value minus the corresponding predicted one. Thus, the null hypothesis H_0 and the alternative hypothesis H_1 of the test are

$$\left\{ \begin{array}{l} H_0 = \left\{ \begin{array}{l} \text{The values of the prediction errors and a series of zero values} \\ \text{come from the same continuous distribution.} \end{array} \right\} \\ H_1 = \left\{ \text{These two series of values are from different continuous distributions.} \right\} \end{array} \right. \quad (2.15)$$

The two-sample Kolmogorov-Smirnov test is applied to complete Hypothesis Test A. The test result, denoted as h , is 1 if the test rejects the null hypothesis at the 5% significance level (also called p-value), and 0 otherwise.

Hypothesis Test B:

Considering that the above hypothesis test is based on a strict requirement, i.e., the prediction error should be zero, we switch to somewhat relaxed requirement for the prediction model now. The requirement is the prediction error should follow a normal distribution with zero mean. So in the second hypothesis test, the null hypothesis H_0 and the alternative hypothesis H_1 are

$$\left\{ \begin{array}{l} H_0 = \left\{ \begin{array}{l} \text{Prediction errors come from a normal distribution} \\ \text{with zero mean and unknown variance.} \end{array} \right\} \\ H_1 = \left\{ \text{Prediction errors do not have a zero mean.} \right\} \end{array} \right. \quad (2.16)$$

The t-test is applied to complete Hypothesis Test B.

One thing worth to mention here is that the above hypothesis tests are based on the assumed performance criterion respectively for the prediction model. The crucial part of an appropriate hypothesis test is the design of the null hypothesis. It is reasonable to presume that an appropriate design of the null hypothesis of the *hit ratio* prediction error should be adapted to the prediction error tolerance of the bank. So the above proposed hypothesis tests are designed for reference only. Some of the hypothesis testing results are presented in Table 2.2 and Table 2.3.

The results of the hypothesis tests in Table 2.2 and Table 2.3 indicate the model performance in terms of the distribution of the prediction errors (or say the residuals). Referred as the assumed criteria of a *hit ratio* prediction model, the hypothesis testing results suggest that both TMM and ATMM fail to meet those criterion.

ATMM Predicted Monthly <i>hit ratio</i>	Hypothesis A- <i>p</i> – value	Hypothesis A- <i>h</i>	Hypothesis B- <i>p</i> – value	Hypothesis B- <i>h</i>
in month T_0	7.1211e-10	1	3.6303e-04	1
in month T_1	1.7617e-18	1	3.5832e-09	1
in month T_2	1.7617e-18	1	5.1460e-09	1
in month T_3	3.6433e-14	1	0.0040	1

Table 2.2: Results of the hypothesis tests for the monthly *hit ratio* prediction by the alternative transition matrix method (ATMM)

TMM Predicted Monthly <i>hit ratio</i>	Hypothesis A- <i>p</i> – value	Hypothesis A- <i>h</i>	Hypothesis B- <i>p</i> – value	Hypothesis B- <i>h</i>
in month T_1	2.9330e-24	1	3.3251e-12	1
in month T_2	1.7617e-18	1	3.9530e-09	1
in month T_3	1.9435e-10	1	0.0165	1

Table 2.3: Results of the hypothesis tests for the monthly *hit ratio* prediction by the transition matrix method (TMM)

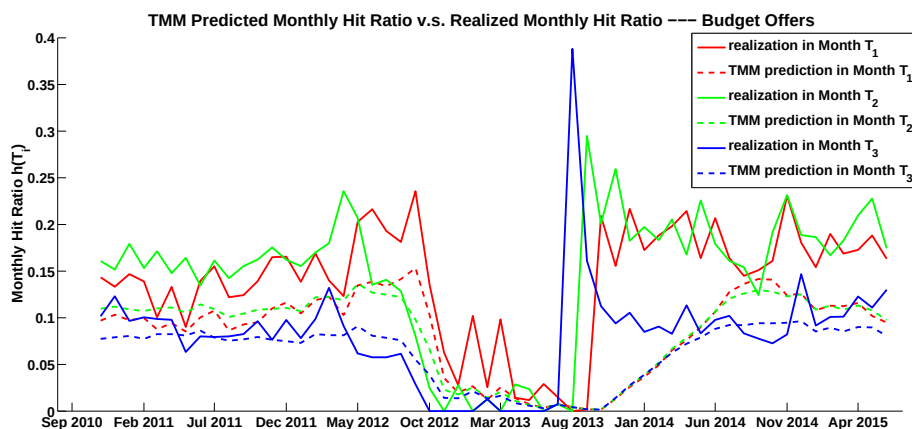


Figure 2.2: Comparison of realized and TMM predicted *hit ratios* for *budget offers*

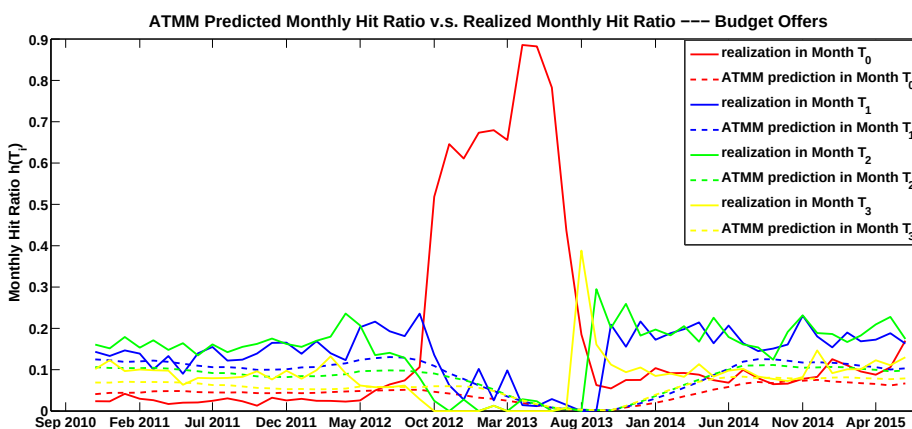


Figure 2.3: Comparison of realized and ATMM predicted *hit ratios* for *budget offers*

A plot of the prediction and the realization may give readers an intuitive check of the model's predictive performance. Figure 2.2 is the plot of the TMM predicted monthly *hit ratios* and the realized monthly *hit ratios* in the case of *budget offers*. Figure 2.3 is the plot of the ATMM predicted monthly *hit ratios* and the realized monthly *hit ratios* in the case of *budget offers*. The x-axis is the month, in which the mortgage offers were sent. From these two figures, first we can get the information that compared to the predicted monthly *hit ratios*, the realized monthly *hit ratios* are much more volatile. Almost all the predicted monthly *hit ratios* are lower than the realized ones. The reason may be the use of the average of the last nine one-month transition matrices as the predicted monthly transition matrix. The average of the last nine one-month transition matrices contains the transition information of some long-lasting offers in their offer processes. Although being still active in the offer process, these offers transit very slowly to a next state, which makes the predicted *hit ratios* biased low. We also see that different monthly realized *hit ratios* behave differently. For example, the month T_1 realized *hit ratios* and the month T_2 ones are relatively close to each other, while being relatively far from the month T_0 realized *hit ratios*. This indicates that it may not be appropriate to use the same predicted transition matrix to predict all the monthly *hit ratios* of the newly sent offers in one month. Hence, the transition matrix method may not be able to capture the features of the realized monthly *hit ratios*. Furthermore, we deduce that the predicted transition matrix should distinguish offer durations. For example, the predicted transition matrix for the month T_0 should be according to the features of the month T_0 transition. By this, we develop an improved transition matrix method (ITMM) in the next section.

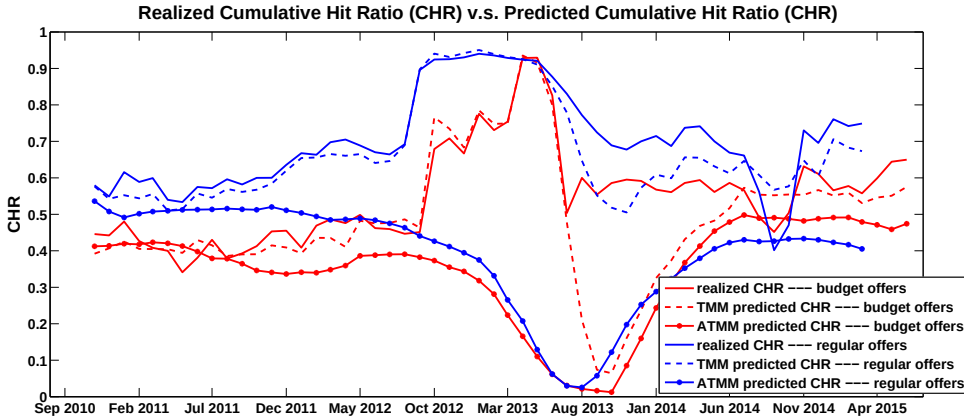


Figure 2.4: comparison of the realized predicted CHRs and the (A)TMM ones

Finally, we present in Figure 2.4, the result of checking the assumed Markov property in the transition matrix method [4]. The cumulative *hit ratio* (CHR) of the offers sent in month T_0 is defined as the sum of all the monthly *hit ratios*, i.e.,

$$\text{CHR} = \sum_{i=0}^n h(T_i),$$

where $h(T_i)$ is the month T_i *hit ratio* of the offers sent in the month T_0 , and the month T_n is the final month in the offer period. For *budget offers*, $n = 3$; for *regular offers*, $n = 9$. If the Markov property of the transition matrix holds in the offer transition process, it will also hold that, for the offers sent in month T_0 , the TMM (or ATMM) predicted CHR is roughly equal to the realized CHR, i.e., in the TMM

$$\sum_{i=0}^n h(T_i) \approx \left(\mathbf{S}^{\text{tr}}(T_1) \cdot \bar{P}^n(T_0) \right)_{320},$$

or in the ATMM

$$\sum_{i=0}^n h(T_i) \approx \left(\mathbf{S}^{\text{tr}}(T_0) \cdot \bar{P}^{n+1}(T_0) \right)_{320},$$

where $h(T_i)$ is the realized month T_i *hit ratio* of the offers sent in the month T_0 , $S(T_j)$ ($j = 0, 1$) is the state ratio at the beginning of month T_j of the offers sent in month T_0 , and $\bar{P}(T_0)$ is the predicted transition matrix by averaging the last nine one-month historical transition matrices.

Figure 2.4 shows a counterexample to the assumed Markov property, since we can see after August 2013, the predicted CHRs are relatively far from the realized CHRs, compared to the distance between the TMM predicted CHR and the realized CHR before August 2013. This counter example implies that the Markov property statement in the transition matrix method [4] is not satisfied, which further implies that the use of the average of the last nine one-month transition matrices to predict monthly *hit ratios* is inappropriate.

To conclude, there are some drawbacks of the transition matrix method, as discussed above. By analyzing those drawbacks and the features of the realized monthly *hit ratios*, we came up with an improved transition matrix method (ITMM) in the next section.

2.3. AN IMPROVED TRANSITION MATRIX METHOD (ITMM)

From the above analysis, we know the monthly offer transition results differ in the offer duration. Based on this empirical result, we modify the predicted transition matrix in TMM such that the obtained predicted transition matrix can not only distinguish the offer duration, but also capture the characteristics of different monthly transitions, which results in the improved transition matrix method (ITMM) in this section.

In the improved transition matrix method (ITMM), we separate the transition matrix according to the offer duration (in months). By Assumption 5, we deduce the month T_0 transition matrix $P(T_0)$ as in (2.17), for the mortgage offers sent in month T_0 . $P(T_i)$ ($i = 0, 1, \dots, n$) is the month T_i transition matrix of the offers sent in month T_0 , where T_n is the final month in the offer period. Here the calculation of the monthly transition matrix in ITMM differentiates mortgage labels and offer options, as required in TMM.

$$P(T_0) = \begin{pmatrix} P_{210,210}(T_0) & P_{210,220}(T_0) & P_{210,230}(T_0) & P_{210,250}(T_0) & P_{210,310}(T_0) & P_{210,320}(T_0) \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad (2.17)$$

where

$$P_{210,j}(T_0) = \frac{M_{210,j}(T_0)}{\sum_{l \in \mathcal{J}} M_{210,l}(T_0)} \quad j \in \mathcal{J}.$$

where $M_{210,j}(T_0)$ is the total principal amount of the month T_0 sent offers which transit to state j at the end of month T_0 . And $\sum_{l \in \mathcal{J}} M_{210,l}(T_0)$ is the total principal amount of the offers sent in month T_0 . The first row of $P(T_0)$ gives all the month T_0 transition information of the offers sent in the month T_0 .

$$P(T_i) = \begin{pmatrix} P_{210,210}(T_i) & P_{210,220}(T_i) & P_{210,230}(T_i) & P_{210,250}(T_i) & P_{210,310}(T_i) & P_{210,320}(T_i) \\ 0 & P_{220,220}(T_i) & P_{220,230}(T_i) & P_{220,250}(T_i) & P_{220,310}(T_i) & P_{220,320}(T_i) \\ 0 & 0 & P_{230,230}(T_i) & P_{230,250}(T_i) & P_{230,310}(T_i) & P_{230,320}(T_i) \\ 0 & 0 & 0 & P_{250,250}(T_i) & P_{250,310}(T_i) & P_{250,320}(T_i) \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad (2.18)$$

where

$$P_{k,j}(T_i) = \frac{M_{k,j}(T_i)}{\sum_{l \in \mathcal{J}} M_{k,l}(T_i)}, \quad k \leq j, \quad k, j \in \mathcal{J} := \{210, 220, 230, 250, 310, 320\},$$

where $M_{k,j}(T_i)$ is the total principal amount of the month T_0 sent offers which stay in state k at the beginning of month T_i and transit to state j at the end of month T_i .

With $\mathbf{S}(T_0)$ as in (2.12), the month T_i ($i = 0, 1, \dots, n$) *hit ratio* $h(T_i)$ can be obtained as follows.

$$\begin{aligned} h(T_0) &= (\mathbf{S}^{\text{tr}}(T_0)P(T_0))_{320} = P_{210,320}(T_0), \\ h(T_1) &= (\mathbf{S}^{\text{tr}}(T_0)P(T_0)P(T_1) - \mathbf{S}^{\text{tr}}(T_0)P(T_0))_{320}, \\ h(T_i) &= \left(\mathbf{S}^{\text{tr}}(T_0) \prod_{z=0}^i P(T_z) - \mathbf{S}^{\text{tr}}(T_0) \prod_{z=0}^{i-1} P(T_z) \right)_{320}. \end{aligned} \quad (2.19)$$

Before predicting the monthly *hit ratios* for the offers sent in month T_0 , we need to obtain a good-quality predicted monthly transition matrix for these offers. After testing the predictive performance of the predicted transition matrix based on different historical periods, we finally settle the predicted transition matrix in ITMM based on the transition in the last one month of month T_0 . So the predicted month T_i ($i = 0, 1, \dots, n$) transition matrix $\hat{P}(T_i)$ for the offers sent in month T_0 , are calculated as (2.17) or (2.18), based on the historical transition information of the offers with the same month duration in the last one month of month T_0 .

Table 2.4 and Table 2.5 present the predictive performance of ITMM in the *budget offer* case. Compared to the results of TMM and ATMM, ITMM has a generally improved predictive performance. Although the **R-squared** results of TMM are higher than the ones of ITMM, TMM has an expensive input which brings relatively higher **R-squared**

Monthly <i>hit ratio</i> Prediction	ITMM – R-squared	ITMM – RMSE
in month T_0	0.86261	0.086428
in month T_1	0.30795	0.048096
in month T_2	0.17069	0.062569
in month T_3	-0.28068	0.060817

Table 2.4: the **R-squared** and the **RMSE** of ITMM in the monthly *hit ratio* prediction for *budget offers*

ITMM Predicted Monthly <i>hit ratio</i>	Hypothesis A- <i>p</i> – value	Hypothesis A- <i>h</i>	Hypothesis B- <i>p</i> – value	Hypothesis B- <i>h</i>
in month T_0	1.57E-08	1	0.8127	0
in month T_1	1.57E-08	1	0.2822	0
in month T_2	1.57E-08	1	0.1859	0
in month T_3	1.32E-07	1	0.1945	0

Table 2.5: Results of the hypothesis tests for the monthly *hit ratio* prediction by ITMM

but causes the loss of the month T_0 *hit ratio* prediction. Therefore, ITMM generally makes an improvement in the predictive performance, compared to (A)TMM.

Some drawbacks of ITMM still make ITMM hard to be accepted in the *hit ratio* prediction. First, ITMM is still inaccurate. For example, although the month T_0 **R-squared** of ITMM in Table 2.4 is more than 80%, the **R-squared** results of the other months are still much lower. Second, ITMM does not involve any state factor, which makes it impossible to calculate the expected loss in (2.3). Considering the above two drawbacks of ITMM, we still need to look for a more appropriate model in predicting the monthly *hit ratios*.

2.4. CORRELATION TEST

One of the state factors which directly inform the offer option holders is the mortgage rate over the offer period. As the way of a product price influencing on clients' choice in the shopping, the mortgage rate may also influence on the offer option holders' decisions in the offer period. So in this section, we test the correlation between *hit ratio* and mortgage rate in order to check whether there is a linear relation between them.

Considering that in the Netherlands the mortgage rate changes on a weekly basis, the *hit ratio* used in the correlation test should also be the weekly rate. The weekly *hit ratio* is defined as follows.

Definition 6. For the offers sent in month T_0 with $M(T_0)$ the total principal amount, the i -th weekly *hit ratio* $h^w(T_0, i)$ is calculated as the ratio of the principal of the i -week-duration closed offers to $M(T_0)$, i.e.,

$$h^w(T_0, i) = \frac{MP(T_0, i)}{M(T_0)} \quad (i = 1, 2, \dots, m) \quad (2.20)$$

where $MP(T_0, i)$ is the total principal of the i -week-duration closed offers among the offers sent in month T_0 .

For an offer sent at time t_1 with initially locked mortgage rate $m(t_1)$, if this offer is closed at time $t_2 > t_1$ and the mortgage rate on the offer closing day is $m(t_2)$, the mortgage rate change of this closed offer is given by

$$\Delta m(t_1, t_2 - t_1) = m(t_1) - m(t_2).$$

The corresponding mortgage rate change of $h^w(T_0, i)$ is a principal-weighted mortgage rate change of all the mortgage rate changes of the i -week-duration closed offers, which is denoted as $\Delta m^w(T_0, i)$. The weight of the mortgage rate change of each i -week-duration closed offer is the ratio of its principal to $M^p(T_0, i)$. Due to the lack of offer details in the transition data sheet (see Appendix A), the mortgage rate during the offer period is matched with the corresponding 10-year mortgage rate (75% LtV) under AAB label during that period, and the weekly *hit ratio* is calculated based on offer number instead of offer principal.

offer duration in weeks	correlation coefficient	p-value
1	0.0500	0.6903
2	-0.1337	0.2846
3	0.2145	0.0838
4	0.0920	0.4625
5	-0.1793	0.1497
6	0.0660	0.5987
7	-0.1607	0.1975
8	-0.0023	0.9853
9	-0.1520	0.2230
10	0.0386	0.7581
11	0.0093	0.9406
12	-0.0405	0.7465
13	0.1244	0.3196
14	-0.1241	0.3207

Table 2.6: The result of the correlation test between the weekly *hit ratios* $\{h_{T_0, i}^w\}_{(i=1,2,\dots,14)}$ and the corresponding mortgage rate change $\{\Delta m_{T_0, i}^w\}_{(i=1,2,\dots,14)}$ for *budget offers*

The correlation result between $\{h^w(T_0, i)\}_{(i=1,2,\dots,14)}$ and $\{\Delta m^w(T_0, i)\}_{(i=1,2,\dots,14)}$ in the *budget offer* case is presented in Table 2.6. The p-value in the correlation test is for the hypothesis testing with the null hypothesis H_0 and the alternative hypothesis H_1 as follows.

$$\begin{cases} H_0 = \{\text{There is no linear relation between the tested variables.}\} \\ H_1 = \{\text{There is non-zero linear relation between the tested variables.}\} \end{cases} \quad (2.21)$$

In Table 2.6, we observe that the p-value is relatively high. And the sign of the correlation coefficient in Table 2.6 interchanges frequently, which implies that the linear relation between the tested variables is inconsistent in the data. So we can deduce from the data that there is no consistent linear relation between $\{h^w(T_0, i)\}_{(i=1,2,\dots,14)}$ and $\{\Delta m^w(T_0, i)\}_{(i=1,2,\dots,14)}$.

By the result in Table 2.6, we start to consider the appropriateness of the variable candidates in the correlation test. It is true that if 99.9% offers are closed in their 1st week

after the offer sending moment, then there will be a very small weekly *hit ratio* for the following weeks even with the most favorable mortgage rate change to the borrowers. Considering that, we replace the weekly *hit ratio* with the conditional weekly *hit ratio* on the remaining active offers in the correlation test. The definition of the conditional weekly *hit ratio* on the remaining active offers is given as follows.

Definition 7. For the offers sent in month T_0 with $M(T_0)$ the total principal amount, the conditional i -th weekly hit ratio $h^{c,w}(T_0, i)$ on the remaining active offers is calculated as the ratio of the total principal of the i -week-duration closed offers to the total principal of the offers of which the offer duration is more than or equal to i weeks, i.e.,

$$h^{c,w}(T_0, i) = \frac{M^p(T_0, i)}{M(T_0) - \sum_{1 \leq d < i} M^p(T_0, d) - \sum_{1 \leq d < i} M^f(T_0, d)} \quad (i = 1, 2, \dots, m) \quad (2.22)$$

where $M^p(T_0, i)$ is the total principal of the i -week-duration closed offers among the offers sent in month T_0 , $M^f(T_0, i)$ is the total principal of the i -week-duration dropped offers among the offers sent in month T_0 .

The correlation result for *budget offers* between $\{h^{c,w}(T_0, i)\}_{(i=1,2,\dots,14)}$ and the corresponding mortgage rate change $\{\Delta m^{c,w}(T_0, i)\}_{(i=1,2,\dots,14)}$ is presented in Table 2.7. Considering that the dropped offers also play a role in determining the conditional weekly *hit ratio* calculation, we investigate the (conditional) fallout rate along with the conditional *hit ratio* investigation. The definitions of weekly fallout rate and conditional weekly fallout rate are given as follows.

Definition 8. For the offers sent in month T_0 with $M(T_0)$ the total principal amount, the i -th weekly fallout rate $f^w(T_0, i)$ is calculated as the ratio of the total principal of the i -week-duration dropped offers to $M(T_0)$, i.e.,

$$f^w(T_0, i) = \frac{M^f(T_0, i)}{M(T_0)}, \quad (i = 1, 2, \dots, m) \quad (2.23)$$

where $M^f(T_0, i)$ is the total principal of the i -week-duration dropped offers among the offers sent in month T_0 .

Definition 9. For the offers sent in month T_0 with $M(T_0)$ the total principal amount, the conditional i -th weekly fallout rate $f^{c,w}(T_0, i)$ on the remaining active offers is calculated as the ratio of the total principal of the i -week-duration dropped offers to the total principal of the offers of which the offer duration is more than or equal to i weeks.

$$f^{c,w}(T_0, i) = \frac{M^d(T_0, i)}{M(T_0) - \sum_{1 \leq d < i} M^p(T_0, d) - \sum_{1 \leq d < i} M^f(T_0, d)} \quad (i = 1, 2, \dots, m) \quad (2.24)$$

where $M^p(T_0, d)$ is the total principal of the d -week-duration closed offers among the offers sent in month T_0 , and $M^f(T_0, d)$ is the total principal of the d -week-duration dropped offers among the offers sent in month T_0 .

The correlation result for *budget offers* between $\{f^w(T_0, i)\}_{(i=1,2,\dots,14)}$ the weekly fallout rate and $\{\Delta \hat{m}^{c,w}(T_0, i)\}_{(i=1,2,\dots,14)}$ the corresponding mortgage rate change is shown

in Table 2.8. The correlation test result for *budget offers* between $\{f^{c,w}(T_0, i)\}_{(i=1,2,\dots,14)}$ the conditional weekly fallout rate and $\{\Delta \hat{m}^{c,w}(T_0, i)\}_{(i=1,2,\dots,14)}$ the corresponding mortgage rate change is shown in Table 2.9.

offer duration in weeks	correlation coefficient	p-value
1	-0.1838	0.1396
2	-0.0119	0.9248
3	0.1375	0.2711
4	0.1531	0.2196
5	0.0962	0.4423
6	0.0117	0.9259
7	-0.1283	0.3046
8	-0.0001	0.9991
9	0.0722	0.5643
10	-0.0003	0.9984
11	0.0474	0.7056
12	-0.1668	0.1808
13	-0.0597	0.6339
14	-0.3174	0.0094

Table 2.7: The correlation test result between the conditional weekly *hit ratio* $\{h^{c,w}(T_0, i)\}_{(i=1,2,\dots,14)}$ and the corresponding mortgage rate change $\{\Delta m^{c,w}(T_0, i)\}_{(i=1,2,\dots,14)}$ for *budget offers*

offer duration in weeks	correlation coefficient	p-value
1	0.0663	0.5969
2	0.0049	0.9691
3	-0.0432	0.7306
4	0.0511	0.6839
5	0.0625	0.6183
6	0.1809	0.1461
7	0.1290	0.3020
8	0.3268	0.0074
9	0.1269	0.3101
10	0.2458	0.0467
11	0.2488	0.0440
12	0.3646	0.0026
13	0.1875	0.1316
14	0.3749	0.0019

Table 2.8: The correlation test result between the weekly fallout rate $\{f^w(T_0, i)\}_{(i=1,2,\dots,14)}$ and the corresponding mortgage rate change $\{\Delta \hat{m}^w(T_0, i)\}_{(i=1,2,\dots,14)}$ for *budget offers*

The inconsistency in the signs of the correlation coefficients still remains in the conditional weekly *hit ratio* test as shown in Table 2.7, although almost all the correlation coefficients are positive in the (conditional) weekly fallout rate test as shown in Table 2.8 and Table 2.9. The p-value is still relatively high. So we can deduce from the data that there is no linear relation between the tested variables. We also tested the correlation between $\{h^{c,w}(T_0, i) - h^{c,w}(T_1, i)\}_{(i=1,2,\dots,14)}$ and $\{\Delta m^{c,w}(T_0, i) - \Delta m^{c,w}(T_1, i)\}_{(i=1,2,\dots,14)}$, between $\{h^{c,w}(T_0, i) - h^{c,w}(T_0, i+1)\}_{(i=1,2,\dots,13)}$ and $\{\Delta m^{c,w}(T_0, i) - \Delta m^{c,w}(T_0, i+1)\}_{(i=1,2,\dots,13)}$, and so on. But still there is no linear relation derived from the data between *hit ratio* (or fallout rate) and mortgage rate.

After the correlation tests in this section, we conclude that there is no linear relation derived from the data between *hit ratio* (or fallout rate) and mortgage rate.

offer duration in weeks	correlation coefficient	p-value
1	0.0495	0.6929
2	0.1278	0.3063
3	0.0401	0.7492
4	0.2411	0.0511
5	0.1873	0.1321
6	0.2730	0.0266
7	0.1254	0.3157
8	0.3008	0.0141
9	0.0956	0.4449
10	0.1827	0.1421
11	0.3000	0.0144
12	0.3273	0.0073
13	0.0657	0.6004
14	0.2072	0.0950

Table 2.9: The correlation test result between the conditional weekly fallout rate $\{f^{c,w}(T_0, i)\}_{(i=1,2,\dots,14)}$ and the corresponding mortgage rate change $\{\Delta \hat{m}^{c,f}(T_0, i)\}_{(i=1,2,\dots,14)}$ for budget offers

2.5. OTHER MODELS

This section presents some other models for the *hit ratio* prediction. And the appropriateness of applying the models into practice is analyzed afterwards.

LOGIT MODEL

The logit model for the offer process comes from the logistic regression model. The form of the logistic model is

$$\ln \left(\frac{p}{1-p} \right) = \beta_0 + x\beta,$$

where p is the probability of an inter-state transition, i.e., the offer transition from one state to another subsequent state, β_0 is the intercept term, and β is the vector of the coefficient β_i associated with an explanatory variables x_i . The maximum likelihood method can be applied to estimate the coefficients in the logistic model.

J. McMurray and T. Thomson [5] used a logistic regression to estimate the closing probability (i.e., the *hit ratio*) of residential mortgages, which includes a wide range of explanatory variables. In analogue, the logit model in [4] uses the regression model to predict the probability of an inter-state transition, which includes the explanatory variables as follows.

x_1 : a binary variable, which is 1 for newly created mortgage offers and 0 otherwise;

x_2 : a binary variable, which is 1 for mortgage offers for newly built houses and 0 otherwise;

x_3 : a binary variable, which is 1 for mortgage offers for refurbishment and 0 otherwise;

x_4 : the negative mortgage rate change, defined as $x_{4,i} = \min(r_{t,i} - r_{0,i}, 0)$ for mortgage offer i , where $r_{t,i}$ is the prevailing mortgage rate at time t for the same type mortgage offer, and $r_{0,i}$ the mortgage rate for offer i at the offer sending moment;

x_5 : the positive mortgage rate change, defined as $x_{5,i} = \max\{r_{t,i} - r_{0,i}, 0\}$.

So the regression formula in the logit model for the i -indexed mortgage offer is

$$\ln\left(\frac{p}{1-p}\right) = \beta_0 + \beta_1 x_{1,i} + \beta_2 x_{2,i} + \beta_3 x_{3,i} + \beta_4 x_{4,i} + \beta_5 x_{5,i} \quad (2.25)$$

Equivalently,

$$p = \frac{1}{1 + \exp\{-(\beta_0 + \beta_1 x_{1,i} + \beta_2 x_{2,i} + \beta_3 x_{3,i} + \beta_4 x_{4,i} + \beta_5 x_{5,i})\}} \quad (2.26)$$

The maximum likelihood method can be applied to estimate the coefficients in (2.26).

The logit model is used to predict the inter-state transition probabilities of an individual offer. To estimate the monthly *hit ratios* of an offer by the logit model, the additional steps we need to take are the multiplication of the probabilities of the inter-state transitions of which the combination makes the offer reach state 320 and the summation of the multiplications of all the feasible transition combinations to get the monthly *hit ratios*. The additional steps may enlarge the prediction error from the inter-state transition prediction by the logit model. More importantly, no obvious evidence proves the reliability of the logit regression model. It can be found that in [4] there is some inconsistency in the signs of the same explanatory variable. Based on the above reasoning, we can not accept the logit model for the *hit ratio* prediction.

HAZARD MODEL

The hazard model describes life time or survival time by modelling the hazard rate. The hazard rate of a mortgage offer is the conditional probability of its transition occurring at time t on the condition of no transition before t , which is given by

$$P(T = t | T \geq t) = \frac{p(t)}{P(T \geq t)} \quad (2.27)$$

where T is the transition time, and $p(t)$ is the probability of transition occurring at time t , i.e.,

$$p(t) = P(T = t) \quad (2.28)$$

where t is the time in the offer period.

Cox Proportional Hazard Model has been applied to model the offer duration by S. Hakim, M. Rashidian and E. Rosenblatt [6]. The model is specified as

$$p(t) = p_0(t) \exp\{\beta^T \mathbf{X}_i\} \quad (2.29)$$

where $\mathbf{X}_i$ is the vector of the explanatory variables, β is the vector of the coefficients, and $p_0(t)$ is the baseline hazard function. The partial likelihood method can be applied to estimate the coefficients β .

Similarly to the logit model [4], the hazard model includes the same explanatory variables and models the hazard rate of the inter-state transition in the mortgage process.

2

The drawbacks of the hazard model in [4] are the same as the ones of the logit model. They both focus on estimating the inter-state transitions of individual offer, while the hazard rate model estimates the time duration before each inter-state transition occurs and the logit model estimates each inter-state transition probability. The advantage of these two models is that they include the state factor, i.e., the mortgage rate change, in the prediction. To conclude, based on the above reasoning, we can not accept these models for the *hit ratio* prediction. In the next part, we turn to another direction in the valuation of mortgage offer options.

II

SECOND PART

3

ENDOGENOUS TERMINATION MODELLING

This chapter is going to present the endogenous termination modelling for the valuation of mortgage offer options. We narrow the scope of this thesis and focus on the *budget option* valuation. First, we come up with a setting of the *budget option* valuation problem based on an optimal exercise strategy in Section 3.1. In view of the setting, three numerical methods are proposed to solve the problem, namely the finite difference method (FDM), the least squares method (LSM), and the stochastic grid bundling method (SGBM). The implementation of FDM, LSM and SGBM in the *budget option* valuation is explained in Section 3.2, Section 3.3, Section 3.4 respectively, which produces some numerical test results presented in Section 3.5. In order to test the convergence of the implemented methods in the *budget option* valuation, we present the convergence study in Section 3.6. As an extension of the interest rate model in the *budget option* valuation, a one-dimensional jump diffusion model of the interest rate is presented and tested in Section 3.7. As a representative of two-dimensional interest rate models, a two-factor interest rate model is applied into the *budget option* valuation, which concludes this chapter.

3.1. PROBLEM SETTING OF THE BUDGET OPTION VALUATION

The problem setting of the *budget option* valuation in this chapter is based on an assumption of the optimal exercise strategy for *budget option* holders, analogously to the contingent-claim models in [3] and [7].

Suppose a *budget offer* is sent at time t_0 and the end of the offer period is time t_M .

Assumption 10. *The fundamental assumption in the budget option framework is that a rational borrower seeks to minimize the market value of the loan during the offer period, thereby strategically exercising the offer option when the prevailing value of the committed loan $PM(m_0, t)$ is lower than the comparable strike price $K(t)$ of the loan ($t_0 \leq t \leq t_M$). At*

mortgage type:	fixed-rate mortgage
offer option:	<i>budget option</i>
locked mortgage rate (yearly compounding):	m_0
amortization scheme:	<i>bullet amortization scheme</i> (or interest-rate only)
mortgage term ¹ :	T (years)
Loan principle:	M

Table 3.1: Contract Specification

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the offer sending moment t_0 , the strike price $K(t)$ is calculated as the t -forward price of the loan, which depends on the term structure of the interest rate at time t_0 .

We assume the forward loan price to be the corresponding strike price $\{K(t)\}_{t_0 \leq t \leq t_M}$ for the offer option. $\{K(t)\}_{t_0 \leq t \leq t_M}$ can be regarded as the initially agreed market cost which the borrower considers to take at each moment during the offer period. The more the strike price outnumbers the prevailing value of this committed loan, the stronger incentive the borrower has to exercise the budget option, which implies that the borrower is able to have a lower market cost of the loan than the initially considered one.

There might be a question why we choose the forward price as the strike price instead of the prevailing mortgage rate. There are two reasons. The first one is that the prevailing mortgage rate is not immediately achievable for *budget option* holders according to the terms in *budget offers*. If a *budget option* holder is a rational investor, at each moment during the offer period (s)he will compare the prevailing market value of the loan in the offer with the previously agreed loan value (or the forward price) correspondingly. The second reason is that the difference between the forward loan price and the prevailing loan value can, to some extent, reflect the difference between the prevailing mortgage rate and the locked mortgage rate, because the prevailing interest rate term structure plays a role in setting the fixed rate of the basic funding cost in the prevailing mortgage rate. We do not include the calculation of mortgage rate over the offer period due to the complexity of mortgage rate settlement.

Now we calculate the prevailing loan price and the strike price for a budget offer, which are the candidates in the optimal exercise strategy. Suppose a *budget offer* is sent at time t_0 which is the beginning of month T_0 . With 3-month offer period at most, this offer will expire at time t_M which is the end of month T_2 , if having not been closed yet. The contract specification of this *budget offer* is described in Table 3.1. If the loan starting time is known for this offer, say the beginning of month $T_i (i \in \{1, 2, 3\})$ (shorthand notation: $T_i (i \in \{1, 2, 3\})$), under the risk-neutral measure $(\Omega, \mathcal{F}, Q)$ with filtration $\{\mathcal{F}_t\}_{t \geq 0}$, the

prevailing market value of the loan at time $t_0 \leq t < T_i$ is ²

$$\begin{aligned}
 PM(m_0, T_i, t) &= \mathbb{E}^Q \left[\sum_{j=1}^N m_0 \cdot \Delta t \cdot M \cdot D(t, T_{i+j}) + M \cdot D(t, T_{i+N}) \middle| \mathcal{F}_t \right] \\
 &= \sum_{j=1}^N m_0 \cdot \Delta t \cdot M \cdot \mathbb{E}^Q [D(t, T_{i+j}) | \mathcal{F}_t] + M \cdot \mathbb{E}^Q [D(t, T_{i+N}) | \mathcal{F}_t] \\
 &= \sum_{j=1}^N m_0 \cdot \Delta t \cdot M \cdot \mathbb{E}^Q \left[\exp \left\{ - \int_t^{T_{i+j}} r(s) ds \right\} \middle| \mathcal{F}_t \right] + M \cdot \mathbb{E}^Q \left[\exp \left\{ - \int_t^{T_{i+N}} r(s) ds \right\} \middle| \mathcal{F}_t \right]
 \end{aligned} \tag{3.1}$$

where $r(s)$ is the risk-free rate at time s ($t \leq s \leq T_N$), $D(t, T_{i+j}) := \exp \left\{ - \int_t^{T_{i+j}} r(s) ds \right\}$ ($j = 1, 2, \dots, N$) is the discount factor to discount cash flows happening at the beginning of month T_{i+j} to the time t , N is the number of payment times, $\Delta t = \frac{T}{N}$ is the time interval (in year fraction) between adjacent mortgage payments, m_0 and M are explained in Table 3.1.

On the other hand, the loan starting time actually is unknown at t , which raises a problem in valuing this offered loan. There is a proposed way to tackle this problem. The prevailing loan value considered at $T_k \leq t < T_{k+1}$, $k = 0, 1, 2$, is based on the assumption that the offer will be closed immediately so that the mortgage on this offer will start at the beginning of month T_{k+1} . So the prevailing market value of the loan considered at $T_k \leq t < T_{k+1}$, $k = 0, 1, 2$, is

$$\begin{aligned}
 PM(m_0, t, r_t) &= \mathbb{E}^Q \left[\sum_{j=1}^N m_0 \cdot \Delta t \cdot M \cdot D(t, T_{k+1+j}) + M \cdot D(t, T_{k+1+N}) \middle| \mathcal{F}_t \right] \\
 &= \sum_{j=1}^N m_0 \cdot \Delta t \cdot M \cdot \mathbb{E}^Q \left[\exp \left\{ - \int_t^{T_{k+1+j}} r(s) ds \right\} \middle| \mathcal{F}_t \right] + M \cdot \mathbb{E}^Q \left[\exp \left\{ - \int_t^{T_{k+1+N}} r(s) ds \right\} \middle| \mathcal{F}_t \right].
 \end{aligned} \tag{3.2}$$

with the corresponding strike price $K(t)$ which is estimated at the offer sending moment t_0 as follows.

$$\begin{aligned}
 K(t) &= \mathbb{E}^Q \left[\sum_{j=1}^N m_0 \cdot \Delta t \cdot M \cdot D(t, T_{k+1+j}) + M \cdot D(t, T_{k+1+N}) \middle| \mathcal{F}_{t_0} \right] \\
 &= \sum_{j=1}^N m_0 \cdot \Delta t \cdot M \cdot \mathbb{E}^Q \left[\exp \left\{ - \int_t^{T_{k+1+j}} r(s) ds \right\} \middle| \mathcal{F}_{t_0} \right] + M \cdot \mathbb{E}^Q \left[\exp \left\{ - \int_t^{T_{k+1+N}} r(s) ds \right\} \middle| \mathcal{F}_{t_0} \right].
 \end{aligned} \tag{3.3}$$

If a holder exercises the *budget option* in month T_0 , (s)he gets a loan which officially starts at the beginning of month T_1 . The same goes in month T_2 , if a holder exercises the *budget option*, (s)he gets a loan which officially starts at the beginning of month T_3 . Different start times of the loan in the offer, i.e., different closing times of the offer, result in different occurring times of the mortgage payments, which forms different underlying assets. In this thesis we first value the *budget option* separately according to the exercise

²Here we do not consider prepayment or a default possibility after the offer period.

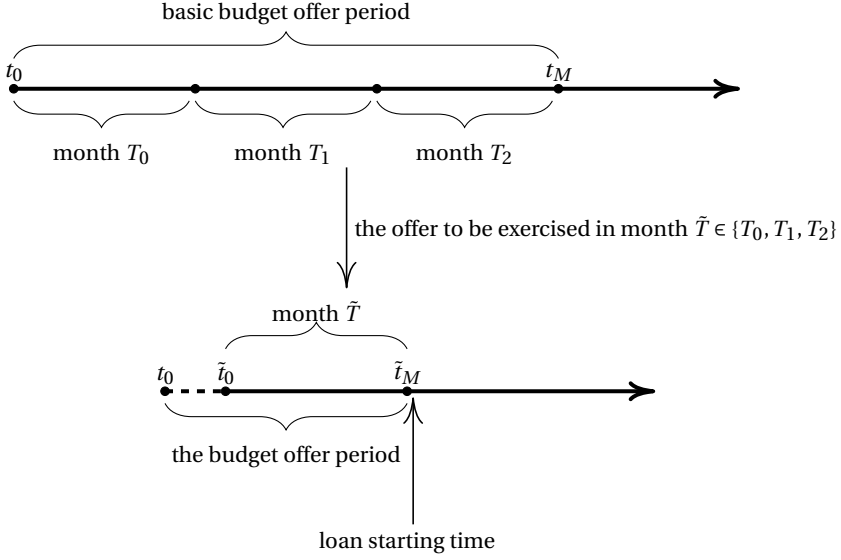


Figure 3.1: budget offer period

month within the offer period. At the offer sending moment t_0 , the price of the *budget option* to be exercised in month T_0 , denoted $V_{t_0}^0$, results from an American-style option exercise in month T_0 . At t_0 , the price of the *budget option* to be exercised in month T_1 , denoted $V_{t_0}^1$, results from holding the *budget option* in month T_0 and then exercising the *budget option* in an American style in month T_1 . At t_0 , the price of *budget option* to be exercised in month T_2 , denoted $V_{t_0}^2$, results from holding the *budget option* in month T_0 and month T_1 , and then exercising the *budget option* in an American style in month T_2 . Figure 3.1 illustrates a general offer period of the budget offer option to be exercised in month $\tilde{T} \in \{T_0, T_1, T_2\}$. After obtaining $V_{t_0}^0, V_{t_0}^1, V_{t_0}^2$, we derive the final price of the *budget option* as

$$V_{t_0} = \max \{V_{t_0}^0, V_{t_0}^1, V_{t_0}^2\}.$$

Rational borrowers only exercise their option when it is in the money, i.e., $PM(m_0, t, r_t) < K(t)$. To fully utilize the early-exercise facility of the *budget option*, at each option-exercisable date $t_0 \leq t \leq t_M$, the holder will optimally compare the immediate exercise payoff

$$I_t = (K(t) - PM(m_0, t, r_t))^+ \quad (3.4)$$

with the expected continuation payoff C_t which is the option value at time t conditional on the option to be exercised after time t . Then the option will be exercised at time t , only if $I_t \geq C_t$, resulting in the value of this *budget option* at t as follows.

$$V_t = \max \{I_t, C_t\}.$$

To calculate V_t , we must know I_t and C_t . Unlike I_t which is known at time t , the expected continuation value C_t is not explicit at time t . The appropriate procedure to estimate

C_t is to adopt a backward approach, in the manner of dynamic programming. There are three proposed methods in pricing V_t , namely the finite difference method (FDM), the least squares method (LSM), and the stochastic grid bundling method (SGBM). In fact, the difference of these three methods in option pricing is the way of calculating C_t . In brief, FDM calculates C_t by a partial differential equation relation; LSM calculates C_t by regressing the simulated continuation value C_t^i on the simulated interest rate r_t^i , where i is the i -th simulated interest rate path in which the *budget option* at t is in the money; SGBM calculates C_t by regressing the one-step-forward simulated option values V_{t+1}^i on the one-step-forward simulated interest rates r_{t+1}^i and combining with the conditional expectation of the applied basis functions of interest rate r_{t+1} on r_t , where i is the i -th simulated interest rate path in one bundled simulation set. After estimating C_t , by dynamic programming on comparing I_t with C_t , the option value V_t is settled by $V_t = \max(I_t, C_t)$ at the exercisable time $t_0 \leq t \leq t_M$. These three methods will be explained in detail in the subsequent sections.

In this thesis, the interest rate model applied in the *budget option* valuation is required to have the analytical solution of $P(t, T) = \mathbb{E}^Q \left[e^{-\int_t^T r_s ds} \middle| \mathcal{F}_t \right]$, ($T > t$), the zero coupon bond price, which aims to facilitate the calculation of the prevailing loan value at each time step during the offer period.

In the basic setting of the *budget option* valuation problem, we apply the Vasicek model [8] defined under the risk-neutral measure $(\Omega, \mathcal{F}, Q)$ by the dynamics

$$dr_t = \kappa(\theta - r_t)dt + \sigma dW_t, \quad (3.5)$$

which results in the solution of the zero coupon bond price $P(t, T)$, $t \leq T$, as follows.

$$\begin{cases} P(t, T) = \mathbb{E}^Q \left[e^{-\int_t^T r_s ds} \middle| \mathcal{F}_t \right] = A(t, T) e^{-B(t, T) r_t} \\ A(t, T) = \exp \left\{ \left(\theta - \frac{\sigma^2}{2\kappa^2} \right) (B(t, T) - T + t) - \frac{\sigma^2}{4\kappa} B(t, T)^2 \right\} \\ B(t, T) = \frac{1}{\kappa} (1 - e^{-\kappa(T-t)}) \end{cases} \quad (3.6)$$

Proof. Under the interest rate dynamics given by (3.5), we get

$$\begin{aligned} d(e^{\kappa s} \cdot r_s) &= e^{\kappa s} \kappa \theta ds + e^{\kappa s} \sigma dW_s \\ \int_t^T d(e^{\kappa s} \cdot r_s) &= \int_t^T e^{\kappa s} \kappa \theta ds + \int_t^T e^{\kappa s} \sigma dW_s \\ e^{\kappa T} r_T - e^{\kappa t} r_t &= \theta (e^{\kappa T} - e^{\kappa t}) + \sigma \int_t^T e^{\kappa s} dW_s \\ r_T &= r_t \cdot e^{-\kappa(T-t)} + \theta (1 - e^{-\kappa(T-t)}) + \sigma e^{-\kappa T} \int_t^T e^{\kappa s} dW_s. \end{aligned} \quad (3.7)$$

We know the distribution of $\left(\int_t^T r_s ds\right)$ by

$$\begin{aligned}
 \int_t^T r_s ds &= \int_t^T r_t \cdot e^{-\kappa(s-t)} ds + \int_t^T \theta (1 - e^{-\kappa(s-t)}) ds + \int_t^T \sigma e^{-\kappa s} \int_t^s e^{\kappa x} dW_x ds \\
 &= \frac{r_t}{\kappa} (1 - e^{-\kappa(T-t)}) + \theta \left(T - t + \frac{e^{-\kappa(T-t)} - 1}{\kappa} \right) + \int_t^T e^{\kappa x} dW_x \int_x^T \sigma e^{-\kappa s} ds \\
 &= \frac{r_t}{\kappa} (1 - e^{-\kappa(T-t)}) + \theta \left(T - t + \frac{e^{-\kappa(T-t)} - 1}{\kappa} \right) + \int_t^T \frac{\sigma (e^{\kappa(x-T)} - 1)}{-\kappa} dW_x \\
 &= \frac{1 - e^{-\kappa(T-t)}}{\kappa} (r_t - \theta) + \theta (T - t) + \int_t^T \frac{\sigma (e^{\kappa(x-T)} - 1)}{-\kappa} dW_x.
 \end{aligned} \tag{3.8}$$

By Ito Isometry, we know

$$\begin{aligned}
 \mathbb{E} \left(\int_t^T \frac{\sigma (e^{\kappa(x-T)} - 1)}{-\kappa} dW_x \right)^2 &= \mathbb{E} \left(\int_t^T \frac{\sigma^2 (e^{\kappa(x-T)} - 1)^2}{\kappa^2} dx \right) \\
 &= -\frac{\sigma^2}{2\kappa} \left(\frac{1 - e^{-\kappa(T-t)}}{\kappa} \right)^2 - \frac{\sigma^2}{\kappa^2} \frac{(1 - e^{-\kappa(T-t)})}{\kappa} + \frac{\sigma^2}{\kappa^2} (T - t).
 \end{aligned} \tag{3.9}$$

Hence, we get

$$\int_t^T r_s ds \sim \mathcal{N} \left(\left(\frac{1 - e^{-\kappa(T-t)}}{\kappa} \right) (r_t - \theta) + \theta (T - t), -\frac{\sigma^2}{2\kappa} \left(\frac{1 - e^{-\kappa(T-t)}}{\kappa} \right)^2 - \frac{\sigma^2}{\kappa^2} \frac{(1 - e^{-\kappa(T-t)})}{\kappa} + \frac{\sigma^2}{\kappa^2} (T - t) \right). \tag{3.10}$$

By the moment generating function of the normal distribution, we get the analytical solution to the zero coupon price $P(t, T)$ in Equation (3.6). ■

3.2. THE FINITE DIFFERENCE METHOD IN THE BUDGET OPTION VALUATION

In this section, the problem of the *budget option* valuation is equivalently formulated at first. In terms of the formulated problem, the finite difference method (FDM) is proposed as an appropriate candidate to solve this problem. And the implementation details of FDM in the *budget option* valuation are explained.

3.2.1. PROBLEM FORMULATION

Without the early exercise facility, the no-arbitrage principle yields that the value of a *budget option* V_t at time t during the offer period should satisfy the PDE [9]:

$$\mathcal{L}(V) \equiv \frac{\partial V}{\partial t} + \kappa(\theta - r_t) \frac{\partial V}{\partial r_t} + \frac{1}{2} \sigma^2 \frac{\partial^2 V}{\partial r_t^2} - r_t V = 0 \tag{3.11}$$

There are two methods to prove Equation (3.11), namely a martingale argument and a riskless portfolio setting method, which are presented below.

Proof by the Martingale Argument Method.

If there is no arbitrage, the process $\left\{ \frac{V_t}{\exp\left\{\int_{t_0}^t r_s ds\right\}} \right\}_{t_0 \leq t \leq t_M}$, i.e., $\{V_t D(t_0, t)\}_{t_0 \leq t \leq t_M}$, should be a martingale under the risk-neutral measure $(\Omega, \mathcal{F}, Q)$ where $r(s)$ (shorthand notation r_s) is the risk-free rate at time s ($t_0 \leq s \leq t_M$). So,

$$\mathbb{E}^Q [d(V_t D(t_0, t)) | \mathcal{F}_t] = 0. \quad (3.12)$$

Now we derive $d(V_t D(t_0, t))$ as follows.

$$\begin{aligned} d(V_t D(t_0, t)) &= D(t_0, t) dV_t + V_t dD(t_0, t) \\ &= D(t_0, t) \left(\frac{\partial V_t}{\partial t} dt + \frac{\partial V}{\partial r_t} dr_t + \frac{1}{2} \frac{\partial^2 V_t}{\partial r_t^2} (dr_t)^2 \right) - V_t D(t_0, t) r_t dt \\ &= D(t_0, t) \left(\frac{\partial V_t}{\partial t} + \kappa(\theta - r_t) \frac{\partial V}{\partial r_t} + \frac{1}{2} \sigma^2 \frac{\partial^2 V_t}{\partial r_t^2} - r_t \cdot V_t \right) dt + D(t_0, t) \cdot \sigma \frac{\partial V}{\partial r_t} dW_t \end{aligned} \quad (3.13)$$

And

$$\mathbb{E}^Q [d(V_t D(t_0, t)) | \mathcal{F}_t] = D(t_0, t) \left(\frac{\partial V_t}{\partial t} + \kappa(\theta - r_t) \frac{\partial V}{\partial r_t} + \frac{1}{2} \sigma^2 \frac{\partial^2 V_t}{\partial r_t^2} - r_t \cdot V_t \right) dt. \quad (3.14)$$

Therefore, Equation (3.11) must hold because of Equation (3.12) under the martingale argument. ■

Proof by the Riskless Portfolio Setting Method.

First, we set up a self-financing portfolio Π_t at $t_0 \leq t \leq t_M$, by holding a one-unit-principal *budget offer* option and Δ_t units of a zero coupon bond portfolio. The budget offer option is sent at the beginning of month T_0 with the parameters specified in Table 3.1 and the mortgage principal $M = 1$. The payments of the zero coupon bond portfolio replicate the mortgage payments of the loan in the *budget offer* closed in month T_2 . We denote the prevailing price of this zero coupon bond portfolio at time $t_0 \leq t \leq t_M$ as $PH(m_0, r_t, t)$ (shorthand notation PH_t), and

$$PH(m_0, r_t, t) = \sum_{j=1}^N m_0 \cdot \Delta t \cdot M \cdot P(t, T_{3+j}) + M \cdot P(t, T_{3+N}), \quad (3.15)$$

where m_0 is the locked mortgage rate in the *budget offer*, the mortgage principal $M = 1$, N represents the number of the payment times of the mortgage specified in Table 3.1, $\Delta t = \frac{T}{N}$. And

$$\Pi_t = V_t - \Delta_t \cdot PH_t.$$

Both V_t and PH_t are functions of time t and instantaneous interest rate r_t . And by

Ito's lemma, the dynamics of Π_t is

$$\begin{aligned}
 d\Pi_t &= \frac{\partial V_t}{\partial t} dt + \frac{\partial V_t}{\partial r_t} dr_t + \frac{1}{2} \frac{\partial^2 V_t}{\partial r_t^2} (dr)^2 - \Delta_t \left(\frac{\partial PH_t}{\partial t} dt + \frac{\partial PH_t}{\partial r_t} dr_t + \frac{1}{2} \frac{\partial^2 PH_t}{\partial r_t^2} (dr)^2 \right) \\
 &= \frac{\partial V_t}{\partial t} dt + \frac{\partial V_t}{\partial r_t} dr_t + \frac{1}{2} \sigma^2 \frac{\partial^2 V_t}{\partial r_t^2} dt - \Delta_t \left(\frac{\partial PH_t}{\partial t} dt + \frac{\partial PH_t}{\partial r_t} dr_t + \frac{1}{2} \sigma^2 \frac{\partial^2 PH_t}{\partial r_t^2} dt \right) \\
 &= \left[\frac{\partial V_t}{\partial t} + \frac{1}{2} \sigma^2 \frac{\partial^2 V_t}{\partial r_t^2} - \Delta_t \left(\frac{\partial PH_t}{\partial t} + \frac{1}{2} \sigma^2 \frac{\partial^2 PH_t}{\partial r_t^2} \right) \right] dt + \left(\frac{\partial V_t}{\partial r_t} - \Delta_t \frac{\partial PH_t}{\partial r_t} \right) dr_t.
 \end{aligned} \tag{3.16}$$

$d\Pi_t$ is random because of dr_t . We eliminate the randomness of $d\Pi_t$ in Equation (3.16) so that Π_t is a riskless portfolio of V_t during the offer period, which is made by setting

$$\Delta_t = \frac{\frac{\partial V_t}{\partial r_t}}{\frac{\partial PH_t}{\partial r_t}}, \quad t_0 \leq t \leq t_M. \tag{3.17}$$

By substituting Equation (3.17) into Equation (3.16), we get

$$d\Pi_t = \left[\frac{\partial V_t}{\partial t} + \frac{1}{2} \sigma^2 \frac{\partial^2 V_t}{\partial r_t^2} - \Delta_t \left(\frac{\partial PH_t}{\partial t} + \frac{1}{2} \sigma^2 \frac{\partial^2 PH_t}{\partial r_t^2} \right) \right] dt. \tag{3.18}$$

By the principle of no arbitrage, we get

$$\begin{aligned}
 d\Delta_t &= r_t \cdot \Pi_t dt \\
 &= r_t (V_t - \Delta_t PH_t) dt,
 \end{aligned}$$

which yields

$$\frac{\partial V_t}{\partial t} + \frac{1}{2} \sigma^2 \frac{\partial^2 V_t}{\partial r_t^2} - r_t V_t + \Delta_t \left(r_t \cdot PH_t - \frac{\partial PH_t}{\partial t} - \frac{1}{2} \sigma^2 \frac{\partial^2 PH_t}{\partial r_t^2} \right) = 0. \tag{3.19}$$

By a martingale argument under the no-arbitrage principle, we know that

$$\frac{\partial PH_t}{\partial t} + \kappa(\theta - r_t) \frac{\partial PH_t}{\partial r_t} + \frac{1}{2} \sigma^2 \frac{\partial^2 PH_t}{\partial r_t^2} = 0. \tag{3.20}$$

By substituting Equation (3.20) into Equation (3.19), we get

$$\frac{\partial V_t}{\partial t} + \frac{1}{2} \sigma^2 \frac{\partial^2 V_t}{\partial r_t^2} - r_t V_t + \Delta_t \cdot \kappa(\theta - r_t) \frac{\partial PH_t}{\partial r_t} = 0. \tag{3.21}$$

Further by substituting Equation (3.17) into Equation (3.21), we get Equation (3.11) and finish the proof. ■

With early exercise facility, however, the story changes. For an American put option, the no-arbitrage principle yields

$$\mathcal{L}(V) \leq 0.$$

At any point (r_t, t) , it will be optimal to either exercise this *budget option*, or to hold on the option. At any time t during the offer period, it holds that (referred to [10])

$$\begin{cases} \mathcal{L}(V) \leq 0, \\ V_t \geq (K(t) - PM(m_0, t, r_t))^+, \\ \mathcal{L}(V) \cdot (V_t - (K(t) - PM(m_0, t, r_t))^+) = 0. \end{cases} \quad (3.22)$$

An explanation of Problem (3.22) is given as follows.

Firstly, if $V_t < (K(t) - PM(m_0, t, r_t))^+$, there is an arbitrage resulting from shorting selling cash $(K(t) - PM(m_0, t, r_t))^+$ to buy the corresponding units of option V_t and then immediately exercising the option by an arbitrageur. Thus, $V_t \geq (K(t) - PM(m_0, t, r_t))^+$.

Secondly, in order to explain $\mathcal{L}(V) \leq 0$, we set up the self-financing portfolio Π_t consisting of one-unit-principal option V_t in long position and Δ_t units of $PH(m_0, t, r_t)$, as the setting in the proof of Equation 3.11 by the riskless portfolio setting method. And we know that

$$\Pi_t = V_t - \Delta_t \cdot PH(m_0, t, r_t),$$

riskless over the option period.

Case 1: $d\Pi_t > r_t \Pi_t dt$

No matter option V_t is European or American style, there is an arbitrage resulting from at time t shorting selling Π_t units of money savings account to buy one unit of the portfolio Π_t by an arbitrageur. One time step (dt) later, i.e., at time $(t + dt)$, the arbitrageur gains the payoff $(d\Pi_t - r_t \Pi_t dt)^+$ by returning the money savings account and selling the portfolio.

Case 2: $d\Pi_t < r_t \Pi_t dt$

If option V_t is European style, there is an arbitrage resulting from at time t shorting selling one unit of the portfolio Π_t to buy Π_t units of money savings account by an arbitrageur. And at time $(t + dt)$, the arbitrageur gains the payoff $(r_t \Pi_t dt - d\Pi_t)^+$ by returning the portfolio and selling the money savings account. However, if option V_t is American style, shorting selling portfolio Π_t puts the arbitrageur at the mercy of the early exercise facility, which no longer guarantees the arbitrageur an arbitrage to beat the bank risklessly [11].

Hence, the American style of the budget option results in $\mathcal{L}(V_t) \leq 0$ over the option period.

Thirdly, as the free boundary problem of an American option valuation indicates, when $V_t > (K(t) - PM(m_0, t, r_t))^+$, the option is held. Otherwise an early exercise causes immediate loss because of

$$-V_t + (K(t) - PM(m_0, t, r_t))^+ < 0.$$

Hence,

$$\mathcal{L}(V_t) = 0, \text{ when } V_t > (K(t) - PM(m_0, t, r_t))^+.$$

On the other hand, when $V_t = (K(t) - PM(m_0, t, r_t))^+$, the holder exercises the option optimally. Hence,

$$\mathcal{L}(V_t) \leq 0, \text{ when } V_t = (K(t) - PM(m_0, t, r_t))^+.$$

Therefore, the *budget option* valuation problem can be formulated into Problem (3.22).

3.2.2. BOUNDARY CONDITION

Now, we consider the boundary condition for Problem (3.22).

At maturity $t = t_M$, if the option is still held, there is no more exercise opportunities left for the holder. The holder either exercises the option at time t_M if it is in the money, or let the option expire. So the boundary condition for V_{t_M} is obvious, i.e.,

$$V_{t_M} = \max\{K(t_M) - PM(m_0, t_M, r_{t_M}), 0\}. \quad (3.23)$$

When $r_t = 0$ ($t_0 < t < t_M$), $PM_t = PM(m_0, t, 0)$. And

$$\max_{r_t \geq 0} \{PM(m_0, t, r_t)\} = PM(m_0, t, 0),$$

because $\frac{\partial PM(m_0, t, r_t)}{\partial r_t} < 0$ under the Vasicek model, which results in the maximum payoff of the American call option at time t as follows.

$$V_t^{\text{call}} = PM(m_0, t, 0) - K(t),$$

under $r_s \geq 0$, $t_0 < s < t_M$. For American options, there is no put-call parity. However, the following relationships hold [12].

$$K(t) \geq V_t^{\text{put}} - V_t^{\text{call}} + PM(m_0, t, r_t)$$

Hence, when $r_t = 0$ ($t_0 < t < t_M$),

$$V_t^{\text{put}} = 0, \text{ when } r_t = 0,$$

i.e., the *budget option* value

$$V_t = 0.$$

When $r_t \rightarrow +\infty$ ($t_0 < t < t_M$), all zero coupon bonds under the Vasicek model equal zero, i.e.,

$$P(t, T) = A(t, T)e^{-B(t, T)r_t} \rightarrow 0, \text{ as } r_t \rightarrow +\infty.$$

So,

$$PM(m_0, t, r_t) \rightarrow 0, \text{ as } r_t \rightarrow +\infty,$$

and

$$\max\{K(t) - PM(m_0, t, r_t), 0\} \rightarrow K(t), \text{ as } r_t \rightarrow +\infty.$$

Hence,

$$V_t \rightarrow K(t), \text{ as } r_t \rightarrow +\infty.$$

3.2.3. THE FINITE DIFFERENCE METHOD IMPLEMENTATION

We transform

$$t = t_M - \tau$$

into Problem (3.22), resulting in the formulated budget option valuation problem as follows.

$$\left\{ \begin{array}{l} \kappa(\theta - r_\tau) \frac{\partial V}{\partial r_\tau} + \frac{1}{2} \sigma^2 \frac{\partial^2 V}{\partial r_\tau^2} - r_\tau V \leq \frac{\partial V}{\partial \tau} \\ V_\tau - (K(\tau) - PM(m_0, \tau, r_\tau))^+ \geq 0 \\ \left(-\frac{\partial V}{\partial \tau} + \kappa(\theta - r_\tau) \frac{\partial V}{\partial r_\tau} + \frac{1}{2} \sigma^2 \frac{\partial^2 V}{\partial r_\tau^2} - r_\tau V_\tau \right) \cdot (V_\tau - (K(\tau) - PM(m_0, \tau, r_\tau))^+) = 0 \\ (t_M - \tau) \in \text{month } \tilde{T} := \left\{ \text{month } T_i \mid \text{the budget option to be exercised in month } T_i, i = 0, 1, 2 \right\}, \end{array} \right. \quad (3.24)$$

and

$$\left\{ \begin{array}{l} \kappa(\theta - r_\tau) \frac{\partial V_\tau}{\partial r_\tau} + \frac{1}{2} \sigma^2 \frac{\partial^2 V_\tau}{\partial r_\tau^2} - r_\tau V_\tau = \frac{\partial V_\tau}{\partial \tau} \\ t_0 \leq (t_M - \tau) < \tilde{t}_0, \end{array} \right.$$

where $\tilde{t}_0$ is the beginning of month $\tilde{T}$, as illustrated in Figure 3.1.

We set up a grid of τ -axis and r -axis. Let $\Delta\tau$ and Δr be the mesh size of τ discretization and the mesh size of r discretization respectively. To avoid unreadable long equations, some shorthand notations are introduced.

$$\begin{aligned} \tau_j &:= j \cdot \Delta\tau, & j &= 0, 1, \dots, \tilde{j} := \frac{t_M - t_0}{\Delta\tau}; \\ r_i &:= i \cdot \Delta r, & i &= 0, 1, \dots, \tilde{i}; \\ V_{i,j} &:= V(r_i, \tau_j), \\ \mathbf{V}_j &:= \mathbf{V}(\tau_j) = \begin{pmatrix} V(r_1, \tau_j) \\ V(r_2, \tau_j) \\ \vdots \\ V(r_{\tilde{i}}, \tau_j) \end{pmatrix}. \end{aligned}$$

The explicit method discretizes $\mathcal{L}(V_{i,j}) = 0$ in Problem (3.24) into

$$\frac{V_{i,j+1} - V_{i,j}}{\Delta\tau} = \kappa(\theta - r_i) \frac{V_{i+1,j} - V_{i-1,j}}{2\Delta r} + \frac{1}{2} \sigma^2 \frac{V_{i+1,j} - 2V_{i,j} + V_{i-1,j}}{(\Delta r)^2} - r_i \cdot V_{i,j}.$$

The implicit method discretizes $\mathcal{L}(V_{i,j}) = 0$ in Problem (3.24) into

$$\frac{V_{i,j+1} - V_{i,j}}{\Delta\tau} = \kappa(\theta - r_i) \frac{V_{i+1,j+1} - V_{i-1,j+1}}{2\Delta r} + \frac{1}{2} \sigma^2 \frac{V_{i+1,j+1} - 2V_{i,j+1} + V_{i-1,j+1}}{(\Delta r)^2} - r_i \cdot V_{i,j+1}.$$

The Crank-Nicolson method discretizes $\mathcal{L}(V_{i,j}) = 0$ in Problem (3.24) into

$$\begin{aligned} \frac{V_{i,j+1} - V_{i,j}}{\Delta\tau} = & \frac{1}{2} \left(\kappa(\theta - r_i) \frac{V_{i+1,j} - V_{i-1,j}}{2\Delta r} + \frac{1}{2}\sigma^2 \frac{V_{i+1,j} - 2V_{i,j} + V_{i-1,j}}{(\Delta r)^2} - r_i \cdot V_{i,j} \right) \\ & \frac{1}{2} \left(\kappa(\theta - r_i) \frac{V_{i+1,j+1} - V_{i-1,j+1}}{2\Delta r} + \frac{1}{2}\sigma^2 \frac{V_{i+1,j+1} - 2V_{i,j+1} + V_{i-1,j+1}}{(\Delta r)^2} - r_i \cdot V_{i,j+1} \right), \end{aligned}$$

which yields

$$\begin{aligned} & \left(\frac{\Delta\tau}{4\Delta r} \kappa(\theta - r_i) - \frac{\Delta\tau\sigma^2}{4(\Delta r)^2} \right) V_{i-1,j+1} + \left(1 + \frac{\Delta\tau\sigma^2}{2(\Delta r)^2} + \frac{\Delta\tau}{2} r_i \right) V_{i,j+1} + \left(-\frac{\Delta\tau}{4\Delta r} \kappa(\theta - r_i) - \frac{\Delta\tau\sigma^2}{4(\Delta r)^2} \right) V_{i+1,j+1} \\ & = \left(-\frac{\Delta\tau}{4\Delta r} \kappa(\theta - r_i) + \frac{\Delta\tau\sigma^2}{4(\Delta r)^2} \right) V_{i-1,j} + \left(1 - \frac{\Delta\tau\sigma^2}{2(\Delta r)^2} - \frac{\Delta\tau}{2} r_i \right) V_{i,j} + \left(\frac{\Delta\tau}{4\Delta r} \kappa(\theta - r_i) + \frac{\Delta\tau\sigma^2}{4(\Delta r)^2} \right) V_{i+1,j}. \end{aligned}$$

By applying the Crank-Nicolson method, we get

$$A \cdot \mathbf{V}_{j+1} = B \cdot \mathbf{V}_j + \mathbf{d}, \quad (3.25)$$

where the iteration matrix A for solving $\mathbf{V}_{j+1}$ is an $\tilde{l} \times \tilde{l}$ matrix, visualized as

$$\begin{pmatrix} a_{1,1} & a_{1,2} & 0 & 0 & \dots & 0 & 0 \\ a_{2,1} & a_{2,2} & a_{2,3} & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \\ 0 & 0 & 0 & 0 & \dots & a_{\tilde{l}-1,\tilde{l}} & a_{\tilde{l},\tilde{l}} \end{pmatrix} \quad (3.26)$$

where

$$a_{i,i-1} = \frac{\Delta\tau}{4\Delta r} \kappa(\theta - r_i) - \frac{\Delta\tau\sigma^2}{4(\Delta r)^2}, \quad (3.27)$$

$$a_{i,i} = 1 + \frac{\Delta\tau\sigma^2}{2(\Delta r)^2} + \frac{\Delta\tau}{2} r_i, \quad (3.28)$$

$$a_{i,i+1} = -\frac{\Delta\tau}{4\Delta r} \kappa(\theta - r_i) - \frac{\Delta\tau\sigma^2}{4(\Delta r)^2}. \quad (3.29)$$

If A is strongly diagonally dominant, the Jacobi and Gauss-Seidel method applied to A converge [13]. By choosing $\Delta\tau$ and Δr satisfying that condition, the convergence of the Jacobi or Gauss-Seidel method applied to A is guaranteed. We give the condition along with its proof below.

Condition 11 (convergence condition of FDM in the *budget option* valuation).

1. $r \geq -\frac{2}{\Delta\tau} - \frac{\sigma^2}{(\Delta r)^2}$;
2. $\Delta r < \kappa$
3. $\frac{\theta\kappa}{\kappa+\Delta r} - \frac{2\Delta r}{\Delta\tau(\kappa+\Delta r)} < r < \frac{\theta\kappa}{\kappa-\Delta r} + \frac{2\Delta r}{\Delta\tau(\kappa-\Delta r)}$.

proof. If A in Equation (3.26) is strictly diagonally dominant, the entries in iteration matrix A satisfy

$$|a_{i,i-1}| + |a_{i,i+1}| < |a_{i,i}|. \quad (3.30)$$

where $1 \leq i \leq \tilde{i}$, and without loss of generality we denote $a_{1,-1} = 0$, $a_{\tilde{i},\tilde{i}+1} = 0$.

So we need to guarantee

$$\left| -\frac{\Delta\tau}{4\Delta r}\kappa(\theta - r_i) + \frac{\Delta\tau\sigma^2}{4(\Delta r)^2} \right| + \left| -\frac{\Delta\tau}{4\Delta r}\kappa(\theta - r_i) - \frac{\Delta\tau\sigma^2}{4(\Delta r)^2} \right| < \left| 1 + \frac{\Delta\tau\sigma^2}{2(\Delta r)^2} + \frac{\Delta\tau}{2}r_i \right|. \quad (3.31)$$

Since

$$\left| -\frac{\Delta\tau}{4\Delta r}\kappa(\theta - r_i) + \frac{\Delta\tau\sigma^2}{4(\Delta r)^2} \right| + \left| -\frac{\Delta\tau}{4\Delta r}\kappa(\theta - r_i) - \frac{\Delta\tau\sigma^2}{4(\Delta r)^2} \right| \leq \left| \frac{\Delta\tau}{2\Delta r}\kappa(\theta - r_i) \right| + \frac{\Delta\tau\sigma^2}{2(\Delta r)^2}, \quad (3.32)$$

as long as it holds that

$$\left| \frac{\Delta\tau}{2\Delta r}\kappa(\theta - r_i) \right| + \frac{\Delta\tau\sigma^2}{2(\Delta r)^2} < \left| 1 + \frac{\Delta\tau\sigma^2}{2(\Delta r)^2} + \frac{\Delta\tau}{2}r_i \right|, \quad (3.33)$$

the iteration matrix A will be strictly diagonally dominant. If $r \geq -\frac{2}{\Delta\tau} - \frac{\sigma^2}{(\Delta r)^2}$, the inequality (3.33) is equivalent to

$$\left| \frac{\Delta\tau}{2\Delta r}\kappa(\theta - r_i) \right| < 1 + \frac{\Delta\tau}{2}r_i. \quad (3.34)$$

The inequality (3.34) is equivalent to

$$\frac{\theta\kappa}{\kappa + \Delta r} - \frac{2\Delta r}{\Delta\tau(\kappa + \Delta r)} < r < \frac{\theta\kappa}{\kappa - \Delta r} + \frac{2\Delta r}{\Delta\tau(\kappa - \Delta r)} \quad (3.35)$$

with $\kappa > \Delta r$. Hence, as long as the relations in (3.35) and $r \geq -\frac{2}{\Delta\tau} - \frac{\sigma^2}{(\Delta r)^2}$ hold, we can guarantee that the iteration matrix A is strictly diagonally dominant and more importantly, the Jacobi and Gauss-Seidel method applied to A converge. ■

The matrix B in Equation (3.25) is visualized as

$$\begin{pmatrix} b_{1,1} & b_{1,2} & 0 & 0 & \dots & 0 & 0 \\ b_{2,1} & b_{2,2} & b_{2,3} & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \\ 0 & 0 & 0 & 0 & \dots & b_{\tilde{i}-1,\tilde{i}} & b_{\tilde{i},\tilde{i}} \end{pmatrix}$$

where

$$b_{i,i-1} = -\frac{\Delta\tau}{4\Delta r}\kappa(\theta - r_i) + \frac{\Delta\tau\sigma^2}{4(\Delta r)^2}, \quad (3.36)$$

$$b_{i,i} = 1 - \frac{\Delta\tau\sigma^2}{2(\Delta r)^2} - \frac{\Delta\tau}{2}r_i, \quad (3.37)$$

$$b_{i,i+1} = \frac{\Delta\tau}{4\Delta r}\kappa(\theta - r_i) + \frac{\Delta\tau\sigma^2}{4(\Delta r)^2}. \quad (3.38)$$

The vector $\mathbf{d}$ in Equation (3.25) contains the boundary conditions for r_0 and r_{i+1} , which is visualized as

$$\mathbf{d} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ d_{\tilde{i}} \end{pmatrix}$$

where

$$\begin{aligned} d_{\tilde{i}} = & \left(\frac{\Delta\tau}{4\Delta r} \kappa(\theta - r_{\tilde{i}-1}) + \frac{\Delta\tau\sigma^2}{4(\Delta r)^2} \right) \cdot (K(\tau_j) - PM(m_0, \tau_j, r_{\tilde{i}})) \\ & + \left(\frac{\Delta\tau}{4\Delta r} \kappa(\theta - r_{\tilde{i}-1}) + \frac{\Delta\tau\sigma^2}{4(\Delta r)^2} \right) \cdot (K(\tau_{j+1}) - PM(m_0, \tau_{j+1}, r_{\tilde{i}})). \end{aligned} \quad (3.39)$$

We repeat the process of using iteration method in solving $\mathbf{V}_j$ until we get $\mathbf{V}_{\tilde{j}}$ which gives the solution of the budget option value at time t_0 against the initial interest rate r_{t_0} .

3.3. THE LEAST-SQUARE METHOD IN THE BUDGET OPTION VALUATION

The least squares method [14] is initialized at the offer maturity t_M and utilizes a backward-induction. Under the risk-neutral measure $(\Omega, \mathcal{F}_t, Q)$, $\{r_t\}_{t \geq t_0}$ represents the stochastic instantaneous interest rate process. We simulate b paths of $\{r_t; t \in \mathfrak{T}\}$ with $\mathfrak{T} = \{t_k\}_{k=0}^M$ the discrete set of simulated time steps at which the interest rate r is sampled. The size of the time size is denoted as $\Delta\tau$, and

$$\Delta\tau = t_{i+1} - t_i, \quad i = 0, 1, \dots, M-1.$$

And r_{t_0} is known at time t_0 for all the simulated paths. Denote $V_k^i = V(m_0, r_{t_k}^i, t_k)$ as the value function of the *budget option* at time t_k on the simulated path i .

during the exercise month $\tilde{T}$

The time $\tilde{t}_M$ denotes the end of month $\tilde{T}$, as illustrated in Figure 3.1. At $\tilde{t}_M$, the value of a yet unexercised *budget option* is the immediate exercise payoff if it is in the money. Consequently, the *budget offer* value for path $i \in \mathfrak{J} := \{1, 2, \dots, b\}$ is obtained by

$$V_{\tilde{t}_M}^i = I_{\tilde{t}_M}^i := \max \left\{ K(\tilde{t}_M) - PM(m_0, \tilde{t}_M, r_{\tilde{t}_M}^i), 0 \right\}.$$

We move one step backward in time. At $(\tilde{t}_M - \Delta\tau)$, the continuation payoff of the option for path i is obtained by

$$\hat{C}_{\tilde{t}_M - \Delta\tau}^i = I_{\tilde{t}_M}^i \cdot \exp \left\{ -\Delta\tau \cdot r_{\tilde{t}_M - \Delta\tau}^i \right\}.$$

The expected continuation payoff function at $(\tilde{t}_M - \Delta\tau)$ takes the form:

$$C_{\tilde{t}_M - \Delta\tau} = \beta_0 + \sum_{j=1}^n \beta_j \cdot \phi_j(r_{\tilde{t}_M - \Delta\tau}), \quad (3.40)$$

which just discounts $V_{t_0}^i$ to one step backward in time.

Repeat the above discounting process until the offer sending moment t_0 is reached. Hence, the estimated value of this *budget option* at t_0 is obtained by

$$\hat{V}_0 = \frac{\sum_{i=1}^b V_{t_0}^i}{b}. \quad (3.41)$$

3

3.4. THE STOCHASTIC GRID BUNDLING METHOD IN THE BUDGET OPTION VALUATION

In addition to the least squares method (LSM), in this section we introduce a newly launched regression-based method in the simulation pricing, the Stochastic Grid Bundling Method (SGBM) by Jain and Oosterlee [15]. We apply SGBM in the *budget option* valuation in order to get comparable solutions to those by FDM and LSM. The convergence performance of SGBM will be presented in Section 3.6.

Applying SGBM aims at getting two estimators to determine the option value. One is called director estimator (DE) which is biased high, so it is regarded as the upper bound of the option value [15]. The other one is called path estimator (PE). Although PE is based on the same optimal exercise strategy as in DE, the regressors for PE in estimating the option continuation value at each time step are obtained from DE, which leads to a bias-high option continuation value at each time step for PE. The bias-high continuation value in PE delays the exercise time from the optimal exercise time, which leads to a bias-low option value in PE. So PE can be regarded as a lower bound of the option value, which will finally converge to the true value as long as the option continuation value obtained by DE converges to the true option continuation value at each time step.

The computing steps to get these two estimators are as follows.

Steps of Director Estimator Computing

Under the risk-neutral measure $(\Omega, \mathcal{F}_t, Q)$, $\{r_t\}_{t \geq t_0}$ represents the stochastic instantaneous interest rate process. We simulate b paths of $\{r_t; t \in \mathfrak{T}\}$ with $\mathfrak{T} = \{t_k\}_{k=1}^M$ the discrete set of simulated time steps at which the interest rate r is sampled. $\Delta\tau = t_{i+1} - t_i$, $(i = 1, \dots, M-1)$ is the length of the time step in each path. And r_{t_0} is known at time t_0 for all the simulated paths. Denote $V_k^i = V(m_0, r_{t_k}^i, t_k)$ as the value function of the *budget option* at time t_k on the simulated interest rate path i .

During the exercise month $\tilde{T}$:

Step 1:

The *budget option* value is initialized at $\tilde{t}_M$, i.e., the end time of month $\tilde{T}$, as illustrated in Figure 3.1. At $\tilde{t}_M$, the value of the *budget option* is the immediate exercise payoff if it is in the money. Consequently, for each simulated path $i \in \mathfrak{J} := \{1, 2, \dots, b\}$,

$$V_{\tilde{t}_M}^i = \max \left\{ K(\tilde{t}_M) - PM(m_0, r_{\tilde{t}_M}^i, \tilde{t}_M), 0 \right\}.$$

Step 2:

We move one step backward in time. At time $(\tilde{t}_M - \Delta\tau)$, we bundle the simulated paths $\{r_{\tilde{t}_M - \Delta\tau}^i, i \in \mathfrak{J}\}$ into $l > 1$ bundles according to a chosen bundling technique. The bundling techniques introduced in [15] are k-means clustering, recursive bifurcation, and recursive bifurcation of reduced state space. In [16], the ‘equal-range bundling’ technique is proposed, which is proved to have comparable computing accuracy to the ‘equal-size’ bundling techniques but more robustness. Although the ‘equal-range bundling’ technique is more efficient in implementation and can guarantee an equal number of simulated paths in each bundle per time step, an overlap problem arises when simulated paths with the same simulated value are across a bundle boundary and separated into two bundles. In that situation, basis functions of the same simulated value contribute to two different regressors, but during the path estimator process, that simulated value is matched with only one of the two regressors, which may induce biasness in regression and estimator. In this paper, we modify the ‘equal-range bundling’ technique by applying a ‘quantile bundling’ technique so that it can guarantee a similar number of simulated paths in each bundle per time step with much efficiency in implementation. The ‘quantile bundling’ technique is that when we bundle the simulated paths $\{r_{\tilde{t}_M - \Delta\tau}^i, i \in \mathfrak{J}\}$ into l bundles, the $\frac{\nu}{l}$ ($\nu = 1, 2, \dots, l$)-th quantiles $Q_{\frac{\nu}{l}}$ of $\{r_{\tilde{t}_M - \Delta\tau}^i, i \in \mathfrak{J}\}$ are used to separate the paths by bundling paths $\{i \in \mathfrak{J} | Q_{\frac{\nu-1}{l}} < r_{\tilde{t}_M - \Delta\tau}^i \leq Q_{\frac{\nu}{l}}\}$, ($\nu = 1, 2, \dots, l$) into the ν -th bundle, where we set $Q_0 = -\infty$ and $Q_l = +\infty$. The number of the bundles is the same for all the time steps, i.e., l bundles per time step. At the meantime, $\{Q_{\frac{\nu}{l}}\}_{\nu=1}^{l-1}$ as the boundary information for each time step is stored so that the same bundling information is applied to bundle the newly simulated paths in the path estimator process.

Step 3:

After bundling $\{r_{\tilde{t}_M - \Delta\tau}^i, i \in \mathfrak{J}\}$ into l bundles, we make regression and get the regressors for each bundle at $(\tilde{t}_M - \Delta\tau)$. For example, for the paths in the α -th bundle ($1 \leq \alpha \leq l$) denoted as path i_α , we regress their option values $V_{\tilde{t}_M}^{i_\alpha}$ at $\tilde{t}_M$ on the basis functions $\phi_j(r_{\tilde{t}_M}^{i_\alpha})$, ($j = 1, \dots, n$) and then get the corresponding coefficients $\beta_{\tilde{t}_M - \Delta\tau}^j(\alpha)$, ($j = 1, \dots, n$) for the α -th bundle at $(\tilde{t}_M - \Delta\tau)$. The basis functions chosen here are (weighted) monomials of the instantaneous interest rate r_t , which are

$$\phi_j(r_t) = r_t^{j-1}, \quad j = 1, 2, 3, 4. \quad (3.42)$$

Numerical studies by Stentoft [17] indicate that using monomials as the basis results have comparable accuracy to Legendre polynomials or to Laguerre polynomials, but higher computational efficiency. More importantly, the choice of monomial basis functions ideally gives us a closed form of $\mathbb{E}^Q[\phi_j(r_t) | r_{t-1}]$. That is because under the Vasicek model (3.5),

the conditional probability of r_t given r_{t-1} is a normal distribution with mean

$$\hat{\mu}_{t-1} = r_{t-1} + \kappa \cdot (\theta - r_{t-1})\Delta\tau$$

and standard deviation

$$\hat{\sigma}_{t-1} = \sigma \cdot \sqrt{\Delta\tau},$$

i.e.,

$$(r_t | r_{t-1}) \sim \mathcal{N}(\hat{\mu}_{t-1}, \hat{\sigma}_{t-1}^2).$$

Hence,

$$\begin{aligned}\mathbb{E}^Q[1 | r_{t-1}] &= 1, \\ \mathbb{E}^Q[r_t | r_{t-1}] &= \hat{\mu}_{t-1}, \\ \mathbb{E}^Q[r_t^2 | r_{t-1}] &= \hat{\mu}_{t-1}^2 + \hat{\sigma}_{t-1}^2, \\ \mathbb{E}^Q[r_t^3 | r_{t-1}] &= \hat{\mu}_{t-1}^3 + 3\hat{\mu}_{t-1} \cdot \hat{\sigma}_{t-1}^2.\end{aligned}$$

Step 4:

After obtaining the coefficients for all the bundles at time $(\tilde{t}_M - \Delta\tau)$, we starts to calculate the expected continuation value $\hat{C}_{\tilde{t}_M - \Delta\tau}$ of the *budget option* at $(\tilde{t}_M - \Delta\tau)$ in order to determine the option value at $(\tilde{t}_M - \Delta\tau)$ by

$$V_{\tilde{t}_M - \Delta\tau} = \max\{I_{\tilde{t}_M - \Delta\tau}, \hat{C}_{\tilde{t}_M - \Delta\tau}\}.$$

The continuation value for a path in the α -th bundle at $(\tilde{t}_M - \Delta\tau)$ with the simulated instantaneous interest rate $r_{\tilde{t}_M - \Delta\tau}^{i_\alpha}$, is

$$\hat{C}_{\tilde{t}_M - \Delta\tau}^{i_\alpha} = \exp\{-\Delta\tau \cdot r_{\tilde{t}_M - \Delta\tau}^{i_\alpha}\} \cdot \sum_{j=1}^4 \mathbb{E}^Q\left[\beta_{\tilde{t}_M - \Delta\tau}^j(\alpha) \cdot \phi_j(r_{\tilde{t}_M}) \middle| r_{\tilde{t}_M - \Delta\tau}^{i_\alpha}\right]. \quad (3.43)$$

By the distribution of $r_{\tilde{t}_M}$ conditional on $r_{\tilde{t}_M - \Delta\tau}^{i_\alpha}$ in the Vasicek model, we can get a closed form approximation of $\hat{C}_{\tilde{t}_M - \Delta\tau}^{i_\alpha}$. The immediate exercise payoff at $(\tilde{t}_M - \Delta\tau)$ on that path is

$$I_{\tilde{t}_M - \Delta\tau}^{i_\alpha} = \max\left\{K(\tilde{t}_M - \Delta\tau) - PM\left(m_0, \tilde{t}_M - \Delta\tau, r_{\tilde{t}_M - \Delta\tau}^{i_\alpha}\right), 0\right\}. \quad (3.44)$$

So, the option value at $(\tilde{t}_M - \Delta\tau)$ on that path is

$$V_{\tilde{t}_M - \Delta\tau}^{i_\alpha} = \max\left\{I_{\tilde{t}_M - \Delta\tau}^{i_\alpha}, \hat{C}_{\tilde{t}_M - \Delta\tau}^{i_\alpha}\right\}. \quad (3.45)$$

We then get the option values for all the simulated paths at $(\tilde{t}_M - \Delta\tau)$.

Step 5:

We move one time step backward and repeat Step 2 to Step 5, until reaching time $\tilde{t}_0$, i.e., the beginning of month $\tilde{T}$.

During the holding time $t_0 \leq t < \tilde{t}_0$:

During $t_0 \leq t < \tilde{t}_0$, the option is not going to be exercised. At $(\tilde{t}_0 - \Delta\tau)$, i.e., one step backward of $\tilde{t}_0$, the option value for the simulated path $i \in \mathfrak{J}$ is

$$V_{\tilde{t}_0 - \Delta\tau}^i = V_{\tilde{t}_0}^i \cdot \exp\left\{-\Delta\tau \cdot r_{\tilde{t}_0 - \Delta\tau}^i\right\},$$

which just discounts $V_{\tilde{t}_0}^i$ to one step backward in time.

We repeat the above discounting process until the offer sending moment t_0 is reached. Finally, the direct estimator for the *budget option* value at t_0 is obtained by

$$V_0(r_{t_0}) = \frac{\sum_{i=1}^b V_{t_0}^i}{b}. \quad (3.46)$$

Steps of Path Estimator Computing

Step 1:

After calculating the direct estimator, we simulate $(2 \times b)$ other Monte Carlo (MC) paths of $\{r_t; t \in \mathfrak{T}\}$ with $\mathfrak{T} = \{t_k\}_{k=1}^M$ the discrete set of simulated time steps. $\Delta\tau = t_{k+1} - t_k$, $(k = 1, \dots, M-1)$ is the length of the time step in each path. Of course, r_{t_0} is known at time t_0 for all the simulated paths. Define $V_k^i = V(m_0, r_{t_k}^i, t_k)$ as the value function of the *budget option* at time t_k on the simulated path i .

Step 2:

For a simulated path $r^i = \{r_{t_0}^i, r_{t_1}^i, \dots, r_{t_{M-1}}^i, r_{t_M}^i\}$ among the $2b$ paths, the exercise time, or called the stopping time for its path estimator, is defined as

$$\hat{\tau}(r^i) = \min\left\{t \mid I^i(t) \geq \hat{C}^i(t)\right\}, \quad (3.47)$$

where $\hat{C}^i(t)$, the expected continuation value of the option at time t for path i , is computed by Equation (3.43). Then the path estimator for path r^i is given by

$$V_{t_0}^i = I_{\hat{\tau}(r^i)}^i \cdot \exp\left\{-\sum_{t=t_0}^{\hat{\tau}(r^i) - \Delta\tau} r_t^i \cdot \Delta\tau\right\}. \quad (3.48)$$

Step 3:

After calculating the path estimators for the simulated $2b$ paths, the path estimator for the *budget option* value at t_0 is obtained by

$$V_0(r_{t_0}) = \frac{\sum_{i=1}^{2b} V_{t_0}^i}{2b}. \quad (3.49)$$

3.5. NUMERICAL RESULTS

The tests in this section are based on the basic setting of a *budget option* valuation problem, which is explained in the previous sections.

In the numerical tests of FDM in the *budget option* valuation, the Gauss-Seidel iteration method for solving Problem (3.22) is applied. The grid sizes in FDM are chosen to be $\Delta\tau = \frac{1}{360}$, $\Delta r = 2 \cdot 10^{-4}$ unless any specification. The stopping criterion in the Gauss-Seidel iteration is chosen to be 10^{-8} .

In the numerical tests of LSM and SGBM, to reduce the variance of the Monte Carlo simulation, the antithetic sampling technique is applied. In each LSM test, there are in total 40,000 simulated paths of the instantaneous interest rate over the offer period, among which 20,000 paths are generated by antithetic sampling next to the other 20,000 paths. In each test of SGBM, there are $N_D = 20,000$ simulated paths for the direct estimator calculation, and $N_P = 2N_D$ ($N_D + N_D(\text{antithetic})$) simulated paths for the path estimator calculation. The solution presented below by LSM or by SGBM is an average of the simulated solutions under 20 different random seeds, and the corresponding standard error (s.e.) is the standard deviation of these 20 simulated solutions. $\Delta\tau = \frac{1}{360}$ is the time step size. Ten bundles are used. The specification for each numerical test is the same, unless stated otherwise.

All the numerical tests presented in this thesis are for the *budget offer* of one-unit principal with the above default parameters.

Some numerical results of FDM, LSM, SGBM in the *budget option* valuation are presented next. To clarify the meaning of notations in the presented tables, we list the explanation below:

V_{FD}^i , ($i = 0, 1, 2$): the value of the *budget option* to be exercised in month T_i solved by FDM;

V_{FD} : the value of the *budget option* value solved by FDM, equal to the maximum of $\{V_{FD}^0, V_{FD}^1, V_{FD}^2\}$;

V_{LSM}^i , ($i = 0, 1, 2$): the value of the *budget option* to be exercised in month T_i solved by LSM;

V_{LSM} : the value of the *budget option* solved by LSM, equal to the maximum of $\{V_{LSM}^0, V_{LSM}^1, V_{LSM}^2\}$;

$s.e.(LSM)$: the standard error of the solution V_{LSM} by LSM;

V_{DE}^i , ($i = 0, 1, 2$): the direct estimator of the *budget option* to be exercised in month T_i solved by SGBM;

V_{DE} : the direct estimator of the *budget option* solved by SGBM, equal to the maximum of $\{V_{DE}^0, V_{DE}^1, V_{DE}^2\}$;

$s.e.(DE)$: the standard error of the direct estimator V_{DE} by SGBM;

V_{PE}^i : the path estimator of the *budget option* to be exercised in month T_i solved by SGBM;

V_{PE} , ($i = 0, 1, 2$): the path estimator of the *budget option* solved by SGBM, equal to the maximum of $\{V_{PE}^0, V_{PE}^1, V_{PE}^2\}$;

$s.e.(PE)$: the standard error of the path estimator V_{PE} by SGBM.

Table 3.2 presents the budget option values V_0 against different locked mortgage rates m_0 in the budget offer. Table 3.3 presents the budget option values V_0 against different values of κ in the Vasicek interest rate model. Table 3.4 presents the budget option values V_0 against different values of θ in the Vasicek interest rate model. Table 3.5 presents the budget option values V_0 against different values of the interest rate r_{t_0} at the offer sending moment t_0 . Table 3.6 presents the budget option values V_0 against different mortgage terms of the *budget offer*. Table 3.7 presents the budget option values V_0 against different values of σ in the Vasicek interest rate model. From the results in the tables, we see that an increase in the mortgage term, or an increase in the volatility σ , results in a relatively significant increase in the *budget option* value, which implies that compared to the other tested parameters, the budget option value is much sensitive to the mortgage term change and the volatility σ change.

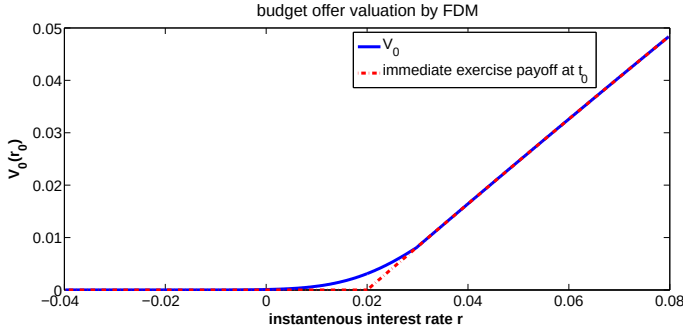


Figure 3.3: *budget option* valuation by FDM (the *budget offer* specified in Table 3.1 with $\kappa = 0.5$, $\theta = 0.02$, $\sigma = 0.02$, $r_{t_0} = 0.02$, $m_0 = 0.02$, 1-year mortgage term).

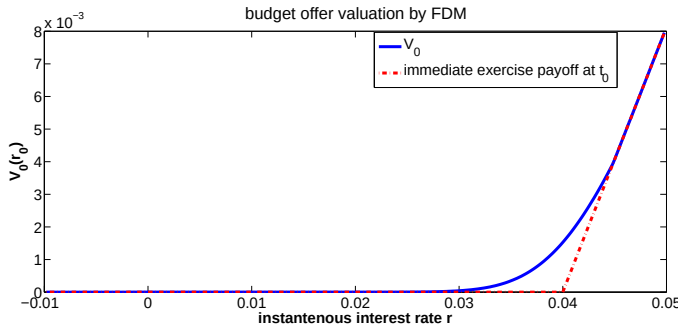


Figure 3.4: *budget option* valuation by FDM (the *budget offer* specified in Table 3.1 with $\kappa = 0.5$, $\theta = 0.02$, $\sigma = 0.01$, $r_{t_0} = 0.04$, $m_0 = 0.02$, 1-year mortgage term).

Figure 3.3 and Figure 3.4 plot the *budget option* value V_0 given by FDM and the immediate exercise payoff against the discretized interest rate axis in the FDM implementation.

m_0	V_{FD}	V_{LSM}	$s.e.(LSM)$	V_{DE}	$s.e.(DE)$	V_{PE}	$s.e.(PE)$
0.015	0.0031	0.003105	1.11E-05	0.003113	2.25E-05	0.003098	1.77E-05
0.02	0.00311	0.003115	1.13E-05	0.003122	2.26E-05	0.003108	1.77E-05
0.025	0.003119	0.003124	1.10E-05	0.003132	2.27E-05	0.003117	1.78E-05

m_0	V_{FD}^0	V_{LSM}^0	V_{DE}^0	V_{PE}^0	V_{FD}^1	V_{LSM}^1	V_{DE}^1	V_{PE}^1	V_{FD}^2	V_{LSM}^2	V_{DE}^2	V_{PE}^2
0.015	0.001775	0.001778	0.001774	0.001777	0.002545	0.00255	0.002547	0.002542	0.0031	0.003105	0.003113	0.003098
0.02	0.00178	0.001784	0.001779	0.001782	0.002553	0.002557	0.002555	0.002549	0.00311	0.003115	0.003122	0.003108
0.025	0.001785	0.001789	0.001784	0.001788	0.00256	0.002565	0.002562	0.002557	0.003119	0.003124	0.003132	0.003117

Table 3.2: *budget option* valuation varying mortgage rate m_0 (the *budget offer* specified in Table 3.1 with $\kappa = 0.5$, $\theta = 0.02$, $\sigma = 0.02$, $r_{t_0} = 0.02$, 1-year mortgage term).

κ	V_{FD}	V_{LSM}	$s.e.(LSM)$	V_{DE}	$s.e.(DE)$	V_{PE}	$s.e.(PE)$
0.3	0.003475	0.003481	1.21E-05	0.003489	2.54E-05	0.003472	1.99E-05
0.6	0.002946	0.002951	1.06E-05	0.002958	2.14E-05	0.002944	1.67E-05
0.9	0.00252	0.002525	9.10E-06	0.002531	1.82E-05	0.002519	1.47E-05

κ	V_{FD}^0	V_{LSM}^0	V_{DE}^0	V_{PE}^0	V_{FD}^1	V_{LSM}^1	V_{DE}^1	V_{PE}^1	V_{FD}^2	V_{LSM}^2	V_{DE}^2	V_{PE}^2
0.3	0.001964	0.001968	0.001963	0.001967	0.002833	0.002838	0.002835	0.002829	0.003475	0.003481	0.003489	0.003472
0.6	0.001696	0.0017	0.001696	0.001699	0.002426	0.002431	0.002429	0.002424	0.002946	0.002951	0.002958	0.002944
0.9	0.001477	0.00148	0.001477	0.00148	0.002096	0.0021	0.002098	0.002094	0.00252	0.002525	0.002531	0.002519

Table 3.3: *budget option* valuation varying κ (the *budget offer* specified in Table 3.1 with $\theta = 0.02$, $\sigma = 0.02$, $r_{t_0} = 0.02$, $m_0 = 0.02$, 1-year mortgage term).

θ	V_{FD}	V_{LSM}	$s.e.(LSM)$	V_{DE}	$s.e.(DE)$	V_{PE}	$s.e.(PE)$
0.02	0.00311	0.003115	1.13E-05	0.003122	2.26E-05	0.003108	1.77E-05
0.04	0.00309	0.003096	1.10E-05	0.003103	2.25E-05	0.003089	1.76E-05
0.06	0.00307	0.003077	1.10E-05	0.003084	2.23E-05	0.003069	1.75E-05

θ	V_{FD}^0	V_{LSM}^0	V_{DE}^0	V_{PE}^0	V_{FD}^1	V_{LSM}^1	V_{DE}^1	V_{PE}^1	V_{FD}^2	V_{LSM}^2	V_{DE}^2	V_{PE}^2
0.02	0.00178	0.001784	0.001779	0.001782	0.002553	0.002557	0.002555	0.002549	0.00311	0.003115	0.003122	0.003108
0.04	0.001769	0.001775	0.001774	0.001774	0.002538	0.002543	0.002541	0.002535	0.00309	0.003096	0.003103	0.003089
0.06	0.001761	0.001767	0.001762	0.001765	0.002524	0.00253	0.002527	0.002521	0.00307	0.003077	0.003084	0.003069

Table 3.4: *budget option* valuation varying θ (the *budget offer* specified in Table 3.1 with $\kappa = 0.5$, $\sigma = 0.02$, $r_{t_0} = 0.02$, $m_0 = 0.02$, 1-year mortgage term).

A *budget option* is a put option on the loan. On the other hand, because of $\frac{\partial PM(m_0, t, r_t)}{\partial r_t} < 0$, a *budget option* can be regarded as a call option on the prevailing instantaneous interest rate r_t . The minimum of the instantaneous interest rates meeting the optimal exercise condition at time $t_0 \leq t \leq t_M$ is called the contact point of the *budget option* at t , denoted as $r_f(t)$. The plot of the contact points is called the early-exercise curve. For a *budget option* sent at the beginning of month T_0 , the early-exercise curve of the *budget option* to be exercised in month T_2 is presented in Figure 3.5. The early-exercise curve of the *budget option* to be exercised in a different month during the offer period is almost the same as the month T_2 one.

Here we define the academic *hit ratio* as a quantity to describe the probability of option holders to optimally exercise their *budget option*.

r_{t_0}	V_{FD}	V_{LSM}	$s.e.(LSM)$	V_{DE}	$s.e.(DE)$	V_{PE}	$s.e.(PE)$
0.02	0.00311	0.003115	1.13E-05	0.003122	2.26E-05	0.003108	1.77E-05
0.04	0.003053	0.003056	1.11E-05	0.003064	2.22E-05	0.00305	1.74E-05
0.06	0.002996	0.002999	1.06E-05	0.003008	2.18E-05	0.002994	1.71E-05

r_{t_0}	V_{FD}^0	V_{LSM}^0	V_{DE}^0	V_{PE}^0	V_{FD}^1	V_{LSM}^1	V_{DE}^1	V_{PE}^1	V_{FD}^2	V_{LSM}^2	V_{DE}^2	V_{PE}^2
0.02	0.00178	0.001784	0.001779	0.001782	0.002553	0.002557	0.002555	0.002549	0.00311	0.003115	0.003122	0.003108
0.04	0.001749	0.001754	0.001749	0.001753	0.002508	0.002513	0.00251	0.002504	0.003053	0.003056	0.003064	0.00305
0.06	0.00172	0.001725	0.00172	0.001723	0.002464	0.002468	0.002465	0.00246	0.002996	0.002999	0.003008	0.002994

Table 3.5: *budget option* valuation varying initial instantaneous interest rates r_{t_0} (the *budget offer* specified in Table 3.1 with $\kappa = 0.5$, $\theta = 0.02$, $\sigma = 0.02$, $m_0 = 0.02$, 1-year mortgage term).

term in years	V_{FD}	V_{LSM}	$s.e.(LSM)$	V_{DE}	$s.e.(DE)$	V_{PE}	$s.e.(PE)$
2	0.004822	0.004832	1.77E-05	0.004841	3.47E-05	0.004819	2.75E-05
5	0.006818	0.006833	2.44E-05	0.006846	4.87E-05	0.006815	4.02E-05
10	0.007333	0.007348	2.75E-05	0.007363	5.23E-05	0.007329	4.30E-05

term in years	V_{FD}^0	V_{LSM}^0	V_{DE}^0	V_{PE}^0	V_{FD}^1	V_{LSM}^1	V_{DE}^1	V_{PE}^1	V_{FD}^2	V_{LSM}^2	V_{DE}^2	V_{PE}^2
2	0.002804	0.002811	0.002803	0.002808	0.003976	0.003985	0.00398	0.003973	0.004822	0.004832	0.004841	0.004819
5	0.003997	0.004008	0.003996	0.004004	0.005636	0.005648	0.005643	0.005633	0.006818	0.006833	0.006846	0.006815
10	0.004304	0.004317	0.004303	0.004312	0.006064	0.006077	0.006071	0.006062	0.007333	0.007348	0.007363	0.007329

Table 3.6: *budget option* valuation varying mortgage term T (the *budget offer* specified in Table 3.1 with $\kappa = 0.5$, $\theta = 0.02$, $\sigma = 0.02$, $m_0 = 0.02$, $r_{t_0} = 0.02$).

σ	θ	V_{FD}	V_{LSM}	$s.e.(LSM)$	V_{DE}	$s.e.(DE)$	V_{PE}	$s.e.(PE)$
0.02	0.02	0.00311	0.003115	1.13E-05	0.003122	2.26E-05	0.003108	1.77E-05
0.05	0.1	0.007583	0.007608	2.59E-05	0.007616	5.48E-05	0.00758	4.29E-05
0.2	0.5	0.026883	0.027227	8.80E-05	0.027049	0.000189	0.026925	0.000155

σ	θ	V_{FD}^0	V_{LSM}^0	V_{DE}^0	V_{PE}^0	V_{FD}^1	V_{LSM}^1	V_{DE}^1	V_{PE}^1	V_{FD}^2	V_{LSM}^2	V_{DE}^2	V_{PE}^2
0.02	0.02	0.00178	0.001784	0.001779	0.001782	0.002553	0.002557	0.002555	0.002549	0.00311	0.003115	0.003122	0.003108
0.05	0.1	0.004361	0.004375	0.004363	0.004371	0.006243	0.006259	0.006249	0.006236	0.007583	0.007608	0.007616	0.00758
0.2	0.5	0.015904	0.015952	0.01591	0.015941	0.022472	0.022585	0.022501	0.022457	0.026883	0.027227	0.027049	0.026925

Table 3.7: *budget option* valuation varying σ (the *budget offer* specified in Table 3.1 with $\kappa = 0.5$, $m_0 = 0.02$, $r_{t_0} = 0.02$).

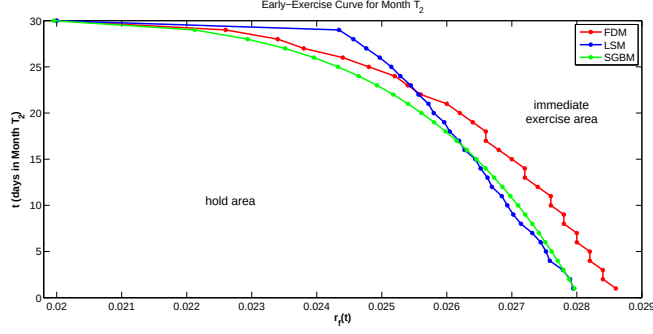


Figure 3.5: month T_2 early-exercise curve derived from FDM, LSM, SGBM (the *budget offer* specified in Table 3.1 with $\kappa = 0.5$, $\theta = 0.02$, $\sigma = 0.02$, $r_{t_0} = 0.02$, $m_0 = 0.02$, 1-year mortgage term, 50,000 simulated paths and 20 bundles per time step)

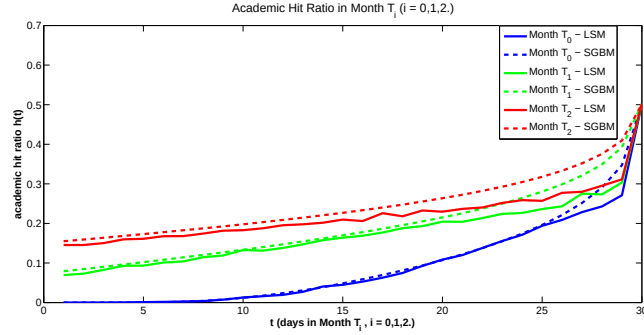


Figure 3.6: academic *hit ratio* curve obtained by LSM, SGBM (the *budget offer* specified in Table 3.1 with $\kappa = 0.5$, $\theta = 0.02$, $\sigma = 0.02$, $r_{t_0} = 0.02$, $m_0 = 0.02$, 1-year mortgage term, 50,000 simulated paths and 20 bundles per time step)

Definition 12 (Academic *hit ratio*). *Academic hit ratio $h(t)$ is defined as the probability of a budget option holder to optimally exercise the budget option at time t within the offer period, which equals to the probability of the prevailing instantaneous interest rate $r(t)$ to be greater than or equal to the contact point $r_f(t)$, i.e.,*

$$h(t) = P(r(t) \geq r_f(t)). \quad (3.50)$$

Figure 3.6 presents the academic *hit ratio* obtained by LSM and SGBM for a *budget option* in each month during the offer period. It can be observed that the academic *hit ratio* curve of month T_2 is higher than the one of month T_1 which is higher than the one of month T_0 . The reason can be derived from the early-exercise curves. For the days with the same order in each exercisable month of the budget offer, say $t_1 < t_2 < t_3$, although the contact points are almost the same, by Equation (3.50) the relatively high offer duration results in a relatively high academic *hit ratio*, i.e., $h(t_1) < h(t_2) < h(t_3)$.

From the numerical results of the *budget option* valuation, it can be concluded that

the *budget option* risk cannot be neglected, especially for mortgage offers of a long mortgage term or of a high interest rate volatility in the offer period.

3.6. CONVERGENCE STUDY

In this section, we are going to check the convergence performances of the applied methods in the *budget option* valuation problem by decreasing grid sizes, increasing the number of simulated paths, and increasing the number of the bundles relatively.

To view the convergence performance of FDM in the *budget option* valuation, we plot the absolute value of its numerical solution error against decreasing the grid sizes ($\Delta r, \Delta \tau$) in a factor of 2 (see Figure 3.7). The absolute value of the error of a numerical solution V_0 is defined as

$$err = |V_0 - V_{ref}|,$$

where V_{ref} is the reference value of V_0 . The path estimator obtained by SGBM of 2^{16} simulated paths in the direct estimator process and 2^{17} in the path estimator process, is used as the reference value in Figure 3.7 which shows that the *budget option* value obtained by FDM converges to the reference one as the grid sizes decrease.

Figure 3.8 shows that for both LSM and SGBM in the *budget option* valuation, as the number of the simulated paths increases, the estimators converge, and the standard error of each estimator decreases.

Figure 3.9 shows that with a relatively large number of the simulated paths in SGBM, as the number of the applied bundles increases, the estimators converge.

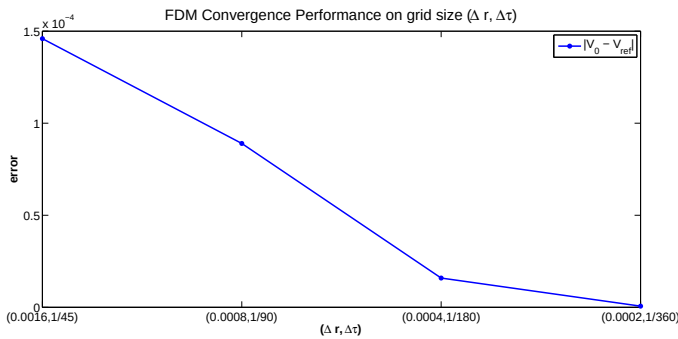


Figure 3.7: FDM convergence performance with respect to the grid size (the *budget offer* specified in Table 3.1 with $\kappa = 0.5$, $\theta = 0.02$, $\sigma = 0.02$, $r_{t_0} = 0.02$, $m_0 = 0.02$, 1-year mortgage term)

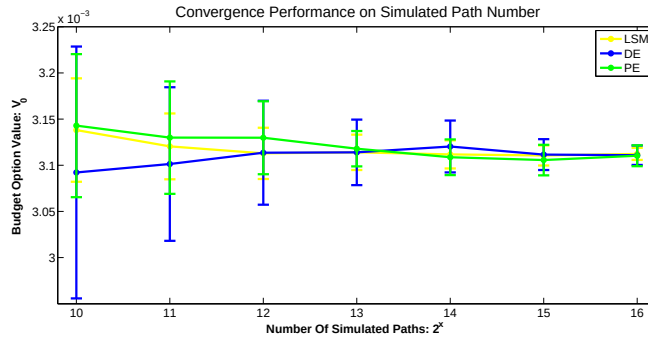


Figure 3.8: LSM and SGBM convergence performance regarding the number of MC paths (the *budget offer* specified in Table 3.1 with $\kappa = 0.5$, $\theta = 0.02$, $\sigma = 0.02$, $r_{t_0} = 0.02$, $m_0 = 0.02$, 1-year mortgage term)

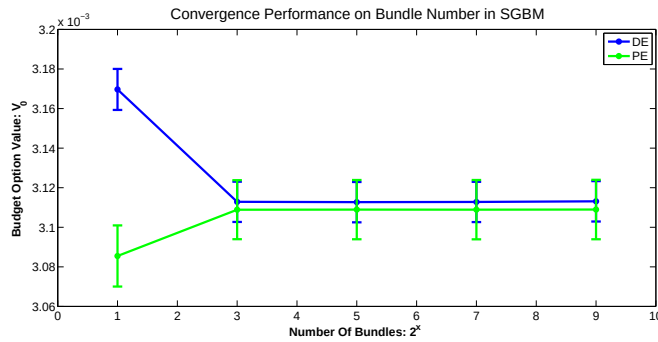


Figure 3.9: SGBM convergence performance regarding the number of bundles (the *budget offer* specified in Table 3.1 with $\kappa = 0.5$, $\theta = 0.02$, $\sigma = 0.02$, $r_{t_0} = 0.02$, $m_0 = 0.02$, 1-year mortgage term)

3.7. ONE-DIMENSIONAL JUMP DIFFUSION PROCESS IN THE BUDGET OPTION VALUATION

Rapid changes in the instantaneous interest rate process can be modelled as jumps. In this section, we model the instantaneous interest rate process as a jump diffusion one, and use LSM and SGBM to value the *budget offer* price based on a jump-diffusion interest rate model.

The time instances for which a jump arrives are denoted as $0 < \tau_1 < \tau_2 < \dots$. The number of jumps is supposed to be counted by the counting variable J_t , where

$$\tau_j = \inf\{t \geq 0, J_t = j\}. \quad (3.51)$$

In order to reduce computational costs of calculating $PM(m_0, t, r_t)$ at each time step for each simulated path, we choose a jump-diffusion model of the analytical solution of zero coupon bond price $P(0, T)$ in this thesis.

According to [18], a jump-extended Vasicek model under the risk neutral measure is given as follows.

$$dr_t = \kappa(\theta - r_t)dt + \sigma dW_t + q_t^u dJ_t^u - q_t^d dJ_t^d, \quad (3.52)$$

where the up-jump variable q_t^u and the down-jump variable q_t^d are exponentially distributed with positive means $\frac{1}{\eta^u}$ and $\frac{1}{\eta^d}$, and up-jump arrival number J_t^u and down-jump arrival number are distributed independently with intensities λ^u and λ^d , i.e.,

$$q_t^u \sim \exp\{\eta^u\}, \quad q_t^d \sim \exp\{\eta^d\}, \quad J_t^u \sim \text{Poisson}\{\lambda^u t\}, \quad J_t^d \sim \text{Poisson}\{\lambda^d t\}.$$

The zero coupon bond price is given by

$$\begin{aligned} P(t, T) &= \exp\{A(\tau) - B(\tau)(r_t - \theta) - H(t, T)\}, \\ \tau &= T - t, \\ H(t, T) &= \int_t^T \theta du = \theta\tau, \\ A(\tau) &= (\tau - B(\tau)) \frac{\sigma^2}{2\kappa^2} - \frac{\sigma^2 \cdot B^2(\tau)}{4\kappa} - \left(\lambda^u + \lambda^d\right)\tau \\ &\quad + \frac{\lambda^u \eta^u}{\kappa \eta^u + 1} \ln \left| \left(1 + \frac{1}{\kappa \eta^u}\right) e^{\kappa\tau} - \frac{1}{\kappa \eta^u} \right| \\ &\quad + \frac{\lambda^d \eta^d}{\kappa \eta^d - 1} \ln \left| \left(1 - \frac{1}{\kappa \eta^d}\right) e^{\kappa\tau} + \frac{1}{\kappa \eta^d} \right|, \\ B(\tau) &= \frac{1 - e^{-\kappa\tau}}{\kappa}. \end{aligned} \quad (3.53)$$

Hence, we are going to price the *budget option* based on the jump-extended Vasicek Model in Equation (3.52). After knowing the analytical solution of zero coupon bond prices, we need to get the conditional expectation of the basis functions in SGBM, i.e., $\mathbb{E}^Q[r_{t+1}^i | r_t]$, $i = 0, 1, 2, 3$. By the independence between the variables in Equation (3.52)

and the moments of a normal distribution, an exponential distribution and a Poisson distribution, we get

$$\begin{aligned}
 \mathbb{E}^Q[1|r_t] &= 1; \\
 \mathbb{E}^Q[r_{t+1}|r_t] &= r_t + \kappa(\theta - r_t)dt + \frac{1}{\eta^u}\lambda^u dt - \frac{1}{\eta^d}\lambda^d dt; \\
 \mathbb{E}^Q[r_{t+1}^2|r_t] &= \left(r_t + \kappa(\theta - r_t)dt\right)^2 + 2\left(r_t + \kappa(\theta - r_t)dt\right)\left(\frac{1}{\eta^u}\lambda^u dt - \frac{1}{\eta^d}\lambda^d dt\right) \\
 &\quad + \sigma^2 dt + \frac{2}{(\eta^u)^2}\left((\lambda^u dt)^2 + \lambda^u dt\right) + \frac{2}{(\eta^d)^2}\left((\lambda^d dt)^2 + \lambda^d dt\right) - 2\frac{1}{\eta^u}\lambda^u dt \cdot \frac{1}{\eta^d}\lambda^d dt; \\
 \mathbb{E}^Q[r_{t+1}^3|r_t] &= \left(r_t + \kappa(\theta - r_t)dt\right)^3 + 3\left(r_t + \kappa(\theta - r_t)dt\right)^2\left(\frac{1}{\eta^u}\lambda^u dt - \frac{1}{\eta^d}\lambda^d dt\right) \\
 &\quad + 3\left(r_t + \kappa(\theta - r_t)dt\right)\left(\sigma^2 dt + \frac{2}{(\eta^u)^2}\left((\lambda^u dt)^2 + \lambda^u dt\right) + \frac{2}{(\eta^d)^2}\left((\lambda^d dt)^2 + \lambda^d dt\right) \right. \\
 &\quad \left. - 2\frac{1}{\eta^u}\lambda^u dt \cdot \frac{1}{\eta^d}\lambda^d dt\right) + 3\sigma^2 dt\left(\frac{1}{\eta^u}\lambda^u dt - \frac{1}{\eta^d}\lambda^d dt\right) \\
 &\quad + \frac{6}{(\eta^u)^3}\left((\lambda^u dt)^3 + 3(\lambda^u dt)^2 + \lambda^u dt\right) - \frac{6}{(\eta^d)^3}\left((\lambda^d dt)^3 + 3(\lambda^d dt)^2 + \lambda^d dt\right) \\
 &\quad - 3\frac{2}{(\eta^u)^2}\left((\lambda^u dt)^2 + \lambda^u dt\right) \cdot \frac{1}{\eta^d}\lambda^d dt \\
 &\quad + 3\frac{1}{\eta^u}\lambda^u dt \frac{2}{(\eta^d)^2}\left((\lambda^d dt)^2 + \lambda^d dt\right).
 \end{aligned}$$

Some numerical test results based on the interest rate model (3.52) are presented below.

Table 3.8 presents the budget option values V_0 under the jump-extended Vasicek model with upward (or/and downward) jumps and without upward (or downward) jumps. It shows that the budget option values under the jump-extended Vasicek model with non-zero jumps are higher than the ones without jumps.

Table 3.9 presents the budget option values V_0 against different values of the expected upward jump number λ^u in the jump-extended Vasicek model. λ^u implies the probability of an upward interest rate jump occurring at time t . The results in Table 3.9 show that a higher expected number of upward interest rate jumps in the offer period is, a higher value of the *budget option* will be. The same conclusion can be derived for the expected number of downward interest rate jumps in the offer period.

Table 3.10 presents the budget option values V_0 against different values of the upward jump size η^u in the jump-extended Vasicek model. η^u implies the jump size of an upward interest rate jump in the offer period. The results in Table 3.10 show that a higher expected jump size of an upward interest rate jump in the offer period is, a higher value of the *budget option* will be. The same conclusion can be derived for the expected jump

size of a downward interest rate jump in the offer period.

	λ^u	λ^d	$\frac{1}{\eta^u}$	$\frac{1}{\eta^d}$	V_{LSM}	$s.e.(LSM)$	V_{DE}	$s.e.(DE)$	V_{PE}	$s.e.(PE)$		
	0	0	0	0	0.003128	2.52E-05	0.003122	2.26E-05	0.003117	2.23E-05		
	3	0	10^{-3}	0	0.003147	2.62E-05	0.003141	2.21E-05	0.003134	2.24E-05		
	0	3	0	10^{-3}	0.003155	2.39E-05	0.003151	2.25E-05	0.003145	2.28E-05		
	3	3	10^{-3}	10^{-3}	0.003173	2.43E-05	0.003169	2.18E-05	0.003161	2.24E-05		
λ^u	λ^d	$\frac{1}{\eta^u}$	$\frac{1}{\eta^d}$	V_{LSM}^0	V_{LSM}^1	V_{LSM}^2	V_{DE}^0	V_{DE}^1	V_{DE}^2	V_{PE}^0	V_{PE}^1	V_{PE}^2
0	0	0	0	0.001785	0.002561	0.003128	0.001779	0.002555	0.003122	0.001778	0.00255	0.003117
3	0	10^{-3}	0	0.001796	0.002576	0.003147	0.00179	0.00257	0.003141	0.001787	0.002565	0.003134
0	3	0	10^{-3}	0.0018	0.002584	0.003155	0.001794	0.002577	0.003151	0.001794	0.002574	0.003145
3	3	10^{-3}	10^{-3}	0.00181	0.002597	0.003173	0.001805	0.002593	0.003169	0.001803	0.002588	0.003161

Table 3.8: under Model (3.52), *budget option* valuation results(the *budget offer* specified in Table 3.1 with $\kappa = 0.5$, $\theta = 0.02$, $\sigma = 0.02$, $r_{t_0} = 0.02$, $m_0 = 0.02$, 1-year mortgage term)

		λ^u	λ^d	$\frac{1}{\eta^u}$	$\frac{1}{\eta^d}$	V_{LSM}	$s.e.(LSM)$	V_{DE}	$s.e.(DE)$	V_{PE}	$s.e.(PE)$			
		1	0	10^{-3}	0	0.003133	2.41E-05	0.003129	2.27E-05	0.003123	2.24E-05			
		5	0	10^{-3}	0	0.003158	2.51E-05	0.003153	2.20E-05	0.003144	2.15E-05			
		9	0	10^{-3}	0	0.003179	2.85E-05	0.003173	2.41E-05	0.003166	2.12E-05			
λ^u	λ^d			$\frac{1}{\eta^u}$	$\frac{1}{\eta^d}$	V_{LSM}^0	V_{LSM}^1	V_{LSM}^2	V_{DE}^0	V_{DE}^1	V_{DE}^2	V_{PE}^0	V_{PE}^1	V_{PE}^2
1	0	10^{-3}		0	0.001788	0.002566	0.003133	0.001783	0.00256	0.003129	0.001781	0.002555	0.003123	
5	0	10^{-3}		0	0.001801	0.002587	0.003158	0.001797	0.00258	0.003153	0.001793	0.002576	0.003144	
9	0	10^{-3}		0	0.001815	0.002606	0.003179	0.001812	0.002599	0.003173	0.001807	0.002595	0.003166	

Table 3.9: under Model (3.52), the *budget option* valuation varying λ^u (the *budget offer* specified in Table 3.1 with $\kappa = 0.5$, $\theta = 0.02$, $\sigma = 0.02$, $r_{t_0} = 0.02$, $m_0 = 0.02$, 1-year mortgage term)

λ^u		λ^d	$\frac{1}{\eta^u}$	$\frac{1}{\eta^d}$	V_{LSM}	$s.e.(LSM)$	V_{DE}	$s.e.(DE)$	V_{PE}	$s.e.(PE)$
3	0	5×10^{-4}	0	0.00313	2.53E-05	0.003126	2.23E-05	0.00312	2.18E-05	
3	0	2×10^{-3}	0	0.003208	2.56E-05	0.003204	2.23E-05	0.003195	2.15E-05	
3	0	5×10^{-3}	0	0.003578	3.28E-05	0.003592	2.60E-05	0.003568	2.59E-05	

λ^u	λ^d	$\frac{1}{\eta^u}$	$\frac{1}{\eta^d}$	V_{LSM}^0	V_{LSM}^1	V_{LSM}^2	V_{DE}^0	V_{DE}^1	V_{DE}^2	V_{PE}^0	V_{PE}^1	V_{PE}^2
3	0	5×10^{-4}	0	0.002564	0.00313	0.001781	0.002558	0.003126	0.00178	0.002553	0.00312	
3	0	2×10^{-3}	0	0.002627	0.003208	0.001827	0.002623	0.003204	0.001821	0.002614	0.003195	
3	0	5×10^{-3}	0	0.002921	0.003578	0.002021	0.002932	0.003592	0.002002	0.002913	0.003568	

Table 3.10: under Model (3.52), the *budget option* valuation varying η^u (the *budget offer* specified in Table 3.1 with $\kappa = 0.5$, $\theta = 0.02$, $\sigma = 0.02$, $r_{t_0} = 0.02$, $m_0 = 0.02$, 1-year mortgage term)

3.8. TWO FACTOR INTEREST RATE MODEL IN THE BUDGET OPTION VALUATION

In this section, we extend the one-factor interest rate model in *budget option* valuation into a two-factor interest rate model. Under the setting of the two-factor interest rate model, we implement some tests of *budget option* valuation and present some numerical test results.

The weakness of one-factor interest rate models is the correlation feature at each time instant between rates for different maturities [19]. For instance, under the Vasicek model (3.5) with the formula of zero coupon bond prices (3.6), the continuous compounding interest rate for the period from t to T is given

$$\begin{aligned} R(t, T) &= -\frac{\ln(P(t, T))}{T-t} \\ &= -\frac{\ln A(t, T)}{T-t} + \frac{B(t, T)}{T-t} r_t \\ &=: a(t, T) + b(t, T) r_t. \end{aligned}$$

For any $T_1, T_2 > t$ and $T_1 \neq T_2$,

$$\begin{aligned} \text{Corr}(R(t, T_1), R(t, T_2)) &= \text{Corr}(a(t, T_1) + b(t, T_1)r_t, a(t, T_2) + b(t, T_2)r_t) \\ &= 1, \end{aligned}$$

which implies at each time instant a perfect-correlation between rates for different maturities. Such feature cannot adapt to the real financial market where the interest rates are known to behave differently.

Based on the principal component analysis (or factor analysis) in [20], we know that historical analysis of the whole yield curve usually suggests that two components can explain 85% to 90% of variations in the yield curve under the real-world measure. The more factors involved in the interest rate model, usually the less numerically-efficient the implementation will be. In the consideration of numerically-efficiency and capability of the model to represent realistic correlation patterns for *budget option* valuation, a two-factor interest rate model is chosen for the *budget option* valuation in this section.

Based on the Vasicek model in [8], a hypothetical two-factor interest rate model [19] under the risk-neutral measure $(\Omega, \mathcal{F}_t, Q)$ is defined by the dynamics

$$\begin{cases} r_t = x_t + y_t \\ dx_t = \kappa_x(\theta_x - x_t)dt + \sigma_x dW_1(t), \\ dy_t = \kappa_y(\theta_y - y_t)dt + \sigma_y dW_2(t), \\ dW_1(t) \cdot dW_2(t) = \rho dt, \end{cases} \quad (3.54)$$

with $\kappa_x > 0, \kappa_y > 0, \sigma_x > 0, \sigma > 0$ and $-1 \leq \rho \leq 1$.

Under the two-factor interest rate model in (3.54), we get

$$\begin{aligned} r_t &= x_0 \cdot e^{-\kappa_x t} + \theta_x (1 - e^{-\kappa_x t}) + \sigma_x e^{-\kappa_x t} \int_0^t e^{\kappa_x s} dW_1(s) \\ &\quad + y_0 \cdot e^{-\kappa_y t} + \theta_y (1 - e^{-\kappa_y t}) + \sigma_y e^{-\kappa_y t} \int_0^t e^{\kappa_y s} dW_2(s), \end{aligned} \quad (3.55)$$

and

$$\begin{aligned}
\int_0^T r_t dt &= \int_0^T \left(x_0 e^{-\kappa_x t} + \theta_x (1 - e^{-\kappa_x t}) + y_0 e^{-\kappa_y t} + \theta_y (1 - e^{-\kappa_y t}) \right) dt \\
&+ \int_0^T \sigma_x e^{-\kappa_x t} \int_0^t e^{\kappa_x s} dW_1(s) dt + \int_0^T \sigma_y e^{-\kappa_y t} \int_0^t e^{\kappa_y s} dW_2(s) dt \\
&= \frac{1 - e^{-\kappa_x T}}{\kappa_x} (x_0 - \theta_x) + \theta_x T + \frac{1 - e^{-\kappa_y T}}{\kappa_y} (y_0 - \theta_y) + \theta_y T \\
&+ \int_0^T \frac{\sigma_x (e^{\kappa_x (s-T)} - 1)}{-\kappa_x} dW_1(s) + \int_0^T \frac{\sigma_y (e^{\kappa_y (s-T)} - 1)}{-\kappa_y} dW_2(s).
\end{aligned} \tag{3.56}$$

If we denote

$$\begin{aligned}
\widehat{W}(T) &:= \int_0^T \frac{\sigma_x (e^{\kappa_x (s-T)} - 1)}{-\kappa_x} dW_1(s) + \int_0^T \frac{\sigma_y (e^{\kappa_y (s-T)} - 1)}{-\kappa_y} dW_2(s); \\
\widehat{W}_1(T) &:= \int_0^T \frac{\sigma_x (e^{\kappa_x (s-T)} - 1)}{-\kappa_x} dW_1(s); \\
\widehat{W}_2(T) &:= \int_0^T \frac{\sigma_y (e^{\kappa_y (s-T)} - 1)}{-\kappa_y} dW_2(s).
\end{aligned}$$

Then we know that the distribution of $\widehat{W}$ is normally distributed, since the sum of normally distributed variables is a normal distribution. The mean and variance of $\widehat{W}(T)$ are given by

$$\mathbb{E}^Q [\widehat{W}(T)] = \mathbb{E}^Q [\widehat{W}_1(T)] + \mathbb{E}^Q [\widehat{W}_2(T)] = 0, \tag{3.57}$$

and

$$\begin{aligned}
\text{Var} [\widehat{W}(T)] &= \text{Var} (\widehat{W}_1(T)) + \text{Var} (\widehat{W}_2(T)) + \text{Cor} (\widehat{W}_1(T), \widehat{W}_2(T)) \\
&= -\frac{\sigma_x^2}{2\kappa_x} \left(\frac{1 - e^{-\kappa_x T}}{\kappa_x} \right)^2 - \frac{\sigma_x^2}{\kappa_x^2} \frac{(1 - e^{-\kappa_x T})}{\kappa_x} + \frac{\sigma_x^2}{\kappa_x^2} T \\
&- \frac{\sigma_y^2}{2\kappa_y} \left(\frac{1 - e^{-\kappa_y T}}{\kappa_y} \right)^2 - \frac{\sigma_y^2}{\kappa_y^2} \frac{(1 - e^{-\kappa_y T})}{\kappa_y} + \frac{\sigma_y^2}{\kappa_y^2} T \\
&+ \int_0^T \frac{\sigma_x (e^{\kappa_x (s-T)} - 1)}{-\kappa_x} \cdot \frac{\sigma_y (e^{\kappa_y (s-T)} - 1)}{-\kappa_y} \cdot \rho ds \\
&= -\frac{\sigma_x^2}{2\kappa_x} \left(\frac{1 - e^{-\kappa_x T}}{\kappa_x} \right)^2 - \frac{\sigma_x^2}{\kappa_x^2} \frac{(1 - e^{-\kappa_x T})}{\kappa_x} + \frac{\sigma_x^2}{\kappa_x^2} T \\
&- \frac{\sigma_y^2}{2\kappa_y} \left(\frac{1 - e^{-\kappa_y T}}{\kappa_y} \right)^2 - \frac{\sigma_y^2}{\kappa_y^2} \frac{(1 - e^{-\kappa_y T})}{\kappa_y} + \frac{\sigma_y^2}{\kappa_y^2} T \\
&+ \frac{\sigma_x \sigma_y \rho}{\kappa_x \kappa_y} \left(\frac{1 - e^{-(\kappa_x + \kappa_y) T}}{\kappa_x + \kappa_y} - \frac{1 - e^{-\kappa_x T}}{\kappa_x} - \frac{1 - e^{-\kappa_y T}}{\kappa_y} + T \right).
\end{aligned} \tag{3.58}$$

Hence, we know that

$$\begin{aligned}
 \int_0^T r_t dt &\sim \mathcal{N}(\mu, \sigma^2) \\
 \mu &:= \frac{1 - e^{-\kappa_x T}}{\kappa_x} (x_0 - \theta_x) + \theta_x T + \frac{1 - e^{-\kappa_y T}}{\kappa_y} (y_0 - \theta_y) + \theta_y T \\
 \sigma^2 &:= -\frac{\sigma_x^2}{2\kappa_x} \left(\frac{1 - e^{-\kappa_x T}}{\kappa_x} \right)^2 - \frac{\sigma_x^2}{\kappa_x^2} \frac{(1 - e^{-\kappa_x T})}{\kappa_x} + \frac{\sigma_x^2}{\kappa_x^2} T \\
 &\quad - \frac{\sigma_y^2}{2\kappa_y} \left(\frac{1 - e^{-\kappa_y T}}{\kappa_y} \right)^2 - \frac{\sigma_y^2}{\kappa_y^2} \frac{(1 - e^{-\kappa_y T})}{\kappa_y} + \frac{\sigma_y^2}{\kappa_y^2} T \\
 &\quad + \frac{\sigma_x \sigma_y \rho}{\kappa_x \kappa_y} \left(\frac{1 - e^{-(\kappa_x + \kappa_y)T}}{\kappa_x + \kappa_y} - \frac{1 - e^{-\kappa_x T}}{\kappa_x} - \frac{1 - e^{-\kappa_y T}}{\kappa_y} + T \right).
 \end{aligned} \tag{3.59}$$

By the moment generating functions of the normal distribution, we get the analytical solution of the zero coupon bond price $P(0, T)$, i.e.,

$$\begin{aligned}
 P(0, T) &= \mathbb{E}^Q \left[\exp \left\{ - \int_0^T r_t dt \right\} \right] \\
 &= e^{-\mu + \frac{\sigma^2}{2}}.
 \end{aligned} \tag{3.60}$$

With the bond price formula under the two-factor interest rate model, the next thing to be considered is to simulate r_t in LSM and SGBM. To facilitate simulating dW_t^i ($i = 1, 2$), we apply the Cholesky decomposition technique to defining the equivalent combinations of independent Wiener processes to the correlated ones. For dW_t^i ($i = 1, 2$),

$$\begin{aligned}
 \begin{pmatrix} dW_1(t) \\ dW_2(t) \end{pmatrix} \begin{pmatrix} dW_1(t) \\ dW_2(t) \end{pmatrix}^{\text{tr}} &= \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix} \cdot dt \\
 &= \begin{pmatrix} 1 & 0 \\ \rho & \sqrt{1 - \rho^2} \end{pmatrix} \begin{pmatrix} d\widetilde{W}_1(t) \\ d\widetilde{W}_2(t) \end{pmatrix} \left(\begin{pmatrix} 1 & 0 \\ \rho & \sqrt{1 - \rho^2} \end{pmatrix} \begin{pmatrix} d\widetilde{W}_1(t) \\ d\widetilde{W}_2(t) \end{pmatrix} \right)^{\text{tr}},
 \end{aligned} \tag{3.61}$$

where $\widetilde{W}_1(t)$ and $\widetilde{W}_2(t)$ are independent Wiener processes. Hence, we define that

$$\begin{pmatrix} dW_1(t) \\ dW_2(t) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \rho & \sqrt{1 - \rho^2} \end{pmatrix} \begin{pmatrix} d\widetilde{W}_1(t) \\ d\widetilde{W}_2(t) \end{pmatrix} = \begin{pmatrix} d\widetilde{W}_1(t) \\ \rho d\widetilde{W}_1(t) + \sqrt{1 - \rho^2} d\widetilde{W}_2(t) \end{pmatrix}. \tag{3.62}$$

Therefore, the two-factor interest rate model in (3.54) is equivalently transformed into

$$\left\{ \begin{aligned} r_t &= x_t + y_t \\ dx_t &= \kappa_x (\theta_x - x_t) dt + \sigma_x d\widetilde{W}_1(t) \\ dy_t &= \kappa_y (\theta_y - y_t) dt + \sigma_y \left(\rho d\widetilde{W}_1(t) + \sqrt{1 - \rho^2} d\widetilde{W}_2(t) \right) \\ d\widetilde{W}_1(t) \cdot d\widetilde{W}_2(t) &= 0 \end{aligned} \right. \tag{3.63}$$

with $\kappa_x > 0$, $\kappa_y > 0$, $\sigma_x > 0$, $\sigma_y > 0$, $-1 \leq \rho \leq 1$, and independent Wiener processes $\widetilde{W}_1$ and $\widetilde{W}_2$.

The basis functions applied in LSM and SGBM are given by

$$1, x_t, x_t^2, y_t, y_t^2, x_t y_t.$$

The conditional expectations of these basis functions of x_{t+1}, y_{t+1} on x_t and y_t are given by

$$\begin{aligned}\mathbb{E}^Q [x_{t+1}|x_t] &= x_t + \kappa_x (\theta_x - x_t) dt, \\ \mathbb{E}^Q [y_{t+1}|y_t] &= y_t + \kappa_y (\theta_y - y_t) dt, \\ \mathbb{E}^Q [x_{t+1}^2|x_t] &= (x_t + \kappa_x (\theta_x - x_t) dt)^2 + \sigma_x^2 dt, \\ \mathbb{E}^Q [y_{t+1}^2|y_t] &= (y_t + \kappa_y (\theta_y - y_t) dt)^2 + \sigma_y^2 dt, \\ \mathbb{E}^Q [x_{t+1} \cdot y_{t+1}|x_t, y_t] &= (x_t + \kappa_x (\theta_x - x_t) dt)(y_t + \kappa_y (\theta_y - y_t) dt) + \sigma_x \sigma_y \rho dt.\end{aligned}$$

Under the two-factor interest rate model, the applied bundling reference at each time step in SGBM is the quantile of the sum of x_t^i and y_t^i , where $i \in \mathfrak{J} := \{1, 2, \dots, b\}$ means the i -th simulated path among the b simulated paths.

We implement the two-factor interest rate model in the *budget option* valuation based on the above derivation. One of the numerical test results is presented in Table 3.11. Table 3.11 shows that a higher value of the correlation coefficient ρ in the two-factor interest rate model (3.54) is, a higher value of the *budget option* will be.

Figure 3.10 presents the month T_2 early-exercise curves under different interest rate models for the *budget offer* specified in Table 3.1 with $m_0 = 0.02$, 1-year mortgage term. Figure 3.11 presents the month T_2 academic hit ratio curves under different interest rate models for the *budget offer* specified in Table 3.1 with $m_0 = 0.02$, 1-year mortgage term. The parameter setting applied in different interest rate models in Figure 3.10 and in Figure 3.11 is given in Table 3.12. Figure 3.11 shows that the month T_2 academic hit ratio curves under different interest rate models are almost the same, although the month T_2 early-exercise curves are different, which implies that with comparable parameter setting, a different interest rate model does not change the academic hit ratio curve. Although the interest rate models change the month T_2 early-exercise curve in 3.10, but the trends of the early-exercise curves are the same, i.e., contact point $r_f(t)$ goes down as the days increase in month T_2 . The same conclusion can be derived for the month T_i ($i = 0, 1, 2$) early-exercise curve and the month T_i ($i = 0, 1, 2$) academic hit ratio curve.

ρ	V_{LSM}	$s.e.(LSM)$	V_{DE}	$s.e.(DE)$	V_{PE}	$s.e.(PE)$
0.8	0.002951	2.49E-05	0.002959	2.12E-05	0.002945	1.10E-05
0.5	0.002693	2.30E-05	0.0027	1.89E-05	0.00269	1.02E-05
0	0.002195	1.93E-05	0.002204	1.47E-05	0.002197	8.32E-06
-0.5	0.001552	1.11E-05	0.001559	9.72E-06	0.001555	6.35E-06
-0.8	0.000983	7.87E-06	0.000987	5.95E-06	0.000985	3.94E-06

ρ	V_{LSM}^0	V_{LSM}^1	V_{LSM}^2	V_{DE}^0	V_{DE}^1	V_{DE}^2	V_{PE}^0	V_{PE}^1	V_{PE}^2
0.8	0.00169	0.002426	0.002951	0.001687	0.002422	0.002959	0.001687	0.002424	0.002945
0.5	0.001542	0.002216	0.002693	0.00154	0.002211	0.0027	0.00154	0.002212	0.00269
0	0.001259	0.001809	0.002195	0.001258	0.001806	0.002204	0.001257	0.001806	0.002197
-0.5	0.000891	0.001278	0.001552	0.00089	0.001278	0.001559	0.000891	0.001279	0.001555
-0.8	0.000565	0.000809	0.000983	0.000563	0.000809	0.000987	0.000564	0.00081	0.000985

Table 3.11: under the two-factor interest rate model (3.54), the *budget option* valuation varying ρ (the *budget offer* specified in Table 3.1 with $\kappa_x = 0.5$, $\kappa_y = 0.5$, $\theta_x = 0.01$, $\theta_y = 0.01$, $\sigma_x = 0.01$, $\sigma_y = 0.01$, $x_0 = 0.01$, $y_0 = 0.01$, $m_0 = 0.02$, 1-year mortgage term)

interest rate model	parameter setting								
Vasicek model	r_0	κ	θ	σ					
	0.02	0.5	0.02	0.02					
upward jump extended Vasicek model	r_0	κ	θ	σ	λ^u	λ^d	$\frac{1}{\eta^u}$	$\frac{1}{\eta^d}$	
	0.02	0.5	0.02	0.02	5	0	10^{-3}	0	
downward jump extended Vasicek model	r_0	κ	θ	σ	λ^u	λ^d	$\frac{1}{\eta^u}$	$\frac{1}{\eta^d}$	
	0.02	0.5	0.02	0.02	0	5	0	10^{-3}	
2-factor interest rate model (positive ρ)	x_0	y_0	κ_x	κ_y	θ_x	θ_y	σ_x	σ_y	ρ
	0.01	0.01	0.5	0.5	0.01	0.01	0.01	0.01	0.5
2-factor interest rate model (negative ρ)	x_0	y_0	κ_x	κ_y	θ_x	θ_y	σ_x	σ_y	ρ
	0.01	0.01	0.5	0.5	0.01	0.01	0.01	0.01	-0.5

Table 3.12: parameters used in different interest rate models in the *budget option* valuation tests

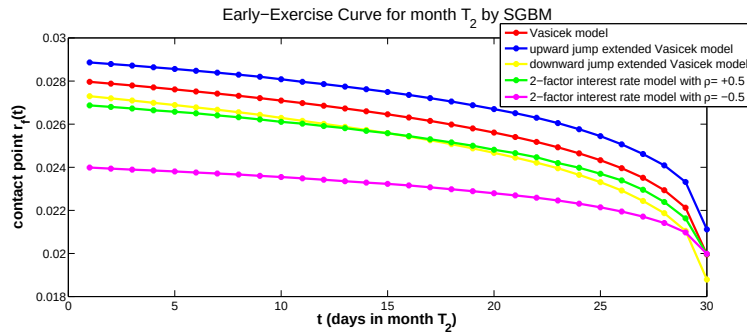


Figure 3.10: month T_2 early-exercise curves by SGBM under different interest rate models with the parameter setting in Table 3.12 (the *budget offer* specified in Table 3.1 with $m_0 = 0.02$, 1-year mortgage term)

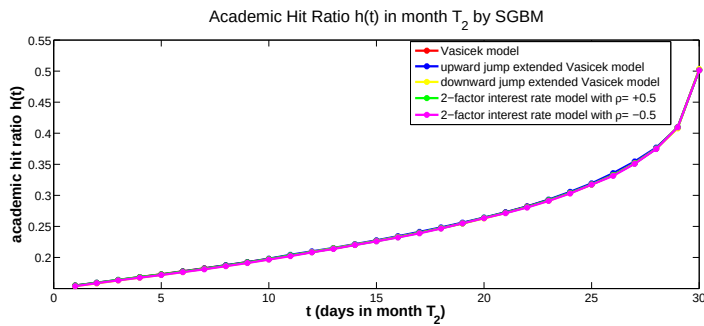


Figure 3.11: month T_2 academic *hit ratio* curves by SGBM under different applied interest rate models with the parameter setting in Table 3.12 (the *budget offer* specified in Table 3.1 with $m_0 = 0.02$, 1-year mortgage term)

4

CONCLUSION AND OUTLOOK

In this thesis, both the exogenous termination modelling and the endogenous termination modelling in the valuation of mortgage offer options have been performed. In the exogenous termination modelling, based on the loss of the hedging portfolio, we explain that the quantity to be estimated is the monthly *hit ratio*. The monthly *hit ratio* prediction is based on the historical data. From our tests and analysis in Chapter 2, we conclude that it is nontrivial to define an appropriate model in the *hit ratio* prediction. In the endogenous termination modelling, we focused on the *budget option* valuation where all the borrowers are assumed rational in exercising their offer options. With the assumption of the strike prices over a offer period, the payoff of option can be calculated. Due to the early exercise facility of offer options, the backward valuation methods with dynamic programming are chosen, namely the finite difference method (FDM), the least squares method (LSM), and the stochastic grid bundling method (SGBM). Despite the lack of benchmark in offer option values, the numerical results from one applied method assist to validate the others. All the numerical test results of the *budget option* valuation match well, which convinces us the validity of the applied methods. In conclusion, we can use the endogenous termination modelling to work on our research question.

Based on the work in this thesis, we present some outlook on the future research in the valuation of mortgage offer options as follows.

The assumption of the strike prices in the offer period can be considered to adapt to individual lender's situation; the interest rate model can be considered to be enriched to adapt individual lender's yield curve model; and the *regular option* valuation can be investigated based on the setting of the endogenous termination modelling.

In summary, it can be said that this thesis forms a foundation of the valuation of mortgage offer options.

APPENDIX



DATA

The data sheets used for the tests in Chapter 2 have been queried based on the data requirement made for this project. We required that the data sheets contain the historical transition information of the ABN AMRO (AAB) label mortgage offers from 01-Jan-2010 to 25-Jan-2016 along with the information of the mortgage contract variables. There are two data sheets prepared for the above data requirement, containing the information of the offers separately. One data sheet called “AanvraagDetails20160127” (AD), contains all the required offer information for the last offer in each mortgage application. The information of transition states and transition dates are not included in AD. The other data sheet called “AanvragenStatusTransities20160127” (AST), contains the transition information for all the offers in each mortgage application. The information of mortgage contract variables is not included in AST.

The problem raised that AD and AST can not be perfectly merged into a completed sheet. The main reason is that the AD data sheet only includes the last offer in each application, while the AST data sheet contains the transition information for all the offers in each application. In order to utilize these two queried data sheets, we assume the offers made in one application are in the same mortgage offer type. By this assumption, we can separate the AST data sheet into the AST data sheet for *budget offers* and the AST data sheet for *regular offers*. Due to the fact that the principal information for each offer in AST is still missing, all the *hit ratio* calculations in the tests presented in Chapter 2 are based on the offer number ratio instead of the principal ratio.

B

INTRODUCTION TO THE MORTGAGE CREDIT DIRECTIVE

On March 21 2016, the Mortgage Credit Directive (MCD) was implemented in Dutch legislation, requiring banks to meet the new rules, under which the mortgage offer process in Figure 1.6 has been changed in MCD adaptation. The changed offer process has not been considered in this thesis, due to the insufficient historical data. For the information of readers, MCD and the changed offer process are described in this appendix.

Learning the lesson from the financial crisis of 2007 to 2008, the European Commission launched a process of identifying and assessing the risks in the EU mortgage credit market. In the Dutch mortgage market, the main risks are high Loan-to-Value (LtV) ratio and Loan-to-Income (LtI) ratio, compared to other European countries. In the UK mortgage market, the risk is that UK mortgages are increasingly extended to high(er) LtI, while the LtV is not an immediate concern as it is much lower than the one in the Dutch mortgage market [21]. The root of all risks in the EU mortgage credit market can be summarized into the lax attitude of responsible lending. As recognizing the importance of responsible behaviors in the credit market, the Mortgage Credit Directive 2014/17/EU (MCD) was adopted on 4 February 2014. The aim of the Directive is to create a Union-wide, transparent and efficient mortgage credit market with a high level of consumer protection¹. In the sense of the high level of consumer protection, the Directive takes effort in preventing the consumers from over indebtedness and imposing transparent information in the market. As acknowledged, consumer protection is necessary in stabilizing the mortgage credit market, which in turn stabilizes the cash flow circulation in the mortgage credit market. It can further strengthen the investors' confidence in the mortgage credit market, which is beneficial to the mortgage lenders'² liquidity position. Such stabilizing and stimulating the money circulation in the mortgage credit market are

¹see, http://ec.europa.eu/finance/financeservices-retail/credit/mortgage/index_en.htm

²Generally, mortgage lenders are banks.

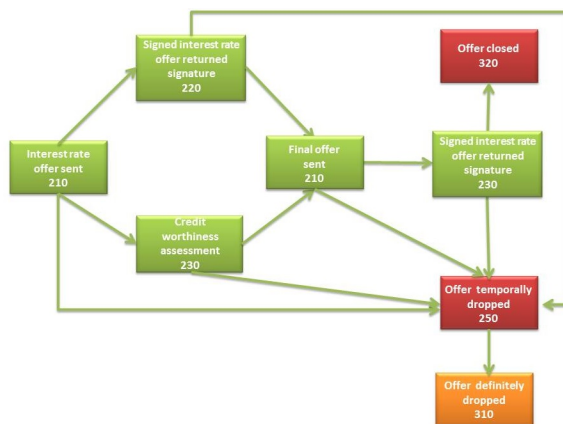


Figure B.1: The changed offer process under MCD

believed capable to bring a stabilized mortgage credit market.

The Directive lays down a common framework for sound underwriting standards and prudential supervisory requirements in the laws, regulations and administrative provisions of the Member States. “The main provisions include consumer information requirements, principle based rules and standards for the performance of services (e.g. conduct of business obligations, competence and knowledge requirements for staff), a consumer creditworthiness assessment obligation, provisions on early repayment, provisions on foreign currency loans, provisions on tying practices, some high-level principles (e.g. those covering financial education, property valuation and arrears and foreclosures) and a passport for credit intermediaries who meet the admission requirements in their home Member State. Member States will have to transpose its provisions into their national law by March 2016.” [22].

Under the requirements of MCD, creditors in the Dutch mortgage markets have launched a series of consistent changes for their mortgage products. In view of the changes in the mortgage offer process, ABN AMRO Hypotheken Groep BV (AAHG) has moved the state of creditworthiness assessment forward to the state of final offer release according to the articles in [22] (see Figure B.1 for the details of the changed mortgage offer process).

C

SPECIAL CASE: $\kappa = 0$ IN THE VASICEK MODEL

After validating the applied methods for the basic setting of the *budget option* valuation, a special case of the interest rate model $\kappa = 0$ in the Vasicek model (3.5) is tested in this section. Under the risk-neutral measure $(\Omega, \mathcal{F}, Q)$ with filtration $\{\mathcal{F}_t\}_{t \geq 0}$, setting $\kappa = 0$ in the Vasicek model (3.5) turns the interest rate model into Equation (C.1).

$$dr_t = \sigma dW_t \quad (\text{C.1})$$

So,

$$r_t = r_0 + \sigma \cdot W_t, \quad (\text{C.2})$$

where r_0 is the instantaneous interest rate at time $t = 0$.

Correspondingly, the zero coupon bond price $P(0, T)$ is

$$P(0, T) = \mathbb{E} \left[\exp \left\{ - \int_0^T r_t dt \right\} \middle| \mathcal{F}_0 \right] = \mathbb{E} \left[\exp \left\{ -r_0 \cdot T - \sigma \int_0^T W_t dt \right\} \middle| \mathcal{F}_0 \right]. \quad (\text{C.3})$$

By Ito's lemma, we get

$$d(W_t \cdot t) = W_t dt + t dW_t.$$

We know that

$$\int_0^T W_t dt = W_T \cdot T - \int_0^T t dW_t, \quad (\text{C.4})$$

which gives that

$$\int_0^T W_t dt \sim \mathcal{N} \left(0, T^2 - \frac{T^3}{3} \right).$$

Hence, by the moment generating functions of a normally distributed variable, Equation (C.3) can be further derived into

$$\begin{aligned} P(0, T) &= e^{-r_0 \cdot T} \mathbb{E} \left[\exp \left\{ -\sigma \int_0^T W_t dt \right\} \right] \\ &= \exp \left\{ -r_0 \cdot T + \frac{\sigma^2}{2} \left(T^2 + \frac{T^3}{3} \right) \right\} \end{aligned} \quad (C.5)$$

Therefore, we get the analytical solution of the zero coupon price $P(0, T)$ under the interest rate model (C.1).

The conditional probability of r_t given r_{t-1} is a normally distributed with mean

$$\hat{\mu}_{t-1} = r_{t-1}$$

and standard deviation

$$\hat{\sigma}_{t-1} = \sigma \cdot \sqrt{\Delta t},$$

i.e.,

$$(r_t | r_{t-1}) \sim \mathcal{N}(\hat{\mu}_{t-1}, \hat{\sigma}_{t-1}^2).$$

Hence, the conditional expectation of the basis functions in SGBM under the interest rate model (C.1) are given by

$$\begin{aligned} \mathbb{E}^Q [1 | r_{t-1}] &= 1, \\ \mathbb{E}^Q [r_t | r_{t-1}] &= \hat{\mu}_{t-1}, \\ \mathbb{E}^Q [r_t^2 | r_{t-1}] &= \hat{\mu}_{t-1}^2 + \hat{\sigma}_{t-1}^2, \\ \mathbb{E}^Q [r_t^3 | r_{t-1}] &= \hat{\mu}_{t-1}^3 + 3\hat{\mu}_{t-1} \cdot \hat{\sigma}_{t-1}^2. \end{aligned}$$

Some numerical test results based on the interest rate model (C.1) are presented below, where V_{FD}^e is the FDM numerical result of the value of the *budget option* which is European style and can only be exercised at the offer maturity date t_M , $\tilde{V}^e$ is the simulation numerical result of the value of the *budget option* which is European style and can only be exercised at the offer maturity date t_M , and $s.e.(\tilde{V}^e)$ is the standard deviation of $\tilde{V}^e$.

Table C.1 presents the budget option values V_0 against different mortgage terms of the *budget offer* under the interest rate model (C.1). Table C.2 presents the budget option values V_0 against different values of the volatility σ in the interest rate model (C.1). Table ?? presents the budget option values V_0 against different locked mortgage rate m_0 of the *budget offer* under the interest rate model (C.1). From the test results in the *budget offer* valuation under the interest rate model (C.1), we see that an increase in the mortgage term, or an increase in the volatility σ , results in a relatively significant increase in the *budget option* value.

term in years	V_{FD}	V_{LSM}	$s.e.(LSM)$	V_{DE}	$s.e.(DE)$	V_{PE}	$s.e.(PE)$	V_{FD}^e	$\tilde{V}^e$	$s.e.(\tilde{V}^e)$
1	0.004182	0.004187	1.48E-05	0.004197	3.06E-05	0.004177	2.39E-05	0.003972	0.003975	2.14E-05
5	0.019474	0.019507	6.48E-05	0.019542	0.000138	0.019451	0.000125	0.019318	0.01933	9.95E-05
10	0.039686	0.039763	0.000137	0.039821	0.000275	0.039641	0.00027	0.039551	0.039573	0.000192

term in years	V_{FD}^0	V_{LSM}^0	V_{DE}^0	V_{PE}^0	V_{FD}^1	V_{LSM}^1	V_{DE}^1	V_{PE}^1	V_{FD}^2	V_{LSM}^2	V_{DE}^2	V_{PE}^2
1	0.002306	0.002311	0.002304	0.002309	0.003364	0.003369	0.003365	0.003359	0.004182	0.004187	0.004197	0.004177
5	0.01102	0.011061	0.01102	0.011038	0.015798	0.015835	0.015809	0.015792	0.019474	0.019507	0.019542	0.019451
10	0.022458	0.022555	0.022476	0.022513	0.032198	0.03227	0.032217	0.032191	0.039686	0.039763	0.039821	0.039641

Table C.1: Under the interest rate model (C.1), *budget option* Valuation varying mortgage term T (the *budget offer* specified in Table 3.1 with $\theta = 0.02$, $\sigma = 0.02$, $r_{t_0} = 0.02$, $m_0 = 0.02$)

σ	θ	V_{FD}	V_{LSM}	$s.e.(LSM)$	V_{DE}	$s.e.(DE)$	V_{PE}	$s.e.(PE)$	V_{FD}^e	$\tilde{V}^e$	$s.e.(\tilde{V}^e)$
0.01	0.02	0.002079	0.002081	7.36E-06	0.002087	1.53E-05	0.002077	1.18E-05	0.001973	0.001975	1.08E-05
0.05	0.1	0.010642	0.010662	3.53E-05	0.010678	7.67E-05	0.010628	6.16E-05	0.010137	0.010144	5.25E-05
0.2	0.5	0.047711	0.04804	0.000137	0.047862	0.000317	0.047656	0.000265	0.045578	0.046027	0.000189

σ	θ	V_{FD}^0	V_{LSM}^0	V_{DE}^0	V_{PE}^0	V_{FD}^1	V_{LSM}^1	V_{DE}^1	V_{PE}^1	V_{FD}^2	V_{LSM}^2	V_{DE}^2	V_{PE}^2
0.01	0.02	0.001148	0.001152	0.001148	0.001151	0.001675	0.001677	0.001675	0.001672	0.002079	0.002081	0.002087	0.002077
0.05	0.1	0.005822	0.005836	0.00582	0.005832	0.00853	0.008544	0.008532	0.008518	0.010642	0.010662	0.010678	0.010628
0.2	0.5	0.025055	0.025121	0.025049	0.025101	0.037481	0.037595	0.037497	0.037442	0.047711	0.04804	0.047862	0.047656

Table C.2: Under the interest rate model (C.1), *budget option* Valuation varying σ (the *budget offer* specified in Table 3.1 with $r_{t_0} = 0.02$, $m_0 = 0.02$, 1-year mortgage term)

m_0	V_{FD}	V_{LSM}	$s.e.(LSM)$	V_{DE}	$s.e.(DE)$	V_{PE}	$s.e.(PE)$	V_{FD}^{eup}	$V_{simulation}^{eup}$	$s.e.(V_{simulation}^{eup})$		
0.015	0.00417	0.004174	1.49E-05	0.004185	3.05E-05	0.004165	2.39E-05	0.003961	0.003964	2.14E-05		
0.02	0.004182	0.004187	1.48E-05	0.004197	3.06E-05	0.004177	2.39E-05	0.003972	0.003975	2.14E-05		
0.025	0.004194	0.004199	1.47E-05	0.004208	3.07E-05	0.004189	2.40E-05	0.003983	0.003986	2.15E-05		
m_0	V_{FD}^0	V_{LSM}^0	V_{DE}^0	V_{PE}^0	V_{FD}^1	V_{LSM}^1	V_{DE}^1	V_{PE}^1	V_{FD}^2	V_{LSM}^2	V_{DE}^2	V_{PE}^2
0.015	0.0023	0.002305	0.002298	0.002303	0.003355	0.003359	0.003356	0.00335	0.00417	0.004174	0.004185	0.004165
0.02	0.002306	0.002311	0.002304	0.002309	0.003364	0.003369	0.003365	0.003359	0.004182	0.004187	0.004197	0.004177
0.025	0.002312	0.002318	0.002311	0.002316	0.003374	0.003378	0.003375	0.003368	0.004194	0.004199	0.004208	0.004189

Table C.3: Under the interest rate model (C.1), *budget option* Valuation varying mortgage rate m_0 (the *budget offer* specified in Table 3.1 with varied $\theta = 0.02$, $\sigma = 0.02$, $r_{t_0} = 0.02$, 1-year mortgage term)

D

REPLICATION TESTS OF BERMUDAN OPTION VALUATION BY SGBM

In this appendix, some numerical tests for the stock option valuation are performed by LSM and SGBM. In Section D.1, the stock price model is chosen as a one-dimensional geometric Brownian motion in the Bermudan stock option valuation. In Section D.2, one-dimensional Merton jump-diffusion (MJD) model is chosen as the underlying model. Afterwards, the case of a two-dimensional Merton jump-diffusion model in the Bermudan stock option valuation is performed in Section D.3. The presented numerical results in this appendix show the validity of the computing codes for the applied numerical methods, which further convinces the readers about the validity in the numerical results of the budget option valuation in this thesis.

Now we start to value a Bermudan put option. The specification of this Bermudan put option is given in Table D.3. On each exercisable date, the option holder has the option but not the obligation to exercise the Bermudan put option. Suppose on an exercisable date t_{M_i} of the stock price S_{M_i} , the option is in the money, i.e., $K - S_{M_i} > 0$. Then the immediate exercise payoff to the option holder is $\max\{K - S_{M_i}, 0\}$.

initial stock price:	S_0
strike price:	K
option period:	T (in years)
number of exercise opportunity:	M

Table D.1: Specification on the Bermudan put option

Table D.2: parameters used in valuing the bermudan put option under GBM. The true option value (TV) is 2.3140 [15].

S_0	K	T	M	r	σ
40	40	1	50	0.06	0.2

D.1. GEOMETRIC BROWNIAN MOTION (GBM)

In the risk-neutral space $(\Omega, \mathcal{F}, Q)$ with filtration $\{\mathcal{F}_t\}_{t \geq 0}$, suppose the stock price S follows a geometric Brownian motion, i.e.,

$$dS_t = r \cdot S_t dt + \sigma \cdot S_t dW_t. \quad (\text{D.1})$$

where r is the risk-free rate. The stock price process is risk-neutral.

The basis functions applied in LSM and SGBM are 1 , $\log(S_t)$, $\log^2(S_t)$, and $\log^3(S_t)$.

From Equation (D.1), we know that the distribution of S_{t+1} conditional on S_t is as follows.

$$\begin{aligned} \left(\log(S_{t+1}) \mid S_t \right) &\sim \mathcal{N}(\hat{\mu}, \hat{\sigma}^2) \\ \hat{\mu} &= \log(S_t) + \left(r - \frac{\sigma^2}{2} \right) dt \\ \hat{\sigma}^2 &= \sigma^2 dt \end{aligned}$$

Hence, we can easily get

$$\begin{aligned} \mathbb{E}[\log(S_{t+1}) \mid S_t] &= \hat{\mu}, \\ \mathbb{E}[\log^2(S_{t+1}) \mid S_t] &= \hat{\mu}^2 + \hat{\sigma}^2, \\ \mathbb{E}[\log^3(S_{t+1}) \mid S_t] &= \hat{\mu}^3 + 3\hat{\mu}\hat{\sigma}^2. \end{aligned}$$

Unless specification, the default implementation set-up is:

In the finite difference method (FDM): time step size $dt = \frac{T}{10 \cdot M}$, the grid size of stock price $\Delta S = 0.1$;

In the least squares method (LSM): The number of simulated paths is $N = 50,000 + 50,000$ antithetic. The result is an average result of 20 tests of different random seeds, and the corresponding standard error (s.e.) is the standard deviation of these 20 test results;

In the stochastic grid bundling method (SGBM): The number of simulated paths is $N_D = 50,000$ in the direct estimator process and $N_p = N_D + N_D$ (antithetic) in the path estimator process. The number of applied bundles is $N_B = 20$ at each time step. The result is an average result of 20 tests of different random seeds, and the corresponding standard error (s.e.) is the standard deviation of these 20 test results.

Table D.3: Numerical Results by FDM, LSM, SGBM with parameters in Table D.2

TV	FDM	LSM	s.e.(LSM)	DE	s.e.(DE)	PE	s.e.(PE)
2.3140	2.3142	2.3116	0.00438	2.3146	0.00226	2.3135	0.00551

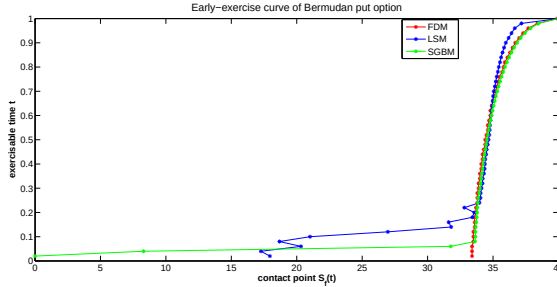


Figure D.1: early-exercise curve for the Bermudan put option with the parameters in Table D.2

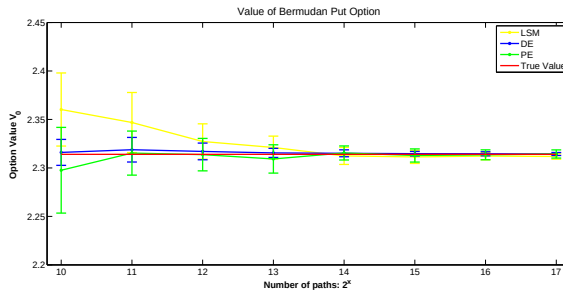


Figure D.2: convergence test on simulated path number for the Bermudan put option with the parameters in Table D.2

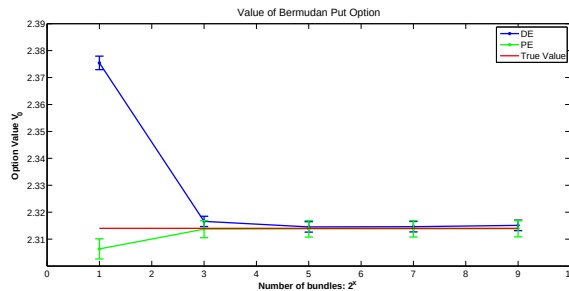


Figure D.3: convergence test on bundle number in SGBM for the Bermudan put option with the parameters in Table D.2 and simulated path number $N_D = 100,000$ in DE and $N_p = N_D + N_{D\text{antithetic}}$ in PE

D.2. 1-D MERTON JUMP-DIFFUSION (MJD) MODEL

In the risk-neutral space $(\Omega, \mathcal{F}, Q)$ with filtration $\{\mathcal{F}_t\}_{t \geq 0}$, suppose the stock price S follows a Merton Jump-diffusion (MJD) Model as follows,

$$dS_t = S_t (\mu dt + \sigma dW_t + (q_t - 1) dJ_t). \quad (D.2)$$

where $\log(q) \sim \mathcal{N}(\mu_j, \sigma_j^2)$, $J_t \sim \text{Poisson}(\lambda t)$, stochastic processes J, q, W are independent of one another, and

$$\mu = r - \lambda \cdot \left(\exp \left[\mu_j + \frac{1}{2} \sigma_j^2 \right] - 1 \right)$$

is chosen such that the risk neutrality of the stock price is achieved. Thus,

$$S_t = S_0 \cdot \exp \left(\left(\mu - \frac{\sigma^2}{2} \right) \cdot t + \sigma W_t \right) \cdot \prod_{j=1}^{J_t} q_j. \quad (D.3)$$

From Equation (D.3), we get

$$\log(S_t) = \log(S_0) + \left(\left(\mu - \frac{\sigma^2}{2} \right) \cdot t + \sigma W_t \right) + \sum_{j=1}^{J_t} \log(q_j). \quad (D.4)$$

The basis functions applied in LSM and SGBM are 1, $\log(S_t)$, $\log^2(S_t)$, and $\log^3(S_t)$.

From Equation (D.4), we know the distribution of S_{t+1} conditional on S_t and dJ_t , which is given by

$$\begin{aligned} \left(\log(S_{t+1}) \mid S_t, dJ_t \right) &\sim \mathcal{N}(\hat{\mu}, \hat{\sigma}^2) \\ dJ_t &\sim \text{Poisson}(\lambda \cdot dt) \\ \hat{\mu} &= \log(S_t) + \left(\mu - \frac{\sigma^2}{2} \right) \cdot dt + \mu_j \cdot dJ_t \\ \hat{\sigma}^2 &= \sigma^2 \cdot dt + \sigma_j^2 \cdot dJ_t. \end{aligned}$$

Hence, we can easily get

$$\begin{aligned} \mathbb{E}[\log(S_{t+1}) \mid S_t] &= \mathbb{E}[\hat{\mu} \mid S_t] \\ &= \log(S_t) + \left(\mu - \frac{\sigma^2}{2} \right) dt + \mu_j \lambda dt, \\ \mathbb{E}[\log^2(S_{t+1}) \mid S_t] &= \mathbb{E}[\hat{\mu}^2 + \hat{\sigma}^2 \mid S_t] \\ &= \left(\log(S_t) + \left(\mu - \frac{\sigma^2}{2} \right) dt \right)^2 + 2\mu_j \lambda dt \left(\log(S_t) + \left(\mu - \frac{\sigma^2}{2} \right) dt \right) \\ &\quad + \mu_j^2 ((\lambda dt)^2 + \lambda dt) + \sigma^2 dt + \sigma_j^2 \lambda dt, \end{aligned}$$

λ	μ_j	σ_j	RV	LSM	s.e.(LSM)	DE	s.e.(DE)	PE	s.e.(PE)
1	-0.2	0.2	4.2742	4.2576	0.015768	4.2957	0.003009	4.2705	0.012899
3	-0.2	0.2	6.8141	6.804	0.020808	6.8303	0.003691	6.8054	0.017242
8	-0.2	0.2	10.745	10.73	0.034074	10.756	0.004351	10.725	0.023866
0.1	-0.9	0.45	3.3696	3.3434	0.029185	3.3236	0.006578	3.3602	0.016422

Table D.4: numerical results of 1D-MJD stock price model by LSM, SGBM with the parameters in Table D.2, simulated paths $N_p^{LSM} = 100,000$ in LSM, $N_p^{DE} = 2^{17}$ in DE, $N_p^{PE} = 2 \times N_p^{DE}$ in PE, and the reference values (RV) obtained in Cong. F's report

$$\begin{aligned}
\mathbb{E}[\log^3(S_{t+1})|S_t] &= \mathbb{E}[\hat{\mu}^3 + 3\hat{\mu} \cdot \hat{\sigma}^2 | S_t] \\
&= \left(\log(S_t) + \left(\mu - \frac{\sigma^2}{2} \right) dt \right)^3 + 3\mu_j \lambda dt \left(\log(S_t) + \left(\mu - \frac{\sigma^2}{2} \right) dt \right)^2 \\
&\quad + 3\mu_j^2 ((\lambda dt)^2 + \lambda dt) \left(\log(S_t) + \left(\mu - \frac{\sigma^2}{2} \right) dt \right) + \mu_j^3 ((\lambda dt)^3 + 3(\lambda dt)^2 + \lambda dt) \\
&\quad + 3\mathbb{E} \left(\left(\log(S_t) + \left(\mu - \frac{\sigma^2}{2} \right) \cdot dt + \mu_j \cdot dJ_t \right) \cdot \left(\sigma^2 \cdot dt + \sigma_j^2 \cdot dJ_t \right) \middle| S_t \right) \\
&= \left(\log(S_t) + \left(\mu - \frac{\sigma^2}{2} \right) dt \right)^3 + 3\mu_j \lambda dt \left(\log(S_t) + \left(\mu - \frac{\sigma^2}{2} \right) dt \right)^2 \\
&\quad + 3\mu_j^2 ((\lambda dt)^2 + \lambda dt) \left(\log(S_t) + \left(\mu - \frac{\sigma^2}{2} \right) dt \right) + \mu_j^3 ((\lambda dt)^3 + 3(\lambda dt)^2 + \lambda dt) \\
&\quad + 3 \left(\log(S_t) + \left(\mu - \frac{\sigma^2}{2} \right) dt \right) \left(\sigma^2 dt + \sigma_j^2 \lambda dt \right) + 3\mu_j \lambda dt \cdot \sigma^2 dt + 3\mu_j \sigma_j^2 ((\lambda dt)^2 + \lambda dt),
\end{aligned} \tag{D.5}$$

As shown in Table D.4, for the rare jump case of $\lambda = 0.1$, SGBM may be inaccurate due to the estimation error from sample distribution (see [16]).

D.3. 2-D MERTON JUMP-DIFFUSION MODEL

In the risk-neutral measure $(\Omega, \mathcal{F}, Q)$ with filtration $\{\mathcal{F}_t\}_{t \geq 0}$, suppose the stock prices S^1 , S^2 follow a Merton jump-diffusion model as follows.

$$\left\{ \begin{array}{l} dS_t^1 = S_t^1 (\mu_1 dt + \sigma_1 dW_t^1 + (q_t^1 - 1) dJ_t) \\ dS_t^2 = S_t^2 (\mu_2 dt + \sigma_2 dW_t^2 + (q_t^2 - 1) dJ_t) \\ dJ_t \sim \text{Poisson}(\lambda \cdot dt) \\ \mathbb{E}^Q [dW_t^1 \cdot dW_t^2] = \rho_{12} dt \\ \log(q_t^i) \sim \mathcal{N}\left(\mu_i^j, (\sigma_i^j)^2\right), i \in \{1, 2\} \\ \mathbb{E}^Q \left[(\log(q_t^1) - \mu_1^j) (\log(q_t^2) - \mu_2^j) \right] = \sigma_1^j \sigma_2^j \rho_{12}^j, \end{array} \right. \quad (\text{D.6})$$

where stochastic processes J, q, W are independent of one another, and

$$\mu_i = r - \lambda \left(\exp \left[\mu_i^j + \frac{1}{2} (\sigma_i^j)^2 \right] - 1 \right), i = 1, 2.$$

is chosen such that the risk neutrality of the stock prices is achieved. Thus,

$$S_t^i = S_0^i \cdot \exp \left(\left(\mu_i - \frac{\sigma_i^2}{2} \right) \cdot t + \sigma_i W_t^i \right) \cdot \prod_{j=1}^{J_t^i} q_j^i, i = 1, 2. \quad (\text{D.7})$$

From Equation (D.7), we get

$$\log(S_t^i) = \log(S_0^i) + \left(\left(\mu_i - \frac{\sigma_i^2}{2} \right) \cdot t + \sigma_i W_t^i \right) + \sum_{j=1}^{J_t^i} \log(q_j^i), i = 1, 2. \quad (\text{D.8})$$

To facilitate simulating dW_t^i ($i = 1, 2$) and $\log(q_t^i)$ ($i = 1, 2$) in LSM and SGBM, we apply the Cholesky decomposition technique to send the equivalent combinations of independent Brownian motion to the correlated ones. For dW_t^i ($i = 1, 2$),

$$\begin{aligned} \begin{bmatrix} dW_t^1 \\ dW_t^2 \end{bmatrix} \begin{bmatrix} dW_t^1 \\ dW_t^2 \end{bmatrix}^T &= \begin{bmatrix} 1 & \rho_{12} \\ \rho_{12} & 1 \end{bmatrix} \cdot dt \\ &= \begin{bmatrix} 1 & 0 \\ \rho_{12} & \sqrt{1 - \rho_{12}^2} \end{bmatrix} \begin{bmatrix} d\widetilde{W}_t^1 \\ d\widetilde{W}_t^2 \end{bmatrix} \left(\begin{bmatrix} 1 & 0 \\ \rho_{12} & \sqrt{1 - \rho_{12}^2} \end{bmatrix} \begin{bmatrix} d\widetilde{W}_t^1 \\ d\widetilde{W}_t^2 \end{bmatrix} \right)^T, \end{aligned} \quad (\text{D.9})$$

where $\widetilde{W}_t^1$ and $\widetilde{W}_t^2$ are Wiener processes and independent of each other. Hence, we define

$$\begin{bmatrix} dW_t^1 \\ dW_t^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \rho_{12} & \sqrt{1 - \rho_{12}^2} \end{bmatrix} \begin{bmatrix} d\widetilde{W}_t^1 \\ d\widetilde{W}_t^2 \end{bmatrix} = \begin{bmatrix} d\widetilde{W}_t^1 \\ \rho_{12} d\widetilde{W}_t^1 + \sqrt{1 - \rho_{12}^2} d\widetilde{W}_t^2 \end{bmatrix}. \quad (\text{D.10})$$

As for $\log(q_t^i)$ ($i = 1, 2$), first we rewrite $\log(q_t^i)$, ($i = 1, 2$) equivalently into

$$\begin{cases} \frac{\log(q_t^i) - \mu_i^j}{\sigma_i^j} = Z_t^i, \quad i = 1, 2. \\ Z_t^i \sim \mathcal{N}(0, 1), \\ \mathbb{E}^Q[Z_t^1 \cdot Z_t^2] = \rho_{12}^j, \end{cases} \quad (\text{D.11})$$

which simplifies our work on the Cholesky decomposition of $\log(q_t^i)$ ($i = 1, 2$) into that of Z_t^i ($i = 1, 2$).

$$\begin{aligned} \begin{bmatrix} Z_t^1 \\ Z_t^2 \end{bmatrix} \begin{bmatrix} Z_t^1 \\ Z_t^2 \end{bmatrix}^T &= \begin{bmatrix} 1 & \rho_{12}^j \\ \rho_{12}^j & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ \rho_{12}^j & \sqrt{1 - (\rho_{12}^j)^2} \end{bmatrix} \begin{bmatrix} \widetilde{Z}_t^1 \\ \widetilde{Z}_t^2 \end{bmatrix} \left(\begin{bmatrix} 1 & 0 \\ \rho_{12}^j & \sqrt{1 - (\rho_{12}^j)^2} \end{bmatrix} \begin{bmatrix} \widetilde{Z}_t^1 \\ \widetilde{Z}_t^2 \end{bmatrix} \right)^T, \end{aligned} \quad (\text{D.12})$$

where $\widetilde{Z}_t^1$ and $\widetilde{Z}_t^2$ are in standard normal distribution and independent of each other. Hence, we define that

$$\begin{bmatrix} Z_t^1 \\ Z_t^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \rho_{12}^j & \sqrt{1 - (\rho_{12}^j)^2} \end{bmatrix} \begin{bmatrix} \widetilde{Z}_t^1 \\ \widetilde{Z}_t^2 \end{bmatrix} = \begin{bmatrix} \widetilde{Z}_t^1 \\ \rho_{12} \widetilde{Z}_t^1 + \sqrt{1 - \rho_{12}^2} \widetilde{Z}_t^2 \end{bmatrix}. \quad (\text{D.13})$$

Therefore, we rewrite Equation (D.6) into

$$\begin{cases} dS_t^1 = S_t^1 \left(\mu_1 dt + \sigma_1 d\widetilde{W}_t^1 + \left(\exp \left\{ \mu_1^j + \sigma_1^j \widetilde{Z}_t^1 \right\} - 1 \right) dJ_t \right), \\ dS_t^2 = S_t^2 \left(\mu_2 dt + \sigma_2 \left(\rho_{12} d\widetilde{W}_t^1 + \sqrt{1 - \rho_{12}^2} d\widetilde{W}_t^2 \right) + \left(\exp \left\{ \mu_2^j + \sigma_2^j \left(\rho_{12} \widetilde{Z}_t^1 + \sqrt{1 - \rho_{12}^2} \widetilde{Z}_t^2 \right) \right\} - 1 \right) dJ_t \right), \\ dJ_t \sim \text{Poisson}(\lambda \cdot dt), \\ \mathbb{E}^Q \left[d\widetilde{W}_t^1 \cdot d\widetilde{W}_t^2 \right] = 0, \\ Z_t^i \sim \mathcal{N}(0, 1), \quad i = 1, 2. \\ \mathbb{E}^Q \left[Z_t^1 \cdot Z_t^2 \right] = 0. \end{cases} \quad (\text{D.14})$$

The basis functions applied in LSM and SGBM are $1, \log(S_t^1), \log(S_t^2), \log^2(S_t^1), \log^2(S_t^2), \log(S_t^1) \log(S_t^2)$. The conditional expectations of these basis functions of S_{t+1}^i ($i = 1, 2$) on

S_t^i ($i = 1, 2$) are given by

$$\begin{aligned}\mathbb{E}^Q \left[\log(S_{t+1}^i) \middle| S_t^i \right] &= \log(S_t^i) + \left(\mu_i - \frac{\sigma_i^2}{2} \right) dt + \mu_i^j \lambda dt, \quad i = 1, 2. \\ \mathbb{E}^Q \left[\log^2(S_{t+1}^i) \middle| S_t^i \right] &= \left(\log(S_t^i) + \left(\mu_i - \frac{\sigma_i^2}{2} \right) dt \right)^2 + 2\mu_i^j \lambda dt \left(\log(S_t^i) + \left(\mu_i - \frac{\sigma_i^2}{2} \right) dt \right) \\ &\quad + \left(\mu_i^j \right)^2 ((\lambda dt)^2 + \lambda dt) + \sigma_i^2 dt + \left(\sigma_i^j \right)^2 \lambda dt, \quad i = 1, 2. \\ \mathbb{E}^Q \left[\log(S_{t+1}^1) \log(S_{t+1}^2) \middle| S_t^1, S_t^2 \right] &= \left(\log(S_t^1) + \left(\mu_1 - \frac{\sigma_1^2}{2} \right) dt + \mu_1^j \lambda dt \right) \left(\log(S_t^2) + \left(\mu_2 - \frac{\sigma_2^2}{2} \right) dt \right) \\ &\quad + \mu_2^j \lambda dt \left(\log(S_t^1) + \left(\mu_1 - \frac{\sigma_1^2}{2} \right) dt \right) + \sigma_1 \sigma_2 \rho_{12} dt \\ &\quad + \left(\mu_1^j \mu_2^j \right) ((\lambda dt)^2 + \lambda dt) + \sigma_1^j \sigma_2^j \rho_{12}^j \lambda dt.\end{aligned}$$

With the strike price K , for the geometric average put option on $S_t = [S_t^1, S_t^2]$, the option intrinsic value

$$h(S_t) = K - \sqrt{S_t^1 \cdot S_t^2}$$

is used to bundle the simulated paths S_t^1, S_t^2 (see [16]).

For the arithmetic average put option on $\mathbf{S}_t = [S_t^1, S_t^2]$, at time t the option intrinsic value

$$h(\mathbf{S}_t) = K - \frac{S_t^1 + S_t^2}{2}$$

is used to bundle the simulated paths S_t^1, S_t^2 (see [16]).

For min put option on $S_t = [S_t^1, S_t^2]$, the option intrinsic value

$$h(\mathbf{S}_t) = K - \frac{S_t^1 + S_t^2}{2}$$

and the difference between S_t^1 and S_t^2 are used to bundle the simulated paths S_t^1, S_t^2 (see [16]).

The numerical test results presented below are obtained under 2D MJD (D.6) with default setting: $N_{LSM} = 2^{17}$ the number of simulated paths in LSM, $N_D = 2^{17}$ the number of simulated paths in DE, $N_P = 2 \times N_D$ the number of simulated paths in PE, and the number of applied bundles $N_b = 64$.

	$S_0 = [S_0^1, S_0^2]$	K	T	M	r	δ	$[\sigma_1, \sigma_2]$	ρ_{12}	λ	$[\mu_1^j, \mu_2^j]$	$[\sigma_1^j, \sigma_2^j]$	ρ_{12}^j
Set 1	[100,100]	100	1	8	0.05	0	[0.12, 0.15]	0.3	0.6	[-0.1, 0.1]	[0.17, 0.13]	-0.2
Set 2		100	3	9	0.05	0.10	[0.20, 0.20]	0				

Table D.5: parameters used for valuing the Bermudan put option under 2D Merton jump-diffusion (MJD) model (δ is dividend rate)

S_0	DE	se(DE)	PE	se(PE)	Reference Value
[90, 90]	8.0726	0.009317	8.0673	0.021006	8.075
[100, 100]	13.903	0.013037	13.901	0.025208	13.902
[110, 110]	21.347	0.017279	21.347	0.026	21.345

Table D.6: numerical results of the max-on-call option valuation under 2-D MJD model by SGBM with Set 2 parameter setting in Table D.5, and the reference values (RV) obtained from [16]

	DE	se(DE)	PE	se(PE)	LSM	se(LSM)	Reference Value
geometric average put option	3.6727	0.006998	3.6679	0.008253	3.66	0.012681	3.6693
arithmetic average put option	3.3878	0.006862	3.3811	0.005586	3.3752	0.012241	3.3825
min put option	9.5981	0.010547	9.544	0.018087	9.45	0.027417	9.5526

Table D.7: numerical results of the Bermudan put option valuation under 2-D MJD model by LSM, SGBM with Set 1 parameter setting in Table D.5, and the reference values (RV) obtained from [16]

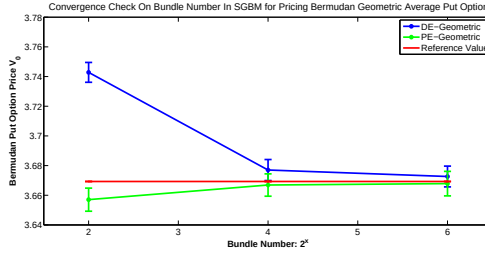


Figure D.4: convergence test on bundle number for Bermudan geometric average put option under 2-D MJD with the parameters in Table D.5

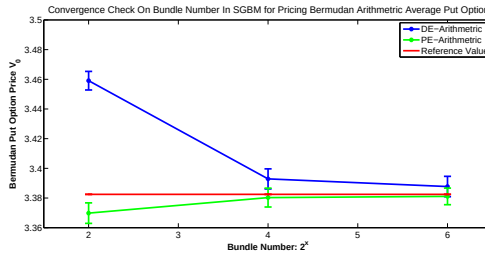


Figure D.5: convergence test on bundle number for Bermudan arithmetic average put option under 2-D MJD with the parameters in Table D.5

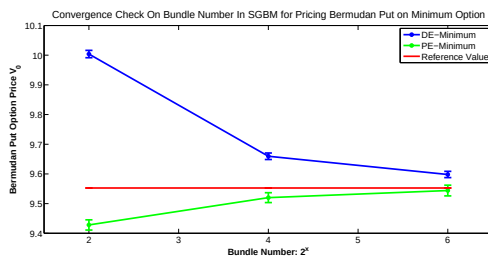


Figure D.6: convergence test on bundle number for Bermudan put-on-minimum option under 2-D MJD with the parameters in Table D.5

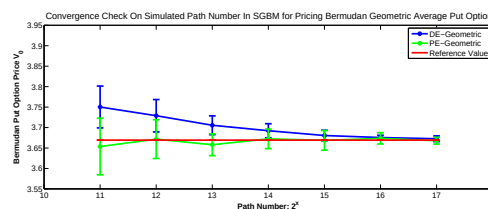


Figure D.7: convergence test on simulated path number for the Bermudan geometric average put option under 2-D MJD with the parameters in Table D.5

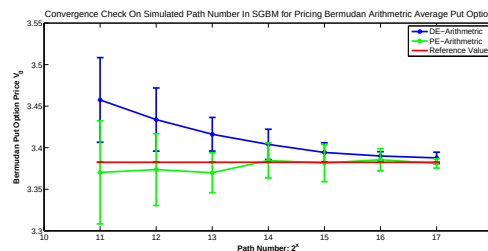


Figure D.8: convergence test on simulated path number for the Bermudan arithmetic average put option under 2-D MJD with parameters in Table D.5

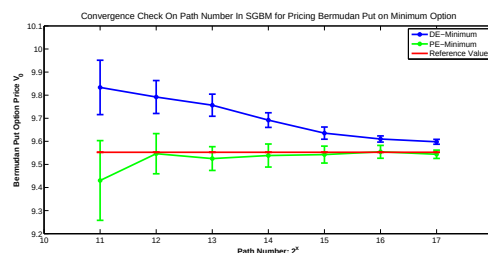


Figure D.9: convergence test on simulated path number for the Bermudan put-on-minimum option under 2-D MJD with the parameters in Table D.5

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