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DOI
10.1007/s40819-021-01214-z

Publication date
2022

Document Version
Final published version

Published in
International Journal of Applied and Computational Mathematics

Int. J. Appl. Comput. Math (2022) 8:26
https://doi.org/10.1007/s40819-021-01214-z

REVIEW ARTICLE

A Convergence Criterion of Newton's Method Based on the Heisenberg Uncertainty Principle

S. Kouhkani[1] · H. Koppelaar[2] · O. Taghipour Birgani[3] · I. K. Argyros[4] · S. Radenović[5]

Accepted: 3 December 2021 / Published online: 15 January 2022

Abstract

The objective in this article is to extend the applicability of Newton's method for solving Banach space valued nonlinear equations. In particular, a new semi-local convergence criterion for Newton's method (NM) based on Kantorovich theorem in Banach space is developed by application of the Heisenberg Uncertainty Principle (HUP). The convergence region given by this theorem is small in general limiting the applicability of NM. But, using HUP and the Fourier transform of the operator involved, we show that it is possible to extend the applicability of NM without additional hypotheses. This is done by enlarging the convergence region of NM and using the concept of epsilon-concentrated operator. Numerical experiments further validate our theoretical results by solving equations in case not covered before by the Newton–Kantorovich theorem.

Keywords Newton's method · Iterative method · Heisenberg uncertainty principle · Banach space · Kantorovich theorem

Mathematics Subject Classification 65G99 · 65T99 · 65Y20 · 47H07 · 45G10

✉ S. Kouhkani
skouhkani@yahoo.com

H. Koppelaar
koppelaar.henk@gmail.com

O. Taghipour Birgani
o.taghipour69@gmail.com

I. K. Argyros
iargyros@cameron.edu

S. Radenović
sradenovic@mas.bg.ac.rs

1 Department of Mathematics, Islamic Azad University branch of Shabestar, Shabestar, Iran

2 Faculty of Electrical Engineering, Mathematics and Computer Science, Delft University of Technology, Delft, Netherlands

3 Department of Mathematics, Iran University of Science and Technology, Tehran, Iran

4 Department of Mathematical Sciences Cameron University, Lawton, OK 73505, USA

5 Faculty of Mechanical Engineering, University of Belgrade, Belgrade, Serbia

Introduction

Newton's method (NM) considers the following equation

$$\hbar(x) = 0 \tag{1}$$

where $\hbar$ is mapping a convex subset S of a Banach space T into a Banach space W. NM, originally designed for polynomials, is to find a solution of Eq. (1) with initial value x_0, by

$$x_{n+1} = x_n - \hbar^{'}(x)^{-1}\hbar(x), \ \ (n \geq 0)(x_0 \in S), \tag{2}$$

where $\hbar^{'}$ is the Frệchet-derivate of $\hbar$ (Kantorovich 1982).

During recent years, various strategies have been developed to improve the convegence of NM by authors such as Traub [24–26] Argyros [1,3–5], Gutiérrez [13–15], Proinov [20] and others. For instance, Traub and Woźniakowski [26] established an optimal convergence condition for Newton iteration in a Banach space. Moreover, Turkyilmazoglu [28] expands the error analysis to obtain an algorithm based on the iterative schemes to improve the convergence of NM.

Werner Karl Heisenberg published a paper in 1927 and described his famous theory which is called the Heisenberg Uncertainty Principle(HUP). We describe this theory by the following formula, $\Delta x \Delta p > \frac{h}{2\pi}$, where x is position and p is momentum.

The Heisenberg uncertainty principle as described by [18] is the formula

$$\int x^2|\hbar(x)|^2 dx \int \xi^2|\hat{\hbar}(\xi)|^2 d\xi \geq \frac{1}{4}\|\hbar\|^2, \tag{3}$$

where $\hat{\hbar}(\xi) = \frac{1}{\sqrt{2\pi}}\int \hbar(x)e^{-i\xi x}dx$ is the Fourier transform of the function $\hbar$. HUP states that a non-zero function and its Fourier transform cannot both be concentrated.

During the past decades, HUP has gained attentions in various fields including signal processing, neuroscience [19,21]. Historically, it goes back to the 1930s when Stewart [22] presented the Uncertainty Principle in the form $\Delta\nu\Delta t$, where $\Delta\nu$ is the frequency and Δt is the time the signal lasts. In 1946, Gabor [11] introduced his remarkable theorem which is called the Gabor transform. Donoho and Strak [8] applied this inequality to signal recovery. Folland and Sitaram [10] surveyed various mathematical aspects of HUP. Yang et al. [29] described the mathematical aspects of HUP using Fourier analysis.

The main objective of the present article is to develop a semilocal criterion convergence of NM under the Newton–Kantorovich theorem by HUP. Notice that the convergence region of NM under the Newton–Kantorovich convergence criteria is small in general, limiting the applicability of the method. We are motivated by this observation. Related work can be found in [27].

The novelty of our article lies in the fact that we extend the applicability of NM (i.e we enlarge the convergence region) using HUP. Our idea can be used to extend the applicability of other methods along the same lines.

The rest of the article contains the background in "Background" section, the semilocal convergence analysis in "Semilocal Convergence" section followed by the numerical examples and the conclusion in "Numerical Examples" and "Conclusion" sections, respectively.

Background

In this section we state some definitions and convergence theorems.

Definition 1 Suppose that $0 < p < \infty$ and $\hbar$ is a complex measurable operator on S. Then the L^p-norm of $\hbar$ is defined by

$$||\hbar||_p = \left(\int_S |\hbar|^p d\mu\right)^{\frac{1}{p}}.$$

Definition 2 We define the Lipschitz condition in the domain S with constant K by

$$\|\hbar(x) - \hbar(y)\| \leq K\|x - y\|, \quad x, y \in S, \tag{4}$$

where $S = B(x_0, r) = x \in T; \|x - x_0\| < r$.

Theorem 1 [16]. *We suppose that $\hbar$ is a Frêchet differentiable operator defined on an open ball then we have*

$$\|\acute{\hbar}(x_0)^{-1}\hbar(x_0)\| \leq a, \tag{5}$$

$$\|\hbar^{'}(x_0)^{-1}[\hbar^{'}(x) - \hbar^{'}(y)]\| \leq b\|x - y\|, \quad x, y \in S, \quad ab < \frac{1}{2}. \tag{6}$$

Definition 3 [9] Suppose that y_n is a sequence in Banach space T and κ_n is a scalar sequence. We say that κ_n is a majorizing sequence of y_n if $\|y_n - y_{n-1}\| \leq \kappa_n - \kappa_{n-1}$, for all $n \in N$.

Theorem 2 ([14]) *Suppose $\hbar$ is a twice differentiable operator defined in a ball $\Omega = B(x_0, r)$. Moreover, suppose that the inverse of $\hbar^{'}(x_0)$ exists and the following conditions hold*

$$\|\hbar^{'}(x_0)^{-1}\hbar(x_0)\| \leq a, \|\hbar^{'}(x_0)^{-1}\hbar^{''}(x_0)\| \leq c$$
$$\|\hbar^{'}(x_0)^{-1}[\hbar^{''}(x) - \hbar^{''}(x_0)]\| \leq b\|x - x_0\|, \quad x \in S.$$

Consider the following polynomial defined by

$$p(x) = \frac{b}{6}x^3 + \frac{c}{2}x^2 - x + a$$

We suppose that this polynomial has two positive roots $r_1, r_2 (r_1 < r_2)$ and $r_1 \leq R$, then NM converges to x^*, solving Eq. (1).

Theorem 3 (The classical uncertainty principle inequality) [6] *Suppose that $\hbar \in L^P(R^d)$. Then, we have*

$$\|\hbar\|_P \leq C\left(\|\hbar\|_P^{-\frac{1}{2}}\|\hat{\hbar}\|_P^{\frac{1}{2}}\right), \tag{7}$$

where $\hat{\hbar}$ is the Fourier transform of $\hbar$ and C is a constant which depends only on P and d. The value of C for $\hbar \in L^P(R^d)$ is $C \geq \frac{(2\sqrt{\pi})^P}{d}$.

Theorem 4 ([12]) *We supose that P and Q are Banach spaces. Define $r : P \to Q, h : Q \to P$ and $g : P \to P$ that are bounded linear operators, where $gx = hx$ for $x \in P \cap Q$. Moreover, suppose $S \subset Q^{'}$. Then, for constants $e_1, e_2 > 0$ the following conditions hold:*

$$\|P^*\beta\|_{P'} \leq e_1, \beta \in S, \tag{8}$$

$$\|hy\|_P \leq e_2 sup|\beta(y)| : \beta \in S, y \in rQ, \tag{9}$$

If $x, y \in P$, $ry \in P$ and $\|x\|_P = 1$,

$$\|x - y\|_P \leq \sigma, \quad \|x - ry\|_P \leq \varepsilon, \quad \|x - gx\|_P \leq \tau, \tag{10}$$

then

$$e_1 e_2 (1+\sigma) \geq 1 - \tau - \|g\|_{\varepsilon}. \tag{11}$$

Proof. *See* [12].

Definition 4 Suppose that $\hbar$ is an operator on a measurable set Γ and $g(t)$ is a function which vanishes outside Γ such that $\|\hbar - g\| \leq \varepsilon$ so, $\hbar$ is ε-concentrated. Moreover, $\hat{\hbar}$ is ε-concentrated on a measurable set Λ if there is an operator $h(w)$ vanishing outside Λ with $\|\hat{\hbar} - h\| \leq \varepsilon$ [8].

Theorem 5 ([8]) *Let* Γ *and* Λ *be measurable sets and suppose there is a Fourier transform pair* $(\hbar, \hat{\hbar})$*, with* $\hbar$ *and* $\hat{\hbar}$ *of unit norm, such that* $\hbar$ *is* ε_Γ*-concentrated on* Γ *and* $\hat{\hbar}$ *is* ε_Λ*-concentrated on* Λ*. Then, the following assertion holds*

$$|\Lambda||\Gamma| \geq (1 - (\varepsilon_\Gamma + \varepsilon_\Lambda))^2. \tag{12}$$

Proof See [8]. □

Lemma 1 *Suppose that* $\hbar \in L^2(R^d)$ *satisfies supp*$(\hat{\hbar}) \subset B_r(0)$*, where* $B_r(a)$ *denotes the ball of radius* r *about* a *for all multi-indices* α*. Then, we have*

$$\|\partial^\alpha \hbar\|_2 \leq (2\pi r)^{|\alpha|} \|\hbar\|_2 \tag{13}$$

Proof See [23]. □

Semilocal Convergence

We explain a method using HUP to weaken the conditions of the NM convergence.

We construct for this purpose a majorizing sequence according to the proposed condition on [4] which is given by

$$\kappa_0 = 0,\ \kappa_1 = \eta,\ \kappa_{n+2} = \kappa_{n+1} + \frac{\int_0^t \omega(\theta(\kappa_{n+1} - \kappa_n)) d(\kappa_{n+1} - \kappa_n)}{1 - \omega_0(\kappa_{n+1})}, n = 0, 1, 2, \ldots, \tag{14}$$

where ω is an operator satisfying the Lipschitz-type condition $\|\hbar'(x_0)^{-1}(\hbar'(u) - \hbar'(v))\| \leq \omega(\|u - v\|)$ which is non-decreasing and continuous.

Generally, two types of $\omega(x)$ are defined

- $\omega(x) = Kx$ for $x \geq 0$, which obtains the Lipschitz case.
- $\omega(x) = Kx^\mu$ for $x \geq 0$ and $\mu \in [0, 1)$, we get the Hö̈lder case.

In this case, we supposed that $\omega(x) = xe^c$ and apply the following majorizing sequence κ_n

$$\kappa_0 = 0,\ \kappa_1 = l,\ \kappa_{n+1} = \kappa_n + \frac{kbe^C(\kappa_n - \kappa_{n+1})^{1+\theta}}{(1+\theta)(1 - zbt_n^\theta)}, \tag{15}$$

where $l = \|\hbar'(x_0)^{-1}\hbar(x_0)\|,\ z = \|\hbar'(x_0)^{-1}\|,\ \theta = 1$ and $C = 2\pi r$.

Theorem 5 describes the common feature of the uncertainty principle and provides the following condition

$$|\Lambda||\Gamma| < 1. \tag{16}$$

We extend this condition to the Newton method on Banach space and applied it on the interval of S to find an optimal radius of the ball of onvergence.We have the following corollary, as a consequence of Theorem 5

Corollary 1 *Suppose that $\hbar$ is an operator and $\hbar : S = [a, b] \subset T \to \Pi = [\hbar(a), \hbar(b)] \subset W$ where T and W are Banach spaces. From the Theorem 5 we deduced that if operator $\hbar$ is ε-concentrated on the interval S and its Fourier transform is not ε-concentrated so the operator $\hbar$ has solution for $\hbar(x) = 0$.*

By applying the Kantorovich theorem [16] and proposed approaches we state the following theorem.

Theorem 6 *Let T and W be Banach spaces and $\hbar : S = [a, b] \subset T \to \Pi = [\hbar(a), \hbar(b)] \subset W$ is a non-zero operator and is ε-concentrated also, we define its Fourier transform as $\hat{\hbar} : \hat{S} \to \hat{\Pi}$. Moreover, there exists a point $x_0 \in S$, such that $\hbar^{'}(x_0)^{-1}$ exists on S, and the following conditions hold:*

$$\|\hbar^{'}(x_0)^{-1}\hbar(x_0)\|_P \le \gamma, \tag{17}$$

$$\|\hbar^{'}(x_0)[\hbar^{'}(x) - \hbar^{'}(y)]\| \le b\|x - y\| \le h,\ h = b\mu,\ \mu = \frac{1 - (\tau + \|g\|_\varepsilon + ab)}{ab}, \tag{18}$$

Then, sequence y_n, $(n \ge 0)$ developed by Eq. (2) exists, converges to $x^{'}$,which is a solution of $\hbar(x) = 0$, and

$$\|y_{n+1} - y_n\| \le \kappa_{n+1} - \kappa_n. \tag{19}$$

Proof Suppose $\bar{B}_r(x_0) \subset S$ and $\hbar$ is ε-concentrated on the measurable set S. Hence, according to the Theorem 5 and Eq. (16) the Fourier transform of $\hbar$ cannot be ε-concentrated on the measurable set $\hat{S}$ and we have $|S||\hat{S}| < 1$. Next, we suppose that $\hbar \in L^P(R^d)$, $\hbar$ is Fréchet differentiable and $\hbar^{'}(x_0)^{-1}$ exists. Then, by substituting the Eqs. (7) and (13) in Kantorovich's condition we have

$$\begin{aligned}
\|\hbar^{'}(x_0)^{-1}\hbar(x_0)\|_P &\le \|\hbar^{'}(x_0)^{-1}\|\|\hbar(x_0)\|_P \\
&\le \frac{1}{(2\pi r)}\|\hbar(x)\|_P \left(C\left(\|\hbar(x)\|_P^{\frac{1}{2}}\|\hat{\hbar}(\xi)\|_P^{\frac{1}{2}}\right)\right) \\
&= \left(\frac{C}{2\pi r}\right)\left(\|\hbar(x)\|_P^{\frac{3}{2}}\|\hat{\hbar}(\xi)\|_P^{\frac{1}{2}}\right).
\end{aligned}$$

So, we get

$$\|\hbar^{'}(x_0)^{-1}\hbar(x_0)\|_P \le \gamma, \quad \gamma = \left(\frac{C}{2\pi r}\right)\left(\|\hbar(x)\|_P^{\frac{3}{2}}\|\hat{\hbar}(\xi)\|_P^{\frac{1}{2}}\right). \tag{20}$$

In Theorem 4 for Eq. (10), we choose $g(x) = \frac{b}{6}x^3 + \frac{c}{2}x^2 - x + a$. Then, by applying the Eq. (6) we have $\|x - y\| \le \sigma$. So, we obtain

$$\|\hbar^{'}(x_0)^{-1}[\hbar^{'}(u) - \hbar^{'}(v)]\| \le \|u - v\| \le b\sigma < 1, \quad x, u, v \in S.$$

We prove Eq. (19) by mathematical induction on n [1,7]. We have for y_n defined in Eq. (2), and κ_n defined in Eq. (15)

$$\|y_n - y_{n-1}\| \le \kappa_n - \kappa_{n-1}, \forall n \in N, \tag{21}$$

we observe that $\|y_1 - y_0\| = \|\hbar^{'}(x_0)^{-1}\hbar(x_0)\| \le \gamma = \kappa_1 - \kappa_0$.

We suppose that y_j are well defined that the Eq. (21) holds for all $j \le n$. Next, we prove $\|y_{n+1} - y_n\| \le \kappa_{n+1} - \kappa_n$ for all $j \ge 0$. We get

$$\|y_{j+1} - y_j\| \le \sum_{i=1}^{j+1}\|y_i - y_{i-1}\| \le \sum_{i=1}^{j+1}(\kappa_i - \kappa_{i-1}) = \kappa_{j+1} - \kappa_0 = \kappa_{j+1},$$

Hence, we obtain

$$\|y_j + \rho(y_{j+1} - y_j) - y_0\| \leq \kappa_j + \rho(\kappa_{j+1} - \kappa_j), \rho \in [0, 1].$$

Using Eq. (2) we have

$$\begin{aligned}\hbar(x_{j+1}) &= \hbar(x_{j+1}) - \hbar(x_j) - \hbar^{'}(x_j)(x_{j+1} - x_j) \\ &= \int_0^1 [\hbar^{'}(x_j + \rho(x_{j+1} - x_j)) - \hbar^{'}(x_j)](x_{j+1} - x_j)d\rho.\end{aligned}$$

Therefore, from Eq. (18) we obtain

$$\begin{aligned}\|\hbar^{'}(x_0)^{-1}\hbar(x_{j+1})\| &\leq \int_0^1 \|\hbar^{'}(x_0)^{-1}[\hbar^{'}(x_j + \rho(x_{j+1} - x_j) - \hbar^{'}(x_j))]\|\|d\rho\|x_{j+1} - x_j\| \\ &\leq \frac{b}{2}\|y_{j+1} - y_j\|^2 \leq \frac{b}{2}(\kappa_{j+1} - \kappa_j)^2.\end{aligned}$$

We have from the Banach Lemma [3,17] that $f^{'}(x)$ is invertible for all $x \in B(x_0, r)$ and $\|\hbar^{'}(x_{j+1})^{-1}\hbar^{'}(x_0)\| \leq \frac{1}{1-b\|y_{j+1}-y_0\|}$.

This Theorem provides conditions which improve the convergence conditions of the Kantorovich theorem. □

Numerical Examples

In this section, we apply the preceding results to solve three equations.

Example 1 Suppose that $S = [-1, 1]$, $\Pi = R$ and $x_0 = 0$. We have the polynomial $\hbar : S \to \Pi$ as the following [13]

$$\hbar(x) = \frac{1}{6}x^3 + \frac{1}{6}x^2 - \frac{5}{6}x + \frac{1}{3}.$$

Then, we obtain

$$r = 1, \|\hbar(x)\| = 0.8168, \|\hbar(0)\| = \frac{\sqrt{2}}{3}, \|\hbar^{'}(0)\| = \frac{5}{3}.$$

The Fourier transform of $\hbar$ can be calculated as

$$\hat{\hbar}(\xi) = 2\pi\frac{\delta(\xi)}{3} - \frac{\pi\delta(1, \xi)5i}{3} - \frac{\pi\delta(2, \xi)}{3} - \frac{\pi\delta(3, \xi)i}{3},$$

where δ denotes the Dirac's delta function.

Besides by discretization of the function and calculating the norm of the discrete Fourier transform, we obtain $\|\hat{\hbar}\| = 2.7165$. We take $C = \sqrt{\pi}$. By substituting these values in condition (17), we obtain

$$\|\hbar^{'}(x_0)^{-1}\hbar(x_0)\|_1 \leq \gamma = 0.150484.$$

Hence, Kantorovich's condition cannot guarantee that Newton's method converges [13]. But, our convergence condition is satisfied. The root on this interval is $\overset{*}{x} = 0.462598$.

Example 2 Consider $T = W = R$, $S = [\sqrt{2} - 1, \sqrt{2} + 1]$, $x_0 = \sqrt{2}$. Define operator $\hbar$ on S by [2]

$$\hbar(x) = \frac{1}{6}x^3 - \left(\frac{2^{\frac{3}{2}}}{6} + 0.23\right).$$

We have $C = \sqrt{\pi}$ and $\|\hbar(x)\| = 0.9470$, $\|hbar(x_0)\| = 0.32527$, $\|\hbar^{'}(x_0)^{-1}\| = \frac{1}{2}$, $\hat{\hbar}(\xi) = -1.4028.\pi\delta(\xi) - \frac{\pi\delta(3,\xi)i}{3}$ and for the discrete Fourier transform, we obtain $\|\hat{\hbar}\| = 3.2617$. So, we get

$$\|\hbar^{'}(x_0)^{-1}\hbar(x_0)\| \leq \gamma = 0.094511.$$

So, our condition is satisfied. The root is $\overset{*}{x} = 1.61450$.

Example 3 Consider $T = W = R$ and operator $\hbar$ defined on $S = [-1, 1]$ as [4]

$$\hbar = e^{-|x|} - 1.$$

For this case the solution is $\overset{*}{x} = 0$. By following Theorem (6) we have $\|\hbar\| = (4e^{-1} - e^{-2} - 1)^{1/2} = 0.5798$, $\hat{\hbar}(\xi) = \frac{2}{\xi^2+1} - 2\pi\delta(\xi)$, $\|\hat{\hbar}\| = 1.9414$ and $\|\hbar^{'}\| = 1 - e^{-2} = 0.8647$ on the interval $S = [-1, 1]$.

Beside, we take $C = \sqrt{\pi}$ and get

$$\|\hbar^{'}(x_0)^{-1}\hbar(x_0)\| \leq \gamma = 0.173528.$$

Conclusion

We applied HUP to extend the applicability of NM. The obtained results in the numerical examples section show the effectiveness of our approach. Despite these precise results, the main issue in applying the suggested method is computing a Fourier transform for the operator. To overcome this problem, we employ mathematical software such as Matlab.

We consider this paper as an early version of our work. Future studies could develop this method by working on measurable sets S and $\hat{S}$ (see Theorem 6) to find an optimal radius of the convergence ball and estimate an optimal function for the function of $g(t)$ in Definition 4.

Acknowledgements We thank all reviewers for the constructive crisism of this article.

Funding No funding.

Data Availability Not Applicable

Declarations

Conflict of interest There is no conflict interest.

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