Scaling Limits of Multi-layer Particle Systems

van Wiechen, H.

DOI
10.4233/uuid:75fe76e2-0459-4e6b-97fe-b2a020b76999

Publication date
2025

Document Version
Final published version

Citation (APA)
van Wiechen, H. (2025). *Scaling Limits of Multi-layer Particle Systems*. [Dissertation (TU Delft), Delft University of Technology]. https://doi.org/10.4233/uuid:75fe76e2-0459-4e6b-97fe-b2a020b76999

Scaling Limits of Multi-layer Particle Systems

Hidde van Wiechen

Scaling Limits of Multi-layer Particle Systems

Dissertation

for the purpose of obtaining the degree of doctor
at Delft University of Technology
by the authority of the Rector Magnificus, prof. dr. ir. H. Bijl,
chair of the Board for Doctorates
to be defended publicly on
Friday 13 February 2026 at 12:30 o'clock

by

Hidde VAN WIECHEN

Master of Science in Applied Mathematics
Delft University of Technology, the Netherlands
born in Gouda, the Netherlands

This dissertation has been approved by the promotors.

Composition of the doctoral committee:

Rector Magnificus	Voorzitter
Prof. dr. F.H.J. Redig	Technische Universiteit Delft, *promotor*
Dr. R.C. Kraaij	Technische Universiteit Delft, *copromotor*
Dr. E. Pulvirenti	Technische Universiteit Delft, *copromotor*

Independent members:

Prof. dr. J.M.A.M. van Neerven	Technische Universiteit Delft
Prof. dr. C. Maes	Katholieke Universiteit Leuven, België
Prof. dr. S. Grosskinsky	Universität Augsburg, Duitsland
Prof. dr. M.A. Peletier,	Technische Universiteit Eindhoven
Prof. dr. A.W. van der Vaart	Technische Universiteit Delft, reservelid

TU Delft — Delft University of Technology

An electronic copy of this dissertation is available at
https://repository.tudelft.nl/.

CONTENTS

1

OVERVIEW OF TOPICS

In this thesis we will study multi-layer particle systems, which are interacting particle systems that live on a geometry consisting of multiple copies of the integers $\mathbb{Z}$. This choice of geometry provides a way of modeling active particles that use internal energy to move in a preferred direction.

The goal of this chapter is to provide an introduction to the field of statistical mechanics and interacting particle systems and to introduce the various topics discussed in this thesis in a mostly intuitive way without going into too much mathematical detail. Afterwards, we motivate the topic of this thesis further and provide a sketch of the content of this thesis. Later, in Chapter 2, we focus on the mathematical background needed for this thesis.

Readers who are familiar with the field of Interacting Particle Systems are advised to skip to Sections 1.5 and 1.6 for the motivation of this thesis and an overview of the following chapters.

1.1 STATISTICAL PHYSICS

Before introducing the field of interacting particle systems as a branch of mathematics, it is only appropriate to first take a step back and begin with an introduction to statistical physics. At its core, statistical physics serves as a bridge between the microscopic world of individual particles and the macroscopic world of systems containing an immense number of particles. For instance, the evolution of a gas or liquid on a macroscopic scale can be described using *partial differential equations* (PDEs). However, when we zoom in to the microscopic scale, we encounter a chaotic system of tiny particles, all colliding and interacting with one another.

The idea of statistical physics is to start at the microscopic scale and use it to infer properties or behaviors at the macroscopic scale. This goes beyond just finding a partial differential equation for the typical evolution of the system (usually referred to as the *hydrodynamic limit*). One of the central questions is: what does the system look like when observed over long periods of time? In many cases, the system evolves toward a stable distribution known as *equilibrium*.

In equilibrium statistical physics, the goal is to understand the macroscopic properties of systems that have reached such a steady state. Although the individual particles continue to move and interact, the system as a whole exhibits no net change in its macroscopic observables. For instance, a gas in a sealed container will reach a point where quantities like temperature, pressure, and density become uniform and constant over time. At this stage, the system is said to be in thermal equilibrium.

But not all systems behave this way. In fact, many real-world systems operate far from equilibrium (weather patterns, epidemics, living organisms). These are referred to as *non-equilibrium systems*. In such cases, instead of settling into a static equilibrium, the system might exhibit complex time-dependent behavior on the macroscopic scale, such as steady fluxes of particles, self-organized patterns, or oscillations. Understanding the emergence of equilibrium, or the mechanisms driving systems away from it, is a major theme in statistical physics.

Building on this, it is important to recognize that statistical physics is not limited to systems of atoms, but also extends to other fields, such as biology. For example, when considering a flock of birds, on the macroscopic scale, we see various patterns emerging, while on the microscopic scale, individual birds adjust their flight patterns in response to those around them. Or in genealogy, the microscopic scale can be viewed as the individuals in a population, while on the macroscopic scale, we observe the evolution and survival of certain genetic traits. "particles" should thus not be considered as necessarily an atom, but rather as an abstract object that can take many forms, both large and small.

But where does the mathematics come into play? (After all, this is a thesis in mathematics.) The actual approach in statistical physics is usually probabilistic in nature. Since we are often considering models with a "large" number of particles (where large can be anything from 100 birds in a flock to 10^{23} atoms in a gram) and every particle can interact with every other particle, it is unfeasible to use a realistic model to describe every particle on the microscopic level. Instead we model them to behave randomly, i.e., at each point in time a particle moves with a certain *velocity*[1] according to a probability distribution, which may depend on its own position and the positions and velocities of other particles.

But is such a random model still realistic enough to say something about the real model? And why would the random model be easier to study? For these questions it is helpful to consider the example of tossing a coin. Although tossing a coin is a complex physical process, in theory, we could predict the outcome if we knew all the initial conditions. However, in practice, this is not feasible. It is way more reasonable to use a random model, where with equal probability the coin lands on heads or tails. While this random model may not be the best model for the individual coin-flips (it will only be correct around 50% of time), it still enables us to draw conclusions on a larger scale.

1 Velocity refers to both the speed and the direction of a particle.

For instance, we can show that in the random model, if the coin is tossed a large number of times, the average number of heads (or tails) will converge to 1/2, as we would expect from the real model as well. This result is called the *Law of Large Numbers* (LLN). Another result, the *Central Limit Theorem*, (CLT) allows us to quantify the probabilities of deviations of the average from 1/2. In statistical physics there are similar types of results, which are referred to as *scaling limits* (these will be discussed further in Section 1.4). The idea here is the same, that while the random model is not a good estimator for the path of individual particles, the average dynamics of a large number of particles still coincide with what we would expect.

In probability theory, the area of study related to statistical physics is often referred to as *Interacting Particle Systems* (IPS). In the next section, we introduce this area and provide examples of some common IPS that will appear throughout this thesis.

1.2 INTERACTING PARTICLE SYSTEMS

The field of IPS is a subfield of Markov Process Theory. We will give a detailed description of Markov processes in Chapter 2. In short, Markov processes are stochastic[2] processes that are memoryless; that is, the future depends only on the present state and not on the past. While this may seem like a strong assumption, it is actually well suited for particle systems. For instance, if you were to know the position and velocity of every particle in your system at a given time, then you would be able to infer something about its future behavior without looking at the past. Furthermore, the theory of Markov processes is a rich field with numerous results that can be directly applied to the field of IPS.

In this thesis we are mostly concerned with particles moving on a lattice, for example on the integers $\mathbb{Z}$. This may seem strange at first since when we think of positions of atoms for example, we think of them moving in the continuum and not on a grid. However, discretizing space has a lot of advantages in defining the interactions between particles: Rather than having to define interaction up to a certain distance in the continuum, we can impose nearest-neighbor interactions on the lattice. Furthermore, we are able to return to the continuum on the macroscopic scale by rescaling space, which will be explained in Section 1.4.

A configuration of particles on this lattice is usually denoted by η, which is a function $\eta : \mathbb{Z} \to \mathbb{N}$ where for every $x \in \mathbb{Z}$ the value $\eta(x) \in \mathbb{N}$ represents the number of particles at position x. Since a configuration is only concerned with the number of particles at every site, we automatically make the assumption that the particles are indistinguishable.

2 stochastic = random

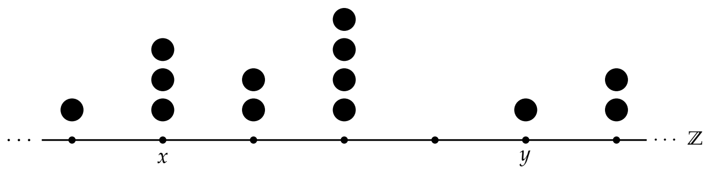

Figure 1: An example of a particle configuration on $\mathbb{Z}$. In this example we have $\eta(x) = 3$ and $\eta(y) = 1$.

The evolution of the system works as follows: Each particle has a set of internal clocks, one for each possible site it can jump to (usually we consider only nearest-neighbor jumps), that ring after a random time. When the first clock rings, the particle jumps to the corresponding site. These random waiting times are exponentially distributed[3] with a parameter that may depend on the entire configuration, the site the particle is jumping from, the site it is jumping to, or other quantities. These parameters are called *rates*, and the general rule is that the higher the rate, the faster the event occurs.

1.2.1 *Examples of IPS*

Below we give a short description of three types of interacting particle systems that are often considered in this thesis.

INDEPENDENT RANDOM WALKERS (IRW)

The first "interacting" particle system we consider is one in which there is no interaction between the particles at all. In the IRW model, particles jump with jump rates that are independent of the rest of the configuration. This model often serves as a good starting point in the analysis of particle systems, before adding any interactions.

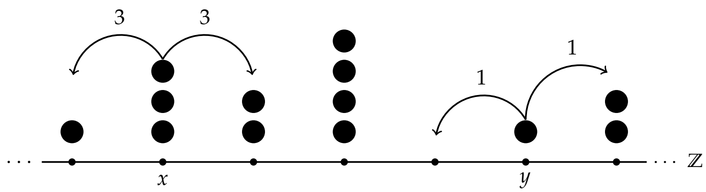

Figure 2: Rates of particles jumping from x and y in the IRW model. Since there are three particles at site x, the total rate of any particle jumping to a neighboring site is 3.

3 The choice of exponential waiting times ensures that the process is Markovian; namely, the exponential distribution has the *memoryless property*, meaning that if the clock has not rung after an arbitrary time, the remaining time until it rings is still exponentially distributed with the same parameter.

For now, we consider the model in which a particle jumps to a nearest-neighbor site at rate 1. This choice of rates makes the model both *simple*[4] and *symmetric*[5].

SIMPLE EXCLUSION PROCESS (SEP)

The SEP is a process that follows the *exclusion rule* of having at most one particle per site. The particles still perform simple symmetric random walk jumps, but any attempt to move to an occupied site is rejected.

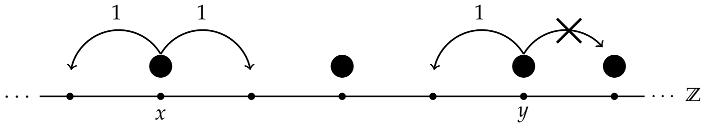

Figure 3: Rates of particles jumping from x and y in the SEP model. One jump from y gets rejected due to the presence of another particle.

The SEP was introduced by Spitzer in [107], whose work actually started the field of IPS. The exclusion dynamic models repulsion between particles and can be used, for instance, to simulate a high-temperature gas where particles interact only through short-range collisions. Another application of the SEP is to simulate traffic flow. In this context, jumps would be restricted to a single direction, leading to what is referred to as the *Totally Asymmetric Simple Exclusion Process* (TASEP).

SIMPLE INCLUSION PROCESS (SIP)

Instead of repulsive dynamics, we could also consider attractive interaction between particles. The SIP does exactly that, where particles are inclined to jump towards sites with more particles. In this process, next to simple symmetric random walk jumps, every particle invites every other particle at nearest neighbor sites to jump to their site with rate 1. Therefore, a single particle will jump to a neighboring site containing n particles with rate $1 + n$.

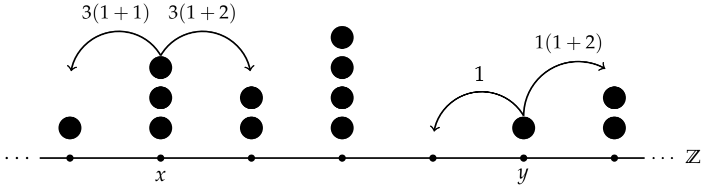

Figure 4: Rates of particles jumping from x and y in the SIP model.

The SIP was introduced in [45] where it arose as a tool to study another process called the *Brownian Energy Process (BEP)*. Afterwards, it has become an object of independent study, explored in more detail in works such as [14, 47, 46].

4 Simple refers to the fact that only nearest-neighbor jumps are possible.
5 Symmetric means that the jump rate between two sites is the same in both directions.

SPIN SYSTEMS

Although not a central topic of this thesis, another well-studied class of IPS are spin systems. Here, unlike particle systems modelling transport (such as IRW, SEP and SIP) the particles are at a fixed position and can change their *state* according to particles around them. Common examples of the spin systems are the *Voter model* (a model to simulate opinion behavior in social networks), the *Contact process* (a model to simulate the spread of an epidemic) and the *Ising model* (a model to simulate magnetic behavior of a material). Especially for these last two, considerable efforts have focused on phase transitions, see for example [55, 88].

1.2.2 *Relevant literature*

The books of Liggett [75, 76] played a key role in developing the mathematical theory around IPS, where many types of IPS are discussed, including some we have mentioned above.

1.3 ERGODIC THEORY

Ergodic theory originated from Ludwig Boltzmann's interpretation of probability (for a detailed record on this, see [122]). Given a process $\{X_t : t \geq 0\}$, Boltzmann's idea was to describe the probability of a state as the proportion of time the process spends in this state, i.e., he defined a probability measure μ as

$$\mu(A) := \lim_{T\to\infty} \frac{1}{T} \int_0^T \mathbb{1}(X_t \in A)\,\mathrm{d}t, \tag{1.3.1}$$

where $\mathbb{1}$ is the indicator function, equal to one if the event in the brackets is true, and zero otherwise.

There were two problems Boltzmann encountered with this definition. The first was existence: does this limit actually produce a probability distribution? In finite systems this is always the case. However in infinite systems the process may escape causing the probability mass to disappear.

The second problem was uniqueness: does the limit always converge to the same value for different initial conditions $X_0 = x$? A simple example illustrating where this can go wrong occurs when X_t is defined on two disjoint sets with no path between them (see Figure 5). The average time spent in one of the two sets is either 1 or 0, depending solely on whether we start in that set or not.

To address this issue, the initial concept of *ergodicity* emerged, defining a system as ergodic if it eventually visits all states of the space. Under this notion of ergodicity, the probability measure defined in (1.3.1) is indeed independent of the initial condition.

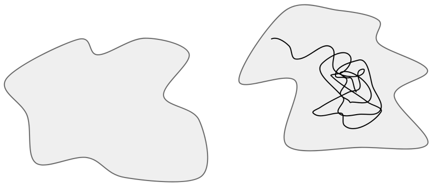

Figure 5: A process defined on two disjoint sets. Once it starts in any of the two sets, it remains within that set indefinitely.

Later, Birkhoff [9] and von Neumann [121] provided a more rigorous mathematical framework for ergodicity, leading to the formal definition of ergodic measures. These are precisely the measures μ for which the equality (1.3.1) holds for μ-almost all initial points x. Consequently, any point in the support of an ergodic measure is accessible from any other point in that support, while points outside the support cannot be reached from points within it. For instance, in the example shown in Figure 5 there are two ergodic measures: one with support in the left set and the other with support in the right set.

These ergodic measures provide insight into the long-term behavior of the process. In this sense, they are closely related to *invariant measures*.

1.3.1 *Invariant measures*

Consider a stochastic process $\{X_t : t \geq 0\}$ on the state space Ω. Suppose the initial position X_0 is drawn from some probability distribution μ on Ω, then one may ask how the distribution of X_t, denoted by μ_t, evolves over time. For Markov processes this result is known, and μ_t solves the *Kolmogorov forward equation*

$$\partial_t \mu_t = \mu_t L. \tag{1.3.2}$$

Here L is the *generator*[6] of the Markov process, which is a linear operator acting on functions f on Ω. For jump processes, this generator is given by

$$Lf(x) = \sum_{y \in \Omega} c(x,y)(f(y) - f(x)) \tag{1.3.3}$$

6 The precise definition of a generator of a Markov process will be given in Chapter 2. For now, intuitively, a generator describes how probabilities evolve over an infinitesimally small time step.

where $c(x,y)$ is the rate of the process jumping from state x to state y. To properly interpret $\mu_t L$ in (1.3.2), we define it as the measure on Ω such that for all f in the *domain*[7] of L we have that

$$\int f \, \mathrm{d}\mu_t L = \int Lf \, \mathrm{d}\mu_t. \tag{1.3.4}$$

An invariant measure, also known as a *stationary distribution*, of the process is a measure μ that does not change over time, i.e., if we start the process from μ then $\partial_t \mu_t = 0$ for all $t \geq 0$. This means that an invariant measure is a fixed point of the Kolmogorov forward equation, satisfying $\mu L = 0$ (meaning that $\int Lf \, \mathrm{d}\mu = 0$ for all f in the domain of L). If the measure μ_t converges to a probability measure as time progresses, then it has to converge to such a fixed point, meaning that knowing the invariant measures gives a certain understanding on the long-term behavior of a process.

Since the generator L is a linear operator, any convex combination of invariant measures is again invariant, making the set of all invariant measures convex. The ergodic measures are the extremal points of this set. Therefore, knowing all the ergodic measures allows us to reconstruct the set of invariant measures.

Another special class of invariant measures is the class of *reversible measures*. A reversible measure μ has to satisfy for all f and g in the domain of L

$$\int gLf \, \mathrm{d}\mu = \int fLg \, \mathrm{d}\mu, \tag{1.3.5}$$

meaning that L is self-dual in the space $L^2(\mu)$. If we were considering a jump process, this is satisfied if the following holds:

$$\mu(x)c(x,y) = \mu(y)c(y,x) \tag{1.3.6}$$

This equality is called the *detailed balance condition*, and it implies that the process is *time-reversible*, meaning that any trajectory is equally likely to occur forward or backward in time. A system exhibiting this property is often described as being in *equilibrium*, which is why reversible measures are sometimes also referred to as *equilibrium measures*.

1.3.2 *Ergodic measures for IPS*

When talking about probabilities on the space of configurations on $\mathbb{Z}$, denoted Ω, the events to consider are so-called *cylinder sets* $C \subset \Omega$ of the form

$$C = \{\eta \in \Omega : \eta(x_1) \in A_1,\ \eta(x_2) \in A_2, ..., \eta(x_k) \in A_k\}, \tag{1.3.7}$$

---

7 In some cases we can not make sense of Lf for all functions f. For instance if $L = \partial_x$ and f is not differentiable then Lf does not exist. The domain of L is precisely those functions f for which Lf is well-defined.

for any $x_1, ..., x_k \in \mathbb{Z}$ and $A_1, ..., A_k \subset \mathbb{N}$. A probability measure on Ω is uniquely defined by their values on these cylinder sets.

An important class of these measures is given by *product measures*

$$\mu = \bigotimes_{x \in \mathbb{Z}} \mu_x, \tag{1.3.8}$$

where every μ_x is a measure on $\mathbb{N}$, which we call the *marginals*. With a product measure, the value of every $\eta(x)$ is distributed according to μ_x, independent of all the other sites. More precisely, if C is defined as in (1.3.7), we have that

$$\mu(C) = \prod_{i=1}^{k} \mu_{x_i}(\{\eta(x_i) \in A_i\}). \tag{1.3.9}$$

For the IRW, SEP and SIP introduced in Section 1.2.1, we find product measures that are ergodic and even reversible for these systems (in Chapter 3 we also give a proof of this). For the IRW the marginals are given by Poisson distributions with equal parameter, for the SEP by Bernoulli distributions, and for the SIP by geometric distributions. The natural question then arises: are these the only ergodic measures?

Two mathematical tools are very useful in the exploration of this question. The first is *duality*, a technique in Markov process theory that allows one to analyze a process through another (often simpler) process. Useful duality relations are however scarce, but for the IRW, SEP and SIP a duality relation exists between the process with infinitely many particles and the same process with a finite number of particles. In particular these processes are dual to the process with one particle, which in all three cases corresponds to a single symmetric random walker.

The second tool is *coupling*. Given two stochastic processes, we can define a coupling, which is a joint process in which each component behaves like the original processes when viewed individually, but together they may exhibit dependencies. A coupling is considered *successful* if the two processes eventually meet and remain together indefinitely. When such a coupling exists, the distributions of the processes converge to each other over time.

In Chapter 2 we give more details on the definitions of these two tools.

1.3.3 *Relevant literature*

The book of Liggett [76] gives an introduction to ergodic theory for the SEP. Later the paper by Kuoch and Redig [66] gives an understanding of the ergodic measures of SIP. [14] gives an overview of well-known duality relations in stochastic models of transport. For an introduction into couplings, we refer to the book of Thorisson [115], the lecture notes of Lindvall [77] and the lecture notes of Den Hollander [27].

1.4 SCALING LIMITS

As discussed in Section 1.1, one of the main motivations of studying statistical physics is to derive macroscopic properties from microscopic dynamics. The process of transitioning from microscopic to macroscopic behavior is referred to as *scaling limits*, and is achieved via an appropriate rescaling of space and time. Intuitively, scaling limits describe how the system behaves when we "zoom out" from individual particle configurations and observe particle densities instead. But how can we approach this "zooming out" mathematically? In this section we start by describing the mathematical idea behind rescaling space. Afterwards, we will discuss the three types of scaling limits that are present in this thesis: the *hydrodynamic limit*, *fluctuations* and *large deviations*.

1.4.1 Rescaling space ("From micro to macro")

The rescaling of space is done using a mathematical object called the *empirical measure*. For a given $N \in \mathbb{N}$, the empirical measure corresponding to a configuration of particles η is then given by

$$\pi^N(\eta) = \frac{1}{N} \sum_{x \in \mathbb{Z}} \eta(x) \cdot \delta_{\frac{x}{N}}, \tag{1.4.1}$$

where $\delta_{\frac{x}{N}}$ is the *Dirac measure*[8].

In words, the empirical measure places for every particle present at the micropoint $x \in \mathbb{Z}$ a point mass at the corresponding macropoint $\frac{x}{N}$. As a result, we obtain, from the micro-configuration η, a measure on the rescaled lattice $\frac{1}{N}\mathbb{Z} \subset \mathbb{R}$. For sufficiently large N, this lattice approximates the continuum $\mathbb{R}$. We now associate each point in the continuum with a point on the lattice $\mathbb{Z}$ through the following relation

$$\text{macropoint: } x \in \mathbb{R} \quad \longleftrightarrow \quad \text{micropoint: } \lceil Nx \rceil \in \mathbb{Z}. \tag{1.4.2}$$

We will refer to $x \in \mathbb{R}$ as a macroscopic point, and to $\lceil Nx \rceil \in \mathbb{Z}$ as the corresponding microscopic point.

The prefactor $\frac{1}{N}$ in the definition of π^N assigns to each particle a "weight" of $\frac{1}{N}$. This weight ensures that the contribution of individual particles vanishes in the limit $N \to \infty$, and only the averages over large blocks surrounding microscopic points are relevant. In this sense, the empirical density should remind the reader of the Law of Large Numbers.

8 The Dirac measure δ_x is a measure with a point mass at x, such that for every function ϕ we have that $\langle \delta_x, \phi \rangle = \phi(x)$.

For appropriate configurations η, as we let N tend to infinity, the empirical measure converges weakly to a measure on $\mathbb{R}$ that is absolutely continuous (with respect to the Lebesgue measure), i.e.,

$$\pi^N(\eta) \to \varrho(x)\,\mathrm{d}x \qquad \text{as } N \to \infty, \tag{1.4.3}$$

for some function $\varrho : \mathbb{R} \to \mathbb{R}_{\geq 0}$. This ϱ can now be viewed as the *density* of particles on the macroscopic scale, corresponding to the configuration η.

Often, instead of the empirical measures of one configuration, we consider a sequence of random configurations $\{\eta^N\}_{N\in\mathbb{N}}$. For any smooth density ϱ, it is possible to find such a sequence of configurations for which π^N converges to $\varrho(x)\,\mathrm{d}x$ weakly in probability, as will become clear from the following example.

Example 1.1. Pick a smooth density $\varrho : \mathbb{R} \to \mathbb{R}_{\geq 0}$. For every $N \in \mathbb{N}$, we draw a random configuration η^N from the product measure (see (1.3.8)) where the marginals are given by Poisson distributions with *slowly varying* parameter $\varrho(\frac{x}{N})$, i.e., $\eta^N(x) \sim \mathrm{Pois}(\varrho(\frac{x}{N}))$. In this way, for every macroscopic point $x \in \mathbb{R}$, the expected number of particles at the corresponding microscopic point $\mathbb{E}[\eta(\lceil Nx \rceil)]$ is approximately equal to $\varrho(x)$. It follows that the averages of large blocks around the microscopic point $\lceil Nx \rceil$ converge to $\varrho(x)$ by the LLN, hence $\pi^N \to \varrho(x)\,\mathrm{d}x$.

In the example above, it is worth noting that any distribution satisfying $\mathbb{E}[\eta(x)] = \varrho(\frac{x}{N})$ could have been chosen instead of the Poisson distribution. However, the choice of the Poisson distribution is deliberate. As discussed in Section 1.3, the product Poisson distributions with a constant parameter serve as the equilibrium measures for the IRW. Thus, we refer to the distribution in the example as a *local equilibrium measure* for this process, as it resembles the equilibrium measure locally (around microscopic points) for large N, but the parameter varies when viewed on a macroscopic scale.

1.4.2 Hydrodynamic limit

The first scaling limit typically considered in the theory of particle systems of transport is the hydrodynamic limit. The idea of the hydrodynamic limit is to provide a description of the average dynamics of the particle density at the macroscopic level. This description is generally expressed as a partial differential equation, which depicts the time-evolution of the particle density. We then refer to this PDE as the *hydrodynamic equation* of the system.

In order to explain this in more detail, we turn to an example. Consider the IRW on $\mathbb{Z}$. For every $N \in \mathbb{N}$, we start our process from a random configuration η^N and we denote by η^N_t the configuration of particles at time t. We can now consider the process of empirical measures corresponding to every configuration η^N_t. However before we do this, we will first need to rescale time. Namely, since we are rescaling space by a factor $\frac{1}{N}$, the particles need to travel on average a distance of order $\mathcal{O}(N)$ in order to affect the density on the macroscopic scale. Therefore, the number of jumps needs to increase by a factor depending on N.

Since we are considering symmetric random walkers on $\mathbb{Z}$, by the Central Limit Theorem, the particles travel a distance of order $\mathcal{O}(\sqrt{t})$ in a finite time t. Consequently, after rescaling space, time must be rescaled by a factor of N^2. This is referred to as the *diffusive time-scale* (the reason for which will be discussed later), and the trajectory of a single particle converges in distribution to that of a Brownian motion as N goes to infinity.

Remark 1.2. If instead of symmetric random walkers, we were considering particles with a drift in a certain direction, the jumps generated by this drift produce a travel distance of order $\mathcal{O}(t)$. In this case, time only needs to be rescaled by a factor of N. If we consider random walkers with both symmetric jumps and jumps with a drift, we can rescale time for both types of jumps using different factors. These types of mixed scalings will appear in Chapter 4, where we examine the scaling limits of *run-and-tumble particles* (see Section 1.5).

We now consider the empirical measure at (rescaled) time $t \geq 0$,

$$\pi_t^N := \pi^N(\eta_{N^2t}^N) = \frac{1}{N} \sum_{x \in \mathbb{Z}} \eta_{N^2t}^N(x) \cdot \delta_{\frac{x}{N}}. \tag{1.4.4}$$

Similarly as in the previous section, for every $t \geq 0$ we expect this measure to converge towards an absolutely continuous measure $\pi_t^N \to \varrho_t(x)\,\mathrm{d}x$ weakly in probability. The hydrodynamic limit is then the time-evolution of this density ϱ_t.

In the case of IRW, it turns out that this density satisfies the *heat equation*

$$\partial_t \varrho_t = \Delta \varrho_t. \tag{1.4.5}$$

The physical intuition behind this equation is the following: when particles (e.g. molecules) in the air collide against each other they create heat, and so more particles in a certain area translates to more heat. Due to the many collisions, the particles keep changing direction, making the trajectory of a single particle resemble a (rescaled) symmetric random walk. Over time, these particles spread out and areas with a high density of particles distribute their mass to areas with a low density of particles, creating what we call a *diffusion*.

As t becomes larger, the density ϱ_t approaches a constant density, i.e., the particles are spread out evenly over $\mathbb{R}$. This limit of the density as $t \to \infty$ under the heat equation dynamics is referred to as a *steady state* of the process, denoted ϱ^s, since it is a state for which $\Delta \varrho^s = 0$. The fact that the steady states of this process are the constant densities is coherent with the fact that the ergodic measures of the system also have a constant parameter. In other words, the steady states of the hydrodynamic equation are equal to the densities corresponding to the ergodic measures of the process.

One could think that by adding interactions between the particles the hydrodynamic limit would change, however this is not always the case. In fact, the hydrodynamic equations of both the SEP and SIP are also equal to the heat equation (see e.g. [61, Chapter 4] for SEP and [89] for SIP). For this reason, one could argue that the hydrodynamic limit may not provide a sufficiently detailed description of the macroscopic behavior of the process, as different processes can lead to the same hydrodynamic limit.

SKETCH OF PROOF

In actual statements regarding hydrodynamic limits, we generally prove a stronger result than the convergence of every π_t^N. Specifically, for fixed (but arbitrary) $T > 0$, we prove that the entire trajectory $\pi_{[0,T]}^N := \{\pi_t^N : t \in [0,T]\}$ converges to the deterministic trajectory of measures $\alpha := \{\varrho_t(x)\,\mathrm{d}x : t \in [0,T]\}$, where ϱ_t solves (1.4.5). Therefore we first set up the framework for the type of convergence we are after.

Denoting by $\mathcal{M}$ the space of *Radon measures*[9] on $\mathbb{R}$, we introduce the *Skorokhod space*, denoted by $D([0,T];\mathcal{M})$, of *càdlàg*[10] trajectories in $\mathcal{M}$. We denote by P^N the probability distribution of $\pi_{[0,T]}^N$, i.e.,

$$P^N(A) = \mathbb{P}(\pi_{[0,T]}^N \in A), \qquad A \subset D([0,T];\mathcal{M}). \tag{1.4.6}$$

We then want to show that $P^N \to \delta_\alpha$ weakly as $N \to \infty$. This is done in two steps.

Step 1: Tightness. Tightness of a sequence of probability measures $\{P^N\}_{N\in\mathbb{N}}$ is a necessary condition for convergence. Intuitively it means that the support of the sequence is mostly contained in a compact set, ensuring that the mass of the probability distributions does not escape to infinity. If the sequence $\{P^N\}_{N\in\mathbb{N}}$ is tight, then *Prokhorov's Theorem* tells us that every subsequence has a convergent (sub)subsequence. If one can further show that all convergent subsequences have the same limit, then the entire sequence converges to that limit − this will constitute Step 2.

---

9 Radon measures on $\mathbb{R}$ are elements of the dual space of continuous functions with compact support $(C_c(\mathbb{R}))^*$. In particular, countable sums of Dirac measures (like π^N) and absolutely continuous measures (like $\varrho(x)\,\mathrm{d}x$) are Radon measures.

10 continue a droite, limite à gauche (right-continuous, left limits)

Step 2: Uniqueness of limits. In order to show that all convergent subsequences $\{P^{N_k}\}_{k\in\mathbb{N}}$ converge to δ_α, it is important to note that α is the unique trajectory of measures in $D([0,T];\mathcal{M})$ such that for all test functions $\phi \in C_c^\infty(\mathbb{R})$

$$\langle \alpha_T, \phi\rangle - \langle \alpha_0, \phi\rangle - \int_0^T \langle \alpha_s, \Delta\phi\rangle \, \mathrm{d}s = 0, \tag{1.4.7}$$

where $\langle \alpha_t, \phi\rangle$ denotes the integral of ϕ with respect to the measure α_t. Therefore, we want to show that the trajectory $\pi^N_{[0,T]}$ solves this equation in the limit. For that we define the *Dynkin martingale*:

$$\langle \pi^N_t, \phi\rangle - \langle \pi^N_0, \phi\rangle - \int_0^t L^N \langle \pi^N_s, \phi\rangle \, \mathrm{d}s =: M_t^{N,\phi}(\pi^N_{[0,T]}), \tag{1.4.8}$$

where L^N is the generator of the IPS. This process is a *martingale* with respect to time $t \in [0,T]$ (see Section 2.3.1 for an introduction to martingales), and looks similar to the left hand side of (1.4.7). We then have to show that

$$L^N \langle \pi^N_s, \phi\rangle = \langle \pi^N_s, \Delta\phi\rangle + o(1), \tag{1.4.9}$$

which follows from direct computations, and that the martingale $M_T^{N,\phi}$ vanishes as $N \to \infty$. This last step can be done by showing that the *predictable quadratic variation* of the process vanishes. For the Dynkin martingale, the predictable quadratic variation has an explicit form given by

$$\langle M^{N,\phi}(\pi^N_{[0,T]})\rangle_t = \int_0^t \Gamma^{N,\phi}(\pi^N_s) \, \mathrm{d}s, \tag{1.4.10}$$

where $\Gamma^{N,\phi}(\pi^N_s)$ is the so-called *Carré du champ* operator, defined by

$$\Gamma^{N,\phi}(\pi^N_s) := L^N \langle \pi^N_s, \phi\rangle^2 - 2\langle \pi^N_s, \phi\rangle L^N \langle \pi^N_s, \phi\rangle. \tag{1.4.11}$$

Through some more direct computations, we can show that this operator indeed vanishes as $N \to \infty$, and so the martingale $M_T^{N,\phi}(\pi^N_{[0,T]})$ vanishes. The rest is an application of the *Portmanteau Theorem*[11].

Remark 1.3. In Chapter 2 we provide an elegant proof of a weaker hydrodynamic result for the IRW, SEP, and SIP – namely the convergence of the expectation $\mathbb{E}[\langle \pi^N_t, \phi\rangle]$. Here we will use duality to prove that the hydrodynamic equation actually follows from the convergence of the single-particle dynamics, further clarifying why the hydrodynamic equations for these three models are equal.

11 The Portmanteau Theorem gives equivalent definitions of weak convergence of measures. Since we have a weakly convergent subsequence $\{P^{N_k}\}_{k\in\mathbb{N}}$, we are able to use this theorem. We state the theorem in Section 2.1.1.

Note that the hydrodynamic equation is derived from equation (1.4.9). For some processes, we do not immediately find a *closed equation* for π_t^N, i.e., after computations, there are terms remaining that depend on η_t^N. The approach is to then replace these terms with averages over large blocks, using what is known as a *replacement lemma*. The proof of such a replacement lemma usually consists of so-called *one block* and *two blocks estimates*. We will see an example of this in Chapter 5 when we are dealing with a *weakly asymmetric* version of a multi-species exclusion process.

Relevant literature

For an overview on hydrodynamic limits, we refer to the book of De Masi and Pressuti [21] and the book of Kipnis and Landim [61]. The latter also gives a good overview of other scaling limits.

One of the first hydrodynamic results was obtained in the following paper by Morrey [84]. Later, the two most common methods for proving a hydrodynamic result were given by Guo, Papanicolou and Varadhan (entropy method) in [51] and by Yau (relative entropy method) in [124]. Furthermore, Seppäläinen gives a clear proof for the hydrodynamic limit of the SEP in [104].

Additionally, there exist hydrodynamic results for extensions of models we have already seen, for instance particle systems with boundary conditions [5, 41], on random environments [39, 85], and on manifolds [116].

1.4.3 Fluctuations

In the previous section we have seen that the hydrodynamic limit can be considered as an infinite-dimensional extension of the LLN in the framework of interacting particle systems. The following natural question then arises: can we find a similar extension for the CLT? It turns out that this extension does exist and is referred to as the fluctuations of the system.

Recall that in order to go from the LLN to the CLT, we subtract the limit of the LLN and introduce a scaling of $\frac{1}{\sqrt{N}}$. Therefore, the fluctuations arise from the limit of the following process

$$Y_t^N = \frac{1}{\sqrt{N}} \sum_{x\in\mathbb{Z}} \left(\eta^N_{N^2 t}(x) - \varrho_t\right) \cdot \delta_{\frac{x}{N}}, \tag{1.4.12}$$

where ϱ_t solves the hydrodynamic equation of the process, with starting value ϱ_0 equal to the density corresponding to the sequence of starting configurations η^N. Often we pick $\varrho_t = \varrho^s$ equal to the steady state of the process, i.e., we assume the starting sequence η^N to be sampled from the ergodic measures of the process. This is to avoid working with time-dependent Markov processes, which can sometimes complicate the analysis. In this case we refer to the limit of Y_t^N as the *stationary fluctuations* (or *equilibrium fluctuations* if the ergodic measures are also equilibrium measures).

Since we expect the limiting process to be very rough, we consider this process in the space of *distributions*[12]. Furthermore, in accordance with the CLT, we expect the process Y_t^N to converge to a *Gaussian* distribution. In the models outlined in Section 1.2, the equilibrium fluctuations are represented by a distribution-valued *Ornstein-Uhlenbeck process* (as we will see in (1.4.14)).

Ornstein-Uhlenbeck process. As a process on $\mathbb{R}$, the Ornstein-Uhlenbeck process is the solution to the *stochastic differential equation* (SDE) given by

$$dx_t = -ax_t\,dt + \sigma\,dB_t. \tag{1.4.13}$$

Here $a > 0$ and $\sigma > 0$ are constants and B_t a standard Brownian motion. The intuition behind this process is as follows: while Brownian motion generates noise, the drift term, $-ax_t\,dt$, pushes the process back toward the origin, with the strength of this drift increasing as the process moves further from the origin. A physical analogy for the Ornstein-Uhlenbeck process is a child playing with a spring: as the child repeatedly disturbs the spring, creating noise, the spring continuously tries to return to its equilibrium position.

Due to its dynamics, one would expect the Ornstein-Uhlenbeck process to remain close to the origin. This is confirmed by the fact that it has a unique invariant probability measure, given by the normal distribution $\mathcal{N}(0, \frac{\sigma^2}{2a})$, which indicates that, at any given time, the process is likely to be found within a compact region around the origin. In this sense it distinguishes itself from a Brownian motion, which has no invariant probability measure and escapes towards infinity as time progresses.

For the IRW, SEP, and SIP, the trajectory of the process Y_t^N in (1.4.12) , with $\varrho_t = \varrho$ constant, converges (in distribution) to the solution of a *stochastic partial differential equation* (SPDE) of the following form:

$$dY_t = \Delta Y_t\,dt + \sqrt{2\chi(\varrho)}\nabla\,d\mathscr{W}_t, \tag{1.4.14}$$

where $d\mathscr{W}_t$ is a so-called *space-time white noise* (a distribution-valued Gaussian process), and $\chi(\varrho)$ is a constant that is dependent on the model. The "gradient" of the white noise $\nabla\,d\mathscr{W}_t$ is the process that satisfies $\langle \nabla\,d\mathscr{W}_t, \phi\rangle = -\langle d\mathscr{W}_t, \nabla\phi\rangle$ for all $\phi \in C_c^\infty(\mathbb{R})$.

12 distributions, or sometimes called *generalized functions*, are elements of the dual space of smooth functions with compact support $(C_c^\infty(\mathbb{R}))^*$. They are more general than Radon measures and allow for rougher "densities".

This form should indeed remind the reader of the SDE of an Ornstein-Uhlenbeck process as written in (1.4.13). Moreover, despite working with a distribution-valued Ornstein-Uhlenbeck process instead of an $\mathbb{R}$-valued one, the intuition remains the same: the noise term drives the process away from the steady state, while the drift term, corresponding to the hydrodynamic equation, pushes it back towards equilibrium.

SKETCH OF PROOF

The proof of the equilibrium fluctuations follows the same approach as that of the hydrodynamic limit: we first establish the tightness of the sequence of probability distributions Q^N for $Y^N_{[0,T]} := \{Y^N_t : t \in [0,T]\}$, and then we prove that any subsequence of Q^N has the same limit Q, which corresponds to the solution $Y_{[0,T]} := \{Y_t : t \in [0,T]\}$ of (1.4.14). However, there are two key differences.

The first difference is that the limiting object is not the Dirac measure of a deterministic trajectory, but instead the probability distribution of a distribution-valued stochastic process. As a consequence, the martingale $M^{N,\phi}_T(Y^N_{[0,T]})$, defined as in (1.4.8), does not vanish as $N \to \infty$, but converges to a new process. This limiting process appears in the *martingale problem* corresponding to Q, which states that Q is the unique probability distribution such that the following two processes are martingales:

$$\begin{aligned} M^{\phi}_t(Y_{[0,T]}) &:= \langle Y_t, \phi\rangle - \langle Y_0, \phi\rangle - \int_0^t \langle Y_s, \Delta\phi\rangle \, \mathrm{d}s, \\ N^{\phi}_t(Y_{[0,T]}) &:= M^{\phi}_t(Y_{[0,T]})^2 - 2t\chi(\varrho)\langle \nabla\phi, \nabla\phi\rangle_{L^2(\mathrm{d}x)}. \end{aligned} \tag{1.4.15}$$

In the proof we then consider the following two processes, both of which are known to be martingales:

$$\begin{aligned} M^{N,\phi}_t(Y^N_{[0,T]}) &:= \langle Y^N_t, \phi\rangle - \langle Y^N_0, \phi\rangle - \int_0^t L^N \langle Y^N_s, \phi\rangle \, \mathrm{d}s, \\ N^{N,\phi}_t(Y^N_{[0,T]}) &:= M^{N,\phi}_t(Y^N_{[0,T]})^2 - \langle M^{N,\phi}(Y^N_{[0,T]})\rangle_t. \end{aligned} \tag{1.4.16}$$

with the predictable quadratic variation $\langle M^{N,\phi}(Y^N_{[0,T]})\rangle_t$ defined as in (1.4.10). The goal is to then prove that these martingales converge to the martingales in (1.4.15).

The second key difference is that instead of working with trajectories in $\mathcal{M}$, we work with trajectories in the space of distributions $(C^\infty_c(\mathbb{R}))^*$. The main issue here is that this space is not metrizable, which is needed in order to use Portmanteau's theorem. However, there is a fix around this problem, as introduced in [117], using the continuity of the process $t \mapsto Y_t$.

Relevant literature

The first results in fluctuation theory were obtained in the papers by Martin-Löf [81] and Rost [99]. The book by Kipnis and Landim [61, Chapter 11] gives a proof of the fluctuations of the SEP on the torus. The first non-stationary fluctuations were obtained in [22], and more recent results of this are given in [40, 60].

1.4.4 Large deviations

Besides the LLN and CLT, one is often interested in the large deviations around the mean value. This concept can once again be extended to the world of IPS, where we consider the large deviations around the hydrodynamic limit. For readers less familiar with large deviation theory, a brief introduction to the topic is first provided in the framework of random variables.

Large deviations for random variables

Consider a sequence of i.i.d. random variables $X_1, X_2, \ldots$ with mean μ and variance σ^2, then the LLN tells us that the averages converge to the mean, i.e.,

$$\frac{1}{N}\sum_{i=1}^{N} X_i \to \mu. \tag{1.4.17}$$

The CLT then quantifies the *normal deviations* (of order $\frac{1}{\sqrt{N}}$) around this mean, and shows that they converge to a normal distribution, i.e.,

$$\frac{1}{\sqrt{N}}\sum_{i=1}^{N}(X_i - \mu) \to \mathcal{N}(0, \sigma^2). \tag{1.4.18}$$

The theory of large deviations, much like the CLT, also concerns itself with deviations around the mean, however it focuses on deviations of order 1 (hence they are referred to as "large"). More specifically, it quantifies the rate at which the probability that the average is close to any value other than μ converges to zero. It is often the case that this rate is exponential and can be written as

$$\mathbb{P}\left[\frac{1}{N}\sum_{i=1}^{N} X_i \approx \alpha\right] \approx e^{-N\mathcal{I}(\alpha)}. \tag{1.4.19}$$

Here the $\approx$-notation can be made more rigorous, however for this introduction it is enough to interpret this as an asymptotic result for the probability that the average is close to α for large N. If the above holds, the sequence of random variables is said to satisfy a *large deviation principle* (LDP) at rate N, with so-called *rate function* $\mathcal{I}$.

Example 1.4. As an introductory example, consider independent coin-flips, where every X_i is 1 if the i'th coin lands on heads and 0 otherwise. The rate function is then given by $\mathcal{I}(\alpha) = \alpha \log 2\alpha + (1-\alpha)\log 2(1-\alpha)$ (see e.g. Dembo and Zeitouni [25]). Note here that $\mathcal{I}(\alpha) = 0$ if and only if $\alpha = \frac{1}{2}$, and positive for any other value $\alpha \in [0,1]$. This is coherent with the fact that $\frac{1}{2}$ is the mean itself, and so the probability should be approximately 1 when $\alpha = \frac{1}{2}$, and 0 otherwise.

A fundamental theorem in the topic of large deviations for random variables is *Cramér's Theorem*. It tells us that an LDP holds for an i.i.d. sequence of random variables if the moment-generating function $\Lambda(t) = \mathbb{E}\left[e^{tX}\right]$ is finite in an open interval around $t = 0$. The rate function is then given by the so-called *Legendre transform* of Λ, i.e.,

$$\mathcal{I}(\alpha) = \sup_{t\in\mathbb{R}} \{\alpha t - \log \Lambda(t)\}. \tag{1.4.20}$$

The proof of Cramér's Theorem relies on tilting the distribution $\mathbb{P}$ of X through the *Cramér transform*

$$\frac{\mathrm{d}\widehat{\mathbb{P}}}{\mathrm{d}\mathbb{P}}(x) = \frac{1}{\Lambda(t^*)} e^{t^* x}, \tag{1.4.21}$$

where t^* is the value of t that maximizes the expression $\alpha t - \log \Lambda(t)$ in the definition of $\mathcal{I}(\alpha)$. It can then be shown that this new distribution $\widehat{\mathbb{P}}$ has expectation α, making the large deviation event $\left\{\frac{1}{N}\sum_{i=1}^N X_i \approx \alpha\right\}$ typical under the tilted measure $\widehat{\mathbb{P}}$. The rate function $\mathcal{I}(\alpha)$ now represents the cost for tilting the measure.

Large deviations for IPS

As we saw in the previous section, large deviations of random variables focus on deviations around the LLN of order 1. The analogue in the context of IPS would then be the deviations of the empirical density around the hydrodynamic limit. For this we again consider the whole trajectory of empirical measures $\pi^N_{[0,T]}$. We are then interested in the rate function $\mathcal{I}$ such that the following holds

$$\mathbb{P}\left[\pi^N_{[0,T]} \approx \hat{\alpha}\right] \approx e^{-N\mathcal{I}(\hat{\alpha})}, \tag{1.4.22}$$

where $\hat{\alpha}$ is now an element of the Skorokhod space $D([0,T];\mathcal{M})$.

Remark 1.5. Even though we are considering $\hat{\alpha} \in D([0,T];\mathcal{M})$, it is typically the case that $\mathcal{I}(\hat{\alpha})$ is only finite if $\hat{\alpha}$ is a trajectory of absolutely continuous measures, i.e., $\hat{\alpha} := \{\widehat{\varrho}_t(x)\,\mathrm{d}x : t \in [0,T]\}$. Therefore, at least for this introduction, we will assume $\hat{\alpha}$ to be of this type.

In general, the rate function $\mathcal{I}$ can be split into two parts: the *static* part and the *dynamic* part,

$$\mathcal{I}(\hat{\alpha}) = h(\hat{\alpha}_0) + \mathcal{I}_{tr}(\hat{\alpha}). \tag{1.4.23}$$

The static part $h(\hat{\alpha}_0)$ deals with the large deviations of the starting measure. For example, suppose we start from a product Poisson measure with slowly varying parameter $\varrho(\frac{x}{N})$ (see Example 1.1). While the typical event is that the empirical measure at time $t = 0$ converges to the measure $\varrho(x)\,\mathrm{d}x$, a large deviation event would be that it converges to some other measure $\widehat{\varrho}(x)\,\mathrm{d}x$. The rate function is then given by

$$h(\hat{\alpha}_0) = \int_{\mathbb{R}} \widehat{\varrho}(x) \log \frac{\widehat{\varrho}(x)}{\varrho(x)}\,\mathrm{d}x - \int_{\mathbb{R}} (\widehat{\varrho}(x) - \varrho(x))\,\mathrm{d}x \tag{1.4.24}$$

The dynamic part $\mathcal{I}_{tr}(\hat{\alpha})$ in (1.4.23) gives the dynamic cost of the deviating trajectory. In the case of the IRW, this part of the rate function has the following *variational formula*:

$$\mathcal{I}_{tr}(\hat{\alpha}) = \sup_G \left\{ \ell(\hat{\alpha}, G) - \frac{1}{2}\int_0^T \langle \hat{\alpha}_t, (\nabla G(\cdot,t))^2 \rangle\,\mathrm{d}t \right\}. \tag{1.4.25}$$

Here, the supremum is taken over smooth functions G, and $\ell(\hat{\alpha}, G)$ is a linear operator given by

$$\ell(\hat{\alpha}, G) = \langle \hat{\alpha}_T, G(\cdot, T)\rangle - \langle \hat{\alpha}_0, G(\cdot, 0)\rangle - \int_0^T \langle \hat{\alpha}_t, (\partial_t + \Delta) G(\cdot, t)\rangle\,\mathrm{d}t. \tag{1.4.26}$$

The integral in (1.4.25) is non-negative and quadratic, and hence can be viewed as the square of a norm:

$$||G||^2_{\mathcal{H}(\hat{\alpha})} := \int_0^T \langle \hat{\alpha}_t, (\nabla G(\cdot,t))^2 \rangle\,\mathrm{d}t, \tag{1.4.27}$$

where $\mathcal{H}(\hat{\alpha})$ is the Hilbert space consisting of L^2-functions where this norm is finite.

Remark 1.6. Analogous to Example 1.4, we would expect that if α solves the heat equation (i.e., the hydrodynamic equation of the IRW) then $\mathcal{I}_{tr}(\alpha) = 0$. Indeed, if α solves the heat equation then $\ell(\alpha, G) = 0$ for every G. Consequently, the supremum in (1.4.25) is achieved for functions G that minimize the norm $||G||_{\mathcal{H}(\alpha)}$. This is certainly the case for $G \equiv 0$, and it follows that $\mathcal{I}_{tr}(\alpha) = 0$. Moreover, it can be shown that this is the only root of $\mathcal{I}_{tr}$, confirming that the heat equation trajectory is the only zero cost trajectory for the dynamics.

SKETCH OF PROOF

A core idea of the proof is based on Cramer's theorem, where we introduced a tilted measure in which the large deviation event becomes typical. A major difference in this context of IPS, is that we also need to identify the possible deviating trajectories. For example, due to conservation of mass, we can not allow trajectories where the total mass of particles changes over time. For these trajectories the rate function will be equal to $\mathcal{I}(\hat{\alpha}) = \infty$. Therefore, for trajectories where $\mathcal{I}(\hat{\alpha}) < \infty$, we need to identify a tilted process with probability distribution $\widehat{P}^N$ such that

$$\widehat{P}^N(\{\pi^N_{[0,T]} \approx \hat{\alpha}\}) \approx 1. \tag{1.4.28}$$

Once this is found, the rate function can be determined, at least heuristically, as follows:

$$\begin{aligned} \mathcal{I}(\hat{\alpha}) &\approx -\lim_{N\to\infty} \frac{1}{N} \log \mathbb{E}_{P^N} \left[\mathbb{1}(\{\pi^N_{[0,T]} \approx \hat{\alpha}\}) \right] \\ &\approx -\lim_{N\to\infty} \frac{1}{N} \log \mathbb{E}_{\widehat{P}^N} \left[\mathbb{1}(\{\pi^N_{[0,T]} \approx \hat{\alpha}\}) \frac{\mathrm{d}P^N}{\mathrm{d}\widehat{P}^N} \right] \\ &\approx \lim_{N\to\infty} \frac{1}{N} \mathbb{E}_{\widehat{P}^N} \left[\log \frac{\mathrm{d}\widehat{P}^N}{\mathrm{d}P^N} \right], \end{aligned} \tag{1.4.29}$$

where $\frac{\mathrm{d}\widehat{P}^N}{\mathrm{d}P^N}$ denotes the *Radon-Nikodym derivative* of the two probability distributions. Although the final step initially appears to provide only a lower bound, it turns out that the two expressions do coincide if we chose the correct tilted process.

The last line of (1.4.29) is called the *relative entropy*[13] of $\widehat{P}^N$ with respect to P^N. The static part of the large deviation rate function, as given in (1.4.24), follows directly from computing the relative entropy of two product Poisson measures with respective densities $\varrho(x)$ and $\widehat{\varrho}(x)$. For the dynamic part of the rate function, we must first introduce the weakly asymmetric version of the model.

Weakly asymmetric model. Recall that the IRW consists of particles that jump to the right or left with an equal rate of 1. In the weakly asymmetric model, we introduce a time-dependent *potential*, which is a smooth function $H : \mathbb{R} \times [0, T] \to \mathbb{R}$, and we perturb the rates at every time $t \in [0, T]$ such that the rate to jump from x to a neighboring site $x \pm 1$ is now equal to $\exp(H(\frac{x\pm1}{N}, t) - H(\frac{x}{N}, t))$.

Note that, since H is continuous, as $N \to \infty$ the rates converge to 1 again (hence "weakly" asymmetric). Nonetheless, the hydrodynamic limit of this model is no longer the heat equation, but is instead given by

$$\partial_t \widehat{\varrho}_t = \Delta \widehat{\varrho}_t + \nabla \left(\widehat{\varrho}_t \cdot \nabla H(\cdot, t) \right). \tag{1.4.30}$$

It turns out that the weakly asymmetric model provides exactly the right perturbations needed for large deviations. Namely, using an argument based on the

---

13 Relative entropy, or also called the *Kullback-Leibler divergence*, is a measure for the difference between two probability distributions. It appears in fields such as information theory, statistics, and also in the study of large deviations (for example in *Sanov's Theorem* [26, Theorem II.2]).

Riesz Representation Theorem, we can show that for trajectories $\hat{\alpha}$ where $\mathcal{I}_{tr}(\hat{\alpha}) < \infty$, there exists a potential $H \in \mathcal{H}(\hat{\alpha})$ such that the density of $\hat{\alpha}$ solves equation (1.4.30). Additionally, it can be shown that the supremum in (1.4.25) is actually achieved for this H, leading to the expression:

$$\mathcal{I}_{tr}(\hat{\alpha}) = \tfrac{1}{2}||H||^2_{\mathcal{H}(\hat{\alpha})}. \tag{1.4.31}$$

Using the *Girsanov formula* (or the main result of the following paper [90]), we are able to derive a formula for the Radon-Nikodym derivative of the probability distribution of $\pi^N_{[0,T]}$ under the weakly asymmetric dynamics, denoted $\widehat{P}^{N,H}$, with respect to that under the normal dynamics P^N:

$$\begin{aligned} \log \frac{\mathrm{d}\widehat{P}^{N,H}}{\mathrm{d}P^N} &= N\langle \pi^N_T, H(\cdot,T)\rangle - N\langle \pi^N_0, H(\cdot,0)\rangle \\ &\quad - \int_0^T e^{-N\langle \pi^N_t, H(\cdot,t)\rangle}(L^N - \partial_t)e^{N\langle \pi^N_t, H(\cdot,t)\rangle}\,\mathrm{d}t. \end{aligned} \tag{1.4.32}$$

The goal is then to establish that

$$e^{-N\langle \pi^N_t, H(\cdot,t)\rangle} L^N e^{N\langle \pi^N_t, H(\cdot,t)\rangle} = N\langle \pi^N_t, \Delta H(\cdot,t)\rangle + \frac{1}{2}N\langle \pi^N_t, (\nabla H(\cdot,t))^2\rangle + \mathcal{O}(1), \tag{1.4.33}$$

so that, by substituting this into (1.4.32), the logarithm of $\frac{\mathrm{d}\widehat{P}^N}{\mathrm{d}P^N}$ resembles the variational formula of $\mathcal{I}_{tr}$ given in (1.4.25).

Relevant literature

An introduction to the field of large deviations can be found in the book by Den Hollander [26]. Furthermore, the book by Feng and Kurtz [35] gives more advanced techniques of proving a large deviation principle.

The first instance of a large deviation principle being proven for an IPS is in the paper by Kipnis, Olla and Varadhan [62], where they prove a large deviation principle for the SEP. This result has also been included in the book by Kipnis and Landim [61, Chapter 10].

More recent results on large deviations are for example given for the SEP with different boundary conditions [42, 43, 69]

1.5 Multi-layer particle systems

The main topic of this thesis is multi-layer particle systems. Instead of looking at particles on one lattice, in a multi-layer particle process we consider multiple copies of the same lattice, which can be viewed as the “layers” of the system.

Particles can exhibit different behavior depending on which layer they occupy, hence we refer to the layer at which a particle resides as the *internal state* of the particle.

For instance, if we consider particles moving on $\mathbb{Z}$, then in a multi-layer setting the particles live on $V := \mathbb{Z} \times S$ where S is referred to as the *internal state space*. A particle at site $(x, \sigma) \in V$ then has position $x \in \mathbb{Z}$ and internal state $\sigma \in S$. Particle configurations on this space are now given by functions $\eta : V \to \mathbb{N}$.

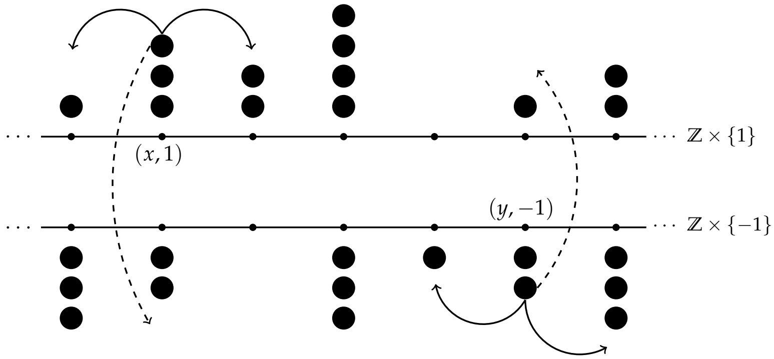

Figure 6: An example of a multi-layer particle process on $\mathbb{Z} \times \{-1, 1\}$. Here $\eta(x, 1) = 3$ and $\eta(y, -1) = 2$. Note that apart from nearest-neighbor jumps on $\mathbb{Z}$, jumps to the other layer are possible.

1.5.1 *Active particles (equilibrium vs. non-equilibrium)*

The primary motivation for studying multi-layer particle systems lies in the investigation of *active particles*, which has been a topic of significant interest in both physics and mathematics (see e.g. [32, 64, 72, 80, 87, 105]). Unlike passive particles, which move only due to external forces (think of dust in the air or ink in water), active particles exhibit *self-propulsion*, i.e., they generate their own motion through energy consumption (think of birds in a flock or people walking in the streets). This consumption of energy drives the system out of equilibrium, as a continuous source of energy is needed in order to maintain the dynamics. Therefore active particles play a role in the study of *non-equilibrium* systems.

For a system in equilibrium, there is a well-established understanding of its behavior, largely due to the existence of the Boltzmann-Gibbs distribution

$$\mathbb{P}(X = x) = \frac{1}{Z_\beta} e^{-\beta E(x)}. \tag{1.5.1}$$

Here β is the inverse temperature, $E(x)$ is the energy of a state x and Z_β is the partition function (which is needed to normalize the distribution). A process corresponding to this distribution is always drawn to states where the energy is minimized.

For systems out of equilibrium, no such general theory exists (yet), and models are analyzed on a case-by-case basis. In addition to introducing activity of particles, alternative methods to produce non-equilibrium behavior include applying an external field that drives the particles in a certain direction, and using "reservoirs" (boundaries that inject and remove particles) with differing densities, thereby creating a current from high to low density.

The reason why people are interested in non-equilibrium systems is that many real-world systems (chemical reactions, living cells, crowd dynamics) operate outside of equilibrium. Furthermore, certain behaviors can emerge in non-equilibrium systems that do not occur in equilibrium (uphill diffusion, flocking, persistent currents). Lately, progress is being made in understanding non-equilibrium systems (see e.g. [11, 113, 114]).

1.5.2 *Run-and-tumble particles*

A toy model for active particles is the *run-and-tumble* particles. These particles jump in a preferred direction according to exponential clocks, but every particle also has an additional exponential clock that, when it rings, causes them to change direction. This results in the motion shown in Figure 7. This type of motion can be found mostly in biological models (for example in E. coli bacteria [91] or motor proteins [54]).

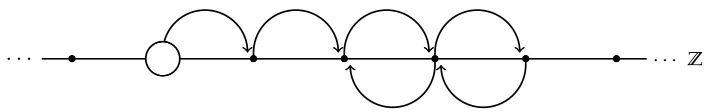

Figure 7: The trajectory of a single run-and-tumble particle on $\mathbb{Z}$. After making several jumps to the right it switches direction and begins jumping to the left.

Remark 1.7. Often simple symmetric random walk jumps are added to the model to represent particle collisions.

This model is not yet a Markov process: knowing the position of a particle alone is not enough to determine its jump rate in a given direction. To do so, we must also account for its current preferred direction. Therefore we model these particles on a multi-layer setting, where every layer (or internal state) corresponds to a possible preferred direction.

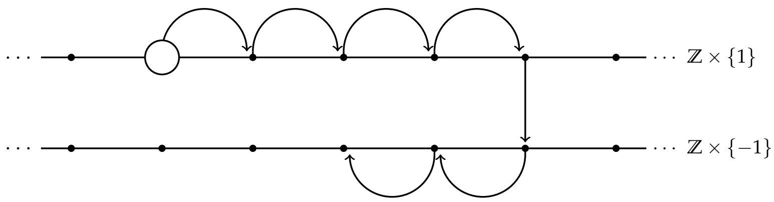

Figure 8: The trajectory of Figure 7 on the multi-layer space $\mathbb{Z} \times \{-1, 1\}$.

When considering particle configurations of run-and-tumble particles, the empirical measure is given by

$$\pi^N = \frac{1}{N} \sum_{(x,\sigma)\in V} \eta(x,\sigma) \cdot \delta_{(\frac{x}{N},\,\sigma)}, \tag{1.5.2}$$

which is a measure on the space $\mathbb{R} \times S$. When letting $N \to \infty$, this may converge to a measure with a density $\varrho(x, \sigma)$. The hydrodynamic limit in the multi-layer setting is then about finding a system of PDE's (one for every $\sigma \in S$) that describes the evolution of this density.

However, the original run-and-tumble particle model of interest did not include any layers but was instead a model of particles on $\mathbb{Z}$ (see Figure 7). In order to return to this scenario, we sum over the layers

$$\eta(x) := \sum_{\sigma\in S} \eta(x,\sigma), \tag{1.5.3}$$

which provides the total number of particles at position $x \in \mathbb{Z}$. The real ambition is to find results for this process, which we hope to extract from the multi-layer process. For example, in the hydrodynamic limit we can try to find a PDE to describe the behavior of the *total density* $\varrho_t(x) := \sum_{\sigma\in S} \varrho_t(x,\sigma)$. In Chapters 4 and 6, we explore this objective in detail and derive results of this type for the case of two layers.

1.5.3 *Multi-species particle systems*

Another type of processes that is closely related to multi-layer systems is *multi-species* systems. Up until this point we have viewed particles as indistinguishable, but in the multi-species case, particles can belong to different species. Every species can exhibit different behavior and may also interact differently with other species.

One example of such a process is the *multi-species stirring process*, which is a multi-species version of the SEP (we will also consider this process in Chapter 5). In this process there is exactly one particle at every site but the particles can be of different species (usually one species, denoted by 0, is seen as the empty sites of the process). Any two nearest-neighbor particles then exchange place with each other with equal rate.

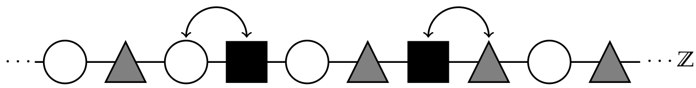

Figure 9: An example of a configuration in the multi-species stirring process with 3 species: 0 (circle), 1 (triangle) and 2 (square). Any two nearest-neighbor particles switch with rate 1.

1.5.4 *Relevant literature*

For an overview of active processes, we refer to the paper by Demaerel and Maes [23] and Gommper et al. [48]. In the book by Tailleur et al. [112] an overview of non-equilibrium statistical physics and active matter is given.

For literature on run-and-tumble particles, there is are numerous physics and mathematics papers on the behavior of a single run-and-tumble particle (see e.g. [53, 105, 106, 118]). Results on multi-layer models are rather limited at present, but some are given in [3, 32, 37, 101].

1.6 OUTLINE OF THIS THESIS

The rest of this thesis is organized as follows:

In Chapter 2 we introduce the mathematical background required for this thesis in a rigorous manner. The topics discussed include Markov semigroups and generators, path-space convergence, ergodic theory, martingales, couplings, and duality.

In Chapter 3 we introduce three types of multi-layer particle systems; the multi-layer exclusion process, the multi-layer inclusion process and the run-and-tumble particle process. We then characterize the ergodic measures with a finite moment condition for these three processes using duality and successful couplings. This chapter is based on [97].

In Chapter 4 we study the hydrodynamic limit and the stationary fluctuations of the multi-layer run-and-tumble particle process, and use them to infer the same scaling limits for the total density. Furthermore, by an application of Schilder's theorem, we find a large deviation result for the fluctuation field of the total density. This chapter is based on [98].

In Chapter 5 we establish a large deviation principle for the multi-species stirring process. The method of proof involves studying the hydrodynamic limit of a weakly asymmetric process and a superexponential estimate. This chapter is based on [18].

In Chapter 6 we return to the multi-layer setting and establish a large deviation principle for the run-and-tumble particle process on two layers with an added mean-field interaction, meaning that the switching between the layers depends on the magnetization of the process. We end with a first step towards an explicit large deviation principle of the total density. This chapter is based on [93].

2

MATHEMATICAL BACKGROUND

In this chapter we introduce the mathematical concepts and tools needed to understand this thesis. This chapter is based on the book of Liggett [76] and the lecture notes of Redig [96]. Readers who are familiar with Markov process theory and ergodic theory can skip to Chapter 3.

2.1 CONTINUOUS-TIME MARKOV PROCESSES

As was mentioned in Section 1.2, the type of models that we consider are Markov processes. In this section we give an introduction to the basics of Markov process theory. Some detailed references on this topic include the book of Blumenthal and Getoor [10] and the book of Ethier and Kurtz [34].

2.1.1 *Markov processes*

Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space and let $X = \{X_t : t \geq 0\}$ be a stochastic process on this space.

Definition 2.1. We call a stochastic process **càdlàg** (continuité a droite, limité a gauche) if it is right-continuous and has left-limits. The space of all càdlàg trajectories on Ω with $t \in [0, T]$ is called the **Skorokhod space** and is denoted by $D([0, T]; \Omega)$.

Throughout this thesis, we assume all our stochastic processes are cádlág. The space of continuous trajectories is a subspace of the Skorokhod space, and is often denoted as $C([0, T]; \Omega)$. More information on the Skorokhod space and the Skorokhod metric turning it into a metric space can be found in e.g. [104, Appendix A.2.2].

Definition 2.2. A **filtration** on $(\Omega, \mathscr{F}, \mathbb{P})$ is a family of σ-algebras $\{\mathscr{F}_t : t \geq 0\}$ such that for every $s < t$ we have that $\mathscr{F}_s \subset \mathscr{F}_t \subset \mathscr{F}$. We call a stochastic process X **adapted** with respect to this filtration if X_t is $\mathscr{F}_t$-measurable for every $t \geq 0$.

Intuitively, a filtration can be understood as an increasing collection of information over time, where $\mathscr{F}_t$ represents all the information up to time t. An adapted process then means that the value of X_t is completely determined by the information contained in $\mathscr{F}_t$ and no further information is needed.

The typical filtration for which the process X is adapted is the so-called **natural filtration** $\mathscr{F}_t = \sigma(X_s : 0 \leq s \leq t)$, i.e., the σ-algebra generated by the process itself up to time t. Unless mentioned otherwise, from now on we assume that we are working with the natural filtration.

Definition 2.3. A **Markov process** is a stochastic process X satisfying the **Markov property**, stating that for every bounded measurable function $f : \Omega \to \mathbb{R}$ and $s < t$,

$$\mathbb{E}[f(X_t)|\mathcal{F}_s] = \mathbb{E}[f(X_t)|X_s], \qquad \mathbb{P} - a.s. \tag{2.1.1}$$

In intuitive terms, given all the information of the past, a Markov process only needs to consider the present state in order to determine the behavior of the process in the future. We therefore also refer to Markov processes as being **memoryless**.

Definition 2.4. A Markov process X is called **time-homogeneous** if for every bounded measurable function $f : \Omega \to \mathbb{R}$, $x \in \Omega$ and $s, t \geq 0$,

$$\mathbb{E}[f(X_{t+s})|X_s = x] = \mathbb{E}[f(X_t)|X_0 = x]. \tag{2.1.2}$$

Most Markov processes considered in this thesis are time-homogeneous, however the weakly asymmetric model introduced in Section 1.4.4 is an example of a process that is not time-homogeneous due to its dependence on a time-dependent potential.

Below we give two examples of Markov processes.

Example 2.1 (Jump processes)**.** A **Markov jump process** on Ω is a Markov process that transitions between states through discrete jumps occuring after exponential waiting times. Given $x, y \in \Omega$, the process is defined through the **transition rates** denoted by $c(x, y)$. If the process is at state x at a given time, each possible transition to $y \neq x$ is governed by an exponential clock with parameter $c(x, y)$. The first clock to ring determines the next jump, and the process transitions from x to the corresponding site y. As a result, the total waiting time at x follows an exponential distribution with parameter $c(x) = \sum_y c(x, y)$, and the probability that, at a jump time, the chain jumps from x to y is given by $\frac{c(x,y)}{c(x)}$.

A special case of a jump process is the **Simple Symmetric Random Walk (SSRW)** on $\mathbb{Z}$. At each site, the random walker can jump one step to the left of the right with equal rate. Usually the rates are taken $c(x, x+1) = c(x, x-1) = \frac{1}{2}$.

Example 2.2 (Brownian motion)**.** A **Brownian motion** $B = \{B_t : t \geq 0\}$ is a stochastic process such that $B_0 = 0$, $B_t - B_s \sim \mathcal{N}(0, t-s)$ for $t > s$, and that increments of non-overlapping intervals are independent. This last property ensures that a Brownian motion is Markov. Unlike a jump process, a Brownian motion is almost surely continuous and has trajectories in the path space $C([0, T]; \mathbb{R})$. However in Example 2.8 we will show that a Brownian motion can be obtained as a limit of a SSRW.

2.1.2 *Markov semigroups*

DEFINITION 2.5. Let X be a time-homogeneous Markov process, then the corresponding **Markov semigroup** is a family of bounded linear operators $\{S_t : t \geq 0\}$ defined by

$$S_t f(x) = \mathbb{E}[f(X_t)|X_0 = x], \tag{2.1.3}$$

acting on functions f in some suitable normed function space $(C, ||\cdot||)$.

The function space $(C, ||\cdot||)$ is dependent on the space Ω and is often some subclass of the continuous functions on Ω (smooth functions, functions with compact support, functions vanishing at infinity, etc.) together with the supremum norm $||f||_\infty = \sup_x |f(x)|$. However, if the process has an invariant measure μ (see Section 2.2 for the definition of an invariant measure), we can always extend the semigroup to $L^p(\mu)$ for $p \geq 1$. As a consequence, we can allow for indicator functions, $\mathbb{1}_A$ for $A \in \mathscr{F}$, in which case the semigroup satisfies

$$S_t \mathbb{1}_A(x) = \mathbb{P}(X_t \in A|X_0 = x). \tag{2.1.4}$$

This relation tells us that the Markov process X uniquely determines the semigroup.

An alternative way of defining a Markov semigroup is as a family of bounded linear operators satisfying the following properties for all $f \in C$

S1. **Identity at** 0: $S_0 f = f$,

S2. **Constant preserving**: $S_t 1 = 1$,

S3. **Positivity preserving**: if $f \geq 0$ then $S_t f \geq 0$,

S4. **Contractivity**: $||S_t f|| \leq ||f||$,

S5. **Strong continuity**: $\lim_{t\downarrow 0} ||S_t f - f|| = 0$

S6. **Semigroup property**: $S_{t+s} f = S_t(S_s f)$.

REMARK 2.3. The strong continuity property S5 is usually not included in the definition, however we will only be working with semigroups satisfying this.

The semigroup defined in (2.1.3) immediately satisfies S1-S5, and S6 follows from the fact that we have a time-homogeneous Markov process, namely

$$\begin{aligned} S_{t+s} f(x) &= \mathbb{E}[f(X_{t+s})|X_0 = x] \\ &= \mathbb{E}[\mathbb{E}[f(X_{t+s})|X_t]|X_0 = x] \\ &= \mathbb{E}[S_s f(X_t)|X_0 = x] = S_t(S_s f)(x). \end{aligned} \tag{2.1.5}$$

The way to go back from a Markov semigroup satisfying S1-S6 to a Markov process is through (2.1.4), where we are able to reconstruct X through the behavior of the semigroup on indicator functions. This means that there is a one-to-one connection between Markov semigroups defined through S1-S6 and time-homogeneous Markov processes.

2.1.3 *Markov generators*

The semigroup property of the Markov semigroup suggests the existence of a linear operator L for which the relation $S_t = e^{tL}$ holds formally. This operator would then be defined as follows,

$$Lf = \lim_{t \downarrow 0} \frac{S_t f - f}{t}, \tag{2.1.6}$$

i.e., you can view L as the 'derivative' of S_t at time 0. This operator is known as a **Markov generator**, and it encodes the rate of change of the process in an infinitesimal time interval.

However, a priori, the limit in (2.1.6) does not necessarily exist. Therefore we define the domain of this operator as the set of functions f for which this limit does exists,

$$D(L) = \left\{ f \in C : \lim_{t \downarrow 0} \frac{S_t f - f}{t} \text{ exists in } ||\cdot|| \right\}. \tag{2.1.7}$$

It is also possible to define a Markov generator without first defining the Markov semigroup. In order for any linear operator $L : D(L) \to C$, with domain $D(L) \subset C$, to be a Markov generator, it has to satisfy the following properties.

G1. $1 \in D(L)$ and $L1 = 0$.

G2. $D(L)$ is dense in C.

G3. L is a closed operator, i.e., $\{(f, Lf) : f \in D(L)\}$ is closed.

G4. For all $\lambda > 0$ the range of $(I - \lambda L)$ is equal to C. Furthermore, for all $f \in D(L)$ we have that

$$||(I - \lambda L)f|| \geq ||f||. \tag{2.1.8}$$

The formal relation between a Markov generator and a Markov semigroup is made rigorous in the Hille-Yosida theorem.

THEOREM 2.1 (Hille-Yosida)**.** There is a one-to-one correspondence between Markov semigroups and Markov generators. The relation between the two is characterized by the following statements.

1. Given a Markov semigroup $\{S_t : t \geq 0\}$, the operator L defined in (2.1.6) and the domain $D(L)$ defined in (2.1.7) satisfy conditions G1-G4.

2. Given a linear generator L satisfying conditions G1-G4, the family of operators $\{S_t : t \geq 0\}$ defined for $f \in C$ by

$$S_t f = \lim_{n \to \infty} (I - \tfrac{t}{n} L)^{-n} f, \tag{2.1.9}$$

is a Markov semigroup.

3. For $f \in D(L)$ we have that $S_t f \in D(L)$ and

$$\partial_t S_t f = S_t L f = L S_t f. \tag{2.1.10}$$

Furthermore, $S_t f$ is the unique solution to the equation $\partial_t f_t = L f_t$ with initial condition $f_0 = f$.

The equation $\partial_t f_t = L f_t$ is called the **Kolmogorov backward equation**. Recall that we already saw the Kolmogorov forward equation in (1.3.2), which encoded the evolution of the distribution over time. The Kolmogorov backward equation encodes the forward evolution of expectations.

Example 2.4 (SSRW)**.** In (1.3.3) we have already mentioned that the generator of general jump processes is given by

$$Lf(x) = \sum_{y \in \Omega} c(x,y)\big(f(y) - f(x)\big), \tag{2.1.11}$$

with $c(x,y)$ the rate of jumping from x to y. In the case of the SSRW, where a single walker jumps one place to the right or left with rate $\frac{1}{2}$, the generator simplifies to

$$Lf(x) = \tfrac{1}{2}\big(f(x+1) + f(x-1) - 2f(x)\big). \tag{2.1.12}$$

Example 2.5 (Brownian motion)**.** The generator of a Brownian motion is given by

$$Lf(x) = \tfrac{1}{2}\Delta f(x). \tag{2.1.13}$$

From the Hille-Yosida theorem, the relation between a Brownian motion and the heat equation becomes clear. Specifically, defining

$$f_t(x) := \mathbb{E}[f(x + B_t)], \tag{2.1.14}$$

then f_t solves the equation $\partial_t f_t = \frac{1}{2}\Delta f_t$.

Example 2.6 (IRW)**.** Recall the IRW process introduced in Section 1.2. The space of particle configuration on $\mathbb{Z}$, given by functions $\eta : \mathbb{Z} \to \mathbb{N}$, is often denoted as $\Omega = \mathbb{N}^{\mathbb{Z}}$. The IRW process is essentially a jump process on this space, where configurations "jump" to other configurations where one particle has jumped to either the left or the right. The generator is then given by

$$Lf(\eta) = \sum_{x \in \mathbb{Z}} \eta(x)\big(f(\eta^{x \to x+1}) + f(\eta^{x \to x-1}) - 2f(\eta)\big), \tag{2.1.15}$$

where $\eta^{x \to x \pm 1}$ denotes the configuration η where one particle has jumped from x to $x \pm 1$.

In many cases, such as in the example above, the domain of the generator $D(L)$ is not easy to characterize. However, for most purposes, it is sufficient to consider only functions in a certain subspace of the domain.

Definition 2.6. A **core** is a subset $\mathscr{D} \subset D(L)$ such that for every $f \in D(L)$ there exists a sequence $(f_N)_{N\in\mathbb{N}} \subset \mathscr{D}$ satisfying $f_N \to f$ and $Lf_N \to Lf$, both with respect to the norm $||\cdot||$.

Example 2.7. When considering $\Omega = \mathbb{N}^{\mathbb{Z}}$, the space of particle configuration on $\mathbb{Z}$, the generator of a jump process is given by

$$Lf(\eta) = \sum_{\eta'\in\Omega} c(\eta,\eta')\big(f(\eta') - f(\eta)\big), \tag{2.1.16}$$

for $f \in D(L) \subset C(\Omega)$. A common core for such a generator is given by the bounded **local functions**, i.e., bounded functions $f(\eta)$ depending only on a finite number of coordinates of η. This space is dense in $C(\Omega)$ with respect to the uniform topology, and if we assume that the process generated by L allows only nearest-neighbor jumps, then we can also find approximating sequences for Lf for every $f \in D(L)$.

2.1.4 *Path-space convergence and tightness*

When considering convergence of stochastic processes, the Trotter-Kurtz theorem [67] explains how convergence of the generator implies convergence of the process.

Theorem 2.2 (Trotter-Kurtz)**.** Let $(X^N)_{N\in\mathbb{N}}$, X be Markov processes with semigroups $(S_t^N)_{N\in\mathbb{N}}$, S_t and generators L_N, L respectively. Furthermore, let $\mathscr{D}$ be a core for L, then the following are equivalent:

- For all $f \in \mathscr{D}$ there exists a sequence of $f_N \in D(L_N)$ such that $f_N \to f$ and $L_N f_N \to Lf$, where both convergences are with respect to $||\cdot||$.
- For all $f \in C$ and $T > 0$ we have that $S_t^N f \to S_t f$ with respect to $||\cdot||$, uniformly in $t \in [0,T]$.
- if $X_0^N \xrightarrow{\text{d}} X_0$, then $X^N \xrightarrow{\text{d}} X$ in $D([0,T];\Omega)$ for all $T > 0$.

Example 2.8. In this example we will show that a rescaled SSRW converges to a Brownian motion in distribution using the Trotter-Kurtz theorem. Recall that the generator of an SSRW X is given by

$$Lf(x) = \tfrac{1}{2}\big(f(x+1) + f(x-1) - 2f(x)\big). \tag{2.1.17}$$

We will perform a diffusive scaling of this process (see Section 1.4.2), where we rescale space by $\frac{1}{N}$ and rescale time by N^2, resulting in the generator of the process $\frac{X_{N^2 t}}{N}$ on $\frac{1}{N}\mathbb{Z} \subset \mathbb{R}$,

$$L_N f(x) = \frac{N^2}{2}\big(f(x+\tfrac{1}{N}) + f(x-\tfrac{1}{N}) - 2f(x)\big) = \tfrac{1}{2}\Delta f(x) + \mathcal{O}(\tfrac{1}{N}). \tag{2.1.18}$$

Note that we can view every L_N as a generator acting on functions in $f \in C^2(\mathbb{R})$, and as $N \to \infty$ we see that $L_N f \to \frac{1}{2}\Delta f$, where the latter is the generator of a Brownian motion. By the Trotter-Kurtz theorem, it follows that $\frac{X_{N^2 t}}{N} \xrightarrow{\text{d}} B_t$ with B_t a standard Brownian motion.

While the Trotter-Kurtz theorem is a powerful tool, it is not always possible to establish convergence of the generator of a process for all functions f. In such cases, in order to still prove a convergence result, the first step is to prove tightness of the sequence of processes.

Definition 2.7. A sequence of stochastic processes X^N is **tight** if for every $\varepsilon > 0$ there exists a compact set $\mathcal{K}_\varepsilon \subset D([0,T];\Omega)$ such that

$$\sup_N \mathbb{P}(X^N \notin \mathcal{K}_\varepsilon) < \varepsilon. \tag{2.1.19}$$

Since in general it is not easy to determine whether a set is compact in the Skorokhod space, we often use the following theorem to check the tightness of a sequence (see e.g., [33, Corollary 7.4]).

Theorem 2.3. The sequence X^N is tight in $D([0,T];\Omega)$ if the following two conditions are satisfied:

1. For every $\varepsilon > 0$ and $t \in [0,T]$ there exists a compact $K_\varepsilon \subset \Omega$ such that

$$\liminf_{N\to\infty} \mathbb{P}(X^N \notin K_\varepsilon) < \varepsilon. \tag{2.1.20}$$

2. For every $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$\lim_{\delta\to 0} \limsup_{N\to\infty} \mathbb{P}(\omega(X^N,\delta) \geq \varepsilon) = 0, \tag{2.1.21}$$

where

$$\omega(X^N,\delta) = \sup\{d(X_t^N, X_s^N) \mid s,t \in [0,T],\ |t-s| < \delta\}. \tag{2.1.22}$$

Tightness is a necessary condition for convergence in distribution. Furthermore, if we have tightness, then by Prokhorov's theorem [92] every subsequence has a further subsequence that converges in distribution.

Theorem 2.4 (Prokhorov)**.** If the sequence X^N is tight, then the set of path space measures $\mathbb{P}(X^N \in \cdot)$ is relatively compact.

If we can then prove that every convergent subsequence has the same limit X, then we have proven convergence of the processes $X^N \xrightarrow{\mathrm{d}} X$ in $D([0,T];\Omega)$. For this final step, Portmanteau's theorem [8, Theorem 2.1] often plays a key role.

Theorem 2.5 (Portmanteau)**.** The following are equivalent

1. $X_N \xrightarrow{\mathrm{d}} X$ in $D([0,T];\Omega)$.
2. $\mathbb{E}[f(X^N)] \to \mathbb{E}[f(X)]$ for all bounded, uniformly continuous f.
3. $\limsup_{N\to\infty} \mathbb{P}(X^N \in \mathcal{C}) \leq \mathbb{P}(X \in \mathcal{C})$ for all closed sets $\mathcal{C} \in D([0,T];\Omega)$.
4. $\liminf_{N\to\infty} \mathbb{P}(X^N \in \mathcal{O}) \leq \mathbb{P}(X \in \mathcal{O})$ for all open sets $\mathcal{O} \in D([0,T];\Omega)$.
5. $\lim_{N\to\infty} \mathbb{P}(X^N \in \mathcal{A}) = \mathbb{P}(X \in \mathcal{A})$ for all sets $\mathcal{A} \in D([0,T];\Omega)$ such that $\mathbb{P}(X \in \partial\mathcal{A}) = 0$.

2.2 INVARIANT AND ERGODIC MEASURES

Let X be a Markov process on Ω with initial state drawn from a probability measure $X_0 \sim \mu$. In this section we will rigorously define the type of initial probability measures discussed in Section 1.3.

2.2.1 *Invariant and reversible measures*

In mathematics, invariance refers to the property of remaining unchanged over time. In Section 1.3 we discussed that the evolution of the initial measure μ, $X_t \sim \mu_t$, was described through the Kolmogorov forward equation

$$\partial_t \mu_t = \mu_t L. \tag{2.2.1}$$

Using Markov semigroups, we are able to express $\mu_t = \mu S_t$ as the unique probability measure such that for every $f \in C$ we have that

$$\int f \, \mathrm{d}\mu S_t = \int S_t f \, \mathrm{d}\mu. \tag{2.2.2}$$

Definition 2.8. A probability measure μ is **invariant** if $\mu S_t = \mu$ for all $t \geq 0$, i.e., if and only if for all $f \in C$ we have that

$$\int S_t f \, \mathrm{d}\mu = \int f \, \mathrm{d}\mu. \tag{2.2.3}$$

However, characterizing that a measure is invariant is usually done via the Markov generator

Proposition 2.1. Let $\mathscr{D}$ be a core for the Markov generator L, then μ is invariant if for all $f \in \mathscr{D}$ we have that

$$\int Lf \,\mathrm{d}\mu = 0. \tag{2.2.4}$$

Indeed, if (2.2.4) holds, then $\mu L = 0$, and the Kolmogorov forward equation (2.2.1) reads $\partial_t \mu_t = 0$, indicating that the measure does not change over time.

A stronger notion than invariance is that of reversibility.

Definition 2.9. A probability measure μ is **reversible** if for all $f, g \in C$ we have that

$$\int (S_t f) g \,\mathrm{d}\mu = \int f (S_t g) \,\mathrm{d}\mu. \tag{2.2.5}$$

Substituting $g \equiv 1$ in (2.2.5), it immediately follows that reversible measures are invariant.

Proposition 2.2. Let $\mathscr{D}$ be a core for the Markov generator L, then μ is reversible if and only if for all $f, g \in \mathscr{D}$ we have that

$$\int (Lf) g \,\mathrm{d}\mu = \int f (Lg) \,\mathrm{d}\mu. \tag{2.2.6}$$

2.2.2 *Ergodic measures and mixing*

In Section 1.3 the ergodic measures were introduced as the measures μ for which

$$\mu(A) = \lim_{T\to\infty} \frac{1}{T} \int_0^T \mathbb{1}(X_t \in A) \,\mathrm{d}t \tag{2.2.7}$$

holds if $X_0 \sim \mu$. We also discussed that the ergodic measures are the extremal points of the set of invariant measures, allowing us to reconstruct the set of invariant measures from the ergodic measures. In this section we introduce the classical definition of ergodic measures.

In order to define ergodicity, we first need to define invariance of sets (note that this notion is distinct from the invariance of probability measures).

Definition 2.10. A set $A \in \mathscr{F}$ is **invariant** if $S_t \mathbb{1}_A = \mathbb{1}_A$ for all $t \geq 0$.

In words, a set A is invariant if the process can never leave A once it enters it, i.e., if $X_0 \in A$ then $X_t \in A$ for all $t \geq 0$ with probability 1. An ergodic measure is then an invariant measure that has its mass in exactly one of these invariant sets.

Definition 2.11. An invariant probability measure μ is **ergodic** if for all invariant sets $A \in \mathscr{F}$ either $\mu(A) = 1$ or $\mu(A) = 0$.

Example 2.9. We consider the example of a jump process on the space {1,2,3,4} where the process can jump between 1 and 2 and between 3 and 4 with rate 1 (see the figure below).

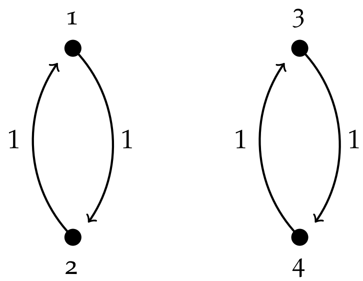

Figure 10: A jump process on the space {1,2,3,4}. The sets {1,2} and {3,4} are the invariant sets of this system.

If we consider the uniform measure ($\mu = (1/4, 1/4, 1/4, 1/4)$), then this is invariant but not ergodic, namely $\mu(\{1,2\}) = \mu(\{3,4\}) = \frac{1}{2}$. The ergodic measures of this process are given by the invariant measures with mass solely on the invariant sets,

$$\mu_0 = (1/2, 1/2, 0, 0), \qquad \mu_1 = (0, 0, 1/2, 1/2). \tag{2.2.8}$$

From the ergodic measures we can recover the invariant measures as convex combinations of the two, i.e.,

$$\mu_a = (1-a)\mu_0 + a\mu_1, \tag{2.2.9}$$

for $a \in [0,1]$. Indeed, the uniform measure is now equal to $\mu = \mu_{\frac{1}{2}}$.

Example 2.10. As mentioned in Section 1.3, the product Bernoulli measures

$$\mu_p = \bigotimes_{x \in \mathbb{Z}} \text{Ber}(p) \tag{2.2.10}$$

are ergodic for the SEP on $\mathbb{Z}$ for every $p \in [0,1]$. However, if we were to restrict ourselves to the SEP on the finite space $\{0, 1, ..., N\}$ for some $N \in \mathbb{N}$, then the product Bernoulli measures remain invariant but are no longer ergodic. In this finite setting, the ergodic measures are indexed by the parameter $k \in \{0, 1, ..., N\}$, and are given by the uniform measures on the subspace of configurations with a fixed number of particles, i.e.,

$$\mu_k(\eta) = \begin{cases} \binom{N}{k}^{-1} & \text{if } |\eta| = k \\ 0 & \text{else} \end{cases} \tag{2.2.11}$$

where $|\eta| = \sum_{x=1}^{N} \eta(x)$ is the number of particles in configuration η. Note that we can recover the product Bernoulli measures as a convex combination from the ergodic measures μ_k as follows,

$$\mu_p = \sum_{k=1}^{N} \mu_p(\{|\eta| = k\}) \cdot \mu_k = \sum_{k=1}^{N} p^k (1-p)^{N-k} \cdot \mu_k. \tag{2.2.12}$$

One of the main results in ergodic theory is the Birkhoff ergodic theorem [9].

Theorem 2.6 (Birkhoff ergodic theorem)**.** Let μ be ergodic, then for all $f \in L^1(\Omega, \mu)$ and μ-almost all $x \in \Omega$ we have that

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T S_t f(x)\,\mathrm{d}t = \int f\,\mathrm{d}\mu. \tag{2.2.13}$$

It is important to note that the ergodic measures are the only measures that satisfy this equality. Namely, taking an invariant set $A \in \mathscr{F}$ and setting $f = \mathbb{1}_A$ we find that

$$\mu(A) = \int \mathbb{1}_A\,\mathrm{d}\mu = \lim_{T\to\infty} \frac{1}{T} \int_0^T S_t \mathbb{1}_A(x)\,\mathrm{d}t = \mathbb{1}_A(x) \tag{2.2.14}$$

where the right-hand side is indeed equal to either 0 or 1.

Lastly we define the mixing probability measures.

Definition 2.12. A probability measure μ is **mixing** if for $f, g \in L^2(\Omega, \mu)$ we have that

$$\lim_{t\to\infty} \int (S_t f) g\,\mathrm{d}\mu = \int f\,\mathrm{d}\mu \int g\,\mathrm{d}\mu \tag{2.2.15}$$

Intuitively, a mixing measure ensures that the process mixes any set in such a way that it is spread out evenly over the whole space. This becomes clearer when we choose $f = \mathbb{1}_A$ and $g = \mathbb{1}_B$, and (2.2.15) reads

$$\lim_{t\to\infty} \mathbb{P}_\mu(X_t \in A, X_0 \in B) = \mu(A)\mu(B), \tag{2.2.16}$$

where $\mathbb{P}_\mu$ denotes the path-space measure of X in $D([0,T];\Omega)$ with initial distribution μ, i.e., $\mathbb{P}_\mu(X_0 \in \cdot) = \mu(\cdot)$. We then see that after a long time, the sets A and B become independent.

Every probability measure μ that is mixing is also ergodic. To see this, take any invariant set $A \in \mathscr{F}$. Using that $\mathbb{1}_A = \mathbb{1}_A^2$ and $S_t \mathbb{1}_A = \mathbb{1}_A$, we find

$$\mu(A) = \int \mathbb{1}_A^2\,\mathrm{d}\mu = \int (S_t \mathbb{1}_A)\mathbb{1}_A\,\mathrm{d}\mu, \tag{2.2.17}$$

for every $t \geq 0$. Since the left-hand side does not depend on t, we can take the limit as $t \to \infty$. By the mixing property we then obtain

$$\mu(A) = \int \mathbb{1}_A\,\mathrm{d}\mu \int \mathbb{1}_A\,\mathrm{d}\mu = \mu(A)^2, \tag{2.2.18}$$

showing that indeed $\mu(A) = 1$ or $\mu(A) = 0$, hence μ is ergodic.

2.3 MATHEMATICAL TOOLS FOR MARKOV PROCESSES

In this section we introduce some useful mathematical tools in the study of Markov processes.

2.3.1 *Dynkin martingales*

Martingales are, alongside Markov processes, among the most studied stochastic processes. The Dynkin martingale is a martingale corresponding to a Markov process. We have already encountered it in (1.4.8), but now we will first give a general introduction to martingales. For a more detailed introduction into martingales, we refer to the book of Gut [52, Chapter 10].

MARTINGALES

DEFINITION 2.13. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $\{\mathcal{F}_t : t \geq 0\}$ a filtration. A **martingale** is a stochastic process $M = \{M_t : t \geq 0\}$ satisfying the following three properties:

i. **Adaptedness**: M_t is $\mathcal{F}_t$-measurable for all $t \geq 0$.

ii. **Integrability**: $\mathbb{E}[|M_t|] < \infty$ for all $t \geq 0$.

iii. **Martingale property**: $\mathbb{E}[M_t|\mathcal{F}_s] = M_s$ for $0 \leq s \leq t$.

The intuitive idea of a martingale comes from the world of gambling, where it can be seen as the profit of a fair game. Here, the best estimation of future winnings is simply the present amount of money held. Furthermore, by the martingale property, a martingale has a constant expectation, namely

$$\mathbb{E}[M_t] = \mathbb{E}[\mathbb{E}[M_t|\mathcal{F}_0]] = \mathbb{E}[M_0]. \tag{2.3.1}$$

EXAMPLE 2.11. A good first example of a martingale is a standard Brownian motion $B = \{B_t : t \geq 0\}$ with the natural filtration $\mathcal{F}_t = \sigma(B_s : 0 \leq s \leq t)$. The adaptedness then follows immediately, and integrability is easily shown. The martingale property follows from the independence of increments

$$\mathbb{E}[B_t|\mathcal{F}_s] = \mathbb{E}[B_t - B_s|\mathcal{F}_s] + \mathbb{E}[B_s|\mathcal{F}_s] = \mathbb{E}[B_t - B_s] + B_s = B_s. \tag{2.3.2}$$

QUADRATIC VARIATION

DEFINITION 2.14. For partitions $P = \{t_0, t_1, ..., t_n\}$ of the interval $[0, t]$ such that $0 = t_0 < t_1 < ... < t_n = t$, the **mesh size** of P is a norm $||\cdot||_m$ given by

$$||P||_m := \max_{1 \leq k \leq n} |t_k - t_{k-1}|. \tag{2.3.3}$$

Definition 2.15. Let M be a real-valued martingale, then the **quadratic variation process** of M is a stochastic process $[M] = \{[M]_t : t \geq 0\}$ defined via the following convergence in probability

$$[M]_t := \lim_{||P||_m \to 0} \sum_{k=1}^{n} (M_{t_k} - M_{t_{k-1}})^2. \tag{2.3.4}$$

The quadratic variation process measures the cumulative variance of the process. Another notion that is related to the quadratic variation is the predictable quadratic variation.

Definition 2.16. Let M be a real-valued martingale, then the **predictable quadratic variation process** of M is the unique predictable (i.e. $\mathcal{F}_{t^-}$-measurable) increasing process $\langle M \rangle = \{\langle M \rangle_t : t \geq 0\}$ such that $\langle M \rangle_0 = 0$ and

$$N_t := M_t^2 - \langle M \rangle_t \tag{2.3.5}$$

is a martingale.

In the case where M is a continuous process, the two notions coincide, as can be found in [111, Proposition 3.8]. However, when M is not continuous, the two may differ, and the process $[M]_t - \langle M \rangle_t$ is again a martingale.

Example 2.12. For a standard Brownian motion, the (predictable) quadratic variation is given by $[B]_t = t$. Verifying this directly using the definition in (2.3.4) involves a rather technical calculation. However, since the standard Brownian motion is continuous, the two notions of quadratic variation coincide and we only need to show that $\{B_t^2 - t : t \geq 0\}$ is a martingale. This follows from the following computation

$$\mathbb{E}[B_t^2 - t | \mathcal{F}_s] = \mathbb{E}[(B_t - B_s)^2] + \mathbb{E}[2B_t B_s - B_s^2 | \mathcal{F}_s] - t = B_s^2 - s. \tag{2.3.6}$$

Example 2.13. For $\lambda > 0$, a Poisson process is the Markov process $\{N_t^\lambda : t \geq 0\}$ satisfying $N_0^\lambda = 0$, and for each $n \in \mathbb{N}$ the process jumps from state n to $n+1$ with rate λ. This results in a process with independent increments and distribution $N_t^\lambda \sim \text{Pois}(\lambda t)$ for all $t > 0$. The compensated process $M_t = N_t^\lambda - \lambda t$ is a martingale and its quadratic variation counts the number of jumps that have occurred in the Poisson process, that is $[M]_t = N_t^\lambda$. Its predictable quadratic variation is given by $\langle M \rangle_t = \lambda t$, which can be checked in a similar way as for the Brownian motion case in (2.3.6). Hence, we indeed find that $[M]_t - \langle M \rangle_t = M_t$ is a martingale.

From 2.3.5 we obtain that if $M_0 = 0$, then

$$\mathbb{E}[M_t^2] = \mathbb{E}[\langle M \rangle_t]. \tag{2.3.7}$$

This relation tells us that if the predictable quadratic variation of a martingale starting at 0 vanishes, then the martingale itself vanishes. This fact is used in the sketch of the proof of the hydrodynamic limit (see Section 1.4.2).

DYNKIN MARTINGALES

The **Dynkin martingales** are a family of martingales corresponding to a Markov process.

THEOREM 2.7. Let $X = \{X_t : t \geq 0\}$ be an $\mathscr{F}_t$-adapted Markov process on $(\Omega, \mathscr{F}, \mathbb{P})$, generated by L. Then for any $f \in D(L)$, the process M defined as

$$M_t^f := f(X_t) - f(X_0) - \int_0^t Lf(X_s)ds \tag{2.3.8}$$

is a real-valued martingale on $(\Omega, \mathscr{F}, \mathbb{P})$ with respect to the filtration $\{\mathscr{F}_t, t \geq 0\}$.

For a Dynkin martingale, we have an explicit formula of the predictable quadratic variation.

THEOREM 2.8. If $f, f^2 \in D(L)$, then the predictable quadratic variation of the Dynkin martingale is given by

$$\langle M^f \rangle_t = \int_0^t \Gamma^f(X_s)ds, \tag{2.3.9}$$

where Γ^f is the **Carré du champ** operator, given by

$$\Gamma^f(x) := Lf^2(x) - 2f(x)Lf(x). \tag{2.3.10}$$

The proofs of Theorem 2.7 and 2.8 can be found in, e.g., [104, Section 8.1]. In the following example we see that for jump processes we obtain an elegant form for the Carré du champ operator, and so correspondingly for the predictable quadratic variation.

EXAMPLE 2.14. Let $X = \{X_t : t \geq 0\}$ be a jump process with generator

$$Lf(x) = \sum_{y \in \Omega} c(x,y)(f(y) - f(x)). \tag{2.3.11}$$

Then, the Carré du champ operator corresponding to this generator is given by

$$\begin{aligned} \Gamma^f(x) &= Lf^2(x) - 2f(x)Lf(x) \\ &= \sum_{y \in \Omega} c(x,y)(f^2(y) - f^2(x)) - 2f(x) \sum_{y \in \Omega} c(x,y)(f(y) - f(x)) \\ &= \sum_{y \in \Omega} c(x,y)(f(y) - f(x))^2. \end{aligned} \tag{2.3.12}$$

2.3.2 *Couplings*

Coupling is a technique in probability used to compare two stochastic processes by constructing them on a shared probability space in a way to highlight their relationship. For more information on couplings, we refer to [26, 77, 115].

Definition 2.17. Let X and Y be two Markov processes. A **coupling** is a joint process $(\check{X}, \check{Y})$ such that $\check{X} \stackrel{d}{=} X$ and $\check{Y} \stackrel{d}{=} Y$.

The power of a coupling lies in the fact that even when X and Y are independent, $\check{X}$ and $\check{Y}$ can be heavily dependent on each other. A standard example where a coupling is used is when X and Y are independent copies of the same process, but $X_0 \sim \mu$ and $Y_0 \sim \nu$. We are then interested in knowing whether these two processes have roughly the same distribution for large t. This means that, if $\{S_t : t \geq 0\}$ is the semigroup of the process, we want that

$$\lim_{t\to\infty} ||\mu S_t - \nu S_t||_{tv} = 0. \tag{2.3.13}$$

Here $||\cdot||_{tv}$ is the **total variation distance** of a signed measure, given by

$$||\mu||_{tv} = \sup_A |\mu(A)|. \tag{2.3.14}$$

We show how couplings can help answering this question. We first define the first meeting time of $\check{X}_t$ and $\check{Y}_t$,

$$\tau := \inf\{t \geq 0 : \check{X}_t = \check{Y}_t\}. \tag{2.3.15}$$

Next we define a coupling $\check{X} = X$ and

$$\check{Y}_t := \begin{cases} Y_t & \text{if } t \leq \tau, \\ X_t & \text{if } t > \tau, \end{cases} \tag{2.3.16}$$

i.e., $\check{Y}_t$ follows the process Y_t up until the meeting time, after which it follows X_t. First note that this indeed produces a coupling of X and Y, namely after the first meeting time they have the same distribution since they are copies of the same Markov process. Furthermore, we have that

$$\mathbb{P}(\check{X}_t = \check{Y}_t | t > \tau) = 1. \tag{2.3.17}$$

Looking at the difference in distribution, we now see that

$$\begin{aligned} ||\mu S_t - \nu S_t||_{tv} &= \sup_A |\mathbb{P}(X_t \in A) - \mathbb{P}(Y_t \in A)| \\ &\leq \sup_A |\mathbb{P}(\check{X}_t \in A,\ t > \tau) - \mathbb{P}(\check{Y}_t \in A,\ t > \tau)| \\ &\leq 2\mathbb{P}(t > \tau). \end{aligned} \tag{2.3.18}$$

In particular, if $\mathbb{P}(\tau < \infty) = 1$ we see that (2.3.13) holds.

Note that in the calculation (2.3.18), we could have considered any coupling $(\check{X}, \check{Y})$ together with the stopping time

$$\check{\tau} := \inf\{t \geq 0 : \check{X}_t = \check{Y}_t\}, \tag{2.3.19}$$

after which we set $\check{X}_t = \check{Y}_t$. We often refer to $\check{\tau}$ as the **coupling time**, and we call such a coupling **successful** if $\mathbb{P}(\check{\tau} < \infty) = 1$. If this is the case, then it again follows that (2.3.13) holds.

In some cases, the stopping time τ in (2.3.15) is not a.s. finite, but we can still show the existence of a successful coupling. Below we give an example of this.

EXAMPLE 2.15. In this example we illustrate a successful coupling, called the **Ornstein coupling**, of two independent simple symmetric random walkers X and Y on $\mathbb{Z}^d$. First note that $X - Y$ is again a simple symmetric random walker on $\mathbb{Z}^d$ and the first meeting time can now be written as

$$\tau = \{t \geq 0 : X_t - Y_t = 0\}. \tag{2.3.20}$$

It is a well known result that a simple symmetric random walker on $\mathbb{Z}^d$ reaches 0 in finite time if $d = 1, 2$, but might never reach 0 if $d \geq 3$ (This is known as Pólya's result [94]). However, for $d \geq 3$ we can still show the existence of a successful coupling.

The trick is to do a coordinate-wise coupling. We are able to write

$$X = (X^{(1)}, X^{(2)}, ..., X^{(d)}), \quad Y = (Y^{(1)}, Y^{(2)}, ..., Y^{(d)}), \tag{2.3.21}$$

where all the individual processes are independent of one another. We then define the first meeting times of every coordinate

$$\tau^{(k)} = \inf\{t \geq 0 : X_t^{(k)} = Y_t^{(k)}\}. \tag{2.3.22}$$

For every $1 \leq k \leq d$ this stopping time is a.s. finite, since it is the first meeting time of two simple symmetric random walkers in one dimension. We then set up our coupling as $\check{X} = X$ and

$$\check{Y}_t^{(k)} = \begin{cases} Y_t^{(k)} & \text{if } t < \tau, \\ X_t^{(k)} & \text{if } t \geq \tau, \end{cases} \tag{2.3.23}$$

i.e., once the two processes meet in a given coordinate, that coordinate remains the same. The coupling time is then equal to $\check{\tau} = \max_{1 \leq k \leq d} \tau^{(k)}$, which is again a.s. finite, therefore the coupling is successful.

2.3.3 *Duality*

DEFINITION 2.18. Let X and Y be two Markov processes on the state spaces Ω_1 and Ω_2 respectively. We say that X and Y are **dual** to one another if there exists a function $\mathfrak{D} : \Omega_1 \times \Omega_2 \to \mathbb{R}$ such that

$$\mathbb{E}\left[\mathfrak{D}(X_t, y) | X_0 = x\right] = \mathbb{E}[\mathfrak{D}(x, Y_t) | Y_0 = y] \tag{2.3.24}$$

for all $x \in \Omega_1$ and $y \in \Omega_2$. We then refer to $\mathfrak{D}$ as the **duality function**.

If the two processes X and Y are the same process in distribution, then we say that X is **self-dual**.

It is important to note that any two processes X and Y can be considered dual to one another with constant duality function. However, such a duality relation is not useful. A useful duality relation enables to reformulate a problem involving X into a problem involving Y, where Y is typically a process that is easier to analyze. Finding a useful duality relation is rare, but once one is found it can enable significant simplifications in deriving properties of X.

In Example 2.17 we will show how duality can help in deriving a weak version of the hydrodynamic limit for IRW. First we present the following proposition, which is useful for establishing that a duality relation holds. The proof can be found in [59, Proposition 1.2].

Proposition 2.3. Let L_X and L_Y be the generators of X and Y respectively. Assume that $\mathfrak{D}(\cdot, y) \in D(L_X)$ for all $y \in \Omega_2$ and $\mathfrak{D}(x, \cdot) \in D(L_Y)$ for all $x \in \Omega_1$, then X and Y are dual to one another if

$$L_X \mathfrak{D}(\cdot, y)(x) = L_Y \mathfrak{D}(x, \cdot)(y) \tag{2.3.25}$$

for all $x \in \Omega_1$ and $y \in \Omega_2$.

Remark 2.16. In the original statement of the Proposition in [59], it is also assumed that $S_{X,t}\mathfrak{D}(\cdot, y) \in D(L_X)$ and $S_{Y,t}\mathfrak{D}(x, \cdot) \in D(L_Y)$ for all $t \geq 0$, with $S_{X,t}, S_{Y,t}$ the Markov semigroups of X and Y respectively. However, since we are working with strongly continuous semigroups, by the Hille-Yosida theorem, this already follows from the conditions $\mathfrak{D}(\cdot, y) \in D(L_X)$ and $\mathfrak{D}(x, \cdot) \in D(L_Y)$.

Example 2.17. In this example we will show that the hydrodynamic limit of the IRW process is the heat equation (in expectation), using duality. I.e., we will show that

$$\mathbb{E}[\langle \pi_t^N, \phi \rangle] \to \int_{\mathbb{R}} \varrho_t(x)\phi(x)\, \mathrm{d}x \tag{2.3.26}$$

for every $\phi \in C_c^\infty(\mathbb{R})$, where π_t^N is the empirical measure defined in (1.4.4) and ϱ_t solves the heat equation $\partial_t \varrho_t = \Delta \varrho_t$.

Recall that the generator of the IRW process defined on local functions $f : \Omega \to \mathbb{R}$ is given by

$$L^{\mathrm{IRW}} f(\eta) = \sum_{y \in \mathbb{Z}} \eta(y) \big(f(\eta^{y \to y+1}) + f(\eta^{y \to y-1}) - 2f(\eta) \big). \tag{2.3.27}$$

The duality relation that we will use involves a single random walker X with generator defined on functions $g : \mathbb{Z} \to \mathbb{R}$ as

$$\mathscr{L} g(x) = g(x+1) + g(x-1) - 2g(x), \tag{2.3.28}$$

with duality function

$$\mathfrak{D}(\eta, x) = \eta(x). \tag{2.3.29}$$

It is a straightforward computation to show that

$$L^{\mathrm{IRW}}\mathfrak{D}(\cdot, x)(\eta) = \mathscr{L}\mathfrak{D}(\eta, \cdot)(x) = \eta(x+1) + \eta(x-1) - 2\eta(x), \tag{2.3.30}$$

proving that we indeed have duality.

Recall that for the hydrodynamic limit we need to perform a rescaling in time of N^2 (see Section 1.4.2). This gives us a process $\{\eta_t^N : t \geq 0\}$ which corresponds to the generator $N^2 L^{\mathrm{IRW}}$. For this process the duality relation still holds with respect to the process X^N corresponding to the generator $N^2\mathscr{L}$.

We start the process $\{\eta_t^N : t \geq 0\}$ from the local equilibrium measure given by the product Poisson measure (See Example 1.1)

$$\mu_N^{\varrho} = \bigotimes_{x \in \mathbb{Z}} \mathrm{Pois}(\varrho(\tfrac{x}{N})), \tag{2.3.31}$$

where $\varrho : \mathbb{R} \to \mathbb{R}$ is a smooth function. We now compute

$$\mathbb{E}[\langle \pi_t^N, \phi \rangle] = \frac{1}{N} \sum_{x \in \mathbb{Z}} \mathbb{E}[\eta_t^N(x)]\phi(\tfrac{x}{N}). \tag{2.3.32}$$

Since $\eta_t^N(x) = \mathfrak{D}(\eta_t^N, x)$, we are able to use duality to compute the expectation

$$\mathbb{E}[\eta_t^N(x)] = \mathbb{E}\left[\eta_0^N(X_t^N) | X_0^N = x\right] = \mathbb{E}\left[\varrho\left(\tfrac{X_t^N}{N}\right) | X_0^N = x\right], \tag{2.3.33}$$

where in the last equality we used that $\eta_0^N \sim \mu_N^{\varrho}$. The rescaled process $\frac{X^N}{N}$ lives on the space $\frac{1}{N}\mathbb{Z}$ with generator given by

$$\mathscr{L}_N g(x) = N^2\left(g(x + \tfrac{1}{N}) + g(x - \tfrac{1}{N}) - 2g(x)\right) = \Delta g(x) + \mathcal{O}(\tfrac{1}{N}). \tag{2.3.34}$$

In particular, by the Trotter-Kurtz Theorem, as $N \to \infty$ the process converges in distribution to the process with generator $L = \Delta$. Denoting the semigroup of this process by $\{S_t : t \geq 0\}$, we find that

$$\mathbb{E}\left[\varrho\left(\tfrac{X_t^N}{N}\right) | X_0^N = x\right] \to S_t\varrho(\tfrac{x}{N}) = \varrho_t(\tfrac{x}{N}), \tag{2.3.35}$$

where by the Hille-Yosida Theorem, ϱ_t solves the heat equation $\partial_t \varrho_t = \Delta \varrho_t$ with initial condition $\varrho_0 = \varrho$. Combining everything, we indeed find that

$$\lim_{N\to\infty} \mathbb{E}[\langle \pi_t^N, \phi \rangle] = \lim_{N\to\infty} \frac{1}{N} \sum_{x \in \mathbb{Z}} \varrho_t(\tfrac{x}{N})\phi(\tfrac{x}{N}) = \int_{\mathbb{R}} \varrho_t(x)\phi(x)\,\mathrm{d}x. \tag{2.3.36}$$

Remark 2.18. The same duality result that holds between the IRW and a single random walker also applies to the SEP and SIP. This further explains why the hydrodynamic equation is the same for all three processes. Moreover, a more general duality result exists for these processes. In [45] it is shown that IRW, SEP and SIP are self-dual, where the dual process only has a finite number of particles. In Chapter 3 we make use of this duality in order to characterize a class of ergodic measures for these processes.

3

ERGODIC THEORY OF MULTI-LAYER INTERACTING PARTICLE SYSTEMS

In this thesis we aim to study multi-layer interacting particle systems. In this chapter[1] we introduce a class of multi-layer interacting particle systems and characterize the set of ergodic probability measures with finite moments. Furthermore, we prove a more general statement that such a characterization is possible under the existence of a polynomial duality relation and a successful coupling.

3.1 INTRODUCTION

In this chapter we study a system of particles performing multi-layer random walks which possibly have interaction of inclusion or exclusion type (see Section 3.2.1 below for a precise description of the models). We characterize the set of invariant probability measures with finite moments. More precisely we prove that under an appropriate condition of moment growth, the only ergodic invariant probability measures are homogeneous product probability measures, indexed by the first moment (particle density). In the case of independent particles, these probability measures are product Poisson measures, in the case of interacting systems they are products of binomial (exclusion) resp. negative binomial (inclusion) distributions.

The characterization of the invariant probability measures of multi-layer exclusion processes has been obtained recently in [3]. The study of the hydrodynamic limit of a system of active particles has been studied in [64], and for a two-layer system with duality in [37].

The important ingredient in our setting here is duality combined with the existence of a successful coupling for the dual process. This road is followed for the symmetric exclusion process in [76, Chapter 8]. Duality allows to characterize the invariant probability measures via the characterization of bounded harmonic functions of the dual process, which is a countable state space Markov chain. If this Markov chain admits a successful coupling, then the bounded harmonic functions are constants, indexed by the number of dual particles. The proof of

1 This chapter is based on [97].

the existence of a successful coupling is a combination of coupling the finite state space internal state process and the Ornstein coupling of random walks. These two ingredients are sufficient in the non-interacting case. In the interacting case, we use the approach of [66] which consists of "spreading out" the particles combined with Ornstein coupling of symmetric random walks.

The rest of this chapter is organized as follows. In Section 3.2 we provide the general setup of multi-layer particle systems, after which we define the three types of processes (exclusion, inclusion and independent walkers) we will study in this chapter. Afterwards, we study the duality properties and invariant probability measures of these processes.

In Section 3.3 we provide a characterization of the ergodic invariant probability measures in a slightly more general setting, where the only assumptions are duality with polynomial duality functions and the existence of a successful coupling. This unifies and generalizes earlier results from chapter 8 of the book of Liggett [76] and Kuoch [66].

Section 3.4 is devoted to the proof of a successful coupling for the models under consideration. For independent particles this amounts to generalize the Ornstein coupling to the multi-layer setting. For interacting particles, it amounts to generalize the approach of Liggett for the exclusion process [76] and Kuoch for the inclusion process [66].

3.2 MODELS AND THEIR DUALITY PROPERTIES

3.2.1 *Models: definitions*

In this chapter we will look at models of configurations where the coordinates of individual particles are of the form (x, σ), with $x \in \mathbb{Z}^d$ the *position* of the particle and $\sigma \in S$ the *internal state*, where S is some finite set. We will denote the single particle state space as $V := \mathbb{Z}^d \times S$, which we will think of as $|S|$ layers of $\mathbb{Z}^d$. For this reason, we will also refer to $\sigma \in S$ as the layer on which a particle at (x, σ) resides.

We consider a configuration process $\{\eta_t : t \geq 0\}$ on a state space Ω_s that will be defined later. The generator of the process is of the following type,

$$\mathscr{L}_s f(\eta) = \sum_{v,w \in V} p(v,w)\eta(v)(\alpha + s\eta(w))\nabla_{v,w} f(\eta). \tag{3.2.1}$$

Here $\eta(v)$ is equal to the number of particles at site $v \in V$ and, if we denote $\eta^{v \to w}$ as the configuration η where a single particle has moved from v to w (if possible), we have

$$\nabla_{v,w} f(\eta) = f(\eta^{v \to w}) - f(\eta). \tag{3.2.2}$$

The value of $s \in \{-1, 0, 1\}$ in (3.2.1) determines the type of the process we consider (exclusion, inclusion or independent particles). For this reason, the parameter s also determines the state space Ω_s that we consider and the single particle transition rates $p(v, w)$ and constants $\alpha \in \mathbb{R}_+$ that we allow. Below we will define the process for each possible value of s.

The main characteristic of our multi-layer particle systems is that the transition rates are determined by the layer on which a particle resides. Therefore, for every $\sigma \in S$ we consider a nearest neighbor symmetric random walk on $\mathbb{Z}^d$ with translation invariant transition rates. We denote by $\pi_\sigma(x)$ the corresponding rate to jump from z to $z + x$. Note that $\pi_\sigma(x) > 0$ if and only if $|x| = 1$ and that $\pi_\sigma(x) = \pi_\sigma(-x)$. Furthermore, we let $\{c(\sigma, \sigma') : \sigma, \sigma' \in S\}$ be transition rates on the set of layers S which we will assume to be symmetric and irreducible. Then we define the following processes:

1. **Symmetric exclusion process ($s = -1$).** Every site contains at most $\alpha \in \mathbb{N}$ particles and jumps to sites where there are already many other particles are less likely. The state space of the multi-layer SEP is given by $\Omega_{-1} = \{0, 1, ..., \alpha\}^V$, and the single particle transition rates we will study for this model are of the following form,

$$p\big((x, \sigma), (y, \sigma')\big) = \pi_\sigma(y - x)\delta_{\sigma,\sigma'} + c(\sigma, \sigma')\delta_{x,y}, \tag{3.2.3}$$

   with $\delta_.$ the Kronecker delta.

2. **Symmetric inclusion process ($s = 1$).** In contrast to the exclusion process, this process actually encourages jumps to sites where other particles already reside. The state space of the multi-layer SIP is given by $\Omega_1 = \mathbb{N}^V$, and the transition rates are again given by (3.2.3). Furthermore, we allow for any $\alpha > 0$.

3. **Independent particles ($s = 0$).** For this chapter, our model for independent particles will be the run-and-tumble particle process (RTP). A run-and-tumble particle is a particle with the following dynamics.

   Random walk jumps. With rate κ, a particle at (x, σ) performs a nearest neighbor symmetric random walk jump on $\mathbb{Z}^d$ according to the transition rates $\pi_\sigma(\cdot)$, i.e., $(x, \sigma) \to (x + y, \sigma)$ with rate $\kappa\pi_\sigma(y)$.

   Active jumps. With rate λ, a particle at (x, σ) performs an active jump in the direction determined by the internal state σ, i.e., there exists a function $v : S \to \mathbb{Z}^d$ such that $(x, \sigma) \to (x + v(\sigma), \sigma)$ with rate λ.

   Internal state jumps. A particle changes its internal state according to the transition rates $\{c(\sigma, \sigma') : \sigma, \sigma' \in S\}$.

The state space of this process is $\Omega_0 = \mathbb{N}^V$, and from the dynamics we conclude that the single particle transition rates are of the following form,

$$p\big((x,\sigma),(y,\sigma')\big) = \kappa\pi_\sigma(y-x)\delta_{\sigma,\sigma'} + \lambda\delta_{\sigma,\sigma'}\delta_{y,x+v(\sigma)} + c(\sigma,\sigma')\delta_{x,y}. \quad (3.2.4)$$

In this case, any choice of $\alpha > 0$ is possible. However, without loss of generality we put $\alpha = 1$. Furthermore, in the special case where $\kappa = 0$ we will assume that $\lambda > 0$ and that the range of v spans the whole of $\mathbb{Z}^d$, i.e.,

$$\text{vct}\{\mathscr{R}(v)\} = \mathbb{Z}^d. \quad (3.2.5)$$

This condition is crucial in order to construct the successful coupling cf. (3.4.18) below.

Remark 3.1. Notice that for the interacting models we only allow for symmetric transitions on every layer. This is because for asymmetric transition rates we only have duality when the particles are independent.

3.2.2 *Duality*

We will state and prove duality results for the processes we just defined. Recall the following definition of duality of Markov processes.

Definition 3.1. Let $\{\eta_t : t \geq 0\}$ and $\{\xi_t : t \geq 0\}$ be two Markov processes on the state spaces Ω and Ω' respectively, and let $\mathfrak{D} : \Omega' \times \Omega \to \mathbb{R}$ be a measurable function. We say that $\{\eta_t : t \geq 0\}$ and $\{\xi_t : t \geq 0\}$ are *dual* to one another, with respect to $\mathfrak{D}$, if

$$\mathbb{E}_\eta\left[\mathfrak{D}(\xi,\eta_t)\right] = \widehat{\mathbb{E}}_\xi\left[\mathfrak{D}(\xi_t,\eta)\right]. \quad (3.2.6)$$

Here $\mathbb{E}_\eta$ denotes the expectation in $\{\eta_t : t \geq 0\}$ starting from η, $\widehat{\mathbb{E}}_\xi$ the expectation in the dual process $\{\xi_t : t \geq 0\}$ starting from ξ, and we assume that both sides are bounded. We then call $\mathfrak{D}$ the *duality function*.

Duality results

Let $|\xi| := \sum_x \xi(x)$ denote the number of particles in ξ and let $\Omega_{s,f} := \{\xi \in \Omega : |\xi| < \infty\}$ be the subspace of Ω_s consisting of only those configurations with a finite number of particles. In the following theorem we will give duality results of the processes defined in Section 3.2.1 with duality functions $\mathfrak{D}_s : \Omega_{s,f} \times \Omega_s \to \mathbb{R}$ of the following form:

$$\mathfrak{D}_s(\xi,\eta) = \prod_{v\in V} d_s(\xi(v),\eta(v)). \quad (3.2.7)$$

The proof of the first two statements can be found in for example [14, Theorem 4.1]. For the third statement, we will make use of another duality result, with the so-called *associated deterministic system*, that is introduced in [13].

THEOREM 3.1. 1. If $s = -1$ then the process generated by $\mathscr{L}_{-1}$ is self-dual with duality function

$$\mathfrak{D}_{-1}(\xi,\eta) = \prod_{v\in V} \frac{\eta(v)!}{(\eta(v)-\xi(v))!} \cdot \frac{(\alpha-\xi(v))!}{\alpha!} \cdot I(\xi(v) \le \eta(v)), \tag{3.2.8}$$

where $I(\cdot)$ denotes the characteristic function.

2. If $s = 1$, then the process generated by $\mathscr{L}_1$ is self-dual with duality function

$$\mathfrak{D}_{1}(\xi,\eta) = \prod_{v\in V} \frac{\eta(v)!}{(\eta(v)-\xi(v))!} \cdot \frac{\Gamma(\alpha)}{\Gamma(\alpha+\xi(v))} \cdot I(\xi(v) \le \eta(v)). \tag{3.2.9}$$

3. If $s = 0$ then the process generated by $\mathscr{L}_0$, with transition rates $p(v,w)$ given by (3.2.4), is dual to its time-reversed process, i.e., the RTP process with single particle transition rates

$$\widehat{p}\big((x,\sigma),(y,\sigma')\big) = \kappa\pi_\sigma(y-x)\delta_{\sigma,\sigma'} + \lambda\delta_{\sigma,\sigma'}\delta_{y,x-v(\sigma)} + c(\sigma,\sigma')\delta_{x=y}, \tag{3.2.10}$$

and with the same parameter $\alpha > 0$. We denote this as the $\widehat{\text{RTP}}$ process. The corresponding duality function is given by

$$\mathfrak{D}_{0}(\xi,\eta) = \prod_{v\in V} \frac{\eta(v)!}{(\eta(v)-\xi(v))!} \cdot I(\xi(v) \le \eta(v)). \tag{3.2.11}$$

REMARK 3.2. If $\xi = \delta_v$ is the configuration containing a single particle at $v \in V$ and no particles elsewhere, then $\mathfrak{D}(\xi,\eta) = c_{\alpha,s}\eta(v)$, with $c_{\alpha,s}$ a positive constant depending on the model ($s \in \{-1,0,1\}$) and the constant α. As a consequence we have that

$$\mathbb{E}_\eta[\eta_t(v)] = \frac{1}{c_{\alpha,s}}\mathbb{E}_\eta[\mathfrak{D}(\delta_v,\eta_t)] = \frac{1}{c_{\alpha,s}}\widehat{\mathbb{E}}_v[\mathfrak{D}(\delta_{v(t)},\eta)] = \widehat{\mathbb{E}}_v[\eta_{v(t)}], \tag{3.2.12}$$

where $\widehat{\mathbb{E}}_v$ denotes the expectation of the dual process starting from δ_v.

Proof of duality for the RTP process

Let $\{v(t) : t \ge 0\}$ be the random path of a single particle in V performing the RTP dynamics starting from $v(0) = v$. The deterministic system we will consider is the following: for a function $f : V \to \mathbb{R}$, define

$$f_t(v) := \sum_{w\in V} p_t(v,w)f(w) = \mathbb{E}\left[f\big(v(t)\big)\right], \tag{3.2.13}$$

where $p_t(v,w)$ is the transition kernel of a single RTP particle. In other words, the process $\{f_t : t \ge 0\}$ follows the Kolmogorov backwards equation of the RTP process. We now have the following duality result:

Proposition 3.1. Let $f : V \to \mathbb{R}$ be such that $f(v) \neq 1$ for only a finite number of $v \in V$. For the deterministic processes $\{f_t : t \geq 0\}$ and the process $\{\eta_t : t \geq 0\}$ generated by $\mathscr{L}_0$, it holds that

$$\mathbb{E}_\eta \left[\prod_{v \in V} f(v)^{\eta_t(v)} \right] = \prod_{v \in V} f_t(v)^{\eta(v)}, \tag{3.2.14}$$

i.e., the two processes are dual to one another with duality function

$$\mathscr{D}(f, \eta) = \prod_{v \in V} f(v)^{\eta(v)}. \tag{3.2.15}$$

The proof of this result is straightforward and only relies on the fact that the particles in the RTP process move independently.
Proof. Define $\{v_i(t) : i \in I, t \geq 0\}$ as the paths of the particles in the configuration η_t with I an arbitrary set of labels, i.e., $\eta_t(v) = \sum_{i \in I} I(v_i(t) = v)$ for all $v \in V$ and $t \geq 0$. We then have that

$$\mathbb{E}\left[\prod_{v \in V} f(v)^{\eta_t(v)} \right] = \mathbb{E}\left[\prod_i f(v_i(t)) \right] = \prod_i \mathbb{E}\left[f(v_i(t)) \right] = \prod_i f_t(v_i) = \prod_{v \in V} f_t(v)^{\eta(v)}, \tag{3.2.16}$$

where in the second step we used the independence of particles. □

We will now prove item 3 of Theorem 3.1 using the duality result in (3.2.14).
Proof. Let $\{\xi_t : t \geq 0\}$ denote an $\widehat{\text{RTP}}$ process and, for a given η, define the sequence of finite configurations $(\eta^N)_{N \in \mathbb{N}}$ as

$$\eta^N(x, \sigma) := \begin{cases} \eta(x, \sigma) & \text{if } x \in [-N, N], \\ 0 & \text{else.} \end{cases} \tag{3.2.17}$$

We will first prove the result for the dual process starting from n particles at position $v_i \in V$, i.e., $\xi = n \cdot \delta_{v_i}$, with δ_{v_i} the configuration with a single particle at v_i, and by replacing the starting configuration η by η^N. By taking the n-th order derivative with respect to $f(v_i)$ on the left-hand side of (3.2.14) and afterwards setting $f \equiv 1$, we find that

$$\begin{aligned} \frac{\partial^n}{\partial f(v_i)^n} \mathbb{E}_{\eta^N} \left[\prod_{v \in V} f(v)^{\eta_t^N(v)} \right] \Bigg|_{f \equiv 1} &= \mathbb{E}_{\eta^N} \left[\frac{\partial^n}{\partial f(v_i)^n} \prod_{v \in V} f(v)^{\eta_t^N(v)} \right] \Bigg|_{f \equiv 1} \\ &= \mathbb{E}_{\eta^N} \left[\frac{\eta_t(v_i)!}{(\eta_t(v_i) - n)!} \cdot I\big(n \leq \eta_t(v_i)\big) \right]. \end{aligned} \tag{3.2.18}$$

Here we were able to interchange the derivatives and the expectation using dominated convergence. Note that the right-hand side is equal to $\mathbb{E}_{\eta^N}\left[\mathfrak{D}_0(\xi, \eta_t^N)\right]$.

Applying the same operations as in (3.2.18) to the right hand side of (3.2.14), we obtain

$$
\begin{aligned}
&\frac{\partial^n}{\partial f(v_i)^n}\prod_{v\in V}\left(\sum_{w\in V}p_t(v,w)f(w)\right)^{\eta^N(v)}\Bigg|_{f\equiv 1}\\
&=\sum_{m=1}^{n}\sum_{\substack{v^{(1)},\dots,v^{(m)}\in V\\ v^{(i)}\neq v^{(j)}}}\sum_{k_1+\dots+k_m=n}\binom{n}{k_1,\dots,k_m}\prod_{j=1}^{m}\frac{\partial^{k_j}}{\partial f(v_i)^{k_j}}\left(\sum_{w\in V}p_t(v^{(j)},w)f(w)\right)^{\eta^N(v^{(j)})}\Bigg|_{f\equiv 1}\\
&=\sum_{m=1}^{n}\sum_{\substack{v^{(1)},\dots,v^{(m)}\in V\\ v^{(i)}\neq v^{(j)}}}\sum_{k_1+\dots+k_m=n}\binom{n}{k_1,\dots,k_m}\prod_{j=1}^{m}d_0\left(\eta^N(v^{(j)}),k_j\right)p_t\left(v^{(j)},v_i\right)^{k_j}\\
&=\sum_{m=1}^{n}\sum_{\substack{v^{(1)},\dots,v^{(m)}\in V\\ v^{(i)}\neq v^{(j)}}}\sum_{k_1+\dots+k_m=n}\binom{n}{k_1,\dots,k_m}\prod_{j=1}^{m}d_0\left(\eta^N(v^{(j)}),k_j\right)\widehat{p}_t\left(v_i,v^{(j)}\right)^{k_j}.
\end{aligned}
\tag{3.2.19}
$$

Here $\widehat{p}_t(w,v)$ is the transition kernel of a single $\widehat{\text{RTP}}$ particle, and we have used that $p_t(v,w)=\widehat{p}_t(w,v)$ for all $v,w\in V$. Notice that the last line in the above formula is the expected value of $\mathfrak{D}_0(\xi_t,\eta^N)$, i.e.,

$$
\frac{\partial^n}{\partial f(v_i)^n}\prod_{v\in V}\left(\sum_{w\in V}p_t(v,w)f(w)\right)^{\eta^N(v)}\Bigg|_{f\equiv 1}=\widehat{\mathbb{E}}_\xi\left[\mathfrak{D}_0(\xi_t,\eta^N)\right]. \tag{3.2.20}
$$

Combining (3.2.14), (3.2.18) and (3.2.20), we find that

$$
\mathbb{E}_{\eta^N}\left[\mathfrak{D}_0(\xi,\eta_t^N)\right]=\widehat{\mathbb{E}}_\xi\left[\mathfrak{D}_0(\xi_t,\eta^N)\right]. \tag{3.2.21}
$$

The claim now follows from monotone convergence as $N\to\infty$.

If we consider any finite configuration of particles $\xi\in\Omega_{0,f}$, i.e., $\xi=\sum_{i=1}^n\delta_{v_i}$ for some $n\in\mathbb{N}$ and $v_i\in V$, then the duality result can be found by taking the derivative with respect to each $f(v_i)$ on both the left-hand side and right-hand side of the equation (3.2.14). □

3.2.3 *Invariant probability measures*

Proposition 3.2. For the processes defined in Section 3.2.1, the following probability measures denoted by μ_s^ρ are invariant.

1. If $s = -1$, then μ_{-1}^{ϱ} with $\varrho \in [0,1]$ is distributed according to a product Binomial distribution, i.e.,

$$\mu_{-1}^{\varrho}(\eta) = \prod_{v \in V} \binom{\alpha}{\eta(v)} \varrho^{\eta(v)} (1-\varrho)^{\alpha-\eta(v)}. \tag{3.2.22}$$

2. If $s = 1$, then μ_1^{ϱ} with $\varrho \in [0,1)$ is distributed according to a product Negative Binomial distribution, i.e.,

$$\mu_1^{\varrho}(\eta) = \prod_{v \in V} \frac{\Gamma(\alpha + \eta(v))}{\Gamma(\alpha) \cdot \eta(v)!} \varrho^{\eta(v)} (1-\varrho)^{\alpha}. \tag{3.2.23}$$

3. If $s = 0$, then μ_0^{ϱ} with $\varrho \geq 0$ is distributed according to a product Poisson distribution, i.e.,

$$\mu_0^{\varrho}(\eta) = \prod_{v \in V} \frac{\varrho^{\eta(v)}}{\eta(v)!} e^{-\varrho}. \tag{3.2.24}$$

Proof. The first two results are well-known and follow from the fact that the probability measures satisfy the detailed balance condition (see e.g. [14]). For the third result, a system of independent walkers on V with single particle transition rates $p(v,w)$, such that for all $w \in V$

$$\sum_{v \in V} (\varrho(v) p(v,w) - \varrho(w) p(w,v)) = 0 \tag{3.2.25}$$

has invariant product Poisson measures $\bigotimes_{v \in V} \mathrm{Pois}(\varrho(v))$ (see e.g. [28]). Note that in our case we have for all $(y,\sigma') \in V$,

$$\sum_{(x,\sigma) \in V} p((x,\sigma),(y,\sigma')) = \sum_{(x,\sigma) \in V} p((y,\sigma'),(x,\sigma)) = \sum_{u \in \mathbb{Z}^d} \pi_{\sigma'}(u) + \lambda + \sum_{\sigma \in S} c(\sigma,\sigma'). \tag{3.2.26}$$

Therefore, the product Poisson measures with constant density $\varrho > 0$ are invariant. □

The following proposition provides the relation between the probability measures of Proposition 3.2 and the duality functions of Theorem 3.1.

Proposition 3.3. Let $\mu \in \mathscr{P}(\Omega_s)$, then $\mu = \mu_s^{\varrho}$ if and only if for every $\xi \in \Omega_{s,f}$ and every $v \in V$,

$$\int \mathfrak{D}_s(\xi,\eta)\, \mathrm{d}\mu(\eta) = \left(\int \mathfrak{D}_s(\delta_v,\eta)\, \mathrm{d}\mu(\eta) \right)^{|\xi|}. \tag{3.2.27}$$

Proof. A straightforward calculation shows that (3.2.27) holds for μ_s^{ϱ}. The uniqueness property follows from the fact that $\mathfrak{D}_s(\xi,\eta)$ is a (multivariate) polynomial of order at most $|\xi|$. This implies that (3.2.27) is actually a moment problem, which in the case of μ_s^ϱ has a unique solution since the marginals have a finite moment generating function (see e.g. [63]). □

From this relation, the invariance of the probability measures also follows from the conservation of particles in the dual process. Namely, by duality and Fubini we have that

$$\int \mathbb{E}_\eta \left[\mathfrak{D}_s(\xi,\eta_t)\right] d\mu_s^\varrho(\eta) = \mathbb{E}_\xi \left[\int \mathfrak{D}_s(\xi_t,\eta) d\mu_s^\varrho(\eta)\right] = \left(\int \mathfrak{D}_s(\delta_v,\eta)\, \mathrm{d}\mu_s^\varrho(\eta)\right)^{|\xi|}. \tag{3.2.28}$$

3.3 ERGODIC THEORY OF PARTICLE SYSTEMS WITH HOMOGENEOUS FACTORIZED DUALITY POLYNOMIALS

In this section we provide a characterization of the ergodic invariant probability measures satisfying a certain moment growth condition in a general setting where we assume the existence of homogenous factorized duality polynomials, and the existence of successful coupling for the dual process. This generalizes earlier results from [76, Chapter 8] for the symmetric exclusion process, and [66] for the inclusion process. The characterization will be applied in Section 3.4 to our models (see Theorem 3.4 below).

3.3.1 *Basic assumptions*

Configurations

We consider a configuration process $\{\eta_t : t \geq 0\}$ on (a subset of) the state space $\Omega = \mathbb{N}^G$, where G is assumed to be an infinite countable set. We denote by S_t the semigroup of this process, i.e., $S_t f(\eta) = \mathbb{E}_\eta f(\eta_t)$. We further denote by Ω_f the set of finite configurations, i.e., elements of $\xi \in \Omega$ such that $|\xi| = \sum_x \xi(x) < \infty$.

Factorized duality functions

We assume that there exists a duality function

$$\mathfrak{D} : \Omega_f \times \Omega \to \mathbb{R}_+ \tag{3.3.1}$$

such that we have the duality relation

$$\mathbb{E}_\eta \mathfrak{D}(\xi,\eta_t) = \widehat{\mathbb{E}}_\xi \mathfrak{D}(\xi_t,\eta). \tag{3.3.2}$$

We assume that $\mathfrak{D}(\emptyset,\eta) = 1$ where $\emptyset$ denotes the empty configuration. Moreover we assume that the duality functions are in homogeneous factorized form, i.e., of the form

$$\mathfrak{D}(\xi,\eta) = \prod_{x\in G} d(\xi(x),\eta(x)), \tag{3.3.3}$$

where $d(0,n) = 1$, i.e., in the product only a finite number of factors is different from one, and where $d(k,\cdot)$ is a non-negative polynomial of degree k. Moreover, we assume that every polynomial $p(n)$ of degree k can be expressed as a linear combination of the polynomials $d(r,n)$ with $0 \leq r \leq k$.

We assume that both in the process $\{\eta_t : t \geq 0\}$ and in the dual process $\{\xi_t : t \geq 0\}$ the number of particles is conserved.

In the examples of this chapter, the duality functions are multivariate polynomials of degree $|\xi|$, and the dual process $\{\xi_t : t \geq 0\}$ is either the same process (for the interacting examples) or the process obtained by reverting the velocities (the $\widehat{\text{RTP}}$ process defined in Theorem 3.1). In this section we take an abstract point of view and prove under general assumptions a structure theorem for the set of (tempered) invariant probability measures.

Tempered probability measures

Given a duality function, we define the $\mathfrak{D}$-transform of a probability measure μ on the configuration space Ω by

$$\widehat{\mu}(\xi) = \int \mathfrak{D}(\xi,\eta)\, \mathrm{d}\mu(\eta), \tag{3.3.4}$$

where we implicitly assume that for all $\xi \in \Omega_f$, $\mathfrak{D}(\xi,\cdot)$ is μ-integrable.

Definition 3.2. We then say that a probability measure μ is tempered if

1. μ satisfies a uniform moment condition, i.e., for all $n \in \mathbb{N}$

$$c_n := \sup_{|\xi| \leq n} \int \mathfrak{D}(\xi,\eta)\, \mathrm{d}\mu(\eta) < \infty. \tag{3.3.5}$$

2. μ is determined by its $\mathfrak{D}$-transform, i.e., $\widehat{\mu} = \widehat{\nu}$ if and only if $\mu = \nu$.

3. The following space

$$\mathcal{D} = \text{vct}\{\mathfrak{D}(\xi,\cdot) : \xi \in \Omega_f\}, \tag{3.3.6}$$

i.e., the vectorspace spanned by the functions $\mathfrak{D}(\xi,\cdot)$, is dense in $L^2(\mu)$.

Notice that by the assumptions on the duality functions, the condition (3.3.5) can be expressed equivalently by the requirement that all moments of the occupation variables are finite uniformly in x, i.e., for all $n \in \mathbb{N}$

$$\sup_{x \in G} \int \eta(x)^n d\mu(\eta) < \infty. \tag{3.3.7}$$

Using Hölder's inequality, we then also obtain that under (3.3.5) we have that for all $n \in \mathbb{N}$

$$\sup_{\xi,\xi' : |\xi| \leq n, |\xi'| \leq n} \int \mathfrak{D}(\xi,\eta)\mathfrak{D}(\xi',\eta)\, \mathrm{d}\mu(\eta) < \infty. \tag{3.3.8}$$

The condition that the $\mathfrak{D}$-transform determines the probability measure uniquely is implied by a growth condition on c_n which implies that the measure μ is uniquely determined by its multivariate moments. Examples of sufficient growth conditions can be found in e.g. [63, Section 3.2]. In these settings, the condition that $\mathcal{D}$ is dense in $L^2(\mu)$ is also natural. In the setting of processes of exclusion type, i.e., when there are at most α particles at each site, the condition of density of $\mathcal{D}$ is natural and follows from the Stone Weierstrass theorem, i.e., $\mathcal{D}$ is uniformly dense in the set of continuous functions $\mathscr{C}(\Omega)$.

Assumptions on the dual process

For the dual process $\{\xi_t : t \geq 0\}$, we assume that it is irreducible on the sets $\Omega_n = \{\xi : |\xi| = n\}$. Moreover we assume that eventually the process $\{\xi_t : t \geq 0\}$ started at ξ with $|\xi| = n$, spreads out over the infinite set Ω_n. This is expressed via the condition that for all $\xi' \in \Omega_f$

$$\lim_{t\to\infty} \widehat{\mathbb{P}}_\xi(\xi_t \perp \xi') = 1, \tag{3.3.9}$$

where we denote $\xi \perp \xi'$ the event that the supports of ξ and ξ' are disjoint. In words, (3.3.9) means that the probability that the configuration at time t has non-zero occupation at fixed sites tends to zero as $t \to \infty$.

Ergodic probability measures

We denote by $\mathcal{I}$ the set of invariant probability measures of the process $\{\eta_t : t \geq 0\}$ and by $\mathcal{T}$ the set of tempered probability measures on Ω. Both $\mathcal{I}$ and $\mathcal{T}$ are convex sets.

We are then interested in characterizing the ergodic probability measures which belong to $\mathcal{T}$. We recall that a probability measure $\mu \in \mathcal{I}$ is ergodic if, for any $f \in L^2(\mu)$, $S_t f = f$ for all $t \geq 0$ implies $f = \int f d\mu$ almost surely. The set of ergodic probability measures coincides with $\mathcal{I}_e$, the set of extreme points of $\mathcal{I}$. Ergodicity is implied by mixing (see e.g. [123, Section 6.3]) which is the property that for all $f, g \in L^2(\mu)$

$$\lim_{t\to\infty} cov_\mu(f, S_t g) = \lim_{t\to\infty} \int \left(f - \int f\,\mathrm{d}\mu\right)\left(S_t g - \int g\,\mathrm{d}\mu\right) \mathrm{d}\mu = 0. \tag{3.3.10}$$

By bilinearity of the covariance and the fact that, for $\mu \in \mathcal{I}$, S_t is a contraction in $L^2(\mu)$, it suffices to show (3.3.10) for a set of functions $f, g \in W$, where W is such that the vectorspace spanned by W is a dense subspace in $L^2(\mu)$.

3.3.2 *Successful coupling*

We say that the dual process admits a *successful coupling* if for all $n \in \mathbb{N}$, $\xi, \xi' \in \Omega_n$ there exists a coupling $\{(\xi_t^{(1)}, \xi_t^{(2)}) : t \geq 0\}$ of the processes $\{\xi_t : t \geq 0\}$ starting from ξ and ξ' such that the following stopping time

$$\tau_{\xi,\xi'} = \inf\{T > 0 : \xi_t^{(1)} = \xi_t^{(2)} \text{ for all } t \geq T\} \tag{3.3.11}$$

is a.s. finite. We call this stopping time the *coupling time*. For this chapter, we will make use of the following consequence of a successful coupling,

$$\lim_{t\to\infty} \widehat{\mathbb{P}}_{\xi,\xi'}(\xi_t^{(1)} \neq \xi_t^{(2)}) = 0, \tag{3.3.12}$$

where $\widehat{\mathbb{P}}_{\xi,\xi'}$ is the path space probability measure of $\{(\xi_t^{(1)}, \xi_t^{(2)}) : t \geq 0\}$ starting from (ξ, ξ').

3.3.3 *Characterization of tempered invariant probability measures*

The following theorem has two parts: the first parts is well-known and appears in various context, e.g. [76, Chapter 2, Chapter 8]. We give its proof in this general context mainly for the sake of completeness. The second part is inspired by [66] in the context of the inclusion process.

Theorem 3.2.
1. If there exists a succesful coupling for the dual process, then for every tempered invariant probability measure μ there exists a function $f : \mathbb{N} \to [0, \infty)$ such that for all $\xi \in \Omega_n$,
$$\widehat{\mu}(\xi) = f(n). \tag{3.3.13}$$
2. If μ is a probability measure on Ω which is tempered, invariant and ergodic then $f(n) = f(1)^n$. As a consequence, μ is a product measure.

Proof. To prove item 1, by duality and the assumption that μ is tempered, we can use Fubini's theorem combined with duality to compute

$$\begin{aligned}\widehat{\mathbb{E}}_\xi \widehat{\mu}(\xi_t) &= \int \widehat{\mathbb{E}}_\xi(\mathfrak{D}(\xi_t, \eta))\, \mathrm{d}\mu(\eta) \\ &= \int \mathbb{E}_\eta \mathfrak{D}(\xi, \eta_t)\, \mathrm{d}\mu(\eta) \\ &= \int \mathfrak{D}(\xi, \eta)\, \mathrm{d}\mu(\eta) = \widehat{\mu}(\xi).\end{aligned} \tag{3.3.14}$$

Here in the last equality we used the invariance of μ. We conclude that $\widehat{\mu}$ is a harmonic function, which is bounded on each Ω_n by the assumption that μ is tempered.

Therefore, using the assumed existence of a successful coupling, by (3.3.12) together with dominated convergence, we obtain for $\xi, \xi' \in \Omega_n$ the following

$$\begin{aligned} \widehat{\mu}(\xi) &= \widehat{\mathbb{E}}_{\xi,\xi'} \widehat{\mu}(\xi_t^{(1)}) \\ &= \widehat{\mathbb{E}}_{\xi,\xi'} \widehat{\mu}(\xi_t^{(1)}) I(\xi_t^{(1)}) = \xi_t^{(2)})) + o(1) \\ &= \widehat{\mathbb{E}}_{\xi,\xi'} \widehat{\mu}(\xi_t^{(2)}) I(\xi_t^{(1)}) = \xi_t^{(2)})) + o(1) \\ &= \widehat{\mathbb{E}}_{\xi'}(\widehat{\mu}(\xi_t)) + o(1) = \widehat{\mu}(\xi') + o(1), \end{aligned} \tag{3.3.15}$$

where $o(1) \to 0$ as $t \to \infty$. This gives that $\widehat{\mu}$ is constant on Ω_n, i.e., $\widehat{\mu}(\xi) = f(n)$ for some $f : \mathbb{N} \to \mathbb{R}$.

To prove item 2, we start by using the ergodicity combined with duality to write

$$\begin{aligned} &\int \mathfrak{D}(\xi, \eta) \, \mathrm{d}\mu(\eta) \int \mathfrak{D}(\xi', \eta) \, \mathrm{d}\mu(\eta) \\ &= \lim_{T \to \infty} \frac{1}{T} \int_0^T \int \mathfrak{D}(\xi, \eta) S_t \mathfrak{D}(\xi', \cdot)(\eta) \, \mathrm{d}\mu(\eta) \, \mathrm{d}t \\ &= \lim_{T \to \infty} \frac{1}{T} \int_0^T \int \mathfrak{D}(\xi, \eta) \widehat{\mathbb{E}}_{\xi'} \mathfrak{D}(\xi_t . \cdot)(\eta) \, \mathrm{d}\mu(\eta) \, \mathrm{d}t, \end{aligned} \tag{3.3.16}$$

Here in the first step we used that $\frac{1}{T} \int_0^T S_t \mathfrak{D}(\xi', \cdot)(\eta) \, \mathrm{d}t \to \int \mathfrak{D}(\xi', \eta) d\mu(\eta)$ holds almost surely and in $L^1(\mu)$, and in the second step we used duality. Now let $\xi \in \Omega_n$, $\xi' \in \Omega_m$ be given. By item 1, we have $f(n) = \widehat{\mu}(\xi)$, $f(m) = \widehat{\mu}(\xi')$. By the homogeneous factorization of $\mathfrak{D}$, we have for $\xi \in \Omega_n$, $\xi' \in \Omega_m$, $\xi \perp \xi'$

$$\mathfrak{D}(\xi, \eta) \mathfrak{D}(\xi', \eta) = \mathfrak{D}(\xi + \xi', \eta), \tag{3.3.17}$$

and therefore, if $\xi \perp \xi'$, we have that

$$\int \mathfrak{D}(\xi, \eta) \mathfrak{D}(\xi', \eta) \, \mathrm{d}\mu(\eta) = f(n + m). \tag{3.3.18}$$

Now combine (3.3.16) and the assumption (3.3.9) with the temperedness of the probability measure μ to conclude

$$\begin{aligned} f(n) f(m) &= \int \mathfrak{D}(\xi, \eta) \, \mathrm{d}\mu(\eta) \int \mathfrak{D}(\xi', \eta) \, \mathrm{d}\mu(\eta) \\ &= \frac{1}{T} \int_0^T \int \mathfrak{D}(\xi, \eta) \widehat{\mathbb{E}}_{\xi'} \mathfrak{D}(\xi_t, \cdot)(\eta) \, \mathrm{d}\mu(\eta) + o(1) \\ &= \frac{1}{T} \int_0^T \int \mathfrak{D}(\xi, \eta) \widehat{\mathbb{E}}_{\xi'} \mathfrak{D}(\xi_t, \cdot)(\eta) I(\xi_t \perp \xi) \, \mathrm{d}\mu(\eta) + o(1) \\ &= \frac{1}{T} \int_0^T \int \widehat{\mathbb{E}}_{\xi'} \mathfrak{D}(\xi + \xi_t, \cdot)(\eta) I(\xi_t \perp \xi) \, \mathrm{d}\mu(\eta) + o(1) \\ &= \frac{1}{T} \int_0^T \mathbb{E}_{\xi'} \left(I(\xi_t \perp \xi) f(n + m) \right) + o(1) \\ &= f(n + m) + o(1), \end{aligned} \tag{3.3.19}$$

where $o(1) \to 0$ as $T \to \infty$ via (3.3.8), (3.3.5). This proves that for all $\xi \in \Omega_n, \xi' \in \Omega_m$ we have

$$\widehat{\mu}(\xi + \xi') = \widehat{\mu}(\xi)\widehat{\mu}(\xi'), \tag{3.3.20}$$

which gives $f(n) = f(1)^n$. This implies that for all $x_1, \ldots, x_n \in G$, $k_1, \ldots, k_n \in \mathbb{N}$,

$$\int \prod_{i=1}^{n} d(k_i, \eta_{x_i})\, \mathrm{d}\mu(\eta) = f(1)^{k_1+\ldots+k_n} = \prod_{i=1}^{n} \int d(k_i, \eta)\, \mathrm{d}\mu(\eta), \tag{3.3.21}$$

which implies that μ is a product measure. □

In the next theorem we prove that invariant tempered product probability measures are ergodic. This, combined with Theorem 3.2, completes the characterization of the set of tempered ergodic probability measures.

We introduce

$$K := \left\{ \int \mathfrak{D}(\delta_x, \eta) d\mu(\eta) : \mu \text{ is an invariant tempered product probability measure} \right\}. \tag{3.3.22}$$

Theorem 3.3.

1. If μ is an invariant tempered product probability measure then it is ergodic.

2. If there exists a successful coupling for the dual process, then the only tempered invariant probability measures which are ergodic are the product probability measures μ_θ for which $\widehat{\mu}_\theta(\xi) = \theta^{|\xi|}$ with $\theta \in K$.

3. If there exists a successful coupling for the dual process, then

$$(\mathcal{T} \cap \mathcal{I})_e = \mathcal{T} \cap \mathcal{I}_e = \{\mu_\theta : \theta \in K\}, \tag{3.3.23}$$

where $(\mathcal{T} \cap \mathcal{I})_e$ are the extreme points of $\mathcal{T} \cap \mathcal{I}$.

Proof. For item 1, as indicated in the section where we defined mixing, it suffices to show that

$$\lim_{t \to \infty} \int \mathfrak{D}(\xi, \eta) \mathbb{E}_\eta \mathfrak{D}(\xi', \eta_t)\, \mathrm{d}\mu(\eta) = \widehat{\mu}(\xi)\widehat{\mu}(\xi') \tag{3.3.24}$$

because by assumption the vectorspace spanned by the $\mathfrak{D}(\xi, \cdot)$ is dense in $L^2(\mu)$.

Using duality, the assumption (3.3.9), the product character of the probability measure μ as well as the assumed temperedness of μ (cf. (3.3.8)), and denoting $o(1)$ for a term which converges to zero as $t \to \infty$, we get

$$\begin{aligned}
\int \mathfrak{D}(\xi,\eta)\mathbb{E}_\eta \mathfrak{D}(\xi',\eta_t)\,\mathrm{d}\mu(\eta) &= \int \mathfrak{D}(\xi,\eta)\widehat{\mathbb{E}}_{\xi'}\mathfrak{D}(\xi_t,\eta)\,\mathrm{d}\mu(\eta) \\
&= \widehat{\mathbb{E}}_{\xi'} \int \mathfrak{D}(\xi,\eta)\mathfrak{D}(\xi_t,\eta) I(\xi_t \perp \xi)\,\mathrm{d}\mu(\eta) + o(1) \\
&= \left(\int \mathfrak{D}(\xi,\eta)\,\mathrm{d}\mu(\eta) \int \widehat{\mathbb{E}}_{\xi'}\mathfrak{D}(\xi_t,\eta)\,\mathrm{d}\mu(\eta) \right) + o(1) \\
&= \int \mathfrak{D}(\xi,\eta)\,\mathrm{d}\mu(\eta) \int \mathbb{E}_\eta(\mathfrak{D}(\xi',\eta_t))\,\mathrm{d}\mu(\eta) + o(1) \\
&= \int \mathfrak{D}(\xi,\eta)\,\mathrm{d}\mu(\eta) \int \mathfrak{D}(\xi',\eta)\,\mathrm{d}\mu(\eta) + o(1) \\
&= \widehat{\mu}(\xi)\widehat{\mu}(\xi') + o(1). \qquad (3.3.25)
\end{aligned}$$

Item 2 follows immediately from item 1 and item 2 of Theorem 3.2. To prove item 3, we only have to prove that

$$(\mathcal{T} \cap \mathcal{I})_e = \mathcal{T} \cap \mathcal{I}_e. \qquad (3.3.26)$$

The implication "$\mu \in \mathcal{T} \cap \mathcal{I}_e$ implies $\mu \in (\mathcal{T} \cap \mathcal{I})_e$" is obvious. To prove the other implication, start from $\mu \in (\mathcal{T} \cap \mathcal{I})_e$ and assume that we have

$$\mu = \lambda \nu_1 + (1-\lambda)\nu_2, \qquad (3.3.27)$$

with $\nu_1, \nu_2 \in \mathcal{I}$ and $0 < \lambda < 1$. Then we have, because $\mu \in \mathcal{T}$, that $\nu_1, \nu_2 \in \mathcal{T}$, and therefore, $\nu_1, \nu_2 \in \mathcal{T} \cap \mathcal{I}$. But then, using that $\mu \in (\mathcal{T} \cap \mathcal{I})_e$ we have $\mu = \nu_1 = \nu_2$, therefore we conclude that $\mu \in \mathcal{I}_e$. □

3.4 Existence of a successful coupling

We can now state the main result of this chapter, i.e., the characterization of the tempered ergodic probability measures for the three models of Section 3.2.1.

Theorem 3.4. For all $s \in \{-1, 0, 1\}$, the probability measures μ_s^ϱ defined in Proposition 3.2 are the only tempered ergodic probability measures of the process generated by $\mathscr{L}_s$.

By Theorem 3.3 we need to show the the dual processes defined in Section 3.2.2 satisfy the assumptions from Section 3.3.1, along with the existence of a successful coupling. We will start by proving that the original assumptions hold.

The irreducibility of the processes on the sets $\Omega_{s,n} = \{\xi \in \Omega_{s,f} : \sum_x \xi(x) = n\}$ is clear from the irreducibility of the single particle random walk. For (3.3.9), let

$\xi' \in \Omega_{s,n}$ and let $(v_i)_{i=1}^n \subset V$ be the coordinates of the particles in the configuration ξ'. Note then that for every $\xi \in \Omega_{s,f}$

$$\widehat{\mathbb{P}}_\xi(\xi_t \not\leq \xi') \leq \sum_{i=1}^n \widehat{\mathbb{P}}_\xi(\xi_t(v_i) \geq 1) \leq \sum_{i=1}^n \widehat{\mathbb{E}}_\xi[\xi_t(v_i)]. \tag{3.4.1}$$

We are able to write $\xi_t(v_i) = \frac{1}{c_{\alpha,s}} \mathfrak{D}_s(\delta_{v_i}, \xi_t)$ cf. Remark 3.2. Hence, using duality

$$\widehat{\mathbb{E}}_\xi[\xi_t(v_i)] = \mathbb{E}_{v_i}[\xi(v(t))] = \sum_{w \in V} p_t(v_i, w)\xi(w), \tag{3.4.2}$$

where $v(t)$ is the path of a particle under the dynamics of the original process starting from v_i, and $p_t(v, w)$ is the corresponding transition kernel. Here we also used that the dual of the dual is the original process (cf. Theorem 3.1, item 3). Because ξ is finite, the sum on the right-hand side is actually a finite sum, and so (3.3.9) follows if $p_t(v, w) \to 0$ as $t \to \infty$ for all $v, w \in V$. To see that this holds, note that for all $x, y, z \in \mathbb{Z}^d$ and $\sigma, \sigma' \in S$ we have that

$$p_t((x, \sigma), (y, \sigma')) = p_t((x + z, \sigma), (y + z, \sigma')). \tag{3.4.3}$$

Therefore, there can not exist an invariant probability measure for the single particle random walk, which means that the random walk is either null-recurrent or transient. Hence we indeed have that $\lim_{t\to\infty} p_t(v, w) = 0$ for all $v, w \in V$ (see e.g. [68, p. 26]).

In order to prove the existence of a successful coupling for the dual processes we proceed as follows:

First, we consider multi-layer symmetric independent random walkers (IRW) on V where the jump rates depend on the layer, i.e., $\widehat{\text{RTP}}$ with $\lambda = 0$ in (3.2.4), which will be needed for the proof of our models. Second, we deal with interacting particles, distinguishing the transient and recurrent cases for both models. Finally, we prove the existence of a successful coupling for a general RTP, distinguishing the cases where there are random walk jumps and the case where there are only active jumps.

In order to prove the successful coupling of finite configurations with identical particle numbers, we pass to a more convenient labeled particle configuration, i.e., when $\xi \in \mathbb{N}^V$ with $\sum_{v\in V} \xi(v) = n$, then $\xi = \sum_{i=1}^n \delta_{(x_i,\sigma_i)}$ and we identify ξ with $\big((x_1, \sigma_1), ..., (x_n, \sigma_n)\big) \in V^n$ where over the course of time, these initially chosen labels remain fixed. With this prescription, the configuration process ξ_t induces a unique process $\big((X_{1,t}, \sigma_{1,t}), ..., (X_{n,t}, \sigma_{n,t})\big)$ on V^n.

3.4.1 *Successful coupling of multi-layer symmetric IRW*

The proof of existence of a successful coupling of multi-layer symmetric IRW makes use of the Ornstein-coupling which is also used for the existence of a successful coupling of simple symmetric IRW on $\mathbb{Z}^d$. The argument can be found in e.g. [57], however for completion we will give a proof here as well.

Proposition 3.4. For all $n \in \mathbb{N}$ and $\mathbf{y}^{(1)}, \mathbf{y}^{(2)} \in \left(\mathbb{Z}^d\right)^n$, there exists a successful coupling $(\mathbf{Y}_t^{(1)}, \mathbf{Y}_t^{(2)})$ of simple symmetric IRW on $\mathbb{Z}^d$ with initial conditions $\mathbf{Y}_0^{(1)} = \mathbf{y}^{(1)}$ and $\mathbf{Y}_0^{(2)} = \mathbf{y}^{(2)}$.

Proof. Since the particles move independently, we only have to show that there exists a successful coupling of two simple symmetric random walkers on $\mathbb{Z}^d$. Namely, if we can successfully couple any two particles in the two configurations $\mathbf{Y}_t^{(1)} = (Y_{1,t}^{(1)}, ..., Y_{n,t}^{(1)})$ and $\mathbf{Y}_t^{(2)} = (Y_{1,t}^{(2)}, ..., Y_{n,t}^{(2)})$, then every stopping time

$$\tau_i := \inf\left\{T > 0 : Y_{i,t}^{(1)} = Y_{i,t}^{(2)} \text{ for all } t \geq T\right\} \tag{3.4.4}$$

is a.s. finite. Note that the coupling time of $\mathbf{Y}_t^{(1)}$ and $\mathbf{Y}_t^{(2)}$ is then equal to $\tau = \max_{1 \leq i \leq n} \tau_i$, which is therefore also a.s. finite.

For the successful coupling of the pair $Y_{i,t}^{(1)}$ and $Y_{i,t}^{(2)}$, let $\{e_1, e_2, ..., e_d\}$ be the standard basis vectors of $\mathbb{Z}^d$. Then we can write

$$Y_{i,t}^{(1)} - Y_{i,t}^{(2)} = a_{1,t}e_1 + a_{2,t}e_2 + ... + a_{d,t}e_d. \tag{3.4.5}$$

Here every $a_{k,t}$ is a simple symmetric random walk on $\mathbb{Z}$. Now define the stopping times

$$\tau_{a_k} := \inf\{t \geq 0 : a_{k,t} = 0\}. \tag{3.4.6}$$

It is clear that every τ_{a_k} is a.s. finite. After time τ_{a_k} we let the processes $Y_{i,t}^{(1)}$ and $Y_{i,t}^{(2)}$ copy each others jumps in the direction of e_k, i.e. $a_{k,t} = 0$ for all $t \geq \tau_{a_k}$. The proof is now finished after the observation that $\tau_i = \max_{1 \leq k \leq d} \tau_{a_k}$. □

Proposition 3.5. For all $n \in \mathbb{N}$ and $\mathbf{y}^{(1)}, \mathbf{y}^{(2)} \in V^n$, there exists a successful coupling $(\mathbf{Y}_t^{(1)}, \mathbf{Y}_t^{(2)})$ of multi-layer symmetric IRW on V, i.e., $\widehat{\text{RTP}}$ with $\lambda = 0$, with initial conditions $\mathbf{Y}_0^{(1)} = \mathbf{y}^{(1)}$ and $\mathbf{Y}_0^{(2)} = \mathbf{y}^{(2)}$.

Proof. Similarly as in the proof of Proposition 3.4, we only have to show that there exists a successful coupling of two random walkers $(Y_t^{(1)}, \sigma_t^{(1)})$ and $(Y_t^{(2)}, \sigma_t^{(2)})$ on V. Initially we let the two random walkers evolve independently, up until the stopping time ς defined as

$$\varsigma := \inf\{t \geq 0 : \sigma_t^{(1)} = \sigma_t^{(2)}\}. \tag{3.4.7}$$

Note that this stopping time is a.s. finite since the set S is finite and the transition rates $\{c(\sigma, \sigma') : \sigma, \sigma' \in S\}$ on S are irreducible. After the stopping time ς, we let the two random walkers copy each others internal state jumps, i.e., we define the processes $(\check{Y}_t^{(i)}, \check{\sigma}_t^{(i)})$ for $i = 1,2$ such that $(\check{Y}_t^{(i)}, \check{\sigma}_t^{(i)}) = (Y_t^{(i)}, \sigma_t^{(i)})$ for $t \leq \varsigma$ and $\check{\sigma}_{\varsigma+t}^{(i)} = \check{\sigma}_t$ for $t \geq 0$, where $\check{\sigma}_t$ is an internal state process starting from $\check{\sigma}_0 = \sigma_\varsigma^{(1)} = \sigma_\varsigma^{(2)}$.

We can again write

$$\check{Y}_{\varsigma+t}^{(1)} - \check{Y}_{\varsigma+t}^{(2)} = a_{1,t}e_1 + a_{2,t}e_2 + ... + a_{d,t}e_d, \tag{3.4.8}$$

where every $a_{k,t}$ is a continuous-time nearest neighbor symmetric random walk on $\mathbb{Z}$ with (time-dependent) transition rates $2\pi_{\check{\sigma}_t}(e_k) > 0$. We again define the stopping times

$$\tau_{a_k} := \inf\{t \geq 0 : a_{k,t} = 0\}, \tag{3.4.9}$$

and after time τ_{a_k} we let the processes $(\check{Y}_t^{(1)}, \check{\sigma}_t^{(1)})$ and $(\check{Y}_t^{(2)}, \check{\sigma}_t^{(2)})$ copy each others jumps in the direction of e_k. Note that the coupling time τ is now equal to $\tau = \varsigma + \max_{1\leq k\leq d} \tau_{a_k}$, which is a.s. finite. □

3.4.2 *Successful coupling of multi-layer SEP*

Let $\mathbf{X}_t^{(1)}$ and $\mathbf{X}_t^{(2)}$ be two finite configurations of multi-layer SEP particles with the same number of particles. We split the proof of the successful coupling up in two parts, namely the transient case and the recurrent case.

Transient case

Assume $d \geq 3$, then the random walk corresponding to the transition rates $\pi_\sigma(\cdot)$ is transient on $\mathbb{Z}^d$ for every $\sigma \in S$. Let $\mathbf{Y}_t$ be an IRW process on $\mathbb{N}^V$ with finitely many particles. Since the transition rates are transient, for any $R > 1$ and any starting position $\mathbf{y} = (y_1, y_2, ..., y_n)$ such that $||y_i - y_j||_1 > R$ for all $i \neq j$, with positive probability $p(R)$ the particles in $\mathbf{Y}_t$ starting from $\mathbf{y}$ will never have *collisions*. Here a collision means that there is a $t > 0$ such that two particles $(Y_{1,t}, \sigma_{1,t})$ and $(Y_{2,t}, \sigma_{2,t})$ in the configuration $\mathbf{Y}_t$ are at neighboring positions of each other, i.e. we either have

$$||Y_{1,t} - Y_{2,t}||_1 = 1 \quad \text{and} \quad \sigma_{1,t} = \sigma_{2,t}, \tag{3.4.10}$$

or

$$Y_{1,t} = Y_{2,t} \qquad \text{and} \quad c(\sigma_{1,t}, \sigma_{2,t}) > 0. \tag{3.4.11}$$

It follows that, conditional on the event that there are no collisions, the multi-layer SEP particles move the same as multi-layer IRW particles.

We now let the configurations $\mathbf{X}_t^{(1)}$ and $\mathbf{X}_t^{(2)}$ move for some time $T > 0$. We denote by $R(T)$ the minimal distance between two particles in the same configuration at time T, i.e.

$$R(T) := \min_{i \neq j}\{||X_{i,t}^{(1)} - X_{j,t}^{(1)}||_1, ||X_{i,t}^{(2)} - X_{j,t}^{(2)}||_1\}, \tag{3.4.12}$$

with $X_{i,t}^{(1)} \in \mathbb{Z}^d$ the position of particle i in configuration $\mathbf{X}_t^{(1)}$. After time T, we start the coupling attempt by letting the SEP particles copy the jumps of IRW particles starting from $\mathbf{X}_T^{(1)}$ and $\mathbf{X}_T^{(2)}$. By Proposition 3.5, this attempt is successful with probability larger than $p(R(T))$. This proof is now finished by noting that in the transient case, we have that $R(T) \to \infty$ as $T \to \infty$ and $p(R) \to 1$ as $R \to \infty$.

Recurrent case

For the case of $d \leq 2$, where every $\pi_\sigma(\cdot)$ is recurrent, we will define the multi-layer SEP process on the *ladder graph* (see e.g. [3]), i.e., we define the state space $\Omega'_{-1} := \{0,1\}^{V \times A}$ with $A = \{1, 2, ..., \alpha\}$. This space can be seen as the space where on every site $v \in V$ there is a *ladder* with α steps, and every particle chooses a step of this ladder if it moves to a new site. We can easily go back from a configuration $\eta' \in \Omega'_{-1}$ to a configuration in $\eta \in \Omega_{-1}$ by setting

$$\eta(x,\sigma) = \sum_{i=1}^{\alpha} \eta'(x,\sigma,i), \quad \text{for all } (x,\sigma) \in V. \tag{3.4.13}$$

We now define the process on Ω'_{-1} through the generator

$$\mathscr{L}'_{-1} f(\eta') = \sum_{j,k=1}^{\alpha} \sum_{v,w \in V} \frac{p(v,w)}{\alpha} \eta'_{(v,j)} \left(\alpha - \eta'_{(w,k)}\right) \nabla_{(v,j),(w,k)} f(\eta'), \tag{3.4.14}$$

i.e., it is the simple symmetric exclusion process on $V \times A$ where particles choose a step on the ladder A uniformly. It is easy to see that $\mathscr{L}'_{-1}$ on Ω'_{-1} corresponds to the generator $\mathscr{L}_{-1}$ on Ω_{-1} through (3.4.13). The successful coupling of the multi-layer SEP now follows from the successful coupling of the simple symmetric exclusion process on Ω'_{-1}. Since the set $V \times A$ is countable, this result is already known, for example in [76, Chapter VIII].

3.4.3 *Successful coupling of multi-layer SIP*

The successful coupling of SIP on $\mathbb{Z}^d$ has already been shown by Kuoch and Redig in [66] and can be extended to our framework of multi-layer particles. In the transient case, this proof uses the same principle as the proof of a successful coupling for multi-layer SEP above, i.e., we let the particles spread out far enough such that there are no collisions with positive probability, after which the particles move like independent random walkers for which there exists a successful coupling by Proposition 3.5. The proof of the recurrent case actually uses a similar approach as in the transient case, in which it lets the particles spread out over time and afterwards makes a coupling attempt. The probability that this coupling attempt is successful has non-zero probability. If the attempt fails, i.e., there is a collision, a new coupling attempt is made. Since these coupling attempts have non-zero probability of success and are independent, there will be a successful coupling eventually. For more details on both proofs, see [66].

3.4.4 *Successful coupling of $\widehat{RTP}$*

For the $\widehat{\text{RTP}}$ process we will also look at two cases, namely the case with random walk jumps of particles, i.e., $\kappa > 0$ in (3.2.4), and without random walk jumps. For the case of $\kappa > 0$, we will see that the successful coupling of $\widehat{\text{RTP}}$ is a corollary of Proposition 3.5 by copying the active and internal jumps of the process. If $\kappa = 0$, then we will need the additional assumption given in (3.2.5). With this assumption we are able to use a similar argument as in the proof of Proposition 3.5 to prove the existence of a successful coupling.

Successful coupling of $\widehat{RTP}$ with $\kappa > 0$

For (X_t, σ_t) a single $\widehat{\text{RTP}}$ particle, we can decouple the dynamics of X_t through the following decomposition,

$$X_t = Y_t + Z_t, \tag{3.4.15}$$

where Y_t is a symmetric random walk starting from X_0 and Z_t are the active jumps starting from 0, both of which are dependent of σ_0.

Since we are dealing with configurations of independent random walkers again, we only have to prove the existence of a successful coupling of two random walkers $(X_t^{(1)}, \sigma_t^{(1)})$ and $(X_t^{(2)}, \sigma_t^{(2)})$. Similarly as in the proof of Proposition 3.5, we let the two random walkers evolve independently up until the stopping time ς defined as in (3.4.7). Afterwards, we let the random walkers copy each others internal state jumps, i.e., we define the processes $(\check{X}_t^{(i)}, \check{\sigma}_t^{(i)})$ for $i = 1, 2$ as $(\check{X}_t^{(i)}, \check{\sigma}_t^{(i)}) = (X_t^{(i)}, \sigma_t^{(i)})$ for $t \leq \varsigma$, and

$$(\check{X}_{\varsigma+t}^{(i)}, \check{\sigma}_{\varsigma+t}^{(i)}) = (\check{Y}_t^{(i)} + \check{Z}_t, \check{\sigma}_t) \tag{3.4.16}$$

where $\check{Y}_t^{(i)}$ is again a symmetric random walk starting from $X_\varsigma^{(i)}$, $\check{Z}_t$ are again the active jumps starting from 0, and $\check{\sigma}_t$ is as defined in the proof of Proposition 3.5. Note that the difference of the positions of the two processes is now equal to

$$\check{X}_{\varsigma+t}^{(1)} - \check{X}_{\varsigma+t}^{(2)} = \check{Y}_t^{(1)} - \check{Y}_t^{(2)}, \tag{3.4.17}$$

i.e., the difference between two symmetric random walkers. The result now follows from Proposition 3.5.

Successful coupling of $\widehat{RTP}$ with $\kappa = 0$ and $\lambda > 0$

Just as in the previous case, for two $\widehat{\text{RTP}}$ processes, denoted $(X_t^{(1)}, \sigma_t^{(1)})$ and $(X_t^{(2)}, \sigma_t^{(2)})$, we define the stopping time ς as in (3.4.7), and set up the processes $(\check{X}_t^{(i)}, \check{\sigma}_t^{(i)})$ for $i = 1, 2$ such that $(\check{X}_t^{(i)}, \check{\sigma}_t^{(i)}) = (X_t^{(i)}, \sigma_t^{(i)})$ for $t \leq \varsigma$ and $\sigma_{\varsigma+t}^{(i)} = \check{\sigma}_t$ for $t \geq 0$.

By (3.2.5) we can now write

$$\check{X}_{\varsigma+t}^{(1)} - \check{X}_{\varsigma+t}^{(2)} = b_{1,t}v(\sigma_1) + b_{2,t}v(\sigma_2) + \ldots + b_{m,t}v(\sigma_m) \tag{3.4.18}$$

for $m = |S|$, $\sigma_k \in S$ and $b_{k,t} \in \mathbb{Z}$ for all k and $t \geq 0$. For every k, the couple $(b_{k,t}, \check{\sigma}_t)$ is a random walk on $\mathbb{Z} \times S$ with the following dynamics:

- If $\check{\sigma}_t = \sigma_k$, $b_{k,t}$ moves as a continuous-time nearest neighbor symmetric random walker on $\mathbb{Z}$ with rate λ.
- If $\check{\sigma}_t \neq \sigma_k$, $b_{k,t}$ does not move.

Since S is finite and the transition rates $c(\sigma, \sigma')$ are irreducible, these random walks are recurrent as their discrete counterparts are recurrent. This implies that the stopping times

$$\tau_{b_k} := \inf\{t \geq 0 : b_{k,t} = 0\} \tag{3.4.19}$$

are almost surely finite. After every time τ_{b_k}, we let the process $\check{X}_{\varsigma+t}^{(2)}$ copy the jumps of $\check{X}_{\varsigma+t}^{(1)}$ in the direction of $v(\sigma_k)$. For the coupling time τ, we then again have that $\tau = \varsigma + \max_{1 \leq k \leq m} \tau_{b_k}$.

Remark 3.3. We are able to extend the results of this section to the case where we take S countable. We would need the additional assumption that the transition rates $c(\sigma, \sigma')$ are positive recurrent, which ensures that we return to any $\sigma \in S$ in almost surely finite time. For the interacting particles we would then distinguish between the cases where the transition rates $p(v, w)$ in (3.2.1) are transient and recurrent (note that the latter need not be the case where $d \leq 2$). For the $\widehat{\text{RTP}}$, the positive recurrence of $c(\sigma, \sigma')$ ensures that every stopping time τ_{b_k} in (3.4.19) is almost surely finite.

4

STATIONARY FLUCTUATIONS OF RUN-AND-TUMBLE PARTICLES

In the last chapter we showed that the product Poisson measures with constant density are ergodic for the run-and-tumble particle system. In this chapter[1] we continue studying run-and-tumble particles and determine the hydrodynamic limit and the stationary fluctuations. For the latter, we need to start our process from an ergodic measure, and we then prove that the fluctuations converge to an infinite dimensional Ornstein Uhlenbeck process. We discuss also an interacting case, where the particles are subjected to exclusion. We then study the fluctuations of the total density, which is a non-Markovian Gaussian process. By considering small noise limits of this process, we obtain in a concrete example a large deviation rate function containing memory terms.

4.1 INTRODUCTION

In this chapter we consider a system of independent run-and-tumble particles on $\mathbb{Z}$ and study the stationary fluctuations of its empirical distribution. Because particles have positions and internal states (which determine the direction in which they move and/or their rate of hopping over lattice edges), the hydrodynamic limit is a system of linear reaction-diffusion equations, describing the macroscopic joint evolution of the densities of particles with a given internal state. In this sense, this chapter can be viewed as a study of macroscopic properties of the multi-layer particle systems which we studied in Chapter 3. The study of hydrodynamic limits and fluctuations around the hydrodynamic limit for particles with internal states, or alternatively, multi-layer systems is quite recent, and to our knowledge at present only a limited set of results is known: see [64], [37], [3], [101].

Our interest in multi-layer systems is motivated from the study of active particles (see e.g. [23]), the study of double diffusivity models (see e.g. [37] and references therein), and finally the study of particle systems described macrosopically by equations containing memory terms. In this chapter we consider multi-layer

1 This chapter is based on [98].

systems in which duality can be applied. Duality is a powerful tool which reduces the study of the hydrodynamic limit to the scaling limit of a single (dual) particle, and as we show in this chapter (see Section 4.3.1 below) also determines uniquely the covariance of the stationary fluctuations of the empirical density of particles. Provided one can show that the stationary fluctuations converge to a Gaussian limiting (distribution-valued) process, this limiting covariance uniquely determines the limiting stationary Gaussian process.

In this chapter we prove that the fluctuation fields of the densities of particles with given internal state converge to a system of stochastic partial differential equations. In these limiting equations, the drift is determined by the hydrodynamic limit, whereas the noise has both a conservative part coming from the transport of particles with a given internal state as well as a non-conservative part coming from the flipping of internal states. We first deal with a system of independent particles, which has a simple dual consisting of independent particles with reversed velocities. Next we indicate how to deal with interacting particles such as layered exclusion processes, where still duality can be used.

One of our motivations of studying fluctuation fields of particles with internal states is to understand fluctuation properties of the total density, i.e., disregarding the internal states of the particles. The configuration which gives at each site the total number of particles is one of the simplest examples of a non-Markovian interacting particle system. The study of the hydrodynamic limit, fluctuations and large deviations around the hydrodynamic limit for non-Markovian particle systems is largely terra incognita. Therefore, we believe that simple examples in which one can have some grip on the explicit form of fluctuations and large deviations are important to obtain.

In our setting, we prove that the fluctuations of the total density of particles converges to a Gaussian distribution-valued process which satisfies a non-Markovian SPDE. We provide a concrete example where we can explicitly characterize the large deviations of the limiting SPDE in the small noise limit. These large deviations give an indication of the large deviations of the total density of particles. The latter can of course also be obtained via a contraction principle from the large deviations of the joint densities of particles with a given internal state. However, the large deviation rate function obtained via this contraction principle is very implicit, and therefore in this chapter we preferred not to follow this road in order to obtain an explicit form of the memory terms of the rate function.

The rest of this chapter is organized as follows. In Section 4.2 we introduce the run-and-tumble particle model and state preliminary results on ergodic measures, duality and hydrodynamic limit, the latter of which will be proven in Section 4.6. In Section 4.3 we state the main result on stationary fluctuations for independent particles, Theorem 4.3, provide a direct proof of the limiting covariance in Section 4.3.1, and consider an interacting case, namely a multi-layer version of the symmetric exclusion process, in Section 4.3.2. In Section 4.4 we study the hydrodynamic limit and the fluctuations of the total density of particles, and prove a large deviations result for the limiting fluctuation process in a particular case. In Section 4.5 we prove the Theorem 4.3.

4.2 BASIC NOTATIONS AND DEFINITIONS

Let $V := \mathbb{Z} \times S$, with $S \subset \mathbb{Z}$ a finite set. The set V is the state space of a single run-and-tumble particle. We see elements $v = (x, \sigma) \in V$ as particles with position $x \in \mathbb{Z}$ and internal state $\sigma \in S$. The dynamics of a single run-and-tumble particle are now as follows

i. At rate κN^2 the particle performs a nearest neighbor jump, i.e., $(x, \sigma) \to (x \pm 1, \sigma)$
ii. At rate λN the particle performs an active jump in the direction of its internal state, i.e., $(x, \sigma) \to (x + \sigma, \sigma)$.
iii. At rate $c(\sigma, \sigma')$ the particle changes its internal state from σ to σ', i.e. $(x, \sigma) \to (x, \sigma')$. Here we assume that the rates $\{c(\sigma, \sigma') : \sigma, \sigma' \in S\}$ are irreducible and symmetric, i.e., $c(\sigma, \sigma') = c(\sigma', \sigma)$.

The run-and-tumble particle process is the Markov process $\{\eta_t : t \geq 0\}$ on the state space $\Omega := \mathbb{N}^V$ consisting of independent random walkers on V where every particle has the dynamics as described above.

From the dynamics we can write down the following generator L_N acting on local functions, i.e., functions $f : \Omega \to \mathbb{R}$ which only depend on a finite number of sites in V.

$$\begin{aligned} L_N f(\eta) = \kappa N^2 &\sum_{(x,\sigma)\in V} \eta(x,\sigma) \left(f\left(\eta^{(x,\sigma)\to(x+1,\sigma)}\right) + f\left(\eta^{(x,\sigma)\to(x-1,\sigma)}\right) - 2f(\eta) \right) \\ &+ \lambda N \sum_{(x,\sigma)\in V} \eta(x,\sigma) \left(f\left(\eta^{(x,\sigma)\to(x+\sigma,\sigma)}\right) - f(\eta) \right) \\ &+ \sum_{(x,\sigma)\in V} \sum_{\sigma'\in S} \eta(x,\sigma) c(\sigma,\sigma') \left(f\left(\eta^{(x,\sigma)\to(x,\sigma')}\right) - f(\eta) \right). \end{aligned} \tag{4.2.1}$$

Here $\eta(x, \sigma)$ denotes the number of particles at site $(x, \sigma) \in V$ in the configuration η, and $\eta^{(x,\sigma)\to(y,\sigma')}$ denotes the configuration η where a single particle has moved from (x, σ) to (y, σ').

With this choice of scaling, in the macroscopic limit, the densities of particles with a given internal state satisfy a system of linear reaction-diffusion equations (see Section 4.4.1 below for the explicit form). Equivalently, one can view the choice of scaling as a diffusive time scale ($t \to N^2 t$), a weak asymmetry (active jumps in the direction of the velocity occur at rate $N = N^{-1}N^2$), and a slow reaction term (changes of internal state happen at rate $1 = N^{-2}N^2$. The scaling is also such that the motion of a single particle converges to a multi-layer Brownian motion with layer-dependent drift (cf. Section 4.2.1 below).

4.2.1 *Scaling limit of the single particle dynamics*

We will denote by $\mathscr{L}_N$ the Markov generator of a single run-and-tumble particle (rescaled in space), more precisely, the generator of the process $(\frac{X_t}{N}, \sigma_t)$ where X_t denotes the position and σ_t the internal state of the particle.

This generator acts on a core consisting of test functions on the space $\mathbb{R} \times S$, which we denote by $C^\infty_{c,S}$, and which is defined via

$$C^\infty_{c,S} := \{\phi : \mathbb{R} \times S \to \mathbb{R} : \phi(\cdot, \sigma) \in C^\infty_c(\mathbb{R}) \text{ for all } \sigma \in S\}. \tag{4.2.2}$$

The generator $\mathscr{L}_N$ then reads as follows:

$$\begin{aligned}\mathscr{L}_N\phi(x,\sigma) = \kappa N^2(&\phi(x + \tfrac{1}{N}, \sigma) + \phi(x - \tfrac{1}{N}, \sigma) - 2\phi(x,\sigma)) \\ &+ \lambda N(\phi(x + \tfrac{\sigma}{N}, \sigma) - \phi(x,\sigma)) \\ &+ \sum_{\sigma' \in S} c(\sigma,\sigma')(\phi(x,\sigma') - \phi(x,\sigma)).\end{aligned} \tag{4.2.3}$$

Corresponding to this generator we have the corresponding Markov semigroup which we denote by S^N_t. Via Taylor approximation we obtain that $\mathscr{L}_N\phi \to A\phi$ uniformly as $N \to \infty$, where A is the differential operator given by

$$A\phi(x,\sigma) = \left(\tfrac{\kappa}{2}\partial_{xx} + \sigma\lambda\partial_x\right)\phi(x,\sigma) + \sum_{\sigma' \in S} c(\sigma,\sigma')\left(\phi(x,\sigma') - \phi(x,\sigma)\right). \tag{4.2.4}$$

Because A generates a Markov semigroup as well, as a consequence of the convergence of the generators we can also obtain $S^N_t\phi \to e^{tA}\phi$ uniformly for all $\phi \in C_{0,S}$, i.e., the functions space consisting of functions $\phi : \mathbb{R} \times S \to \mathbb{R}$ such that $\phi(\cdot,\sigma) \in C_0(\mathbb{R})$ for all $\sigma \in S$.

The operator A above is also an operator on (a subset of) the Hilbert space $L^2(\mathrm{d}x \times |\cdot|_S)$, where $|\cdot|_S$ is the counting measure over S. The inner product on this Hilbert space, denoted by $\langle\langle\cdot,\cdot\rangle\rangle$, is the following

$$\langle\langle\phi,\psi\rangle\rangle := \sum_{\sigma\in S}\int_{\mathbb{R}} \phi(x,\sigma)\psi(x,\sigma)\,\mathrm{d}x. \tag{4.2.5}$$

Later on we will need the adjoint of the operator A with respect to this inner product, which acts on $\phi \in C^\infty_{c,S}$ as follows:

$$A^*\phi(x,\sigma) = \left(\tfrac{\kappa}{2}\partial_{xx} - \sigma\lambda\partial_x\right)\phi(x,\sigma) + \sum_{\sigma' \in S} c(\sigma,\sigma')\left(\phi(x,\sigma') - \phi(x,\sigma)\right). \tag{4.2.6}$$

4.2.2 *Basic properties of independent run-and-tumble particles*

Before we state the theorem of the stationary fluctuations, we first review a few known results on run-and-tumble particles which we need.

Stationary ergodic product measures

We define the measures μ^{ϱ}, with $\varrho \in [0,\infty)$, as the product Poisson measure with density ϱ, i.e.

$$\mu^{\varrho} := \bigotimes_{(x,\sigma)\in V} \mathrm{Pois}(\varrho). \tag{4.2.7}$$

In Chapter 3 we proved that these measures are stationary and ergodic with respect for run-and-tumble particle process $\{\eta_t : t \geq 0\}$. For this reason, when we study the stationary fluctuations of the densities of particles with given internal state, we will start the process $\{\eta_t : t \geq 0\}$ from the measure μ^{ϱ}.

Duality

Definition 4.1. We say that two Markov processes $\{\eta_t : t \geq 0\}$ and $\{\xi_t : t \geq 0\}$, on the state spaces Ω and Ω' respectively, are *dual* to one another with respect to a duality function $\mathfrak{D} : \Omega \times \Omega' \to \mathbb{R}$ if

$$\mathbb{E}_{\eta}\left[\mathfrak{D}(\xi,\eta_t)\right] = \widehat{\mathbb{E}}_{\xi}\left[\mathfrak{D}(\xi_t,\eta)\right] < \infty, \tag{4.2.8}$$

where $\mathbb{E}_{\eta}$ denotes the expectation in $\{\eta_t : t \geq 0\}$ starting from η and $\widehat{\mathbb{E}}_{\xi}$ the expectation in the dual process $\{\xi_t : t \geq 0\}$ starting from ξ.

In Chapter 3 we proved that the run-and-tumble particle process is dual to its time-reversed process where the active jumps are in the reverse direction, i.e., the process corresponding to the following generator

$$\begin{aligned}\widehat{L}_N f(\eta) = \kappa N^2 &\sum_{(x,\sigma)\in V} \eta(x,\sigma)\left(f\left(\eta^{(x,\sigma)\to(x+1,\sigma)}\right) + f\left(\eta^{(x,\sigma)\to(x-1,\sigma)}\right) - 2f(\eta)\right)\\ &+ \lambda N \sum_{(x,\sigma)\in V} \eta(x,\sigma)\left(f\left(\eta^{(x,\sigma)\to(x-\sigma,\sigma)}\right) - f(\eta)\right)\\ &+ \sum_{(x,\sigma)\in V}\sum_{\sigma'\in S} \eta(x,\sigma)c(\sigma,\sigma')\left(f\left(\eta^{(x,\sigma)\to(x,\sigma')}\right) - f(\eta)\right).\end{aligned} \tag{4.2.9}$$

The duality function is then given by

$$\mathfrak{D}(\xi,\eta) = \prod_{(x,\sigma)\in V} \frac{\eta(x,\sigma)!}{\xi(x,\sigma)!(\eta(x,\sigma)-\xi(x,\sigma))!} \cdot I\big(\xi(x,\sigma) \leq \eta(x,\sigma)\big), \tag{4.2.10}$$

where I denotes the indicator function, and where ξ is assumed to be a finite configuration, i.e.,

$$\sum_{(x,\sigma)} \xi(x,\sigma) < \infty \tag{4.2.11}$$

In this chapter we will mostly need this duality relation in the form of duality with a single dual particle, i.e.,

$$\mathbb{E}_\eta[\eta_t(x,\sigma)] = \widehat{\mathbb{E}}_{(x,\sigma)}[\eta(\widehat{X}_t,\widehat{\sigma}_t)], \tag{4.2.12}$$

where $(\frac{\widehat{X}_t}{N},\widehat{\sigma}_t)$ is the process corresponding to the (time-reversed) generator $\widehat{\mathscr{L}_N}$ given by

$$\begin{aligned}\widehat{\mathscr{L}_N}\phi(x,\sigma) = \kappa N^2(&\phi(x+\tfrac{1}{N},\sigma)+\phi(x-\tfrac{1}{N},\sigma)-2\phi(x,\sigma))\\ &+\lambda N(\phi(x-\tfrac{\sigma}{N},\sigma)-\phi(x,\sigma))\\ &+\sum_{\sigma'\in S} c(\sigma,\sigma')(\phi(x,\sigma')-\phi(x,\sigma)).\end{aligned} \tag{4.2.13}$$

We denote the corresponding Markov semigroup of this process as $\widehat{S}_t^N$. By a Taylor expansion, we obtain that $\widehat{\mathscr{L}_N}\phi \to A^*\phi$, with A^* defined as in (4.2.6), uniformly in N for all $\phi \in C^\infty_{c,S}$, and therefore we are able to write for all $\phi \in C_{0,S}$ that $\widehat{S}_t^N\phi \to e^{tA^*}\phi$ uniformly.

4.2.3 *Hydrodynamic limit*

In this section we will briefly mention the hydrodynamic limit of the run-and-tumble particle process. For the proof, which follows standard methodology, we refer to the appendix.

Given a function $\varrho : \mathbb{R}\times S \to \mathbb{R}$ such that $\varrho(\cdot,\sigma) \in C_b^2(\mathbb{R})$ for all $\sigma \in S$, we start by defining the product Poisson measures μ_N^ϱ for every $N \in \mathbb{N}$ as follows

$$\mu_N^\varrho := \bigotimes_{(x,\sigma)\in V} \text{Pois}\big(\varrho(\tfrac{x}{N},\sigma)\big). \tag{4.2.14}$$

This is the local equilibrium distribution corresponding to the macroscopic profile ϱ.

Furthermore, for every $N \in \mathbb{N}$, the process $\{\eta_t^N : t \geq 0\}$ is the run-and-tumble particle process started from $\eta_0^N \sim \mu_N^\varrho$. We can now define the empirical measures of the process, denoted by $\pi^N_{[0,T]} = \{\pi_t^N : t \geq 0\}$, as follows

$$\pi_t^N := \frac{1}{N}\sum_{(x,\sigma)\in V} \eta_t^N(x,\sigma)\delta_{(\frac{x}{N},\sigma)}, \tag{4.2.15}$$

where δ is the dirac measure. We think of π_t^N as the macroscopic profile corresponding to the microscopic configuration η_t. In the rhs of (4.2.15) every particle of type σ contributes a mass $1/N$ at the "macro spatial location" x/N.

For every $t \geq 0$, π_t^N is a positive measure in the space of Radon measures on $\mathbb{R} \times S$, denoted $\mathcal{M}$, such that when paired with a test function $\phi \in C_{c,S}^\infty$ we obtain

$$\left\langle \pi_t^N, \phi \right\rangle = \frac{1}{N} \sum_{(x,\sigma)\in V} \eta_t^N(x,\sigma)\phi(\tfrac{x}{N},\sigma). \tag{4.2.16}$$

By the choice of the initial distribution, we have at time $t = 0$ zero that

$$\langle \pi_0^N, \phi \rangle \to \int \varrho(x,\sigma)\phi(x,\sigma)dx \tag{4.2.17}$$

We then have the following result for the hydrodynamic limit.

Theorem 4.1. For every $t \geq 0$, $\varepsilon > 0$ and $\phi \in C_{c,S}^\infty$, we have that

$$\lim_{N\to\infty} \mathbb{P}\left(\left| \pi_t^N(\phi) - \sum_{\sigma\in S} \int \varrho_t(x,\sigma)\phi(x,\sigma)dx \right| > \varepsilon \right) = 0, \tag{4.2.18}$$

where $\varrho_t(x,\sigma)$ solves the PDE $\dot{\varrho}_t = A^*\varrho_t$ with initial condition $\varrho_0(x,\sigma) = \varrho(x,\sigma)$.

This results is actually a corollary of an stronger theorem which shows convergence of the trajectories $\pi_{[0,T]}^N$ in the path space $D([0,T];\mathcal{M})$ equipped with the Skorokhod topology. Let $\alpha = \{\alpha_t : t \geq 0\}$ denote the trajectory of measures on $\mathbb{R} \times S$ such that for all $t \geq 0, \phi \in C_{c,S}^\infty$ we have that $\langle \alpha_t, \phi \rangle = \langle \langle \phi, \varrho_t \rangle \rangle$, where ϱ_t solves the PDE in the above theorem. The trajectory α is then the unique continuous path in $D([0,T];\mathcal{M})$ such that for all $\phi \in C_{c,S}^\infty$

$$\mathscr{M}_t^\phi(\alpha) = \langle \alpha_t, \phi \rangle - \langle \alpha_0, \phi \rangle - \int_0^t \langle \alpha_s, A\phi \rangle \, \mathrm{d}s = 0. \tag{4.2.19}$$

Theorem 4.2. For any $N \in \mathbb{N}$, let P^N be the law of the process $\pi_{[0,T]}^N$. Then $P^N \to \delta_\alpha$ weakly in $D([0,T];\mathcal{M})$ for any $T > 0$, with α the unique continuous path solving (4.2.19).

For the sake of self-containedness, the proof of Theorem 4.2 is provided in the appendix. The method of proof is standard and it follows Seppäläinen, in [104, Chapter 8].

4.2.4 *Fluctuation fields*

For every $N \in \mathbb{N}$ we start $\{\eta_t^N : t \geq 0\}$ from the ergodic measure μ^ϱ with $\varrho > 0$ constant. We then define the fluctuation field $Y_{[0,T]}^N := \{Y_t^N : t \in [0,T]\}$ as

$$Y_t^N = \frac{1}{\sqrt{N}} \sum_{x\in\mathbb{Z}} (\eta_t^N(x,\sigma) - \varrho)\delta_{(\frac{x}{N},\sigma)}. \tag{4.2.20}$$

This process takes values in the space of distributions on $\mathbb{R} \times S$, denoted by $(C^\infty_{c,S})^*$. We expect the fluctuation field $Y^N_{[0,T]}$ to converge weakly to a generalized stationary Ornstein-Uhlenbeck process. Before we can state the result we first recall some basic definitions of space-time white noise (see e.g. [56] for a detailed account).

Definition 4.2. A random distribution $\mathscr{W}$ is called a *white noise* on $\mathbb{R} \times S$ if $\{\langle \phi, \mathscr{W} \rangle : \phi \in C^\infty_{c,s}\}$ is jointly centered Gaussian with covariance

$$\mathbb{E}[\langle \mathscr{W}, \phi \rangle \langle \mathscr{W}, \psi \rangle] = \langle\langle \phi, \psi \rangle\rangle . \tag{4.2.21}$$

where $\langle\langle , \rangle\rangle$ denotes the inner product defined in (4.2.5). We denote by $\mathrm{d}\mathscr{W}_t$ the time-differential of space-time white noise. This object is such that when paired with a test function $\phi \in C^\infty_{c,S}$ and integrated over time gives a Brownian motion, i.e.,

$$\int_0^t \langle \mathrm{d}\mathscr{W}_s, \phi \rangle = B(\langle\langle \phi, \phi \rangle\rangle\, t), \tag{4.2.22}$$

where $B(\cdot)$ is a standard Brownian motion on $\mathbb{R}$. We denote by $\frac{\mathrm{d}\mathscr{W}_t}{\mathrm{d}t}$ the corresponding space-time white noise. This random space-time distribution is such that for all $\phi : [0,T] \times \mathbb{R} \times S \to \mathbb{R}$, with $\phi(t,\cdot)$ a test function $\langle \frac{\mathrm{d}\mathscr{W}_t}{\mathrm{d}t}, \phi \rangle$ is jointly Gaussian with covariance

$$\mathbb{E}\left[\left\langle \frac{\mathrm{d}\mathscr{W}_t}{\mathrm{d}t}, \phi \right\rangle \left\langle \frac{\mathrm{d}\mathscr{W}_t}{\mathrm{d}t}, \psi \right\rangle\right] = \int_0^T \langle\langle \phi(t,\cdot), \psi(t,\cdot) \rangle\rangle \,\mathrm{d}t. \tag{4.2.23}$$

Remark 4.1. Informally speaking, a white noise on $\mathbb{R} \times S$ is a Gaussian field $W(x,\sigma)$ with covariance $\delta(x-y)\delta_{\sigma,\sigma'}$, and a space-time white noise on $\mathbb{R} \times S$ is a Gaussian field $W(t,x,\sigma)$ with covariance $\delta(t'-t)\delta(x-y)\delta_{\sigma,\sigma'}$.

4.3 STATIONARY FLUCTUATIONS

We are now ready to state our result on stationary fluctuations. We start with the case of independent particles; in Section 4.3.2 below we will consider an interacting case.

Theorem 4.3. Assume that η_0 is distributed according to the Poisson product measure μ^ϱ. For every $N \in \mathbb{N}$, let Q^N denote the law of the process $Y^N_{[0,T]}$ defined in (4.2.20). Then $Q^N \to Q$ weakly in $D([0,T]; (C^\infty_{c,S})^*)$ for any $T > 0$, where Q is the law of the stationary Gaussian process Y satisfying the following SPDE

$$\mathrm{d}Y_t = A^* Y_t \,\mathrm{d}t + \sqrt{2\kappa\varrho}\,\partial_x \,\mathrm{d}\mathscr{W}_t + \sqrt{2\varrho\Sigma}\,\mathrm{d}\widetilde{\mathscr{W}_t}. \tag{4.3.1}$$

Here $\mathrm{d}\mathscr{W}_t$ and $\mathrm{d}\widetilde{\mathscr{W}_t}$ are two independent space-time white noises on the space $\mathbb{R} \times S$, and Σ is the operator working on test functions $\phi \in C^\infty_{c,S}$ as

$$(\Sigma\phi)(x,\sigma) = -\sum_{\sigma' \in S} c(\sigma,\sigma')\big(\phi(x,\sigma') - \phi(x,\sigma)\big). \tag{4.3.2}$$

By the assumed symmetry of the rates $c(\sigma,\sigma')$, for $\phi,\psi \in C^\infty_{c,S}$ we have $\langle\langle \Sigma\phi,\psi\rangle\rangle = \langle\langle \phi,\Sigma\psi\rangle\rangle$, and moreover $\langle\langle \Sigma\phi,\psi\rangle\rangle \geq 0$. Hence the operator is bounded, self-adjoint and non-negative and therefore its square root $\sqrt{\Sigma}$ is well-defined. The process $\partial_x \, \mathrm{d}\mathscr{W}_t$ is defined as the process of distributions such that for all $\phi \in C^\infty_{c,S}$

$$\langle \partial_x \, \mathrm{d}\mathscr{W}_t, \phi\rangle = -\langle \mathrm{d}\mathscr{W}_t, \partial_x\phi\rangle \,. \tag{4.3.3}$$

The rigorous meaning of the SPDE in (4.3.1) is defined in terms of a martingale problem as in [61]. More precisely, the map $\phi \mapsto \langle Y_t,\phi\rangle$ is the solution of the following martingale problem: for every $\phi \in C^\infty_{c,S}$, the following two processes

$$\begin{aligned}\mathscr{M}^\phi_t(Y_{[0,T]}) &= \langle Y_t,\phi\rangle - \langle Y_0,\phi\rangle - \int_0^t \langle Y_s, A\phi\rangle ds,\\ \mathscr{N}^\phi_t(Y_{[0,T]}) &= \mathscr{M}^\phi_t(Y_{[0,T]})^2 - 2t\kappa\varrho\,\langle\langle \partial_x\phi,\partial_x\phi\rangle\rangle - 2t\varrho\,\langle\langle\phi,\Sigma\phi\rangle\rangle\end{aligned} \tag{4.3.4}$$

are martingales with respect to the natural filtration $\mathscr{F}_t = \sigma(Y_s : 0 \leq s \leq t)$.

4.3.1 *Stationary covariance of the fluctuation fields via duality*

We will first compare the covariance structure of the limiting process of $Y^N_{[0,T]}$ with the covariance structure of the process solving the SPDE in (4.3.1). This covariance uniquely characterizes the process. More precisely, if we can prove that $Y^N_t \to Y_t$ where Y_t is a distribution-valued stationary Gaussian process, then the covariance $\mathbb{E}(\langle Y_t,\phi\rangle\langle Y_0,\psi\rangle)$ uniquely determines this process. In that sense, the computation of the covariance already determines the only possible candidate limit Y_t. As we show below, the covariance is in turn completely determined by the scaling limit of a single dual particle. This shows that for systems with duality, both the hydrodynamic limit and the stationary fluctuations are uniquely determined by the scaling limit of a single dual particle.

Proposition 4.1. For all $\phi,\psi \in C^\infty_c(\mathbb{R}\times S)$

$$\lim_{N\to\infty} \mathbb{E}[\langle Y^N_t,\phi\rangle\langle Y^N_0,\psi\rangle] = \mathbb{E}[\langle Y_t,\phi\rangle\langle Y_0,\psi\rangle] = \varrho\left\langle\left\langle e^{tA}\phi,\psi\right\rangle\right\rangle. \tag{4.3.5}$$

Here $\mathbb{E}$ denotes the stationary expectation starting from the initial configuration distributed according to $\eta_0 \sim \mu^\varrho$.

Proof. If Y is a solution to the SPDE in (4.3.1), then we can write

$$\langle Y_t,\phi\rangle = \mathscr{M}^\phi_t(Y_{[0,T]}) + \langle Y_0,\phi\rangle + \int_0^t \langle Y_s, A\phi\rangle \,\mathrm{d}s, \tag{4.3.6}$$

where $\mathscr{M}_t^{\phi}(Y_{[0,T]})$ is a martingale with respect to the filtration $\mathscr{F}_t = \sigma(Y_s : 0 \le s \le t)$ such that $\mathscr{M}_0^{\phi}(Y_{[0,T]}) = 0$. By the martingale property we have that

$$\begin{aligned}\mathbb{E}[\mathscr{M}_t^{\phi}(Y_{[0,T]})\langle Y_0, \psi\rangle] &= \mathbb{E}\left[\mathbb{E}[\mathscr{M}_t^{\phi}(Y_{[0,T]})\langle Y_0, \psi\rangle | \mathscr{F}_0]\right] \\ &= \mathbb{E}\left[\langle Y_0, \psi\rangle \mathbb{E}[\mathscr{M}_t^{\phi}(Y_{[0,T]})|\mathscr{F}_0]\right] = 0, \end{aligned} \tag{4.3.7}$$

and so

$$\mathbb{E}[\langle Y_t, \phi\rangle\langle Y_0, \psi\rangle] = \mathbb{E}[\langle Y_0, \phi\rangle\langle Y_0, \psi\rangle] + \int_0^t \mathbb{E}[\langle Y_s, A\phi\rangle\langle Y_0, \psi\rangle]\, \mathrm{d}s. \tag{4.3.8}$$

Therefore, using that $\mathbb{E}[\langle Y_0, \phi\rangle\langle Y_0, \psi\rangle] = \varrho \langle\langle \phi, \psi\rangle\rangle$ we obtain that if Y is a solution of (4.3.1), then we have

$$\mathbb{E}[\langle Y_t, \phi\rangle\langle Y_0, \psi\rangle] = \mathbb{E}[\langle Y_0, e^{tA}\phi\rangle\langle Y_0, \psi\rangle] = \varrho \left\langle\left\langle e^{tA}\phi, \psi\right\rangle\right\rangle. \tag{4.3.9}$$

On the other hand, for any $N \in \mathbb{N}$ we have that

$$\begin{aligned}&\mathbb{E}\left[\langle Y_t^N, \phi\rangle\langle Y_0^N, \psi\rangle\right] \\ &= \frac{1}{N}\sum_{(x,\sigma)\in V}\sum_{(y,\sigma')\in V} \phi(\tfrac{x}{N}, \sigma)\psi(\tfrac{y}{N}, \sigma') \int \mathbb{E}_{\eta}\left[(\eta_t(x,\sigma) - \varrho)(\eta(y,\sigma') - \varrho)\right] \mathrm{d}\mu^{\varrho}(\eta) \\ &= \frac{1}{N}\sum_{(x,\sigma)\in V}\sum_{(y,\sigma')\in V} \phi(\tfrac{x}{N}, \sigma)\psi(\tfrac{y}{N}, \sigma') \int \widehat{\mathbb{E}}_{(x,\sigma)}\left[(\eta(\widehat{X}_t, \widehat{\sigma}_t) - \varrho)(\eta(y,\sigma') - \varrho)\right] \mathrm{d}\mu^{\varrho}(\eta) \\ &= \frac{1}{N}\sum_{(x,\sigma)\in V}\sum_{(y,\sigma')\in V} \phi(\tfrac{x}{N}, \sigma)\psi(\tfrac{y}{N}, \sigma')\widehat{\mathbb{E}}_{(x,\sigma)}\left[\mathrm{Cov}_{\mu^{\varrho}}\left(\eta(\widehat{X}_t, \widehat{\sigma}_t), \eta(y,\sigma')\right)\right],\end{aligned} \tag{4.3.10}$$

where we used duality for the second equality and Fubini for the last equality. Now note that, because μ^{ϱ} is a product of Poisson measures, the covariance term is equal to ϱ if and only if $(\widehat{X}_t, \widehat{\sigma}_t) = (y, \sigma')$ and zero otherwise. Therefore

$$\begin{aligned}&\sum_{(y,\sigma')\in V} \psi(\tfrac{y}{N}, \sigma')\widehat{\mathbb{E}}_{(x,\sigma)}\left[\mathrm{Cov}_{\mu^{\varrho}}\left(\eta(\widehat{X}_t, \widehat{\sigma}_t), \eta(y,\sigma')\right)\right] \\ &\qquad = \varrho \sum_{(y,\sigma')\in V} \psi(\tfrac{y}{N}, \sigma')\widehat{\mathbb{E}}_{(x,\sigma)}\left[I\left((\widehat{X}_t, \widehat{\sigma}_t) = (y,\sigma')\right)\right] \\ &\qquad = \varrho \cdot (\widehat{S}_t^N \psi)(\tfrac{x}{N}, \sigma).\end{aligned} \tag{4.3.11}$$

Here $\widehat{S}_t^N$ is the semigroup of the Markov process $(\frac{\widehat{X}_t}{N}, \widehat{\sigma}_t)$, for which we have the following uniform convergence $\widehat{S}_t^N \psi \to e^{tA^*}\psi$ (see Section 4.2.2) . By now combining (4.3.10) and (4.3.11), we find that

$$\begin{aligned}\mathbb{E}\left[\langle Y_t^N, \phi\rangle\langle Y_0^N, \psi\rangle\right] &= \varrho \cdot \frac{1}{N} \sum_{(x,\sigma)\in V} \sum_{(y,\sigma')\in V} \phi(\tfrac{x}{N}, \sigma)(\widehat{S}_t^N \psi)(\tfrac{x}{N}, \sigma) \\ &\to \varrho \cdot \left\langle\left\langle \phi, e^{tA^*}\psi \right\rangle\right\rangle \\ &= \varrho \cdot \left\langle\left\langle e^{tA}\phi, \psi \right\rangle\right\rangle, \end{aligned} \tag{4.3.12}$$

which concludes the proof. □

Remark 4.2. In Proposition 3.2, the only place where the independence of the particles is manifest is in the pre-factor ϱ which corresponds to the limiting variance of the fluctuation field at time zero, because η_0 is distributed as μ^ϱ. When considering any other system which satisfies duality, when A is the scaling limit of the single particle generator, and $\chi(\varrho)$ is the limiting variance of the fluctuation field at time zero, we find that the limiting covariance is given by

$$\mathbb{E}[\langle Y_t, \phi\rangle\langle Y_0, \psi\rangle] = \chi(\varrho) \left\langle\left\langle e^{tA}\phi . \psi \right\rangle\right\rangle. \tag{4.3.13}$$

E.g. for the exclusion process studied in the section below, $\chi(\varrho) = \varrho(\alpha - \varrho)$.

4.3.2 *Interacting case: the multi-layer SEP*

The multi-layer symmetric exclusion process, or multi-layer SEP, is a generalization of the symmetric exclusion process on $\mathbb{Z}$ to the multi-layered setting on $\mathbb{Z} \times S$. For this process we look at configurations $\eta \in \{0, 1, ..., \alpha\}^V$ with $\alpha \in \mathbb{N}$, i.e., there are at most α particles per site $v \in V$. Instead of having an active component on every layer $\sigma \in S$ like the run-and-tumble particle system, multi-layer SEP switches to a different diffusion coëfficient, denoted by κ_σ, between the layers. The generator of this process is then as follows

$$\begin{aligned} L_N^{SEP} f(\eta) = N^2 &\sum_{(x,\sigma)\in V} \kappa_\sigma \sum_{|x-y|=1} \eta(x,\sigma)\left(\alpha - \eta(y,\sigma)\right)\left(f\left(\eta^{(x,\sigma)\to(y,\sigma)}\right) - f(\eta)\right) \\ &+ \sum_{(x,\sigma)\in V} \sum_{\sigma'\in S} c(\sigma,\sigma')\eta(x,\sigma)(\alpha - \eta(x,\sigma'))\left(f\left(\eta^{(x,\sigma)\to(x,\sigma')}\right) - f(\eta)\right). \end{aligned} \tag{4.3.14}$$

In Chapter 3 we proved that this process is self-dual and has ergodic measures given by product Binomial measures $\nu_\varrho = \bigotimes_{v\in V} \mathrm{Bin}(\alpha, \varrho)$ where $\varrho \in (0, 1)$ is constant.

The corresponding single-particle generator is then given by

$$\begin{aligned}\mathscr{L}_N^{SEP}\phi(x,\sigma) = \alpha\kappa_\sigma\big((\phi(x+\tfrac{1}{N},\sigma)+\phi(x-\tfrac{1}{N},\sigma)-2\phi(x,\sigma)\big)\\ +\sum_{\sigma'\in S} c(\sigma,\sigma')\big(\phi(x,\sigma')-\phi(x,\sigma)\big),\end{aligned} \tag{4.3.15}$$

and $\mathscr{L}_N^{SEP}\phi \to B\phi$ uniformly, where

$$(B\phi)(x,\sigma) = \frac{\alpha\kappa_\sigma}{2}\partial_{xx}\phi(x,\sigma) + \sum_{\sigma'\in S} \alpha c(\sigma,\sigma')\big(\phi(x,\sigma')-\phi(x,\sigma)\big). \tag{4.3.16}$$

Since we took the rates $c(\sigma,\sigma')$ symmetric, this operator is self-adjoint in the Hilbert space $L^2(\mathrm{d}x \times |\cdot|_S)$.

Using the same line of proof as in Section 4.5 below, we obtain the following SPDE for the stationary fluctuation field,

$$\mathrm{d}Y_t = BY_t\,\mathrm{d}t + \sqrt{2\varrho(\alpha-\varrho)K}\partial_x\,\mathrm{d}\mathscr{W}_t + \sqrt{2\varrho(\alpha-\varrho)\Sigma}\,\mathrm{d}\widetilde{\mathscr{W}_t}. \tag{4.3.17}$$

Here K is the operator given by $(K\phi)(x,\sigma) = \kappa_\sigma\phi(x,\sigma)$. Note in the noise terms the appearance of the terms $\varrho(\alpha-\varrho)$ instead of ϱ as in (4.3.1). This comes from the fact that for $(x,\sigma)\neq(y,\sigma')$

$$\mathbb{E}_{\nu_\varrho}[\eta_s(x,\sigma)(\alpha-\eta_s(y,\sigma'))] = \varrho(\alpha-\varrho), \tag{4.3.18}$$

which plays a role in the calculation of the expectation of the Carré du champ operator.

4.4 SCALING LIMITS OF THE TOTAL DENSITY

If we sum over the layers, i.e., over the σ-variables, then the resulting configuration which gives the total number of particles at each site is no longer a Markov process. Therefore, both in the hydrodynamic limit as well as in the fluctuations we expect memory terms to appear in the form of higher order time derivatives in the limiting equations. The stationary fluctuations of the empirical distribution of the total number of particles will then become a non-Markovian Gaussian process which we can identify explicitly.

Next, we consider the small-noise limit of these fluctuations. We then obtain a large deviation principle via large deviations of Schilder's type for Gaussian processes (i.e., small variance limit of Gaussian processes, see e.g. [30] p. 88, and also [73]), and we have memory terms in the corresponding large deviation rate function. We will make these memory effects explicit in the simplest possible setting where $\kappa = 0$ in (4.2.1). To our knowledge, this is the first example of an explicit expression for a large deviation rate function of the empirical distribution of particles in a non-Markovian context. In general such rate functions can be obtained from the contraction principle of the Markovian multi-layer system, but this expression in the form of an infimum is implicit, can rarely be made explicit, and therefore does not make manifest the effect of memory terms.

In the whole of this section, for notational simplicity, we further restrict to $S = \{-1, 1\}$ (two layers) and put $c(1, -1) = c(-1, 1) =: \gamma$. The aim is then to study the fluctuations of the total density of particles, where we sum up the particles in both layers. This produces an empirical measure and fluctuation field on $\mathbb{R}$ given by

$$\zeta_t^N = \frac{1}{N} \sum_{(x,\sigma)\in V} \eta_t^N(x,\sigma)\delta_{\frac{x}{N}}, \qquad Z_t^N = \frac{1}{\sqrt{N}} \sum_{(x,\sigma)\in V} (\eta_t(x,\sigma) - \varrho)\delta_{\frac{x}{N}}. \tag{4.4.1}$$

4.4.1 *Hydrodynamic equation for the total density*

From Theorem 4.1 we can deduce that ζ_t^N converges in probability to $\varrho_t(x)\,\mathrm{d}x$, where the density $\varrho_t(x)$ is the sum of the densities on both layers, i.e., $\varrho_t(x) = \varrho_t(x,1) + \varrho_t(x,-1)$ with $\varrho_t(x,\sigma)$ the solution to the hydrodynamic equation $\dot{\varrho}_t = A^*\varrho_t$. We can rewrite this equation as a coupled system of linear PDE's given by

$$\begin{cases} \dot{\varrho}_t(x,1) = \left(\frac{\kappa}{2}\partial_{xx} - \lambda\partial_x\right)\varrho_t(x,1) + \gamma(\varrho_t(x,-1) - \varrho_t(x,1)), \\ \\ \dot{\varrho}_t(x,-1) = \left(\frac{\kappa}{2}\partial_{xx} + \lambda\partial_x\right)\varrho_t(x,-1) + \gamma(\varrho_t(x,1) - \varrho_t(x,-1)). \end{cases} \tag{4.4.2}$$

Summing up both equations gives us a PDE for the total density $\varrho_t(x)$ depending on the difference of the densities, denoted by $\Delta_t(x) := \varrho_t(x,1) - \varrho_t(x,-1)$.

$$\begin{cases} \dot{\varrho}_t(x) = \frac{\kappa}{2}\partial_{xx}\varrho_t(x) - \lambda\partial_x\Delta_t(x), \\ \\ \dot{\Delta}_t(x) = \frac{\kappa}{2}\partial_{xx}\Delta_t(x) - \lambda\partial_x\varrho_t(x) - 2\gamma\Delta_t(x). \end{cases} \tag{4.4.3}$$

From this system we can actually find a closed equation for $\varrho(x)$. Namely, by first taking a second derivative in time of the upper equation we find that

$$\begin{aligned} \ddot{\varrho}_t(x) &= \frac{\kappa}{2}\partial_{xx}\dot{\varrho}_t(x) - \lambda\partial_x\dot{\Delta}_t(x) \\ &= \frac{\kappa}{2}\partial_{xx}\dot{\varrho}_t(x) - \lambda\partial_x\left(\frac{\kappa}{2}\partial_{xx}\Delta_t(x) - \lambda\partial_x\varrho_t(x) - 2\gamma\Delta_t(x)\right). \end{aligned} \tag{4.4.4}$$

Now we use that from the first equation in (4.4.3) we have $-\lambda\partial_x\Delta_t(x) = \dot{\varrho}_t(x) - \frac{\kappa}{2}\partial_{xx}\varrho_t(x)$. Substituting this in (4.4.4), we find the following closed equation for the total density

$$\ddot{\varrho}_t(x) - (\kappa\partial_{xx} + 2\gamma)\dot{\varrho}_t(x) = \left((\lambda^2 - \gamma\kappa)\partial_{xx} - \frac{\kappa^2}{4}(\partial_x)^4\right)\varrho_t(x). \tag{4.4.5}$$

4.4.2 *Fluctuations of the total density*

For the analysis of the fluctuation field of the total density we first define the fluctuation fields of each layer, and then by taking higher order derivatives as in the previous subsection, we obtain a second order SPDE for the fluctuations of the total density (cf. (4.4.17) below). We first set up a framework where we can rigorously deal with the various distributions coming from the SPDE given in (4.3.1) corresponding to both layers. We start by defining a fluctuation field for each layer individually.

$$Y_{t,\sigma}^N = \frac{1}{\sqrt{N}} \sum_{x\in\mathbb{Z}} (\eta_t(x,\sigma) - \varrho)\delta_{\frac{x}{N}}, \; \sigma \in \{-1,1\} \tag{4.4.6}$$

The relation between these fluctuation fields and Z_t^N is as follows: for every $\phi \in C_c^\infty$ we have that

$$\left\langle Z_t^N, \phi \right\rangle = \left\langle Y_{t,1}^N, \phi \right\rangle + \left\langle Y_{t,-1}^N, \phi \right\rangle. \tag{4.4.7}$$

However, there is also a direct relation between the fluctuation fields on both layers and the fluctuation field Y_t^N on $\mathbb{R} \times S$ defined in (4.2.20): for every $\phi \in C_{c,S}^\infty$ the following holds

$$\left\langle Y_t^N, \phi \right\rangle = \left\langle Y_{t,1}^N, \phi(\cdot,1) \right\rangle + \left\langle Y_{t,-1}^N, \phi(\cdot,-1) \right\rangle. \tag{4.4.8}$$

In this way Y_t^N, but more importantly its limiting process Y_t, can be interpreted as a row vector of distributions, $Y_t = \begin{pmatrix} Y_{t,1} & Y_{t,-1} \end{pmatrix}$, working on a column vector of functions, $\phi = \begin{pmatrix} \phi(\cdot,1) & \phi(\cdot,-1) \end{pmatrix}^T$. With this in mind, we can look at the vector representation of the measure A^*Y_t. We have that

$$\begin{aligned}
\langle A^* Y_t, \phi \rangle &= \langle Y_t, A\phi \rangle \\
&= \left\langle Y_{t,1}, (\tfrac{\kappa}{2}\partial_{xx} + \lambda\partial_x)\phi(\cdot,1) \right\rangle + \langle \gamma(Y_{t,-1} - Y_{t,1}), \phi(\cdot,1) \rangle \\
&\quad + \left\langle Y_{t,-1}, (\tfrac{\kappa}{2}\partial_{xx} - \lambda\partial_x)\phi(\cdot,-1) \right\rangle + \langle \gamma(Y_{t,1} - Y_{t,-1}), \phi(\cdot,-1) \rangle \\
&= \left\langle (\tfrac{\kappa}{2}\partial_{xx} - \lambda\partial_x) Y_{t,1} + \gamma(Y_{t,-1} - Y_{t,1}), \phi(\cdot,1) \right\rangle \\
&\quad + \left\langle (\tfrac{\kappa}{2}\partial_{xx} + \lambda\partial_x) Y_{t,-1} + \gamma(Y_{t,1} - Y_{t,-1}), \phi(\cdot,-1) \right\rangle.
\end{aligned} \tag{4.4.9}$$

Therefore A^*Y_t corresponds to the following vector of distributions

$$A^* Y_t = \begin{pmatrix} (\tfrac{\kappa}{2}\partial_{xx} - \lambda\partial_x) Y_{t,1} + \gamma(Y_{t,-1} - Y_{t,1}) \\ (\tfrac{\kappa}{2}\partial_{xx} + \lambda\partial_x) Y_{t,-1} + \gamma(Y_{t,1} - Y_{t,-1}) \end{pmatrix}^T. \tag{4.4.10}$$

In a similar way we can find a vector representation of the noise part in the SPDE (4.3.1), namely

$$\sqrt{2\kappa\varrho}\partial_x \,\mathrm{d}\mathscr{W}_t + \sqrt{2\varrho\Sigma}\,\mathrm{d}\widetilde{\mathscr{W}_t} = \sqrt{2\kappa\varrho}\partial_x \begin{pmatrix} \mathrm{d}W_{t,1} \\ \mathrm{d}W_{t,-1} \end{pmatrix}^T + \sqrt{2\varrho\Sigma} \begin{pmatrix} \mathrm{d}\widetilde{W}_{t,1} \\ \mathrm{d}\widetilde{W}_{t,-1} \end{pmatrix}^T$$
$$= \begin{pmatrix} \sqrt{2\kappa\varrho}\partial_x \,\mathrm{d}W_{t,1} + \sqrt{\gamma\varrho}\left(\mathrm{d}\widetilde{W}_{t,-1} - \mathrm{d}\widetilde{W}_{t,1}\right) \\ \sqrt{2\kappa\varrho}\partial_x \,\mathrm{d}W_{t,-1} + \sqrt{\gamma\varrho}\left(\mathrm{d}\widetilde{W}_{t,1} - \mathrm{d}\widetilde{W}_{t,-1}\right) \end{pmatrix}^T, \tag{4.4.11}$$

where all the $\mathrm{d}W_{t,i}, \mathrm{d}\widetilde{W}_{t,i}$ are independent space-time white noises on $\mathbb{R}$. Denoting $\widetilde{W}_t = \frac{1}{\sqrt{2}}(\widetilde{W}_{t,-1} - \widetilde{W}_{t,1})$, then $\widetilde{W}_t$ is again a space-time white noise in $\mathbb{R}$. In this notation, the SPDE in (4.3.1) actually gives us a system of SPDE's given by

$$\begin{cases} \mathrm{d}Y_{t,1} = \left[\frac{\kappa}{2}\partial_{xx}Y_{t,1} - \lambda\partial_x Y_{t,1} + \gamma\left(Y_{t,-1} - Y_{t,1}\right)\right]\mathrm{d}t + \sqrt{2\kappa\varrho}\partial_x \,\mathrm{d}W_{t,1} + \sqrt{\gamma\varrho}\,\mathrm{d}\widetilde{W}_t, \\ \mathrm{d}Y_{t,-1} = \left[\frac{\kappa}{2}\partial_{xx}Y_{t,-1} + \lambda\partial_x Y_{t,-1} + \gamma\left(Y_{t,1} - Y_{t,-1}\right)\right]\mathrm{d}t + \sqrt{2\kappa\varrho}\partial_x \mathrm{d}W_{t,-1} + \sqrt{\gamma\varrho}\,\mathrm{d}\widetilde{W}_t. \end{cases} \tag{4.4.12}$$

Now we are able to sum up these equations to get an SPDE for the fluctuation process of the total density Z_t. Just like in the hydrodynamic limit, this will again depend on the difference of the two processes above, denoted by $R_t := Y_{t,1} - Y_{t,-1}$. This gives us the following system of coupled SPDE's

$$\begin{cases} \mathrm{d}Z_t = \left[\frac{\kappa}{2}\partial_{xx}Z_t - \lambda\partial_x R_t\right]\mathrm{d}t + 2\sqrt{\kappa\varrho}\partial_x \,\mathrm{d}W_{t,Z}, \\ \mathrm{d}R_t = \left[\frac{\kappa}{2}\partial_{xx}R_t - \lambda\partial_x Z_t - 2\gamma R_t\right]\mathrm{d}t + 2\sqrt{\kappa\varrho}\partial_x \,\mathrm{d}W_{t,R} + 2\sqrt{2\gamma\varrho}\,\mathrm{d}\widetilde{W}_t, \end{cases} \tag{4.4.13}$$

where

$$W_{t,Z} = \frac{1}{\sqrt{2}}\left(W_{t,1} + W_{t,-1}\right), \quad W_{t,R} = \frac{1}{\sqrt{2}}\left(W_{t,1} - W_{t,-1}\right) \tag{4.4.14}$$

which are independent space-time white noises on $\mathbb{R}$.

4.4.3 *Covariance of the total density*

The process Z_t introduced as in (4.4.13) is a (non-Markovian) stationary Gaussian processes. Therefore, we can characterize Z_t through its covariances. Using (4.4.7) and (4.4.8), we can actually relate this covariance to the covariance structure of Y_t, which we have already calculated in Proposition 4.1. In order to do so, for a given $\phi, \psi \in C_c^\infty$ we define the functions $\bar{\phi}, \bar{\psi} \in C_{c,S}^\infty$ via $\bar{\phi}(x,\sigma) = \phi(x)$ and $\bar{\psi}(x,\sigma) = \psi(x)$. The covariance can then be computed as follows

$$\begin{aligned}
&\mathbb{E}[\,\langle Z_t, \phi\rangle \,\langle Z_0, \psi\rangle] \\
&\quad = \mathbb{E}[(\langle Y_{t,1}, \phi\rangle + \langle Y_{t,-1}, \phi\rangle)\,(\langle Y_{0,1}, \psi\rangle + \langle Y_{0,-1}, \psi\rangle)] \\
&\quad = \mathbb{E}[(\langle Y_{t,1}, \bar{\phi}(\cdot,1)\rangle + \langle Y_{t,-1}, \bar{\phi}(\cdot,-1)\rangle)\,(\langle Y_{0,1}, \bar{\psi}(\cdot,1)\rangle + \langle Y_{0,-1}, \bar{\psi}(\cdot,-1)\rangle)] \\
&\quad = \mathbb{E}[\langle Y_t, \bar{\phi}\rangle\,\langle Y_0, \bar{\psi}\rangle] \\
&\quad = \varrho \cdot \left\langle\left\langle e^{tA}\bar{\phi}, \bar{\psi}\right\rangle\right\rangle. \qquad (4.4.15)
\end{aligned}$$

This covariance strongly resembles the covariance of a stationary Ornstein-Uhlenbeck process, but notice that the semigroup e^{tA} works on the "extended" functions $\bar{\phi}, \bar{\psi}$, which corresponds to the non-Markovianity of the process $\{Z_t, t \geq 0\}$.

Notice that the formula for the covariance obtained in (4.4.15) is solely based on duality, and is therefore valid as long as we have duality for the multi-layer system, i.e., beyond the case of two internal states and including also interacting cases such as the multi-layer SEP.

4.4.4 *Closed form equation and large deviations for the case $\kappa = 0$*

In the case of $\kappa = 0$ the noise term vanishes in the upper equation of (4.4.13) and therefore we we can solve the system explicitly. Namely, we then find that

$$\begin{cases} \mathrm{d}Z_t = -\lambda \partial_x R_t \,\mathrm{d}t, \\ \\ \mathrm{d}R_t = -\left[\lambda \partial_x Z_t + 2\gamma R_t\right] \mathrm{d}t + 2\sqrt{\gamma\varrho}\,\mathrm{d}\widetilde{W}_t. \end{cases} \qquad (4.4.16)$$

Just like for the hydrodynamic limit of the total density, by now taking a second derivative in time in the first equation we find that $\mathrm{d}^2 Z_t = -\lambda\partial_x \,\mathrm{d}R_t \,\mathrm{d}t$. By now filling in $\mathrm{d}R_t$ from the lower equation, we have that

$$\begin{aligned} \frac{\mathrm{d}^2 Z_t}{\mathrm{d}t^2} &= \lambda^2 \partial_{xx} Z_t - 2\gamma\lambda\partial_x R_t + 2\lambda\sqrt{\gamma\varrho}\partial_x \frac{\mathrm{d}\widetilde{W}_t}{\mathrm{d}t} \qquad (4.4.17) \\ &= \lambda^2 \partial_{xx} Z_t + 2\gamma \frac{\mathrm{d}Z_t}{\mathrm{d}t} + 2\lambda\sqrt{\gamma\varrho}\partial_x \frac{\mathrm{d}\widetilde{W}_t}{\mathrm{d}t}. \end{aligned}$$

From the expression above we are also able to obtain the rate function for the large deviations of $Z_t^{(\varepsilon)}$ in the small noise regime, i.e., where we add a factor ε before the noise $\widetilde{W}_t$ which we will send to zero. I.e., we are interested in the large deviations of Schilder type (see [30], [73]) for the family of Gaussian process given by

$$\frac{\mathrm{d}^2 Z_t^{(\varepsilon)}}{\mathrm{d}t^2} = \lambda^2 \partial_{xx} Z_t^{(\varepsilon)} + 2\gamma \frac{\mathrm{d}Z_t^{(\varepsilon)}}{\mathrm{d}t} + \varepsilon 2\lambda\sqrt{\gamma\varrho}\partial_x \frac{\mathrm{d}\widetilde{W}_t}{\mathrm{d}t}. \qquad (4.4.18)$$

We use that

$$\mathbb{P}\left(\varepsilon\partial_x\frac{\mathrm{d}\widetilde{W}_t}{\mathrm{d}t}\asymp\Gamma(t,x)\right)\asymp\exp\left(-\varepsilon^{-2}\frac{1}{2}\int_0^T||\Gamma(t,\cdot)||^2_{H_{-1}}\,\mathrm{d}t\right),\tag{4.4.19}$$

which has to be interpreted in the sense of the large deviation principle in the space of space-time distributions. The rate function in the above equation can be derived from the log-moment-generating function of a space-time white noise on $\mathbb{R}$, which for a test function $\phi\in C_c^\infty([0,T]\times\mathbb{R})$ is equal to

$$\Lambda(\phi)=\lim_{\varepsilon\to 0}\varepsilon^2\log\left(\mathbb{E}[e^{\varepsilon^{-1}\left\langle\phi,\partial_x\frac{\mathrm{d}W_t}{\mathrm{d}t}\right\rangle}]\right)=\frac{1}{2}\left\langle\partial_x\phi,\partial_x\phi\right\rangle_{L^2(\mathbb{R}\times[0,T])}.\tag{4.4.20}$$

The Legendre transform of Λ then yields the rate function,

$$\begin{aligned}\Lambda^*(\Gamma(t,x))&=\sup_{\phi\in C_c^\infty([0,T]\times\mathbb{R})}\left\{\langle\phi,\Gamma\rangle_{L^2([0,T]\times\mathbb{R})}-\frac{1}{2}\langle\partial_x\phi,\partial_x\phi\rangle_{L^2([0,T]\times\mathbb{R})}\right\}\\&=\frac{1}{2}\int_0^T||\Gamma(t,\cdot)||^2_{H^{-1}}\,\mathrm{d}t.\end{aligned}\tag{4.4.21}$$

As a consequence, we obtain the large deviation principle for the random space-time distribution $Z_t^{(\varepsilon)}$, namely from (4.4.18) it follows that

$$\begin{aligned}&\mathbb{P}\left(Z_t^{(\varepsilon)}\asymp\Gamma(t,x)\right)\\&\quad=\mathbb{P}\left(\varepsilon\partial_x\frac{\mathrm{d}\widetilde{W}_t}{\mathrm{d}t}\asymp\frac{1}{2\lambda\sqrt{\gamma\varrho}}\left(\ddot{\Gamma}(t,x)-2\gamma\dot{\Gamma}(t,x)-\lambda^2\partial_{xx}\Gamma(t,x)\right)\right)\\&\quad\asymp\exp\left(-\varepsilon^{-2}\frac{1}{4\lambda\sqrt{\gamma\varrho}}\int_0^T\left|\left|\ddot{\Gamma}(t,\cdot)-2\gamma\dot{\Gamma}(t,\cdot)-\lambda^2\partial_{xx}\Gamma(t,\cdot)\right|\right|^2_{H_{-1}}\mathrm{d}t\right).\end{aligned}\tag{4.4.22}$$

4.5 PROOF OF THEOREM 4.3

In this section we prove Theorem 4.3, following the line of proof of Van Ginkel and Redig in [117]. For the readers convenience we sketch the main steps.

We start by introducing the Dynkin martingales of $\langle Y_t^N,\phi\rangle$. For every $\phi\in C_{c,S}^\infty$ and $N\in\mathbb{N}$, let $\{\mathscr{F}_t^N:t\geq 0\}$ be the filtration generated by $\{Y_t^N:t\geq 0\}$. Because the configuration process $\{\eta_t:t\geq 0\}$ is a Markov processes the following processes

$$\begin{aligned}\mathscr{M}_t^{N,\phi}(Y^N_{[0,T]})&=\langle Y_t,\phi\rangle-\langle Y_0^N,\phi\rangle-\int_0^t L_N\langle Y_s^N,\phi\rangle\,\mathrm{d}s,\\\mathscr{N}_t^{N,\phi}(Y^N_{[0,T]})&=\mathscr{M}_t^{N,\phi}(Y^N_{[0,T]})^2-\int_0^t\Gamma^{N,\phi}(Y_s^N)\,\mathrm{d}s,\end{aligned}\tag{4.5.1}$$

are $\mathscr{F}_t^N$-martingales, where $\Gamma_s^{N,\phi}$ is the so-called Carré du champ operator given by

$$\Gamma^{N,\phi}(Y_s^N) := L_N\big(\langle Y_s^N,\phi\rangle^2\big) - 2\langle Y_s^N,\phi\rangle L_N\langle Y_s^N,\phi\rangle. \tag{4.5.2}$$

The aim is then to prove that as $N \to \infty$, the martingales in (4.5.1) converge to the martingales from (4.3.4). This fact, complemented with a proof of tightness and the fact that the martingale problem (4.3.4) has a unique solution, then completes the proof. In Section 4.5.1 we prove the convergence of the martingales, in Section 4.5.2 we prove the tightness, and in Section 4.5.3 we prove the uniqueness of the solution of the martingale problem (4.3.4).

4.5.1 *Substituting the martingales*

Our goal for this section is to show that in the limit as $N \to \infty$, we can substitute $\mathscr{M}_t^\phi(Y_{[0,T]}^N)$ and $\mathscr{N}_t^\phi(Y_{[0,T]}^N)$ (with $\mathscr{M}_t^\phi$ and $\mathscr{N}_t^\phi$ defined as in (4.3.4)) for $\mathscr{M}_t^{N,\phi}(Y_{[0,T]}^N)$ and $\mathscr{N}_t^{N,\phi}(Y_{[0,T]}^N)$ respectively. We do so in the Propositions 4.2 and 4.3. We recall the reader that the expectation $\mathbb{E}$ stands for the stationary expectation starting from the initial configuration distributed according to $\eta_0 \sim \mu^\varrho$.

Proposition 4.2. For all $\phi \in C_{c,S}^\infty$ we have

$$\lim_{N\to\infty} \mathbb{E}\left[\left|\mathscr{M}_t^{N,\phi}(Y_{[0,T]}^N) - \mathscr{M}_t^\phi(Y_{[0,T]}^N)\right|^2\right] = 0. \tag{4.5.3}$$

Proof. First of all, note that by definition

$$\mathbb{E}\left[\left|\mathscr{M}_t^{N,\phi}(Y_{[0,T]}^N) - \mathscr{M}_t^\phi(Y_{[0,T]}^N)\right|^2\right] = \mathbb{E}\left[\left|\int_0^t L_N\langle Y_s^N,\phi\rangle\,\mathrm{d}s - \int_0^t \langle Y_s^N, A\phi\rangle\,\mathrm{d}s\right|^2\right]. \tag{4.5.4}$$

For a given $(x,\sigma) \in V$ we have that

$$\begin{aligned} L_N\eta(x,\sigma) &= \kappa N^2[\eta(x+1,\sigma) + \eta(x-1,\sigma) - 2\eta(x,\sigma)] \\ &\quad + \lambda N[\eta(x-\sigma,\sigma) - \eta(x,\sigma)] \\ &\quad + \sum_{\sigma'\in S} c(\sigma,\sigma')[\eta(x,\sigma') - \eta(x,\sigma)], \end{aligned} \tag{4.5.5}$$

and so in particular we find that

$$L_N\langle Y_s^N,\phi\rangle = \frac{1}{\sqrt{N}}\sum_{(x,\sigma)\in V}\big(L_N\eta_s(x,\sigma)\big)\phi(\tfrac{x}{N},\sigma) = \frac{1}{\sqrt{N}}\sum_{(x,\sigma)\in V}\eta_s(x,\sigma)(\mathscr{L}_N\phi)(\tfrac{x}{N},\sigma), \tag{4.5.6}$$

where we remind the reader that $\mathscr{L}_N$ is the generator of a single run-and-tumble particle on the rescaled space $\frac{1}{N}\mathbb{Z} \times S$. Now, using that for any $\phi \in C^\infty_{c,S}$ we have that

$$\sum_{(x,\sigma)\in V} \varrho \cdot (\mathscr{L}_N \phi)(\tfrac{x}{N}, \sigma) = 0, \tag{4.5.7}$$

we are able to write

$$L_N \langle Y^N_s, \phi \rangle = \frac{1}{\sqrt{N}} \sum_{(x,\sigma)\in V} (\eta_s(x,\sigma) - \varrho) \cdot (\mathscr{L}_N \phi)(\tfrac{x}{N}, \sigma). \tag{4.5.8}$$

Since $\mathscr{L}_N \phi \to A\phi$ uniformly, where A is defined in (4.2.4), we have that

$$L_N \langle Y^N_s, \phi \rangle = \frac{1}{\sqrt{N}} \sum_{(x,\sigma)\in V} (\eta_s(x,\sigma) - \varrho) \cdot (A\phi)(\tfrac{x}{N}, \sigma) + R_1(\phi, N, s), \tag{4.5.9}$$

where $R_1(\phi, N, s)$ is an error term produced by the Taylor approximations. Since ϕ is compactly supported, if we define $V^N_\phi := \{(x,\sigma) \in V, \phi(\frac{x}{N}, \sigma) \neq 0\}$ then $|V^N_\phi| = \mathcal{O}(N)$. Furthermore, the error term is bounded in the following way

$$|R_1(\phi, N, s)| \leq \frac{1}{N^{3/2}} \sum_{v\in V^N_\phi} (\eta_s(v) - \varrho)(\kappa ||\partial_{xxx}\phi||_\infty + \lambda\sigma^2 ||\partial_{xx}\phi||_\infty). \tag{4.5.10}$$

Therefore we find that for every $\phi \in C^\infty_{c,S}$ and $t \geq 0$,

$$\begin{aligned} &\mathbb{E}\left[R_1(\phi, N, s)^2\right] \\ &\quad \leq \frac{1}{N^3} \mathbb{E}\left[\sum_{v,w\in V^N_\phi} (\eta_s(v) - \varrho)(\eta_s(w) - \varrho)(\kappa ||\partial_{xxx}\phi||_\infty + \lambda\sigma^2 ||\partial_{xx}\phi||_\infty)^2\right] \\ &\quad = \frac{1}{N^3} \sum_{v,w\in V^N_\phi} \mathrm{Cov}\big(\eta_s(v), \eta_s(w)\big)(\kappa ||\partial_{xxx}\phi||_\infty + \lambda\sigma^2 ||\partial_{xx}\phi||_\infty)^2. \end{aligned} \tag{4.5.11}$$

Since we are starting the process η_t from the invariant product measure μ^ϱ, we have that

$$\mathrm{Cov}\big(\eta_s(v), \eta_s(w)\big) = \varrho \cdot I\big(v = w\big). \tag{4.5.12}$$

Therefore,

$$\mathbb{E}\left[R_1(\phi, N, s)^2\right] \leq \frac{1}{N^3} |V^N_\phi| \varrho (\kappa ||\partial_{xxx}\phi||_\infty + \lambda\sigma^2 ||\partial_{xx}\phi||_\infty)^2 \to 0, \tag{4.5.13}$$

where we used the fact that $|V^N_\phi| = \mathcal{O}(N)$. Note that the above convergence is uniform in s, and therefore by dominated convergence we find that

$$\lim_{N\to\infty} \mathbb{E}\left[\left|\mathscr{M}^{N,\phi}_t(Y^N_{[0,T]}) - \mathscr{M}^{\phi}_t(Y^N_{[0,T]})\right|^2\right] = \lim_{N\to\infty} \int_0^t \mathbb{E}\left[R_1(\phi, N, s)^2\right] \mathrm{d}s = 0, \tag{4.5.14}$$

which concludes the proof. □

The substitution of $\mathscr{N}_t^{\phi}(Y^N_{[0,T]})$ is a bit more work and requires a fourth moment estimate. We start by proving two lemmas. The proof of the substitution result in Proposition 4.3 immediately follows from these lemmas.

Lemma 4.1. For all $\phi \in C^\infty_{c,S}$ we have the following

$$\lim_{k\to\infty} \mathbb{E}\left[\left(\mathscr{M}_t^{N,\phi}(Y^N_{[0,T]})^2 - \mathscr{M}_t^{\phi}(Y^N_{[0,T]})^2\right)^2\right] = 0. \tag{4.5.15}$$

Proof. We start with the following application of Hölder's inequality

$$\begin{aligned}
&\mathbb{E}\left[\left(\mathscr{M}_t^{N,\phi}(Y^N_{[0,T]})^2 - \mathscr{M}_t^{\phi}(Y^N_{[0,T]})^2\right)^2\right] \\
&= \mathbb{E}\left[\left(\mathscr{M}_t^{N,\phi}(Y^N_{[0,T]}) - \mathscr{M}_t^{\phi}(Y^N_{[0,T]})\right)^2\left(\mathscr{M}_t^{N,\phi}(Y^N_{[0,T]}) + \mathscr{M}_t^{\phi}(Y^N_{[0,T]})\right)^2\right] \\
&\le \left(\mathbb{E}\left[\left(\mathscr{M}_t^{N,\phi}(Y^N_{[0,T]}) - \mathscr{M}_t^{\phi}(Y^N_{[0,T]})\right)^4\right]\cdot \mathbb{E}\left[\left(\mathscr{M}_t^{N,\phi}(Y^N_{[0,T]}) + \mathscr{M}_t^{\phi}(Y^N_{[0,T]})\right)^4\right]\right)^{\frac{1}{2}}.
\end{aligned} \tag{4.5.16}$$

We will first show that the first expectation in the last line vanishes as $N \to \infty$, and afterwards we will show that the second expectation is uniformly bounded in N. Note that by (4.5.9)

$$\begin{aligned}
\mathbb{E}\left[\left(\mathscr{M}_t^{N,\phi}(Y^N_{[0,T]}) - \mathscr{M}_t^{\phi}(Y^N_{[0,T]})\right)^4\right] &= \mathbb{E}\left[\left(\int_0^t [R_1(\phi,N,s)]\,\mathrm{d}s\right)^4\right] \\
&\le t^3 \int_0^T \mathbb{E}\left[R_1(\phi,N,s)^4\right]\mathrm{d}s.
\end{aligned} \tag{4.5.17}$$

Using the bound in (4.5.10) we find that

$$\mathbb{E}\left[R_1(\phi,N,s)^4\right] \le \frac{1}{N^6}\sum_{\substack{v_i\in V^N_\phi\\ 1\le i\le 4}} \mathbb{E}\left[\prod_{i=1}^4(\eta_s(v_i)-\varrho)\right](\kappa||\partial_{xxx}\phi||_\infty + \lambda\sigma^2||\partial_{xx}\phi||_\infty)^4. \tag{4.5.18}$$

Since we start from the product Poisson measure μ^ϱ, we only get non-zero contributions in the expectation on the right-hand side when all v_i are equal or when we have two distinct pairs, given by

$$\mathbb{E}\left[(\eta_s(v)-\varrho)^4\right] = 3\varrho^2+\varrho, \quad \mathbb{E}\left[(\eta_s(v)-\varrho)^2(\eta_s(w)-\varrho)^2\right] = \varrho^2. \tag{4.5.19}$$

Therefore, it follows that

$$\mathbb{E}\left[R_1(\phi,N,s)^4\right] \le \frac{1}{N^6}\left(|V^N_\phi|(3\varrho^2+\varrho) + |V^N_\phi|^2\varrho^2\right)(\kappa||\partial_{xxx}\phi||_\infty + \lambda\sigma^2||\partial_{xx}\phi||_\infty)^4, \tag{4.5.20}$$

and so $R_1(\phi, N, s, \sigma) \xrightarrow{L^4} 0$ uniformly in s. From this we can conclude that

$$\mathbb{E}\left[\left(\mathscr{M}_t^{N,\phi}(Y_{[0,T]}^N) - \mathscr{M}_t^{\phi}(Y_{[0,T]}^N)\right)^4\right] \leq t^3 \int_0^t \mathbb{E}\left[R_1(\phi, N, s)^4\right] \mathrm{d}s \to 0. \tag{4.5.21}$$

To now show that the second expectation in the last line of (4.5.16) is uniformly bounded in N, note that

$$\begin{aligned}\mathbb{E}&\left[\left(\mathscr{M}_t^{N,\phi}(Y_{[0,T]}^N) + \mathscr{M}_t^{\phi}(Y_{[0,T]}^N)\right)^4\right] \\ &\leq 8\left(\mathbb{E}\left[\left(\mathscr{M}_t^{N,\phi}(Y_{[0,T]}^N)\right)^4\right] + \mathbb{E}\left[\left(\mathscr{M}_t^{\phi}(Y_{[0,T]}^N)\right)^4\right]\right),\end{aligned} \tag{4.5.22}$$

and similarly

$$\mathbb{E}\left[\left(\mathscr{M}_t^{\phi}(Y_{[0,T]}^N)\right)^4\right] \leq 27\left(\mathbb{E}\left[\langle Y_t^N, \phi\rangle^4\right] + \mathbb{E}\left[\langle Y_0^N, \phi\rangle^4\right] + \mathbb{E}\left[\left(\int_0^t \langle Y_s^N, A\phi\rangle \,\mathrm{d}s\right)^4\right]\right). \tag{4.5.23}$$

Now we need to show that three expectations on the right-hand-side are uniformly bounded. For the first expectation, we find that

$$\mathbb{E}\left[\langle Y_t^N, \phi\rangle^4\right] \leq \frac{1}{N^2} \cdot \sum_{v_1 \in V_\phi^N} \cdots \sum_{v_4 \in V_\phi^N} \mathbb{E}\left[\prod_{i=1}^4 (\eta_t(v_i) - \varrho)\right] ||\phi||_\infty. \tag{4.5.24}$$

Similarly as in (4.5.20), we find that

$$\mathbb{E}\left[\langle Y_t^N, \phi\rangle^4\right] \leq \frac{1}{N^2}\left(|V_\phi^N|(3\varrho^2 + \varrho) + |V_\phi^N|^2\varrho^2\right) ||\phi||_\infty = \mathcal{O}(1), \tag{4.5.25}$$

hence it is uniformly bounded, and similar approaches can be used for $\mathbb{E}\left[\langle Y_0^N, \phi\rangle^4\right]$ and $\mathbb{E}\left[\langle Y_s^N, A\phi\rangle^4\right]$. The fact that the last expectation in (4.5.23) is uniformly bounded now follows from an application of Hölder's inequality, namely

$$\mathbb{E}\left[\left(\int_0^t \langle Y_s^N, A\phi\rangle \,\mathrm{d}s\right)^4\right] \leq t^3 \int_0^T \mathbb{E}\left[\langle Y_s^N, A\phi\rangle^4\right] \mathrm{d}s. \tag{4.5.26}$$

Therefore we know that $\mathbb{E}\left[\left(\mathscr{M}_t^{\phi}(Y_{[0,T]}^N)\right)^4\right]$ is uniformly bounded. The proof for $\mathbb{E}\left[\left(\mathscr{M}_t^{N,\phi}(Y_{[0,T]}^N)\right)^4\right]$ works the same way if we use that

$$\mathbb{E}\left[\left(L_N\langle Y_s^N, \phi\rangle\right)^4\right] = 8\left(\mathbb{E}\left[\langle Y_s^N, A\phi\rangle^4\right] + \mathbb{E}\left[R_1(\phi, N, t, \sigma)^4\right]\right), \tag{4.5.27}$$

where by (4.5.20) we already know that $\mathbb{E}\left[R_1(\phi, N, t, \sigma)^4\right]$ is uniformly bounded. Hence we can conclude that (4.5.15) holds. □

Lemma 4.2. For all $\phi \in C^\infty_{c,S}$ we have the following

$$\lim_{N\to\infty} \mathbb{E}\left[\left(\int_0^t \Gamma^{N,\phi}(Y^N_s)\,\mathrm{d}s - 2t\kappa\varrho\,\langle\langle\partial_x\phi,\partial_x\phi\rangle\rangle - 2t\varrho\,\langle\langle\phi,\Sigma\phi\rangle\rangle\right)^2\right] = 0, \quad (4.5.28)$$

with Σ defined as in (4.3.2).

Proof. First we recall that for a Markov process with generator L determined by the transition rates $r(\eta,\eta')$ the carré du champ operator is computed as follows.

$$\begin{aligned} Lf^2(\eta) - 2f(\eta)\cdot Lf(\eta) &= \sum_{\eta'\in\Omega} r(\eta,\eta')\Big(\big(f^2(\eta') - f^2(\eta)\big) - 2\big(f(\eta)f(\eta') - f^2(\eta)\big)\Big) \\ &= \sum_{\eta'\in\Omega} r(\eta,\eta')\big(f(\eta') - f(\eta)\big)^2, \end{aligned} \quad (4.5.29)$$

Translating this to our setting with $L = L_N$ and $f = Y^N$ we obtain

$$\begin{aligned} \Gamma^{N,\phi}(Y^N_s) &= \kappa N \sum_{(x,\sigma)\in V} \eta_s(x,\sigma)\,\Big(\phi(\tfrac{x+1}{N},\sigma) - \phi(\tfrac{x}{N},\sigma))^2 + (\phi(\tfrac{x-1}{N},\sigma) - \phi(\tfrac{x}{N},\sigma))^2\Big) \\ &\quad + \lambda \sum_{(x,\sigma)\in V} \eta_s(x,\sigma)\,\big(\phi(\tfrac{x+\sigma}{N},\sigma) - \phi(\tfrac{x}{N},\sigma)\big)^2 \\ &\quad + \frac{1}{N}\sum_{(x,\sigma)\in V}\sum_{\sigma'\in S} c(\sigma,\sigma')\eta_s(x,\sigma)\big(\phi(x,\sigma') - \phi(x,\sigma)\big)^2. \end{aligned} \quad (4.5.30)$$

Using Taylor expansion with rest term, we can write

$$\begin{aligned} \Gamma^{N,\phi}(Y^N_s) &= \frac{2\kappa}{N}\sum_{(x,\sigma)\in V} \eta_s(x,\sigma)\,\big(\partial_x\phi(\tfrac{x}{N},\sigma)\big)^2 \\ &\quad + \frac{1}{N}\sum_{(x,\sigma)\in V}\sum_{\sigma'\in S} c(\sigma,\sigma')\eta_s(x,\sigma)(\phi(\tfrac{x}{N},\sigma') - \phi(\tfrac{x}{N},\sigma)^2 \\ &\quad + R_2(\phi,s,N), \end{aligned} \quad (4.5.31)$$

with $R_2(\phi,s,N)$ the error term, which is bounded as follows

$$|R_2(\phi,s,N)| \le \kappa\frac{1}{N^3}\sum_{(x,\sigma)\in V^N_\phi} \eta_s(x,\sigma)\kappa||\partial_{xx}\phi||_\infty + \frac{1}{N^2}\sum_{(x,\sigma)\in V^N_\phi} \eta_s(x,\sigma)\lambda\sigma||\phi'||_\infty. \quad (4.5.32)$$

Following the line of thought leading to (4.5.11), we obtain that $R_2(\phi, s, N) \xrightarrow{L^2} 0$. Therefore, for the expectation we find that

$$\begin{aligned}\mathbb{E}\left[\Gamma^{N,\phi}(Y_s^N)\right] &= \frac{2\kappa\varrho}{N} \sum_{(x,\sigma)\in V} (\partial_x \phi(\tfrac{x}{N}, \sigma))^2 + \frac{2\varrho}{N} \sum_{\sigma' \in S} c(\sigma, \sigma')(\phi(\tfrac{x}{N}, \sigma') - \phi(\tfrac{x}{N}, \sigma))^2 \\ &\quad + \mathbb{E}\left[R_2(\phi, s, N)\right] \\ &\to 2\kappa\varrho \left\langle\left\langle \partial_x\phi, \partial_x\phi\right\rangle\right\rangle + 2\varrho \left\langle\left\langle \phi, \Sigma\phi \right\rangle\right\rangle, \end{aligned} \tag{4.5.33}$$

and for the variance

$$\begin{aligned}\operatorname{Var}\left[\Gamma^{N,\phi}(Y_s^N)\right] &\leq \frac{C(\phi,s)}{N^2} \sum_{v,w\in V_\phi^N} \operatorname{Cov}\big(\eta_s(v), \eta_s(w)\big) \\ &= \frac{C(\phi,s)}{N^2} |V_\phi^N| \varrho \to 0, \end{aligned} \tag{4.5.34}$$

with $C(\phi, s)$ some constant and where we have used (4.5.12) for the equality. Since the variance converges to zero, this means that $\Gamma^{N,\phi}(Y_s^N)$ converges to its mean in L^2. Therefore

$$\begin{aligned}&\lim_{N\to\infty} \mathbb{E}\left[\left(\int_0^t \Gamma^{N,\phi}(Y_s^N)\,\mathrm{d}s - 2t\kappa\varrho \left\langle\left\langle \partial_x\phi, \partial_x\phi\right\rangle\right\rangle - 2t\varrho \left\langle\left\langle \phi, \Sigma\phi \right\rangle\right\rangle\right)^2\right] \\ &\quad \leq \lim_{N\to\infty} \int_0^t \mathbb{E}\left[\left(\Gamma^{N,\phi}(Y_s^N) - 2\kappa\varrho \left\langle\left\langle \partial_x\phi, \partial_x\phi\right\rangle\right\rangle - 2\varrho \left\langle\left\langle \phi, \Sigma\phi \right\rangle\right\rangle\right)^2\right] \mathrm{d}s \\ &\quad = 0, \end{aligned} \tag{4.5.35}$$

where we used dominated convergence for the last equality. □

PROPOSITION 4.3. For all $\phi \in C_{c,S}^\infty$

$$\lim_{N\to\infty} \mathbb{E}\left[\left|\mathscr{N}_t^{N,\phi}(Y_{[0,T]}^N) - \mathscr{N}_t^{\phi}(Y_{[0,T]}^N)\right|^2\right] = 0. \tag{4.5.36}$$

Proof. We have that

$$\begin{aligned}&\mathbb{E}\left[\left|\mathscr{N}_t^{N,\phi}(Y_{[0,T]}^N) - \mathscr{N}_t^{\phi}(Y_{[0,T]}^N)\right|^2\right] \\ &\quad \leq 2\mathbb{E}\left[\left(\mathscr{M}_t^{N,\phi}(Y_{[0,T]}^N)^2 - \mathscr{M}_t^{\phi}(Y_{[0,T]}^N)^2\right)^2\right] \\ &\qquad + 2\mathbb{E}\left[\left(\int_0^t \Gamma^{N,\phi}(Y_s^N)\,\mathrm{d}s - 2t\kappa\varrho \left\langle\left\langle \partial_x\phi, \partial_x\phi\right\rangle\right\rangle - 2t\varrho \left\langle\left\langle \phi, \Sigma\phi \right\rangle\right\rangle\right)\right]. \end{aligned} \tag{4.5.37}$$

The proof now follows from Lemma 4.1 and 4.2. □

4.5.2 *Tightness*

In this section we will show the tightness of the collection $\{Y^N_{[0,T]} : N \in \mathbb{N}\}$.

Proposition 4.4. $\{Y^N_{[0,T]} : N \in \mathbb{N}\}$ is tight in $D([0,T];(C^\infty_{c,S})^*)$.

Proof. Since $C^\infty_{c,S}$ is a nuclear space, by Mitoma [83, Theorem 4.1] it suffices to prove that for a fixed $\phi \in C^\infty_{c,S}$ we have that $\{\langle Y^N_{[0,T]}, \phi\rangle : N \in \mathbb{N}\}$ is tight in the path space $D([0,T];\mathbb{R})$. Aldous' criterion, as stated in [2, Theorem 1], tells us that it suffices to show the following two things:

A.1 For all $t \in [0,T]$ and $\varepsilon > 0$ there exists a compact $K(t,\varepsilon) \in \mathbb{R}$ such that

$$\sup_{N\in\mathbb{N}} \mathbb{P}\big(\langle Y^N_t, \phi\rangle \notin K(t,\varepsilon)\big) \le \varepsilon. \tag{4.5.38}$$

A.2 For all $\varepsilon > 0$

$$\lim_{\delta\to 0}\limsup_{N\to\infty}\sup_{\substack{\tau\in\mathscr{T}_T\\ \theta\le\delta}} \mathbb{P}\big(|\langle Y^N_\tau,\phi\rangle - \langle Y^N_{\tau+\theta},\phi\rangle| > \varepsilon\big) = 0, \tag{4.5.39}$$

with $\mathscr{T}_T$ the set of all stopping times bounded by T.

Fix $t \in [0,T]$ and $\phi \in C^\infty_{c,S}$. Then, for every $\sigma \in S$ we have that

$$\mathbb{E}[\langle Y^N_t,\phi\rangle] = \frac{1}{\sqrt{N}}\sum_{(x,\sigma)\in V} \mathbb{E}\left[\eta_t(x,\sigma) - \varrho\right]\phi(\tfrac{x}{N},\sigma) = 0, \tag{4.5.40}$$

and

$$\mathrm{Var}[\langle Y^N_t,\phi\rangle] = \frac{1}{\sqrt{N}}\sum_{(x,\sigma)\in V} \mathrm{Var}\left[\eta_t(x,\sigma) - \varrho\right]\phi(\tfrac{x}{N},\sigma) = \frac{1}{N}\varrho\sum_{(x,\sigma)\in V}\phi^2(\tfrac{x}{N},\sigma). \tag{4.5.41}$$

By the central limit theorem, we therefore see that every $\langle Y^N_t,\phi\rangle$ converges in distribution to the normal distribution $\mathcal{N}\big(0,\varrho\,\langle\langle\phi,\phi\rangle\rangle\big)$. This implies the tightness of the real-valued random variables $\{\langle Y^N_t,\phi\rangle : N\in\mathbb{N}\}$, and therefore also **A.1**.

To prove **A.2**, we note that for every bounded stopping time $\tau \in \mathscr{T}_T$ we have that

$$\langle Y^N_\tau,\phi\rangle = \mathscr{M}^{N,\phi}_\tau(Y^N_{[0,T]}) + \langle Y^N_0,\phi\rangle + \int_0^\tau L_N\langle Y^N_s,\phi\rangle ds, \tag{4.5.42}$$

with $\mathscr{M}_\tau^{N,\phi}(Y_{[0,T]}^N)$ the Dynkin martingale of $\langle Y_\tau^N, \phi\rangle$. Using the Markov inequality, we can then deduce that

$$\begin{aligned}\mathbb{P}\big(|\langle Y_\tau^N, \phi\rangle - \langle Y_{\tau+\theta}^N, \phi\rangle| > \varepsilon\big) &\leq \frac{1}{\varepsilon^2}\mathbb{E}\left[\Big(\langle Y_\tau^N, \phi\rangle - \langle Y_{\tau+\theta}^N, \phi\rangle\Big)^2\right] \\ &\leq \frac{2}{\varepsilon^2}\left(\mathbb{E}\left[\Big(\mathscr{M}_\tau^{N,\phi}(Y_{[0,T]}^N) - \mathscr{M}_{\tau+\theta}^{N,\phi}(Y_{[0,T]}^N)\Big)^2\right]\right. \\ &\quad \left.+ \mathbb{E}\left[\left(\int_\tau^{\tau+\theta} L_N\langle Y_s^N, \phi\rangle \,\mathrm{d}s\right)^2\right]\right). \end{aligned} \tag{4.5.43}$$

For the integral term, note that by the Cauchy-Schwarz inequality and Fubini we have that

$$\begin{aligned}\mathbb{E}\left[\left(\int_\tau^{\tau+\theta} L_N\langle Y_s^N, \phi\rangle dr\right)^2\right] &\leq \sqrt{\theta}\cdot\left(\mathbb{E}\left[\int_0^{T+\theta}\Big(L_N\langle Y_s^N, \phi\rangle\Big)^2 \mathrm{d}s\right]\right)^{\frac{1}{2}} \\ &= \sqrt{\theta}\cdot\left(\int_0^{T+\theta}\mathbb{E}\left[\Big(L_N\langle Y_s^N, \phi\rangle\Big)^2\right]\mathrm{d}s\right)^{\frac{1}{2}}. \end{aligned} \tag{4.5.44}$$

In the proof of Lemma 4.1 we have shown that $\{L_N\langle Y_s^N, \phi\rangle : N \in \mathbb{N}\}$ is uniformly bounded in L^4, hence it is also uniformly bounded in L^2, i.e.

$$C := \sup_{N\in\mathbb{N}} \mathbb{E}\left[\Big(L_N\langle Y_s^N, \phi\rangle\Big)^2\right] < \infty. \tag{4.5.45}$$

Combining (4.5.44) and (4.5.45), we find that

$$\lim_{\delta\to 0}\limsup_{N\to\infty}\sup_{\substack{\tau\in\mathscr{T}_T\\ \theta\leq\delta}} \mathbb{E}\left[\left(\int_\tau^{\tau+\theta} L_N\langle Y_s^N, \phi\rangle dr\right)^2\right] \leq \lim_{\delta\to 0}\sqrt{\delta C T} = 0. \tag{4.5.46}$$

For the martingale, by the martingale property we have that

$$\mathbb{E}\left[\mathscr{M}_\tau^{N,\phi}(Y_{[0,T]}^N)\mathscr{M}_{\tau+\theta}^{N,\phi}(Y_{[0,T]}^N)\right] = \mathbb{E}\left[\Big(\mathscr{M}_\tau^{N,\phi}(Y_{[0,T]}^N)\Big)^2\right], \tag{4.5.47}$$

hence we see that

$$\mathbb{E}\left[\Big(\mathscr{M}_\tau^{N,\phi}(Y_{[0,T]}^N) - \mathscr{M}_{\tau+\theta}^{N,\phi}(Y_{[0,T]}^N)\Big)^2\right] = \mathbb{E}\left[\Big(\mathscr{M}_{\tau+\theta}^{N,\phi}(Y_{[0,T]}^N)\Big)^2 - \Big(\mathscr{M}_\tau^{N,\phi}(Y_{[0,T]}^N)\Big)^2\right]. \tag{4.5.48}$$

Since $\mathbb{E}\left[\mathscr{M}_0^{N,\phi}(Y_{[0,T]}^N)\right] = 0$, we can use that

$$\mathbb{E}\left[\Big(\mathscr{M}_t^{N,\phi}(Y_{[0,T]}^N)\Big)^2\right] = \mathbb{E}\left[\int_0^t \Gamma^{N,\phi}(Y_s^N)\right]\mathrm{d}s, \tag{4.5.49}$$

because $\int_0^t \Gamma^{N,\phi}(Y_s^N)\,ds$ is the predictable quadratic variation of the process $\mathscr{M}_t^{N,\phi}(Y_{[0,T]}^N)$. Furthermore, $\mathbb{E}\left[\left(\Gamma^{N,\phi}(Y_s^N)\right)^2\right]$ is uniformly bounded since $\Gamma^{N,\phi}(Y_s^N)$ converges in L^2, hence

$$\begin{aligned}
\sup_{N\in\mathbb{N}} \mathbb{E}&\left[\left(\mathscr{M}_\tau^{N,\phi}(Y_{[0,T]}^N) - \mathscr{M}_{\tau+\theta}^{N,\phi}(Y_{[0,T]}^N)\right)^2\right] \\
&= \sup_{N\in\mathbb{N}} \mathbb{E}\left[\int_\tau^{\tau+\theta} \Gamma^{N,\phi}(Y_s^N)\right] ds, \\
&\le \sqrt{\theta}\cdot\left(\int_0^{T+\theta} \sup_{N\in\mathbb{N}} \mathbb{E}\left[\left(\Gamma^{N,\phi}(Y_s^N)\right)^2\right] ds\right)^{\frac{1}{2}} < \infty,
\end{aligned} \tag{4.5.50}$$

where we used Cauchy Schwarz in the second line. From this we can again conclude that

$$\lim_{\delta\to 0}\limsup_{N\to\infty}\sup_{\substack{\tau\in\mathscr{T}_T\\ \theta\le\delta}} \mathbb{E}\left[\left(\mathscr{M}_\tau^{N,\phi}(Y_{[0,T]}^N) - \mathscr{M}_{\tau+\theta}^{N,\phi}(Y_{[0,T]}^N)\right)^2\right] = 0. \tag{4.5.51}$$

Combining (4.5.46) and (4.5.51) with (4.5.43), we indeed find that (**A.2**) holds. □

4.5.3 *Uniqueness of limits*

By the tightness, there exists a subsequence N_k and a process $Y \in D([0,T];(C_{c,S}^\infty)^*)$ such that $Y^{N_k} \to Y$ in distribution.

Lemma 4.3. For each $\phi \in C_{c,S}^\infty$ we have that $t \mapsto \langle Y_t, \phi\rangle$ is a.s. continuous.

Proof. We define the following functions

$$w_\delta(X) = \sup_{|t-s|<\delta} |X_t - X_s|, \qquad w'_\delta(X) = \inf_{\substack{0=t_0<t_1<\ldots<t_r=1\\ t_i-t_{i-1}<\delta}} \max_{1\le i\le r} \sup_{t_{i-1}\le s<t\le t_i} |X_t - X_s|, \tag{4.5.52}$$

then we have the following inequality

$$w_\delta(X) \le 2w'_\delta(X) + \sup_t |X_t - X_{t-}|. \tag{4.5.53}$$

From **A.2** it follows for all $\varepsilon > 0$ and all $\sigma \in S$ we have that

$$\lim_{\delta\to 0}\limsup_{N\to\infty} \mathbb{P}(w'_\delta(\langle Y_{[0,T]}^N, \phi\rangle) \ge \varepsilon) = 0. \tag{4.5.54}$$

Now note that

$$\sup_t \left|\langle Y_t^N, \phi\rangle - \langle Y_{t-}^N, \phi\rangle\right| \le \sup_t \frac{1}{\sqrt{N}} \sum_{v\in V} |(\eta_t(v) - \eta_{t-}(v))\phi(v)| \le \frac{1}{\sqrt{N}}||\phi||_\infty \to 0, \tag{4.5.55}$$

where we used that there can be at most one jump between the times t and t^- for the second inequality. Therefore, by combining (4.5.54) and (4.5.55) with (4.5.53) we can conclude that

$$\lim_{\delta\to 0}\limsup_{N\to\infty}\mathbb{P}(w_\delta(\langle Y^N_{[0,T]},\phi\rangle)\geq\varepsilon)=0. \tag{4.5.56}$$

Therefore we find that $t\mapsto\langle Y_t,\phi\rangle$ is a.s. continuous. □

Finally we show that Y solves the martingale problem in (4.3.4).

Proposition 4.5. For every $\phi\in C^\infty_{c,S}$ the processes $\mathscr{M}^\phi_t(Y_{[0,T]})$ and $\mathscr{N}^\phi_t(Y_{[0,T]})$ defined in (4.3.4) are martingales with respect to the filtration $\{\mathscr{F}_t : t\geq 0\}$ generated by Y.

Proof. Fix arbitrary $n\in\mathbb{N}$, $s\geq 0$, $0\leq s_1\leq\ldots\leq s_n\leq s$, $\psi_1,\ldots,\psi_n\in C^\infty_{c,S}$ and $\Psi\in C_b(\mathbb{R}^n)$, and define the function $\mathcal{I}: D([0,T];(C^\infty_{c,S})^*)\to\mathbb{R}$ as

$$\mathcal{I}(X):=\Psi\left(X_{s_1}(\psi_1),\ldots,X_{s_n}(\psi_n)\right). \tag{4.5.57}$$

To show that $\mathscr{M}^\phi_t(Y_{[0,T]})$ and $\mathscr{N}^\phi_t(Y_{[0,T]})$ are $\mathscr{F}_t$-martingales, it suffices to show that

$$\begin{aligned}\lim_{k\to\infty}\mathbb{E}\left[\mathscr{M}^{N_k,\phi}_t(Y^{N_k}_{[0,T]})\mathcal{I}(Y^{N_k}_{[0,T]})\right]&=\mathbb{E}\left[\mathscr{M}^\phi_t(Y_{[0,T]})\mathcal{I}(Y_{[0,T]})\right],\\ \lim_{k\to\infty}\mathbb{E}\left[\mathscr{N}^{N_k,\phi}_t(Y^{N_k}_{[0,T]})\mathcal{I}(Y^{N_k}_{[0,T]})\right]&=\mathbb{E}\left[\mathscr{N}^\phi_t(Y_{[0,T]})\mathcal{I}(Y_{[0,T]})\right],\end{aligned} \tag{4.5.58}$$

with $\mathscr{M}^{N,\phi}_t$ and $\mathscr{N}^{N,\phi}_t$ the Dynkin martingales defined in (4.5.1). Namely, by the martingale property we then have that

$$\begin{aligned}\mathbb{E}\left[\mathscr{M}^\phi_t(Y_{[0,T]})\mathcal{I}(Y_{[0,T]})\right]&=\lim_{k\to\infty}\mathbb{E}\left[\mathscr{M}^{N_k,\phi}_t(Y^{N_k}_{[0,T]})\mathcal{I}(Y^{N_k}_{[0,T]})\right]\\&=\lim_{k\to\infty}\mathbb{E}\left[\mathscr{M}^{N_k,\phi}_s(Y^{N_k}_{[0,T]})\mathcal{I}(Y^{N_k}_{[0,T]})\right]\\&=\mathbb{E}\left[\mathscr{M}^\phi_s(Y_{[0,T]})\mathcal{I}(Y_{[0,T]})\right],\end{aligned} \tag{4.5.59}$$

and analogous for $\mathscr{N}^\phi_t(Y_{[0,T]})$.

We start by proving $\mathscr{M}^\phi_t(Y_{[0,T]})$ is a martingale. First of all, note that from Proposition 4.2 we can conclude

$$\lim_{k\to\infty}\mathbb{E}\left[\mathscr{M}^{N_k,\phi}_t(Y^{N_k}_{[0,T]})\mathcal{I}(Y^{N_k}_{[0,T]})\right]=\lim_{k\to\infty}\mathbb{E}\left[\mathscr{M}^\phi_t(Y^{N_k}_{[0,T]})\mathcal{I}(Y^{N_k}_{[0,T]})\right]. \tag{4.5.60}$$

Furthermore, in Lemma 4.1 we have shown that the process $\mathscr{M}^\phi_t(Y^N_{[0,T]})$ is uniformly bounded in L^4, hence it is also uniformly bounded in L^2, therefore

$$\sup_{k\in\mathbb{N}}\mathbb{E}\left[\left|\mathscr{M}^\phi_t(Y^{N_k}_{[0,T]})\mathcal{I}(Y^{N_k}_{[0,T]})\right|^2\right]\leq||\Psi||^2_\infty\sup_{k\in\mathbb{N}}\sum_{\sigma\in S}\mathbb{E}\left[\left(\mathscr{M}^\phi_t(Y^{N_k}_{[0,T]})\right)^2\right]<\infty. \tag{4.5.61}$$

This implies that we have uniform integrability of $\mathscr{M}_t^\phi(Y^{N_k}_{[0,T]})\mathcal{I}(Y^{N_k}_{[0,T]})$. It now suffices to show that $\mathscr{M}_t^\phi(Y^{N_k}_{[0,T]})\mathcal{I}(Y^{N_k}_{[0,T]})$ converges to $\mathscr{M}_t^\phi(Y_{[0,T]})\mathcal{I}(Y_{[0,T]})$ in distribution. One usually concludes this using the Portmanteau theorem, but because the path space $D([0,T];(C^\infty_{c,S})^*_S)$ is not metrizable, we cannot directly use this. Instead, using the exact same method as introduced in [117, Proposition 5.2], one can work around the problem of non-metrizability via the continuity of $t \mapsto \langle Y_t, \phi\rangle$.

The proof that $\mathscr{N}_t^\phi(Y_{[0,T]})$ is a martingale works in the same way. First we note that by Proposition 4.3 we have that

$$\lim_{k\to\infty} \mathbb{E}\left[\mathscr{N}_t^{N_k,\phi}(Y^{N_k}_{[0,T]})\mathcal{I}(Y^{N_k}_{[0,T]})\right] = \lim_{k\to\infty} \mathbb{E}\left[\mathscr{N}_t^{\phi}(Y^{N_k}_{[0,T]})\mathcal{I}(Y^{N_k}_{[0,T]})\right]. \tag{4.5.62}$$

Therefore we only need to show that

$$\sup_{k\in\mathbb{N}} \mathbb{E}\left[\left|\mathscr{N}_t^{\phi}(Y^{N_k}_{[0,T]})\mathcal{I}(Y^{N_k}_{[0,T]})\right|^2\right] < \infty. \tag{4.5.63}$$

Afterwards the convergence of $\mathscr{N}_t^{\phi}(Y^{N_k}_{[0,T]})\mathcal{I}(Y^{N_k}_{[0,T]})$ to $\mathscr{N}_t^{\phi}(Y_{[0,T]})\mathcal{I}(Y_{[0,T]})$ in distribution follows from the same arguments as above.

To see that (4.5.63) holds, note that

$$\mathbb{E}\left[\left(\mathscr{N}_t^{\phi}(Y^{N_k}_{[0,T]})\right)^2\right] \leq 2\mathbb{E}\left[\left(\mathscr{M}_t^{\phi}(Y^{N_k}_{[0,T]})\right)^4\right] + 8t^2\varrho^2\left(\kappa\langle\langle\partial_x\phi,\partial_x\phi\rangle\rangle + \langle\langle\phi,\Sigma\phi\rangle\rangle\right)^2. \tag{4.5.64}$$

In the proof of Lemma 4.1, we have already shown that $\mathbb{E}\left[\left(\mathscr{M}_t^{\phi}(Y^{N}_{[0,T]})\right)^4\right]$ is uniformly bounded in N, hence the result follows. □

4.6 HYDRODYNAMIC LIMIT

In this section we give the proof of the hydrodynamic limit, i.e., of Theorem 4.2. We follow the standard methodology of [104].

4.6.1 *Preliminary results*

Before we start the proof of Theorem 4.2, we first show the following lemma which, using duality, provides uniform upper bounds for the first and second moment of the expected particle number when starting from the local equilibrium distribution (4.2.14).

Lemma 4.4. For all $N \in \mathbb{N}$, $t \geq 0$ and $(x,\sigma) \in V$ we have that

$$\mathbb{E}_{\mu^\varrho_N}\left[\eta^N_t(x,\sigma)\right] \leq ||\varrho||_\infty, \tag{4.6.1}$$

and

$$\mathbb{E}_{\mu_N^{\varrho}}\left[\eta_t^N(x,\sigma)^2\right] \leq ||\varrho||_\infty^2 + ||\varrho||_\infty. \tag{4.6.2}$$

Proof. For (4.6.1), note that by duality we have that

$$\begin{aligned}\mathbb{E}_{\mu_N^{\varrho}}\left[\eta_t^N(x,\sigma)\right] &= \int \mathbb{E}_{\eta^N}\left[\mathfrak{D}(\delta_{(x,\sigma)},\eta_t^N)\right] \mathrm{d}\mu_N^{\varrho}(\eta^N)\\ &= \int \widehat{\mathbb{E}}_{(x,\sigma)}\left[\mathfrak{D}(\delta_{(\widehat{X}_t,\widehat{\sigma}_t)},\eta^N)\right] \mathrm{d}\mu_N^{\varrho}(\eta^N)\\ &= \widehat{\mathbb{E}}_{(x,\sigma)}\left[\varrho(\tfrac{\widehat{X}_t}{N},\widehat{\sigma}_t)\right] \leq ||\varrho||_\infty. \end{aligned} \tag{4.6.3}$$

Similarly for (4.6.2), we have that

$$\begin{aligned}&\mathbb{E}_{\mu_N^{\varrho}}\left[\eta_t^N(x,\sigma)^2\right]\\ &\quad= \int \mathbb{E}_{\eta^N}\left[\mathfrak{D}(2\delta_{(x,\sigma)},\eta_t^N) + \mathfrak{D}(\delta_{(x,\sigma)},\eta_t^N)\right] \mathrm{d}\mu_N^{\varrho}(\eta^N)\\ &\quad= \int \widehat{\mathbb{E}}_{(x,\sigma),(x,\sigma)}\left[\mathfrak{D}(\delta_{\widehat{X}_t^{(1)},\widehat{\sigma}_t^{(1)})} + \delta_{(\widehat{X}_t^{(2)},\widehat{\sigma}_t^{(2)})},\eta^N) + \mathfrak{D}(\delta_{(\widehat{X}_t^{(1)},\widehat{\sigma}_t^{(1)})},\eta^N)\right] \mathrm{d}\mu_N^{\varrho}(\eta^N)\\ &\quad= \widehat{\mathbb{E}}_{(x,\sigma),(x,\sigma)}\left[\varrho(\tfrac{\widehat{X}_t^{(1)}}{N},\widehat{\sigma}_t^{(1)})\varrho(\tfrac{\widehat{X}_t^{(2)}}{N},\widehat{\sigma}_t^{(2)}) + \varrho(\tfrac{\widehat{X}_t^{(1)}}{N},\widehat{\sigma}_t^{(1)})\right] \leq ||\varrho||_\infty^2 + ||\varrho||_\infty. \end{aligned} \tag{4.6.4}$$

□

Now we define the processes $\mathscr{M}_t^{\phi}(\pi_{[0,T]}^N)$ and $\mathscr{M}_t^{N,\phi}(\pi_{[0,T]}^N)$ as in (4.3.4) and (4.5.1) respectively, and show that we can exchange these processes in the limit.

Proposition 4.6. For all $t \geq 0$ and $\phi \in C_{c,S}^{\infty}$, we have that

$$\lim_{N\to\infty} \mathbb{E}\left[\left|\mathscr{M}_t^{\phi}(\pi_{[0,T]}^N) - \mathscr{M}_t^{N,\phi}(\pi_{[0,T]}^N)\right|\right] = 0. \tag{4.6.5}$$

Proof. Through similar calculations as in the proof of Proposition 4.2, we find that

$$L_N\langle \pi_s^N,\phi\rangle = \langle \pi_s^N, A\phi\rangle + R_3(\phi,N,s). \tag{4.6.6}$$

Here $R_3(\phi,N,s)$ is the error term of the Taylor approximations, which is bounded as follows

$$|R_3(\phi,N,s)| \leq \frac{1}{N^2}\sum_{(x,\sigma)\in V_N} \eta_s^N(x,\sigma)\big(\kappa||\phi_{xxx}||_\infty + \lambda\sigma^2||\phi_{xx}||_\infty\big), \tag{4.6.7}$$

and so by (4.6.1)

$$\begin{aligned}&\lim_{N\to\infty} \mathbb{E}\left[\left|\mathscr{M}_t^{\phi}(\pi_{[0,T]}^N) - \mathscr{M}_t^{N,\phi}(\pi_{[0,T]}^N)\right|\right]\\ &\qquad= \lim_{N\to\infty}\int_0^t \mathbb{E}\big[\,|R_3(\phi,N,s)|\,\big]\,\mathrm{d}s\\ &\qquad\leq \lim_{N\to\infty}\frac{1}{N^2} t|V_N|\cdot||\varrho||_\infty\big(\kappa||\phi_{xxx}||_\infty + \lambda\sigma^2||\phi_{xx}||_\infty\big) = 0 \end{aligned} \tag{4.6.8}$$

which concludes the proof. □

Lastly we will prove that the martingale $\mathscr{M}_t^{N,\phi}(\pi^N_{[0,T]})$ actually vanishes in the limit.

Lemma 4.5. For any $\phi \in C^\infty_{c,S}$ we have that

$$\lim_{N\to\infty} \mathbb{E}\left[\sup_{t\in[0,T]} \left|\mathscr{M}_t^{N,\phi}(\pi^N_{[0,T]})\right|^2\right] = 0. \tag{4.6.9}$$

Proof. First of all, by Doob's maximal inequality, we have that

$$\mathbb{E}\left[\sup_{t\in[0,T]} \left|\mathscr{M}_t^{N,\phi}(\pi^N_{[0,T]})\right|^2\right] \leq 4\mathbb{E}\left[\left(\mathscr{M}_T^{N,\phi}(\pi^N_{[0,T]})\right)^2\right]. \tag{4.6.10}$$

Since $\mathscr{M}_t^{N,\phi}(\pi^N_{[0,T]})$ is a mean-zero martingale, this expectation is equal to the expectation of the predictable quadratic variation of $\mathscr{M}_t^{N,\phi}(\pi^N_{[0,T]})$, i.e.,

$$\mathbb{E}\left[\left(\mathscr{M}_T^{N,\phi}(\pi^N_{[0,T]})\right)^2\right] = \mathbb{E}\left[\int_0^T \Gamma^{N,\phi}(\pi^N_s)\,\mathrm{d}s\right] = \int_0^T \mathbb{E}\left[\Gamma^{N,\phi}(\pi^N_s)\right]\mathrm{d}s, \tag{4.6.11}$$

where $\Gamma_s^{N,\phi}$ is as defined in (4.5.2). By using the same calculations to get (4.5.31) we find that

$$\begin{aligned}\Gamma^{N,\phi}(\pi^N_s) = \frac{2\kappa}{N^2} &\sum_{(x,\sigma)\in V} \eta^N_s(x,\sigma)(\partial_x\phi(\tfrac{x}{N},\sigma))^2 \\ &+ \frac{1}{N^2}\sum_{(x,\sigma)\in V}\sum_{\sigma'\in S} c(\sigma,\sigma')\eta^N_s(x,\sigma)(\phi(\tfrac{x}{N},\sigma') - \phi(\tfrac{x}{N},\sigma))^2 \\ &+ R_4(\phi,s,N,\sigma),\end{aligned} \tag{4.6.12}$$

with $R_4(\phi,s,N)$ bounded as follows

$$|R_4(\phi,s,N)| \leq \kappa\frac{1}{N^4}\sum_{(x,\sigma)\in V_N} \eta^N_s(x,\sigma)(\kappa||\phi_{xx}||_\infty + \lambda\sigma||\phi_x||_\infty). \tag{4.6.13}$$

By dominated convergence and (4.6.1) we can then conclude that

$$\lim_{N\to\infty} \mathbb{E}\left[\left(\mathscr{M}_T^{N,\phi}(\pi^N_{[0,T]})\right)^2\right] = \lim_{N\to\infty}\int_0^T \mathbb{E}\left[\Gamma^{N,\phi}(\pi^N_s)\right]\mathrm{d}s = 0, \tag{4.6.14}$$

and the result follows. □

4.6.2 *Tightness*

We now prove the tightness result for the sequence $\{\pi^N_{[0,T]} : N \in \mathbb{N}\}$.

Proposition 4.7. $\{\pi^N_{[0,T]} : N \in \mathbb{N}\}$ is tight in $D([0,T];\boldsymbol{M})$.

Proof. In the space $D([0,T];\mathcal{M})$ we can prove tightness by showing that the following two assertions hold.

B.1 For all $t \in [0,T]$ and $\varepsilon > 0$ there exists a compact $K(t,\varepsilon) \subset \mathcal{M}$ such that

$$\sup_{N\in\mathbb{N}} \mathbb{P}\big(\pi^N_t \notin K(t,\varepsilon)\big) \leq \varepsilon. \tag{4.6.15}$$

B.2 For all $\varepsilon > 0$

$$\lim_{\delta\to 0}\limsup_{N\to\infty} \mathbb{P}\big(\omega(\pi^N_{[0,T]},\delta) \geq \varepsilon\big) = 0, \tag{4.6.16}$$

where

$$\omega(\alpha,\delta) = \sup\{d\big(\alpha_s,\alpha_t\big) : s,t \in [0,T], |t-s| < \delta\}, \tag{4.6.17}$$

and d is the metric on $\mathcal{M}$ given by

$$d\big(\alpha,\beta\big) = \sum_{j=1}^{\infty} 2^{-j}\left(1 \wedge \big|\alpha(\phi_j) - \beta(\phi_j)\big|\right) \tag{4.6.18}$$

for some specific choice of test functions $\phi_j \in C^\infty_{c,S}$.

We start by proving B.1. Fix $\varepsilon > 0$ and $t \geq 0$, and for some $C > 0$ let K_C be the following set

$$K_C = \Big\{\mu \in \mathcal{M} : \mu([-k,k]\times S) \leq C(2k+1)k^2 \text{ for all } k \in \mathbb{N}\Big\}. \tag{4.6.19}$$

By [104, Proposition A.25], this is a compact set in $\mathcal{M}$, and by Markov's inequality we now have that

$$\begin{aligned}
&\mathbb{P}(\pi^N_t([-k,k]\times S) \geq C(2k+1)k^2) \\
&\qquad \leq \frac{1}{C(2k+1)k^2}\mathbb{E}\left[\pi^N_t([-k,k]\times S)\right] \\
&\qquad = \frac{1}{C(2k+1)k^2 N}\sum_{(x,\sigma)\in[-kN,kN]\times S}\mathbb{E}\left[\eta^N_t(x,\sigma)\right] \\
&\qquad \leq \frac{1}{Ck^2}|S|\cdot||\varrho||_\infty.
\end{aligned} \tag{4.6.20}$$

Here we have used the inequality in (4.6.1). Therefore

$$\mathbb{P}(\pi_t^N \notin K_C) \leq \sum_{k=1}^{\infty} \mathbb{P}(\pi_t^N([-k,k]) \geq C(2k+1)k) \leq \frac{1}{C}|S| \cdot ||\varrho||_\infty \sum_{k=1}^{\infty} \frac{1}{k^2} < \infty. \tag{4.6.21}$$

By now taking C big enough, we then have that for all $N \in \mathbb{N}$ that $\mathbb{P}(\pi_t^N \notin K_C) \leq \varepsilon$, which finishes the proof of **B.1**.

In order to prove that **B.2** holds, note first that

$$\begin{aligned}\omega(\pi_{[0,T]}^N, \delta) &= \sup_{\substack{s,t \in [0,T] \\ |t-s|<\delta}} \sum_{j=1}^{\infty} 2^{-j} \left(1 \wedge \left|\langle \pi_t^N, \phi_j\rangle - \langle \pi_s^N, \phi_j\rangle\right|\right) \\ &\leq 2^{-m} + \sum_{j=1}^{m} \sup_{\substack{s,t \in [0,T] \\ |t-s|<\delta}} \left|\langle \pi_t^N, \phi_j\rangle - \langle \pi_s^N, \phi_j\rangle\right|. \end{aligned} \tag{4.6.22}$$

Here we have taken m arbitrarily, so the first term can be made as small as we want. We now want to show that the expecation of the sum vanishes as we let $N \to \infty$ and $\delta \downarrow 0$. Afterwards, the claim can be shown by using the Markov inequality.

Note first that we have the following,

$$\begin{aligned}&\mathbb{E}\left[\sup_{\substack{s,t \in [0,T] \\ |t-s|<\delta}} \left|\langle \pi_t^N, \phi_j\rangle - \langle \pi_s^N, \phi_j\rangle\right|^2\right] \\ &= \mathbb{E}\left[\sup_{\substack{s,t \in [0,T] \\ |t-s|<\delta}} \left|\mathscr{M}_t^{N,\phi_j}(\pi_{[0,T]}^N) - \mathscr{M}_s^{N,\phi_j}(\pi_{[0,T]}^N) - \int_s^t L_N\langle \pi_r^N, \phi_j\rangle \, \mathrm{d}r\right|^2\right] \\ &\leq 4\mathbb{E}\left[\sup_{t \in [0,T]} \left(\mathscr{M}_t^{N,\phi_j}(\pi_{[0,T]}^N)\right)^2\right] + 2\mathbb{E}\left[\sup_{\substack{s,t \in [0,T] \\ |t-s|<\delta}} \left|\int_s^t L_N\langle \pi_r^N, \phi_j\rangle \, \mathrm{d}r\right|^2\right]. \end{aligned} \tag{4.6.23}$$

By Lemma 4.5, the first term goes to zero as $N \to \infty$.

For the second term in (4.6.23), by filling in (4.6.6) we find that

$$\begin{aligned}\left|\int_s^t L_N\langle \pi_r^N, \phi_j\rangle \, \mathrm{d}r\right|^2 &= \left(\int_s^t \left(\langle \pi_r^N, A\phi_j\rangle + R_3(\phi_j, N, r)\right) \mathrm{d}r\right)^2 \\ &\leq 2\left(\int_s^t \langle \pi_r^N, A\phi_j\rangle \, \mathrm{d}r\right)^2 + 2\left(\int_s^t R_3(\phi_j, N, r) \, \mathrm{d}r\right)^2. \end{aligned} \tag{4.6.24}$$

By the upper bound on $R_3(\phi_j, N, r)$ in (4.6.7) and by (4.6.1), we can see that the last term vanishes in expectation when $N \to \infty$. For the other term we have that

$$\left(\int_s^t \langle \pi_r^N, A\phi_j \rangle \, \mathrm{d}r\right)^2 = \frac{1}{N^2}\left[\int_s^t \sum_{(x,\sigma)\in V} \eta_r^N(x,\sigma)\cdot (A\phi_j)(\tfrac{x}{N},\sigma)\,\mathrm{d}r\right]^2. \tag{4.6.25}$$

Using that $|t-s| < \delta$ and applying Hölder a number of times, we find that

$$\left(\int_s^t \langle \pi_r^N, A\phi_j \rangle \, \mathrm{d}r\right)^2 \leq \frac{1}{N^2}|V_{\phi_j}^N|\delta \cdot ||A\phi||_\infty \sum_{(x,\sigma)\in V_{\phi_j}^N} \int_0^T \left(\eta_r^N(x,\sigma)\right)^2 \mathrm{d}r. \tag{4.6.26}$$

Using the inequality in (4.6.2), we find that

$$\mathbb{E}\left[\sup_{\substack{s,t\in[0,T]\\|t-s|<\delta}} \left(\int_s^t \langle \pi_r^N, A\phi_j \rangle \, \mathrm{d}r\right)^2\right] \leq \frac{1}{N^2}|V_{\phi_j}^N|^2 \delta T \cdot ||A\phi||_\infty (||\varrho||_\infty^2 + ||\varrho||_\infty) = \mathcal{O}(\delta). \tag{4.6.27}$$

Therefore

$$\lim_{\delta\downarrow 0}\limsup_{N\to\infty} \mathbb{E}\left[\sup_{\substack{s,t\in[0,T]\\|t-s|<\delta}} \left|\langle \pi_t^N, \phi_j\rangle - \langle \pi_s^N, \phi_j\rangle\right|^2\right]$$
$$= \lim_{\delta\downarrow 0}\limsup_{N\to\infty} \mathbb{E}\left[\sup_{\substack{s,t\in[0,T]\\|t-s|<\delta}} \left(\int_s^t \langle \pi_r^N, A\phi_j \rangle \, \mathrm{d}r\right)^2\right] = 0. \tag{4.6.28}$$

So, by going back to (4.6.22) and using the Markov inequality, we get the following:

$$\mathbb{P}(\omega(\pi_{[0,T]}^N, \delta) \geq \varepsilon) \leq \frac{1}{\varepsilon}\left(2^{-m} + \sum_{j=1}^m \mathbb{E}\left[\sup_{\substack{s,t\in[0,T]\\|t-s|<\delta}} \left|\langle \pi_t^N, \phi_j\rangle - \langle \pi_s^N, \phi_j\rangle\right|\right]\right). \tag{4.6.29}$$

By now taking m such that $2^{-m} < \varepsilon^2$ and using (4.6.28) we see that

$$\lim_{\delta\downarrow 0}\limsup_{N\to\infty} \mathbb{P}(\omega(\pi_{[0,T]}^N, \delta) \geq \varepsilon) < \varepsilon, \tag{4.6.30}$$

which ultimately proves the tightness result. $\square$

4.6.3 *Proof of hydrodynamic limit*

Now we have everything needed to prove the result.

Proof of Theorem 4.2.
From the tightness of the sequence $\{P^N : N \in \mathbb{N}\}$ we know that there exists a subsequence $\{P^{N_k} : k \in \mathbb{N}\}$ that converges weakly in the Skorokhod topology, i.e., $P^{N_k} \xrightarrow{w} P$ for some probability measure P on $D([0,T];\mathcal{M})$. If we can show that every convergent subsequence converges to the dirac measure $P = \delta_\pi$ with π the unique continuous path solving (4.2.19), then the result follows.

First of all, by **B.2**, we immediately know that P is concentrated on continuous paths in $D([0,T];\mathcal{M})$. Now define for $\phi \in C^\infty_{c,S}$, $\varepsilon > 0$ and $T > 0$ the following set

$$H(\phi,\varepsilon) := \left\{ \beta \in D([0,T];\mathcal{M}) : \sup_{t\in[0,T]} \left| \langle \beta_t, \phi\rangle - \langle \beta_0, \phi\rangle - \int_0^t \langle \beta_s, A\phi\rangle \, \mathrm{d}s \right| \le \varepsilon \right\}. \tag{4.6.31}$$

Analogously as in [104, Lemma 8.7] one can prove that this set is closed in the Skorokhod topology. Since the set $H(\phi,\varepsilon)$ is closed, we can apply the Portmanteau Theorem to see that

$$\begin{aligned} P\big(H(\phi,\varepsilon)\big) &\ge \limsup_{k\to\infty} P^{N_k}\big(H(\phi,\varepsilon)\big) \\ &= \limsup_{k\to\infty} \mathbb{P}\left(\sup_{t\in[0,T]} \left| \langle \pi^{N_k}_t, \phi\rangle - \langle \pi^{N_k}_0, \phi\rangle - \int_0^t \langle \pi^{N_k}_s, A\phi\rangle \, \mathrm{d}s \right| \le \varepsilon \right) \\ &= \limsup_{k\to\infty} \mathbb{P}\left(\sup_{t\in[0,T]} \left| \mathscr{M}^{\phi}_t(\pi^{N_k}_{[0,T]}) \right| \le \varepsilon \right) \\ &= \limsup_{k\to\infty} \mathbb{P}\left(\sup_{t\in[0,T]} \left| \mathscr{M}^{N_k,\phi}_t(\pi^{N_k}_{[0,T]}) \right| \le \varepsilon \right). \end{aligned} \tag{4.6.32}$$

Here we have used Proposition 4.6 for the last equality. By Lemma 4.5 and the Markov inequality we then have that

$$\mathbb{P}\left(\sup_{t\in[0,T]} \left| \mathscr{M}^{N_k,\phi}_t(\pi^{N_k}_{[0,T]}) \right| > \varepsilon \right) \le \frac{1}{\varepsilon^2} \mathbb{E}\left[\sup_{t\in[0,T]} \left| \mathscr{M}^{N_k,\phi}_t(\pi^{N_k}_{[0,T]}) \right|^2 \right] \to 0, \tag{4.6.33}$$

so $P\big(H(\phi,\varepsilon)\big) = 1$. Since we took $\varepsilon > 0$ arbitrarily, we indeed find that $P = \delta_\pi$. □

5

LARGE DEVIATIONS OF THE MULTI-SPECIES STIRRING PROCESS

In this chapter[1] we deviate from the multi-layer setting and consider a multi-species particle system, namely the multi-species stirring process. We prove a large deviation principle for the trajectory of the vector of densities of the different species. The method involves extending the method of the paper by Kipnis, Olla and Varadhan [62] based on the superexponential estimate to the multi-species setting. This requires a careful choice of the corresponding weakly asymmetric dynamics, which is parametrized by potentials depending on the various species. We also prove the hydrodynamic limit of this weakly asymmetric dynamics, which is similar to the ABC model in [49, 12]. Using the appropriate asymmetric dynamics, we also obtain that the mobility matrix relating the drift currents to the potentials coincides with the covariance matrix of the reversible multinomial distribution, which then further leads to the Einstein relation.

5.1 INTRODUCTION

Interacting particle systems [76, 107] are used to study how macroscopic equations emerge from microscopic stochastic dynamics, as well as in the study of driven non-equilibrium systems and their non-equilibrium steady states. Among these, a well-studied process is the so-called Symmetric Simple Exclusion Process (SSEP), where particle interactions are governed by an exclusion constraint that permits at most one particle per site. This model (and various modifications of it) has been extensively studied in the literature, both in the study of scaling limits [61, 21, 104, 108] as well as in the understanding of microscopic properties of non-equilibrium steady states [103, 45]. The study of large deviations of the trajectory of the empirical density field for the SSEP was initiated in [62] (see also [51] where the gradient method was introduced in the context of Ginzburg-Landau models). The method developed there, valid for so-called gradient systems is based on the superexponential estimate, which allows to replace empirical averages of local

1 This chapter is based on [18].

functions by functions of the density field. This implies that one can prove with the same method at the same time the hydrodynamic limit of the weakly asymmetric exclusion process as well as the large deviations from the hydrodynamic limit for the SSEP.

The study of systems with multiple conserved quantities and their hydrodynamic limits has gained substantial interest in recent times see e.g. [12, 16, 49, 98] and references therein (see also e.g. [95] for an earlier reference). In particular, these results constitute rigorous versions of fluctuating hydrodynamics or mode coupling theory, see e.g. [108, 109, 110]. Another motivation for multi-species (and also connected multi-layer) models and their scaling limits is the phenomenon of uphill diffusion [17, 37] and systems of active particles.

The process we study in our paper is the multi-species analogue of the SSEP, known as the multi-species stirring process [15, 119, 125], on the geometry of the torus. In this process, at every site there is at most one particle, which can be of type $a \in \{1, \ldots, n\}$. The absence of a particle is called a particle of type zero. To each nearest neighbor edge is associated a Poisson clock of rate 1, different Poisson clocks being independent. When the clock of an edge rings, the occupancies of that edge are exchanged. An exchange between a particle of type $a \in \{1, \ldots, n\}$ at site x and an empty site at site $x + 1$ is of course the same as a jump of the particle from x to $x + 1$. It is well-known that the hydrodynamic limit for the densities of the n types of particles is a system of uncoupled heat equations, and in [16] it is also proved that the fluctuations around this hydrodynamic limit is an infinite dimensional Ornstein-Uhlenbeck process. Other results on the multi-species stirring process include duality, and exact formulas for the moments in the non-equilibrium steady state of a boundary driven version using duality combined with integrability [15].

To our knowledge, no explicit formula exists for the large deviation rate function for the density profile in the non-equilibrium steady state, as is the case e.g. for the SSEP, see [29]. In the setting of the macroscopic fluctuation theory, the rate function in the non-equilibrium steady state is strongly related to the rate function for the trajectory of the empirical density profile, i.e., the large deviations around the hydrodynamic limit. Therefore, in order to make progress in the understanding of non-equilibrium large deviations in multi-species models, it is natural to study the large deviations around the hydrodynamic limit for the multi-species stirring process. To the best of our knowledge, no rigorous results have been established in the context of dynamic large deviations for the multi-species stirring process.

In this paper we implement the method of [62] for gradient systems, based on the superexponential estimate, (see also [61, Chapter 10]) in our multi-species setting. The study of the large deviation principle for the multi-species stirring process relies on the introduction of a well-chosen weakly asymmetric process, where the rates are deformed by an exponential tilting, i.e., by introducing weak and slowly varying (in space) external fields that introduce a drift on the particles of various types. To understand the probability of deviating trajectories for the densities, one has to choose these potentials governing the asymmetry in such a way that in the modified dynamics the deviating trajectory becomes typical. The large deviation rate function is then roughly the relative entropy of the modified dynamics w.r.t. original dynamics which can be computed with the Girsanov formula. In particular, exactly as is done in [62] for the single species case, we also prove as byproduct the hydrodynamic limit of this weakly asymmetric multi-species process, which is a system of nonlinear coupled parabolic equations, closely related to the ABC model [49]. In this limiting partial differential equation (PDE), in addition to diffusion, a drift term is introduced into the currents, which makes this system appealing for describing multi-component diffusion processes in applications [100, 19]. The relation between the drift currents and the fields is via the symmetric Onsager matrix, which coincides with the covariance matrix of the multinomial reversible measures.

As a perspective towards further research, this work could serve as a starting point for various questions. These include exploring the extension of large deviation principles and hydrodynamic limits to boundary-driven systems in the multi-species setup, as previously done for the single species case [43]. Additionally one can investigate the density field fluctuations in the weakly asymmetric multi-species stirring process, analogous to what has been done for the ABC model in the context of the Kardar-Parisi-Zhang (KPZ) universality class [12]. Moreover, it would be of interest to apply these techniques to the multi-layer exclusion process defined in Chapter 3.

Starting from the literature, in Section 5.2 we first recall the definition of the multi-species stirring process, on the geometry of a torus, reporting also its reversible measure. Then, in Section 5.2.2 we define a weakly asymmetric version of the multi-species stirring process where the transition rates are perturbed by a family of potentials, indexed by the species involved in the transition and dependent on space and time.

Finally, in Section 5.2.3 we state the so-called *superexponential estimate*. This estimate turns out to be a useful tool in the proof of the hydrodynamic limit of the weakly asymmetric model and in the proof of the large deviation principle as well. The proof of this estimate goes beyond the main scope of this paper, therefore we report it in Appendix 5.6.

In Section 5.3 we state the hydrodynamic limit of the weakly asymmetric model. We postpone the proof to Appendix 5.5 since it can be shown by standard methods. Then, in Section 5.3.1, we make a specific choice of potentials needed for the proof of the large deviation principle. This choice is further motivated by Einstein relations between diffusion, mobility and compressibility matrices.

In Section 5.4, we proceed to state and prove the large deviation principle. With both the original model and the weakly asymmetric model established, we first obtain the Radon-Nikodym derivative of their respective path-space measures in Section 5.4.1. This can be computed using the Girsanov formula and will be equal to exponential martingale associated with the original model. For the upper bound, we first establish the exponential tightness of the path-space measures in Section 5.4.2 (which reduces the proof to verifying the upper bound for compact sets instead of closed sets). The upper bound is then derived in Section 5.4.3 using the martingale property of the Radon-Nikodym derivative.

For the lower bound, which we prove in Section 5.4.4, we demonstrate the relationship between the large deviation rate function and the hydrodynamic limit of the weakly asymmetric model. Specifically, for every deviating path, we show the existence of a potential such that this path becomes typical under the weakly asymmetric dynamics. This leads to a new formulation of the large deviation rate functional, expressed as the norm of this potential in an appropriate Sobolev space. Finally, using this relationship, we are able to demonstrate the lower bound.

5.2 THE MULTI-SPECIES STIRRING PROCESS

In this section, we describe the multi-species stirring process. We first examine the symmetric case, then we define a weakly asymmetric version in which the transition rates are "deformed" through a potential.

In both cases, we consider the geometry of a one-dimensional torus with N sites, denoted by $\mathbb{T}_N = \mathbb{Z}/N\mathbb{Z}$. Additionally, for simplicity, we consider the scenario with two types of particles, in addition to vacancies, also called holes (the general case of n types of particles will be considered in Remarks 5.4 and 5.9). The occupation variable is denoted by $\boldsymbol{\eta}(x) = (\eta_0(x), \eta_1(x), \eta_2(x))$, where $\eta_a(x) \in \{0,1\}$ represents the presence or absence of a particle of type a at site x. For any time $t \geq 0$, the configuration of the process is denoted by $\boldsymbol{\eta}_t$.

As a convention, we use the labels $a = 1,2$ to distinguish particles of species 1 and 2, and we use the label $a = 0$ to denote the holes. The term "holes" is motivated by the fact that its occupation variable is determined once we know the occupation variable of the species of particles 1 and 2, due to the so-called "exclusion constraint"

$$\eta_0(x) = 1 - \eta_1(x) - \eta_2(x) \qquad \forall x \in \mathbb{T}_N. \tag{5.2.1}$$

Therefore, the configuration space reads

$$\Omega_N = \left\{ \boldsymbol{\eta} = (\eta_0, \eta_1, \eta_2) \ : \ \sum_{a=0}^{2} \eta_a(x) = 1 \text{ for all } x \in \mathbb{T}_N \right\}. \tag{5.2.2}$$

In the literature, the multi-species stirring process has also been considered with maximal occupancy per site higher than 1 (see [125, 15]), and also the boundary driven case has been considered (see [119, 15]).

In this paper, on the same geometry and configuration space, we introduce two types of dynamics: the symmetric and the weakly asymmetric ones. In the symmetric dynamics, each transition occurs at the same rate to both the left and the right. In contrast, the weakly asymmetric dynamics introduces a weak asymmetry in the rates, resulting in a "drift" in the particles' jumps.

5.2.1 *The symmetric case*

In the symmetric case, the dynamics consists in swapping occupancies of nearest neighbor sites according to independent rate 1 Poisson processes. More precisely, considering any bond $(x, x+1)$, any particle or hole present at site x is exchanged with any particle or hole present at site $x+1$. For any $a, b \in \{0, 1, 2\}$ such that $\eta_a(x)\eta_b(x) = 1$ we now define the configuration $\boldsymbol{\eta}_{a,b}^{x,x+1}$ obtained by swapping the occupancies at x and $x+1$, i.e.,

$$\boldsymbol{\eta}_{a,b}^{x,x+1} = \boldsymbol{\eta} - \delta_a^x + \delta_b^x - \delta_b^{x+1} + \delta_a^{x+1} \tag{5.2.3}$$

where $\pm\delta_a^x$ indicates that a particle or vacancy of type a is added or removed at site x. If $\eta_a(x)\eta_b(x+1) \neq 1$ then we make the convention that $\boldsymbol{\eta}_{a,b}^{x,x+1} = \boldsymbol{\eta}$. The infinitesimal generator of this process given by

$$L_N f(\boldsymbol{\eta}) = N^2 \sum_{x \in \mathbb{T}_N} \sum_{a,b=0}^{2} \eta_a(x)\eta_b(x+1) \left(f(\boldsymbol{\eta}_{a,b}^{x,x+1}) - f(\boldsymbol{\eta}) \right). \tag{5.2.4}$$

We denote by $\mathbb{T} = [0, 1]$ the one-dimensional torus. For $\boldsymbol{\varrho} = (\varrho_1, \varrho_2)$ where $\varrho_a : \mathbb{T} \to [0, 1]$ for $a \in \{1, 2\}$ are smooth functions, we introduce the local equilibrium product measures associated to $\boldsymbol{\varrho}$:

$$\nu_N^{\boldsymbol{\varrho}} := \bigotimes_{x \in \mathbb{T}_N} \text{Multinomial}(1, \varrho_1(\tfrac{x}{N}), \varrho_2(\tfrac{x}{N})), \tag{5.2.5}$$

i.e., $\varrho_1(\frac{x}{N})$ and $\varrho_2(\frac{x}{N})$ are the probabilities of having a particle of type 1 respectively 2 at x. Furthermore, the probability to have no particle at any site $x \in \mathbb{T}_N$ is equal to $\varrho_0(\frac{x}{N}) := 1 - \varrho_1(\frac{x}{N}) - \varrho_2(\frac{x}{N})$. These measures are called the local equilibrium measures since for constant profiles these measures are reversible, which follows from the detailed balance condition. In the following, it will be useful to denote by $\nu_N^{1/3}$ the reversible measure with multinomial densities given by $\varrho_1 = \varrho_2 = \frac{1}{3}$.

5.2.2 *The weakly asymmetric stirring process*

We introduce a weakly asymmetric version of the multi-species stirring process, which will play a crucial role in the study of large deviations. We parametrize the weak asymmetry by three smooth functions $\boldsymbol{H} = (H_{01}, H_{02}, H_{12})$. Moreover, we define for $a < b$ and fixed $T > 0$

$$H_{ba}(x,t) := -H_{ab}(x,t) \qquad \forall x \in \mathbb{T}, \quad \forall t \in [0,T]. \tag{5.2.6}$$

The reason for this antisymmetric choice (5.2.6) will be clarified later. The time-dependent generator of the weakly asymmetric multi-species stirring process parametrized by $\boldsymbol{H}$ is then given by

$$L^{\boldsymbol{H}}_{N,t} f(\boldsymbol{\eta}) = N^2 \sum_{x \in \mathbb{T}_N} \sum_{a,b=0}^{2} c^{\boldsymbol{H},ab}_{(x,x+1)}(t) \left(f(\boldsymbol{\eta}^{x,x+1}_{a,b}) - f(\boldsymbol{\eta}) \right), \tag{5.2.7}$$

where

$$c^{\boldsymbol{H},ab}_{(x,x+1)}(t) = \exp\left(\nabla_N H_{ab}(\tfrac{x}{N}, t) \right) \eta_a(x) \eta_b(x+1) \tag{5.2.8}$$

Here ∇_N denotes the discrete gradient, i.e.,

$$\nabla_N H_{ab}(\tfrac{x}{N}, t) = H_{ab}(\tfrac{x+1}{N}, t) - H_{ab}(\tfrac{x}{N}, t). \tag{5.2.9}$$

Later on we will omit the explicit dependence on t in (5.2.7) for notational simplicity.

Remark 5.1. In this remark we explain the choice imposed by (5.2.6). In general, the weakly asymmetric rate of exchanging a particle of type α at x and a particle of type β at y with $x \sim y$ nearest neighbors, determined by the potential $H_{\alpha\beta}$, is given by

$$\exp\left(H_{\alpha\beta}(\tfrac{y}{N}, t) - H_{\alpha\beta}(\tfrac{x}{N}, t) \right) \eta^x_\alpha \eta^y_\beta \tag{5.2.10}$$

However, exchanging occupancy of type α at site x with type β at site y via the potential $H_{\alpha\beta}$ has to be identical with exchanging occupancy of type β at site y with type α at site x via the potential $H_{\beta\alpha}$. Therefore, the following has to hold

$$\exp\left(H_{\alpha\beta}(\tfrac{y}{N}, t) - H_{\alpha\beta}(\tfrac{x}{N}, t) \right) \eta^x_\alpha \eta^y_\beta = \exp\left(H_{\beta\alpha}(\tfrac{x}{N}, t) - H_{\beta\alpha}(\tfrac{y}{N}, t) \right) \eta^y_\beta \eta^x_\alpha. \tag{5.2.11}$$

This is satisfied if and only if $H_{\alpha\beta} = -H_{\beta\alpha}$.

We introduce some further notation. For all $T > 0$, we consider the Skorokhod space $D([0,T], \Omega)$, which consists of the càdlàg trajectories taking values in Ω. On this space, we define the following path space measures:

- $\mathbb{P}^{1/3}_N$: path space measure of the symmetric process with generator (5.2.4), initialized with the distribution $\nu^{1/3}_N$.

- $\mathbb{P}_N^{\varrho}$: path space measure of the symmetric process with generator (5.2.4), initialized with the distribution ν_N^{ϱ}.
- $\mathbb{P}_N^{\varrho,\boldsymbol{H}}$: path space measure of the weakly asymmetric process with generator (5.2.7), initialized with the distribution ν_N^{ϱ}.

For each species $a \in \{1,2\}$ we introduce the corresponding empirical density field

$$\pi_a^N(\boldsymbol{\eta}) := \frac{1}{N} \sum_{x \in \mathbb{T}_N} \eta_a(x) \delta_{\frac{x}{N}}, \tag{5.2.12}$$

When considering the process $\{\boldsymbol{\eta}_t : t \geq 0\}$, we denote the empirical measure by $\pi_{a,t}^N := \pi_a^N(\boldsymbol{\eta}_t)$. This process $\{\pi_{a,t}^N : t \geq 0\}$ takes values in $D([0,T], \mathcal{M}_1)$, where $\mathcal{M}_1$ denotes the space of measures over Ω with total mass bounded by 1, i.e., $\sup_{||f|| \leq 1} \langle \pi_{a,t}^N, f \rangle \leq 1$. Additionally, we define the row vector of density fields

$$\boldsymbol{\pi}_t^N = \begin{pmatrix} \pi_{1,t}^N & \pi_{2,t}^N \end{pmatrix} \tag{5.2.13}$$

taking values in the space $D([0,T], \mathcal{M}_1 \times \mathcal{M}_1)$. We consider two functions $G_1, G_2 \in C^{2,1}(\mathbb{T} \times [0,T])$ and we list them in a vector denoted by

$$\boldsymbol{G}(x,t) := \begin{pmatrix} G_1(x,t) \\ G_2(x,t) \end{pmatrix}. \tag{5.2.14}$$

Then, we denote the pairing

$$\langle \boldsymbol{\pi}_t^N, \boldsymbol{G}(\cdot,t) \rangle = \int_{\mathbb{T}} G_1(x,t) \pi_{1,t}^N(\mathrm{d}x) + \int_{\mathbb{T}} G_2(x,t) \pi_{2,t}^N(\mathrm{d}x). \tag{5.2.15}$$

5.2.3 *Superexponential estimate*

In this section we state the so-called *superexponential estimate*. This is a crucial tool initially introduced in [51], [62], which allows to replace macroscopic averages of local observables by an appropriate function of the local density. This is crucial both in the derivation of the hydrodynamic limit of the weakly asymmetric model as well as in the large deviations of the symmetric model. In the latter it becomes important that the replacement is superexponentially good, i.e., can still be performed e.g. in exponential martingales containing local averages. This replacement is carried out within a space interval constructed around a microscopic point. Eventually, the size of this interval shrinks as the system size increases.

We consider a local function ϕ defined on Ω_N for every large enough N, meaning that ϕ only depends on a fixed number of sites. For example, in the main part of this paper we only consider functions of the type $\phi(\boldsymbol{\eta}) = \eta_a(x)\eta_b(x+1)$ for $a, b \in \{0,1,2\}$ and $x \in \mathbb{T}_N$. Furthermore, we define

$$\widetilde{\phi}(\boldsymbol{p}) := \mathbb{E}_{\nu_N^{\boldsymbol{p}}}[\phi]. \tag{5.2.16}$$

namely the expectation with respect to the product over sites of multinomial distribution $\nu_N^{\boldsymbol{p}}$ with constant parameters $\boldsymbol{p} = (p_1, p_2)$.

Next, we introduce a function that will play a key role in the superexponential estimate. This function relates to the behavior of occupation variables in a small neighborhood around a microscopic point and it reads

$$V_{N,\varepsilon}(\boldsymbol{\eta}) = \sum_{x\in\mathbb{T}_N} \left| \sum_{|x-y|\le\varepsilon N} \tau_y \phi(\boldsymbol{\eta}) - \widetilde{\phi}\left(\sum_{|x-y|\le\varepsilon N} \eta_1(y), \sum_{|x-y|\le\varepsilon N} \eta_2(y) \right) \right|, \tag{5.2.17}$$

where τ_y is the shift operator, and we used the averaged sum notation

$$\sum_{i\in I} a_i := \frac{1}{|I|} \sum_{i\in I} a_i. \tag{5.2.18}$$

The superexponential estimate is then the following result.

Theorem 5.1. For any $\delta > 0$, for all $T > 0$ and $\phi : \Omega_N \to \mathbb{R}$

$$\overline{\lim_{\varepsilon\to 0}}\ \overline{\lim_{N\to\infty}} \frac{1}{N} \log\ \mathbb{P}_N^{1/3} \left(\frac{1}{N} \int_0^T V_{N,\varepsilon}(\boldsymbol{\eta}_t)\, \mathrm{d}t \ge \delta \right) = -\infty. \tag{5.2.19}$$

Since the proof of this Theorem is rather long and involved, and it is not the main result of this paper, we postpone it to appendix 5.6.

In the following corollary we show that the superexponential estimate also holds when we start from a local equilibrium distribution.

Corollary 5.1. Given a profile $\boldsymbol{\varrho} = (\varrho_1, \varrho_2)$, (5.2.19) holds also for the path space measure $\mathbb{P}_N^{\boldsymbol{\varrho}}$.

Proof. the proof follows from Theorem 5.1 and from the following upper bound for all sets $A \subset \Omega_N$

$$\mathbb{P}_N^{\boldsymbol{\varrho}}(A) = \sum_{\boldsymbol{\eta}\in\Omega_N} \frac{d\nu_N^{\boldsymbol{\varrho}}}{d\nu_N^{1/3}}(\boldsymbol{\eta}) \mathbb{P}_N^{\boldsymbol{\eta}}(A) \nu_N^{1/3}(\boldsymbol{\eta}) \le 3^N \mathbb{P}_N^{1/3}(A). \tag{5.2.20}$$

□

5.3 HYDRODYNAMIC LIMIT OF THE WEAKLY ASYMMETRIC MODEL

In this section we state the hydrodynamic limit of the weakly asymmetric version of the multi-species stirring model with generator (5.2.7).

REMARK 5.2. Sometimes, in this section and in the following one, in order to alleviate the notation, we do not explicitly write the space and time dependence of the densities. Namely, when this dependence is understood we only write $\boldsymbol{\varrho}, \varrho_1, \varrho_2$ in place of $\boldsymbol{\varrho}_t(x), \varrho_{1,t}(x), \varrho_{2,t}(x)$. The same convention is used for the potentials $H_{ab}(x,t)$.

THEOREM 5.2. As N tends to infinity, the density fields for the species $a = 1, 2$ converge in probability $\mathbb{P}_N^{\varrho,H}$ to the unique weak solution $(\varrho_{1,t}(x), \varrho_{2,t}(x))$ of the following system of hydrodynamic equations

$$\begin{cases} \partial_t \varrho_1 & = \Delta \varrho_1 - 2\nabla\left(\varrho_1(1-\varrho_1-\varrho_2)\nabla H_{10}\right) - 2\nabla\left(\varrho_1\varrho_2\nabla H_{12}\right), \\ \\ \partial_t \varrho_2 & = \Delta \varrho_2 - 2\nabla\left(\varrho_2(1-\varrho_1-\varrho_2)\nabla H_{20}\right) + 2\nabla\left(\varrho_1\varrho_2\nabla H_{12}\right), \end{cases} \tag{5.3.1}$$

with initial conditions $\varrho_{a,0}(x) = \varrho_a(x)$.

For the proof we refer to Section 5.5. In particular in the case where every $H_{ab} = 0$, we recover the uncoupled heat equations

$$\begin{aligned} \partial_t \varrho_1 &= \Delta \varrho_1, \\ \partial_t \varrho_2 &= \Delta \varrho_2. \end{aligned} \tag{5.3.2}$$

which in matrix form reads

$$\partial_t \begin{pmatrix} \varrho_1 \\ \varrho_2 \end{pmatrix} = D(\varrho_1, \varrho_2) \begin{pmatrix} \Delta\varrho_1 \\ \Delta\varrho_2 \end{pmatrix}, \tag{5.3.3}$$

where

$$D(\varrho_1, \varrho_2) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{5.3.4}$$

is the diffusion matrix.

5.3.1 *Potentials for large deviations*

In order to prove the large deviations for the trajectory of the empirical densities, we need appropriate perturbations of the dynamics which make these deviating trajectories typical. As will become clear in Section 5.4, these perturbations correspond to the weakly asymmetric stirring process, with potentials which we denote by

$$H_1(x,t) := H_{10}(x,t), \qquad H_2(x,t) := H_{20}(x,t) \qquad \forall x \in \mathbb{T} \quad \text{and} \quad t \in [0,T], \tag{5.3.5}$$

and where moreover, the potential H_{12} satisfies

$$H_{12}(x,t) = H_1(x,t) - H_2(x,t) \qquad \forall x \in \mathbb{T} \quad \text{and} \quad t \in [0,T]. \tag{5.3.6}$$

Therefore, the resulting hydrodynamic equations read

$$\begin{cases} \partial_t \varrho_1 &= \Delta \varrho_1 - 2\nabla \left(\varrho_1(1-\varrho_1)\nabla H_1\right) + 2\nabla \left(\varrho_1 \varrho_2 \nabla H_2\right), \\ \\ \partial_t \varrho_2 &= \Delta \varrho_2 - 2\nabla \left(\varrho_2(1-\varrho_2)\nabla H_2\right) + 2\nabla \left(\varrho_1 \varrho_2 \nabla H_1\right). \end{cases} \tag{5.3.7}$$

The intuitive interpretation of this choice of potentials is the following. Particles of type 1 and 2 are driven across the holes (particles of type 0) by the force depending on the potentials H_1 and H_2 (namely the external fields are given by the gradient of the potentials) respectively. When two particles of type 1 and 2 are adjacent, a competition between the fields generated by the potentials H_1 and H_2 sets in. As a result, the net field acting on each species is given by $\pm\nabla(H_1 - H_2)$ respectively. Moreover, as we will point out later, this choice of the fields allows the system to satisfy the Einstein relation connecting diffusion, mobility and compressibility matrices.

5.3.2 *Currents and the Einstein relation*

MACROSCOPIC CURRENTS. The hydrodynamic equations (5.3.7) can be interpreted as conservation laws. To illustrate this, we compute the macroscopic currents for each species. These currents represent the net flux crossing an infinitesimal volume surrounding a point $u \in \mathbb{T}$ at any time $t \in [0,T]$. We identify two types of currents:

1. *Fick's currents*: These currents are proportional to minus the density gradients via the diffusion matrix as given in (5.3.4). The currents are expressed as

$$\begin{pmatrix} J_1^F \\ J_2^F \end{pmatrix} = -D(\varrho_1, \varrho_2) \begin{pmatrix} \nabla \varrho_1 \\ \nabla \varrho_2 \end{pmatrix}. \tag{5.3.8}$$

   Generally, the diffusivity matrix (5.3.4) may depend on the densities, but in this case, it simplifies to the identity matrix.

2. *Drift currents*: these currents are defined as the product of (twice)[2] the mobility matrix

$$\chi(\varrho_1, \varrho_2) = \begin{pmatrix} \varrho_1(1-\varrho_1) & -\varrho_1\varrho_2 \\ -\varrho_1\varrho_2 & \varrho_2(1-\varrho_2) \end{pmatrix} \tag{5.3.9}$$

and the external field, which is the gradient of the potential (H_1, H_2). Specifically, these currents are given by

$$\begin{pmatrix} J_1^D \\ J_2^D \end{pmatrix} = 2\chi(\varrho_1, \varrho_2) \begin{pmatrix} \nabla H_1 \\ \nabla H_2 \end{pmatrix}. \tag{5.3.10}$$

It is important to note that the mobility matrix (5.3.9) is symmetric and corresponds to the covariance matrix of the multinomial distribution with parameters ϱ_1, ϱ_2. This matrix also appears in the study of fluctuations as proved in [16].

We now compute the total currents, which are given by the sum of Fick's and of the drift currents for each species. Namely they read

$$\begin{pmatrix} J_1 \\ J_2 \end{pmatrix} = \begin{pmatrix} J_1^F \\ J_2^F \end{pmatrix} + \begin{pmatrix} J_1^D \\ J_2^D \end{pmatrix}. \tag{5.3.11}$$

Therefore, equation (5.3.7) can be obtained by substituting the total currents (5.3.11) in the continuity equations of the densities, i.e.

$$\begin{aligned} \partial_t \varrho_1 &= -\nabla J_1, \\ \partial_t \varrho_2 &= -\nabla J_2. \end{aligned} \tag{5.3.12}$$

Einstein's relation. We introduce the free energy functional F, that is defined as the large deviation functional of a multinomial random variable with number of trials equal to 1 and probabilities all equal to $1/3$. Namely, we have that

$$F(\varrho_1, \varrho_2) = \varrho_1 \log(\varrho_1) + \varrho_2 \log(\varrho_2) + (1-\varrho_1-\varrho_2)\log(1-\varrho_1-\varrho_2) + \log(3). \tag{5.3.13}$$

We compute the Hessian matrix of $F(\varrho_1, \varrho_2)$, sometimes called the inverse of the compressibility matrix, obtaining

$$F''(\varrho_1, \varrho_2) = \begin{pmatrix} \frac{1}{\varrho_1} + \frac{1}{1-\varrho_1-\varrho_2} & \frac{1}{1-\varrho_1-\varrho_2} \\ \frac{1}{1-\varrho_1-\varrho_2} & \frac{1}{\varrho_2} + \frac{1}{1-\varrho_1-\varrho_2} \end{pmatrix}. \tag{5.3.14}$$

2 The factor 2 in front is due to the fact that in the generator (5.2.4) both jumps, to the left and to the right, have rate 1, instead of $1/2$.

Then we see that by combining (5.3.4), (5.3.9) and (5.3.14), the following relation holds.

$$D(\varrho_1,\varrho_2) = F''(\varrho_1,\varrho_2)\chi(\varrho_1,\varrho_2). \tag{5.3.15}$$

This equality is called the Einstein relation (see [6, 108] for details). Notice that we used the specific form of the potentials described in (5.3.5) and (5.3.6) to obtain the Einstein relation (5.3.15), which provides another physical motivation for these conditions.

Remark 5.3. We can recover the hydrodynamic limit of the single species weakly asymmetric exclusion process from equation (5.3.7) as given in [62, Theorem 3.1]. Namely, if we choose the same potential $H_1 = H_2 = H$, then we obtain the following.

$$\begin{cases} \partial_t \varrho_1 &= \Delta\varrho_1 - 2\nabla\left(\varrho_1(1-\varrho_1-\varrho_2)\nabla H\right), \\ \\ \partial_t \varrho_2 &= \Delta\varrho_2 - 2\nabla\left(\varrho_2(1-\varrho_1-\varrho_2)\nabla H\right). \end{cases} \tag{5.3.16}$$

By now defining $\varrho := \varrho_1 + \varrho_2$, i.e., ϱ does not distinguish between particles of type 1 and type 2, then ϱ satisfies

$$\partial_t \varrho = \Delta\varrho - 2\nabla(\varrho(1-\varrho)\nabla H). \tag{5.3.17}$$

This result is to be expected, since the process defined as $\eta := \eta_1 + \eta_2$ is a standard (weakly asymmetric) exclusion process.

Remark 5.4. At the cost of more notational complexity, but no additional mathematical difficulties, one can generalize the hydrodynamic limit of Theorem 5.2 to any number of species, i.e., $a \in \{0,1,...,n\}$ for any $n \in \mathbb{N}$.

The hydrodynamic limit of the weakly asymmetric model with the general potentials $H_{ab} = -H_{ba}$ is now given by a system of n dependent partial differential equations

$$\partial_t \varrho_a = \Delta\varrho_a - 2\sum_{b\neq a}\nabla\left(\varrho_a\varrho_b\nabla H_{ab}\right), \tag{5.3.18}$$

with the convention that $\varrho_0 = 1 - \sum_{a=1}^n \varrho_a$. For the large deviations of the trajectories of the densities we only need n potentials. The choice of potentials, which is the analogue of the conditions (5.3.5) and (5.3.6), then reads

$$H_a := H_{a0}, \qquad H_{ab} := H_a - H_b. \tag{5.3.19}$$

This choice of potentials then results in the following hydrodynamic limit

$$\partial_t \varrho_a = \Delta\varrho_a - 2\nabla\left(\varrho_a(1-\varrho_a)\nabla H_a\right) - 2\sum_{b\neq a}\nabla\left(\varrho_a\varrho_b\nabla H_b\right). \tag{5.3.20}$$

5.4 LARGE DEVIATIONS

In this section we aim to prove the large deviation principle of the multi-species stirring process. We start by defining the rate function $\mathcal{I}^{\varrho} : D([0,T], \mathcal{M}_1 \times \mathcal{M}_1) \to [0,\infty]$ which consists of two parts

$$\mathcal{I}^{\varrho}(\widehat{\boldsymbol{\alpha}}) = h^{\varrho}(\widehat{\boldsymbol{\alpha}}_0) + \mathcal{I}_{tr}(\widehat{\boldsymbol{\alpha}}), \tag{5.4.1}$$

where $\widehat{\boldsymbol{\alpha}}_0$ denotes the trajectory $\widehat{\boldsymbol{\alpha}}$ evaluated at the initial time $t=0$. Here $h^{\varrho}(\widehat{\boldsymbol{\alpha}}_0)$ is the static part of the large deviation functional, i.e. the one due to the initial product measure $\nu_N \boldsymbol{\varrho}$ as defined in (5.2.5). It is given by the formula

$$h^{\varrho}(\widehat{\boldsymbol{\alpha}}_0) := \sup_{\boldsymbol{\phi}} h^{\varrho}(\widehat{\boldsymbol{\alpha}}_0;\boldsymbol{\phi}), \quad h^{\varrho}(\widehat{\boldsymbol{\alpha}}_0;\boldsymbol{\phi}) := \sum_{a=0}^{2} \langle \alpha_{a,0}, \phi_a \rangle - \int_{\mathbb{T}} \log\left(\sum_{a=0}^{2} \varrho_a(x) e^{\phi_a(x)} \right) \mathrm{d}x, \tag{5.4.2}$$

where the supremum is taken over all continuous $\boldsymbol{\phi} = (\phi_0, \phi_1, \phi_2)$ and we use that $\varrho_0 := 1 - \varrho_1 - \varrho_2$.

$\mathcal{I}_{tr}(\widehat{\boldsymbol{\alpha}})$ is the dynamic part of the large deviation functional, i.e., the one due to the dynamics of the trajectory $\boldsymbol{\varrho}$ over time. It has the following form,

$$\mathcal{I}_{tr}(\widehat{\boldsymbol{\alpha}}) := \sup_{\boldsymbol{G}} \mathcal{I}_{tr}(\widehat{\boldsymbol{\alpha}};\boldsymbol{G}), \qquad \mathcal{I}_{tr}(\widehat{\boldsymbol{\alpha}};\boldsymbol{G}) := \ell(\widehat{\boldsymbol{\alpha}};\boldsymbol{G}) - \tfrac{1}{2}||\boldsymbol{G}||^2_{\mathcal{H}(\widehat{\boldsymbol{\alpha}})}. \tag{5.4.3}$$

Here the supremum is taken over vectors of functions $\boldsymbol{G} = \binom{G_1}{G_2}$ where both $G_1, G_2 \in C^{2,1}(\mathbb{T} \times [0,T])$. The operator ℓ is the linear operator corresponding to the hydrodynamic limit of the multi-species SEP, i.e., it is given by

$$\ell(\widehat{\boldsymbol{\alpha}};\boldsymbol{G}) = \langle \widehat{\boldsymbol{\alpha}}_T, \boldsymbol{G}(\cdot,T)\rangle - \langle \widehat{\boldsymbol{\alpha}}_0, \boldsymbol{G}(\cdot,0)\rangle - \int_0^T \langle \widehat{\boldsymbol{\alpha}}_t, (\partial_t + \Delta)\boldsymbol{G}(\cdot,t)\rangle \,\mathrm{d}t, \tag{5.4.4}$$

which is equal to zero for all $\boldsymbol{G}$ iff $\widehat{\boldsymbol{\alpha}}$ solves the PDE $\partial_t \boldsymbol{\varrho}_t = \Delta \boldsymbol{\varrho}_t$ in the sense of distributions. Lastly, the norm in the definition of the rate function (5.4.3) is the norm corresponding to the following inner product

$$\begin{aligned}\langle \boldsymbol{G}, \boldsymbol{H}\rangle_{\mathcal{H}(\widehat{\boldsymbol{\alpha}})} &= 2\int_0^T \langle \widehat{\alpha}_{1,t}(1-\widehat{\alpha}_{1,t}), \nabla G_1(\cdot,t)\nabla H_1(\cdot,t)\rangle \,\mathrm{d}t \\ &+ 2\int_0^T \langle \widehat{\alpha}_{2,t}(1-\widehat{\alpha}_{2,t}), \nabla G_2(\cdot,t)\nabla H_2(\cdot,t)\rangle \,\mathrm{d}t \\ &- 2\int_0^T \langle \widehat{\alpha}_{1,t}\widehat{\alpha}_{2,t}, \nabla G_1(\cdot,t)\nabla H_2(\cdot,t) + \nabla G_2(\cdot,t)\nabla H_1(\cdot,t)\rangle \,\mathrm{d}t.\end{aligned} \tag{5.4.5}$$

Through this norm, and its action on smooth functions, we can then define a Hilbert space $\mathcal{H}(\widehat{\boldsymbol{\alpha}})$ as the completion of the set of smooth functions.

Remark 5.5. In Lemmas 5.3 and 5.4 we give more explicit forms of the functionals h^{ϱ} and $\mathcal{I}_{tr}$ respectively. Namely, we find functions $\boldsymbol{\phi}$ and $\boldsymbol{G}$ such that $h^{\varrho}(\widehat{\boldsymbol{\alpha}}_0) = h^{\varrho}(\widehat{\boldsymbol{\alpha}};\boldsymbol{\phi})$ and $\mathcal{I}_{tr}(\widehat{\boldsymbol{\alpha}}) = \mathcal{I}_{tr}(\widehat{\boldsymbol{\alpha}};\boldsymbol{G})$. Furthermore, we will see that $h^{\varrho}(\widehat{\boldsymbol{\alpha}}_0)$ can be written as the limit of relative entropies of multinomials with respective densities $\widehat{\varrho}_0$ and ϱ, with $\widehat{\varrho}_0$ the density of $\widehat{\boldsymbol{\alpha}}_0$, and $\mathcal{I}_{tr}(\widehat{\boldsymbol{\alpha}}) = \frac{1}{2}||\boldsymbol{H}||^2_{\mathcal{H}(\widehat{\boldsymbol{\alpha}})}$ where $\boldsymbol{H} \in \mathcal{H}(\widehat{\boldsymbol{\alpha}})$ is the unique function such that $\widehat{\boldsymbol{\alpha}}$ satisfies (5.3.7) in the weak sense.

In order for a large deviation principle to hold, we need to show that we have the following two inequalities:

- **Upper bound**: For every closed $\mathcal{C} \subset D([0,T];\mathcal{M}_1 \times \mathcal{M}_1)$ we have that

$$\overline{\lim_{N\to\infty}} \frac{1}{N} \log \mathbb{P}^{\varrho}_N \left(\boldsymbol{\pi}^N_{[0,T]} \in \mathcal{C}\right) \leq -\inf_{\widehat{\boldsymbol{\alpha}}\in\mathcal{C}} \mathcal{I}^{\varrho}(\widehat{\boldsymbol{\alpha}}) \tag{5.4.6}$$

- **Lower bound**: For every open $\mathcal{O} \subset D([0,T];\mathcal{M}_1 \times \mathcal{M}_1)$ we have that

$$\underline{\lim_{N\to\infty}} \frac{1}{N} \log \mathbb{P}^{\varrho}_N \left(\boldsymbol{\pi}^N_{[0,T]} \in \mathcal{O}\right) \geq -\inf_{\widehat{\boldsymbol{\alpha}}\in\mathcal{O}} \mathcal{I}^{\varrho}(\widehat{\boldsymbol{\alpha}}) \tag{5.4.7}$$

We give a proof for the upper and lower bound in Sections 5.4.3 and 5.4.4 respectively. First we calculate the Radon-Nikodym derivative $\frac{d\mathbb{P}^{\varrho,H}_N}{d\mathbb{P}^{\varrho}_N}$ of the path space measures of the weakly asymmetric process relative to the original process in Section 5.4.1. Additionally, we establish exponential tightness in Section 5.4.2 which allows for the substitution of closed sets with compact sets in the derivation of the upper bound.

5.4.1 *Radon-Nikodym derivative and the exponential martingale*

The goal of this section is to obtain an explicit expression of the Radon-Nikodym derivative of the path space measure $\mathbb{P}^{\varrho,\boldsymbol{H}}_N$ with respect to the path space measure $\mathbb{P}^{\varrho}_N$. From the literature (see [61, Proposition 2.6] and [78, Chapter 19]) the Girsanov formula states that

$$\log\left(\frac{dP^{\varrho \boldsymbol{H}}_N}{d\mathbb{P}^{\varrho}_N}\right) = \sum_{x\in\mathbb{T}_N}\sum_{a,b=0}^{2}\int_0^T \log\left(\frac{c^{\boldsymbol{H},ab}_{(x,x+1)}(t)}{\eta_{a,t}(x)\eta_{b,t}(x+1)}\right) dJ^{x,x+1}_{ab}(t)$$
$$- N^2 \sum_{x\in\mathbb{T}_N}\sum_{a,b=0}^{2}\int_0^T \eta_{a,t}(x)\eta_{b,t}(x+1)\left(\exp\left\{\nabla_N H_{ab}\left(\tfrac{x}{N},t\right)\right\} - 1\right)dt. \tag{5.4.8}$$

Here, we represent by $J^{x,x+1}_{ab}(t)$ the number of transitions occurred up to time $t \in [0,T]$ that swap the occupancies of species a,b between sites x and $x+1$.

Under the path space measure $\mathbb{P}_N^{\varrho,\boldsymbol{H}}$ the random process $\{J_{a,b}^{x,x+1}(t) : t \geq 0\}$ is a Poisson process with time-dependent intensity $c_{(x,x+1)}^{\boldsymbol{H},ab}(t)$. In the following result we provide an alternative formula for the Radon-Nikodym derivative defined in (5.4.8).

Lemma 5.1. For all $T \geq 0$, for all $N \in \mathbb{N}$ and for all $H_1, H_2 \in C^{2,1}(\mathbb{T} \times [0,T])$ under conditions (5.2.6), (5.3.5) and (5.3.6) we have that

$$Z_{N,T}^{\boldsymbol{H}}(\boldsymbol{\pi}_{[0,T]}^N) := \frac{\mathrm{d}\mathbb{P}_N^{\varrho,\boldsymbol{H}}}{\mathrm{d}\mathbb{P}_N^{\varrho}} = \exp\Big(N\langle \boldsymbol{\pi}_T^N, \boldsymbol{H}(\cdot,T)\rangle - N\langle \boldsymbol{\pi}_0^N, \boldsymbol{H}(\cdot,0)\rangle\Big) \cdot \exp\left(-\int_0^T e^{-N\langle \boldsymbol{\pi}_t^N, \boldsymbol{H}(\cdot,t)\rangle}\left(\partial_t + L_N\right) e^{N\langle \boldsymbol{\pi}_t^N, \boldsymbol{H}(\cdot,t)\rangle}\, \mathrm{d}t\right). \tag{5.4.9}$$

Proof. We consider the first term in the right hand side of equation (5.4.8) and we write

$$\begin{aligned}
\sum_{x\in\mathbb{T}_N}\sum_{a,b=0}^{2}\int_0^T \log\left(\frac{c_{(x,x+1)}^{\boldsymbol{H},ab}(t)}{\eta_{a,t}(x)\eta_{b,t}(x+1)}\right)\mathrm{d}J_{ab}^{x,x+1}(t) \\
= \sum_{x\in\mathbb{T}_N}\int_0^T \nabla_N H_{10}\left(\tfrac{x}{N},t\right)\left[\mathrm{d}J_{10}^{x,x+1}(t) - \mathrm{d}J_{01}^{x,x+1}(t)\right] \\
+ \sum_{x\in\mathbb{T}_N}\int_0^T \nabla_N H_{20}\left(\tfrac{x}{N},t\right)\left[\mathrm{d}J_{20}^{x,x+1}(t) - \mathrm{d}J_{02}^{x,x+1}(t)\right] \\
+ \sum_{x\in\mathbb{T}_N}\int_0^T \nabla_N H_{12}\left(\tfrac{x}{N},t\right)\left[\mathrm{d}J_{12}^{x,x+1}(t) - \mathrm{d}J_{21}^{x,x+1}(t)\right].
\end{aligned} \tag{5.4.10}$$

We use conditions (5.2.6), (5.3.5) and (5.3.6). Moreover, we denote by $\mathrm{d}\eta_{a,t}(x)$ the infinitesimal net current of particles of type a crossing the site x at time $t \in [0,T]$ i.e., it is defined through $\int_0^t \mathrm{d}\eta_{\alpha,s}(x) = \eta_{\alpha,t}(x) - \eta_{\alpha,0}(x)$ for any $t > 0$.

$$\begin{aligned}
\mathrm{d}\eta_{1,t}(x) &= \mathrm{d}J_{01}^{x,x+1}(t) - \mathrm{d}J_{10}^{x,x+1}(t) - \mathrm{d}J_{12}^{x,x+1}(t) + \mathrm{d}J_{21}^{x,x+1} \\
&\quad - \mathrm{d}J_{01}^{x-1,x}(t) + \mathrm{d}J_{10}^{x-1,x}(t) + \mathrm{d}J_{12}^{x-1,x}(t) - \mathrm{d}J_{21}^{x-1,x}(t), \\
\mathrm{d}\eta_{2,t}(x) &= \mathrm{d}J_{02}^{x,x+1}(t) - \mathrm{d}J_{20}^{x,x+1}(t) - \mathrm{d}J_{21}^{x,x+1}(t) + \mathrm{d}J_{12}^{x,x+1}(t) \\
&\quad - \mathrm{d}J_{02}^{x-1,x}(t) + \mathrm{d}J_{20}^{x-1,x}(t) + \mathrm{d}J_{21}^{x-1,x}(t) - \mathrm{d}J_{12}^{x-1,x}(t),
\end{aligned} \tag{5.4.11}$$

and so

$$
\begin{aligned}
&\sum_{x\in\mathbb{T}_N}\Big\{\int_0^T H_1\left(\frac{x}{N},t\right)\Big[\mathrm{d}J_{01}^{x,x+1}(t)-\mathrm{d}J_{10}^{x,x+1}(t)-\mathrm{d}J_{12}^{x,x+1}(t)+\mathrm{d}J_{21}^{x,x+1}\\
&\qquad\qquad -\mathrm{d}J_{01}^{x-1,x}(t)+\mathrm{d}J_{10}^{x-1,x}(t)+\mathrm{d}J_{12}^{x-1,x}(t)-\mathrm{d}J_{21}^{x-1,x}(t)\Big]\\
&\quad+\int_0^T H_2\left(\frac{x}{N},t\right)\Big[\mathrm{d}J_{02}^{x,x+1}(t)-\mathrm{d}J_{20}^{x,x+1}(t)-\mathrm{d}J_{21}^{x,x+1}(t)+\mathrm{d}J_{12}^{x,x+1}(t)\\
&\qquad\qquad -\mathrm{d}J_{02}^{x-1,x}(t)+\mathrm{d}J_{20}^{x-1,x}(t)+\mathrm{d}J_{21}^{x-1,x}(t)-\mathrm{d}J_{12}^{x-1,x}(t)\Big]\Big\}\\
&=\sum_{x\in\mathbb{T}_N}\left\{\int_0^T H_1\left(\frac{x}{N},t\right)\mathrm{d}\eta_{1,t}(x)+\int_0^T H_2\left(\frac{x}{N},t\right)\mathrm{d}\eta_{2,t}(x)\right\}\\
&=N\langle\pi_{1,T}^N,H_1(\cdot,T)\rangle+N\langle\pi_{2,T}^N,H_2(\cdot,T)\rangle-N\langle\pi_{1,0}^N,H_1(\cdot,0)\rangle-N\langle\pi_{2,0}^N,H_2(\cdot,0)\rangle\\
&\quad-N\int_0^T\langle\pi_{1,t}^N,\partial_t H_1(\cdot,t)\rangle\,\mathrm{d}t-N\int_0^T\langle\pi_{1,t}^N,\partial_s H_2(\cdot,t)\rangle\,\mathrm{d}t,
\end{aligned}
\tag{5.4.12}
$$

where in the last equality of (5.4.12) we have integrated by parts. On the other hand, by applying the generator (5.2.4) we have that

$$
\begin{aligned}
&e^{-N\langle\boldsymbol{\pi}^N(\boldsymbol{\eta}),\boldsymbol{H}(\cdot,t)\rangle}L_N e^{N\langle\boldsymbol{\pi}^N(\boldsymbol{\eta}),\boldsymbol{H}(\cdot,t)\rangle}\\
&=N^2\sum_{x\in\mathbb{T}_N}\sum_{a,b=0}^{2}\eta_a(x)\eta_b(x+1)\Big(\exp\Big\{N\langle\pi_1^N(\boldsymbol{\eta}_{a,b}^{x,x+1}),H_1(\cdot,t)\rangle-N\langle\pi_1^N(\boldsymbol{\eta}),H_1(\cdot,t)\rangle\Big\}\\
&\qquad\cdot\exp\Big\{N\langle\pi_2^N((\boldsymbol{\eta}_{a,b}^{x,x+1})_s),H_2(\cdot,t)\rangle-N\langle\pi_2^N(\boldsymbol{\eta}),H_2(\cdot,t)\rangle\Big\}-1\Big)\\
&=\sum_{x\in\mathbb{T}_N}\sum_{a,b=0}^{2}\eta_a(x)\eta_b(x+1)\left[\exp\left\{\nabla_N H_{ab}\left(\tfrac{x}{N},t\right)\right\}-1\right],
\end{aligned}
\tag{5.4.13}
$$

where we have used the conditions on the potentials expressed in equations (5.2.6), (5.3.5) and (5.3.6). Combining (5.4.12) and (5.4.13), we indeed get (5.4.9). □

We can further expand the exponential function on the right hand side of (5.4.13) by using the Taylor series and the constraint that $\eta_0(x) = 1-\eta_1(x)-\eta_2(x)$. Namely, for the functions H_1, H_2 satisfying (5.2.6), (5.3.5) and (5.3.6), we obtain

$$
\begin{aligned}
&e^{-N\langle\boldsymbol{\pi}^N(\boldsymbol{\eta}),\boldsymbol{H}(\cdot,t)\rangle}L_N e^{N\langle\boldsymbol{\pi}^N(\boldsymbol{\eta}),\boldsymbol{H}(\cdot,t)\rangle}\\
&=N^2\sum_{x\in\mathbb{T}_N}\sum_{a,b=0}^{2}\eta_a(x)\eta_b(x+1)\left\{\nabla_N H_{ab}\left(\tfrac{x}{N},t\right)+\frac{1}{2}\left(\nabla_N H_{ab}\left(\tfrac{x}{N},t\right)\right)^2\right\}+\mathcal{O}(1)\\
&=\sum_{x\in\mathbb{T}_N}\Big\{\eta_1(x)\Delta H_1\left(\tfrac{x}{N},t\right)+\eta_2(x)\Delta H_2\left(\tfrac{x}{N},t\right)\\
&\quad+\eta_1(x)\left(1-\eta_1(x+1)\right)\left(\nabla H_1\left(\tfrac{x}{N},t\right)\right)^2+\eta_2(x)\left(1-\eta_2(x+1)\right)\left(\nabla H_2\left(\tfrac{x}{N},t\right)\right)^2\\
&\quad-\left(\eta_1(x)\eta_2(x+1)+\eta_2(x)\eta_1(x+1)\right)\left(\nabla H_1\left(\tfrac{x}{N},t\right)\nabla H_2\left(\tfrac{x}{N},t\right)\right)\Big\}+\mathcal{O}(1).
\end{aligned}
\tag{5.4.14}
$$

COROLLARY 5.2. The super exponential estimate (5.2.19) holds also for the path space measure $\mathbb{P}_N^{\varrho,\boldsymbol{H}}$.

Proof. Note that from (5.4.14) we have that

$$\frac{d\mathbb{P}_N^{\varrho,\boldsymbol{H}}}{d\mathbb{P}_N^{\varrho}} \leq \exp\{cN\}, \tag{5.4.15}$$

since $\eta_a(x) \leq 1$ for all $a \in \{1,2\}$ and $x \in \mathbb{T}_N$ and because the functions $H_1(\cdot,\cdot), H_2(\cdot,\cdot) \in C^{2,1}(\mathbb{T}\times[0,T])$.

For any measurable set $A \subset D([0,T], \mathcal{M}_1 \times \mathcal{M}_1)$ we have the following chain of inequalities

$$\begin{aligned}\frac{1}{N}\log \mathbb{P}_N^{\varrho,\boldsymbol{H}}(A) &= \frac{1}{N}\log \mathbb{E}_N^{\varrho}\left[\mathbb{1}_A \frac{d\mathbb{P}_N^{\varrho,\boldsymbol{H}}}{d\mathbb{P}_N^{\varrho}}\right] \\ &\leq \frac{1}{N}\log \mathbb{E}_N^{\varrho}\left[\mathbb{1}_A\right] + c. \end{aligned}\tag{5.4.16}$$

Here we have changed the path space measure from $\mathbb{P}_N^{\varrho,\boldsymbol{H}}$ to $\mathbb{P}_N^{\varrho}$ and we have used the estimate (5.4.15). Therefore, by taking the limit $N \to \infty$ and by using Corollary 5.1 we have the result. □

5.4.2 *Exponential tightness*

THEOREM 5.3 (Exponential Tightness)**.** For any $n \in \mathbb{N}$ there exists a compact set $\mathcal{K}_n \subset D([0,T], \mathcal{M}_1 \times \mathcal{M}_1)$ such that

$$\overline{\lim_{N\to\infty}} \frac{1}{N}\log \mathbb{P}_N^{\varrho}(\boldsymbol{\pi}_{[0,T]}^N \notin \mathcal{K}_n) = -n. \tag{5.4.17}$$

With exponential tightness, the large deviation upper bound for closed sets $\mathcal{C} \in D([0,T], \mathcal{M}_1 \times \mathcal{M}_1)$ follows from the upper bound for compact sets. Namely, for every $n \in \mathbb{N}$ we have that

$$\overline{\lim_{N\to\infty}} \frac{1}{N}\log \mathbb{P}_N^{\varrho}(\boldsymbol{\pi}_{[0,T]}^N \in \mathcal{C}) \leq \overline{\lim_{N\to\infty}} \frac{1}{N}\log\left[\mathbb{P}_N^{\varrho}(\boldsymbol{\pi}_{[0,T]}^N \in \mathcal{C}\cap\mathcal{K}_n) \vee \mathbb{P}_N^{\varrho}(\boldsymbol{\pi}_{[0,T]}^N \notin \mathcal{K}_n)\right], \tag{5.4.18}$$

where $\mathcal{C} \cap \mathcal{K}_n$ is a compact set.

We will prove Theorem 5.3 following the same approach as in [61, Section 10.4]. We start with the following Lemma.

LEMMA 5.2. For every $\varepsilon > 0$ and $\boldsymbol{G} \in C^2(\mathbb{T}) \times C^2(\mathbb{T})$

$$\lim_{\delta\to 0}\overline{\lim_{N\to\infty}} \frac{1}{N}\log \mathbb{P}_N^{\varrho}\left(\sup_{|s-t|<\delta}\left|\langle \boldsymbol{\pi}_t^N, \boldsymbol{G}\rangle - \langle \boldsymbol{\pi}_s^N, \boldsymbol{G}\rangle\right| \geq \varepsilon\right) = -\infty. \tag{5.4.19}$$

Proof. First note that by (5.2.20), it is enough to show that (5.4.19) holds for the equilibrium measure $\mathbb{P}_N^{1/3}$. We then use the following inclusion

$$\left\{\sup_{|s-t|<\delta}\left|\langle\boldsymbol{\pi}_t^N,\boldsymbol{G}\rangle-\langle\boldsymbol{\pi}_s^N,\boldsymbol{G}\rangle\right|\geq\varepsilon\right\}\subset\bigcup_{k=0}^{[T\delta^{-1}]}\left\{\sup_{k\delta\leq t<(k+1)\delta}\left|\langle\boldsymbol{\pi}_t^N,\boldsymbol{G}\rangle-\langle\boldsymbol{\pi}_{k\delta}^N,\boldsymbol{G}\rangle\right|>\tfrac{1}{4}\varepsilon\right\},\tag{5.4.20}$$

in order to find that

$$\begin{aligned}\overline{\lim_{N\to\infty}}\frac{1}{N}\log\mathbb{P}_N^{1/3}&\left(\sup_{|s-t|<\delta}\left|\langle\boldsymbol{\pi}_t^N,\boldsymbol{G}\rangle-\langle\boldsymbol{\pi}_s^N,\boldsymbol{G}\rangle\right|\geq\varepsilon\right)\\&\leq\overline{\lim_{N\to\infty}}\frac{1}{N}\log\sup_{k=0}^{[T\delta^{-1}]}\mathbb{P}_N^{1/3}\left(\sup_{k\delta\leq t<(k+1)\delta}\left|\langle\boldsymbol{\pi}_t^N,\boldsymbol{G}\rangle-\langle\boldsymbol{\pi}_{k\delta}^N,\boldsymbol{G}\rangle\right|\geq\tfrac{1}{4}\varepsilon\right)\\&=\overline{\lim_{N\to\infty}}\frac{1}{N}\log\mathbb{P}_N^{1/3}\left(\sup_{0\leq t<\delta}\left|\langle\boldsymbol{\pi}_t^N,\boldsymbol{G}\rangle-\langle\boldsymbol{\pi}_0^N,\boldsymbol{G}\rangle\right|\geq\tfrac{1}{4}\varepsilon\right),\end{aligned}\tag{5.4.21}$$

where we used that $\mathbb{P}_N^{1/3}$ is an invariant measure for the last equality.

Since we are considering every $\boldsymbol{G}$, we can neglect the absolute value. Furthermore, recalling the definition of $Z_{t,N}^{\boldsymbol{G}}(\boldsymbol{\pi}_{[0,T]}^N)$ in (5.4.9), we have that for any $\lambda>0$

$$\begin{aligned}&\mathbb{P}_N^{1/3}\left(\sup_{0\leq t<\delta}\langle\boldsymbol{\pi}_t^N,\boldsymbol{G}\rangle-\langle\boldsymbol{\pi}_0^N,\boldsymbol{G}\rangle\geq\tfrac{1}{4}\varepsilon\right)\\&=\mathbb{P}_N^{1/3}\left(\sup_{0\leq t<\delta}\frac{1}{N}\log Z_{t,N}^{\lambda\boldsymbol{G}}(\boldsymbol{\pi}_{[0,T]}^N)+\frac{1}{N}\int_0^t e^{-\lambda N\langle\boldsymbol{\pi}_s^N,\boldsymbol{G}\rangle}\left(\partial_s+L_N\right)e^{\lambda N\langle\boldsymbol{\pi}_s^N,\boldsymbol{G}\rangle}\,\mathrm{d}s\geq\tfrac{1}{4}\lambda\varepsilon\right)\\&\leq\mathbb{P}_N^{1/3}\left(\sup_{0\leq t<\delta}\frac{1}{N}\log Z_{t,N}^{\lambda\boldsymbol{G}}(\boldsymbol{\pi}_{[0,T]}^N)\geq\tfrac{1}{8}\lambda\varepsilon\right)\\&\quad+\mathbb{P}_N^{1/3}\left(\sup_{0\leq t<\delta}\frac{1}{N}\int_0^t e^{-\lambda N\langle\boldsymbol{\pi}_s^N,\boldsymbol{G}\rangle}\left(\partial_s+L_N\right)e^{\lambda N\langle\boldsymbol{\pi}_s^N,\boldsymbol{G}\rangle}\,\mathrm{d}s\geq\tfrac{1}{8}\lambda\varepsilon\right).\end{aligned}\tag{5.4.22}$$

Note that by (5.4.14) and the fact that there is at most one particle per site,

$$\sup_{0\leq t<\delta}\frac{1}{N}\int_0^t e^{-\lambda N\langle\boldsymbol{\pi}_s^N,\boldsymbol{G}\rangle}\left(\partial_s+L_N\right)e^{\lambda N\langle\boldsymbol{\pi}_s^N,\boldsymbol{G}\rangle}\,\mathrm{d}s=\mathcal{O}(\delta).\tag{5.4.23}$$

Furthermore, by Doob's martingale inequality

$$\begin{aligned}\mathbb{P}_N^{1/3}\left(\sup_{0\leq t<\delta}\frac{1}{N}\log Z_{t,N}^{\lambda\boldsymbol{G}}(\boldsymbol{\pi}_{[0,T]}^N)\geq\tfrac{1}{8}\lambda\varepsilon\right)&=\mathbb{P}_N^{1/3}\left(\sup_{0\leq t<\delta}Z_{t,N}^{\lambda\boldsymbol{G}}(\boldsymbol{\pi}_{[0,T]}^N)\geq e^{\frac{1}{8}N\lambda\varepsilon}\right)\\&\leq e^{-\frac{1}{8}N\lambda\varepsilon},\end{aligned}\tag{5.4.24}$$

where we used that $Z_{t,N}^{\lambda G}(\boldsymbol{\pi}_{[0,T]}^N)$ is a mean 1 martingale. Therefore we find that

$$\overline{\lim_{\delta\to 0}}\ \overline{\lim_{N\to\infty}} \frac{1}{N}\log \mathbb{P}_N^{1/3}\left(\sup_{0\le t<\delta}\langle \boldsymbol{\pi}_t^N, \boldsymbol{G}\rangle - \langle \boldsymbol{\pi}_0^N, \boldsymbol{G}\rangle \ge \varepsilon\right) \le -\tfrac{1}{8}\lambda\varepsilon, \tag{5.4.25}$$

and since we took $\lambda > 0$ arbitrary this concludes the proof. □

With this Lemma we are able to prove the exponential tightness of the empirical distributions.

Proof of Theorem 5.3. Consider a countable uniformly dense family $\{\boldsymbol{H}_k\}_{k\in\mathbb{N}} \subset C^2(\mathbb{T}) \times C^2(\mathbb{T})$. Then, for each $\delta > 0$, $\varepsilon > 0$ we define the following set

$$\mathcal{C}_{k,\delta,\varepsilon} = \left\{\boldsymbol{\alpha} \in D([0,T], \mathcal{M}_1 \times \mathcal{M}_1);\ \sup_{|t-s|\le\delta} |\langle \boldsymbol{\alpha}_t, \boldsymbol{H}_k\rangle - \langle \boldsymbol{\alpha}_s, \boldsymbol{H}_k\rangle| \le \varepsilon\right\}. \tag{5.4.26}$$

First of all, note that $\mathcal{C}_{k,\delta,\varepsilon}$ is closed. Furthermore, by Lemma 5.2 we know that we can find a $\delta = \delta(k, m, n)$ such that

$$\mathbb{P}_N^{\varrho}(\boldsymbol{\pi}_{[0,T]}^N \notin \mathcal{C}_{k,\delta,1/m}) \le \exp(-Nnmk) \tag{5.4.27}$$

for N large enough. We then define

$$\mathcal{K}_n = \bigcap_{k\ge 1, m\ge 1} \mathcal{C}_{k,\delta(k,m,n),1/m}. \tag{5.4.28}$$

Then we find that

$$\mathbb{P}_N^{\varrho}(\boldsymbol{\pi}_{[0,T]}^N \notin \mathcal{K}_n) \le \sum_{k\ge 1, m\ge 1} \exp(-Nnmk) \le C\exp(-Nn) \tag{5.4.29}$$

where $C > 0$ is some constant, and so

$$\overline{\lim_{N\to\infty}} \frac{1}{N}\log \mathbb{P}_N^{\varrho}(\boldsymbol{\pi}_{[0,T]}^N \notin \mathcal{K}_n) \le -n. \tag{5.4.30}$$

Since $\mathcal{K}_n$ is closed, we now only have to show that $\mathcal{K}_n$ is relatively compact for every $n \in \mathbb{N}$, i.e., we need to show that the following two things holds [61, Proposition 4.1.2]

1. $\{\boldsymbol{\alpha}_t : \boldsymbol{\alpha} \in \mathcal{K}_n, t \in [0,T]\}$ is relatively compact in $\mathcal{M}_1 \times \mathcal{M}_1$.
2. $\lim_{\delta\to 0} \sup_{\boldsymbol{\alpha}\in\mathcal{K}_n} w_\delta(\boldsymbol{\alpha}) = 0$ where

$$w_\delta(\boldsymbol{\alpha}) := \sup_{|t-s|\le\delta} \sum_{k=1}^{\infty} \frac{1}{2^k} \frac{|\langle \boldsymbol{\alpha}_t, \boldsymbol{H}_k\rangle - \langle \boldsymbol{\alpha}_s, \boldsymbol{H}_k\rangle|}{1 + |\langle \boldsymbol{\alpha}_t, \boldsymbol{H}_k\rangle - \langle \boldsymbol{\alpha}_s, \boldsymbol{H}_k\rangle|} = 0. \tag{5.4.31}$$

Note here that (1) is satisfied since $\mathcal{M}_1 \times \mathcal{M}_1$ itself is compact, and (2) is satisfied by the definition of $\mathcal{K}_n$, hence $\mathcal{K}_n$ is compact. □

5.4.3 *Proof of the upper bound*

Theorem 5.4 (Upper bound for compact sets).
For any compact set $\mathcal{K} \subset D([0,T];\mathcal{M}_1 \times \mathcal{M}_1)$ we have that

$$\overline{\lim_{N\to\infty}} \frac{1}{N} \log \mathbb{P}_N^{\varrho}\left(\boldsymbol{\pi}_{[0,T]}^N \in \mathcal{K}\right) \leq - \inf_{\widehat{\boldsymbol{\alpha}}\in\mathcal{K}} I^{\varrho}(\widehat{\boldsymbol{\alpha}}). \tag{5.4.32}$$

Proof. For any given $G_1, G_2 \in C^{2,1}(\mathbb{T}\times[0,T])$, and $\varepsilon > 0$, $\delta > 0$, we introduce the following

$$B^{\delta,a,b}_{\varepsilon,N,G_a,G_b} := \left\{ \{\boldsymbol{\eta}_t : 0 \leq t \leq T\} \,:\, \left| \frac{1}{N} \sum_{x\in\mathbb{T}_N} \int_0^T \nabla G_a\left(\tfrac{x}{N},t\right) \nabla G_b\left(\tfrac{x}{N},t\right) \cdot \left(\eta_{a,t}(x)\eta_{b,t}(x+1) - \left[\sum_{|x-y|\leq\varepsilon N} \eta_{a,t}(y)\right]\left[\sum_{|x-y|\leq\varepsilon N} \eta_{b,t}(y)\right]\right) \mathrm{d}t \right| \leq \delta \right\}. \tag{5.4.33}$$

Moreover, we denote by

$$B^{\delta}_{\varepsilon,N,\boldsymbol{G}} = \cap_{a,b=1}^{2} B^{\delta,a,b}_{\varepsilon,N,G_a,G_b}. \tag{5.4.34}$$

By the superexponential estimate with $\phi(\boldsymbol{\eta}) = \eta_{a,t}(x)\eta_{b,t}(x+1)$, we then have that

$$\overline{\lim_{N\to\infty}} \frac{1}{N} \log \mathbb{P}_N^{\varrho}\left(\boldsymbol{\pi}_{[0,T]}^N \in \mathcal{K}\right) \leq \overline{\lim_{\delta\to 0}}\ \overline{\lim_{N\to\infty}} \frac{1}{N} \log \mathbb{P}_N^{\varrho}\left(\left\{\boldsymbol{\pi}_{[0,T]}^N \in \mathcal{K}\right\} \cap B^{\delta}_{\varepsilon,N,\boldsymbol{G}}\right). \tag{5.4.35}$$

We now define the measures $q_\varepsilon := \frac{1}{2\varepsilon}\mathbb{1}_{\{[-\varepsilon,+\varepsilon]\}}$ and $\boldsymbol{q}_\varepsilon = (q_\varepsilon \quad q_\varepsilon)$, and we introduce the following

$$\begin{aligned}
\widetilde{Z}^{\boldsymbol{G}}_{T,N}(\boldsymbol{\pi}^N_{[0,T]} * \boldsymbol{q}_\varepsilon) := &\exp\left(\ell(\boldsymbol{\pi}^N_{[0,T]} * \boldsymbol{q}_\varepsilon; \boldsymbol{G}) - \tfrac{1}{2}||\boldsymbol{G}||^2_{\mathcal{H}(\boldsymbol{\pi}^N_{[0,T]}*\boldsymbol{q}_\varepsilon)}\right) \\
= &\exp\left\{N\langle(\boldsymbol{\pi}^N_T * \boldsymbol{q}_\varepsilon), \boldsymbol{G}(\cdot,T)\rangle - N\langle(\boldsymbol{\pi}^N_0 * \boldsymbol{q}_\varepsilon), \boldsymbol{G}(\cdot,0)\rangle\right\} \\
&\cdot \exp\left\{-N\int_0^T \langle(\boldsymbol{\pi}^N_t * \boldsymbol{q}_\varepsilon), (\partial_s + \Delta)\boldsymbol{G}(\cdot,t)\rangle\,\mathrm{d}t\right\} \\
&\cdot \exp\left\{-N\int_0^T \langle(\pi^N_{1,t} * q_\varepsilon)(1-(\pi^N_{1,t} * q_\varepsilon)), (\nabla G_1(\cdot,t))^2\rangle\,\mathrm{d}t\right\} \\
&\cdot \exp\left\{-N\int_0^T \langle(\pi^N_{2,t} * q_\varepsilon)(1-(\pi^N_{2,t} * q_\varepsilon)), (\nabla G_2(\cdot,t))^2\rangle\,\mathrm{d}t\right\} \\
&\cdot \exp\left\{-2N\int_0^t \langle(\pi^N_{1,t} * q_\varepsilon)(\pi^N_{2,t} * q_\varepsilon), \nabla G_1(\cdot,t)\nabla G_2(\cdot,t)\rangle\,\mathrm{d}t\right\}.
\end{aligned} \tag{5.4.36}$$

Recalling the definition of the exponential martingale $Z^{\boldsymbol{G}}_{T,N}(\boldsymbol{\mu}_N)$ defined in (5.4.9), by (5.4.13) we have that for all N and all $\{\boldsymbol{\eta}_t : 0 \le t \le T\} \in B^{\delta}_{\varepsilon,N,\boldsymbol{G}}$,

$$\widetilde{Z}^{\boldsymbol{G}}_{T,N}(\boldsymbol{\pi}^N_{[0,T]} * \boldsymbol{q}_\varepsilon) \le Z^{\boldsymbol{G}}_{N,T}(\boldsymbol{\pi}^N_{[0,T]}) \exp\{N(c(\varepsilon)+\delta)\}. \tag{5.4.37}$$

with $c(\varepsilon)$ a constant that vanishes as $\varepsilon \to 0$. Using (5.4.35) and recalling the definition of $h^{\varrho}(\cdot;\boldsymbol{\phi})$ in (5.4.2) we then find that

$$\begin{aligned}
&\varlimsup_{N\to\infty} \log \mathbb{P}^{\varrho}_N\left(\left\{\boldsymbol{\pi}^N \in \mathcal{K}\right\} \cap B^{\delta}_{\varepsilon,N,\boldsymbol{G}}\right) \\
&\quad = \varlimsup_{N\to\infty} \frac{1}{N} \log \mathbb{E}^{\varrho}_N\left[\mathbb{1}_{\left\{\boldsymbol{\pi}^N_{[0,T]}\in\mathcal{K}\right\}\cap B^{\delta}_{\varepsilon,N,G}} \frac{\widetilde{Z}^{\boldsymbol{G}}_{T,N}(\boldsymbol{\pi}^N_{[0,T]} * \boldsymbol{q}_\varepsilon)}{\widetilde{Z}^{\boldsymbol{G}}_{T,N}(\boldsymbol{\pi}^N_{[0,T]} * \boldsymbol{q}_\varepsilon)} \cdot \frac{e^{Nh^{\varrho}(\pi^N_0;\boldsymbol{\phi})}}{e^{Nh^{\varrho}(\pi^N_0;\boldsymbol{\phi})}}\right] \\
&\quad \le \varlimsup_{N\to\infty} \frac{1}{N} \log \mathbb{E}^{\varrho}_N\left[Z^{\boldsymbol{G}}_{N,T}(\boldsymbol{\pi}^N_{[0,T]}) \cdot e^{Nh^{\varrho}(\pi^N_0;\boldsymbol{\phi})}\right] + c(\varepsilon) + \delta \\
&\qquad - \inf_{\widehat{\boldsymbol{\alpha}}\in\mathcal{K}}\left\{h^{\varrho}(\widehat{\boldsymbol{\alpha}}_0;\boldsymbol{\phi}) + \ell(\widehat{\boldsymbol{\alpha}} * \boldsymbol{q}_\varepsilon;\boldsymbol{G}) - ||\boldsymbol{G}||_{\mathcal{H}(\widehat{\boldsymbol{\alpha}}*\boldsymbol{q}_\varepsilon)}\right\}.
\end{aligned} \tag{5.4.38}$$

Since $Z^{\boldsymbol{G}}_{N,T}(\boldsymbol{\pi}^N)$ is a martingale with $Z^{\boldsymbol{G}}_{0,N}(\boldsymbol{\pi}^N) = 1$

$$\mathbb{E}^{\varrho}_N\left[Z^{\boldsymbol{G}}_{N,T}(\boldsymbol{\pi}^N_{[0,T]}) e^{Nh^{\varrho}(\pi^N_0;\boldsymbol{\phi})}\right] = \mathbb{E}_{\nu^{\varrho}_N}\left[e^{Nh^{\varrho}(\pi^N_0;\boldsymbol{\phi})}\right] = 1. \tag{5.4.39}$$

By taking the limsup for $\varepsilon \to 0$ and $\delta \to 0$, by optimizing over $\boldsymbol{G}$ and over $\boldsymbol{\phi}$ and by exchanging the supremum and the infimum (by using the argument of Lemma 11.3 of [120]) we obtain that

$$\varlimsup_{\varepsilon\to 0} \varlimsup_{\delta\to 0} \varlimsup_{N\to\infty} \frac{1}{N} \log \mathbb{P}^{\varrho}_N\left(\left\{\boldsymbol{\pi}^N_{[0,T]} \in \mathcal{K}\right\} \cap B^{\delta}_{\varepsilon,N,\boldsymbol{G}}\right) \le -\inf_{\widehat{\boldsymbol{\alpha}}\in\mathcal{K}} \mathcal{I}^{\varrho}(\widehat{\boldsymbol{\alpha}}), \tag{5.4.40}$$

and the Theorem follows. □

5.4.4 *Proof of the lower bound*

Lemma 5.3. Assume that $h^{\varrho}(\widehat{\boldsymbol{\alpha}}_0) < \infty$, then there exists a density $\widehat{\varrho}_0 := \frac{d\widehat{\boldsymbol{\alpha}}_0}{d\lambda}$, with λ the Lebesgue measure, and

$$h^{\varrho}(\widehat{\boldsymbol{\alpha}}_0) = \lim_{N\to\infty} \frac{1}{N} \mathbb{E}_{\nu^{\widehat{\varrho}_0}_N}\left[\log\left(\frac{d\nu^{\widehat{\varrho}_0}_N}{d\nu^{\varrho}_N}\right)\right]. \tag{5.4.41}$$

Proof. If $\widehat{\boldsymbol{\alpha}}_0$ is not absolutely continuous with respect to the Lebesgue measure, then there exists a $A \subset \mathbb{T}$ and $a \in \{1,2\}$ such that $\lambda(A) = 0$ and $\varrho_{a,0}(A) > 0$. Then, for every $n \in \mathbb{N}$ we choose a sequence $(\phi^{(n)}_{a,k})_{k\in\mathbb{N}}$ such that $\phi^{(n)}_{a,k} \to n\mathbb{1}_A$, one can show that $h^{\varrho}(\widehat{\boldsymbol{\alpha}}_0) = \infty$ by letting $n \to \infty$. Hence if $h^{\varrho}(\widehat{\boldsymbol{\alpha}}_0) < \infty$ we have that $\widehat{\varrho}_0$ exists. The rest of a proof is just a calculation.

$$\begin{aligned} h^{\varrho}(\widehat{\boldsymbol{\alpha}}_0) &= \sup_{\boldsymbol{\phi}} \left\{ \sum_{a=0}^{2} \langle \widehat{\varrho}_{a,0}, \phi_a \rangle_{L^2(\mathbb{T})} - \int_{\mathbb{T}} \log \left(\sum_{a=0}^{2} \varrho_a(x) e^{\phi_a(x)} \right) \mathrm{d}x \right\} \\ &= \sum_{a=0}^{2} \langle \widehat{\varrho}_{a,0}, \log \left(\tfrac{\widehat{\varrho}_{a,0}}{\varrho_a} \right) \rangle_{L^2(\mathbb{T})} \\ &= \lim_{N\to\infty} \frac{1}{N} \mathbb{E}_{\nu_N^{\widehat{\varrho}_0}} \left[\log \left(\frac{\mathrm{d}\nu_N^{\widehat{\varrho}_0}}{\mathrm{d}\nu_N^{\varrho}} \right) \right], \end{aligned} \tag{5.4.42}$$

where the supremum was attained for the function $\boldsymbol{\phi} = (\phi_a)_{a=0,1,2}$ such that $\phi_a = \log(\frac{\widehat{\varrho}_{a,0}}{\varrho_a})$ and we used that $\sum_{a=0}^{2} \widehat{\varrho}_{a,0} = 1$. □

Lemma 5.4. Assume that $\mathcal{I}_{tr}(\widehat{\boldsymbol{\alpha}}) < \infty$, then there exists an $\boldsymbol{H} \in \mathcal{H}(\widehat{\boldsymbol{\alpha}})$ such that for all smooth $\boldsymbol{G}$ we have that

$$\ell(\widehat{\boldsymbol{\alpha}}; \boldsymbol{G}) = \langle \boldsymbol{G}, \boldsymbol{H} \rangle_{\mathcal{H}(\widehat{\boldsymbol{\alpha}})}. \tag{5.4.43}$$

Moreover, the following holds

$$\mathcal{I}_{tr}(\widehat{\boldsymbol{\alpha}}) = \mathcal{I}_{tr}(\widehat{\boldsymbol{\alpha}}; \boldsymbol{H}) = \tfrac{1}{2} ||\boldsymbol{H}||^2_{\mathcal{H}(\widehat{\boldsymbol{\alpha}})}. \tag{5.4.44}$$

Remark 5.6. Observe that if (5.4.43) holds for all $\boldsymbol{G}$, then $\widehat{\boldsymbol{\alpha}}$ has a density $\widehat{\boldsymbol{\varrho}}$ that satisfies the equations (5.3.7). This will be used in the proof of the large deviation lower bound. Indeed, by choosing a non-typical trajectory, we can find an $\boldsymbol{H}$ that makes it typical, i.e. that makes it solve the weakly asymmetric hydrodynamic equations.

Proof. By definition, we have that

$$\mathcal{I}_{tr}(\widehat{\boldsymbol{\alpha}}) \geq \lambda \ell(\widehat{\boldsymbol{\alpha}}; \boldsymbol{G}) - \frac{1}{2}\lambda^2 ||\boldsymbol{G}||^2_{\mathcal{H}(\widehat{\boldsymbol{\alpha}})} \tag{5.4.45}$$

for any $\lambda > 0$. Optimizing over λ we have that

$$\lambda^* = \frac{\ell(\widehat{\boldsymbol{\alpha}}; \boldsymbol{G})}{||\boldsymbol{G}||^2_{\mathcal{H}(\widehat{\boldsymbol{\alpha}})}}, \tag{5.4.46}$$

and so

$$\ell(\widehat{\boldsymbol{\alpha}}; \boldsymbol{G})^2 \leq 2\mathcal{I}_{tr}(\widehat{\boldsymbol{\alpha}}) ||\boldsymbol{G}||^2_{\mathcal{H}(\widehat{\boldsymbol{\alpha}})}. \tag{5.4.47}$$

This means that the linear functional $\ell(\varrho;\cdot)$ is bounded in the Hilbert space $\mathcal{H}(\widehat{\boldsymbol{\alpha}})$ and so, by the Riesz representation Theorem, there exists an $\boldsymbol{H} \in \mathcal{H}(\widehat{\boldsymbol{\alpha}})$ such that (5.4.43) holds for all $\boldsymbol{G}$. Using this, we find that

$$\begin{aligned}\mathcal{I}_{tr}(\widehat{\boldsymbol{\alpha}}) &= \sup_{\boldsymbol{G}}\left\{\langle \boldsymbol{G},\boldsymbol{H}\rangle_{\mathcal{H}(\widehat{\boldsymbol{\alpha}})} - \tfrac{1}{2}||\boldsymbol{G}||^2_{\mathcal{H}(\widehat{\boldsymbol{\alpha}})}\right\}\\ &= \sup_{\boldsymbol{G}}\left\{\tfrac{1}{2}||\boldsymbol{H}||^2_{\mathcal{H}(\widehat{\boldsymbol{\alpha}})} - \tfrac{1}{2}||\boldsymbol{H}-\boldsymbol{G}||^2_{\mathcal{H}(\widehat{\boldsymbol{\alpha}})}\right\}\\ &= \tfrac{1}{2}||\boldsymbol{H}||^2_{\mathcal{H}(\widehat{\boldsymbol{\alpha}})}. \end{aligned} \tag{5.4.48}$$

Note that the supremum is uniquely attained for $\boldsymbol{G} = \boldsymbol{H}$, hence we indeed have that $\mathcal{I}_{tr}(\widehat{\boldsymbol{\alpha}}) = \mathcal{I}_{tr}(\widehat{\boldsymbol{\alpha}};\boldsymbol{H})$. □

Remark 5.7. We have shown that if $\mathcal{I}_{tr}(\widehat{\boldsymbol{\alpha}}) < \infty$, then there exists an $\boldsymbol{H} \in \mathcal{H}(\varrho)$ such that the density of $\widehat{\boldsymbol{\alpha}}$, denoted $\widehat{\varrho}$, satisfies the equations (5.3.7). However, for the proof of the hydrodynamic limit of the weakly asymmetric model we need a stronger regularity condition on $\boldsymbol{H}$, namely $H_{ab} \in C^{2,1}([0,T]\times\mathbb{T})$ for each a,b. We denote the subset of all trajectories that satisfy this extra regularity condition by $D_o\left([0,T],\mathcal{M}_1\times\mathcal{M}_1\right)$. By the convexity and lower semi-continuity of $\mathcal{I}_0$, it can then be shown that every $\widehat{\boldsymbol{\alpha}} \in D_o\left([0,T],\mathcal{M}_1\times\mathcal{M}_1\right)$ can be approximated by trajectories $\widehat{\boldsymbol{\alpha}}_n \in D\left([0,T],\mathcal{M}_1\times\mathcal{M}_1\right)$, such that

$$\lim_{n\to\infty}\mathcal{I}_{tr}(\widehat{\boldsymbol{\alpha}}_n) = \mathcal{I}_{tr}(\widehat{\boldsymbol{\alpha}}). \tag{5.4.49}$$

A detailed proof of such a result can be found e.g. in [61, Lemma 10.5.5].

Lemma 5.5. Let $\widehat{\boldsymbol{\alpha}} \in D_o\left([0,T],\mathcal{M}_1\times\mathcal{M}_1\right)$ have density $\widehat{\varrho}$ that satisfies (5.3.7) for some $\boldsymbol{H} \in C^{2,1}(\mathbb{T}\times[0,T])$, then

$$\lim_{N\to\infty}\frac{1}{N}\mathbb{E}_N^{\widehat{\varrho}_0,H}\left[\log\left(\frac{d\mathbb{P}_N^{\varrho}}{d\mathbb{P}_N^{\widehat{\varrho}_0,\boldsymbol{H}}}\right)\right] = -\mathcal{I}^{\varrho}(\widehat{\boldsymbol{\alpha}}). \tag{5.4.50}$$

Proof. First note that

$$\frac{d\mathbb{P}_N^{\widehat{\varrho}_0,\boldsymbol{H}}}{d\mathbb{P}_N^{\varrho}} = \frac{d\nu_N^{\widehat{\varrho}_0}}{d\nu_N^{\varrho}}\frac{d\mathbb{P}_N^{\varrho,\boldsymbol{H}}}{d\mathbb{P}_N^{\varrho}}. \tag{5.4.51}$$

Therefore, by Lemma 5.3 we only have to show that

$$\lim_{N\to\infty}\frac{1}{N}\mathbb{E}_N^{\widehat{\varrho}_0,\boldsymbol{H}}\left[\log\left(\frac{d\mathbb{P}_N^{\widehat{\varrho}_0}}{d\mathbb{P}_N^{\widehat{\varrho}_0,\boldsymbol{H}}}\right)\right] = -\mathcal{I}_{tr}(\widehat{\boldsymbol{\alpha}}). \tag{5.4.52}$$

Recall the definition of the set $B^{\delta}_{\varepsilon,N,\boldsymbol{H}}$ in (5.4.34). By Corollary 5.2 and the upper bound on the Radon-Nikodym derivative (5.4.15), we find that

$$\lim_{N\to\infty}\frac{1}{N}\mathbb{E}_N^{\widehat{\varrho}_0,\boldsymbol{H}}\left[\log\left(\frac{d\mathbb{P}_N^{\widehat{\varrho}_0}}{d\mathbb{P}_N^{\widehat{\varrho}_0,\boldsymbol{H}}}\right)\right] = \lim_{N\to\infty}\frac{1}{N}\mathbb{E}_N^{\widehat{\varrho}_0,\boldsymbol{H}}\left[\log\left(\frac{d\mathbb{P}_N^{\widehat{\varrho}_0}}{d\mathbb{P}_N^{\widehat{\varrho}_0,\boldsymbol{H}}}\right)\mathbb{1}_{B^{\delta}_{\varepsilon,N,\boldsymbol{H}}}\right] \tag{5.4.53}$$

for all $\delta > 0$ and ε small enough. On this set $B^{\delta}_{\varepsilon,N,\boldsymbol{H}}$, using the explicit form of the Radon-Nikodym derivative in (5.4.9) and (5.4.14), we can write that

$$\frac{1}{N}\log\left(\frac{\mathrm{d}\mathbb{P}_N^{\widehat{\varrho}_0}}{\mathrm{d}\mathbb{P}_N^{\widehat{\varrho}_0,\boldsymbol{H}}}\right) = -\ell(\boldsymbol{\pi}^N_{[0,T]};\boldsymbol{H}) - ||\boldsymbol{H}||_{\mathcal{H}(\boldsymbol{\pi}^N_{[0,T]}*\boldsymbol{q}_\varepsilon)} + \mathcal{O}(\delta) + \mathcal{O}(\tfrac{1}{N}). \tag{5.4.54}$$

Since $\boldsymbol{\pi}^N_{[0,T]} \to \widehat{\boldsymbol{\alpha}}$ in distribution, we find that

$$\lim_{\delta\to 0}\lim_{\varepsilon\to 0}\lim_{N\to\infty}\frac{1}{N}\mathbb{E}_N^{\widehat{\varrho}_0,\boldsymbol{H}}\left[\log\left(\frac{\mathrm{d}\mathbb{P}_N^{\widehat{\varrho}_0}}{\mathrm{d}\mathbb{P}_N^{\widehat{\varrho}_0,\boldsymbol{H}}}\right)\mathbb{1}_{B^{\delta}_{\varepsilon,N,\boldsymbol{H}}}\right] = -\mathcal{I}_{tr}(\widehat{\boldsymbol{\alpha}};\boldsymbol{H}). \tag{5.4.55}$$

By Lemma 5.4 we have that $\mathcal{I}_{tr}(\widehat{\boldsymbol{\alpha}};\boldsymbol{H}) = \mathcal{I}_{tr}(\widehat{\boldsymbol{\alpha}})$, concluding the proof. □

Theorem 5.5. Fix $\widehat{\boldsymbol{\alpha}} \in D_o\left([0,T],\mathcal{M}_1\times\mathcal{M}_1\right)$, then for any open neighborhood $\mathcal{O}$ around $\widehat{\boldsymbol{\alpha}}$ we have that

$$\underline{\lim}_{N\to\infty}\frac{1}{N}\log\mathbb{P}_N^{\varrho}\left(\boldsymbol{\pi}^N_{[0,T]}\in\mathcal{O}\right) \geq -\mathcal{I}^{\varrho}(\widehat{\boldsymbol{\alpha}}). \tag{5.4.56}$$

Proof. If $\mathcal{I}^{\varrho}(\widehat{\boldsymbol{\alpha}}) = \infty$ then the result is immediate, hence we can assume that $\mathcal{I}^{\varrho}(\boldsymbol{\varrho}) < \infty$. Therefore, by Lemma 5.4, there exists a smooth $\boldsymbol{H}$ such that 5.3.7 holds weakly. Fix this $\boldsymbol{H}$ and recall that $\widehat{\boldsymbol{\varrho}}_t$ denotes the density of $\widehat{\boldsymbol{\alpha}}_t$ for every $t \geq 0$. We then have that

$$\mathbb{P}_N^{\varrho}\left(\boldsymbol{\pi}^N_{[0,T]}\in\mathcal{O}\right) = \mathbb{E}_N^{\widehat{\varrho}_0,\boldsymbol{H}}\left[\mathbb{1}_{\{\boldsymbol{\pi}^N_{[0,T]}\in\mathcal{O}\}}\frac{\mathrm{d}\mathbb{P}_N^{\varrho}}{\mathrm{d}\mathbb{P}_N^{\widehat{\varrho}_0,\boldsymbol{H}}}\right]. \tag{5.4.57}$$

From Theorem 5.2 it follows that $\widehat{\boldsymbol{\alpha}}$ is the typical trajectory of the new dynamics, and so

$$\lim_{N\to\infty}\mathbb{P}_N^{\widehat{\varrho}_0,\boldsymbol{H}}\left(\boldsymbol{\pi}^N_{[0,T]}\in\mathcal{O}\right) = 1. \tag{5.4.58}$$

It then follows that

$$\begin{aligned}\underline{\lim}_{N\to\infty}\frac{1}{N}\log\mathbb{P}_N^{\varrho}(\boldsymbol{\pi}^N_{[0,T]}\in\mathcal{O}) &= \underline{\lim}_{N\to\infty}\frac{1}{N}\log\mathbb{E}_N^{\widehat{\varrho}_0,\boldsymbol{H}}\left[\frac{\mathrm{d}\mathbb{P}_N^{\varrho}}{\mathrm{d}\mathbb{P}_N^{\widehat{\varrho}_0,\boldsymbol{H}}}\right]\\ &\geq \underline{\lim}_{N\to\infty}\frac{1}{N}\mathbb{E}_N^{\widehat{\varrho}_0,\boldsymbol{H}}\left[\log\left(\frac{\mathrm{d}\mathbb{P}_N^{\varrho}}{\mathrm{d}\mathbb{P}_N^{\widehat{\varrho}_0,\boldsymbol{H}}}\right)\right]\\ &= -\mathcal{I}^{\varrho}(\widehat{\boldsymbol{\alpha}}).\end{aligned} \tag{5.4.59}$$

where we used Lemma 5.5 in the last step. □

THEOREM 5.6. For any open set $\mathcal{O} \subset D([0,T], \mathcal{M}_1 \times \mathcal{M}_1)$ we have that

$$\varliminf_{N\to\infty} \frac{1}{N} \log \mathbb{P}_N^{\varrho}\left(\pi_{[0,T]}^N \in \mathcal{O}\right) \geq -\inf_{\widehat{\boldsymbol{\alpha}}\in\mathcal{O}} \mathcal{I}^{\varrho}(\widehat{\boldsymbol{\alpha}}). \tag{5.4.60}$$

Proof. The proof is a straightforward consequence of Theorem 5.5 and Remark 5.7. □

REMARK 5.8. In parallel with Remark 5.3, by choosing the same potential $H_1 = H_2 = H$ we can recover the large deviation rate function of the dynamics of the single species SEP as given in [62]. Namely, by putting $\widehat{\varrho} = \widehat{\varrho}_1 + \widehat{\varrho}_2$ the rate function $\mathcal{I}_{tr}(\widehat{\boldsymbol{\alpha}})$ from (5.4.48) becomes a function of $\widehat{\varrho}$ only, i.e.,

$$\mathcal{I}_{tr}(\widehat{\boldsymbol{\alpha}}) = \frac{1}{2}||\boldsymbol{H}||^2_{\mathcal{H}(\widehat{\boldsymbol{\alpha}})} = \int_0^T \langle (\widehat{\varrho}_t(1-\widehat{\varrho}_t), (\nabla H(\cdot,t))^2\rangle_{L^2(\mathbb{T})}\, \mathrm{d}t. \tag{5.4.61}$$

REMARK 5.9. In parallel with Remark 5.4, the large deviation result reported in this section can be generalized to an arbitrary number of species, namely $a \in \{0, 1, \ldots, n\}$. In this case, we consider a n-dimensional vector of densities denoted by $\boldsymbol{\varrho} = (\varrho_1, \ldots, \varrho_n)$. Moreover, we consider n-potentials denoted by H_a, that we list in the vector $\boldsymbol{H} = (H_1, \ldots, H_n)$. Therefore, the large deviation functional reads

$$\mathcal{I}^{\varrho,(n)}(\widehat{\boldsymbol{\alpha}}) = \mathcal{I}_{tr}^{(n)}(\widehat{\boldsymbol{\alpha}}) + h^{\varrho,(n)}(\widehat{\boldsymbol{\alpha}}_0). \tag{5.4.62}$$

Here $h^{\varrho,(n)}(\widehat{\boldsymbol{\alpha}}_0)$ is the relative entropy between the multinomial distributions with densities corresponding to the density of $\widehat{\boldsymbol{\alpha}}$, denoted $\widehat{\boldsymbol{\varrho}}$, evaluated at time $t = 0$ and the original starting density given by $\boldsymbol{\varrho} = (\varrho_1, \ldots, \varrho_n)$. Moreover, we have that

$$\mathcal{I}_{tr}^{(n)}(\widehat{\boldsymbol{\alpha}}) = \frac{1}{2}\|\boldsymbol{H}\|^2_{\mathcal{H}(\widehat{\boldsymbol{\alpha}})}, \tag{5.4.63}$$

where the norm is given by

$$\begin{aligned}\|\boldsymbol{H}\|^2_{\mathcal{H}(\widehat{\boldsymbol{\alpha}})} = 2\sum_{a=1}^{n}\int_0^T \langle \widehat{\varrho}_{a,t}(1-\widehat{\varrho}_{a,t}), (\nabla H_a(\cdot,t))^2\rangle_{L^2(\mathbb{T})}\, \mathrm{d}t \\ -2\sum_{a=1}^{n}\sum_{b\neq a}\int_0^T \langle \widehat{\varrho}_{a,t}\widehat{\varrho}_{b,t}, \nabla H_a(\cdot,t)\nabla H_b(\cdot,t)\rangle_{L^2(\mathbb{T})}\, \mathrm{d}t.\end{aligned} \tag{5.4.64}$$

For any $k \leq n$ there exists a relation between the dynamic part of the large deviation rate function of the n-species model $\mathcal{I}_{tr}^{(n)}$ and of the k-species model $\mathcal{I}_{tr}^{(k)}$. Namely, for any partition $\{A_1, ..., A_k\}$ of the set $\{1, ..., n\}$, by choosing the same potentials within each partition, i.e., $H_j = H_\ell$ for every $j \in A_\ell$, we find that

$$\mathcal{I}_{tr}^{(n)}(\alpha_1, ..., \alpha_n) = \mathcal{I}_{tr}^{(k)}(\tilde{\alpha}_1, ..., \tilde{\alpha}_k) \tag{5.4.65}$$

where $\tilde{\alpha}_\ell = \sum_{j\in A_\ell} \alpha_j$. This generalizes the result in Remark 5.8.

5.5 PROOF OF THE HYDRODYNAMIC LIMIT OF THE WEAKLY-ASYMMETRIC PROCESS

We consider the Dynkin martingale

$$\begin{aligned} M^{\boldsymbol{G}}_{N,t}(\boldsymbol{\pi}^N_{[0,T]}) &= \langle \boldsymbol{\pi}^N_t, \boldsymbol{G}(\cdot,t)\rangle - \langle \boldsymbol{\pi}^N_0, \boldsymbol{G}(\cdot,0)\rangle - \int_0^t (\partial_t + L^{\boldsymbol{H}}_{N,s})\langle \boldsymbol{\pi}^N_s, \boldsymbol{G}(\cdot,s)\rangle \, \mathrm{d}s \\ &= M^{G_1}_{1,N,t}(\boldsymbol{\pi}^N_{[0,T]}) + M^{G_2}_{2,N,t}(\boldsymbol{\pi}^N_{[0,T]}), \end{aligned} \tag{5.5.1}$$

where

$$M^{G}_{a,N,t}(\boldsymbol{\pi}^N_{[0,T]}) = \langle \pi^N_{a,t}, G(\cdot,t)\rangle - \langle \pi^N_{a,0}, G(\cdot,0)\rangle - \int_0^t (\partial_t + L^{\boldsymbol{H}}_{N,s})\langle \pi^N_{a,s}, G(\cdot,s)\rangle \, \mathrm{d}s. \tag{5.5.2}$$

We see that we need to apply the generator $L^{\boldsymbol{H}}_{N,t}$ to the density field $\langle \pi^N_{a,t}, G(\cdot,t)\rangle$. This can be derived from the effect of the generator applied to the function $f(\boldsymbol{\eta}) = \eta_a(x)$. We start in the case of $a = 1$. If we look at $L^{\boldsymbol{H}}_{N,t}\eta_1(x)$ we get a positive (resp. negative) contribution of the rates where a particle of type 1 is added (resp. subtracted) at position x, i.e.,

$$\begin{aligned} L^{\boldsymbol{H}}_{N,s}\eta_1(x) = c^{\boldsymbol{H},01}_{(x,x+1)}(s) + c^{\boldsymbol{H},21}_{(x,x+1)}(s) + c^{\boldsymbol{H},10}_{(x-1,x)}(s) + c^{\boldsymbol{H},12}_{(x-1,x)}(s) \\ - c^{\boldsymbol{H},10}_{(x,x+1)}(s) - c^{\boldsymbol{H},12}_{(x,x+1)}(s) - c^{\boldsymbol{H},01}_{(x-1,x)}(s) - c^{\boldsymbol{H},21}_{(x-1,x)}(s). \end{aligned} \tag{5.5.3}$$

Using that

$$\begin{aligned} c^{\boldsymbol{H},ab}_{(x,x+1)}(s) &= \exp\Big(\nabla_N H_{ab}(\tfrac{x}{N},s)\Big)\eta_a(x)\eta_b(x+1) \\ &= \Big(1 + \nabla_N H_{ab}(\tfrac{x}{N},s) + \big(\nabla_N H_{ab}(\tfrac{x}{N},s)\big)^2\Big)\,\eta_a(x)\eta_b(x+1) + \mathcal{O}(\tfrac{1}{N^3}), \end{aligned} \tag{5.5.4}$$

we find that

$$\begin{aligned} &L^{\boldsymbol{H}}_{N,s}\eta_1(x) \\ &= \big(1 + \nabla_N H_{01}(\tfrac{x}{N},s)\big)\,\eta_0(x)\eta_1(x+1) + \big(1 + \nabla_N H_{21}(\tfrac{x}{N},s)\big)\,\eta_2(x)\eta_1(x+1) \\ &\quad + \Big(1 + \nabla_N H_{10}(\tfrac{x-1}{N},s)\Big)\,\eta_1(x-1)\eta_0(x) + \Big(1 + \nabla_N H_{12}(\tfrac{x-1}{N},s)\Big)\,\eta_1(x-1)\eta_2(x) \\ &\quad - \big(1 + \nabla_N H_{10}(\tfrac{x}{N},s)\big)\,\eta_1(x)\eta_0(x+1) - \big(1 + \nabla_N H_{12}(\tfrac{x}{N},s)\big)\,\eta_1(x)\eta_2(x+1) \\ &\quad - \Big(1 + \nabla_N H_{01}(\tfrac{x-1}{N},s)\Big)\,\eta_0(x-1)\eta_1(x) - \Big(1 + \nabla_N H_{21}(\tfrac{x-1}{N},s)\Big)\,\eta_2(x+1)\eta_1(x) \\ &\quad + R(N,x,s), \end{aligned} \tag{5.5.5}$$

with $R(N,x,s)$ a remainder term which we will show vanishes once combined with the test function G as we send $N \to \infty$.

First we will focus on the terms that do not depend on $\boldsymbol{H}$ in the above equation. Using the fact that $\eta_0 = 1 - \eta_1 - \eta_2$, after some calculation we find that

$$\begin{aligned}\eta_0(x)\eta_1(x+1) + \eta_2(x)\eta_1(x+1) + \eta_1(x-1)\eta_0(x) + \eta_1(x-1)\eta_2(x)\\ - \eta_1(x)\eta_0(x+1) - \eta_1(x)\eta_2(x+1) - \eta_0(x-1)\eta_1(x) - \eta_2(x+1)\eta_1(x)\\ = \eta_1(x+1) + \eta_1(x-1) - 2\eta_1(x),\end{aligned} \tag{5.5.6}$$

i.e., we recover the discrete Laplacian of $\eta_1(x)$.

For the terms depending on the potential $H_{10} = -H_{01}$ we then have that

$$\begin{aligned}&(\eta_1(x-1)\eta_0(x) + \eta_0(x-1)\eta_1(x))\, \nabla_N H_{10}(\tfrac{x-1}{N}, s)\\ &\quad - (\eta_0(x)\eta_1(x+1) + \eta_1(x)\eta_0(x+1))\, \nabla_N H_{10}(\tfrac{x}{N}, s).\end{aligned} \tag{5.5.7}$$

and the terms depending on the potential $H_{12} = -H_{21}$

$$\begin{aligned}&(\eta_1(x-1)\eta_2(x) + \eta_2(x+1)\eta_1(x))\, \nabla_N H_{12}(\tfrac{x-1}{N}, s)\\ &\quad - (\eta_2(x)\eta_1(x+1) + \eta_1(x)\eta_2(x+1))\, \nabla_N H_{12}(\tfrac{x}{N}, s).\end{aligned} \tag{5.5.8}$$

With these calculations, we then find that

$$\begin{aligned}&L^{\boldsymbol{H}}_{N,s}\langle \pi^N_{1,s}, G(\cdot, s)\rangle\\ &= N \sum_{x\in\mathbb{T}_N} (\eta_1(x+1) + \eta_1(x-1) - 2\eta_1(x))G(\tfrac{x}{N}, s)\\ &\quad + N \sum_{x\in\mathbb{T}_N} \Big(\big(\eta_1(x-1)\eta_0(x) + \eta_0(x-1)\eta_1(x)\big)\nabla_N H_{10}(\tfrac{x-1}{N}, s)\\ &\qquad - \big(\eta_0(x)\eta_1(x+1) + \eta_1(x)\eta_0(x+1)\big)\nabla_N H_{10}(\tfrac{x}{N}, s)\Big) G(\tfrac{x}{N}, s)\\ &\quad + N \sum_{x\in\mathbb{T}_N} \Big(\big(\eta_1(x-1)\eta_2(x) + \eta_2(x+1)\eta_1(x)\big)\nabla_N H_{12}(\tfrac{x-1}{N}, s)\\ &\qquad - \big(\eta_2(x)\eta_1(x+1) + \eta_1(x)\eta_2(x+1)\big)\nabla_N H_{12}(\tfrac{x}{N}, s)\Big) G(\tfrac{x}{N}, s)\\ &\quad + N \sum_{x\in\mathbb{T}_N} R(N, x, s)G(\tfrac{x}{N}, s),\end{aligned} \tag{5.5.9}$$

where by reordering the terms, we have

$$\begin{aligned}&L^{\boldsymbol{H}}_{N}\langle \pi^N_{1,s}, G(\cdot, s)\rangle\\ &= N \sum_{x\in\mathbb{T}_N} \eta_1(x)\Delta_N G(\tfrac{x}{N}, s)\\ &\quad + N \sum_{x\in\mathbb{T}_N} (\eta_0(x)\eta_1(x+1) + \eta_1(x)\eta_0(x+1))\, \nabla_N H_{10}(\tfrac{x}{N}, s)\nabla_N G(\tfrac{x}{N}, s)\\ &\quad + N \sum_{x\in\mathbb{T}_N} (\eta_2(x)\eta_1(x+1) + \eta_1(x)\eta_2(x+1))\, \nabla_N H_{12}(\tfrac{x}{N}, s)\nabla_N G(\tfrac{x}{N}, s)\\ &\quad + N \sum_{x\in\mathbb{T}_N} R(N, x, s)G(\tfrac{x}{N}, s).\end{aligned} \tag{5.5.10}$$

The remainder term $R(N,x,s)$ is given by

$$\begin{aligned}R(N,x,s) = &(\eta_0(x)\eta_1(x+1) - \eta_1(x)\eta_0(x+1))\left(\nabla_N H_{10}(\tfrac{x}{N},s)\right)^2\\&-(\eta_0(x-1)\eta_1(x) - \eta_1(x-1)\eta_0(x))\left(\nabla_N H_{10}(\tfrac{x-1}{N},s)\right)^2\\&+(\eta_2(x)\eta_1(x+1) - \eta_1(x)\eta_2(x+1))\left(\nabla_N H_{12}(\tfrac{x}{N},s)\right)^2\\&-(\eta_2(x+1)\eta_1(x) - \eta_1(x-1)\eta_2(x))\left(\nabla_N H_{12}(\tfrac{x-1}{N},s)\right)^2\\&+\mathcal{O}(\tfrac{1}{N^3}),\end{aligned} \tag{5.5.11}$$

and when combined with a test function, we see that

$$\begin{aligned}&N\sum_{x\in\mathbb{T}_N} R(N,x,s)G(\tfrac{x}{N},s)\\&=-N\sum_{x\in\mathbb{T}_N}(\eta_0(x)\eta_1(x+1) - \eta_1(x)\eta_0(x+1))\left(\nabla_N H_{10}(\tfrac{x}{N},s)\right)^2\nabla_N G(\tfrac{x}{N},s)\\&\quad-N\sum_{x\in\mathbb{T}_N}(\eta_2(x)\eta_1(x+1) - \eta_1(x)\eta_2(x+1))\left(\nabla_N H_{12}(\tfrac{x}{N},s)\right)^2\nabla_N G(\tfrac{x}{N},s)\\&\quad+\mathcal{O}(\tfrac{1}{N}).\end{aligned} \tag{5.5.12}$$

Note that this vanishes as $N\to\infty$ since the discrete derivative ∇_N is an operator of order $\frac{1}{N}$.

Thus, all in all:

$$\begin{aligned}&L_N^{\boldsymbol{H}}\langle\pi^N_{1,s},G(\cdot,s)\rangle\\&=\frac{1}{N}\sum_{x\in\mathbb{Z}}\eta_1(x)\Delta G\left(\tfrac{x}{N}\right)\\&\quad+\frac{1}{N}\sum_{x\in\mathbb{T}_N}(\eta_0(x)\eta_1(x+1)+\eta_1(x)\eta_0(x+1))\,\nabla H_{10}(\tfrac{x}{N},s)\nabla G(\tfrac{x}{N},s)\\&\quad+\frac{1}{N}\sum_{x\in\mathbb{T}_N}(\eta_2(x)\eta_1(x+1)+\eta_1(x)\eta_2(x+1))\,\nabla H_{12}(\tfrac{x}{N},s)\nabla G(\tfrac{x}{N},s)+\mathcal{O}(\tfrac{1}{N}).\end{aligned} \tag{5.5.13}$$

By choosing: $\phi(\boldsymbol{\eta}) = \eta_1(x)\eta_1(x+1)$ and $\phi(\boldsymbol{\eta}) = \eta_1(x)\eta_2(x+1)$ we now use the superexponential estimate in Theorem 5.1 twice and we replace

$$\frac{2}{N}\sum_{x\in\mathbb{Z}}\eta_1(x)\eta_1(x+1) \longrightarrow \frac{2}{N}\sum_{x\in\mathbb{Z}}\left(\sum_{|x-y|\le\varepsilon N}\eta_1(y)\right)\left(\sum_{|x-y|\le\varepsilon N}\eta_1(y)\right), \tag{5.5.14}$$

and

$$\frac{2}{N}\sum_{x\in\mathbb{Z}}\eta_1(x)\eta_2(x+1) \longrightarrow \frac{2}{N}\sum_{x\in\mathbb{Z}}\left(\sum_{|x-y|\le\varepsilon N}\eta_1(y)\right)\left(\sum_{|x-y|\le\varepsilon N}\eta_2(y)\right). \quad (5.5.15)$$

Indeed, to prove equation (5.5.14) one writes that, for all $G, H_{10} \in C^{2,1}(\mathbb{T}\times[0,T])$ and for all $a>0$, there exists $\varepsilon_0>0$ such that for all $\boldsymbol{\eta}\in\Omega$ and for all $\varepsilon<\varepsilon_0$ we have that

$$\Bigg|\frac{1}{N}\int_0^T\sum_{x=0}^{N}\nabla G\left(\frac{x}{N},t\right)\nabla H_{10}\left(\frac{x}{N},t\right)\eta_1(x)\eta_1(x+1)\,\mathrm{d}t$$
$$-\frac{1}{N}\int_0^T\nabla G\left(\frac{x}{N},t\right)\nabla H_{10}\left(\frac{x}{N},t\right)\left(\sum_{|x-y|\le\varepsilon N}\eta_1(y)\eta_1(y+1)\right)\mathrm{d}t\Bigg|\le a. \quad (5.5.16)$$

Therefore, using the superexponential estimate of Theorem 5.1 we have that

$$\begin{aligned}\overline{\lim_{\varepsilon\to0}}\,\overline{\lim_{N\to\infty}}\,\frac{1}{N}\log\mathbb{P}_N^{\varrho,\boldsymbol{H}}\Bigg(\Bigg|&\frac{1}{N}\int_0^T\sum_{x=0}^{N}\nabla G\left(\frac{x}{N},t\right)\nabla H_{10}\left(\frac{x}{N},t\right)\eta_1(x)\eta_1(x+1)\,\mathrm{d}t\\&-\frac{1}{N}\int_0^T\nabla G\left(\frac{x}{N},t\right)\nabla H_{10}\left(\frac{x}{N},t\right)\\&\times\frac{2}{N}\sum_{x\in\mathbb{Z}}\left(\sum_{|x-y|\le\varepsilon N}\eta_1(y)\right)\left(\sum_{|x-y|\le\varepsilon N}\eta_1(y)\right)\mathrm{d}t\Bigg|\ge a\Bigg)\\&=-\infty.\end{aligned} \quad (5.5.17)$$

With the same argument one can prove (5.5.15) as well. Moreover, using $q_\varepsilon=\frac{1}{2\varepsilon}\mathbb{1}_{\{[-\varepsilon,+\varepsilon]\}}$, then we can write

$$\frac{1}{N}\sum_{x\in\mathbb{Z}}\left(\sum_{|x-y|\le\varepsilon N}\eta_{1,s}(y)\right)\left(\sum_{|x-y|\le\varepsilon N}\eta_{1,s}(y)\right)=\left(\pi_{1,s}^N*q_\varepsilon\right)\left(\pi_{1,s}^N*q_\varepsilon\right), \quad (5.5.18)$$

and

$$\frac{1}{N}\sum_{x\in\mathbb{Z}}\left(\sum_{|x-y|\le\varepsilon N}\eta_{1,s}(y)\right)\left(\sum_{|x-y|\le\varepsilon N}\eta_{2,s}(y)\right)=\left(\pi_{1,t}^N*q_\varepsilon\right)\left(\pi_{2,s}^N*q_\varepsilon\right). \quad (5.5.19)$$

Combining (5.5.13) with equations (5.5.18) and (5.5.19), we have that the Dynkin martingale $M_{1,N,t}^G(\boldsymbol{\pi}_{[0,T]}^N)$ is written as a function of the empirical density, namely

$$
\begin{aligned}
&M^{G}_{1,N,t}(\boldsymbol{\pi}^N_{[0,T]}) \\
&= \langle \pi^N_{1,t}, G(\cdot,t)\rangle - \langle \pi^N_{1,0}, G(\cdot,0)\rangle - \int_0^t \langle \pi^N_{1,s}, (\partial_t + \Delta) G(\cdot,s)\rangle \,\mathrm{d}s \\
&\quad - 2\int_0^t \left\langle \left(\pi^N_{1,s} * q_\varepsilon\right)\left(1 - \left(\pi^N_{1,s} * q_\varepsilon\right) - \left(\pi^N_{2,s} * q_\varepsilon\right)\right), \nabla H_{10}(\cdot,s)\nabla G(\cdot,s)\right\rangle \mathrm{d}s \\
&\quad - 2\int_0^t \left\langle \left(\pi^N_{1,s} * q_\varepsilon\right)\left(\pi^N_{2,s} * q_\varepsilon\right), \nabla H_{12}(\cdot,s)\nabla G(\cdot,s)\right\rangle \mathrm{d}s + R(\varepsilon, N), \qquad (5.5.20)
\end{aligned}
$$

where the remainder term $R(\varepsilon, N)$ goes to zero in probability as $N \to \infty$ and $\varepsilon \to 0$. A similar result can be found for $M^G_{2,N,t}(\boldsymbol{\pi}^N_{[0,T]})$.

Now we show that the martingale $M^{\boldsymbol{G}}_{N,t}(\boldsymbol{\pi}^N_{[0,T]})$ vanishes as $N \to \infty$. The predictable quadratic variation is computed by the Carré-du-Champ formula as

$$
\begin{aligned}
&\Gamma^{\boldsymbol{G}}_{N,t}(\boldsymbol{\pi}^N(\boldsymbol{\eta})) \\
&\quad = L^{\boldsymbol{H}}_{N,t}\langle \boldsymbol{\pi}^N(\boldsymbol{\eta}), \boldsymbol{G}(\cdot,t)\rangle^2 - 2\langle \boldsymbol{\pi}^N(\boldsymbol{\eta}), \boldsymbol{G}(\cdot,t)\rangle L^{\boldsymbol{H}}_N \langle \boldsymbol{\pi}^N(\boldsymbol{\eta}), \boldsymbol{G}(\cdot,t)\rangle \\
&\quad = \sum_{x\in\mathbb{T}_N}\sum_{a,b=0}^{2} c^{\boldsymbol{H},ab}_{(x,x+1)}(t)\left[\langle \boldsymbol{\pi}^N(\boldsymbol{\eta}^{x,x+1}_{a,b}), \boldsymbol{G}(\cdot,t)\rangle - \langle \boldsymbol{\pi}^N(\boldsymbol{\eta}), \boldsymbol{G}(\cdot,t)\rangle\right]^2 \\
&\quad = \sum_{x\in\mathbb{T}_N}\left(c^{\boldsymbol{H},01}_{(x,x+1)}(t) + c^{\boldsymbol{H},10}_{(x,x+1)}(t)\right)\left(\nabla_N G_1\left(\tfrac{x}{N},t\right)\right)^2 \\
&\qquad + \sum_{x\in\mathbb{T}_N}\left(c^{\boldsymbol{H},02}_{(x,x+1)}(t) + c^{\boldsymbol{H},20}_{(x,x+1)}(t)\right)\left(\nabla_N G_2\left(\tfrac{x}{N},t\right)\right)^2 \\
&\qquad + \sum_{x\in\mathbb{T}_N}\left(c^{\boldsymbol{H},12}_{(x,x+1)}(t) + c^{\boldsymbol{H},21}_{(x,x+1)}(t)\right)\left(\nabla_N G_1\left(\tfrac{x}{N},t\right) - \nabla_N G_2\left(\tfrac{x}{N},t\right)\right)^2,
\end{aligned}
\qquad (5.5.21)
$$

which is of order $1/N$ and goes to 0 as $N \to \infty$. This implies that for all $\delta > 0$

$$
\begin{aligned}
&\lim_{N\to\infty}\mathbb{P}^{\varrho,\boldsymbol{H}}_N\left(\sup_{t\in[0,T]}|M^{\boldsymbol{G}}_{N,t}(\boldsymbol{\pi}^N_{[0,T]})| \geq \delta\right) \\
&\qquad\qquad \leq \lim_{N\to\infty}\frac{1}{\delta^2}\mathbb{E}^{\varrho,\boldsymbol{H}}_N\left[\sup_{t\in[0,T]}|M^{\boldsymbol{G}}_{N,t}(\boldsymbol{\pi}^N_{[0,T]})|^2\right] \\
&\qquad\qquad \leq \lim_{N\to\infty}\frac{4}{\delta^2}\mathbb{E}^{\varrho,\boldsymbol{H}}_N\left[|M^{\boldsymbol{G}}_{N,t}(\boldsymbol{\pi}^N_{[0,T]})|^2\right] \\
&\qquad\qquad = \lim_{N\to\infty}\frac{4}{\delta^2}\mathbb{E}^{\varrho,\boldsymbol{H}}_N\left[\int_0^T \Gamma^{\boldsymbol{G}}_{N,t}(\boldsymbol{\pi}^N_t)\,\mathrm{d}t\right] = 0. \qquad (5.5.22)
\end{aligned}
$$

Next, one can show tightness of the sequence of random processes $\boldsymbol{\pi}^N_{[0,T]}$ is tight by using (5.5.22). The argument is standard and we refer to [61, 104]. By tightness we have the existence of convergent subsequences, and by combining this with (5.5.20) and (5.5.22) we observe that these convergent subsequences are concentrated on the set of trajectories $\boldsymbol{\alpha} \in D([0,T];\mathcal{M}_1 \times \mathcal{M}_1)$ such that for all $\delta > 0$ there exists an $\hat{\varepsilon}$ such that for all $\varepsilon \leq \hat{\varepsilon}$ and for all $t \in [0,T]$ we have that

$$\Big| \langle \boldsymbol{\alpha}_t, \boldsymbol{G}(\cdot,t)\rangle - \langle \boldsymbol{\alpha}_0, \boldsymbol{G}(\cdot,0)\rangle - \int_0^t \langle \boldsymbol{\alpha}_s, (\partial_t + \Delta)\,, \boldsymbol{G}(\cdot,s)\rangle \,\mathrm{d}s$$
$$+ \int_0^t \langle (\alpha_{1,s} * q_\varepsilon)\,(1 - (\alpha_{1,s} * q_\varepsilon) - (\alpha_{2,s} * q_\varepsilon)), \nabla G_1(\cdot,s) H_{10}(\cdot,s)\rangle \,\mathrm{d}s$$
$$+ \int_0^t \langle (\alpha_{2,s} * q_\varepsilon)\,(1 - (\alpha_{1,s} * q_\varepsilon) - (\alpha_{2,s} * q_\varepsilon)), \nabla G_2(\cdot,s) \nabla H_{20}(\cdot,s)\rangle \,\mathrm{d}s$$
$$- \int_0^t \langle (\alpha_{1,s} * q_\varepsilon)\,(\alpha_{2,s} * q_\varepsilon)\,, (\nabla G_1(\cdot,s) - \nabla G_2(\cdot,s))\,\nabla H_{12}(\cdot,s)\rangle \,\mathrm{d}s \Big| \leq \delta. \tag{5.5.23}$$

Finally, letting $\hat{\varepsilon}$ tend to 0, we observe that $\boldsymbol{\alpha}$ has density ϱ which solves equation (5.3.1).

5.6 PROOF OF THE SUPEREXPONENTIAL ESTIMATE

The objective of this appendix is to prove the superexponential estimate presented in Theorem 5.1. We follow here the road of the original paper [62], i.e., reducing the problem to one and two blocks estimates which then boil down to a uniform equivalence of ensembles. For the convenience of the reader and self-consistency of the paper, we nevertheless prefer to provide full details.

5.6.1 *Equivalence of ensembles*

In the following, for $\gamma_1, \gamma_2 > 0$ two constants, we denote by $\nu_N^{\gamma_1,\gamma_2}$ the measure

$$\nu_N^{\gamma_1,\gamma_2} = \bigotimes_{x \in \mathbb{T}_N} \mathrm{Multinomial}(1, \gamma_1, \gamma_2). \tag{5.6.1}$$

Furthermore, given $k_1, k_2 \in \mathbb{N}_0$ such that $k_1 + k_2 \leq N$, we define the event

$$\Omega_N^{k_1,k_2} := \Big\{ \boldsymbol{\eta} \in \Omega_N \;\Big|\; |\eta_1| = k_1, |\eta_2| = k_2 \Big\} \tag{5.6.2}$$

where $|\eta_a| = \sum_{x\in\mathbb{T}_N} \eta_a(x)$ is the total number of particles of type a. Since the marginals are independent, the measure $\nu_N^{\gamma_1,\gamma_2}$ conditioned on this event $\Omega_N^{k_1,k_2}$ is given by the uniform measure on $\Omega_N^{k_1,k_2}$, i.e.,

$$\nu_N^{\gamma_1,\gamma_2}\left(\boldsymbol{\eta} \,\middle|\, \Omega_N^{k_1,k_2}\right) = \begin{cases} \binom{N}{k_1,k_2}^{-1} & \text{if } \boldsymbol{\eta} \in \Omega_N^{k_1,k_2} \\ 0 & \text{else} \end{cases} \tag{5.6.3}$$

Note that this is independent of the choice of γ_1, γ_2, and so we set

$$\nu_N(\cdot \,|\, \Omega_N^{k_1,k_2}) := \nu_N^{\gamma_1,\gamma_2}(\cdot \,|\, \Omega_N^{k_1,k_2}) \tag{5.6.4}$$

Lemma 5.6. For any $\ell \leq N$ let $\{i_1, ..., i_\ell\} \subset \mathbb{T}_N$, and take $m_1, m_2 \in \mathbb{N}$ such that $m_1 + m_2 \leq N$. Define $\boldsymbol{\zeta}(i_k) = (\zeta_0(i_k), \zeta_1(i_k), \zeta_2(i_k))$ with $\zeta_a(i_k) \in \{0, 1\}$ such that $\zeta_0(i_k) + \zeta_1(i_k) + \zeta_2(i_k) = 1$ for all $1 \leq k \leq \ell$ and $\sum_{k=1}^{\ell} \zeta_a(i_k) = m_a$ for $a = 1, 2$. Furthermore, let $k_{1,N}, k_{2,N}$ be such that $\frac{k_{1,N}}{N} \to \gamma_1$ and $\frac{k_{2,N}}{N} \to \gamma_2$ as $N \to \infty$. Then

$$\begin{aligned}\lim_{N\to\infty} \nu_N\Big(\boldsymbol{\eta}(i_k) = \boldsymbol{\zeta}(i_k) \text{ for } k = 1, ..., \ell \,|\, \Omega_N^{k_{1,N},k_{2,N}}\Big)& \\ = \gamma_1^{m_1}\gamma_2^{m_2}(1-\gamma_1-\gamma_2)^{\ell-m_1-m_2}.&\end{aligned} \tag{5.6.5}$$

Proof. By (5.6.3), the statement follows from a direct computation,

$$\begin{aligned}\lim_{N\to\infty} \nu_N\Big(\boldsymbol{\eta}(i_k) = \boldsymbol{\zeta}(i_k) \text{ for } k = 1, ..., \ell \,|\, \Omega_N^{k_{1,N},k_{2,N}}\Big)& \\ = \lim_{N\to\infty} \frac{\binom{N-\ell}{k_1-m_1,k_2-m_2}}{\binom{N}{k_1,k_2}}& \\ = \gamma_1^{m_1}\gamma_2^{m_2}(1-\gamma_1-\gamma_2)^{\ell-m_1-m_2}.&\end{aligned} \tag{5.6.6}$$

□

We call a function $\phi : \Omega \to \mathbb{R}$, with Ω defined as in (5.2.2), local if $\phi(\boldsymbol{\eta})$ depends only on $\eta_1, ..., \eta_\ell$ for some fixed ℓ not dependent on N

Theorem 5.7 (Equivalence of ensembles)**.** For every local function ϕ, we have that

$$\overline{\lim_{N\to\infty}} \sup_{0\leq k_1,k_2\leq N \,:\, k_1+k_2\leq N} \left| \mathbb{E}_{\nu_N(\cdot|\Omega_N^{k_1,k_2})}[\phi] - \mathbb{E}_{\nu_N^{k_1/N,k_2/N}}[\phi] \right| = 0 \tag{5.6.7}$$

Proof. For every finite N, the supremum over k_1 and k_2 is reached. Denote by $k_{1,N}^*$ and $k_{2,N}^*$ a value of k's where the supremum is attained. Since

$$0 \leq \frac{k_{1,N}^*}{N} \leq 1, \tag{5.6.8}$$

there exists a convergent subsequence of $k_1^*(N)/N$. By consequence, as ϕ is local and by Lemma 5.6, we have

$$\overline{\lim_{i\to\infty}} \left| \mathbb{E}_{\nu_N\left(\cdot \,\middle|\, \Omega_{N_i}^{k_{1,N_i}^*,k_{2,N_i}^*}\right)}[\phi] - \mathbb{E}_{\nu_N^{k_{1,N_i}^*/N_i,\, k_{2,N_i}^*/N_i}}[\phi] \right| = 0. \tag{5.6.9}$$

This holds for every possible converging subsequence of $\frac{k_{1,N}^*}{N}$ and hence the statement follows. □

5.6.2 *One and two blocks estimates*

In this section, our goal is to show that proving Theorem 5.1 can be reduced to establishing two key lemmas, referred to as the one block and two blocks estimates, respectively. We hereby follow verbatim the steps of the proof of Theorem 2.1 of [62], with necessary adaptations to cover the multi-species case. The crucial aspect of this approach lies in the application of the Feynman-Kac formula (c.f. [61, Proposition A.7.1]).

We focus on the quantity $\mathbb{P}_N^{1/3}\left(\frac{1}{N}\int_0^t V_{N,\varepsilon}(\boldsymbol{\eta}(s))\,\mathrm{d}s \geq \delta\right)$ and we apply the exponential Chebyshev inequality, obtaining

$$\mathbb{P}_N^{1/3}\left(\frac{1}{N}\int_0^t V_{N,\varepsilon}(\boldsymbol{\eta}(s))\,\mathrm{d}s \geq \delta\right) \leq e^{-\delta Na}\mathbb{E}_N^{1/3}\left[\exp\left(a\int_0^t V_{N,\varepsilon}(\boldsymbol{\eta}(s))\,\mathrm{d}s\right)\right], \tag{5.6.10}$$

where we have denoted by $\mathbb{E}_N^{1/3}$ the expectation with respect to $\mathbb{P}_N^{1/3}$. To estimate the right hand side of (5.6.10) we define the operator

$$\mathcal{K} = \mathcal{L} + aV, \tag{5.6.11}$$

i.e.,

$$\mathcal{K}f(\boldsymbol{\eta}) = \mathcal{L}f(\boldsymbol{\eta}) + aV(\boldsymbol{\eta})f(\boldsymbol{\eta}). \tag{5.6.12}$$

Using Feynman-Kac formula (see [61, Proposition A.7.1, Lemma A.7.2])

$$\begin{aligned}\mathbb{E}_N^{1/3}\left[\exp\left(a\int_0^t V_{N,\varepsilon}(\boldsymbol{\eta}_s)\,\mathrm{d}s\right)\right] &= \left\langle 1, e^{t\mathcal{K}}1\right\rangle_{L^2(\nu^N_{1/3,1/3})} \\ &\leq \exp\left(t\lambda_{max}(\mathcal{K})\right),\end{aligned} \tag{5.6.13}$$

where $\lambda_{max}(\mathcal{K})$ is the largest eigenvalue of the operator $\mathcal{K}$. It follows that

$$e^{-\delta Na}\mathbb{E}_N^{1/3}\left[\exp\left(a\int_0^t V_{N,\varepsilon}(\boldsymbol{\eta}(s))\,\mathrm{d}s\right)\right] \leq \exp\left(N\left(\frac{t}{N}\lambda_{max}(\mathcal{K})\right) - \delta a\right). \tag{5.6.14}$$

Therefore, to prove the superexponential estimate, it is enough to show that

$$\varlimsup_{\varepsilon\to 0}\ \varlimsup_{N\to\infty}\ \frac{1}{N}\lambda_{max}(\mathcal{K}) = 0. \tag{5.6.15}$$

For the largest eigenvalue we have the variational representation

$$\lambda_{max}(\mathcal{K}) = \sup_{f_N}\left\{a\langle V_{N,\varepsilon}(\boldsymbol{\eta}), f_N\rangle_{\nu^N_{1/3,1/3}} - N^2 D_N(f_N)\right\}, \tag{5.6.16}$$

where the supremum is taken over probability densities f_N, i.e., $f_N \geq 0$ and $\sum_{\eta\in\Omega} f_N(\eta)3^{-N} = 1$. Furthermore, D_N is the so-called *Dirichlet form* associated with the generator $\mathcal{L}$, and is given by

$$\begin{aligned} D_N(f_N) &= \frac{1}{2}\sum_{\eta,\xi\in\Omega} \nu^N_{1/3,1/3}(\eta)\mathcal{L}(\eta,\xi)\left(\sqrt{f_N(\xi)} - \sqrt{f_N(\eta)}\right)^2 \\ &= \frac{3^{-N}}{2}\sum_{\eta\in\Omega}\sum_{x\in\mathbb{T}_N}\sum_{a,b=0}^{2} \eta_a(x)\eta_b(x+1)\left(\sqrt{f_N(\eta^{x,x+1}_{a,b})} - \sqrt{f_N(\eta)}\right)^2, \end{aligned} \tag{5.6.17}$$

and where

$$\langle V_{N,\varepsilon}(\eta), f_N\rangle_{\nu^N_{1/3,1/3}} = \sum_{\eta\in\Omega} V_{N,\varepsilon}(\eta) f_N(\eta)3^{-N}. \tag{5.6.18}$$

Since $\phi(\eta)$ is bounded, there exists a positive constant C such that

$$\langle V_{N,\varepsilon}(\eta), f_N\rangle_{\nu^N_{1/3,1/3}} \leq CN. \tag{5.6.19}$$

As a result, we restrict the supremum to the set of densities f_N that satisfy $D_N(f_N) \leq C/N$. Furthermore, we consider only the densities f_N that are translation invariant (since $D_N(\cdot)$ is convex, for details see Appendix 10 of [61]). Consequently, we obtain the estimate

$$\begin{aligned} &\sup_{f_N : D_N(f_N)\leq C/N} \left\{ a\sum_{\eta\in\Omega} V_{N,\varepsilon}(\eta) f_N(\eta)3^{-N} - N^2 D_N(f_N)\right\} \\ &\leq \sup_{f_N : D_N(f_N)\leq C/N} \left\{ a\sum_{\eta\in\Omega} V_{N,\varepsilon}(\eta) f_N(\eta)3^{-N}\right\}. \end{aligned} \tag{5.6.20}$$

This implies that is enough to show that

$$\overline{\lim_{\varepsilon\to 0}}\ \overline{\lim_{N\to\infty}} \sup_{f_N : D_N(f_N)\leq C/N} \frac{1}{N}\left\{\sum_{\eta\in\Omega} V_{N,\varepsilon}(\eta) f_N(\eta)3^{-N}\right\} = 0. \tag{5.6.21}$$

Writing out the definition of $V_{N,\varepsilon}(\eta)$ given in definition (5.2.17), we obtain, using translation invariance of f and recalling the notation of the averaged sum in (5.2.18),

$$\begin{aligned} &\overline{\lim_{\varepsilon\to 0}}\ \overline{\lim_{N\to\infty}} \sup_{f_N : D_N(f_N)\leq C/N} \\ &\left\{\sum_{\eta\in\Omega}\left| \sum_{|x|\leq\varepsilon N} \tau_x\phi(\eta) - \widetilde{\phi}\left(\sum_{|x|\leq\varepsilon N}\eta_1(x), \sum_{|x|\leq\varepsilon N}\eta_2(x)\right)\right| f_N(\eta)3^{-N}\right\} = 0. \end{aligned} \tag{5.6.22}$$

For any fixed $y \in \mathbb{T}_N$, we consider a neighborhood of discrete points $\{y-k, y-k+1, \ldots, y+k-1, y+k\}$. Within this neighborhood, we have the following approximation

$$\frac{1}{N}\sum_{y} \tau_y \phi(\boldsymbol{\eta}) = \frac{1}{N}\sum_{y} \sum_{|z-y|\le k} \tau_z \phi(\boldsymbol{\eta}) + \mathcal{O}\left(\frac{k}{N}\right). \tag{5.6.23}$$

Next we add and subtract the quantity $\widetilde{\phi}\left(\sum_{|z-x|\le k} \eta_1(z), \sum_{|z-x|\le k} \eta_2(z)\right)$ inside the absolute value of equation (5.6.22), obtaining

$$\begin{aligned}
&\left| \sum_{|x|\le \varepsilon N} \tau_x \phi(\boldsymbol{\eta}) - \widetilde{\phi}\left(\sum_{|y|\le \varepsilon N} \eta_1(y), \sum_{|y|\le \varepsilon N} \eta_2(y) \right) \right| \\
&\le \sum_{|x|\le \varepsilon N} \left| \sum_{|z-x|\le k} \tau_z \phi(\boldsymbol{\eta}) - \widetilde{\phi}\left(\sum_{|x-z|\le k} \eta_1(z), \sum_{|x-z|\le k} \eta_2(z) \right) \right| \\
&\quad + \sum_{|x|\le \varepsilon N} \left| \widetilde{\phi}\left(\sum_{|x-z|\le k} \eta_1(z), \sum_{|x-z|\le k} \eta_2(z) \right) - \widetilde{\phi}\left(\sum_{|y|\le \varepsilon N} \eta_1(y), \sum_{|y|\le \varepsilon N} \eta_2(y) \right) \right| \\
&\quad + \mathcal{O}\left(\frac{k}{N}\right).
\end{aligned} \tag{5.6.24}$$

We consider the second addend in the right-hand-side of (5.6.24). By exploiting the multi-variable mean-value theorem and (5.6.23) we have that

$$\begin{aligned}
&\sum_{|x|\le \varepsilon N} \left| \widetilde{\phi}\left(\sum_{|x-z|\le k} \eta_1(z), \sum_{|x-z|\le k} \eta_2(z) \right) - \widetilde{\phi}\left(\sum_{|y|\le \varepsilon N} \eta_1(y), \sum_{|y|\le \varepsilon N} \eta_2(y) \right) \right| \\
&\le \|\nabla \widetilde{\phi}\|_\infty \sum_{|x|\le \varepsilon N} \left\| \left(\sum_{|x-z|\le k} \eta_1(z), \sum_{|x-z|\le k} \eta_2(z) \right) - \left(\sum_{|y|\le \varepsilon N} \eta_1(y), \sum_{|y|\le \varepsilon N} \eta_2(y) \right) \right\|_2 \\
&\le \|\nabla \widetilde{\phi}\|_\infty \sum_{|x|\le \varepsilon N} \sum_{|y|\le \varepsilon N} \left\| \left(\sum_{|x-z|\le k} \eta_1(z), \sum_{|x-z|\le k} \eta_2(z) \right) - \left(\sum_{|y-z|\le k} \eta_1(z), \sum_{|y-z|\le k} \eta_2(z) \right) \right\|_2 \\
&\quad + \mathcal{O}\left(\frac{k}{N}\right).
\end{aligned} \tag{5.6.25}$$

Using furthermore that $||x||_2 \leq ||x||_1$, it follows that

$$\begin{aligned}
&\left| \sum_{|x|\leq \varepsilon N} \tau_x \phi(\boldsymbol{\eta}) - \widetilde{\phi}\left(\sum_{|y|\leq \varepsilon N} \eta_1(y), \sum_{|y|\leq \varepsilon N} \eta_2(y) \right) \right| \\
&\leq \sum_{|x|\leq \varepsilon N} \left| \sum_{|z-x|\leq k} \tau_z \phi(\boldsymbol{\eta}) - \widetilde{\phi}\left(\sum_{|x-z|\leq k} \eta_1(z), \sum_{|x-z|\leq k} \eta_2(z) \right) \right| \\
&+ \|\nabla \widetilde{\phi}\|_\infty \sum_{|x|\leq \varepsilon N} \sum_{|y|\leq \varepsilon N} \left\| \left(\sum_{|x-z|\leq k} \eta_1(z), \sum_{|x-z|\leq k} \eta_2(z) \right) - \left(\sum_{|y-z|\leq k} \eta_1(z), \sum_{|y-z|\leq k} \eta_2(z) \right) \right\|_1 \\
&+ \mathcal{O}\left(\frac{k}{N}\right).
\end{aligned} \tag{5.6.26}$$

Arrived at this point, in order to obtain (5.6.21) it is sufficient to prove the following two lemmas:

Lemma 5.7 (One block estimate)**.** For all $c > 0$

$$\varlimsup_{k\to\infty} \varlimsup_{N\to\infty} \sup_{f_N : D(f_N) \leq c/N} \sum_{\boldsymbol{\eta}\in\Omega} \left| \sum_{|z|\leq k} \tau_z \phi(\boldsymbol{\eta}) - \widetilde{\phi}\left(\sum_{|z|\leq k} \eta_1(z), \sum_{|z|\leq k} \eta_2(z) \right) \right| f_N(\boldsymbol{\eta}) 3^{-N} = 0. \tag{5.6.27}$$

Lemma 5.8 (Two blocks estimate)**.** For all $c > 0$

$$\varlimsup_{k\to\infty} \varlimsup_{\varepsilon\to 0} \varlimsup_{N\to\infty} \sup_{|r|\leq 2\varepsilon N+1} \sup_{f_N : D(f_N) \leq c/N} \sum_{\boldsymbol{\eta}\in\Omega} \left\| \left(\sum_{|z|\leq k} \eta_1(z), \sum_{|z|\leq k} \eta_2(z) \right) - \left(\sum_{|z+r|\leq k} \eta_1(z), \sum_{|z+r|\leq k} \eta_2(z) \right) \right\|_1 f_N(\boldsymbol{\eta}) 3^{-N} = 0. \tag{5.6.28}$$

5.6.3 *Proof of the one block estimate*

Fix $k \in \mathbb{N}$ such that $k \leq N$, and consider the set $\{x \in \mathbb{T}_N : |x| \leq k\}$. We introduce the subspace $\Omega_{2k+1} \subset \Omega$, which represents the state space restricted to these $2k+1$ sites. Then, for any function $g : \Omega_{2k+1} \to \mathbb{R}$, we define the "restricted" Dirichlet form as follows:

$$D^*_{2k+1}(g) = \frac{1}{2} \sum_{\boldsymbol{\eta}\in\Omega_{2k+1}} 3^{-(2k+1)} \sum_{x=-k}^{k-1} \sum_{a,b=0}^{2} \eta_a(x)\eta_b(x+1) \left(\sqrt{g(\boldsymbol{\eta}^{x,x+1}_{a,b})} - \sqrt{g(\boldsymbol{\eta})} \right)^2. \tag{5.6.29}$$

Next, we define the marginal of the density f_N over Ω_{2k+1} as

$$f_N^k(\boldsymbol{\eta}) = 3^{-N+2k+1} \sum_{\boldsymbol{\eta}(x)\,:\,|x|>k} f_N(\boldsymbol{\eta}). \tag{5.6.30}$$

Using the following inequality

$$\left(\sqrt{\sum_j a_j} - \sqrt{\sum_j b_j}\right)^2 \leq \sum_j \left(\sqrt{a_j} - \sqrt{b_j}\right)^2, \tag{5.6.31}$$

we have that

$$\begin{aligned}
&D^*_{2k+1}(f_N^k) \\
&= \frac{1}{2} \sum_{\boldsymbol{\eta}\in\Omega_{2k+1}} 3^{-N} \sum_{x=-k}^{k-1} \sum_{a,b=0}^{2} \eta_a(x)\eta_b(x+1) \left(\sqrt{\sum_{(\eta^x)_{|x|>k}} f_N(\boldsymbol{\eta}^{x,x+1}_{a,b})} - \sqrt{\sum_{(\eta^x)_{|x|>k}} f_N(\boldsymbol{\eta})} \right)^2 \\
&\leq \frac{1}{2} \sum_{\boldsymbol{\eta}\in\Omega} 3^{-N} \sum_{x=-k}^{k-1} \sum_{a,b=0}^{2} \eta_a(x)\eta_b(x+1) \left(\sqrt{f_N(\boldsymbol{\eta}^{x,x+1}_{a,b})} - \sqrt{f_N(\boldsymbol{\eta})} \right)^2 \\
&= \frac{1}{2} \sum_{x=-k}^{k-1} \sum_{\boldsymbol{\eta}\in\Omega} 3^{-N} \sum_{a,b=0}^{2} \eta^0_a \eta^1_b \left(\sqrt{f_N(\boldsymbol{\eta}^{0,1}_{a,b})} - \sqrt{f_N(\boldsymbol{\eta})} \right)^2 \\
&= \frac{2k}{N} D(f_N).
\end{aligned} \tag{5.6.32}$$

Here, in the up to last equality we have used the translation invariance. All in all, we obtain the upper bound

$$D^*_{2k+1}(f_N^k) \leq \frac{2k}{N} D(f_N). \tag{5.6.33}$$

As a consequence

$$\begin{aligned}
&\sup_{f_N\,:\,D(f_N)\leq c/N} \sum_{\boldsymbol{\eta}\in\Omega} \left| \sum_{|z|\leq k} \tau_z\phi(\boldsymbol{\eta}) - \widetilde{\phi}\left(\sum_{|z|\leq k} \eta_1(z), \sum_{|z|\leq k} \eta_2(z) \right) \right| f_N(\boldsymbol{\eta}) 3^{-N} \\
&\leq \sup_{g_k\,:\,D^*_{2k+1}(g_k)\leq (2ck)/N^2} \sum_{\boldsymbol{\eta}\in\Omega} \left| \sum_{|z|\leq k} \tau_z\phi(\boldsymbol{\eta}). - \widetilde{\phi}\left(\sum_{|z|\leq k} \eta_1(z), \sum_{|z|\leq k} \eta_2(z) \right) \right| g_k(\boldsymbol{\eta}) 3^{-2k-1} \\
&\quad + \mathcal{O}\left(\frac{k}{N}\right).
\end{aligned} \tag{5.6.34}$$

Taking the limsup as $N \to \infty$ and using the compactness of the level sets of the Dirichlet form (for details see Appendix 10 of [61]), we have

$$\begin{aligned}
&\overline{\lim_{N\to\infty}} \sup_{g_k : D^*_{2k+1}(g_k) \le (2ck)/N^2} \sum_{\eta\in\Omega} \left| \sum_{|z|\le k} \tau_z\phi(\eta) - \tilde{\phi}\left(\sum_{|z|\le k} \eta_1(z), \sum_{|z|\le k} \eta_2(z) \right) \right| g_k(\eta) 3^{-2k-1} \\
&\le \sup_{g_k : D^*_{2k+1}(g_k) = 0} \sum_{\eta\in\Omega} \left| \sum_{|z|\le k} \tau_z\phi(\eta) - \tilde{\phi}\left(\sum_{|z|\le k} \eta_1(z), \sum_{|z|\le k} \eta_2(z) \right) \right| g_k(\eta) 3^{-2k-1}.
\end{aligned} \tag{5.6.35}$$

The set of probability distribution with density g_k such that $D^*_{2k+1}(g_k) = 0$ is the set of uniform distributions over Ω_{2k+1} with fixed number of particles k_1, k_2 of species 1 and 2 respectively. Therefore, taking the supremum in equation (5.6.35) is equivalent to taking the supremum over all configurations η in the space Ω_{2k+1} with fixed number of particles k_1 and k_2 of the two species. As a consequence, by taking the limsup for $k \to \infty$, we have that

$$\begin{aligned}
&\overline{\lim_{k\to\infty}} \sup_{g_k : D^*_{2k+1}(g_k) = 0} \sum_{\eta\in\Omega} \left| \sum_{|z|\le k} \tau_z\phi(\eta) - \tilde{\phi}\left(\sum_{|z|\le k} \eta_1(z), \sum_{|z|\le k} \eta_2(z) \right) \right| g_k(\eta) 3^{-2k-1} \\
&= \overline{\lim_{k\to\infty}} \sup_{\substack{k_1,k_2 = 0,\dots,2k+1 \\ k_1+k_2 \le 2k+1}} \sum_{\eta \in \Omega^{k_1,k_2}_{2k+1}} \frac{\left| \sum_{|z|\le k} \tau_z\phi(\eta) - \tilde{\phi}\left(\sum_{|z|\le k} \eta_1(z), \sum_{|z|\le k} \eta_2(z) \right) \right|}{\binom{2k+1}{k_1,k_2}} \\
&= \overline{\lim_{k\to\infty}} \sup_{\substack{k_1,k_2 = 0,\dots,2k+1 \\ k_1+k_2 \le 2k+1}} \left| \mathbb{E}_{\nu(\cdot | \Omega^{k_1,k_2}_{2k+1})}[\phi] - \mathbb{E}_{\nu_{2k+1}^{\frac{k_1}{2k+1}, \frac{k_2}{2k+1}}}[\phi] \right| \\
&= 0.
\end{aligned} \tag{5.6.36}$$

Here, in the last step, we used Theorem 5.7.

5.6.4 *Proof of the two blocks estimate*

In analogy to the approach used in the proof of Lemma 5.7, we now consider two blocks of size $2k+1$: the first centered around the microscopic point $0 \in \mathbb{T}_N$ and the second centered around the microscopic point $r \in \mathbb{T}_N$. The centers of these two blocks are separated by a distance of at most $2\varepsilon N + 1$. We denote by ζ, ξ the configurations in the first and second block respectively, both belonging to the sub-space Ω_{2k+1}. We consider an arbitrary function $g : \Omega_{2k+1} \times \Omega_{2k+1} \to \mathbb{R}$ and we define the following "restricted" Dirichlet-forms:

$$D_k^1(g) = \frac{1}{2} \sum_{\zeta,\xi \in \Omega_{2k+1}} 3^{-4k-2} \sum_{x=-k}^{k-1} \sum_{a,b=0}^{2} \zeta_a^x \zeta_b^{x+1} \left(\sqrt{g(\zeta_{a,b}^{x,x+1}, \xi)} - \sqrt{g(\zeta,\xi)} \right)^2$$

$$D_k^2(g) = \frac{1}{2} \sum_{\zeta,\xi \in \Omega_{2k+1}} 3^{-4k-2} \sum_{x=-k}^{k-1} \sum_{a,b=0}^{2} \zeta_a^x \zeta_b^{x+1} \left(\sqrt{g(\zeta, \xi_{a,b}^{x,x+1})} - \sqrt{g(\zeta,\xi)} \right)^2$$

$$\Delta_k(g) = \frac{1}{2} \sum_{\zeta,\xi \in \Omega_{2k+1}} 3^{-4k-2} \left(\sqrt{g(\zeta,\xi)^0} - \sqrt{g(\zeta,\xi)} \right)^2 \tag{5.6.37}$$

where $(\zeta,\xi)^0$ indicates the configurations where the occupation variables at the center points of the two blocks have been exchanged. Intuitively, the first Dirichlet-form concerns the first block; the second Dirichlet-form the second block; the third Dirichlet-form takes into account the transfer of particles from one block to the other. We now introduce the marginal over the two blocks

$$f_N^{r,k}(\eta) = 3^{-N+4k+2} \sum_{\eta(x)\,:\,|x|>k,\ |x-r|>k} f_N(\eta). \tag{5.6.38}$$

Arguing as in the proof of Lemma 5.7, one can show the follwing estimates:

$$D_k^1(f_N^{r,k}) \leq \frac{2k}{N} D(f_N)$$

$$D_k^2(f_N^{r,k}) \leq \frac{2k}{N} D(f_N). \tag{5.6.39}$$

We now aim to find an upper bound for the Dirichlet-form $\Delta_k(\cdot)$ in terms of ε. Obtaining the configuration $(\zeta,\xi)^0$ from the configuration (ζ,ξ) is equivalent to permuting the occupation variables η^0 and η^r. We introduce the permutation operator $P_{x,y}$ between sites x and y, defined as follows:

$$P_{x,y}\eta = (\eta(0),\ldots,\eta(x-1),\eta(y),\eta(x+1),\ldots,\eta(y-1),\eta(x),\eta(y+1),\ldots,\eta(N)). \tag{5.6.40}$$

By applying (5.6.31) and by the definition of the marginal over the two blocks written in (5.6.38) we obtain

$$\Delta_k(f_N^{r,k}) = \frac{1}{2} \sum_{\zeta,\xi \in \Omega_{2k+1}} 3^{-4k-2} \left(\sqrt{f_N^{r,k}(\zeta,\xi)^0} - \sqrt{f_N^{r,k}(\zeta,\xi)} \right)^2$$

$$\leq \frac{1}{2} \sum_{\eta \in \Omega} 3^{-N} \left(\sqrt{f_N(P_{0,r}\eta)} - \sqrt{f_N(\eta)} \right)^2. \tag{5.6.41}$$

This permutation operator satisfies the property[3]

$$P_{1,2}P_{3,2}P_{2,1} = P_{1,2}P_{2,1}P_{1,3} = P_{1,3}. \tag{5.6.42}$$

Therefore, we have that

$$\begin{aligned}
&\left(\sqrt{f_N(P_{0,r}\boldsymbol{\eta})} - \sqrt{f_N(\boldsymbol{\eta})}\right)^2 \\
&= \Bigg(\sqrt{f_N(P_{0,1}\boldsymbol{\eta})} - \sqrt{f_N(\boldsymbol{\eta})} + \sqrt{f_N(P_{0,1}P_{1,2}\boldsymbol{\eta})} - \sqrt{f_N(P_{0,1}\boldsymbol{\eta})} \\
&\quad + \sqrt{f_N(P_{2,3}P_{1,2}P_{0,1}\boldsymbol{\eta})} - \sqrt{f_N(P_{1,2}P_{0,1}\boldsymbol{\eta})} + \dots \\
&\quad + \sqrt{f_N(P_{1,0}\cdots P_{r-1,r-2}P_{r-1,r}\cdots P_{0,1}\boldsymbol{\eta})} - \sqrt{f_N(P_{2,1}\cdots P_{r-1,r-2}P_{r-1,r}\cdots P_{0,1}\boldsymbol{\eta})}\Bigg)^2 \\
&\leq (2r-1)\Bigg\{\left(\sqrt{f_N(P_{0,1}\boldsymbol{\eta})} - \sqrt{f_N(\boldsymbol{\eta})}\right)^2 + \left(\sqrt{f_N(P_{0,1}P_{1,2}\boldsymbol{\eta})} - \sqrt{f_N(P_{0,1}\boldsymbol{\eta})}\right)^2 \\
&\quad + \left(\sqrt{f_N(P_{2,3}P_{1,2}P_{0,1}\boldsymbol{\eta})} - \sqrt{f_N(P_{1,2}P_{0,1}\boldsymbol{\eta})}\right)^2 + \dots \\
&\quad + \left(\sqrt{f_N(P_{1,0}\cdots P_{r-1,r-2}P_{r-1,r}\cdots P_{0,1}\boldsymbol{\eta})} - \sqrt{f_N(P_{2,1}\cdots P_{r-1,r-2}P_{r-1,r}\cdots P_{0,1}\boldsymbol{\eta})}\right)^2\Bigg\}.
\end{aligned} \tag{5.6.43}$$

Consequently, we find that

$$\begin{aligned}
&\frac{1}{2}\sum_{\boldsymbol{\eta}\in\Omega} 3^{-N}\left(\sqrt{f_N(P_{0,r}\boldsymbol{\eta})} - \sqrt{f_N(\boldsymbol{\eta})}\right)^2 \\
&\qquad \leq (2r-1)^2\frac{1}{2}\sum_{\boldsymbol{\eta}\in\Omega} 3^{-N}\sum_{a,b=0}^{2}\eta_a^0\eta_b^1\left(\sqrt{f_N(\boldsymbol{\eta}_{a,b}^{0,1})} - \sqrt{f_N(\boldsymbol{\eta})}\right)^2,
\end{aligned} \tag{5.6.44}$$

where we used the translation invariance of f_N. Therefore, using (5.6.41), it follows that

$$\Delta_k(f_N^{r,k}) \leq \frac{(2r-1)^2}{N}D(f_N). \tag{5.6.45}$$

Finally, for fixed $c > 0$, $\varepsilon > 0$ and $N \in \mathbb{N}$, we define the set

$$A_{N,\varepsilon} := \left\{g \ : \ D_k^1(g) \leq \frac{2ck}{N^2},\ D_k^2(g) \leq \frac{2ck}{N^2},\ \Delta_k(g) \leq \varepsilon^2 c\right\}. \tag{5.6.46}$$

Arguing as in the proof of Lemma 5.7 it follows that

$$\{f_N \ : \ D(f_N) \leq c/N,\} \cap \{r \ : \ |r| \leq \varepsilon N\} \subset \{g \ : \ g \in A_{N,\varepsilon}\}. \tag{5.6.47}$$

3 that can be proved by using the fact that $P_{i,j} = P_{j,i}$ and $P_{i,j}P_{j,k} = P_{j,k}P_{i,k}$.

The above inclusion relation implies that

$$\sup_{|r|\le \varepsilon N}\ \sup_{f_N\,:\,D(f_N)\le c/N}$$
$$\sum_{\boldsymbol{\eta}\in\Omega}\left\|\left(\sum_{|z|\le k}\eta_1(z),\sum_{|z|\le k}\eta_2(z)\right)-\left(\sum_{|z+r|\le k}\eta_1(z),\sum_{|z+r|\le k}\eta_2(z)\right)\right\|_1 f_N(\boldsymbol{\eta})3^{-N}$$
$$\le \sup_{g\in A_{N,\varepsilon}}\sum_{\boldsymbol{\zeta},\boldsymbol{\xi}\in\Omega_{2k+1}}\left\|\left(\sum_{|x|\le k}\zeta_1^x,\sum_{|x|\le k}\zeta_2^x\right)-\left(\sum_{|x|\le k}\xi_1^x,\sum_{|x|\le k}\xi_2^x\right)\right\|_1 g(\boldsymbol{\zeta},\boldsymbol{\xi})3^{-4k-2}. \tag{5.6.48}$$

By taking the limsup for $N\to\infty$ and $\varepsilon\to 0$, and by exploiting the compactness of the level sets of the Dirichlet-forms we eventually take the supremum over all g in the following set

$$A := \{g : D_k^1(g) = D_k^2(g) = \Delta_k(g) = 0\}. \tag{5.6.49}$$

The set of distributions that satisfy this is the set of uniform distributions over $\Omega_{2k+1}\times\Omega_{2k+1}$, with fixed numbers k_1 and k_2 of particles of species 1 and 2 respectively. Choosing $\phi_1,\phi_2 : \Omega_{2k+1}\to\mathbb{R}$ as $\phi_1(\boldsymbol{\zeta}) = \zeta_1(0)$, $\phi_2(\boldsymbol{\zeta}) = \zeta_2(0)$, we find $\widetilde{\phi}^1(\gamma_1,\gamma_2) = \gamma_1$, $\widetilde{\phi}^2(\gamma_1,\gamma_2) = \gamma_2$. As a consequence, we obtain

$$\overline{\lim_{k\to\infty}}\sup_{g\in A}\sum_{\boldsymbol{\zeta},\boldsymbol{\xi}\in\Omega_{2k+1}}\left|\left(\sum_{|x|\le k}\zeta_1(x)\right)-\left(\sum_{|x|\le k}\xi_1(x)\right)\right| g(\boldsymbol{\zeta},\boldsymbol{\xi})3^{-4k-2}$$
$$+\overline{\lim_{k\to\infty}}\sup_{g\in A}\sum_{\boldsymbol{\zeta},\boldsymbol{\xi}\in\Omega_{2k+1}}\left|\left(\sum_{|x|\le k}\zeta_2(x)\right)-\left(\sum_{|x|\le k}\xi_2(x)\right)\right| g(\boldsymbol{\zeta},\boldsymbol{\xi})3^{-4k-2}$$
$$\le \overline{\lim_{k\to\infty}}\sup_{\substack{0\le k_1,k_2\le 4k+2\\ k_1+k_2\le 4k+2}}\sum_{\boldsymbol{\zeta}\in\Omega_{2k+1}^{k_1,k_2}}\frac{\left|\left(\sum_{|x|\le k}\zeta_1(x)\right)-\widetilde{\phi}^1\left(\frac{k_1}{4k+2},\frac{k_2}{4k+2}\right)\right|}{\binom{4k+2}{k_1,k_2}}$$
$$+\overline{\lim_{k\to\infty}}\sup_{\substack{0\le k_1,k_2\le 4k+2\\ k_1+k_2\le 4k+2}}\sum_{\boldsymbol{\xi}\in\Omega_{2k+1}^{k_1,k_2}}\frac{\left|\left(\sum_{|x|\le k}\xi_1(x)\right)-\widetilde{\phi}^1\left(\frac{k_1}{4k+2},\frac{k_2}{4k+2}\right)\right|}{\binom{4k+2}{k_1,k_2}}$$
$$+\overline{\lim_{k\to\infty}}\sup_{\substack{0\le k_1,k_2\le 4k+2\\ k_1+k_2\le 4k+2}}\sum_{\boldsymbol{\zeta}\in\Omega_{2k+1}^{k_1,k_2}}\frac{\left|\left(\sum_{|x|\le k}\zeta_2(x)\right)-\widetilde{\phi}^2\left(\frac{k_2}{4k+2},\frac{k_2}{4k+2}\right)\right|}{\binom{4k+2}{k_1,k_2}}$$
$$+\overline{\lim_{k\to\infty}}\sup_{\substack{0\le k_1,k_2\le 4k+2\\ k_1+k_2\le 4k+2}}\sum_{\boldsymbol{\xi}\in\Omega_{2k+1}^{k_1,k_2}}\frac{\left|\left(\sum_{|x|\le k}\xi_2(x)\right)-\widetilde{\phi}^2\left(\frac{k_2}{4k+2},\frac{k_2}{4k+2}\right)\right|}{\binom{4k+2}{k_1,k_2}} = 0. \tag{5.6.50}$$

The last equality follows from Theorem 5.7.

6

LARGE DEVIATIONS OF MEAN-FIELD RUN-AND-TUMBLE PARTICLES

In this chapter[1] we return to run-and-tumble particles, considering particles on two layers that only perform active jumps. We furthermore introduce a mean-field interacting to the process, where the rate to jump to the other layer depends on the magnetization of the process. We then study the large deviations of this process, for which we again need to introduce a weakly perturbed process. As a first step towards establishing a large deviation principle for the total density, we apply the contraction principle and characterize the multi-layer trajectories with finite rate function that are compatible with the given trajectory of the total density.

6.1 INTRODUCTION

Run-and-tumble particles serve as simple models of active matter [105]. Systems of active particles constitute an important class of non-equilibrium systems where at the microscopic scale energy is dissipated in order to produce directed motion. They show a rich phenomenology such as clustering and long-range order [20, 102, 105].

In the mathematics literature on interacting particle systems, results on active particles are not very abundant (in contrast with the physics literature). To our knowledge, the first rigorous result on hydrodynamic limits for active particles is [64], where a system of locally interacting active particles was studied. In [65], [64], the authors prove the hydrodynamic limit (using the non-gradient method) and identify a motility-induced phase separation and a transition to collective motion from the equations obtained in the hydrodynamic limit. See also e.g. [86] for recent results in the physics literature on locally interacting active particles and collective effects therein.

In this paper we consider a simpler model where the interaction between the run-and-tumble particles is of mean-field type, i.e., via their empirical distribution. For this model we can then both prove the hydrodynamic limit and the large

1 this chapter is based on [CITE].

deviations from the hydrodynamic limit. In our model, the particles move on the one-dimensional torus, and have an internal state which takes values ± 1 and determines the direction of motion. The interaction between the particles arises implicitly via the flip rate at which particles flip their internal state which depends on the magnetization.

First, we start by deriving the hydrodynamic limit, which is a coupled system of partial differential equations for the densities of particles with internal state ± 1, and an ordinary differential equation for the "magnetization". Second, we consider a weakly perturbed model where the influence of an external field which weakly depends on time and (microscopic) space and is added. This field influences the rate at which particles flip their internal state and the rate at which they jump in the direction of their internal state. This is reminiscent of the weakly asymmetric exclusion process [62] which is an essential tool to study the large deviations for the trajectory of the density in the symmetric exclusion process (SEP). Contrary to the situation of the SEP, in our model, the perturbation does not act on the direction of the motion, which is always in the direction of the internal state.

We prove the hydrodynamic limit of this weakly perturbed model and use it to prove a large deviation principle for the trajectory of the densities in the original model. The technique of proof is based on a change of measure between the original and the weakly perturbed model and the associated exponential martingale (the Radon-Nikodym derivative of the perturbed model w.r.t. the original model). Due to the mean-field character of the interaction, no super-exponential replacement lemmas are needed, i.e., the quantities appearing in the Radon-Nikodym derivative between the perturbed and unperturbed model are a function of the empirical densities and the magnetization.

The rest of our paper is organized as follows. In Section 6.2 we introduce the model, the weakly perturbed model and state the hydrodynamic limit of both. In Section 6.3 we prove large deviations for the trajectory of the densities. In Section 6.4 we provide the proof of the hydrodynamic limits stated in Section 6.2.

6.2 RUN-AND-TUMBLE PARTICLES WITH MEAN-FIELD SWITCHING RATES

In this section we describe the run-and-tumble particle model with mean-field switching rates. Later on we will also define a weakly perturbed version of this model which we will need for the large deviations. For both models, we will consider particles on the two-layered torus $V_N := \mathbb{T}_N \times S$, where $\mathbb{T}_N = \mathbb{Z}/N\mathbb{Z}$ is the discrete torus, and $S = \{-1, 1\}$ which we will call the internal state space. We then say that a particle $(x, \sigma) \in V_N$ has position $x \in \mathbb{T}_N$ and internal state $\sigma \in S$. The parameter N is a scaling parameter, and we will be interested in the limiting dynamics when $N \to \infty$.

We will consider processes of particle configurations $\{\eta_t^N : t \geq 0\}$ on the state space $\Omega_N = \mathbb{N}^{V_N}$. We denote by $\eta_t^N(x,\sigma)$ the number of particles at site $(x,\sigma) \in V_N$ at time $t \geq 0$. The Markovian dynamics is then described as follows:

i) Active jump: with rate N a particle jumps from (x,σ) to $(x+\sigma,\sigma)$.

ii) Internal state flip: a particle jumps from (x,σ) to $(x,-\sigma)$, with a mean-field rate denoted by $c(\sigma, m_N(\eta_t^N))$

We assume $c(\sigma,\cdot)$ to be continuous in the second variable and bounded from above and below (away from zero). Here by "mean-field" we mean that the flip rate $c(\sigma, m_N(\eta_t^N))$ depends on the whole configuration η only via its magnetization $m_N(\eta_t^N)$. For $\eta \in \Omega_N$, this magnetization is defined via

$$m_N(\eta) := \frac{1}{|\eta|} \sum_{x\in\mathbb{T}_N} \big(\eta(x,1) - \eta(x,-1)\big), \tag{6.2.1}$$

where $|\eta| := \sum_{(x,\sigma)\in V_N} \eta(x,\sigma)$ denotes the total number of particles in the configuration η. Here we also use the convention that if $|\eta| = 0$ then $m_N(\eta) = 0$.

More precisely, the process is defined by its generator working on functions $f : \Omega_N \to \mathbb{R}$ as follows.

$$\begin{aligned}\mathscr{L}_N f(\eta) = N &\sum_{(x,\sigma)\in V_N} \eta(x,\sigma)\big[f(\eta^{(x,\sigma)\to(x+\sigma,\sigma)}) - f(\eta)\big] \\ &+ \sum_{(x,\sigma)\in V_N} c(\sigma, m_N(\eta))\eta(x,\sigma)\big[f(\eta^{(x,\sigma)\to(x,-\sigma)}) - f(\eta)\big],\end{aligned} \tag{6.2.2}$$

where $\eta^{(x,\sigma)\to(y,\sigma')}$ denotes the configuration η where a single particle has jumped from (x,σ) to (y,σ'), if possible.

Remark 6.1. If we choose the rates $c(\sigma,m) \equiv 1$, then the particles do not interact with each other, and we recover the run-and-tumble particle process, studied, for instance, in [105, 97, 98, 118, 23]. An actual example of mean-field rates where particles do interact with one another is, for instance, given by the Curie-Weiss Glauber rates $c(\sigma,m) = e^{-\sigma\beta m}$ with $\beta > 0$.

6.2.1 *Hydrodynamic limit*

For $N \in \mathbb{N}$, we define the empirical measure of a configuration $\eta \in \Omega_N$ by

$$\pi^N(\eta) := \frac{1}{N} \sum_{(x,\sigma)\in V_N} \eta(x,\sigma)\delta_{(\frac{x}{N},\sigma)}, \tag{6.2.3}$$

where δ denotes the Dirac measure. For a given η, $\pi^N(\eta)$ is a positive Radon measure on the macroscopic space $V := \mathbb{T} \times S$, where $\mathbb{T} = [0,1]$ is the torus. We denote the space of positive Radon measures on V by $\mathcal{M}_V$. Given $t \geq 0$, we further denote the empirical measure of the configuration η_t^N as

$$\pi_t^N := \pi^N(\eta_t^N) = \frac{1}{N} \sum_{(x,\sigma)\in V_N} \eta_t^N(x,\sigma)\delta_{(\frac{x}{N},\sigma)}, \tag{6.2.4}$$

For every $N \in \mathbb{N}$, this produces a process $\{\pi_t^N : t \in [0,T]\}$ with trajectories in the Skorokhod space $D([0,T];\mathcal{M}_V)$. The corresponding space of test functions is given by,

$$C^\infty(V) := \{\phi : V \to \mathbb{R} \mid \phi(\cdot,\sigma) \in C^\infty(\mathbb{T}) \text{ for all } \sigma \in S\}. \tag{6.2.5}$$

For $\phi \in C^\infty(V)$ we now denote the pairing

$$\left\langle \pi_t^N, \phi \right\rangle = \frac{1}{N} \sum_{(x,\sigma)\in V_N} \eta_t^N(x,\sigma)\phi(\tfrac{x}{N},\sigma). \tag{6.2.6}$$

For $N \in \mathbb{N}$ and smooth $\varrho(x,\sigma) : V \to \mathbb{R}_{\geq 0}$ we define the product Poisson measures

$$\mu_N^\varrho = \bigotimes_{(x,\sigma)\in V_N} \text{Pois}(\varrho\left(\tfrac{x}{N},\sigma\right)), \tag{6.2.7}$$

which is the local equilibrium measure for the non-interacting ($c(\sigma,m) \equiv 1$) case (see [97]). We assume that the process $\{\eta_t^N : t \geq 0\}$ starts at $t = 0$ from the configuration $\eta^N = \eta_0^N$ distributed according to μ_N^ϱ, which fixes the initial density profile. More precisely, π_0^N converges as $N \to \infty$ to the positive measure with density $\varrho(x,\sigma) : V \to \mathbb{R}_{\geq 0}$, i.e., we have that for every $\phi \in C^\infty(V)$ and $\varepsilon > 0$

$$\lim_{N\to\infty} \mathbb{P}_N^\varrho \left(\left| \langle \pi_0^N, \phi \rangle - \sum_{\sigma\in S} \int \phi(x,\sigma)\varrho(x,\sigma)\, \mathrm{d}x \right| > \varepsilon \right) = 0. \tag{6.2.8}$$

Here $\mathbb{P}_N^\varrho$ denotes the path-space measure of the process with initial distribution μ_N^ϱ, where we omit the dependence on the terminal time T since we assume that this is fixed.

The question of the hydrodynamic limit is to find the limiting PDE for π_t^N as $N \to \infty$. We start with the following preliminary computation

$$\begin{aligned} &\mathscr{L}_N \langle \pi^N(\eta), \phi \rangle \\ &= N \sum_{(x,\sigma)\in V_N} \eta(x,\sigma) \left(\langle \pi^N(\eta^{(x,\sigma)\to(x+\sigma,\sigma)}), \phi \rangle - \langle \pi^N(\eta), \phi \rangle \right) \\ &\quad + \sum_{(x,\sigma)\in V_N} \eta(x,\sigma)c(\sigma,m_N(\eta)) \left(\langle \pi^N(\eta^{(x,\sigma)\to(x,-\sigma)}), \phi \rangle - \langle \pi^N(\eta), \phi \rangle \right) \\ &= \frac{1}{N} \sum_{(x,\sigma)\in V_N} \eta(x,\sigma) \left[N\big(\phi(\tfrac{x+\sigma}{N},\sigma) - \phi(\tfrac{x}{N},\sigma)\big) + c(\sigma,m_N(\eta))\big(\phi(x,-\sigma) - \phi(x,\sigma)\big) \right]. \end{aligned} \tag{6.2.9}$$

From this computation, we observe that the evolution of the empirical measure depends on the evolution of the magnetization $m_N(\eta_t^N)$. Note however that the magnetization, defined in (6.2.1), can be expressed in terms of the empirical measure as follows,

$$m_N(\eta) = m(\pi^N(\eta)) := \frac{\langle \pi^N(\eta), \mathbb{1}_{\sigma=1} - \mathbb{1}_{\sigma=-1}\rangle}{\langle \pi^N(\eta), 1\rangle}. \tag{6.2.10}$$

This representation of the magnetization motivates the following definition of the magnetization corresponding to a density $\varrho(x,\sigma) : V \to \mathbb{R}_{\geq 0}$ as

$$m(\varrho) := \frac{\langle \varrho, \mathbb{1}_{\sigma=1} - \mathbb{1}_{\sigma=-1}\rangle_{L^2(V)}}{\langle \varrho, 1\rangle_{L^2(V)}}, \tag{6.2.11}$$

where $\langle \cdot, \cdot \rangle_{L^2(V)}$ denotes the inner product of L^2-functions on the space V, given by

$$\langle \psi, \phi \rangle_{L^2(V)} = \sum_{\sigma \in S} \int_{\mathbb{T}} \psi(x,\sigma)\phi(x,\sigma)\,\mathrm{d}x, \tag{6.2.12}$$

for any $\psi, \phi \in L^2(V)$.

Theorem 6.1. $\mathbb{P}_N^{\varrho}(\pi_{[0,T]}^N \in \cdot) \to \delta_\alpha$, where $\alpha \in D([0,T];\mathcal{M}_V)$ is the trajectory of measures with density $\varrho_t(x,\sigma)$ which solves the following equation,

$$\dot{\varrho}_t(x,\sigma) = -\sigma\partial_x\varrho_t(x,\sigma) + c(-\sigma, m(\varrho_t))\varrho_t(x,-\sigma) - c(\sigma, m(\varrho_t))\varrho_t(x,\sigma), \tag{6.2.13}$$

with initial condition $\varrho_0(x,\sigma) = \varrho(x,\sigma)$.

This theorem will be a consequence of a more general hydrodynamic limit of a weakly perturbed modification of the model, which we will introduce in Section 6.2.2. Apart from the hydrodynamic limit of the empirical measure, we can also find a limiting equation of the magnetization.

Corollary 6.1. $\mathbb{P}_N^{\varrho}(m(\pi_{[0,T]}^N) \in \cdot) \to \delta_{m(\varrho_\cdot)}$, where ϱ_t solves (6.2.13). Furthermore, $m_t := m(\varrho_t)$ solves the following equation,

$$\dot{m}_t = c(-1, m_t)\cdot(1 - m_t) - c(1, m_t)\cdot(1 + m_t), \tag{6.2.14}$$

with initial condition $m_0 = m(\varrho)$.

Remark 6.2. When considering the Curie-Weiss Glauber rates $c(\sigma, m) = e^{-\sigma\beta m}$ with inverse temperature $\beta > 0$, the evolution of the magnetization is given by

$$\dot{m}_t = 2\sinh(\beta m_t) - 2m_t\cosh(\beta m_t). \tag{6.2.15}$$

As $t \to \infty$ the process m_t converges to a solution of the mean-field equation

$$m^* = \tanh(\beta m^*). \tag{6.2.16}$$

For $\beta \leq 1$ the only solution is $m^* = 0$. However, for $\beta > 1$, there exist two nonzero solutions, and if $m_0 \neq 0$, the process m_t converges to either the positive or negative solution, depending on the initial value m_0. This simulates a type of flocking behavior of the particles, where particles tend to move in the same direction.

6.2.2 *Weakly perturbed model*

We furthermore introduce a weakly perturbed version of our model, which will be a key tool for the study of large deviations. The weak perturbation will be parametrized by a time-dependent potential $H : [0, T] \times V \to \mathbb{R}$, which we will assume to be differentiable in time and continuous in space, and which we will denote by $H \in C^{1,0}([0, T] \times V)$. The time-dependent generator of this model, acting on functions $f : \Omega_N \to \mathbb{R}$, is given as follows:

$$\begin{aligned} \mathscr{L}^H_{N,t} f(\eta) = N \sum_{(x,\sigma)\in V_N} \eta(x,\sigma) e^{H_t(\frac{x+\sigma}{N},\sigma) - H_t(\frac{x}{N},\sigma)} \left[f(\eta^{(x,\sigma)\to(x+\sigma,\sigma)}) - f(\eta) \right] \\ + \sum_{(x,\sigma)\in V_N} \eta(x,\sigma) c(\sigma, m_N(\eta)) e^{H_t(\frac{x}{N},-\sigma) - H_t(\frac{x}{N},\sigma)} \left[f(\eta^{(x,\sigma)\to(x,-\sigma)}) - f(\eta) \right]. \end{aligned} \tag{6.2.17}$$

Note that for $H = 0$ we recover the original model. We will denote by $\mathbb{P}^{\varrho,H}_N$ the path-space measure of this process, where η^N_0 is distributed as μ^ϱ_N (cf. (6.2.7)). We further abbreviate $\widetilde{H}_t(x) := H_t(x, 1) - H_t(x, -1)$. The hydrodynamic limit of this process is then given in the following theorem.

Theorem 6.2. $\mathbb{P}^{\varrho,H}_N(\pi^N_{[0,T]} \in \cdot) \to \delta_{\alpha^H}$, where $\alpha^H \in D([0, T]; \mathcal{M}_V)$ is the trajectory of measures with density $\varrho^H_t(x, \sigma)$ which solves the following equation,

$$\begin{aligned} \dot{\varrho}^H_t(x,\sigma) = -\sigma\partial_x \varrho^H_t(x,\sigma) + c(-\sigma, m(\varrho^H_t)) e^{\sigma \widetilde{H}_t(x)} \varrho^H_t(x,-\sigma) \\ - c(\sigma, m(\varrho^H_t)) e^{-\sigma \widetilde{H}_t(x)} \varrho^H_t(x,\sigma), \end{aligned} \tag{6.2.18}$$

with initial condition $\varrho^H_0(x, \sigma) = \varrho(x, \sigma)$.

Note that Theorem 6.1 follows from this Theorem 6.2 by choosing $H \equiv 0$. The proof of Theorem 6.2 will be postponed to Section 6.4. Below we give the evolution of the magnetization under the perturbed dynamics.

Corollary 6.2. $\mathbb{P}^{\varrho,H}_N(m(\pi^N_{[0,T]}) \in \cdot) \to \delta_{m(\varrho^H)}$, where ϱ^H_t solves (6.2.18). Furthermore, $m^H_t := m(\varrho^H_t)$ solves the following equation,

$$\dot{m}^H_t = \frac{1}{\langle \varrho, 1 \rangle_{L^2(V)}} \left\langle \varrho^H_t, -2\sigma e^{-\sigma \widetilde{H}_t(x)} c(\sigma, m^H_t) \right\rangle_{L^2(V)}. \tag{6.2.19}$$

with initial condition $m_0 = m(\varrho)$.

Proof. Note that if $\alpha_N \to \alpha$ in $D([0,T] : \mathcal{M}_V)$ then $m(\alpha_N) \to m(\alpha)$ in $D([0,T] : [-1,1])$, hence $m(\pi^N_{[0,T]}) \xrightarrow{d} m.$ under $\mathbb{P}_N^{\varrho,H}$. Therefore, we can find an equation for the evolution of the magnetization from the evolution of ϱ_t^H. First note that $\langle \varrho_t^H, 1\rangle_{L^2(V)} = \langle \varrho, 1\rangle_{L^2(V)}$ for all $t \geq 0$, which is due to the conservation of particles. Therefore, using the definition (6.2.11), we find

$$\dot{m}_t = \frac{\langle \dot{\varrho}_t^H, \mathbb{1}_{\sigma=1} - \mathbb{1}_{\sigma=-1}\rangle_{L^2(V)}}{\langle \varrho, 1\rangle_{L^2(V)}} \tag{6.2.20}$$

By the periodic boundary conditions, we have that

$$\langle -\sigma\partial_x \varrho_t^H, \mathbb{1}_{\sigma=1} - \mathbb{1}_{\sigma=-1}\rangle_{L^2(V)} = 0, \tag{6.2.21}$$

hence (6.2.19) follows by filling in (6.2.18) into (6.2.20). □

Corollary 6.1 follows from Corollary 6.2 by choosing $H \equiv 0$ and using that for any density ϱ we have that

$$\frac{\int_{\mathbb{T}} \varrho(x,\sigma)\,\mathrm{d}x}{\langle \varrho, 1\rangle_{L^2(V)}} = \tfrac{1}{2}(1 + \sigma m(\varrho)). \tag{6.2.22}$$

Note that, unlike in the unperturbed model, under the perturbed dynamics the evolution of the magnetization is not a closed equation, but depends on the density ϱ_t^H. This is because under the these dynamics the magnetization process is no longer a Markov process on its own, and additional information on the positions of the particles is required.

6.3 Large deviations

In this section we will prove a large deviation principle for the run-and-tumble particle process with mean-field switching rates. We start by defining the rate function $\mathcal{I}^\varrho : D([0,T]; \mathcal{M}_V) \to [0,\infty]$, which is given in two parts

$$\mathcal{I}^\varrho(\widehat{\alpha}) = h_0^\varrho(\widehat{\alpha}_0) + \mathcal{I}_{tr}(\widehat{\alpha}) \tag{6.3.1}$$

Here $h_0^\varrho(\widehat{\alpha}_0)$ is the static part of the large deviation rate function, only depending on the measure at time $t = 0$. It can be informally written as

$$h_0^\varrho(\widehat{\alpha}_0) = \mu_N^\varrho(\pi_0^N \approx \widehat{\alpha}_0), \tag{6.3.2}$$

i.e, it corresponds to the large deviation principle of the initial density profile π_0^N under the starting distribution μ_N^ϱ. Since μ_N^ϱ is given by a product Poisson measure, the corresponding rate function is known, and given by

$$h_0^\varrho(\widehat{\alpha}_0) = \sup_\phi h_0^\varrho(\widehat{\alpha}_0;\phi), \qquad h_0^\varrho(\widehat{\alpha}_0;\phi) = \langle \widehat{\alpha}_0, \phi\rangle - \langle \varrho, e^\phi - 1\rangle_{L^2(V)}. \tag{6.3.3}$$

Here the supremum is taken over all $\phi \in C^\infty(V)$.

The term $\mathcal{I}_{tr}(\widehat{\alpha})$ in (6.3.1) is the dynamic part of the rate function, and depends on the whole trajectory $\widehat{\alpha}$. It is given by the following:

$$\mathcal{I}_{tr}(\widehat{\alpha}) = \sup_G \mathcal{I}_{tr}(\widehat{\alpha}; G), \tag{6.3.4}$$

where

$$\mathcal{I}_{tr}(\widehat{\alpha}; G) = \ell(\widehat{\alpha}; G) - \int_0^T \left\langle \widehat{\alpha}_t, c(\sigma, m(\widehat{\alpha}_t)) \left(e^{-\sigma \widetilde{G}_t(x)} - 1 \right) \right\rangle \mathrm{d}t. \tag{6.3.5}$$

Here the supremum in (6.3.4) is taken over all $G \in C^\infty([0,T] \times V)$, and we recall the notation $\widetilde{G}_t(x) = G_t(x,1) - G_t(x,-1)$. Furthermore, $\ell(\widehat{\alpha}; G)$ is a linear functional, defined as follows

$$\ell(\widehat{\alpha}; G) := \langle \widehat{\alpha}_T, G_T \rangle - \langle \widehat{\alpha}_0, G_0 \rangle - \int_0^T \langle \widehat{\alpha}_t, (\partial_t + \sigma \partial_x) G_t \rangle \, \mathrm{d}t, \tag{6.3.6}$$

and $m(\widehat{\alpha}_t)$ is the magnetization of the measure, defined as

$$m(\widehat{\alpha}_t) = \frac{\langle \widehat{\alpha}_t, \mathbb{1}_{\sigma=1} - \mathbb{1}_{\sigma=-1} \rangle}{\langle \widehat{\alpha}_t, 1 \rangle}. \tag{6.3.7}$$

Remark 6.3. In Lemmas 6.4 and 6.5 we derive more explicit expressions of the static part $h_0^\varrho(\widehat{\alpha}_0)$ and the dynamic part $\mathcal{I}_{tr}(\widehat{\alpha})$. Specifically, we identify the functions ϕ and H for which $h_0^\varrho(\widehat{\alpha}_0) = h_0^\varrho(\widehat{\alpha}_0; \phi)$ and $\mathcal{I}_{tr}(\widehat{\alpha}) = \mathcal{I}_{tr}(\widehat{\alpha}; H)$.

In order to prove the large deviation principle with rate function $\mathcal{I}$, we need to establish the upper and lower bound.

i. **Upper bound**: For every closed set $\mathcal{C} \subset D([0,T]; \mathcal{M}_V)$ we have

$$\overline{\lim_{N\to\infty}} \frac{1}{N} \log \mathbb{P}_N^\varrho(\pi_{[0,T]}^N \in \mathcal{C}) \le - \inf_{\widehat{\alpha} \in \mathcal{C}} \mathcal{I}^\varrho(\widehat{\alpha}) \tag{6.3.8}$$

ii. **Lower bound**: For every open set $\mathcal{O} \subset D([0,T]; \mathcal{M}_V)$ we have

$$\underline{\lim_{N\to\infty}} \frac{1}{N} \log \mathbb{P}_N^\varrho(\pi_{[0,T]}^N \in \mathcal{O}) \ge - \inf_{\widehat{\alpha} \in \mathcal{O}} \mathcal{I}^\varrho(\widehat{\alpha}) \tag{6.3.9}$$

6.3.1 *Radon-Nikodym derivatives*

The goal of this section is to obtain an explicit form for the Radon-Nikodym derivative of the path-space measure of the weakly perturbed process $\mathrm{d}\mathbb{P}_N^{\varrho,H}$ with respect to the path-space measure of the original process $\mathrm{d}\mathbb{P}_N^\varrho$. This Radon-Nikodym derivative is given by the so-called exponential martingale of the process, and is given in the following lemma.

Lemma 6.1. For all $T > 0$, $N \in \mathbb{N}$ and $H \in C^\infty(V)$, we have that

$$\frac{\mathrm{d}\mathbb{P}_N^{\varrho,H}}{\mathrm{d}\mathbb{P}_N^{\varrho}} = \mathcal{Z}_{N,T}^H(\pi_{[0,T]}^N)$$
$$:= \exp\left(N\langle \pi_T^N, H_T\rangle - N\langle \pi_0^N, H_0\rangle - \int_0^T e^{-N\langle \pi_t^N, H_t\rangle}(\partial_t + \mathscr{L}_N)e^{N\langle \pi_t^N, H_t\rangle}\,\mathrm{d}t\right). \tag{6.3.10}$$

Proof. By Palmowski and Rolski [90], the exponential martingale $Z_{N,T}^H(\pi_{[0,T]}^N)$ is equal to the Radon-Nikodym derivative $\frac{\mathrm{d}\widetilde{\mathbb{P}}_N}{\mathrm{d}\mathbb{P}_N^\varrho}$, where $\widetilde{\mathbb{P}}_N$ is the path-space measure (up to time T) of the process corresponding to the time-dependent Markov generator on $\widehat{\varrho}_N$ given by

$$\widetilde{\mathscr{L}}_{N,t} f(\eta) = e^{-N\langle \pi^N(\eta), H_t\rangle}\left[\mathscr{L}_N\big(f(\eta)\cdot e^{N\langle \pi^N(\eta), H_t\rangle}\big) - f(\eta)\cdot \mathscr{L}_N e^{N\langle \pi^N(\eta), H_t\rangle}\right]. \tag{6.3.11}$$

where π^N now denotes the empirical measure as a function of $\eta \in \widehat{\varrho}_N$, as defined in (6.2.3). We will prove that $\widetilde{\mathbb{P}}_N = \mathbb{P}_N^{\varrho,H}$ by showing that $\widetilde{\mathscr{L}}_{N,t}$ is equal to the generator of the weakly perturbed process $\mathscr{L}_{N,t}^H$ as defined in (6.2.17). We compute, using (6.2.2)

$$\begin{aligned} &e^{-N\langle \pi^N(\eta), H_t\rangle}\mathscr{L}_N\big(f(\eta)\cdot e^{N\langle \pi^N(\eta), H_t\rangle}\big)\\ &\quad= N\sum_{(x,\sigma)\in V_N}\eta(x,\sigma)\big[f(\eta^{(x,\sigma)\to(x+\sigma,\sigma)})e^{H_t(\frac{x+\sigma}{N},\sigma)-H_t(\frac{x}{N},\sigma)} - f(\eta)\big]\\ &\qquad+ \sum_{(x,\sigma)\in V_N}\eta(x,\sigma)c(\sigma,m(\eta))\big[f(\eta^{(x,\sigma)\to(x,-\sigma)})e^{H_t(\frac{x}{N},-\sigma)-H_t(\frac{x}{N},\sigma)} - f(\eta)\big], \end{aligned} \tag{6.3.12}$$

where we used that

$$\langle X_N(\eta^{(x,\sigma)\to(y,\sigma')}), H_t\rangle - \langle X_N(\eta), H_t\rangle = H_t(\tfrac{y}{N},\sigma') - H_t(\tfrac{x}{N},\sigma). \tag{6.3.13}$$

Similarly we compute

$$\begin{aligned} &e^{-N\langle \pi^N(\eta), H_t\rangle}\mathscr{L}_N e^{N\langle \pi^N(\eta), H_t\rangle}\\ &\quad= N\sum_{(x,\sigma)\in V_N}\eta(x,\sigma)\big[e^{H_t(\frac{x+\sigma}{N},\sigma)-H_t(\frac{x}{N},\sigma)} - 1\big]\\ &\qquad+ \sum_{(x,\sigma)\in V_N}\eta(x,\sigma)c(\sigma,m(\eta))\big[e^{H_t(\frac{x}{N},-\sigma)-H_t(\frac{x}{N},\sigma)} - 1\big]. \end{aligned} \tag{6.3.14}$$

Substituting (6.3.12) and (6.3.14) into (6.3.11), we indeed find that $\widetilde{\mathscr{L}}_{N,t} = \mathscr{L}_{N,t}^H$, completing the proof. □

Corollary 6.3.

$$\frac{1}{N}\log\left(\mathcal{Z}^H_{N,T}(\pi^N_{[0,T]})\right) = \mathcal{I}_{tr}(\pi^N_{[0,T]};H) + \mathcal{O}(\tfrac{1}{N}). \tag{6.3.15}$$

Proof. Making use of the following approximation

$$e^{H_t(\frac{x+\sigma}{N},\sigma)-H_t(\frac{x}{N},\sigma)} - 1 = \sigma\partial_x H_t(\tfrac{x}{N},\sigma) + \mathcal{O}(\tfrac{1}{N}), \tag{6.3.16}$$

we are able to write (6.3.14) as

$$e^{-N\langle\pi^N_t,H_t\rangle}\mathscr{L}_N e^{N\langle\pi^N_t,H_t\rangle} = N\langle\pi^N_t,\sigma\partial_x H_t\rangle + N\left\langle \pi^N_t, c(\sigma,m(\pi^N_t))\left(e^{-\sigma\widetilde{H}_t(x)}-1\right)\right\rangle \\ + \mathcal{O}(1). \tag{6.3.17}$$

By now plugging this into (6.3.10), we find that

$$\begin{aligned}
&\mathcal{Z}^H_{N,T}(\pi^N_{[0,T]}) \\
&= \exp\left(N\langle\pi^N_T,H_T\rangle - N\langle\pi^N_0,H_0\rangle - N\int_0^T\langle\pi^N_t,(\partial_t+\sigma\partial_x)H_t\rangle\,\mathrm{d}t + \mathcal{O}(1)\right) \\
&\quad\times\exp\left(-N\int_0^T\left\langle\pi^N_t,c(\sigma,m(\pi^N_t))\left(e^{-\sigma\widetilde{H}_t(x)}-1\right)\right\rangle\mathrm{d}t\right) \\
&= \exp\left(N\ell(\pi^N_{[0,T]};H) - N\int_0^T\left\langle\pi^N_t,c(\sigma,m(\pi^N_t))\left(e^{-\sigma\widetilde{H}_t(x)}-1\right)\right\rangle\mathrm{d}t + \mathcal{O}(1)\right),
\end{aligned} \tag{6.3.18}$$

finishing the proof. □

6.3.2 *Upper bound*

In this section we will prove the large deviation upper bound (6.3.8). A crucial ingredient is to prove that the path-space measures of $\pi^N_{[0,T]}$ are exponentially tight, which then reduces the proof of (6.3.8) to compact sets.

Theorem 6.3 (Exponential Tightness)**.** For any $n \in \mathbb{N}$ there exists a compact set $\mathcal{K}_n \subset D([0,T],\mathcal{M}_V)$ such that

$$\varlimsup_{N\to\infty}\frac{1}{N}\log\mathbb{P}^{\varrho}_N(\pi^N_{[0,T]}\notin\mathcal{K}_n) = -n. \tag{6.3.19}$$

Before proving exponential tightness, we first give the proof of the upper bound for compact sets.

Theorem 6.4 (Upper bound for compact sets)**.** For every compact set $\mathcal{K}\subset D([0,T];\mathcal{M}_V)$ we have that

$$\varlimsup_{N\to\infty}\frac{1}{N}\log\mathbb{P}^{\varrho}_N(\pi^N_{[0,T]}\in\mathcal{K}) \le -\inf_{\widehat{\alpha}\in\mathcal{K}}\mathcal{I}^{\varrho}(\widehat{\alpha}). \tag{6.3.20}$$

Proof. We start with the following computation

$$\begin{aligned}
&\frac{1}{N}\log \mathbb{P}_N^{\varrho}(\pi_{[0,T]}^N \in \mathcal{K}) \\
&= \frac{1}{N}\log \mathbb{E}_N^{\varrho}\left[\mathbb{1}_{\pi_{[0,T]}^N \in \mathcal{K}} \cdot \frac{e^{Nh_0^{\varrho}(\pi_0^N;\phi)}}{e^{Nh_0^{\varrho}(\pi_0^N;\phi)}} \cdot \frac{\mathcal{Z}_{N,T}^G(\pi_{[0,T]}^N)}{\mathcal{Z}_{N,T}^G(\pi_{[0,T]}^N)}\right] \\
&\leq -\inf_{\widehat{\alpha}\in\mathcal{K}} \frac{1}{N}\log\left[e^{Nh_0^{\varrho}(\widehat{\alpha}_0;\phi)} \cdot \mathcal{Z}_{N,T}^G(\widehat{\alpha})\right] + \frac{1}{N}\log \mathbb{E}_N^{\varrho}\left[e^{Nh_0^{\varrho}(\pi_0^N;\phi)} \cdot \mathcal{Z}_{N,T}^G(\pi_{[0,T]}^N)\right],
\end{aligned} \tag{6.3.21}$$

where we recall the definition of h_0^{ϱ} in (6.3.3) and of $\mathcal{Z}_{N,T}^G$ in (6.3.10). Since $\mathcal{Z}_{N,T}^G(\pi_{[0,T]}^N)$ is a martingale with $\mathcal{Z}_{N,0}^G(\pi_{[0,T]}^N) = 1$, we actually find that

$$\begin{aligned}
\mathbb{E}_N^{\varrho}\left[e^{Nh_0^{\varrho}(\pi_0^N;\phi)} \cdot \mathcal{Z}_{N,T}^G(\pi_{[0,T]}^N)\right] &= \mathbb{E}_{\mu_N^{\varrho}}\left[e^{Nh_0^{\varrho}(\pi_0^N;\phi)}\right] \\
&= \mathbb{E}_{\mu_N^{\varrho}}\left[e^{N\left(\langle \pi_0^N,\phi\rangle - \langle \varrho, e^{\phi}-1\rangle_{L^2(V)}\right)}\right] = 1,
\end{aligned} \tag{6.3.22}$$

where we used that μ_N^{ϱ} is a product Poisson distribution. Therefore the second term in (6.3.21) vanishes. For the first term, note that we took ϕ and G arbitrarily, so we have

$$\overline{\lim_{N\to\infty}} \frac{1}{N}\log \mathbb{P}_N^{\varrho}(\pi_{[0,T]}^N \in \mathcal{K}) \leq -\sup_{\phi,G} \overline{\lim_{N\to\infty}} \inf_{\widehat{\alpha}\in\mathcal{K}} \frac{1}{N}\log\left[e^{Nh_0^{\varrho}(\widehat{\alpha}_0;\phi)} \cdot \mathcal{Z}_{N,T}^G(\widehat{\alpha})\right] \tag{6.3.23}$$

using Corollary 6.3, we find that

$$\begin{aligned}
-\sup_{\phi,G} \overline{\lim_{N\to\infty}} \inf_{\widehat{\alpha}\in\mathcal{K}} \frac{1}{N}\log\left[e^{Nh_0^{\varrho}(\widehat{\alpha}_0;\phi)} \cdot \mathcal{Z}_{N,T}^G(\widehat{\alpha})\right] &= -\sup_{\phi,G} \inf_{\widehat{\alpha}\in\mathcal{K}} h_0^{\varrho}(\widehat{\alpha}_0;\phi) + \mathcal{I}_{tr}(\widehat{\alpha};G) \\
&= -\inf_{\widehat{\alpha}\in\mathcal{K}} \mathcal{I}^{\varrho}(\widehat{\alpha}),
\end{aligned} \tag{6.3.24}$$

where we were able to interchange the supremum over ϕ and G with the infimum over $\widehat{\alpha}$ using the argument of Lemma 11.3 in [120], using that $\mathcal{K}$ is compact. □

In order to prove the exponential tightness, we want to use the method used in [61, pages 271-273]. However, since we do not know the invariant measure of this system, we first turn to a perturbed model of which we do know the invariant measures, namely the independent run-and-tumble particle system, as defined in [98] (without diffusive jumps). The generator of this process corresponds to setting $c(\sigma, m) \equiv 1$ in the generator $\mathscr{L}_N$ defined in (6.2.2), and we will denote it by $\mathscr{L}_N^{\mathrm{RTP}}$. For this process, we know that the Product Poisson measures with constant density ϱ_c are invariant. We denote the path-space measure of this process, started from μ_N^{ϱ}, by $\mathbb{P}_N^{\mathrm{RTP},\varrho}$.

Lemma 6.2. There exists a constant $C > 0$ such that

$$\mathbb{E}_{\mathbb{P}_N^{\mathrm{RTP},\varrho}}\left[\left(\frac{\mathrm{d}\mathbb{P}_N^{\varrho}}{\mathrm{d}\mathbb{P}_N^{\mathrm{RTP},\varrho_c}}\right)^2\right] \leq e^{CN}. \tag{6.3.25}$$

Proof. Note that we can split the Radon-Nikodym derivative in the following way

$$\frac{\mathrm{d}\mathbb{P}_N^{\varrho}}{\mathrm{d}\mathbb{P}_N^{\mathrm{RTP},\varrho_c}} = \frac{\mathrm{d}\mu_N^{\varrho}}{\mathrm{d}\mu_N^{\varrho_c}} \cdot \frac{\mathrm{d}\mathbb{P}_N^{\varrho}}{\mathrm{d}\mathbb{P}_N^{\mathrm{RTP},\varrho}}. \tag{6.3.26}$$

The Radon-Nikodym derivative of the Poisson distributions can be bounded in the following way

$$\begin{aligned}\frac{\mathrm{d}\mu_N^{\varrho}}{\mathrm{d}\mu_N^{\varrho_c}}(\eta) &= \exp\left(\sum_{(x,\sigma)\in V_N} \eta(x,\sigma)\log\left(\frac{\varrho(\frac{x}{N},\sigma)}{\varrho_c}\right) - \sum_{(x,\sigma)\in V_N}(\varrho(\tfrac{x}{N},\sigma)-\varrho_c)\right)\\ &\leq \exp\left(\log\left(\frac{||\varrho||_\infty}{\varrho_c}\right)|\eta^N| + 2\varrho_c N\right).\end{aligned} \tag{6.3.27}$$

Since the jump rates of the processes corresponding to $\mathbb{P}_N^{\mathrm{RTP},\varrho}$ and $\mathbb{P}_N^{\varrho}$ only differ at the internal state jumps, by the Girsanov formula, we find that the Radon-Nikodym derivative is given by

$$\begin{aligned}\frac{\mathrm{d}\mathbb{P}_N^{\varrho}}{\mathrm{d}\mathbb{P}_N^{\mathrm{RTP},\varrho}} = \exp\Bigg(&\sum_{(x,\sigma)\in V_N}\int_0^T \log\left(c(\sigma, m_N(t))\right)\mathrm{d}J_t^{(x,\sigma)\to(x,-\sigma)}\\ &- \sum_{(x,\sigma)\in V_N}\int_0^T \eta_t^N(x,\sigma)\left(c(\sigma,m_N(t))-1\right)\mathrm{d}t\Bigg),\end{aligned} \tag{6.3.28}$$

where $J_t^{(x,\sigma)\to(x,-\sigma)}$ is the number of jumps made from (x,σ) to $(x,-\sigma)$ up to time t. Since $c(\sigma, m_N(t))$ is bounded from above and below, we can find constants $c_1, c_2 > 0$ such that

$$\left(\frac{\mathrm{d}\mathbb{P}_N^{\varrho}}{\mathrm{d}\mathbb{P}_N^{\mathrm{RTP},\varrho}}\right)^2 \leq \exp\left(c_1\sum_{(x,\sigma)\in V_N} J_T^{(x,\sigma)\to(x,-\sigma)} + c_2 T|\eta^N|\right), \tag{6.3.29}$$

where we recall that $|\eta^N|$ is the total number of particles in the configuration η^N. Note that $\sum_{(x,\sigma)\in V_N} J_T^{(x,\sigma)\to(x,-\sigma)}$ is the total number of internal state jumps up to time T, which under $\mathbb{P}_N^{\mathrm{RTP},\varrho}$ is a Poisson process with intensity $|\eta^N|$. Combining (6.3.27) and (6.3.29), we can find a constant $c_3 > 0$ such that

$$\mathbb{E}_{\mathbb{P}_N^{\mathrm{RTP},\varrho}}\left[\left(\frac{\mathrm{d}\mathbb{P}_N^{\varrho}}{\mathrm{d}\mathbb{P}_N^{\mathrm{RTP},\varrho_c}}\right)^2\right] \le e^{4\varrho_c N}\mathbb{E}_{\mathbb{P}_N^{\mathrm{RTP},\varrho}}\left[\exp\left(c_3 T|\eta^N|\right)\right] \le \exp\left(4\varrho_c N + 2||\varrho||_\infty\left(e^{c_3 T}-1\right)N\right), \tag{6.3.30}$$

hence (6.3.25) holds. □

Now we can proceed with the proof of exponential tightness, using the approach from [61]. We will need the following result.

Lemma 6.3. For every $\varepsilon > 0$ and $G \in C^\infty(V)$

$$\lim_{\delta\to 0}\lim_{N\to\infty}\frac{1}{N}\log\mathbb{P}_N^{\varrho}\left(\sup_{|s-t|<\delta}\left|\left\langle \pi_t^N, G\right\rangle - \left\langle \pi_s^N, G\right\rangle\right| \ge \varepsilon\right) = -\infty. \tag{6.3.31}$$

Proof. Let $\varepsilon > 0$ be given. First note that we have the following

$$\mathbb{P}_N^{\varrho}\left(\sup_{|s-t|<\delta}\left|\left\langle \pi_t^N, G\right\rangle - \left\langle \pi_s^N, G\right\rangle\right| \ge \varepsilon\right) \le \mathbb{P}_N^{\varrho}\left(\sup_{|s-t|<\delta}\left\langle \pi_t^N, G\right\rangle - \left\langle \pi_s^N, G\right\rangle \ge \varepsilon\right) + \mathbb{P}_N^{\varrho}\left(\sup_{|s-t|<\delta}\left\langle \pi_t^N, -G\right\rangle - \left\langle \pi_s^N, -G\right\rangle \ge \varepsilon\right). \tag{6.3.32}$$

Since we are considering every $G \in C^\infty(V)$, we can neglect the absolute value in (6.3.31). Furthermore, by Hölder's inequality we have that for a general event A

$$\mathbb{P}_N^{\varrho}(A) = \mathbb{E}_{\mathbb{P}_N^{\mathrm{RTP},\varrho_c}}\left[\mathbb{1}_A\frac{\mathrm{d}\mathbb{P}_N^{\varrho}}{\mathrm{d}\mathbb{P}_N^{\mathrm{RTP},\varrho_c}}\right] \le \left(\mathbb{P}_N^{\mathrm{RTP},\varrho_c}(A)\right)^{\frac{1}{2}}\left(\mathbb{E}_{\mathbb{P}_N^{\mathrm{RTP},\varrho}}\left[\left(\frac{\mathrm{d}\mathbb{P}_N^{\varrho}}{\mathrm{d}\mathbb{P}_N^{\mathrm{RTP},\varrho_c}}\right)^2\right]\right)^{\frac{1}{2}}. \tag{6.3.33}$$

Therefore, by Lemma 6.2, it is enough to prove the result for $\mathbb{P}_N^{\mathrm{RTP},\varrho_c}$.

Using the following inclusion.

$$\left\{\sup_{|s-t|<\delta}\langle \pi_t^N, G\rangle - \langle \pi_s^N, G\rangle \ge \varepsilon\right\} \subset \bigcup_{k=0}^{[T\delta^{-1}]}\left\{\sup_{0\le t<\delta}\langle \pi_{k\delta+t}^N, G\rangle - \langle \pi_{k\delta}^N, G\rangle > \frac{\varepsilon}{4}\right\}, \tag{6.3.34}$$

we are able to find that

$$\begin{aligned}
&\overline{\lim_{N\to\infty}} \frac{1}{N} \log \mathbb{P}_N^{\mathrm{RTP},\varrho_c}\left(\sup_{|s-t|<\delta} \langle \pi_t^N, G\rangle - \langle \pi_s^N, G\rangle \geq \varepsilon \right) \\
&\qquad \leq \overline{\lim_{N\to\infty}} \frac{1}{N} \log \max_{k=0}^{\lceil T\delta^{-1}\rceil} \mathbb{P}_N^{\mathrm{RTP},\varrho_c}\left(\sup_{0\leq t<\delta} \langle \pi_{k\delta+t}^N, G\rangle - \langle \pi_{k\delta}^N, G\rangle \geq \tfrac{1}{4}\varepsilon \right) \\
&\qquad = \overline{\lim_{N\to\infty}} \frac{1}{N} \log \mathbb{P}_N^{\mathrm{RTP},\varrho_c}\left(\sup_{0\leq t<\delta} \langle \pi_t^N, G\rangle - \langle \pi_0^N, G\rangle \geq \tfrac{1}{4}\varepsilon \right), \qquad (6.3.35)
\end{aligned}$$

where in the last step we used that $\mu_N^{\varrho_c}$ is invariant for the RTP system. By denoting the exponential martingale corresponding to $\mathscr{L}_N^{\mathrm{RTP}}$ as $\mathcal{Z}_{N,t}^{\mathrm{RTP},G}(\pi_{[0,T]}^N)$ (recall the definition of the exponential martingale in (6.3.10)), and multiplying both sides by a constant $\lambda > 0$, we find that

$$\begin{aligned}
&\mathbb{P}_N^{\mathrm{RTP},\varrho_c}\left(\sup_{0\leq t<\delta} \langle \pi_t^N, \lambda G\rangle - \langle \pi_0^N, \lambda G\rangle \geq \tfrac{1}{4}\lambda\varepsilon \right) \\
&\quad \leq \mathbb{P}_N^{\mathrm{RTP},\varrho_c}\Bigg(\sup_{0\leq t<\delta} \frac{1}{N} \log \mathcal{Z}_{N,t}^{\mathrm{RTP},\lambda G}(\pi_{[0,T]}^N) \\
&\qquad\qquad + \frac{1}{N}\int_0^t e^{-N\langle \pi_s^N, \lambda G_s\rangle}(\partial_s + \mathscr{L}_N^{\mathrm{RTP}}) e^{N\langle \pi_s^N, \lambda G_s\rangle}\, \mathrm{d}s \geq \tfrac{1}{4}\lambda\varepsilon \Bigg) \\
&\quad \leq \mathbb{P}_N^{\mathrm{RTP},\varrho_c}\left(\sup_{0\leq t<\delta} \mathcal{Z}_{N,t}^{\mathrm{RTP},\lambda G}(\pi_{[0,T]}^N) \geq e^{\frac{1}{8}N\lambda\varepsilon} \right) \\
&\qquad + \mathbb{P}_N^{\mathrm{RTP},\varrho_c}\left(\sup_{0\leq t<\delta} \frac{1}{N}\int_0^t e^{-N\langle \pi_s^N, \lambda G_s\rangle}(\partial_s + \mathscr{L}_N^{\mathrm{RTP}}) e^{N\langle \pi_s^N, \lambda G_s\rangle}\, \mathrm{d}s \geq \tfrac{1}{8}\lambda\varepsilon \right).
\end{aligned}$$

(6.3.36)

Recalling that $\mathcal{Z}_{N,t}^{\mathrm{RTP},\lambda G}(\pi_{[0,T]}^N)$ is a non-negative martingale, by Doob's martingale inequality we can upper bound the first part by

$$\mathbb{P}_N^{\mathrm{RTP},\varrho_c}\left(\sup_{0\leq t<\delta} \mathcal{Z}_{N,t}^{\mathrm{RTP},\lambda G}(\pi_{[0,T]}^N) \geq e^{\frac{1}{8}N\lambda\varepsilon} \right) \leq \mathbb{E}_N^{\mathrm{RTP},\varrho_c}\left[\mathcal{Z}_{N,\delta}^{\mathrm{RTP},\lambda G} \right] \cdot e^{-\frac{1}{8}N\lambda\varepsilon} = e^{-\frac{1}{8}N\lambda\varepsilon}.$$

(6.3.37)

For the second part, using (6.3.17), we are able to find the following upper bound for the integrand

$$\frac{1}{N} e^{-N\langle \pi_s^N, \lambda G_s\rangle}(\partial_s + \mathscr{L}_N^{\mathrm{RTP}}) e^{N\langle \pi_s^N, \lambda G_s\rangle} \leq \frac{1}{N}|\eta^N| \left(\lambda ||\partial_x G||_\infty + M\left(e^{\lambda ||\widetilde{G}||_\infty} - 1 \right) \right).$$

(6.3.38)

Therefore, using that $|\eta^N|$ is Poisson distributed with parameter $N\varrho_c$ under $\mathbb{P}_N^{\mathrm{RTP},\varrho_c}$, we find that by the Markov inequality,

$$\mathbb{P}_N^{\mathrm{RTP},\varrho_c}\left(\sup_{0\le t<\delta}\frac{1}{N}\int_0^t e^{-N\langle\pi_s^N,\lambda G_s\rangle}(\partial_s+\mathscr{L}_N^{\mathrm{RTP}})e^{N\langle\pi_s^N,\lambda G_s\rangle}\,\mathrm{d}s\ge\tfrac{1}{8}\lambda\varepsilon\right)=\mathcal{O}(\delta). \tag{6.3.39}$$

Combining (6.3.37) and (6.3.39), we find that

$$\lim_{\delta\to 0}\lim_{N\to\infty}\frac{1}{N}\log\mathbb{P}_N^{\mathrm{RTP},\varrho_c}\left(\sup_{|s-t|<\delta}\left|\left\langle\pi_t^N,G\right\rangle-\left\langle\pi_s^N,G\right\rangle\right|\ge\varepsilon\right)=-\frac{1}{8}\lambda\varepsilon, \tag{6.3.40}$$

and since we can take λ arbitrarily large, this concludes the proof. □

We are now ready to give a proof of the exponential tightness.

Proof of Theorem 6.3. We start by defining the following set

$$\mathcal{E}_K=\left\{\widehat{\alpha}\in D([0,T];\mathcal{M}_V):\ \sup_{t\in[0,T]}\widehat{\alpha}_t(V)\le K\right\}. \tag{6.3.41}$$

For this set, by the Chernoff inequality, we find that

$$\mathbb{P}_N^{\varrho}(\pi_{[0,T]}^N\notin\mathcal{E}_K)=\mathbb{P}_N^{\varrho}\left(\sup_{t\in[0,T]}\pi_t^N(V)>K\right)\le e^{-NK}\mathbb{E}_N^{\varrho}\left[\exp\left(\sup_{t\in[0,T]}N\pi_t^N(V)\right)\right], \tag{6.3.42}$$

where the expectation can be upper bounded in the following way

$$\mathbb{E}_N^{\varrho}\left[\exp\left(\sup_{t\in[0,T]}N\pi_t^N(V)\right)\right]=\mathbb{E}_N^{\varrho}\left[\exp\left(|\eta^N|\right)\right]\le e^{N||\varrho||_\infty(e-1)}. \tag{6.3.43}$$

Combining (6.3.42) and (6.3.43), we can find a sequence of numbers $(K_n)_{n\in\mathbb{N}}$ such that for every $n\in\mathbb{N}$

$$\varlimsup_{N\to\infty}\frac{1}{N}\log\mathbb{P}_N^{\varrho}(\pi_{[0,T]}^N\notin\mathcal{E}_{K_n})\le -n \tag{6.3.44}$$

Next we consider a countable uniformly dense family $\{\phi_j\}_{j\in\mathbb{N}}\subset C^\infty(V)$ and define for each $\delta>0$ and $\varepsilon>0$ the following set

$$\mathcal{C}_{j,\delta,\varepsilon}=\left\{\widehat{\alpha}\in D([0,T];\mathcal{M}_V):\ \sup_{|t-s|<\delta}\left|\langle\widehat{\alpha}_t,\phi_j\rangle-\langle\widehat{\alpha}_s,\phi_j\rangle\right|\le\varepsilon\right\}. \tag{6.3.45}$$

Note that for any choice of the parameters the set $\mathcal{C}_{j,\delta,\varepsilon}$ is closed, and by Lemma 6.3 there exists a $\delta = \delta(j,m,n)$ such that

$$\mathbb{P}_N^{\varrho}(\pi_{[0,T]}^N \notin \mathcal{C}_{j,\delta,1/m}) \leq \exp(-Nnmj) \tag{6.3.46}$$

for large enough N, and so

$$\mathbb{P}_N^{\varrho}\left(\pi_{[0,T]}^N \notin \bigcap_{j\geq 1, m\geq 1} \mathcal{C}_{j,\delta(j,m,n),1/m}\right) \leq \sum_{j\geq 1, m\geq 1} \exp(-Nnmk) \leq C\exp(-Nn), \tag{6.3.47}$$

for some constant $C > 0$. By now considering the set

$$\mathcal{K}_n = \mathcal{E}_K \cap \bigcap_{j\geq 1, m\geq 1} \mathcal{C}_{j,\delta(j,m,n),1/m}, \tag{6.3.48}$$

it follows that (6.3.19) holds for this choice of $\mathcal{K}_n$. Since we furthermore know that it is closed we only need to show that it is relatively compact, which can be done by proving the following two things [61, Proposition 4.1.2]:

1. $\{\widehat{\alpha}_t : \widehat{\alpha} \in \mathcal{K}_n, t \in [0,T]\}$ is relatively compact in $\mathcal{M}_V$.
2. $\lim_{\delta\to 0} \sup_{\widehat{\alpha}\in\mathcal{K}_n} w_\delta(\widehat{\alpha}) = 0$, where

$$w_\delta(\widehat{\alpha}) := \sup_{|t-s|\leq\delta} \sum_{k=1}^{\infty} \frac{1}{2^k} \left(1 \wedge |\langle\widehat{\alpha}_t, \phi_j\rangle - \langle\widehat{\alpha}_s, \phi_j\rangle|\right) = 0. \tag{6.3.49}$$

Here item 1. is satisfied since $\mathcal{K}_n \subset \mathcal{E}_{K_n}$ and closed balls are compact in $\mathcal{M}_V$, and 2. follows from the definition of the sets $C_{j,\delta,\varepsilon}$. □

6.3.3 *Lower bound*

In this section we will prove the large deviation lower bound, as given in (6.3.9). The main idea is to show that if $\mathcal{I}^{\varrho}(\widehat{\alpha}) < \infty$, then there exists a function H such that

$$\mathcal{I}^{\varrho}(\widehat{\alpha}) = \lim_{N\to\infty} \frac{1}{N} \mathbb{E}_N^{\widehat{\alpha}_0}\left[\frac{d\mathbb{P}_N^{\widehat{\alpha}_0,H}}{d\mathbb{P}_N^{\varrho}}\right]. \tag{6.3.50}$$

To achieve this, we have to show two things. First we have to show that if $h_0^{\varrho}(\widehat{\alpha}_0) < \infty$, then $\widehat{\alpha}_0$ has a density $\widehat{\varrho}_0$, and h_0^{ϱ} can be written as the relative entropy of product Poisson distributions with the respective densities ϱ and $\widehat{\varrho}_0$. After that, we have to show that if $\mathcal{I}_{tr}(\widehat{\alpha}) < \infty$, then there exists a measurable

$H : [0,T] \times V \to \mathbb{R}$ such that $\widehat{\alpha}$ satisfies the hydrodynamic equation of the weakly perturbed model, given in (6.2.18), and that H is then a function for which the supremum in the definition of $\mathcal{I}_{tr}$ in (6.3.4) is attained.

The first step follows from the following Lemma

Lemma 6.4. If $h_0^{\varrho}(\widehat{\alpha}_0) < \infty$, then $\widehat{\alpha}_0$ has a density $\widehat{\varrho}_0 : V \to \mathbb{R}$, and

$$h_0^{\varrho}(\widehat{\alpha}_0) = \lim_{N\to\infty} \frac{1}{N} \mathbb{E}_{\mu_N^{\widehat{\varrho}_0}} \left[\log \frac{d\mu_N^{\widehat{\varrho}_0}}{d\mu_N^{\varrho}} \right]. \tag{6.3.51}$$

Proof. Assume $h_0^{\varrho}(\widehat{\alpha}_0) < \infty$ If $\widehat{\alpha}_0$ is not absolutely continuous, then there exists a Borel set $A \subset V$ such that $\widehat{\alpha}_0(A) > 0$ and $\lambda_V(A) = 0$, with λ_V the Lebesgue measure on V. By the definition of $h_0^{\varrho}(\widehat{\alpha}_0)$ in (6.3.3), we have that for $\phi \in C^\infty(V)$

$$\langle \widehat{\alpha}_0, \phi \rangle \leq h_0^{\varrho}(\widehat{\alpha}_0) + \langle \varrho, e^{\phi} - 1 \rangle_{L^2(V)}. \tag{6.3.52}$$

For $n \in \mathbb{N}$ now take a sequence $(\phi_k^{(n)})_{k\in\mathbb{N}} \subset C^\infty(V)$ such that $\phi_k^{(n)} \to n\mathbb{1}_A$ pointwise as $k \to \infty$. It then follows that

$$\langle \widehat{\alpha}_0, \phi_k^{(n)} \rangle \to n\widehat{\alpha}_0(A), \qquad \langle \varrho, e^{\phi_k^{(n)}} - 1 \rangle_{L^2(V)} \to 0, \tag{6.3.53}$$

as $k \to \infty$. By taking n large enough this contradicts (6.3.52), hence we can conclude that $\widehat{\alpha}_0$ has a density $\widehat{\varrho}_0$.

The rest of the proof of (6.3.51) follows then from calculating the supremum.

$$\begin{aligned} h_0^{\varrho}(\widehat{\alpha}_0) &= \sup_{\phi} \left\{ \langle \widehat{\varrho}_0, \phi \rangle_{L^2(V)} - \langle \varrho, e^{\phi} - 1 \rangle_{L^2(V)} \right\} \\ &= \langle \widehat{\varrho}_0, \log(\tfrac{\widehat{\varrho}_0}{\varrho}) \rangle_{L^2(V)} - \langle \widehat{\varrho}_0 - \varrho, 1 \rangle_{L^2(V)} \\ &= \lim_{N\to\infty} \frac{1}{N} \mathbb{E}_{\mu_N^{\widehat{\varrho}_0}} \left[\log \frac{d\mu_N^{\widehat{\varrho}_0}}{d\mu_N^{\varrho}} \right]. \end{aligned} \tag{6.3.54}$$

where the supremum is attained for $\phi = \log(\frac{\widehat{\varrho}_0}{\varrho})$. □

By a similar argument as in the previous lemma, we can show that if $\mathcal{I}_{tr}(\widehat{\alpha}) < \infty$ then there exists a density $\widehat{\varrho} : [0,T] \times V \to \mathbb{R}$ for the whole trajectory $\widehat{\alpha}$. In the following Lemma we prove an alternative formula for the dynamic part of the rate function in the case that $\widehat{\varrho}_t(v) > 0$ for all $t \in [0,T], v \in V$.

Lemma 6.5. If $\mathcal{I}_{tr}(\widehat{\alpha}) < \infty$ and $\widehat{\varrho} > 0$ then there exists a bounded measurable function H such that $\widehat{\alpha}$ satisfies the equation (6.2.18) in the weak sense. Furthermore, $\mathcal{I}_{tr}(\widehat{\alpha}) = \mathcal{I}_{tr}(\widehat{\alpha}; H)$ and

$$\mathcal{I}_{tr}(\widehat{\alpha}) = \int_0^T \left\langle \widehat{\alpha}_t, \left(e^{-\sigma \widetilde{H}_t(x)} (-\sigma \widetilde{H}_t(x) - 1) + 1 \right) c(\sigma, m(\widehat{\alpha}_t)) \right\rangle dt. \tag{6.3.55}$$

Proof. By the definition of $\mathcal{I}_{tr}$ in (6.3.4), we have the following

$$\sup_{\substack{G\in C^\infty([0,T]\times V)\\ ||G||_\infty\leq 1}} \left\{ \ell(\widehat{\alpha};G) - \int_0^T \left\langle \widehat{\alpha}_t, c(\sigma, m(\widehat{\alpha}_t)) \left(e^{-\sigma \widetilde{G}_t(x)} - 1 \right) \right\rangle \mathrm{d}t \right\} \leq \mathcal{I}_{tr}(\widehat{\alpha}) \tag{6.3.56}$$

and so

$$\sup_{\substack{G\in C^\infty([0,T]\times V)\\ ||G||_\infty\leq 1}} \ell(\widehat{\alpha};G) \leq \mathcal{I}_{tr}(\widehat{\alpha}) + \int_0^T \langle \widehat{\alpha}_t, c(\sigma, m(\widehat{\alpha}_t))\,(e-1)\rangle \,\mathrm{d}t < \infty. \tag{6.3.57}$$

Consequently, by the Hahn-Banach theorem, we can extend the linear functional $\ell(\widehat{\alpha};\cdot)$ to a bounded linear functional in $C([0,T]\times V)$. Therefore, by the Riesz representation theorem, there exists a signed measure $\nu \in \mathcal{M}_{[0,T]\times V}$ such that

$$\ell(\widehat{\alpha};G) = \langle \nu, G\rangle := \int_{[0,T]\times V} G d\nu. \tag{6.3.58}$$

Again, since we assume that $\mathcal{I}_{tr}(\widehat{\alpha}) < \infty$, this measure ν has a density $g : [0,T]\times V \to \mathbb{R}$. By then plugging in the definition of $\ell(\widehat{\alpha};G)$ in (6.3.6), we find that $\widehat{\alpha}$ satisfies

$$\langle \widehat{\alpha}_T, G_T\rangle - \langle \widehat{\alpha}_0, G_0\rangle - \int_0^T \langle \widehat{\alpha}_t, (\partial_t + \sigma\partial_x)G_t\rangle \,\mathrm{d}t = \int_0^T \langle g_t, G_t\rangle_{L^2(V)} \,\mathrm{d}t, \tag{6.3.59}$$

i.e., it satisfies the following PDE in the weak sense

$$\dot{\widehat{\alpha}}_t(x,\sigma) = -\sigma\partial_x \widehat{\alpha}_t(x,\sigma) + g_t(x,\sigma). \tag{6.3.60}$$

We can now split up $g_t(x,\sigma) = \sigma f_t(x) + h_t(x)$, and we will show that $h \equiv 0$ almost everywhere. To see this, note that

$$\begin{aligned} \mathcal{I}_{tr}(\widehat{\alpha}) = \sup_G \Bigg\{ &\int_0^T \langle \sigma f_t, G_t\rangle_{L^2(V)} \,\mathrm{d}t + \int_0^T \langle h_t, G_t\rangle_{L^2(V)} \,\mathrm{d}t \\ &- \int_0^T \left\langle \widehat{\alpha}_t, c(\sigma, m(\widehat{\alpha}_t)) \left(e^{-\sigma\widetilde{G}_t(x)} - 1\right)\right\rangle \mathrm{d}t \Bigg\} \\ = \sup_{\widetilde{G},\overline{G}} \Bigg\{ &\int_0^T \langle f_t, \widetilde{G}_t\rangle_{L^2(\mathbb{T})} \,\mathrm{d}t + \int_0^T \langle h_t, \overline{G}_t\rangle_{L^2(\mathbb{T})} \,\mathrm{d}t \\ &- \int_0^T \left\langle \widehat{\alpha}_t, c(\sigma, m(\widehat{\alpha}_t)) \left(e^{-\sigma\widetilde{G}_t(x)} - 1\right)\right\rangle \mathrm{d}t \Bigg\} \end{aligned} \tag{6.3.61}$$

where $\overline{G}_t(x) = G_t(x,1) + G_t(x,-1)$. By considering functions where $G_t(x,1) = G_t(x,-1)$, it follows that if $h \not\equiv 0$ almost everywhere then the last supremum is infinite, which contradicts $\mathcal{I}^{\varrho}(\widehat{\alpha}) < \infty$. Therefore,

$$\dot{\widehat{\alpha}}_t(x,\sigma) = -\sigma\partial_x \widehat{\alpha}_t(x,\sigma) + \sigma f_t(x) \tag{6.3.62}$$

holds weakly, with $f : [0,T] \times \mathbb{T} \to \mathbb{R}$ some bounded measurable function.

Setting $\Psi_t(x,\sigma) = e^{-\sigma \widetilde{H}_t(x)}$ in (6.2.18), we solve the following equation for Ψ_t,

$$\sigma f_t(x) = \frac{1}{\Psi_t(x,\sigma)} c(-\sigma, m(\widehat{\alpha}_t))\widehat{\varrho}_t(x,-\sigma) - \Psi_t(x,\sigma)c(\sigma, m(\widehat{\alpha}_t))\widehat{\varrho}_t(x,\sigma) \quad (6.3.63)$$

This is a quadratic equation in Ψ_t, with the following positive and bounded solution

$$\Psi_t(x,\sigma) = \frac{-\sigma f_t(x) + \sqrt{f_t(x)^2 + 4\widehat{\varrho}_t(x,\sigma)c(\sigma,m(\widehat{\alpha}_t))\widehat{\varrho}_t(x,-\sigma)c(-\sigma,m(\widehat{\alpha}_t))}}{2\widehat{\varrho}_t(x,\sigma)c(\sigma,m(\widehat{\alpha}_t))}. \quad (6.3.64)$$

It is a straightforward calculation to show that $\Psi(x,\sigma) \cdot \Psi(x,-\sigma) = 1$, and so (6.2.18) holds for $\widetilde{H}_t(x) = -\log(\Psi(x,1))$.

Now, assuming that $\widehat{\alpha}$ satisfies (6.2.18) we find that

$$\begin{aligned}
\mathcal{I}_{tr}(\widehat{\alpha}) &= \sup_G \left\{ \ell(\widehat{\alpha};G) - \int_0^T \langle \widehat{\alpha}_t, \left(e^{-\sigma\widetilde{G}_t(x)} - 1\right) c(\sigma,m(\widehat{\alpha}_t))\rangle \,\mathrm{d}t \right\} \\
&= \sup_G \left\{ \int_0^T \left\langle \widehat{\alpha}_t, \left(-e^{-\sigma\widetilde{H}_t(x)}\sigma\widetilde{G}_t(x) - e^{-\sigma\widetilde{G}_t(x)} + 1\right) c(\sigma,m(\widehat{\alpha}_t)) \right\rangle \mathrm{d}t \right\} \\
&= \int_0^T \left\langle \widehat{\alpha}_t, \sup_{p\in\mathbb{R}} \left(-e^{-\sigma\widetilde{H}_t(x)}\sigma p - e^{-\sigma p} + 1\right) c(\sigma,m(\widehat{\alpha}_t)) \right\rangle \mathrm{d}t, \qquad (6.3.65)
\end{aligned}$$

where we can interchange the supremum in the last integral by dominated convergence. This supremum is attained for $p = \widetilde{H}_t(x)$, indeed showing that $\mathcal{I}_{tr}(\widehat{\alpha}) = \mathcal{I}_{tr}(\widehat{\alpha};H)$. After filling this in, it follows that (6.3.55) holds. □

We now have a clear formulation of the rate function when $\widehat{\alpha}$ has positive density, however if the density can be zero then the formulation in (6.3.64) is not well-defined. Furthermore, in order for the hydrodynamic limit of the weakly perturbed model to hold in Theorem 6.1, we need to assume that $H \in C^{1,0}([0,T] \times V)$. We therefore define the following space

$$\mathcal{D} := \left\{ \widehat{\alpha} \in D([0,T];\mathcal{M}_V) : \widehat{\varrho} > 0,\ \widehat{\alpha} \text{ satisfies (6.2.18) with } H \in C^{1,0}([0,T] \times V) \right\}. \quad (6.3.66)$$

We will now show that the rate function of trajectories outside this set can be approximated by the rate function of trajectories within this set.

Lemma 6.6. Let $\widehat{\alpha} \in D([0,T];\mathcal{M}_V)$ such $\mathcal{I}_{tr}(\widehat{\alpha}) < \infty$, then there exists a sequence $(\widehat{\alpha}_k)_{k\in\mathbb{N}} \subset \mathcal{D}$ such that $\widehat{\alpha}_k \to \widehat{\alpha}$ weakly and

$$\mathcal{I}_{tr}(\widehat{\alpha}) = \lim_{k\to\infty} \mathcal{I}_{tr}(\widehat{\alpha}_k). \quad (6.3.67)$$

Proof. We first show that $\widehat{\alpha} \in D([0,T];\mathcal{M}_V)$ can be approximated by trajectories with positive density. We define the following measure for any $\varepsilon > 0$

$$\widehat{\alpha}_\varepsilon = (1-\varepsilon)\widehat{\alpha} + \varepsilon 1 \tag{6.3.68}$$

where 1 on the right-hand side denotes the measure with constant density equal to 1. It follows that $\widehat{\alpha}_\varepsilon$ has positive density and that $\widehat{\alpha}_\varepsilon \to \widehat{\alpha}$ weakly as $\varepsilon \to 0$. Therefore, by convexity and lower semicontinuity of the rate function $\mathcal{I}^\varrho$, we then find that

$$\overline{\lim_{\varepsilon\to 0}} \mathcal{I}_{tr}(\widehat{\alpha}_\varepsilon) \leq \mathcal{I}_{tr}(\widehat{\alpha}) \leq \underline{\lim}_{\varepsilon\to 0} \mathcal{I}_{tr}(\widehat{\alpha}_\varepsilon). \tag{6.3.69}$$

Hence we have indeed found a good approximation.

Now assume that $\widehat{\alpha} \in D([0,T];\mathcal{M}_V)$ has density $\widehat{\varrho} > 0$ and that $\mathcal{I}_{tr}(\widehat{\alpha}) < \infty$. By Lemma 6.5, there exists a bounded measurable $H : [0,T] \times V \to \mathbb{R}$ such that (6.2.18) holds weakly. Now find a sequence $\widehat{\alpha}_k$ with densities $\widehat{\varrho}_k \in C^{2,1}([0,T] \times V)$ such that $\widehat{\varrho}_k \to \widehat{\varrho}$ pointwise as $k \to \infty$. It then follows that $\widehat{\alpha}_k \to \widehat{\alpha}$ weakly and, by (6.3.62) and (6.3.64), each $\widehat{\alpha}_k$ satisfies (6.2.18) for some function $H_k \in C^{1,0}([0,T] \times V)$ where $H_k \to H$ pointwise. By the formulation of $\mathcal{I}_{tr}(\widehat{\alpha})$ in (6.3.55), we can then indeed conclude that (6.3.67) holds. □

Theorem 6.5. Given a $\widehat{\alpha} \in D([0,T];\mathcal{M}_V)$, for every neighborhood $\mathcal{O} \subset D([0,T];\mathcal{M}_V)$ of $\widehat{\alpha}$ we have that

$$\underline{\lim}_{N\to\infty} \frac{1}{N} \log \mathbb{P}_N^\varrho(\pi_{[0,T]}^N \in \mathcal{O}) \geq -\mathcal{I}^\varrho(\widehat{\alpha}). \tag{6.3.70}$$

As a consequence, for every open set $\mathcal{O} \subset D([0,T];\mathcal{M}_V)$ we have that

$$\underline{\lim}_{N\to\infty} \frac{1}{N} \log \mathbb{P}_N^\varrho(\pi_{[0,T]}^N \in \mathcal{O}) \geq -\inf_{\widehat{\alpha}\in\mathcal{O}} \mathcal{I}^\varrho(\widehat{\alpha}). \tag{6.3.71}$$

Proof. If $\mathcal{I}^\varrho(\widehat{\alpha}) = \infty$, the result is immediate, therefore we can assume that $\mathcal{I}^\varrho(\widehat{\alpha}) < \infty$. By Lemma 6.6 it is then enough to prove it for $\widehat{\alpha} \in \mathcal{D}$, and so by Theorem 6.2 there exists an $H \in C^{1,0}([0,T] \times V)$ such that

$$\mathbb{P}_N^{\widehat{\varrho}_0,H}(\pi_{[0,T]}^N \in \cdot) \to \delta_{\widehat{\alpha}}. \tag{6.3.72}$$

where $\widehat{\varrho}_0$ is the density of $\widehat{\alpha}_0$. Therefore, we have that

$$\begin{aligned} \underline{\lim}_{N\to\infty} \frac{1}{N} \log \mathbb{P}_N^\varrho(\pi_{[0,T]}^N \in \mathcal{O}) &= -\underline{\lim}_{N\to\infty} \frac{1}{N} \log \mathbb{E}_N^{\widehat{\varrho}_0,H}\left[\mathbb{1}_{\pi_{[0,T]}^N\in\mathcal{O}} \frac{d\mu_N^{\widehat{\varrho}_0}}{d\mu_N^\varrho} \cdot \frac{d\mathbb{P}_N^{\widehat{\varrho}_0,H}}{d\mathbb{P}_N^{\widehat{\varrho}_0}}\right] \\ &\geq -\underline{\lim}_{N\to\infty} \frac{1}{N} \mathbb{E}_N^{\widehat{\varrho}_0,H}\left[\log \frac{d\mu_N^{\widehat{\varrho}_0}}{d\mu_N^\varrho} \cdot \frac{d\mathbb{P}_N^{\widehat{\varrho}_0,H}}{d\mathbb{P}_N^{\widehat{\varrho}_0}}\right] \\ &= -\mathcal{I}^\varrho(\widehat{\alpha}), \end{aligned} \tag{6.3.73}$$

where we used Lemma 6.4, Corollary 6.3 and that $\mathcal{I}_{tr}(\widehat{\alpha}) = \mathcal{I}_{tr}(\widehat{\alpha}; H)$. □

6.4 PROOF OF THE HYDRODYNAMIC LIMIT

In this section we prove the hydrodynamic limit of the weakly perturbed process defined in Section 6.2.2. We first prove that the PDE given in (6.2.18) is well-posed, and afterwards we prove Theorem (6.2).

6.4.1 *Well-posedness of the PDE*

The approach is to define a sequence of densities $\varrho^{(n)}$ through the following incursive relation

$$\partial_t \varrho_t^{(n+1)}(x,\sigma) = -\sigma \partial_x \varrho_t^{(n+1)}(x,\sigma) + c(-\sigma, m(\varrho_t^{(n)}))e^{\sigma H_t(x)} \varrho_t^{(n+1)}(x,-\sigma) \\ - c(\sigma, m(\varrho_t^{(n)}))e^{-\sigma H_t(x)} \varrho_t^{(n+1)}(x,\sigma), \quad (6.4.1)$$

where every $\varrho^{(n)}$ starts from $\varrho_0^{(n)} = \varrho$. Setting $\mathcal{T}(\varrho^{(n)}) = \varrho^{(n+1)}$, it is enough to show that $\mathcal{T}$ is a contraction (up to some finite time $T > 0$) in the space $L^\infty([0,T]; L^1(V))$. First note that for every trajectory $\varrho^{(n)}$, the trajectory $\varrho^{(n+1)}$ solving 6.4.1 satisfies the conservation of particles, hence for any $t \geq 0$ we have that $||\varrho_t^{(n+1)}||_{L^1(V)} = ||\varrho||_{L^1(V)}$ with ϱ the initial profile. Therefore, we indeed have that $\mathcal{T} : L^\infty([0,T]; L^1(V)) \to L^\infty([0,T]; L^1(V))$. Now let ϱ_1, ϱ_2 be any two trajectories of densities, and denote the following

$$\begin{aligned} \psi_1 &= \mathcal{T}(\varrho_1), & \psi_2 &= \mathcal{T}(\varrho_2), \\ m_{1,t} &= m(\varrho_{1,t}), & m_{2,t} &= m(\varrho_{2,t}). \end{aligned} \quad (6.4.2)$$

Furthermore, denote $\delta = \psi_1 - \psi_2$ and $\varepsilon = m_1 - m_2$. For $[m] = \{m_t : t \in [0,T]\}$ a deterministic trajectory of magnetizations, we define the mapping

$$Q_{[m]}[\psi](x,\sigma,t) = c(-\sigma, m_t)e^{\sigma H_t(x)}\psi_t(x,-\sigma) - c(\sigma, m_t)e^{-\sigma H_t(x)}\psi_t(x,\sigma). \quad (6.4.3)$$

Note that this mapping is linear in ψ. We then have that

$$\partial_t \delta_t(x,\sigma) = -\sigma \partial_x \delta_t(x,\sigma) + Q_{[m_1]}[\psi_1](x,\sigma,t) - Q_{[m_2]}[\psi_2](x,\sigma,t). \quad (6.4.4)$$

We can rewrite the difference of the last two terms in a linear and non-linear part as follow

$$Q_{[m_1]}[\psi_1] - Q_{[m_2]}[\psi_2] = Q_{[m_1]}[\psi_1 - \psi_2] + \left(Q_{[m_1]}[\psi_2] - Q_{[m_2]}[\psi_2]\right). \quad (6.4.5)$$

For the linear part, we have that

$$||Q_{[m_1]}[\delta](\cdot,\cdot,t)||_{L^1(V)} = \left|\left|c(-\sigma, m_{1,t})e^{\sigma H_t(x)}\delta_t(x,-\sigma) - c(\sigma, m_{1,t})e^{-\sigma H_t(x)}\delta_t(x,\sigma)\right|\right|_{L^1(V)} \\ \leq C_1 ||\delta_t||_{L^1(V)}, \quad (6.4.6)$$

with C_1 some constant depending on the bounded functions c and H. For the non-linear part, we have that

$$\begin{aligned} Q_{[m_1]}[\psi_2](x,\sigma,t) - Q_{[m_2]}[\psi_2](x,\sigma,t) \\ = [c(-\sigma, m_{1,t}) - c(-\sigma, m_{2,t})]e^{\sigma H_t(x)}\psi_{2,t}(x,-\sigma) \\ - [c(\sigma, m_{1,t}) - c(\sigma, m_{2,t})]e^{-\sigma H_t(x)}\psi_{2,t}(x,\sigma). \end{aligned} \tag{6.4.7}$$

By the Lipschitz-continuity of $c(\sigma, m)$ we now have that

$$||Q_{[m_1]}[\psi_2](\cdot,\cdot,t) - Q_{[m_2]}[\psi_2](\cdot,\cdot,t)||_{L^1(V)} \leq L|\varepsilon_t| \cdot C_2||\psi_{2,t}||_{L^1(V)}, \tag{6.4.8}$$

with L the Lipschitz-constant and C_2 some constant depending on H. Here $||\psi_{2,t}||_{L^1(V)} = ||\varrho||_{L^1(V)}$ by conservation of particles. Furthermore, we have that

$$|\varepsilon_t| = |m_{1,t} - m_{2,t}| = \frac{1}{||\varrho||_{L^1(V)}} \cdot ||\varrho_{1,t} - \varrho_{2,t}||_{L^1(V)}. \tag{6.4.9}$$

From (6.4.6) and (6.4.8), we find that

$$\begin{aligned} \partial_t||\delta_t||_{L^1(V)} &\leq ||Q_{[m_1]}[\delta](\cdot,\cdot,t)||_{L^1(V)} + ||Q_{[m_1]}[\psi_2](\cdot,\cdot,t) - Q_{[m_2]}[\psi_2](\cdot,\cdot,t)||_{L^1(V)} \\ &\leq C_1||\delta_t||_{L^1(V)} + LC_2||\varrho_{1,t} - \varrho_{2,t}||_{L^1(V)} \\ &\leq C_1||\delta_t||_{L^1(V)} + LC_2||\varrho_1 - \varrho_2||_{L^\infty([0,T];L^1(V))}. \end{aligned} \tag{6.4.10}$$

Note that the transport term vanishes since $\int_{\mathbb{T}} \partial_x|\delta_t(x,\sigma)|dx = 0$ by periodic boundary conditions. We can rewrite this in integral form as

$$||\delta_t||_{L^1(V)} \leq \int_0^t C_1||\delta_s||_{L^1(V)}\, \mathrm{d}s + LC_2 t||\varrho_1 - \varrho_2||_{L^\infty([0,T];L^1(V))}. \tag{6.4.11}$$

By Gronwall's inequality, we find that

$$||\delta_t||_{L^1(V)} \leq LC_2 t e^{C_1 t}||\varrho_1 - \varrho_2||_{L^\infty([0,T];L^1(V))}, \tag{6.4.12}$$

and so we can conclude that

$$||\mathcal{T}(\varrho_1) - \mathcal{T}(\varrho_2)||_{L^\infty([0,T];L^1(V))} \leq LC_2 T e^{C_1 T}||\varrho_1 - \varrho_2||_{L^\infty([0,T];L^1(V))}. \tag{6.4.13}$$

Taking $T > 0$ small enough, this is a contraction on $L^\infty([0,T];L^1(V))$, hence the sequence $\varrho^{(n)}$ converges locally to a unique solution of the PDE (**??**).

6.4.2 *Proof of Theorem* (6.2)

Recall that $\alpha^H \in D([0,T];\mathcal{M}_V)$ denotes the process such that for every $t \in [0,T]$ the measure α_t^H has density $\varrho_t^H(x,\sigma)$, which is the solution the the PDE given by

(6.2.18). In this way, α^H is the unique trajectory of measures measure such that for every $G \in C^\infty([0,T]\times V)$ and $t \in [0,T]$ we have that

$$\mathscr{M}_t^{H,G}(\alpha^H) := \left\langle \alpha_t^H, G_t \right\rangle - \left\langle \alpha_0^H, G_0 \right\rangle - \int_0^t \left\langle \alpha_s^H, \left(\partial_s + \left(A^H_{s,\alpha^H}\right)^*\right) G_s \right\rangle ds = 0, \tag{6.4.14}$$

where $\left(A^H_{s,\alpha^H}\right)^*$ is the differential operator given by

$$\left(A^H_{s,\alpha^H}\right)^* G_s(x,\sigma) = \sigma\partial_x G_s(x,\sigma) + c(\sigma, m(\alpha_s^H))e^{-\sigma\widetilde{H}_s(x)}\big(G_s(x,-\sigma) - G_s(x,\sigma)\big), \tag{6.4.15}$$

which is the action of the adjoint of A^H_{s,α^H} on smooth functions, with A^H_{s,α^H} given by

$$\begin{aligned} A^H_{s,\alpha^H} G(x,\sigma) = -\sigma\partial_x G(x,\sigma) + c(-\sigma, m(\alpha_s^H))e^{\sigma\widetilde{H}_s(x)}G(x,-\sigma) \\ - c(\sigma, m(\alpha_s^H))e^{-\sigma\widetilde{H}_s(x)}G(x,\sigma). \end{aligned} \tag{6.4.16}$$

For a given $G \in C^\infty([0,T]\times V)$ we define the Dynkin Martingale

$$\mathscr{M}_{N,t}^{H,G}(\pi^N_{[0,T]}) := \langle \pi_t^N, G_t\rangle - \langle \pi_0^N, G_0\rangle - \int_0^t (\partial_s + \mathscr{L}^H_{N,s})\langle \pi_s^N, G_s\rangle\, ds. \tag{6.4.17}$$

Lemma 6.7. For every $G \in C^\infty(V)$ we have

$$\lim_{N\to\infty} \mathbb{E}_N^{\varrho,H}\left[\sup_{t\in[0,T]} \left|\mathscr{M}_{N,t}^{H,G}(\pi^N_{[0,T]}) - \mathscr{M}_t^{H,G}(\pi^N_{[0,T]})\right|\right] = 0. \tag{6.4.18}$$

Proof. Note that

$$\mathscr{M}_{N,t}^{H,G}(\pi^N_{[0,T]}) - \mathscr{M}_t^{H,G}(\pi^N_{[0,T]}) = \int_0^t \left(\mathscr{L}^H_{N,s}\langle \pi_s^N, G_s\rangle - \left\langle \pi_s^N, \left(A^H_{s,\pi^N_{[0,T]}}\right)^* G_s \right\rangle\right) ds. \tag{6.4.19}$$

So we need to calculate $\mathscr{L}^H_{N,s}\langle \pi_s^N, G_s\rangle$. In order to do that, we start with the preliminary calculation

$$\begin{aligned} &\mathscr{L}^H_{N,s}\langle \pi_s^N, G_s\rangle \\ &= N \sum_{(x,\sigma)\in V_N} \eta_s^N(x,\sigma) e^{H_s(\frac{x+\sigma}{N},\sigma) - H_s(\frac{x}{N},\sigma)} \left[\left\langle \pi^N\left((\eta_s^N)^{(x,\sigma)\to(x+\sigma,\sigma)}\right), G_s\right\rangle - \langle \pi_s^N, G_s\rangle\right] \\ &\quad + \sum_{(x,\sigma)\in V_N} \eta_s^N(x,\sigma) c(\sigma, m(\pi_s^N)) e^{-\sigma\widetilde{H}_s(x)} \left[\left\langle \pi^N\left((\eta_t^N)^{(x,\sigma)\to(x,-\sigma)}\right), G_s\right\rangle - \langle \pi_s^N, G_s\rangle\right] \\ &= \sum_{(x,\sigma)\in V_N} \eta_s^N(x,\sigma) e^{H_s(\frac{x+\sigma}{N},\sigma) - H_s(\frac{x}{N},\sigma)} \left(G_s(\tfrac{x+\sigma}{N},\sigma) - G_s(\tfrac{x}{N},\sigma)\right) \\ &\quad + \frac{1}{N}\sum_{(x,\sigma)\in V_N} \eta_s^N(x,\sigma) c(\sigma, m(\pi_s^N)) e^{-\sigma\widetilde{H}_s(x)} \big(G_s(x,-\sigma) - G_s(x,\sigma)\big), \end{aligned} \tag{6.4.20}$$

Using Taylor approximation, we can now write

$$\begin{aligned}
&\mathscr{L}^H_{N,s}\langle \pi^N_s, G_s\rangle \\
&= \frac{1}{N}\sum_{(x,\sigma)\in V_N} \eta^N_s(x,\sigma)\left[\sigma\partial_x G_s(\tfrac{x}{N},\sigma) + c(\sigma, m(\pi^N_s))e^{-\sigma\widetilde{H}_s(x)}\big(G_s(x,-\sigma)-G_s(x,\sigma)\big)\right] \\
&\qquad\qquad + R(N,H,G,s) \\
&= \left\langle \pi^N_s, \left(A^H_{s,\pi^N_{[0,T]}}\right)^* G_s \right\rangle + R(N,H,G,s),
\end{aligned} \tag{6.4.21}$$

where

$$\begin{aligned}
&R(N,H,G,s) \\
&\le \frac{1}{N}\sum_{(x,\sigma)\in V_N} \eta(x,\sigma)||\partial_x G||_\infty \sup_{(y,\sigma')\in V_N}\left|H_s(\tfrac{y+\sigma'}{N},\sigma') - H_s(\tfrac{y}{N},\sigma')\right| \\
&\quad + \frac{1}{N^2}\sum_{(x,\sigma)\in V_N} \eta(x,\sigma)||\partial_{xx}G||_\infty e^{2||H||_\infty} \\
&= \frac{1}{N}|\eta^N|\left(||\partial_x G||_\infty \sup_{(y,\sigma')\in V_N}\left|H_s(\tfrac{y+\sigma'}{N},\sigma') - H_s(\tfrac{y}{N},\sigma')\right| + \frac{1}{N}||\partial_{xx}G||_\infty e^{2||H||_\infty}\right).
\end{aligned} \tag{6.4.22}$$

We then conclude that

$$\begin{aligned}
&\mathbb{E}^{\varrho,H}_N\left[\sup_{t\in[0,T]}\left|\mathscr{M}^{H,G}_{N,t}(\pi^N_{[0,T]}) - \mathscr{M}^{H,G}_t(\pi^N_{[0,T]})\right|\right] \\
&= \mathbb{E}^{\varrho,H}_N\left[\sup_{t\in[0,T]}\left|\int_0^t R(N,H,G,s)\,\mathrm{d}s\right|\right] \\
&\le \frac{T}{N}\left(||\partial_x G||_\infty \sup_{(y,\sigma')\in V_N}\left|H_s(\tfrac{y+\sigma'}{N},\sigma') - H_s(\tfrac{y}{N},\sigma')\right| + \frac{1}{N}||\partial_{xx}G||_\infty e^{2||H||_\infty}\right)\mathbb{E}^{\varrho,H}_N\left[|\eta^N|\right] \\
&\to 0,
\end{aligned} \tag{6.4.23}$$

where we used that $\mathbb{E}^{\varrho,H}_N\left[|\eta^N|\right] \le ||\varrho||_\infty N$ and that H_s is continuous on $\mathbb{T}$. □

Lemma 6.8. For every $G \in C^\infty([0,T]\times V)$ we have that

$$\lim_{N\to\infty}\mathbb{E}^{\varrho,H}_N\left[\sup_{t\in[0,T]}\left(\mathscr{M}^{H,G}_{N,t}(\pi^N_{[0,T]})\right)^2\right] = 0. \tag{6.4.24}$$

Proof. By Doob's maximal inequality we find that

$$\mathbb{E}_N^{\varrho,H}\left[\sup_{t\in[0,T]}\left(\mathscr{M}_{N,t}^{H,G}(\pi_{[0,T]}^N)\right)^2\right] \le 4\mathbb{E}_N^{\varrho,H}\left[\left(\mathscr{M}_{N,T}^{H,G}(\pi_{[0,T]}^N)\right)^2\right]$$
$$= 4\mathbb{E}_N^{\varrho,H}\left[\int_0^T \Gamma_{N,t}^{H,G}(\pi_t^N)\,\mathrm{d}t\right], \tag{6.4.25}$$

where in the last equality we use that the predictable quadratic variation of the martingale $\mathscr{M}_{N,t}^{H,G}$ is given by the integral of the carré du champ operator $\Gamma_{H,t}^{N,G}$ defined by

$$\Gamma_{N,t}^{H,G}(\pi_t^N) = \mathscr{L}_{N,t}^H(\langle \pi_t^N, G_t\rangle)^2 - 2\langle \pi_t^N, G_t\rangle \cdot \mathscr{L}_{N,t}^H\langle \pi_t^N, G_t\rangle. \tag{6.4.26}$$

For a general jump process generator $Lf(\eta) = \sum_{\eta'} r(\eta,\eta')(f(\eta') - f(\eta))$ we have that

$$Lf^2(\eta) - 2f(\eta)\cdot Lf(\eta) = \sum_{\eta'} r(\eta,\eta')\big(f(\eta') - f(\eta)\big)^2, \tag{6.4.27}$$

and so we find that

$$\Gamma_{N,t}^{H,G}(\pi_t^N) = \frac{1}{N}\sum_{(x,\sigma)\in V_N} \eta_t^N(x,\sigma) e^{H_t(\frac{x+\sigma}{N},\sigma) - H_t(\frac{x}{N},\sigma)}\big(G_t(\tfrac{x+\sigma}{N},\sigma) - G_t(\tfrac{x}{N},\sigma)\big)^2$$
$$+ \frac{1}{N^2}\sum_{(x,\sigma)\in V_N} \eta_t^N(x,\sigma) c(\sigma, m(\pi_t^N)) e^{-\sigma\widetilde{H}_t(x)}\big(G_t(x,-\sigma) - G_t(x,\sigma)\big)^2. \tag{6.4.28}$$

By the mean value theorem and since $c(\sigma, m)$ is bounded, we can find an upper bound given by

$$\Gamma_{N,t}^{H,G}(\pi_{[0,T]}^N) \le \mathcal{O}(\tfrac{1}{N^2})\cdot|\eta^N|. \tag{6.4.29}$$

Using again that $\mathbb{E}_N^{\varrho,H}\left[|\eta^N|\right] \le ||\varrho||_\infty N$, we then find that

$$\mathbb{E}_N^{\varrho,H}\left[\sup_{t\in[0,T]}\left(\mathscr{M}_{N,t}^{H,G}(\pi_{[0,T]}^N)\right)^2\right] \le 4T\mathcal{O}(\tfrac{1}{N^2})\cdot \mathbb{E}_N^{\varrho,H}\left[|\eta^N|\right] \to 0. \tag{6.4.30}$$

□

Proposition 6.1. $\{\pi_{[0,T]}^N : N \in \mathbb{N}\}$ is tight in $D([0,T];\mathcal{M}_V)$.

Proof. By Aldous' criteria, we have to show the following:

B.1 For all $t \in [0,T]$ and $\varepsilon > 0$ there exists a compact $K(t,\varepsilon) \subset \mathcal{M}_V$ such that

$$\sup_{N \in \mathbb{N}} \mathbb{P}_N^{\varrho,H}\big(\pi_t^N \notin K(t,\varepsilon)\big) \leq \varepsilon. \tag{6.4.31}$$

B.2 For all $\varepsilon > 0$

$$\lim_{\delta \to 0} \limsup_{N \to \infty} \mathbb{P}_N^{\varrho,H}\big(\omega(\pi_{[0,T]}^N, \delta) \geq \varepsilon\big) = 0, \tag{6.4.32}$$

where

$$\omega(\pi_{[0,T]}^N, \delta) = \sup\{d(\pi_t^N, \pi_s^N)| : s,t \in [0,T], |t-s| < \delta\}, \tag{6.4.33}$$

with d the metric on $\mathcal{M}_V$ defined for $\alpha, \beta \in \mathcal{M}_V$ as

$$d(\alpha,\beta) = \sum_{j=1}^{\infty} 2^{-j}\Big(1 \wedge \big|\,\langle \alpha, \phi_j \rangle - \langle \beta, \phi_j \rangle\,\big|\Big). \tag{6.4.34}$$

We start with proving B.1. For every $C > 0$, we have that the set

$$K_C := \{\mu \in \mathcal{M}_V : \mu(V) \leq C\} \tag{6.4.35}$$

is compact in $\mathcal{M}_V$. Furthermore,

$$\mathbb{P}_N^{\varrho,H}(\pi_t^N \notin K_C) = \mathbb{P}_N^{\varrho,H}(\pi_t^N(V) > C) \leq \frac{1}{C}\mathbb{E}_N^{\varrho,H}[\pi_t^N(V)], \tag{6.4.36}$$

where we used the Markov inequality in the last step. Here

$$\mathbb{E}_N^{\varrho,H}[\pi_t^N(V)] = \mathbb{E}_N^{\varrho,H}\left[\frac{1}{N}\sum_{(x,\sigma)\in V_N} \eta_t^N(x,\sigma)\delta_{(\frac{x}{N},\sigma)}(V)\right] = \frac{1}{N}\mathbb{E}_N^{\varrho,H}[|\eta^N|] \leq ||\varrho||_\infty. \tag{6.4.37}$$

Therefore

$$\mathbb{P}_N^{\varrho,H}(\pi_t^N \notin K_C) \leq \frac{1}{C}||\varrho||_\infty. \tag{6.4.38}$$

Since we took C arbitrarily, we can take $C > ||\varrho||_\infty \varepsilon^{-1}$, and B.1 follows.

To prove B.2, take $\varepsilon' < \varepsilon$ and note that by the Markov inequality we have that

$$\begin{aligned}\lim_{\delta \to 0} \limsup_{N \to \infty} \mathbb{P}_N^{\varrho,H}(\omega(\pi_{[0,T]}^N, \delta) > \varepsilon) &\leq \lim_{\delta \to 0} \limsup_{N \to \infty} \frac{1}{\varepsilon}\mathbb{E}_N^{\varrho,H}\left[\omega(\pi_{[0,T]}^N, \delta)\right] \\ &\leq \lim_{\delta \to 0} \limsup_{N \to \infty} \frac{1}{\varepsilon'}\mathbb{E}_N^{\varrho,H}\left[\omega(\pi_{[0,T]}^N, \delta)\right].\end{aligned} \tag{6.4.39}$$

Now for $\omega(\pi^N_{[0,T]},\delta)$ we have that

$$\omega(\pi^N_{[0,T]},\delta) = \sup_{\substack{s,t\in[0,T]\\|t-s|<\delta}} \sum_{j=1}^{\infty} 2^{-j}\Big(1\wedge\big|\langle\pi^N_t,\phi_j\rangle - \pi^N_s(\phi_j)\big|\Big)$$
$$\leq 2^{-m} + \sup_{\substack{s,t\in[0,T]\\|t-s|<\delta}} \sum_{j=1}^{m} \big|\langle\pi^N_t,\phi_j\rangle - \pi^N_s(\phi_j)\big|, \tag{6.4.40}$$

where we took $m\in\mathbb{N}$ arbitrarily. Using the martingale $\mathscr{M}^{H,\phi_j}_{N,t}(\pi^N_{[0,T]})$, we find that

$$\begin{aligned}
&\mathbb{E}^{\varrho,H}_N\Big[\big|\langle\pi^N_t,\phi_j\rangle - \pi^N_s(\phi_j)\big|\Big]\\
&= \mathbb{E}^{\varrho,H}_N\left[\left|\mathscr{M}^{H,\phi_j}_{N,t}(\pi^N_{[0,T]}) - \mathscr{M}^{H,\phi_j}_{N,s}(\pi^N_{[0,T]}) - \int_s^t \mathscr{L}^H_{N,s}\langle\pi^N_r,\phi_j\rangle\,\mathrm{d}r\right|\right]\\
&\leq 2\mathbb{E}^{\varrho,H}_N\left[\sup_{t\in[0,T]}\left|\mathscr{M}^{H,\phi_j}_{N,t}(\pi^N_{[0,T]})\right|\right] + \mathbb{E}^{\varrho,H}_N\left[\left|\int_s^t \mathscr{L}^H_{N,s}\langle\pi^N_r,\phi_j\rangle\,\mathrm{d}r\right|\right].
\end{aligned} \tag{6.4.41}$$

By Lemma 6.8 the first expectation vanishes as $N\to\infty$. By (6.4.21) we can upper bound the second expectation by

$$\begin{aligned}
\mathbb{E}^{\varrho,H}_N\left[\left|\int_s^t \mathscr{L}^H_{N,s}\langle\pi^N_r,\phi_j\rangle\,\mathrm{d}r\right|\right] &\leq \mathbb{E}^{\varrho,H}_N\left[\left|\int_s^t \left\langle\pi^N_r,\left(A^H_{s,\pi^N_{[0,T]}}\right)^*\phi_j\right\rangle\mathrm{d}r\right|\right]\\
&\quad + \mathbb{E}^{\varrho,H}_N\left[\left|\int_s^t R(N,\phi_j,r)\,\mathrm{d}r\right|\right].
\end{aligned} \tag{6.4.42}$$

From (6.4.23) we see that the second expectation also vanishes as $N\to\infty$ (uniformly in s and t). For the first expectation note that

$$\begin{aligned}
&\mathbb{E}^{\varrho,H}_N\left[\left|\int_s^t \left\langle\pi^N_r,\left(A^H_{s,\pi^N_{[0,T]}}\right)^*\phi_j\right\rangle\mathrm{d}r\right|\right]\\
&\leq \mathbb{E}^{\varrho,H}_N\left[\left|\int_s^t \frac{1}{N}\sum_{(x,\sigma)\in V_N}\eta^N_r(x,\sigma)\left|\left|\left(A^H_{s,\pi^N_{[0,T]}}\right)^*\phi_j\right|\right|_\infty\mathrm{d}r\right|\right]\\
&\leq \frac{1}{N}\left|\left|\left(A^H_{s,\pi^N_{[0,T]}}\right)^*\phi_j\right|\right|_\infty |t-s|\cdot\mathbb{E}^{\varrho,H}_N\left[|\eta^N|\right]\\
&\leq \left|\left|\left(A^H_{s,\pi^N_{[0,T]}}\right)^*\phi_j\right|\right|_\infty ||\varrho||_\infty\delta.
\end{aligned} \tag{6.4.43}$$

Combining all of the above, we find that

$$\begin{aligned}
&\lim_{\delta\to 0}\limsup_{N\to\infty}\mathbb{P}_N^{\varrho,H}(\omega(\pi^N_{[0,T]},\delta)>\varepsilon)\\
&\quad\le \frac{1}{\varepsilon'}2^{-m}+\lim_{\delta\to 0}\limsup_{N\to\infty}\frac{1}{\varepsilon'}\mathbb{E}_N^{\varrho,H}\left[\sup_{\substack{s,t\in[0,T]\\|t-s|<\delta}}\sum_{j=1}^{m}\left|\langle\pi_t^N,\phi_j\rangle-\pi_s^N(\phi_j)\right|\right]\\
&\quad\le \frac{1}{\varepsilon'}2^{-m}+\lim_{\delta\to 0}\frac{1}{\varepsilon'}\left|\left|\left(A^H_{s,\pi^N_{[0,T]}}\right)^*\phi_j\right|\right|_\infty ||\varrho||_\infty\delta\\
&\quad=\frac{1}{\varepsilon'}2^{-m}.
\end{aligned}\tag{6.4.44}$$

Since we took m arbitrarily, we can choose it such that $2^{-m}\le(\varepsilon')^2$, i.e.,

$$\lim_{\delta\to 0}\limsup_{N\to\infty}\mathbb{P}_N^{\varrho,H}(\omega(\pi^N_{[0,T]},\delta)>\varepsilon)<\varepsilon',\tag{6.4.45}$$

and since we took ε' arbitrarily small, we indeed find that B.2 holds. □

We are now ready to give the proof of the hydrodynamic limit of the weakly perturbed model.

Proof of Theorem 6.2. By Prokhorov's theorem, the tightness of the sequence $\{\pi^N_{[0,T]}:N\in\mathbb{N}\}$ implies that the sequence is sequentially compact. If we then prove that every convergent subsequence converges to δ_α, then the theorem holds.

Take such a convergent subsequence $\mathbb{P}^{\varrho,H}_{N_k}(\pi^N_{[0,T]}\in\cdot)\to P^*$, with P^* a probability measure on $D([0,T];\mathcal{M}_V)$. For a given $\varepsilon>0$ and $G\in C^\infty(V)$, define the set

$$\Xi^{H,G}_\varepsilon=\left\{\beta\in D([0,T];\mathcal{M}_V):\sup_{t\in[0,T]}\left|\mathscr{M}^{H,G}_t(\beta)\right|\le\epsilon\right\},\tag{6.4.46}$$

which is closed in the Skorokhod topology. By Portmanteau's theorem, we now have that

$$\begin{aligned}
P^*(\Xi^{H,G}_\varepsilon)&\ge\lim_{k\to\infty}\mathbb{P}^{\varrho,H}_{N_k}(\pi^{N_k}_\cdot\in\Xi^{H,G}_\varepsilon)\\
&=\lim_{k\to\infty}\mathbb{P}^{\varrho,H}_{N_k}\left(\sup_{t\in[0,T]}\left|\mathscr{M}^{H,G}_t(\pi^{N_k}_\cdot)\right|\le\varepsilon\right)\\
&=\lim_{k\to\infty}\mathbb{P}^{\varrho,H}_{N_k}\left(\sup_{t\in[0,T]}\left|\mathscr{M}^{H,G}_{N_k,t}(\pi^{N_k}_\cdot)\right|\le\varepsilon\right),
\end{aligned}\tag{6.4.47}$$

where we used Lemma 6.7 for the last step. Now by using Chebyshev's inequality together with Lemma 6.8, we find that

$$\mathbb{P}^{\varrho,H}_{N_k}\left(\sup_{t\in[0,T]}\left|\mathscr{M}^{H,G}_{N_k,t}(\pi^{N_k}_{\cdot})\right|>\varepsilon\right)\leq\frac{1}{\varepsilon^2}\mathbb{E}^{\varrho,H}_{N_k}\left[\sup_{t\in[0,T]}\left|\mathscr{M}^{H,G}_{N_k,t}(\pi^{N_k}_{\cdot})\right|^2\right]\to 0, \quad (6.4.48)$$

and so indeed

$$P^*(\Xi^{H,G}_{\varepsilon})\geq\lim_{k\to\infty}\mathbb{P}^{\varrho,H}_{N_k}\left(\sup_{t\in[0,T]}\left|\mathscr{M}^{H,G}_{N_k,t}(\pi^{N_k}_{\cdot})\right|\leq\varepsilon\right)=1. \quad (6.4.49)$$

Since this is true for all $\varepsilon>0$ and $G\in C^\infty(V)$, it follows that $P^*=\delta_{\alpha^H}$. □

6.5 A NOTE ON THE TOTAL DENSITY OF THE RTP PROCESS

For this section we focus on the case where $c(\sigma,m)\equiv 1$, where particles evolve independently. We are interested in the evolution of the total density of particles, i.e., we consider the process $\eta^N_t(x):=\eta^N_t(x,1)+\eta^N_t(x,-1)$. The empirical measure of this process is given by

$$\zeta_{N,t}=\frac{1}{N}\sum_{x\in\mathbb{T}_N}\eta^N_t(x)\delta_{\frac{x}{N}}, \quad (6.5.1)$$

which has trajectories in $D([0,T];\mathcal{M})$, with $\mathcal{M}$ the space of Radon measures on the torus $\mathbb{T}$. We will give a result on the hydrodynamic limit of the total density of the weakly perturbed model and find a large deviation rate function through the contraction principle.

6.5.1 *Hydrodynamic limit of the weakly perturbed total density*

We can deduce that the empirical measure $\zeta_{N,t}$ converges in probability to the measure $\varrho^H_t(x)\,\mathrm{d}x$, with $\varrho^H_t(x)=\varrho^H_t(x,1)+\varrho^H_t(x,-1)$ where $\varrho^H_t(x,\sigma)$ solves the hydrodynamic limit of the weakly perturbed multi-layer model in Theorem 6.2 with $c(\sigma,m)\equiv 1$. We can rewrite this as a coupled system of linear PDE's given by

$$\begin{cases}\dot{\varrho}^H_t(x,1)=-\partial_x\varrho^H_t(x,1)+e^{\widetilde{H}_t(x)}\varrho^H_t(x,-1)-e^{-\widetilde{H}_t(x)}\varrho^H_t(x,1),\\[2ex]\dot{\varrho}^H_t(x,-1)=\partial_x\varrho^H_t(x,-1)+e^{-\widetilde{H}_t(x)}\varrho^H_t(x,1)-e^{\widetilde{H}_t(x)}\varrho^H_t(x,-1).\end{cases} \quad (6.5.2)$$

From this, we obtain a PDE for the total density $\varrho_t^H(x) = \varrho_t^H(x,1) + \varrho_t^H(x,-1)$, depending on the difference of the two densities $\Delta_t^H(x) = \varrho_t^H(x,1) - \varrho_t^H(x,-1)$, given by

$$\begin{cases} \dot{\hat{\varrho}}_t(x) = -\partial_x \Delta_t^H(x), \\ \\ \dot{\Delta}_t^H(x) = -\partial_x \varrho_t^H(x) + 2\sinh(\widetilde{H}_t(x))\varrho_t^H(x) - 2\cosh(\widetilde{H}_t(x))\Delta_t(x). \end{cases} \tag{6.5.3}$$

The goal is to obtain a closed equation for $\varrho_t^H(x)$. Taking a second derivative in time in the first equation and substituting the second equation gives us

$$\begin{aligned} \ddot{\varrho}_t^H(x) = \partial_{xx}\varrho_t^H(x) - 2\partial_x \left(\sinh(\widetilde{H}_t(x))\varrho_t^H(x)\right) \\ + 2\left(\partial_x \cosh(\widetilde{H}_t(x))\right)\Delta_t(x) + 2\cosh(\widetilde{H}_t(x))\partial_x\Delta_t(x) \end{aligned} \tag{6.5.4}$$

Here $\partial_x \Delta_t^H(x) = -\dot{\varrho}_t^H(x)$ and so if $\partial_x \cosh(\widetilde{H}_t(x)) = 0$ then we have a closed equation for $\varrho_t^H(x)$ given by

$$\ddot{\varrho}_t^H(x) = \partial_{xx}\varrho_t^H(x) - 2\partial_x \left(\sinh(\widetilde{H}_t(x))\varrho_t^H(x)\right) - 2\cosh(\widetilde{H}_t(x))\dot{\varrho}_t^H(x). \tag{6.5.5}$$

On the other hand, if $\partial_x \cosh(\widetilde{H}_t(x)) \neq 0$, then we can find from (6.5.4) an equation of $\Delta_t^H(x)$ written in terms of $\varrho_t^H(x)$, given by

$$\begin{aligned} \Delta_t^H(x) = \frac{1}{2\partial_x \cosh(\widetilde{H}_t(x))}\Big(\ddot{\varrho}_t^H(x) - \partial_{xx}\varrho_t^H(x) + 2\partial_x \left(\sinh(\widetilde{H}_t(x))\varrho_t^H(x)\right) \\ - 2\cosh(\widetilde{H}_t(x))\dot{\varrho}_t^H(x)\Big). \end{aligned} \tag{6.5.6}$$

Substituting this back into the equation $\dot{\hat{\varrho}}_t(x) - \partial_x \Delta_t(x)$, we find that

$$\begin{aligned} \dot{\varrho}_t^H(x) = -\partial_x\Bigg(\frac{1}{2\partial_x \cosh(\widetilde{H}_t(x))}\Big(\ddot{\varrho}_t^H(x) - \partial_{xx}\varrho_t^H(x) + 2\partial_x \left(\sinh(\widetilde{H}_t(x))\varrho_t^H(x)\right) \\ - 2\cosh(\widetilde{H}_t(x))\dot{\varrho}_t^H(x)\Big)\Bigg). \end{aligned} \tag{6.5.7}$$

6.5.2 *A large deviation principle for the total density*

For this section we assume that we start from a deterministic sequence of configurations η^N such that $\zeta_{N,0}$ converges to $\widehat{\varrho}(x)\,\mathrm{d}x$ for some density $\widehat{\varrho}(x)$. We are then interested in the dynamical large deviations of the trajectory $\zeta_{N,t}$, with rate function $\mathcal{J}_{tr} : D([0,T];\mathcal{M}) \to [0,\infty]$. Given that in the multi-layer case the rate function is only finite for trajectories of measures with density, the same must also hold for the total density.

Let $\widehat{\beta} \in D([0,T];\mathcal{M})$ have density $\widehat{\varrho}_t(x)$. For $\widehat{\alpha} \in D([0,T];\mathcal{M}_V)$, we denote $\widehat{\alpha} \sim \widehat{\beta}$ if $\widehat{\alpha}$ has a density $\widehat{\varrho}_t(x,\sigma)$ such that $\widehat{\varrho}_t(x) = \widehat{\varrho}_t(x,1) + \widehat{\varrho}_t(x,-1)$. The contraction principle then tells us that

$$\mathcal{J}_{tr}(\widehat{\beta}) = \inf_{\widehat{\alpha}\sim\widehat{\beta}} \mathcal{I}_{tr}(\widehat{\alpha}). \tag{6.5.8}$$

However, for most $\widehat{\alpha}$ this rate function will be infinite. Therefore we would like to restrict this infimum to only those trajectories for which the rate function is finite.

By Lemma 6.5, we know that if $\mathcal{J}_{tr}(\widehat{\beta}) < \infty$, then the density $\widehat{\varrho}_t(x)$ must satisfy the hydrodynamic limit from Section 6.5.1 for some $\widetilde{H}_t^{(1)}(x)$. However, a priori, this function $\widetilde{H}_t^{(1)}(x)$ is not unique. Therefore we need to establish a relation between the different possible solutions.

Given a trajectory $\widehat{\varrho}_t(x)$, then by (6.5.3) a corresponding multi-layer trajectory $\widehat{\varrho}_t(x,\sigma)$ has to satisfy

$$\begin{cases} \widehat{\varrho}_t(x) = \widehat{\varrho}_t(x,1) + \widehat{\varrho}_t(x,-1), \\ \dot{\widehat{\varrho}}_t(x) = -\partial_x(\widehat{\varrho}_t(x,1) - \widehat{\varrho}_t(x,-1)). \end{cases} \tag{6.5.9}$$

Let $\widehat{\varrho}_t^{(1)}(x,\sigma)$ and $\widehat{\varrho}_t^{(2)}(x,\sigma)$ be two such trajectories that satisfy this. We then find that

$$\begin{cases} \widehat{\varrho}_t^{(1)}(x,1) - \widehat{\varrho}_t^{(2)}(x,1) = \widehat{\varrho}_t^{(2)}(x,-1) - \widehat{\varrho}_t^{(1)}(x,-1), \\ \partial_x\left(\widehat{\varrho}_t^{(1)}(x,1) - \widehat{\varrho}_t^{(2)}(x,1)\right) = \partial_x\left(\widehat{\varrho}_t^{(1)}(x,-1) - \widehat{\varrho}_t^{(2)}(x,-1)\right). \end{cases} \tag{6.5.10}$$

Combining both, we find that

$$\partial_x\left(\widehat{\varrho}_t^{(1)}(x,\sigma) - \widehat{\varrho}_t^{(2)}(x,\sigma)\right) = 0, \tag{6.5.11}$$

meaning that the difference between two solutions can differ at most by a time-dependent constant, i.e.,

$$\widehat{\varrho}_t^{(2)}(x,\sigma) = \widehat{\varrho}_t^{(1)}(x,\sigma) + \sigma c_t \tag{6.5.12}$$

for some function $c_t : [0,T] \to \mathbb{R}$ such that $\widehat{\varrho}_t^{(1)}(x,\sigma) + \sigma c_t \geq 0$ (the last part since we can not allow for negative densities). Furthermore, for the rate functions to be finite, both trajectories have to satisfy the weakly perturbed hydrodynamic equation, i.e.,

$$\begin{cases} \dot{\widehat{\varrho}}_t^{(1)}(x,\sigma) = -\sigma\partial_x\widehat{\varrho}_t^{(1)}(x,\sigma) + e^{\sigma\widetilde{H}_t^{(1)}(x)}\widehat{\varrho}_t^{(1)}(x,-\sigma) - e^{-\sigma\widetilde{H}_t^{(1)}(x)}\widehat{\varrho}_t^{(1)}(x,\sigma), \\ \dot{\widehat{\varrho}}_t^{(2)}(x,\sigma) = -\sigma\partial_x\widehat{\varrho}_t^{(2)}(x,\sigma) + e^{\sigma\widetilde{H}_t^{(2)}(x)}\widehat{\varrho}_t^{(2)}(x,-\sigma) - e^{-\sigma\widetilde{H}_t^{(2)}(x)}\widehat{\varrho}_t^{(2)}(x,\sigma). \end{cases} \tag{6.5.13}$$

for some $\widetilde{H}^{(1)}$ and $\widetilde{H}^{(2)}$. Plugging in (6.5.12), we find that

$$\begin{aligned} &\sigma\dot{c}_t + 2\sigma\cosh(\widetilde{H}_t^{(2)}(x))c_t \\ &\quad= \left(e^{\sigma\widetilde{H}_t^{(2)}(x)} - e^{\sigma\widetilde{H}_t^{(1)}(x)}\right)\widehat{\varrho}_t^{(1)}(x,-\sigma) - \left(e^{-\sigma\widetilde{H}_t^{(2)}(x)} - e^{-\sigma\widetilde{H}_t^{(1)}(x)}\right)\widehat{\varrho}_t^{(1)}(x,\sigma). \end{aligned} \tag{6.5.14}$$

This has the following solution for $\widetilde{H}_t^{(2)}(x)$

$$\widetilde{H}_t^{(2)}(x) = -\log\left(\frac{-b_t(x) + \sqrt{b_t(x)^2 + 4(\widehat{\varrho}_t^{(1)}(x,1)+c_t)(\widehat{\varrho}_t^{(1)}(x,-1)-c_t)}}{2(\widehat{\varrho}_t^{(1)}(x,1)+c_t)}\right), \tag{6.5.15}$$

where

$$b_t(x) = \dot{c}_t + e^{\widetilde{H}_t^{(1)}(x)}\widehat{\varrho}_t^{(1)}(x,-1) - e^{-\widetilde{H}_t^{(1)}(x)}\widehat{\varrho}_t^{(1)}(x,1). \tag{6.5.16}$$

The large deviation rate function of the sum $\widehat{\varrho}_t(x)$ is then given by

$$\mathcal{J}_{tr}(\widehat{\beta}) = \inf_{\substack{c_t \\ \widehat{\varrho}_t^{(1)}(x,\sigma)+\sigma c_t \geq 0}} \int_0^T \left\langle \widehat{\varrho}_t^{(1)}(x,\sigma) + \sigma c_t, e^{-\sigma\widetilde{H}_t^{(2)}(x)}(-\sigma\widetilde{H}_t^{(2)}(x) - 1) + 1 \right\rangle_{L^2(V)} \mathrm{d}t \tag{6.5.17}$$

where $\widehat{\varrho}_t^{(1)}(x,\sigma)$ is any trajectory satisfying the weakly perturbed hydrodynamic equation and such that $\widehat{\varrho}_t(x) = \widehat{\varrho}_t^{(1)}(x,1) + \widehat{\varrho}_t^{(1)}(x,-1)$.

BIBLIOGRAPHY

[1] N.I. Akhiezer and N. Kemmer. *The Classical Moment Problem and Some Related Questions in Analysis*. University mathematical monographs. Oliver & Boyd, 1965.

[2] D. Aldous. Stopping Times and Tightness. *The Annals of Probability*, 6(2):335 – 340, 1978.

[3] G. Amir, C. Bahadoran, O. Busani, and E. Saada. Invariant measures for multilane exclusion process. ., arXiv preprint arXiv:2105.12974, 2021.

[4] G. Amir, C. Bahadoran, O. Busani, and E. Saada. Invariant measures for multilane exclusion process. 2021.

[5] R. Baldasso, O. Menezes, A. Neumann, and R.R. Souza. Exclusion Process with Slow Boundary. *Journal of Statistical Physics*, 167(5):1112–1142, June 2017.

[6] L. Bertini, A. De Sole, D. Gabrielli, G. Jona-Lasinio, and C. Landim. Stochastic interacting particle systems out of equilibrium. *Journal of Statistical Mechanics: Theory and Experiment*, 2007(07):P07014, 2007.

[7] L. Bertini, A. De Sole, D. Gabrielli, G. Jona-Lasinio, and C. Landim. Stochastic interacting particle systems out of equilibrium. *Journal of Statistical Mechanics: Theory and Experiment*, 2007(07):P07014, jul 2007.

[8] P. Billingsley. *Convergence of probability measures*. Wiley Series in Probability and Statistics: Probability and Statistics. John Wiley & Sons Inc., New York, second edition, 1999. A Wiley-Interscience Publication.

[9] G.D. Birkhoff. Proof of the ergodic theorem. *Proc. Natl. Acad. Sci. USA*, 17:656–660, 1931.

[10] R.M.C. Blumenthal and R.K. Getoor. *Markov Processes and Potential Theory*. Dover books on mathematics. Dover Publications, 2007.

[11] B. Buča, K. Klobas, and T. Prosen. Rule 54: exactly solvable model of nonequilibrium statistical mechanics. *Journal of Statistical Mechanics: Theory and Experiment*, 2021(7):074001, jul 2021.

[12] G. Cannizzaro, P. Gonçalves, R. Misturini, and A. Occelli. From ABC to KPZ. *arXiv preprint arXiv:2304.02344*, 2023.

[13] G. Carinci, S. Floreani, C. Giardinà, and F. Redig. Boundary driven Markov gas: duality and scaling limits. arXiv preprint arXiv:2112.12698, 2021.

[14] G. Carinci, C. Giardinà, C. Giberti, and F. Redig. Duality for stochastic models of transport. *Journal of Statistical Physics*, 152(4):657–697, Jul 2013.

[15] F. Casini, R. Frassek, and C. Giardinà. Duality for the multispecies stirring process with open boundaries. *Journal of Physics A: Mathematical and Theoretical*, 2023.

[16] F. Casini, C. Giardinà, and F. Redig. Density fluctuations for the multi-species stirring process. *Journal of Theoretical Probability*, pages 1–38, 2024.

[17] F. Casini, C. Giardinà, and C. Vernia. Uphill in reaction-diffusion multi-species interacting particles systems. *Journal of Statistical Physics*, 190(8):132, 2023.

[18] F. Casini, F. Redig, and H. van Wiechen. A large deviation principle for the multispecies stirring process. *Electronic Journal of Probability*, 30:1–36, 2025.

[19] M. Colangeli, C. Giberti, and C. Vernia. Uphill diffusions in single and multi-species systems. *Journal of Physics A: Mathematical and Theoretical*, 56(39):393001, 2023.

[20] A. Czirók, E. Ben-Jacob, I. Cohen, and T. Vicsek. Formation of complex bacterial colonies via self-generated vortices. *Physical Review E*, 54(2):1791, 1996.

[21] A. De Masi and E. Presutti. *Mathematical methods for hydrodynamic limits*. Springer, 2006.

[22] A. De Masi, E. Presutti, and E. Scacciatelli. The weakly asymmetric simple exclusion process. *Annales de l'I.H.P. Probabilités et statistiques*, 25(1):1–38, 1989.

[23] T. Demaerel and C. Maes. Active processes in one dimension. *Phys. Rev. E*, 97:032604, Mar 2018.

[24] A. Dembo and O. Zeitouni. *Large Deviations Techniques and Applications*. Stochastic Modelling and Applied Probability. Springer Berlin Heidelberg, 2009.

[25] A. Dembo and O. Zeitouni. *Large Deviations Techniques and Applications*. Springer Berlin Heidelberg, Berlin, Heidelberg, 2010.

[26] F. den Hollander. *Large Deviations*. Fields Institute monographs. American Mathematical Soc., 2000.

[27] F. den Hollander. *Probability Theory: The Coupling Method*. Leiden university, 2012.

[28] C. Derman. Some contributions to the theory of denumerable Markov chains. *Transactions of the American Mathematical Society*, 79:541–555, 1955.

[29] B. Derrida, J.L. Lebowitz, and E.R. Speer. Large deviation of the density profile in the steady state of the open symmetric simple exclusion process. *Journal of statistical physics*, 107(3):599–634, 2002.

[30] J.D. Deuschel and D.W. Stroock. *Large Deviations*. AMS Chelsea Publishing Series. American Mathematical Society, 2001.

[31] A. Dhar, A. Kundu, S.N. Majumdar, S. Sabhapandit, and G. Schehr. Run-and-tumble particle in one-dimensional confining potentials: Steady-state, relaxation, and first-passage properties. *Physical Review E*, 99(3), Mar 2019.

[32] C. Erignoux. Hydrodynamic limit for an active exclusion process. *Mémoires de la Société Mathématique de France*, (169), 2021.

[33] S.N. Ethier and T.G. Kurtz. *Markov processes – characterization and convergence*. Wiley Series in Probability and Mathematical Statistics: Probability and Mathematical Statistics. John Wiley & Sons Inc., New York, 1986.

[34] S.N. Ethier and T.G. Kurtz. *Markov Processes: Characterization and Convergence*. Wiley Series in Probability and Statistics. Wiley, 2009.

[35] J. Feng and T.G. Kurtz. *Large Deviations for Stochastic Processes*. Mathematical surveys and monographs. American Mathematical Society, 2006.

[36] K. Fleischmann, J. Gärtner, and I. Kaj. A schilder type theorem for super-brownian motion. *Canadian Journal of Mathematics*, 48(3):542–568, 1996.

[37] S. Floreani, C. Giardinà, F. den Hollander, S. Nandan, and F. Redig. Switching interacting particle systems: Scaling limits, uphill diffusion and boundary layer. *Journal of Statistical Physics*, 186(3), Jan 2022.

[38] S. Floreani, C. Giardinà, F. den Hollander, S. Nandan, and F. Redig. Switching Interacting Particle Systems: Scaling Limits, Uphill Diffusion and Boundary Layer. *Journal of Statistical Physics*, 186(3):33, January 2022.

[39] S. Floreani, F. Redig, and F. Sau. Hydrodynamics for the partial exclusion process in random environment. *Stochastic Processes and their Applications*, 142:124–158, 2021.

[40] C. Franceschini, P. Gonçalves, M. Jara, and B. Salvador. Non-equilibrium fluctuations for sep with open boundary. *Stochastic Processes and their Applications*, 178:104463, 2024.

[41] C. Franceschini, P. Gonçalves, and B. Salvador. Hydrodynamical Behavior for the Symmetric Simple Partial Exclusion with Open Boundary. *Mathematical Physics, Analysis and Geometry*, 26(2):11, May 2023.

[42] T. Franco, P. Gonçalves, C. Landim, and A. Neumann. Dynamical large deviations for the boundary driven symmetric exclusion process with robin boundary conditions. *Latin American Journal of Probability and Mathematical Statistics*, 19, 2022.

[43] T. Franco, P. Gonçalves, and A. Neumann. Large deviations for the ssep with slow boundary: the non-critical case. *Latin American Journal of Probability and Mathematical Statistics*, 20:359, 01 2023.

[44] R. Garcia-Millan and G. Pruessner. Run-and-tumble motion in a harmonic potential: field theory and entropy production. *Journal of Statistical Mechanics: Theory and Experiment*, 2021(6):063203, Jun 2021.

[45] C. Giardina, J. Kurchan, F. Redig, and K. Vafayi. Duality and hidden symmetries in interacting particle systems. *Journal of Statistical Physics*, 135:25–55, 2009.

[46] C. Giardinà, J. Kurchan, and F. Redig. Duality and exact correlations for a model of heat conduction. *Journal of Mathematical Physics*, 48(3):033301, 03 2007.

[47] C. Giardinà, F. Redig, and K. Vafayi. Correlation Inequalities for Interacting Particle Systems with Duality. *Journal of Statistical Physics*, 141(2):242–263, October 2010.

[48] G. Gompper et al. The 2020 motile active matter roadmap. *Journal of Physics: Condensed Matter*, 32(19):193001, feb 2020.

[49] P. Gonçalves, R. Misturini, and A. Occelli. Hydrodynamics for the ABC model with slow/fast boundary. *Stochastic Processes and their Applications*, 161:350–384, 2023.

[50] R. Großmann, F. Peruani, and M. Bär. Diffusion properties of active particles with directional reversal. *New Journal of Physics*, 18(4):043009, Apr 2016.

[51] M. Guo, G.C. Papanicolaou, and S.R.S. Varadhan. Nonlinear diffusion limit for a system with nearest neighbor interactions. *Communications in Mathematical Physics*, 118(1):31–59, 1988.

[52] A. Gut. *Probability: A Graduate Course*. Springer texts in statistics. Springer, New York, NY, 2 edition, October 2012.

[53] M. Guéneau and L. Touzo. Relating absorbing and hard wall boundary conditions for a one-dimensional run-and-tumble particle. *Journal of Physics A: Mathematical and Theoretical*, 57(22):225005, may 2024.

[54] A.E. Hafner, L. Santen, H. Rieger, and M.R. Shaebani. Run-and-pause dynamics of cytoskeletal motor proteins. *Scientific Reports*, 6(1):37162, November 2016.

[55] T.E. Harris. Contact interactions on a lattice. *Annals of Probability*, 2:969–988, 1974.

[56] T. Hida. *Brownian Motion*. Stochastic Modelling and Applied Probability. Springer New York, 2012.

[57] F. den Hollander. *Probability Theory: The Coupling Method*. Mathematical Institute, Leiden University, 2012.

[58] A. Jakubowski. On the Skorokhod topology. *Annales de l'Institut Henri Poincaré. Probabilités et Statistiques*, 22, 01 1986.

[59] S. Jansen and N. Kurt. On the notion(s) of duality for Markov processes. *Probability Surveys*, 11:59 – 120, 2014.

[60] M. Jara and O. Menezes. Non-equilibrium fluctuations of interacting particle systems. *arXiv preprint arXiv:1810.09526*, 2018.

[61] C. Kipnis and C. Landim. *Scaling Limits of Interacting Particle Systems*. Grundlehren der mathematischen Wissenschaften. Springer Berlin Heidelberg, 1998.

[62] C. Kipnis, S. Olla, and S.R.S. Varadhan. Hydrodynamics and large deviation for simple exclusion processes. *Communications on Pure and Applied Mathematics*, 42(2):115–137, March 1989.

[63] C. Kleiber and J. Stoyanov. Multivariate distributions and the moment problem. *Journal of Multivariate Analysis*, 113:7–18, 2013. Special Issue on Multivariate Distribution Theory in Memory of Samuel Kotz.

[64] M. Kourbane-Houssene, C. Erignoux, T. Bodineau, and J. Tailleur. Exact hydrodynamic description of active lattice gases. *Physical Review Letters*, 120(26), Jun 2018.

[65] M. Kourbane-Houssene, C. Erignoux, T. Bodineau, and J. Tailleur. Exact hydrodynamic description of active lattice gases. *Physical review letters*, 120(26):268003, 2018.

[66] K. Kuoch and F. Redig. Ergodic theory of the symmetric inclusion process. *Stochastic Processes and their Applications*, 126(11):3480–3498, 2016.

[67] Thomas G Kurtz. Extensions of trotter's operator semigroup approximation theorems. *Journal of Functional Analysis*, 3(3):354–375, 1969.

[68] K. Lai Chung. *The main limit theorem*. Springer Berlin Heidelberg, Berlin, Heidelberg, 1960.

[69] C. Landim, J. Mangi, and B. Salvador. Exclusion processes with non-reversible boundary: hydrodynamics and large deviations. *arXiv preprint arXiv:2411.17653*, 11 2024.

[70] C. Landim and S. Velasco. Dynamic and static large deviations of a one dimensional ssep in weak contact with reservoirs. *arXiv preprint arXiv:2308.10895*, 2023.

[71] P. Le Doussal, S.N. Majumdar, and G. Schehr. Noncrossing run-and-tumble particles on a line. *Physical Review E*, 100(1), Jul 2019.

[72] D. Li. Hydrodynamic limit for an active-passive exclusion process. *arXiv preprint arXiv:2501.15862*, 2025.

[73] M. Lifshits. *Lectures on Gaussian Processes*. SpringerBriefs in Mathematics. Springer Berlin Heidelberg, 2012.

[74] T.M. Liggett. Coupling the Simple Exclusion Process. *The Annals of Probability*, 4(3):339 – 356, 1976.

[75] T.M. Liggett. *Stochastic Interacting Systems: Contact, Voter and Exclusion Processes*. Grundlehren der mathematischen Wissenschaften. Springer, Berlin, Heidelberg, 1999.

[76] T.M. Liggett. *Interacting Particle Systems*. Classics in Mathematics. Springer Berlin Heidelberg, 2004.

[77] T. Lindvall. *Lectures on Coupling Method*. Dover, 1992.

[78] R.S. Liptser and A.N. Shiryaev. *Statistics of Random Processes II: Applications*. Stochastic Modelling and Applied Probability. Springer Berlin Heidelberg, 2010.

[79] K. Malakar, V. Jemseena, A. Kundu, K. Vijay Kumar, S. Sabhapandit, S.N. Majumdar, S. Redner, and A. Dhar. Steady state, relaxation and first-passage properties of a run-and-tumble particle in one-dimension. *Journal of Statistical Mechanics: Theory and Experiment*, 2018(4):043215, Apr 2018.

[80] D. Martin, J. O'Byrne, M. E. Cates, É. Fodor, C. Nardini, J. Tailleur, and F. van Wijland. Statistical mechanics of active ornstein-uhlenbeck particles. *Phys. Rev. E*, 103:032607, Mar 2021.

[81] A. Martin-Löf. Limit theorems for the motion of a Poisson system of independent Markovian particles with high density. *Zeitschrift für Wahrscheinlichkeitstheorie und Verwandte Gebiete*, 34(3):205–223, September 1976.

[82] J. Mason, C. Erignoux, R.L. Jack, and M. Bruna. Exact hydrodynamics and onset of phase separation for an active exclusion process. *Proceedings of the Royal Society A*, 479(2279):20230524, 2023.

[83] I. Mitoma. Tightness of Probabilities On $C([0,1];\mathscr{Y}')$ and $D([0,1];\mathscr{Y}')$. *The Annals of Probability*, 11(4):989 – 999, 1983.

[84] C.B. Morrey. On the derivation of the equations of hydrodynamics from statistical mechanics. *Proceedings of the National Academy of Sciences of the United States of America*, 40(5):317–322, 1954.

[85] K. Nagy. Symmetric random walk in random environment in one dimension. *Periodica Mathematica Hungarica*, 45(1):101–120, September 2002.

[86] H. Noguchi. Spatiotemporal patterns in active four-state potts models. *Scientific Reports*, 15(1):674, 2025.

[87] J. O'Byrne, Y. Kafri, J. Tailleur, and F. van Wijland. Time irreversibility in active matter, from micro to macro. *Nature Reviews Physics*, 4:167 – 183, 2021.

[88] L. Onsager. Crystal statistics. I. a two-dimensional model with an order-disorder transition. *Phys. Rev.*, 65:117–149, Feb 1944.

[89] A. Opoku and F. Redig. Coupling and hydrodynamic limit for the inclusion process. *Journal of Statistical Physics*, 160(3):532–547, 2015.

[90] Z. Palmowski and T. Rolski. A technique for exponential change of measure for Markov processes. *Bernoulli*, 8(6):767 – 785, 2002.

[91] A.E. Patteson, A. Gopinath, M. Goulian, and P.E. Arratia. Running and tumbling with E. coli in polymeric solutions. *Scientific Reports*, 5(1):15761, October 2015.

[92] Y.V. Prokhorov. Convergence of random processes and limit theorems in probability theory. *Theory of Probability & Its Applications*, 1(2):157–214, 1956.

[93] E. Pulvirenti, F. Redig, and H. van Wiechen. Hydrodynamic limit and large deviations for run-and-tumble particles with mean-field switching rates. 2025.

[94] G. Pólya. Über eine aufgabe der wahrscheinlichkeitsrechnung betreffend die irrfahrt im straßennetz. *Mathematische Annalen*, 84(1–2):149–160, March 1921.

[95] J. Quastel. Diffusion of color in the simple exclusion process. *Communications on pure and applied mathematics*, 45(6):623–679, 1992.

[96] F. Redig. *Basic techniques in interacting particle systems*. Lecture notes. Ravello, 2014.

[97] F. Redig and H. van Wiechen. Ergodic theory of multi-layer interacting particle systems. *Journal of Statistical Physics*, 190(4):88, 2023.

[98] F. Redig and H. van Wiechen. Stationary fluctuations of run-and-tumble particles. *Markov Processes and Related Fields*, 30(2):297–331, 2024.

[99] H. Rost. Hydrodynamik gekoppelter diffusionen: Fluktuationen im gleichgewicht. In *Dynamics and Processes*, pages 97–107. Springer Berlin Heidelberg, 1983.

[100] M. Rudan et al. *Physics of semiconductor devices*. Springer, 2015.

[101] M. Sasada. Hydrodynamic limit for two-species exclusion processes. *Stochastic processes and their applications*, 120(4):494–521, 2010.

[102] V. Schaller, C. Weber, C. Semmrich, E. Frey, and A.R. Bausch. Polar patterns of driven filaments. *Nature*, 467(7311):73–77, 2010.

[103] G. Schütz and S. Sandow. Non-abelian symmetries of stochastic processes: Derivation of correlation functions for random-vertex models and disordered-interacting-particle systems. *Physical Review E*, 49(4):2726, 1994.

[104] T. Seppäläinen. Translation invariant exclusion processes (book in progress). *University of Wisconsin, Department of Mathematics*, 2008.

[105] A.P. Solon, M. E. Cates, and J. Tailleur. Active brownian particles and run-and-tumble particles: A comparative study. *The European Physical Journal Special Topics*, 224(7):1231–1262, July 2015.

[106] A.P. Solon and J. Tailleur. Revisiting the flocking transition using active spins. *Phys. Rev. Lett.*, 111:078101, Aug 2013.

[107] F. Spitzer. Interaction of Markov processes. *Advances in Mathematics*, 5(2):246–290, 1970.

[108] H. Spohn. *Large scale dynamics of interacting particles*. Springer Science & Business Media, 2012.

[109] H. Spohn. Nonlinear fluctuating hydrodynamics for anharmonic chains. *Journal of Statistical Physics*, 154:1191–1227, 2014.

[110] H. Spohn and G. Stoltz. Nonlinear fluctuating hydrodynamics in one dimension: the case of two conserved fields. *Journal of Statistical Physics*, 160:861–884, 2015.

[111] P.J.C. Spreij. *Stochastic integration*. Lecture notes. 2021.

[112] J. Tailleur, G. Gompper, M.C. Marchetti, J.M. Yeomans, and C. Salomon. *Active Matter and Nonequilibrium Statistical Physics: Lecture Notes of the Les Houches Summer School: Volume 112, September 2018*. Oxford University Press, 11 2022.

[113] J. Tailleur, J. Kurchan, and V. Lecomte. Mapping out-of-equilibrium into equilibrium in one-dimensional transport models. *Journal of Physics A: Mathematical and Theoretical*, 41(50):505001, nov 2008.

[114] A.G. Thompson, J. Tailleur, M.E. Cates, and R.A. Blythe. Lattice models of nonequilibrium bacterial dynamics. *Journal of Statistical Mechanics: Theory and Experiment*, 2011(02):P02029, feb 2011.

[115] H. Thorisson. *Coupling, Stationarity, and Regeneration*. Springer Verlag, Germany, 2000.

[116] B. van Ginkel and F. Redig. Hydrodynamic Limit of the Symmetric Exclusion Process on a Compact Riemannian Manifold. *Journal of Statistical Physics*, 178(1):75–116, January 2020.

[117] B. van Ginkel and F. Redig. Equilibrium fluctuations for the symmetric exclusion process on a compact riemannian manifold. *Markov Processes and Related Fields*, 28(1):29–52, 2022.

[118] B. van Ginkel, B. van Gisbergen, and F. Redig. Run-and-tumble motion: The role of reversibility. *Journal of Statistical Physics*, 183(3):44, 2021.

[119] M. Vanicat. Exact solution to integrable open multi-species SSEP and macroscopic fluctuation theory. *Journal of Statistical Physics*, 166(5):1129–1150, 2017.

[120] S.R.S. Varadhan. Large deviations and applications. *Society for Industrial and Applied Mathematics (SIAM)*, 1984.

[121] J. von Neumann. Proof of the quasi-ergodic hypothesis. *Proc. Natl. Acad. Sci. USA*, 18:70–82, 1932.

[122] J. von Plato. Boltzmann's ergodic hypothesis. *Archive for History of Exact Sciences*, 42(1):71–89, 1991.

[123] P. Walters. *An introduction to ergodic theory*. Springer-Verlag, 1982.

[124] H. Yau. Relative entropy and hydrodynamics of Ginzburg-Landau models. *Letters in Mathematical Physics*, 22(1):63–80, May 1991.

[125] Z. Zhou et al. Orthogonal polynomial stochastic duality functions for multi-species SEP (2j) and multi-species IRW. *SIGMA. Symmetry, Integrability and Geometry: Methods and Applications*, 17:113, 2021.

SUMMARY

In this thesis we study scaling limits and other properties of multi-layer particle systems. The idea of a multi-layer particle system is to consider particles on multiple copies of a spatial domain, where particles can exhibit different behavior on each layer. In this way, the layers represent internal states that a particle can have and transition into or out of.

The multi-layer particle systems we consider are a multi-layer version of the Simple Exclusion Process (SEP), a multi-layer version of the Simple Inclusion Process (SIP) and run-and-tumble particles. The latter serves as a toy model for active particles, i.e., particles that generate their own motion through energy consumption.

In the run-and-tumble particle model, particles tend to move in a certain direction on each layer. This active component of the dynamics drives the system out of equilibrium, which motivates the study of such systems. Moreover, when we sum over the layers to obtain the total number of particles at a given location, we obtain a new process which is no longer Markovian. However, after having derived scaling limits for the multi-layer model, we can extract results on the total density of particles by summing the equations over the layers and deriving a closed-form equation.

In Chapter 3 we aim to characterize the ergodic measures of multi-layer particle systems on multiple copies of the lattice $\mathbb{Z}^d$. The particle systems we consider are the multi-layer versions of the SEP and SIP and of the run-and-tumble particle process. We prove a general result stating that, under the existence of a successful coupling and a polynomial duality result, one can characterize the tempered (i.e., having finite duality moments) ergodic measures of the system. We apply this result on the aforementioned systems and find that the ergodic measures are given by product measures with marginals of constant density: binomial measures for the multi-layer SEP, negative binomial measures for the multi-layer SIP and Poisson measures for run-and-tumble particles.

In Chapter 4 we further investigate the run-and-tumble particle process, this time on the multi-layered version of the one-dimensional lattice $\mathbb{Z}$. We derive the hydrodynamic limit and the stationary fluctuations. For the latter, we consider the process started from the ergodic product Poisson measures. The results are coupled systems of (S)PDE's, one for each layer. We then turn to the case of two layers and sum up the equation in order to find a hydrodynamic and fluctuation result for the total density. Finally, by applying Schilder's theorem, we derive a

large deviation rate function for the SPDE describing these fluctuations, which includes memory terms.

In Chapter 5 we consider a multi-species (a concept closely related to multi-layer) version of the SEP and prove a large deviation principle, generalizing the result of Kipnis, Olla and Varadhan in [62]. The proof relies on the introduction of a weakly asymmetric version of the process and a replacement lemma, which is needed to close the evolution equation for the empirical density under the weakly asymmetric dynamics.

In Chapter 6 we return to run-and-tumble particles with an added mean-field interaction. In this model we look at two layers of the discrete torus where the rate for a particle to jump to the other layer depends on the magnetization of the system, i.e., the relative difference in the number of particles on each layer. In this way, we can model particles that tend to cluster. We derive a hydrodynamic equation dependent on the evolution of the magnetization and afterwards we prove a large deviation principle. Lastly, as a first step toward establishing a large deviation principle for the total density, we apply the contraction principle and restrict the set of admissible multi-layer trajectories corresponding to a given deviation in the total density.

SAMENVATTING

In deze scriptie bestuderen we schalingslimieten en andere eigenschappen van meerlaagse deeltjesystemen. Het idee van een meerlaags deeltjesysteem is om deeltjes te beschouwen op meerdere kopieën van een ruimtelijk domein, waarbij de deeltjes op elke laag verschillend gedrag kunnen vertonen. Op deze manier vertegenwoordigen de lagen interne toestanden waarin een deeltje zich kan bevinden en waar het naar kan overgaan of uit kan terugkeren.

De meerlaagse deeltjesystemen die we beschouwen zijn een meerlaagse versie van het Simpele Exclusieprocess (SEP), een meerlaagse versie van het Simpele Inclusieprocess (SIP), en run-and-tumble-deeltjes. Het laatstgenoemde model dient als een voorbeeldmodel voor actieve deeltjes, dat wil zeggen deeltjes die hun eigen beweging genereren door energieverbruik.

In het run-and-tumble-deeltjesmodel hebben deeltjes de neiging zich in een bepaalde richting te bewegen op elke laag. Deze actieve component van de dynamica drijft het systeem uit evenwicht, wat de motivatie vormt om dit soort systemen te bestuderen. Bovendien, wanneer we over de lagen sommeren om het totale aantal deeltjes op een gegeven locatie te verkrijgen, resulteert dit in een nieuw proces dat niet langer Markoviaans is. Nadat we echter schalingslimieten voor het meerlaagse model hebben afgeleid, kunnen we resultaten verkrijgen over de totale dichtheid van deeltjes door de vergelijkingen over de lagen te sommeren en een gesloten vergelijking af te leiden.

In Hoofdstuk 3 richten we ons op het karakteriseren van de ergodische maten van meerlaagse deeltjesystemen op meerdere kopieën van het rooster $\mathbb{Z}^d$. De systemen die we beschouwen zijn de meerlaagse versies van het SEP, het SIP en het run-and-tumble-deeltjesproces. We bewijzen een algemeen resultaat dat stelt dat, onder het bestaan van een geslaagde koppeling en een polynoom-dualiteitsresultaat, men de getemperde (dat wil zeggen met eindige dualiteitsmomenten) ergodische maten van het systeem kan karakteriseren. We passen dit resultaat toe op de bovengenoemde systemen en vinden dat de ergodische maten worden gegeven door productmaten met marginalen van constante dichtheid; binomiale maten voor het meerlaagse SEP, negatieve binomiale maten voor het meerlaagse SIP, en Poisson-maten voor run-and-tumble-deeltjes.

In Hoofdstuk 4 onderzoeken we het run-and-tumble-deeltjesproces verder, ditmaal op de meerlaagse versie van het eendimensionale rooster $\mathbb{Z}$. We leiden de hydrodynamische limiet en de stationaire fluctuaties af. Voor het laatste beschouwen we het proces gestart vanuit de ergodische product-Poisson-maten. De resultaten zijn gekoppelde systemen van (S)PDE's, één voor elke laag. Vervolgens bekijken

we het geval van twee lagen en sommeren we de vergelijkingen om een hydrodynamisch resultaat en een fluctuatie-resultaat voor de totale dichtheid te verkrijgen. Ten slotte, door gebruik te maken van de stelling van Schilder, leiden we een grote afwijkingen 'rate' functie af voor de SPDE die deze fluctuaties beschrijft, en die geheugentermen bevat.

In Hoofdstuk 5 beschouwen we een 'multi-species' versie (een concept nauw verwant aan meerlagen) van het SEP en bewijzen we een grote afwijkingen principe, waarmee we het resultaat van Kipnis, Olla en Varadhan in [62] generaliseren. Het bewijs is gebaseerd op de introductie van een zwak asymmetrische versie van het proces en een vervangingslemma, dat nodig is om de evolutievergelijking voor de empirische dichtheid onder de zwak asymmetrische dynamica te sluiten.

In Hoofdstuk 6 keren we terug naar run-and-tumble-deeltjes, nu met een toegevoegde 'mean field' interactie. In dit model bekijken we twee lagen van de discrete torus waarbij de sprongkans van een deeltje naar de andere laag afhangt van de magnetisatie van het systeem, dat wil zeggen het relatieve verschil in het aantal deeltjes op elke laag. Op deze manier kunnen we deeltjes modelleren die de neiging hebben te clusteren. We leiden een hydrodynamische vergelijking af die afhangt van de evolutie van de magnetisatie en bewijzen vervolgens een grote afwijkingen principe. Tot slot, als eerste stap in het opstellen van een grote afwijkingen principe voor de totale dichtheid, passen we het contractieprincipe toe en beperken we de verzameling toegestane meerlaagse trajecten die overeenkomen met een gegeven afwijking in de totale dichtheid.

ACKNOWLEDGEMENTS

Writing the acknowledgements of this thesis is both easy and difficult, since there are just so many people to thank.

First of all, I want to express my gratitude to my promotor and supervisor. Frank, thank you for piquing my interest in interacting particle systems many years ago, for supervising my Master's thesis, and for giving me the opportunity to continue this research in a PhD. I have learned a lot over these past years and a lot of that is due to your knowledge and your ability to make me feel motivated and refocused after our meetings, especially whenever I felt lost.

I would also like to thank my copromotors, Richard and Elena.

Richard, I really enjoyed our mathematical conversations where we started with nothing and just let ideas run. It did not always lead to anything concrete, but it was always interesting to hear your intuition behind a difficult problem. Furthermore, I will never forget the day you took some PhDs water-skiing.

Elena, although neither of us knew you were involved in my PhD until after half a year, I am very happy that you were. I always enjoyed that, for every meeting, we first spent half an hour talking about anything but mathematics but just allowed ourselves to vent over the silliest things.

Furthermore I would like to thank the rest of my committee — Jan van Neerven, Christian Maes, Stefan Grosskinsky, Mark Peletier and Aad van der Vaart — for taking the time to read my thesis. Christian, thank you for allowing me to work in Leuven for a month and for your wonderful sense of humor.

Then I want to thank all the people in the probability department which made the office the friendly and welcoming place that it is. Starting with the staff — Dorota, Cor, Rik, Julia, Tina, Antonis, Özge, Robert, Fenghui, Ludolf, André and Erdal — and the PhDs and postdocs I've had the pleasure of working with throughout the years — Serena, Vicente, Simone, Bart, Jan-Tino, Anne, Yanyan, Chenguang, Jonathan, Jasper, Berend, Stefan, Yago, Victor. Thank you for all the lunches shared and for all the interesting conversations that came with them. I want to give some special thanks to Serena and Vicente for being there with me from the beginning until the end, and to Jan-Tino for the same enthusiasm for board games. Lastly, I want to thank Francesco Casini for visiting Delft multiple times and for a great collaboration.

Also the PhDs in the Statistics department always brought the necessary fun (and the necessary drama). Thank you Ardjen, Marc, Andrea, Sebastiano, Lasse, Thorben, Francesco, Chris, Wieger, Dominique and Koen. Especially Marc for the fun and Ardjen for the drama.

Furthermore, I want to give a special thanks to Andreas and Jeroen. If it weren't for all those long days we spent together at Pulse, I'm not sure this thesis would ever have been completed.

Outside of my PhD life, there were a lot of people who made sure I was able to enjoy my life beyond my research.

Thank you Bouke, Casper, Rob and Stefan for the much-needed vacations and for describing, mimicking and humming words out of a pan.

Thank you Nick, Victor and Joël for letting me escape reality from time to time through D&D.

Thank you to DWH/Outsite for providing me with a group of friends larger than I ever dared to dream of — Niv, Jelle, Dante, Govert, Quinten, Rory, Wouter, Leon, Wessel, Ward, Cas, Sharwin, Ingmar, Max, Sander, Thomas, Alexandru, Didi, Wieke and many others.

Thank you Andries, Ronald, Reinier, Carlos, René, Sander, Friso, Bart, Chris and Roger for being willing to spend 7 hours with me in a single evening to play just one boardgame.

Thank you Isabelle, Bente, Casper, Sjoerd, Joris, Stijn and Thijs-Jan for keeping the tradition alive and still meeting once a year for an amazing Christmas dinner.

Thank you Femke, Michael and Merel (the Smatjes) for being the most fun housemates ever.

Thank you to my colleagues from Stanislascollege Westplantsoen for warmly welcoming me back and making me feel at home again after leaving to start this PhD.

Thank you Dennis & Colin for letting me take care of your cats while you are away.

Thank you Scott for all the nerdy times we spent together.

And finally thank you to my family — pap, mam, Wietske, Jelle, Miguel, Bente and Humpie. Thank you for always being there for me when I needed you. And thank you Daan, for sharing your life with me these past years.

CURRICULUM VITAE

Hidde van Wiechen was born in Gouda on the 4th of September 1996 and lived all of his youth in Boskoop, a village nearby. After completing high school in 2014 at the Coornhert Gymnasium in Gouda, he moved to Delft to study Applied Mathematics at the TU Delft.

During his Bachelor, he acquired a 2$^{\text{nd}}$-degree teaching qualification as part of his minor after succeeding his internship at the Stanislascollege Westplantsoen in Delft. The next year he became a substitute teacher at this school. He then obtained his Bachelor's degree in 2018 with his thesis "On different characterizations of a normal distribution" under supervision of Prof.dr. Mark Veraar.

His studies continued with the master in Applied Mathematics, during which he was a part-time teacher at the Stanislascollege Westplantsoen, and he also acquired his 1$^{\text{st}}$-degree teaching qualification through another internship at this school. In 2021 he graduated under supervision of Prof.dr. Frank Redig with his thesis "Ergodic theory and hydrodynamic limit for run-and-tumble particle processes".

In September 2021 he started his PhD in the Applied Probability group of TU Delft under the joint supervision of Prof.dr. Frank Redig, Dr. Richard Kraaij and Dr. Elena Pulvirenti. In August 2025 he returned to the Stanislascollege Westplantsoen as a full-time mathematics teacher.

PUBLICATIONS

Published

1. **"Ergodic theory for multi-layer particle systems"**
   with F. Redig (TU Delft)
   Published in: Journal of Statistical Physics, 190(4) (2023), article 88

2. **"Stationary fluctuation for run-and-tumble particles"**
   with F. Redig (TU Delft)
   Published in: Markov Processes and Related Fields, 30(2) (2024), 297-331

3. **"Large deviations of the multi-species stirring process"**
   with F. Casini (KU Leuven) and F. Redig (TU Delft)
   Published in: Electronic Journal of Probability, 30 (2025), 1-36

Submitted

1. **"Large deviations of mean-field run-and-tumble particles"**
   with E. Pulvirenti (TU Delft) and F. Redig (TU Delft)