Semilinear Elliptic Eigenvalue Problems

G. Sweers

SEMILINEAIRE ELLIPTISCHE EIGENWAARDE PROBLEMEN

GUIDO SWEERS

foto: dori steeneken

SEMILINEAIR ELLIPTIC EIGENVALUE PROBLEMS

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Overige leden van de commissie:

Prof. H. Berestycki
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Prof. L.S. Frank
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PREFACE.

This thesis consists of an introduction and nine chapters which are divided into two parts. Part A contains several aspects of one particular semilinear elliptic eigenvalue problem. Part B is concerned with three independent elliptic problems.

Chapter 1a appeared:

Ph. Clément, G. Sweers, Existence et multiplicité des solutions d'un problème aux valeurs propres elliptique semilinéaires, C.R. Acad. Sc. Paris 302, Série I, 19 (1986), 681-683.

1b appeared:

Ph. Clément, G. Sweers, Existence and multiplicity results for a semilinear elliptic eigenvalue problem, Annali della Scuola Normale Superiore di Pisa 14 (1987),---.

Chapter 2 will appear:

G. Sweers, On the maximum of solutions for a semilinear elliptic problem, Proceedings of the Royal Society of Edinburgh.

Chapter 3 appeared:

G. Sweers, Some results for a semilinear elliptic problem with a large parameter, Proceedings ICIAM 87, Contributions from the Netherlands, Paris-La Villette, June 29-July 3 1987.

Chapter 4 will appear:

Ph. Clément, G. Sweers, Getting a solution between sub and supersolutions without monotone iteration, Rendiconti dell' Instituto di Matematica dell'Università di Trieste.

Chapter 5 appeared as report 87-83 Dept. Math. T.U.Delft.

Chapter 7 appeared:

B. Kawohl, G. Sweers, Remarks on eigenvalues and eigenfunctions of a special elliptic system, Journal of Appl.Math.Ph.(ZAMP) 38 (1987), 730-740.

Chapter 8 will appear:

G. Sweers, A counterexample with convex domain to a conjecture of De Saint Venant, Journal of Elasticity.

CONTENTS

INTRODUCTION.

Statement of the problem.

In this thesis we are mainly interested in the following semilinear elliptic eigenvalue problem:

$$(1)\qquad \begin{cases} -\Delta u = \lambda f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with a smooth boundary and where f is a C^1-function. A survey of results and methods up to 1981, for (1) with general f, can be found in [23]. For a more recent survey we refer to [19].

A solution of (1) is a pair $(\lambda,u) \in \mathbb{R}^+ \times C^2(\bar{\Omega})$ satisfying (1). Let (λ,u) be solution of (1) with $\max u = m = u(x_0)$ for some $x_0 \in \Omega$. Then the second derivatives $(\partial_i)^2 u$ in x_0 are nonpositive and hence $f(m) = -\lambda^{-1}\Delta u(x_0) \geq 0$ holds. Using the strong maximum principle, see [27], one can show $f(m) > 0$ for positive m ([5]). Indeed, let (λ,u) be a solution with $\max u = u(x_0) = \rho > 0$ and $f(\rho) = 0$. Take $\omega \geq 0$ such that $\omega + \lambda f'(s) \geq 0$ for $s \in [\min u, \rho]$. Hence $s \to \omega s + \lambda f(s)$ is an increasing function on this interval and

$$(2)\qquad (\Delta-\omega)(u-\rho) = \omega\rho + \lambda f(\rho) - (\omega u + \lambda f(u)) \geq 0 \quad \text{in } \Omega.$$

Since $\max(u-\rho) = u(x_0) - \rho = 0$, Theorem 2.6 of [27] implies that $u - \rho = 0$ in Ω, which, together with $u \in C(\bar{\Omega})$, violates the boundary condition.

The main object of part A is to study solutions the maximum of which is close to a positive zero of f. We shall make the following structural assumption on f: f possesses a "falling" zero ρ

(3) $f(\rho) = 0$ and $f(u) > 0$ for $u \in (\rho-\epsilon, \rho)$,

where ϵ is some positive number.

Recently Angenent, [7], established the existence of a curve of positive solutions with $\max u < \rho$, which are unique for λ large, under the additional assumptions

(4)
$$\begin{cases} f(u) > 0 & \text{on } (0,\rho), \\ f(0) > 0 \text{ or } f'(0) > 0, \end{cases}$$

(5) $f'(\rho) < 0$.

We shall consider functions f which may change sign on $(0,\rho)$ and we shall also weaken assumption (5) to

(6) $f'(u) \leq 0$ on $(\rho-\epsilon, \rho)$

for some positive number ϵ.

In addition to this we will investigate consequences of lack of smoothness for both f and the boundary $\partial\Omega$.

A necessary and sufficient condition for existence of solutions.

A first question concerning (1) is whether there are solutions (λ, u) with $\max u = m$ for every positive m with $f(m) > 0$.

Consider for example $f(u) = \sin u$. In the one-dimensional case it is possible to answer this question by direct computations. Multiply the differential equation by u_x and integrate. Suppose $\max u = u(x_0) = m$ and $m > \pi$. Let $x_1 \in \Omega$ be such that $u(x_1) = \pi$, then

(7)
$$0 < (u_x(x_1))^2 = (u_x(x_1))^2 - (u_x(x_0))^2 =$$
$$= -2\lambda \int_{x_0}^{x_1} \sin(u(x))\, u_x(x)\, dx =$$
$$= 2\lambda \int_{u(x_1)}^{u(x_0)} \sin(s)\, ds = 2\lambda \int_{\pi}^{m} \sin(s)\, ds \leq 0,$$

which is a contradiction. Similarly one finds that

$$(8) \qquad \int_u^\rho f(s)\, ds > 0 \qquad \text{for all } u \in [0,\rho)$$

is a necessary condition in the one-dimensional case for the existence of a solution with maximum near a "falling" zero ρ.

In 1973 Fife showed, [12], using the method of asymptotic expansions, that (8) is a sufficient condition for the existence of such a solution on a domain in $\mathbb{R}^N$. In 1981 Hess, [18], showed, when $f(0)$ is positive, the existence of such a solution under a slightly stronger condition, by using a variational argument. De Figueiredo proved in [14], under additional assumptions on the domain, that condition (8) is also necessary. We will show in chapter 1 that condition (8) is necessary for the existence of a solution which has its maximum near the falling zero ρ. We shall not assume that this solution is positive. For a recent improvement of De Figueiredo's result see [11]. In this paper the necessity is proved only for positive solutions. Since Gidas-Ni-Nirenberg's result, [16], about radial symmetry of positive solutions on the ball is used, their proof does not apply to the case of nonpositive solutions.

The main tool in chapter 1 is the so-called sweeping principle.

The sweeping principle.

The definition of the classical version of a subsolution for (1) is, see [24, ch.10.B]:

$$(9) \qquad \begin{cases} u \in C^2(\Omega) \cap C(\bar{\Omega}), & \\ -\Delta u \le \lambda f(u) & \text{in } \Omega, \\ u \le 0 & \text{on } \partial\Omega. \end{cases}$$

In [30], [9] a weaker version is used, which increases the range of applications. We shall call (λ,u) a subsolution of (1) if

$$(10) \qquad \begin{cases} u \in C(\bar{\Omega}), & \\ \int_\Omega \left(-\Delta\varphi\, u - \lambda\, \varphi\, f(u) \right) dx \le 0 & \text{for all } \varphi \in \mathfrak{D}^+(\Omega), \\ u \le 0 & \text{on } \partial\Omega. \end{cases}$$

The function space $\mathfrak{D}^+(\Omega)$ consists of all nonnegative functions in

$C_0^\infty(\Omega)$. Supersolutions are defined by reversing the inequality signs. For short we call u a (sub/super)solution of (1) if (λ, u) is.
Similar to the classical maximum principle one can prove, for $u_1 \leq u_2$ being a sub-, respectively a supersolution, that when $f \in C^1$ either $u_1 < u_2$ in Ω or $u_1 = u_2$ in Ω. Serrin in [32] obtains a very useful result for classical sub- and supersolutions by combining the maximum principle with a connectedness argument. Sattinger in [31] called it Serrin's Sweeping Principle. This principle plays an essential role in almost all chapters of part A. The version which is used for the proof of the necessity of condition (8) reads as follows:

> Let $\{ \bar{u}(t) \; ; \; t \in [0,1] \}$ be a family of supersolutions defined as in (10) and let $\underline{u}$ be a subsolution defined in (10). Suppose $\bar{u}(t) > 0$ on $\partial\Omega$ for all $t \in [0,1]$ and suppose $\bar{u} \in C([0,1];C(\bar{\Omega}))$.
>
> If $\quad \bar{u}(0) \geq \underline{u}$ in Ω,
>
> then $\quad \bar{u}(t) > \underline{u}$ in Ω for all $t \in [0,1]$.

For a more general version we refer to Lemma A.2 in the appendix of chapter 1.

Solutions between sub- and supersolutions.

Suppose there exist a subsolution u_1 and a supersolution u_2 of (1) with $u_1 \leq u_2$ in Ω. If $f \in C^1$ one can construct a solution u by a monotone iteration scheme, see e.g. [31]. When f is only continuous one can no longer use this method. Ako was able to show in [2] the existence of a solution u (in an appropriate sense), with $u_1 \leq u \leq u_2$, for a quasilinear elliptic problem. For (1) we give a proof in chapter 4. A similar proof holds for $f \in C(\bar{\Omega}\times\mathbb{R})$. We call u a solution if it is both a sub- and a supersolution.

Positivity.

For $f \in C^1$ it can be shown that the maximum of two subsolutions in $H^1(\Omega) \cap C(\bar{\Omega})$ is again a supersolution; see the proof of Lemma 2.6 in chapter 1. This result can be improved for $f \in C^0$ without assuming more regularity of the subsolution than $C(\bar{\Omega})$. In chapter 5 this is proved using Kato's inequality; see [21], [8]. If one is interested in positive solutions and $f(0) \geq 0$, then $\underline{u} = 0$ is a useful subsolution. From the result just mentioned one finds that for any subsolution u_1, the function $u_1^+ = \max(0,u_1)$ is also a subsolution. Then there exists a solution $u \in [u_1,\rho]$ which is positive. This argument cannot be used for f with $f(0) < 0$, since $\underline{u} = 0$ is not a subsolution. The problem of finding a positive solution when $f(0) < 0$ is considered in chapter 1, therein assuming that the domain Ω satisfies a uniform interior sphere condition, see [1], [17]. We then have

$$\Omega = \cup \{ B(y,\epsilon) \; ; \; y \in \Omega \text{ with } d(y,\partial\Omega) < \epsilon \} \tag{11}$$

for some $\epsilon > 0$, where $B(y,\epsilon)$ is the ball with center y and radius ϵ. In particular a domain with C^2-boundary satisfies (11). If condition (8) is satisfied one can prove the existence of a positive radially symmetric solution $(\tilde{\lambda},\tilde{u})$ on the ball $B(0,\epsilon)$. Set $\tilde{u} = 0$ outside of $B(0,\epsilon)$ and define

$$\underline{u}(x) = \sup \{ \tilde{u}(x-y) \; ; \; y \in \Omega \text{ with } d(y,\partial\Omega) > \epsilon \}. \tag{12}$$

It can be shown, see the appendix of chapter 5, that $\underline{u}$ is a positive subsolution of (1) with $\lambda = \tilde{\lambda}$. By rescaling one obtains a positive subsolution for all $\lambda > \tilde{\lambda}$. A related result is found in [35].

One may ask whether a condition like (11) is necessary. In particular Professor W. Jäger, [20], raised the question of the existence of a positive solution, having its maximum near a falling zero, when $f(0) < 0$ and Ω is a square. When $f(0) < 0$ we establish in chapter 5 a critical angle for the domain in order to obtain positive solutions. If a domain has a corner with a subcritical angle, there is no positive solution for any λ. As an example consider $\Omega = (-1,1)^N$ with $N \geq 2$, the (hyper)cube. Then, for every f with $f(0) < 0$, there is no positive solution. Hence there exist sign-changing stable solutions.

Stability.

Untill now it may seem that most solutions can be found by using sub- and supersolutions. This is however not true. In general one can only find stable solutions in this way.

A solution (λ,u) of (1) is called stable if, for all $\epsilon > 0$ there exists $\delta > 0$ such that every solution U of the related parabolic problem

$$(13)\qquad \begin{cases} U_t - \Delta U = \lambda f(U) & \text{in } \mathbb{R}_+ \times \Omega, \\ U = 0 & \text{on } \mathbb{R}_+ \times \partial\Omega, \end{cases}$$

with $\| U(0) - u \|_\infty < \delta$ and $U(0) \in C(\bar{\Omega})$, satisfies for all $t \in \mathbb{R}_+$ $\| U(t) - u \|_\infty < \epsilon$.

Let u_1, u_2 be a sub- and a supersolution respectively, with $u_1 < u_2$ in Ω and $u_1 < 0 < u_2$ on $\partial\Omega$. Then, see [30], [25] and the appendix of chapter 5, there is a stable solution u, such that $u_1 < u < u_2$ in Ω. Consider for example $f(u) = (1-u)(u-2)(u-4)$. Using sub- and supersolutions as in chapter 1 one obtains, for λ large, two solutions; one with its maximum near 1 and another with its maximum near 4. These solutions will be local minimizers of the variational problem $\min\{ I(\lambda,u) \ ; u \in H_0^1(\Omega) \}$, see [18], with the functional I defined by

$$(14)\qquad I(\lambda,u) = \int_\Omega \left(\tfrac{1}{2}|\nabla u(x)|^2 - \lambda \int_0^{u(x)} f(s)\, ds \right) dx.$$

The first subsolution in chapter 1 is obtained by minimizing $I(\lambda,u)$ with for Ω the unit ball. By using the Mountain Pass Lemma, [6], [28], or the Leray-Schauder degree, see [3], one can show that there is a third solution, which is generally unstable.

A stronger notion of stability is the following. Consider for a solution (λ,u) of (1) the linearized eigenvalue problem:

$$(15)\qquad \begin{cases} -\Delta v - \lambda f_u(u)\, v = \mu\, v & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases}$$

If the principal eigenvalue μ_0, [4], is strictly positive, then we call the solution (λ,u) strongly stable. A solution which is strongly stable is also stable. For a solution to be stable μ_0 must be nonnegative but not necessarily strictly positive.

If a solution (λ,u) is strongly stable then $\mu = 0$ is not an eigenvalue of (15) and one can use the implicit function theorem in an appropriate function space in order to obtain a curve of strongly stable solutions parametrized by λ in a neighbourhood of (λ,u). The existence of such a curve is obtained in Theorem 2' of chapter 1.

Uniqueness.

For $\partial\Omega \in C^3$, $f \in C^{1,\gamma}$ satisfying (4) and (5) Angenent obtained, for λ large, a unique positive solution with maximum below ρ. By allowing $f'(0) = f(0) = 0$, or introducing positive zeros of f, one can no longer expect this to hold; see [29]. Nevertheless, assuming (6) instead of (4) and (5), we prove in chapter 1 the uniqueness of solutions for λ large in an order interval $[z,\rho] \subset C(\bar{\Omega})$, where $z \in C_0^{\infty}(\Omega)$ is a nonnegative function such that $f > 0$ in $[\max z,\rho)$.

The question was raised by Professor E.N. Dancer, [10], whether there is a unique positive solution u with $\max u \in (\rho-\varepsilon,\rho)$ for λ large. In order to answer this question affirmatively we use results for the so-called P-functions. A reference on this subject is [33].

We use the following P-function:

$$P(x) = |\nabla u(x)|^2 + 2\lambda \int_0^{u(x)} f(s)\,ds. \tag{16}$$

First for the torsion problem, $\lambda f \equiv 1$, Payne showed in 1968, [26], that when the domain is convex, P attains its maximum where the solution becomes maximal. For nonconvex but smooth domains and arbitrary f, P attains its maximum at a critical point of u or at a boundary point. For a solution (λ,u) of (1) it is shown in chapter 2, using the boundary layer behaviour, that at the boundary $P(x) =$
$= |\nabla u(x)|^2 = |\frac{\partial}{\partial n}u(x)|^2 = 2\lambda \int_0^{\rho} f(s)\,ds + o(\lambda)$ for λ large. At an interior critical point $P(x) = 2\lambda \int_0^{u(x)} f(s)\,ds \leq 2\lambda \int_0^{\rho} f(s)\,ds$ holds by condition (8). The bound, obtained in this way, on the gradient of a solution together with the fact that ρ is a zero of f, is

sufficient to show that a solution with maximum near ρ lies above a special subsolution. One concludes the proof by using the sweeping principle and the results from chapter 1.

Nonautonomous nonlinearities.

In chapter 6 we obtain results for functions f which also depend on x. Existence results for solutions with boundary layers, and possibly interior layers, are established for the case in which f is only continuous by the method of sub- and supersolutions. For a more regular f the existence of strongly stable solutions (with boundary layer only) is also obtained. In [12], [13] similar results were proved by using the method of asymptotic expansions and Schauder's fixed point theorem. Angenent in [7] considered nonautonomous nonlinearities assuming (4) and (5) for every $x \in \bar{\Omega}$.

Part B.

The second part of this thesis contains results for three independent elliptic problems. In chapter 7 variational arguments are used to obtain various results for the eigenfunctions and eigenvalues of the Lamé-system, which arises in the theory of elasticity. Among them we mention the fact that the first eigenvalue on the ball is not simple. Chapter 8 contains another problem from the theory of elasticity, which was raised by De Saint Venant. Consider the boundary value problem:

$$(4) \qquad \begin{cases} -\Delta u = 1 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

for domains $\Omega \subset \mathbb{R}^2$, which have the orthogonal axes as axes of symmetry. The function ∇u contains the stress components of an elastic bar with cross section Ω under torsion. Details of the history of this problem can be found in [22]. In a paper of 1856 De Saint Venant observed that in the domains he considered $|\nabla u(x)|$ becomes maximal in those points on $\partial\Omega$ which have minimal distance from the origin. In 1859 he knew of a counterexample with nonconvex domain. For convex domains the conjecture remained that $|\nabla u(x)|$ becomes maximal on the intersection of $\partial\Omega$ and the largest inscribed circle of $\bar{\Omega}$. Kawohl proved the conjecture for some convex domains with additional assumptions on the boundary. In chapter 8 we show that the conjecture is not true for every convex symmetric domain. For more detailed knowledge of the solution one uses a maximum principle for the function P, due to [24], defined by:

$$(18) \qquad P(x) = \det \begin{bmatrix} \frac{\partial}{\partial x_1}\frac{\partial}{\partial x_1} u(x) & \frac{\partial}{\partial x_1}\frac{\partial}{\partial x_2} u(x) & \frac{\partial}{\partial x_1} u(x) \\ \frac{\partial}{\partial x_1}\frac{\partial}{\partial x_2} u(x) & \frac{\partial}{\partial x_2}\frac{\partial}{\partial x_2} u(x) & \frac{\partial}{\partial x_2} u(x) \\ \frac{\partial}{\partial x_1} u(x) & \frac{\partial}{\partial x_2} u(x) & 2u(x) \end{bmatrix} .$$

It is unclear whether there are generalizations for other right hand sides.

Finally, in chapter 9, a maximum principle is proved for a linear elliptic system for which the classical maximum principle, see [27, p.189], is not applicable. The system considered here is not cooperative. A weakly coupled linear elliptic system is called cooperative if all the off-diagonal terms have nonnegative coefficients. By direct but tedious computations we prove that at least on a ball a maximum principle holds for the following system:

$$(19) \qquad \begin{cases} -\Delta u = f - \lambda v & \text{in } \Omega, \\ -\Delta v = f & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases}$$

when λ is small but positive. Indeed, if $f \geq 0$ in Ω is not identically 0 then $u > 0$ in Ω. This system cannot be reduced to a cooperative system by the argument in [15]. The result for (19) can also be used to find positivity of solutions for other noncooperative systems. For example for the same positive λ as in (19) one may show that the functions u and v are positive in Ω if they satisfy:

$$(20) \qquad \begin{cases} -\Delta u = f - \lambda v & \text{in } \Omega, \\ -\Delta v = \lambda u & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases}$$

for $f \geq 0$ in Ω not identically zero.

It would be very surprising if this maximum principle, which is not directly provable by the classical version, would just hold for the ball. However, a proof for arbitrary domains, or a counterexample, is still missing.

References:

[1] R.A. Adams, Sobolev spaces, Academic Press, New York, 1975.

[2] K. Ako, On the Dirichlet problem for quasilinear elliptic differential equations of the second order, J. Math. Soc. Japan 13 (1961), 45-62.

[3] H. Amann, Existence of multiple solutions for nonlinear elliptic boundary value problems, Indiana Univ.Math.Journal 21 (1972), 925-935.

[4] H. Amann, Nonlinear operators in ordered Banach spaces and some applications to nonlinear boundary value problems, Lecture Notes in Math. 543, Springer-Verlag 1975, 1-55.

[5] A. Ambrosetti, P. Hess, Positive solutions of asymptotically linear elliptic eigenvalue problems, J.Math.Anal.Appl. 73 (1980), 441-422.

[6] A. Ambrosetti, P. Rabinowitz, Dual variational methods in critical point theory and applications, J.Funct.Anal. 14 (1973), 349-381.

[7] S.B. Angenent, Uniqueness of the soluton of a semilinear boundary value problem, Math.Ann. 272 (1985), 129-138.

[8] W. Arendt, Kato's inequality; a characterization of generators of positive semigroups, Proc.Roy.Irish Acad.Sect A 84 (1984), 155-174.

[9] H. Berestycki, P.-L. Lions, Some applications of the method of super and subsolutions, in Bifurcation and Nonlinear Eigenvalue Problems, Lecture Notes in Mathematics 782, Springer-Verlag, Berlin, 1980, 16-41.

[10] E.N. Dancer, private communication.

[11] E.N. Dancer, K. Schmitt, On positive solutions of semilinear elliptic equations, Proc.A.M.S. 101 (1987), 445-452.

[12] P.C. Fife, Semilinear boundary value problems with small parameters, Arch. Rat. Mech. Anal. 52 (1973), 205-232.

[13] P.C. Fife, W.M. Greenlee, Interior transition layers for elliptic boundary value problems with a small parameter, Russian Math. Surveys 29:4 (1974), 103-131, from Успехи Матем.Наук. 29:4 (1974), 103-130.

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[14] D.G. De Figueiredo, On the uniqueness of positive solutions of the Dirichlet problem $-\Delta u = \lambda \sin u$, Nonl.P.D.E. and Appl., Collège de France Seminar Vol.7, Pitman 1985, 80-83.

[15] D.G. De Figueiredo, E. Mitidieri, A maximum principle for an elliptic system and applications to semilinear problems, SIAM J. Math. Anal. 17 (1986), 836-849.

[16] B. Gidas, W.M. Ni and L. Nirenberg, Symmetry and related properties via the maximum principle, Comm.Math.Phys. 68 (1979), 209-243.

[17] D. Gilbarg, N.S. Trudinger, Elliptic partial differential equations of second order, Springer-Verlag, Berlin/Heidelberg/ New York, 1977.

[18] P. Hess, On multiple solutions of nonlinear elliptic eigenvalue problems, Comm. Part. Diff. Eq. 6 (1981), 951-961.

[19] J. Hernandez, Qualitative methods for nonlinear diffusion equations, Lecture Notes in Math. 1224, Springer-Verlag 1986, 47-118.

[20] W. Jäger, private communication.

[21] T. Kato, Schrödinger operators with singular potentials, Israel J.Math. 13 (1972), 135-148.

[22] B. Kawohl, On the location of maxima of the gradient for solutions to qausilinear elliptic problems and a problem raised by Saint Venant, Journal of Elasticity 17 (1987), 195-206.

[23] P.-L. Lions, On the existence of positive solutions of semilinear elliptic equations, SIAM Review 24 (1982), 441-467.

[24] L.G. Makar-Limanov, Solution of Dirichlet's problem for the equation $\Delta u = -1$ in a convex region, Math. Notes Acad. Sci. USSR 9 (1971), 52-53.

[25] H. Matano, Asymptotic behaviour and stability of solutions of semilinear diffusion equations, Publ.RIMS. Kyoto Univ. 15 (1979), 401-454.

[26] L.E. Payne, Bounds for the maximum stress in the Saint Venant torsion problem, Indian J.Mech.Math., special issue (1968), 51-59.

[27] M. Protter, H. Weinberger, Maximum principles in differential equations, Prentice Hall, Englewood Cliffs N.J., 1967.

[28] P. Pucci, J. Serrin, Extensions of the mountain pass theorem, J.Funct.Anal. 59 (1984), 185-210.

[29] P.H. Rabinowitz, Pairs of positive solutions of nonlinear elliptic partial differential equations, Indiana Univ. Math. Journal 23 (1973), 172-185.

[30] D.H. Sattinger, Monotone methods in nonlinear elliptic and parabolic boundary value problems, Indiana Univ. Math. Journal 21 (1972), 979-1000.

[31] D.H. Sattinger, Topics in stability and bifurcation theory, Lecture Notes in Mathematics 309, Springer-Verlag, Berlin/Heidelberg/New York, 1973.

[32] J. Serrin, Nonlinear equations of second order, A.M.S. symposium in Partial Diff. Eq., Berkeley, August 1971.

[33] R. Sperb, Maximum principles and their applications, Mathematics in Science and Engineering 157, Academic Press, New York, 1981.

[34] J. Smoller, Shock waves and reaction-diffusion equations, Springer-Verlag, Berlin/Heidelberg/New York, 1983.

[35] J. Smoller, A. Wasserman, Existence of positive solutions for semilinear elliptic equations in general domains, Arch. Rat.Mech.Anal. 98 (1987), 229-249.

A

Chapter 1a

Existence et multiplicité des solutions d'un problème aux valeurs propres elliptique semilinéaire.

Chapter 1b

Existence and multiplicity results for a semilinear elliptic eigenvalue problem.

THÉORÈME 2. — *Soient* $\Gamma \in C^3$, $f \in C^{1,\mu}$, $\mu \in (0,1)$ *satisfaisant* (2), (3) *et*

(4) *il existe* $\varepsilon > 0$ *tel que* $f'(\rho) < 0$ *pour* $\rho \in (\rho_2 - \varepsilon, \rho_2)$. *Alors :*

(i) *il existe* $\lambda_1 > 0$ *et* $\varphi \in C^1([\lambda_1, \infty); C^2(\bar{\Omega}))$ *tels que* $(\lambda, \varphi(\lambda))$ *est solution de* (1) *pour* $\lambda \geqq \lambda_1$, *avec* $\varphi(\lambda) > 0$ *dans* Ω, $\max \varphi(\lambda) \in (\rho_1, \rho_2)$ *et* $\lim_{\lambda \to \infty} \max \varphi(\lambda) = \rho_2$;

(ii) *pour toute fonction* $z \in C_0^\infty(\Omega)$ *non négative telle que* $\max z \in (\rho_1, \rho_2)$, *il existe* $\lambda(z) \geqq \lambda_1$ *tel que si* (λ, u) *est solution de* (1) *avec* $\lambda \geqq \lambda(z)$, *et* $z \leqq u \leqq \rho_2$, *alors* $u = \varphi(\lambda)$;

(iii) *si* $\mu_0(\lambda)$ *désigne la valeur propre principale de*

$$\begin{cases} -\Delta h - \lambda f'(\varphi(\lambda)) h = \mu h & \text{dans } \Omega \\ h = 0 & \text{sur } \Gamma, \end{cases}$$

alors $\mu_0(\lambda) > 0$, *pour* $\lambda \geqq \lambda_1$.

Remarque 2. — Lorsque $\rho_1 = 0$ et $f'(0) = 0$ ou $\rho_1 > 0$, on démontre comme dans [9] que pour λ assez grand, il existe au moins deux solutions de (1) satisfaisant $\max u \in (\rho_1, \rho_2)$.

Dans le cas où Ω est la boule unité, $\rho_1 = 0$ et

$$(5) \qquad f(u) = |u|^\alpha g(u) \qquad \text{avec} \quad \alpha \in \left(1, \frac{N+2}{N-2}\right)$$

nous obtenons le résultat de multiplicité suivant :

THÉORÈME 3. — *Soit f satisfaisant* (2), (4), (5) *avec* $g \in C^{1,\mu}$ *et* $g(0) > 0$. *Si* $N = 1$, 2 *ou* $N > 2$ *et* $\alpha \in (1, N/(N-2))$ *ou*

$$(6) \qquad \left(\frac{N+2}{N-2} - \alpha\right). u^{\alpha+1} g(u) \geqq \frac{2N}{N-2}. \int_0^u s^{\alpha+1} g'(s)\, ds \quad \text{pour tout } u \in [0, \rho_2],$$

alors il existe $\lambda_0 > 0$ *tel que pour* $\lambda > \lambda_0$, (1) *possède exactement deux solutions positives inférieures à* ρ_2, *lorsque* Ω *est la boule unité.*

Remarque 3. — La condition (6) est empruntée à [8], th. 3.1, p. 13.

Remarque 4. — Dancer [3] a montré que si g ne satisfait pas aux conditions du théorème 3, alors il peut exister plus de deux solutions.

2. DÉMONSTRATION DU THÉORÈME 1. — *Suffisance.* — Dans le cas où $f(0) > 0$ ou $f(0) = 0$ (en prolongeant f par imparité pour $u < 0$), on peut appliquer les arguments de Hess [6]. Si $f(0) < 0$, on modifie f de sorte que $f(-a) = 0$ et $J(\rho) > 0$ pour $\rho \in [-a, \rho_2)$ où a est un nombre positif suffisamment petit. En utilisant le résultat d'existence pour $f(0) \geqq 0$, on démontre qu'il existe une solution radiale (μ, v) de $-\Delta v = \mu f(v)$ dans B, $v = -a$ sur ∂B et $v \geqq -a$ dans B où B désigne la boule unité, et l'on définit $v = -a$ pour $x \in \mathbb{R}^N \setminus \bar{B}$. Il existe $\theta \in (0, 1)$ tel que $v(r) > 0$ pour $r \in [0, \theta)$ et $v(r) < 0$ pour $r \in (\theta, 1]$. Puisque Ω satisfait la « condition de sphère intérieure uniforme » (uniform interior sphere condition), il existe $\varepsilon > 0$ tel que $\Omega = \bigcup_{x \in A} B(x, \varepsilon)$ où $A = \{x \in \Omega \mid \text{dist}(x, \partial\Omega) > \varepsilon\}$. On pose $\lambda = \mu.(\theta\varepsilon^{-1})^2$, et l'on définit $w(y; x) = v(\theta\varepsilon^{-1}|x - y|)$, pour $x \in \bar{\Omega}$ et $y \in A$. On vérifie que $w(y; .)$ est une famille de sous-solutions de (1) inférieures à ρ_2, qui est une sursolution de (1). Choisissant $y_0 \in A$, on en déduit l'existence d'une solution (λ, u) avec $w(y_0, x) \leqq u(x) \leqq \rho_2$, $x \in \bar{\Omega}$. On note que $w(y_0, x) < 0 = u(x)$, pour tout $x \in \partial\Omega$ et tout $y \in A$. Puisque la famille $w(y, .)$ dépend continûment de y [dans la topologie $C(\bar{\Omega})$] et puisque A est connexe par arc, il s'en suit par un argument de balayage dû à Serrin ([12], [11]) que $w(y, x) < u(x)$ pour tout $x \in \Omega$ et pour tout $y \in A$. D'où l'on conclut que $u(x) > 0$ pour $x \in \Omega$ et $\partial u/\partial n < 0$

sur $\partial\Omega$ où n désigne la normale extérieure. Ceci achève la démonstration de la suffisance de la condition (2).

NÉCESSITÉ. — Supposons que la condition ne soit pas satisfaite et qu'il existe une solution positive (λ, u) de (1), telle que max $u \in (\rho_1, \rho_2)$. Si min $\{J(\rho) \mid \rho \in [0, \rho_1]\} = 0$, on peut modifier f sur l'intervalle (max u, ρ_2) de telle sorte que

$$J^* := \min \{J(\rho) \mid \rho \in [0, \rho_1]\} < 0$$

et que $f(\rho)$ reste positive sur (ρ_1, ρ_2). (λ, u) est aussi solution de (1) pour la fonction modifiée. Soit v l'unique solution du problème

$$\begin{cases} -v'' = f(v) & \text{pour} \quad r \geqq 0 \\ v(0) = \rho_2 \\ v'(0) = -(-J^*)^{1/2}. \end{cases}$$

On montre facilement que, soit v existe pour tout $r > 0$ et est positive, décroissante avec $0 < \inf_{r>0} v(r) < \rho_1$, soit v possède un premier minimum positif en r_0, tel que $v(r_0) < \rho_1$. Dans ce cas v est symétrique par rapport à r_0 et $v(2r_0) = \rho_2$.

Dans le premier cas on définit $V(r) = v(r)$ pour $r > 0$ et $V(r) = \rho_2$ pour $r \leqq 0$. Dans le deuxième cas, $V(r) = v(r)$ pour $r \in (0, 2r_0)$ et ρ_2 ailleurs. Finalement on définit $w(\lambda, t; x) = V(\lambda^{-1/2} . (x_1 - t))$ pour $x \in \mathbb{R}^N$ et $t \in \mathbb{R}$. On vérifie que $w(\lambda, t; .)$ est une famille de sursolutions de (1) satisfaisant $w(\lambda, t; x) = \rho_2 > u$ pour t assez grand, $x \in \Omega$, et $w(\lambda, t; x) > 0$ pour $x \in \partial\Omega$ et pour tout $t \in \mathbb{R}$. En utilisant l'argument de balayage de Serrin on montre que $u(x) \leqq \inf_{t \in \mathbb{R}} w(\lambda, t, x) < \rho_1$ pour tout $x \in \Omega$, contredisant le fait que max $u > \rho_1$. □

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RÉFÉRENCES BIBLIOGRAPHIQUES

[1] S. B. ANGENENT, Uniqueness of the solution of semilinear boundary value problem, *Math. Ann.*, 272, 1985, p. 129-138.

[2] E. N. DANCER, Uniqueness for elliptic equations when a parameter is large, *Nonl. Anal. Theory, Methods and Appl.*, 8, 1984, p. 835-836.

[3] E. N. DANCER, On the number of positive solutions of weakly nonlinear elliptic equations when a parameter is large, *Proc. Roy. Soc. Edingburgh* (à paraitre).

[4] E. N. DANCER, *Multiple fixed points of positive mappings*, preprint.

[5] D. DE FIGUEIREDO, On the uniqueness of positive solutions of the Dirichlet problem $-\Delta u = \lambda \sin u$, *Nonl. P.D.E. and Appl. Collège de France Seminar*, 7, Pitman, 1985, p. 80-83.

[6] P. HESS, On multiple solutions of nonlinear elliptic eigenvalue problems, *Comm. Part. Diff. Eq.*, 6, 1981, p. 951-961.

[7] P.-L. LIONS, On the existence of positive solutions of semilinear elliptic equations, *S.I.A.M. Rev.*, 24, 1982, p. 441-467.

[8] W. M. NI et J. SERRIN, *Existence and nonexistence theorems for ground states of quasilinear partial differential equations. The anomalous case*, Univ. of Minnesota, Math. Report 84-150, 1984.

[9] P. H. RABINOWITZ, Pairs of positive solutions of nonlinear elliptic partial differential equations, *Indiana Univ. Math. J.*, 23, 1973, p. 172-185.

[10] P. H. RABINOWITZ, Some aspects of nonlinear eigenvalue problems, *Rocky Mountain J. Math.*, 3, 1973, p. 161-202.

[11] D. H. SATTINGER, *Topics in stability and bifurcation theory*, *Lecture Notes in Math.*, 309, Berlin-Heidelberg-New York, Springer, 1973.

[12] J. SERRIN, Nonlinear equations of second order, *A.M.S. Symp. in Partial Diff. Eq.*, Berkeley, August 1971.

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EXISTENCE AND MULTIPLICITY RESULTS FOR A SEMILINEAR ELLIPTIC EIGENVALUE PROBLEM

1. INTRODUCTION.

The following eigenvalue problem will be considered :

$$(P)\quad \begin{cases} -\Delta u = \lambda f(u) \text{ in } \Omega \subset \mathbb{R}^N, \\ u = 0 \text{ on } \partial\Omega = \Gamma, \end{cases}$$

for $\lambda > 0$. The domain Ω is assumed to be bounded and to have a smooth boundary of class C^3.

The function f will satisfy appropriate smoothness conditions. A positive solution of (P) will be a pair (λ,u) in $\mathbb{R}^+ \times C^2(\bar{\Omega})$ satisfying (P) with $u > 0$ in Ω. We shall call u a solution of (P_λ).

It is a consequence of the strong maximum principle, see [2], that if such a solution exists, then f(max u) is positive. The main goal of this paper is to study positive solutions having their maximum close to a zero of f. Therefore we assume :

(F1) there are two numbers ρ_1 and ρ_2 such that $\rho_1 < \rho_2$, $0 < \rho_2$, $f(\rho_1) = f(\rho_2) = 0$ and $f > 0$ in (ρ_1,ρ_2).

In [13] Hess proves the existence of solutions (λ,u) of (P), satisfying $\max u \in (\rho_1,\rho_2)$, when $f(0) > 0$ under the following condition :

$$(F2)\quad J(\rho) = \int_\rho^{\rho_2} f(s)ds > 0 \text{ for every } \rho \in [0,\rho_2).$$

In theorem 1 we prove that (F2) is a *necessary* and sufficient condition for the existence of such a solution even without the condition $f(0) \geq 0$.

Theorem 1. *Let $f \in C^1$ satisfy (F1). Then problem (P) possesses a positive solution (λ,u) with $\max u \in (\rho_1,\rho_2)$, if and only if (F2) holds.*

Theorem 1 improves a result of De Figueiredo in [10], since it does not use the inheritance condition or even the starshapedness of Ω. It also answers a question of Dancer in [9].

Next to this existence result we will prove a uniqueness result for positive solutions having their maximum close to ρ_2. We need the following condition :

(F3) there exists an $\varepsilon > 0$ such that $f' \leq 0$ in $(\rho_2 - \varepsilon, \rho_2)$.

Theorem 2. *Let* $f \in C^{1,\gamma}$ *, for some* $\gamma \in (0,1)$*, satisfy* (F1), F(2) *and* (F3). *Let* $\Gamma \in C^3$. *Then there are* $\lambda_0 > 0$ *and a nonnegative function* $z_0 \in C_0^\infty(\bar{\Omega})$ *with* $\max z_0 \in (\rho_1, \rho_2)$, *such that for all* $\lambda > \lambda_0$, (P_λ) *possesses exactly one solution* u_λ *with* $z_0 < u_\lambda < \rho_2$. *Moreover,* $\lim\limits_{\lambda\to\infty} \max u_\lambda = \rho_2$.

Remarks 1. We will state and prove a sharper version of this theorem in section 4 (theorem 2').

2. If $\rho_1 < 0$, or $\rho_1 = 0$ and $f'(0) > 0$, theorem 2 was proven in a recent paper, [3], by Angenent. For $\rho_1 \leq 0$ there are also related results in [8].

3. If $\rho_1 = 0$ and $f'(0) = 0$, Rabinowitz showed in [19] the existence of pairs of solutions for λ large enough by a degree argument.

When $\rho_1 = 0$ and $f'(0) = 0$ the question arises, whether or not there are exactly two positive solutions of (P_λ), with maximum less than ρ_2, for λ large enough. We shall consider this problem only for $\Omega = B$, the unit ball in $\mathbb{R}^N$.

It is known, [12], that positive solutions for $\Omega = B$ are radially symmetric, and can be parametrized by $u(0)$. If f satisfies (F1) to (F3), it follows from theorem 1 and 2' that λ is a monotone increasing function of $u(0)$, for $u(0) \in (\rho_2-\varepsilon, \rho_2)$, where ε is some small positive number. Let $\mathcal{C}$ denote the component of solutions of (P) containing these solutions (λ, u) with $u(0) \in (\rho_2-\varepsilon, \rho_2)$.

Set $\rho^* := \inf\{u(0)\ ;\ (\lambda,u) \in \mathcal{C}\}$. If $\rho^* > 0$, it can be shown that more than one component of solutions (λ,u), with $u(0) \in (0,\rho_2)$ may exist, implying the existence of at least four solutions for λ large enough.

In theorem 3 we find a sufficient condition on f, which guarantees the existence of a component $\mathcal{D}$ of solutions (λ,u) of (P) satisfying $\inf\{u(0)\ ;\ (\lambda,u) \in \mathcal{D}\} = 0$.

Theorem 3. *If in problem* (P) *,* Ω *is the unit ball in* $\mathbb{R}^N$*, with* $N > 2$*, and* f *satisfies the condition*

(G1)
$$f(u) = |u|^\alpha \,.\, g(u) \quad \text{for some } \alpha \in (1, \tfrac{N+2}{N-2}) \text{ and } g \in C^{1,\gamma} \text{ with } g(0) > 0$$

then the following holds.
There is an $\varepsilon_0 > 0$ *such that for every* $\varepsilon \in (0,\varepsilon_0)$ *there exists a positive solution* (λ,u) *of* P *with* $u(0) = \varepsilon$.
Moreover λ *is a decreasing function of* ε*, and* $\lim_{\varepsilon \downarrow 0} \lambda(\varepsilon) = \infty$.

If f satisfies (G1), (F1) and (F3), there is one branch of solutions $\lambda \to (\lambda,\bar{u}_\lambda)$ with $\lim_{\lambda\to\infty} \bar{u}_\lambda(0) = \rho_2$, and one branch of solutions $\lambda \to (\lambda,\underline{u}_\lambda)$ with $\lim_{\lambda\to\infty} \underline{u}_\lambda(0) = 0$. Then, since $u(0) \in (\rho^*,\rho_2)$ parametrizes the solutions of (P) on the ball, which are radially symmetric, [12], one finds the following. For λ large enough, (P_λ) possesses exactly two positive solutions, with maximum less than ρ_2, if and only if $\rho^* = 0$. If $\rho^* > 0$, there exists a positive radially symmetric solution of

(P*)
$$\begin{cases} -\Delta u = f(u) \text{ in } \mathbb{R}^N, \\ \lim_{|x|\to\infty} u(x) = 0, \end{cases}$$

satisfying $u(0) = \rho^*$.
For the sake of completeness this will be shown in section 5. Ni and Serrin, in [15], found conditions on f which exclude the existence of a positive solution of (P^*).
Combining these results we obtain :

Corollary. *If in problem* (P) *on the unit ball in* $\mathbb{R}^N$*, with* $N > 2$, f *satisfies condition* (G1) , (F1) , (F3) *and* for α and g defined in (G1) either holds $\alpha \le \frac{N}{N-2}$ or

(G2) $$(\frac{N+2}{N-2} - \alpha) \cdot u^{\alpha+1} \cdot g(u) \ge \frac{2N}{N-2} \int_0^u s^{\alpha+1} \cdot g'(s)ds \text{ for all } u \in [0,\rho_2].$$

then for λ large enough problem (P_λ) possesses exactly two positive solutions with maximum less than ρ_2.

Remarks. 1. If $N \le 2$, theorem 3 and corollary still hold if one replaces in (G1), $(1, \frac{N+2}{N-2})$ by $(1,\infty)$. Condition (G2) is no longer needed.

2. In [11], Gardner and Peletier prove a similar result when $\rho_1 > 0$, by using different techniques.

3. For every $\alpha \in (\frac{N}{N-2}, \frac{N+2}{N-2})$ a function f exists, for which $\rho^* > 0$ Such f can be found by using the example on page 2 of [15]. This construction is done in [7].

Concerning the proofs, the main tools will be the sweeping principle of Serrin,see [22],[21], and the construction of appropriate super- and subsolutions. For the sake of completeness we define in the appendix a notion of super-and subsolutions and we prove a suitable version of the sweeping principle. Some basic ideas for the proof of theorem 2 are contained in [3].
The results of this paper where announced in [6].
We learned that Dancer and Schmitt, [24], have independently found a different proof of the necessity of (F2) in theorem 1.

2. PRELIMINARY RESULTS.

In this section we collect some preliminary results, which will be useful in the coming proofs. The first result for $f(0) > 0$ is contained in [13].

Lemma 2.1. *Let* $f \in C^1$ *satisfy* (F1), (F2) *and* $f(0) \geq 0$. *Then problem* (P) *possesses a positive solution* (λ, u), *with* $\max u \in (\rho_1, \rho_2)$.

Proof. First modify the function f outside of $[0,\rho_2]$ by setting $f(\rho) = 0$ for $\rho > \rho_2$ and $f(\rho) = 2f(0) - f(-\rho)$ for $\rho < 0$. Note that f is bounded on $\mathbb{R}$. As in [13] we want to minimize

$$I(u,\lambda) = \tfrac{1}{2} \int_\Omega |Du|^2\, dx - \lambda \int_\Omega F(u)dx \quad \text{in } W_0^{1,2}(\Omega),$$

where $F(u) = \int_0^u f(s)ds$.

For $\lambda > 0$, $I(u,\lambda)$ is bounded below.

Let u_n be a minimizing sequence for a fixed λ, then

$$I(|u_n|,\lambda) = \tfrac{1}{2} \int_\Omega |D|u_n||^2\, dx - \lambda \int_\Omega F(|u_n|)\, dx \leq$$

$$\leq \tfrac{1}{2} \int_\Omega |Du_n|^2\, dx - \lambda \int_\Omega \left\{ \int_0^{|u_n|} (f(s) - f(0))\, ds + \int_0^{|u_n|} f(0)\, ds \right\} dx \leq$$

$$\leq \tfrac{1}{2} \int_\Omega |Du_n|^2\, dx - \lambda \int_\Omega \left\{ \int_0^{u_n} (f(s) - f(0))\, ds + \int_0^{u_n} f(0)\, ds \right\} dx =$$

$$= I(u_n,\lambda).$$

Since $I(\cdot,\lambda)$ is sequentially weakly lower semicontinuous and coercive in $W_0^{1,2}(\Omega)$, $I(\cdot,\lambda)$ possesses a nonnegative minimizer, which we denote by u_λ. It is standard that (λ,u_λ) is a solution of (P), with the modified f.
By applying the strong maximum principle, we deduce as in [2], that either $f(\| u_\lambda \|_\infty) > 0$ or $u_\lambda = 0$.
Thus $\| u_\lambda \|_\infty < \rho_2$, hence (λ,u) is a solution of (P).

Set $\alpha = \min\left\{ \int_{\rho}^{\rho_2} f(s)ds \; ; \; 0 \le \rho \le \max(0,\rho_1) \right\}$

$\beta = \max\left\{ \int_{\rho}^{\rho_2} f(s)ds \; ; \; 0 \le \rho \le \rho_2 \right\}$.

Suppose that for all positive λ, $\| u_\lambda \|_\infty \le \rho_1$ holds, then we will obtain a contradiction.

We choose $\delta > 0$ such that $2 \, . |\Omega^\delta| . \beta < |\Omega| \, . \alpha$, with $\Omega^\delta = \{x \in \Omega; d(x,\Gamma) < \delta\}$ and $|\Omega|$ denoting the Lebesgue-measure of Ω. This is possible since $\alpha > 0$ and $\lim_{\delta \downarrow 0} |\Omega^\delta| = 0$.

Next we choose $w \in C_0^\infty(\Omega)$, satisfying $0 \le w \le \rho_2$ in Ω^δ and $w = \rho_2$ in $\Omega - \Omega^\delta$; then

$$I(w,\lambda) - I(u_\lambda,\lambda) =$$

$$= \tfrac{1}{2} \int_\Omega (|Dw|^2 - |Du_\lambda|^2) dx - \lambda \int_\Omega (F(w) - F(u_\lambda)) dx \le$$

$$\le \tfrac{1}{2} \int_\Omega |Dw|^2 dx - \lambda \left(\int_\Omega F(\rho_2) dx + \int_{\Omega^\delta} (F(w) - F(\rho_2)) dx - \int_\Omega F(u_\lambda) dx \right) \le$$

$$\le \tfrac{1}{2} \int_\Omega |Dw|^2 dx + 2 \lambda |\Omega^\delta| \beta - \lambda \int_\Omega (F(\rho_2) - F(u_\lambda)) dx =$$

$$= \tfrac{1}{2} \int_\Omega |Dw|^2 dx + 2 \lambda |\Omega^\delta| \beta - \lambda \int_\Omega \int_{u_\lambda}^{\rho_2} f(s) \, ds \, dx \le$$

$$\le \tfrac{1}{2} \int_\Omega |Dw|^2 dx + \lambda (2 |\Omega^\delta| \beta - |\Omega| \alpha) < 0$$

for λ large enough, since $2|\Omega^\delta| . \beta - |\Omega| . \alpha < 0$.

Then $I(w,\lambda) < I(u_\lambda,\lambda)$, contradicting the fact that u_λ is a minimizer. This completes the proof of the lemma.

In what follows it will be convenient to modify f outside of $[0,\rho_2]$ in an appropriate way.

Let $f \in C^1$, respectively $C^{1,\gamma}$ for some $\gamma \in (0,1)$, satisfy (F1) and (F2). Then there is a function $f^* \in C^1$, respectively $C^{1,\gamma}$, satisfying $f^* = f$ on $[0,\rho_2]$ and

$$(F^*) \begin{cases} f^* \text{ is bounded,} \\ f^* < 0 \text{ in } (\rho_2,\infty). \\ f^* = 0 \text{ in } (-\infty,-1], \\ \int_u^{\rho_2} f^*(s)ds > 0 \text{ for } u \in [-1,0]. \end{cases}$$

Since we are interested in solutions (λ,u) of (P) with $0 \le u < \rho_2$, we may assume without loss of generality that f satisfies (F^*). Then we have

$$(2.1) \qquad \inf\{\int_u^{\rho_2} f(s)ds \ ; \ |\rho_2-u| > \delta\} > 0, \text{ for all } \delta > 0.$$

Lemma 2.2. *Let* $f \in C^1$ *satisfy* (F1),F(2) *and* (F^*).
Then there exist $\mu > 0$ *and* $v \in C^2(\mathbb{R}^N)$ *, radially symmetric, which satisfy :*

$$\begin{cases} -\Delta v = \mu . f(v) \text{ in } \mathbb{R}^N, \\ v(0) \in (\rho_1,\rho_2), \\ v(1) = -1, \\ v'(r) < 0 \text{ for } r > 0. \end{cases}$$

Proof. Since f(u-1) satisfies F(1) and (F2) it follows from lemma 2.1 that there exists a positive solution (μ,w) of

$$\begin{cases} -\Delta u = \lambda . f(u-1) \text{ in } B, \\ u = 0 \quad \text{in } \partial B, \end{cases}$$

where B is the unit ball in $\mathbb{R}^N$, satisfying $\max w \in (\rho_1+1, \rho_2+1)$. By [12] w is radially symmetric and $w'(r) < 0$ for $r \in (0,1)$. Set $v(r) = w(r)-1$ for $r \in [0,1]$ and

$$v(r) = \begin{cases} -1 + (r^{2-N}-1) \ . \ (2-N)^{-1} \ . \ w'(1) \quad \text{for } r \in (1,\infty) \text{ if } N \neq 2, \\ -1 + \log r \ . \ w'(1) \text{ for } r \in (1,\infty) \quad \text{if } N = 2. \end{cases}$$

Since $f = 0$ on $(-\infty,-1]$ one verifies that v is the required function. This completes the proof of the lemma.

Corollary 2.3. *Let* (μ,v) *be like in lemma 2.2, and let* $\alpha \in (0,1)$ *be the unique zero of* v.
Then for $y \in \Omega$ *and* $\lambda > \mu \,.\, \alpha^2 .\, d(y,\Gamma)^{-2}$

$$w(\lambda, y ; x) := v((\lambda/\mu)^{\frac{1}{2}} \,.\, (x-y)) \,,\; x \in \Omega, \tag{2.2}$$

is a subsolution of (P_λ).

Proof. The function $w(\lambda,y) \in C^2(\mathbb{R}^N)$ satisfies $-\Delta w = \lambda \,.\, f(w)$ in $\mathbb{R}^N$, hence $\int_\Omega (w \,.\, (-\Delta\phi) - \lambda \,.\, f(w) \,.\, \phi)dx = 0$ for all $\phi \in \mathcal{D}^+(\Omega)$, where $\mathcal{D}^+(\Omega)$ consists of all nonnegative functions in $C_0^\infty(\Omega)$. Since $w(\lambda,y) < 0$ on Γ for $\lambda > \mu \,.\, \alpha^2 \,.\, d(y,\Gamma)^{-2}$, $w(\lambda,y)$ satisfies the definition of subsolution given in the appendix. This shows the corollary.

Next we establish some results for the one-dimensional problem

$$\text{(2.3)} \quad \begin{cases} -u'' = f(u), \; x > 0, \\ u(0) = 0, \\ u'(0) = \delta, \end{cases}$$

where $f \in C^1$ satisfies (F1), (F2) and $(F^\star)$.

Lemma 2.4. *Problem (2.3) possesses a unique solution* u_δ *in* $\mathbb{R}_+$ *for all* $\delta \in \mathbb{R}$. *The function* $\delta \to u_\delta \in C[0,r]$ *is continuous for every* $r > 0$.
Moreover, set $\delta_1 = (2 \,.\, \int_0^{\rho_2} f(s)ds)^{\frac{1}{2}}$ *and*
$\delta_2 = (\max\{ -2 \,.\, \int_\rho^0 f(s)ds \;;\; \rho \in [-1,0] \})^{\frac{1}{2}}$,

1) *if* $\delta > \delta_1$, *then* $u_\delta(x) > (\delta-\delta_1) \,.\, x$ *for* $x \in \mathbb{R}_+$,
2) *if* $\delta = \delta_1$, *then* $u'_\delta > 0$ *on* $\mathbb{R}_+$ *and* $\lim_{x\to\infty} u_\delta(x) = \rho_2$,
3) *if* $-\delta_2 \le \delta < \delta_1$, *then* $\sup \{u_\nu(x) ; x \in \mathbb{R}_+ , \nu \in [-\delta_2,\delta]\} <$
4) *if* $\delta < -\delta_2$, *then* $u_\delta < 0$ *on* $\mathbb{R}_+$.

Proof. Since f is C^1 and bounded, the first assertion of the lemma is standard.

Note that a solution of (2.3) satisfies

$$(2.4) \qquad (u'(x))^2 = \delta^2 - 2 \cdot \int_0^{u(x)} f(s)ds .$$

1) If $\delta > \delta_1$, then using (2.1) and (2.4) we have

$$(u_\delta'(x))^2 > (\delta-\delta_1)^2 + 2 \cdot \int_{u_\delta(x)}^{\rho_2} f(s)ds \geq (\delta-\delta_1)^2.$$

Since $u_\delta'(0) > 0$, we obtain $u_\delta(x) > (\delta-\delta_1) \cdot x$ for $x \in \mathbb{R}_+$.

2) If $\delta = \delta_1 = (2 \cdot \int_0^{\rho_2} f(s)ds)^{1/2}$, we have

$$(2.5) \qquad (u_\delta'(x))^2 = 2 \cdot \int_{u_\delta(x)}^{\rho_2} f(s)ds .$$

It follows from (2.5), $f(\rho_2) = 0$ and the uniqueness for the initial value problem that $u_\delta(x) \neq \rho_2$ for all $x \in \mathbb{R}_+$, and thus $u_\delta < \rho_2$ on $\mathbb{R}_+$. Since u_δ is monotonically increasing and bounded there exists a sequence $\{x_n\}$, with $\lim_{n\to\infty} x_n = \infty$ and $\lim_{n\to\infty} u_\delta'(x_n) = 0$. From (2.1) and (2.5) it follows that $\lim_{x\to\infty} u_\delta(x) = \rho_2$.

3) Note that $\delta_1^2 - \delta_2^2 = 2 \cdot \int_0^{\rho_2} f(s)ds - \max\{-2 \cdot \int_\rho^0 f(s)ds \; ; \; \rho \in [-1,0]\} =$

$$= 2 \cdot \min\{\int_\rho^{\rho_2} f(s)ds; \; \rho \in [-1,0]\}.$$

Hence by (2.1), $\delta_1 > \delta_2$.

If $-\delta_2 \leq \nu \leq \delta < \delta_1$, one has

$$0 \leq (u_\nu'(x))^2 = \nu^2 - 2 \cdot \int_0^{u_\nu(x)} f(s)ds \leq$$

$$\leq \max(\delta_2^2, \delta^2) - 2 \cdot \int_0^{u_\nu(x)} f(s)ds =$$

$$= \max(\delta_2^2-\delta_1^2, \delta^2-\delta_1^2) + 2 \cdot \int_{u_\nu(x)}^{\rho_2} f(s)ds.$$

Since $\max(\delta_2^2-\delta_1^2, \delta^2-\delta_1^2) < 0$, one finds, by using (2.1) again, that $|u_\nu(x)-\rho_2| \geq m > 0$ for all $x \in \mathbb{R}_+$. From $u_\nu(0) = 0$ it follows $u_\nu < \rho_2 - m$ on $\mathbb{R}_+$.

4) If $\delta < -\delta_2$, then

$$(u_\delta'(x))^2 > \max\{-2 \cdot \int_\rho^0 f(s)dx \; ; \; \rho \in [-1,0]\} -2 \cdot \int_0^{u_\delta(x)} f(s)ds \geq 0$$

for all $u_\delta(x) \leq 0$.

Since $u_\delta'(0) < 0$, one finds $u_\delta' < 0$ on $\mathbb{R}_+$. Hence $u_\delta < 0$ on $\mathbb{R}_+$.

This completes the proof of lemma 2.4.

Lemma 2.4 will be used to establish some results for the problem on the halfspace $D = \{(x_1,\ldots,x_N) \in \mathbb{R}^N \; ; \; x_1 > 0\}$.

Proposition 2.5. *Let $f \in C^{1,\gamma}$, for some $\gamma \in (0,1)$, satisfy (F1), (F2) and (F3). Let $u \in C^2(D) \cap C(\bar{D})$ be a solution of*

$$\begin{cases} -\Delta u = f(u) \text{ in } D, \\ u = 0 \text{ on } \partial D, \end{cases}$$

with $0 \leq u < \rho_2$ in D and $\lim_{x_1 \to \infty} u(x_1,x') = \rho_2$ uniformly for $x' \in \mathbb{R}^{N-1}$.

Then $u(x_1,x') = u_{\delta_1}(x_1)$ for $x_1 \geq 0$ and $x' \in \mathbb{R}^{N-1}$ where u_{δ_1} is defined in lemma 2.4.

In order to prove proposition 2.5 we also need

Lemma 2.6. *Let $(x_1,u) \to g(x_1,u)$ be a function such that $g, \frac{\partial}{\partial u} g \in C^{0,\gamma}(\overline{\mathbb{R}}_+ \times$ for some $\gamma \in (0,1)$, and $|g(x_1,u)| < h(u)$ for some $h \in C^0(\mathbb{R})$. Let $U \in C^2(D) \cap C^0(\bar{D})$ be a bounded solution of*

$$\begin{cases} -\Delta u = g(x_1,u) \text{ in } D, \\ u = 0 \text{ on } \partial D. \end{cases}$$

Then S, defined by $S(x_1) = \sup\{U(x_1,x') \; ; \; x' \in \mathbb{R}^{N-1}\}$, is continuous on $[0,\infty)$, with $S(0) = 0$, and satisfies

(2.6) $$\int_{\mathbb{R}_+} (S.(-\phi'') - g(x_1,S) \cdot \phi)dx_1 \leq 0 \text{ for all } \phi \in \mathcal{D}^+(\mathbb{R}_+).$$

$\mathcal{D}^+(\mathbb{R}_+)$ consists of all nonnegative functions in $C_0^\infty(\mathbb{R}_+)$.

Proof of lemma 2.6. Since U and ΔU are bounded and $U = 0$ on ∂D, it follows from standard regularity properties that U and all first-order derivatives are uniformly bounded and uniformly Hölder continuous with exponent γ. Let $\{\Omega_n\}$ be an increasing sequence of bounded subdomains of D, with smooth boundary and such that $\bigcup_{n \in \mathbb{N}} \Omega_n = D$. We first prove that for each $n \in \mathbb{N}$, if $u_1, u_2 \in C(\bar{\Omega}_n) \cap H^1(\Omega_n)$ satisfy

$$(2.7) \qquad \int_D (u.(-\Delta\phi) - g(x_1,u) . \phi)dx \le 0 \text{ for all } \phi \in \mathcal{D}^+(\Omega_n),$$

then $u_3 = \sup(u_1,u_2)$ also satisfies (2.7).
Let $\omega \in \mathbb{R}_+$ be such that $u \to g(x_1,u) + \omega . u$ is increasing on $[\min u_1 \wedge \min u_2, \max u_1 \vee \max u_2]$ for every $x \in \bar{\Omega}_n$.
We obtain

$$\int_D (u_i . (-\Delta\phi) + \omega . u_i . \phi)dx \le \int_D (g(x_1,u_3) + \omega . u_3)\phi . dx$$

for all $\phi \in \mathcal{D}^+(\Omega_n)$, $i = 1,2$.
Set $h = g(x_1,u_3) + \omega . u_3$ and let w satisfy

$$\begin{cases} -\Delta w + \omega . w = h \text{ in } \Omega_n, \\ w = 0 \text{ on } \partial\Omega_n. \end{cases}$$

Note that $w \in C(\bar{\Omega}_n) \cap H^1(\Omega_n)$. Then $w_i = u_i - w$, $i = 1,2$, satisfies

$$(2.8) \qquad \int_D (w .(-\Delta\phi) + \omega . w . \phi)dx \le 0 \text{ for all } \phi \in \mathcal{D}^+(\Omega_n).$$

It is known that $\sup(w_1,w_2)$ also satisfies (2.8), see [23,Th. 28.1]. Therefore u_3 satisfies (2.7). Note that $u_3 \in C(\bar{\Omega}_n) \cap H^1(\Omega_n)$. By induction it follows that if $u_i \in C(\bar{\Omega}_n) \cap H^1(\Omega_n)$, $i = 1,\ldots,k$, satisfies (2.7), then $\sup\{u_i\ ;\ i = 1,\ldots,k\}$ also satisfies (2.7). Let u_i be translates of U perpendicular to $(1,0,\ldots,0)$. Since $U \in C(\bar{D}) \cap H^1_{loc}(D)$, $\sup\{u_i\ ;\ i = 1,\ldots,k\}$ will satisfy (2.7).

Then by using the Lebesgue dominated convergence theorem and the fact that U is bounded, one shows that

$S(x_1) = \sup \{U(x_1,x') ; x' \in \mathbb{R}^{N-1}\} = \sup \{U(x_1,x') ; x' \in \mathbb{Q}^{N-1}\}$

also satisfies (2.7) for each n. From the choice of the Ω_n it follows

$$\int_D (S(-\Delta\phi) - g(x_1,S) \cdot \phi)dx \leq 0 \text{ for all } \phi \in \mathcal{D}^+(D).$$

By choosing ϕ of the form $\phi_1.\phi_2$, with $\phi_1 \in \mathcal{D}^+(\mathbb{R}_+)$ and $\phi_2 \in \mathcal{D}^+(\mathbb{R}^{N-1})$, $\phi_2 \neq 0$, one gets (2.6), since S only depends on x_1.
Note that S, as the supremum of continuous functions, is lower semi-continuous on $[0,\infty)$. From (2.6) and the fact that $g(x_1,S)$ is bounded, we deduce that S is the sum of a convex function on $(0,\infty)$ and a C^1-function on $[0,\infty)$. Hence $S \in C(0,\infty)$. Since $U(0,x') = 0$ and since $\frac{\partial}{\partial x_1} U(0,x')$ is uniformly bounded, $S(0) = 0$ and S is continuous in 0.
This completes the proof of lemma 2.6.

Proof of proposition 2.5. Without loss of generality we assume that f satisfies (F^*). Define

$I(x_1) = \inf\{U(x_1,x') ; x' \in \mathbb{R}^{N-1}\}$ and
$S(x_1) = \sup\{U(x_1,x') ; x' \in \mathbb{R}^{N-1}\}$.

It is sufficient to prove that

(2.9) $\quad I \geq u_\delta \quad$ on $\mathbb{R}_+$, and

(2.10) $\quad S \leq u_\delta \quad$ on $\mathbb{R}_+$,

for $\delta = \delta_1$.
We first prove (2.9) for $\delta = \delta_1$. By lemma 2.4,4), (2.9) holds with $\delta < -\delta_2$, since $I \geq 0$ on $\overline{\mathbb{R}}_+$. We will use a sweeping argument to prove (2.9) for every $\delta \in (-\delta_2-1,\delta_1)$. Let $\delta \in (-\delta_2-1,\delta_1)$. By lemma 2.4,3) and 4), there exists $\rho < \rho_2$ such that

(2.11) $\quad \sup\{u_\theta(x_1) ; x_1 \in \mathbb{R}_+, \theta \leq \delta\} \leq \rho.$

For some $R > 0$ one has $I > \rho$ on $[R,\infty)$. It follows from lemma 2.6, with $g(x_1,u) = -f(-u)$, that $I \in C[0,\infty)$, $I(0) = 0$ and

$$\int_{\mathbb{R}_+} (I.(-\phi'') - f(I) \, . \, \phi)dx \geq 0 \text{ for all } \phi \in \mathcal{D}^+(\mathbb{R}_+).$$

Hence I is a supersolution of

$$(2.12) \begin{cases} -u'' = f(u) \text{ in } (0,R), \\ u(0) = 0 \, , \\ u(R) = \rho. \end{cases}$$

For $\theta \in [-\delta_2-1, \delta]$, (2.11) shows that u_θ is a subsolution of (2.12). We are now in the position to use lemma A.2 and we obtain $I \geq u_\delta$ on $(0,R)$, hence on $\mathbb{R}_+$. For $x_1 \geq 0$ one has

$$I(x_1) \geq \lim_{\delta \uparrow \delta_1} u_\delta(x_1) = u_{\delta_1}(x_1).$$

This completes the proof of (2.9), with $\delta = \delta_1$.

Next we give a sketch of the proof of (2.10). Since $\frac{\partial}{\partial x_1} U$ is uniformly bounded, there exists $c > 0$ such that

$$S(x_1) < c \, . \, x_1 \text{ for } x_1 \in \mathbb{R}_+.$$

By lemma 2.4, 1), one has (2.10) with $\delta = \delta_1 + c$. Let $\delta \in (\delta_1,\delta_1+c)$. Also from lemma 2.4, 1), it follows

$$u_\theta(x_1) > \rho_2+1 \text{ for } x_1 > R := (\delta-\delta_1)^{-1}. \, (\rho_2+1) \text{ and } \theta \in [\delta,\delta_1+c].$$

Note that $S \leq \rho_2$. Then one concludes as above after using a sweeping argument for the problem

$$\begin{cases} -u'' = f(u) \quad \text{in } (0,R), \\ u(0) = 0, \\ u(R) = \rho_2. \end{cases}$$

This completes the proof of proposition 2.5.

3. PROOF OF THE FIRST THEOREM.

NECESSITY : With $J(\rho) = \int_{\rho}^{\rho_2} f(s)ds$, and assuming $\rho_1 > 0$, define

$$J^* := \min \{J(\rho);\ \rho \in [0,\rho_1]\}.$$

Suppose condition (F2) is not satisfied, that is $J^* \le 0$.
Let (λ,u) be a positive solution of (P) satisfying $\max u \in (\rho_1,\rho_2)$.
We will obtain a contradiction.
First, if $J^* = 0$, modify f to f^* in C^1 such that $f > f^* > 0$ in $(\max u,\ \rho_2)$ and $f = f^*$ elsewhere. Still u is a solution of (P_λ), but now $J^* < 0$.
Hence we may assume without loss of generality that $J^* < 0$.
Consider the initial value problem

$$(3.1) \qquad -v'' = f(v),$$

$$(3.2) \quad \begin{cases} v(0) = \rho_2, \\ v'(0) = -(-J^*)^{\frac{1}{2}}. \end{cases}$$

For a solution of (3.1), (3.2) one has :

$$(3.3) \qquad (v'(r))^2 = -J^* + 2 \cdot \int_{v(r)}^{\rho_2} f(s)ds.$$

Set $\rho^* := \max\{\rho \in [0,\rho_1]\ ;\ J(\rho) = J^*\}$.
Because of (3.3), $(v'(r))^2 \ge -J^* + 2 \int_{v(r)}^{\rho_2} f(s)ds \ge -J^* > 0$ holds for $v(r)$ in $[\rho_1,\rho_2]$, and hence $\inf v < \rho_1$.
Next we show that v remains positive. If not, there exists an r^* such that $v(r^*) = \rho^*$, and since (3.3) holds, one finds

$$(v'(r^*))^2 = -J^* + 2\int_{\rho^*}^{\rho_2} f(s)ds = +J^* < 0,$$

a contradiction.
So either $v(r) \downarrow \tilde{\rho} \in (\rho^*,\rho_1)$ if $r \to \infty$, or v has a first positive minimum, say in $\tilde{r}$, and v is symmetric with respect to $\tilde{r}$.

In the first case define $V(r) := \begin{cases} v(r) & \text{for } r > 0 \\ \rho_2 & \text{for } r \leq 0 \end{cases}$, and in the second case $V(r) := \begin{cases} v(r) \text{ for } r \in (0,2\tilde{r}) \\ \rho_2 \text{ elsewhere in } \mathbb{R} \end{cases}$.

Set $w(\lambda,t;x) = V(\lambda^{\frac{1}{2}}.(x_1-t))$, where $x = (x_1,\dots\dots,x_N)$.
Then $\{w(\lambda,t;\cdot) \ ; \ t \in \mathbb{R}\}$ is a family of supersolutions, and for t large enough $w(\lambda,t;\cdot) = \rho_2$ in Ω.
By the sweeping principle $u < w(\lambda,t;\cdot)$ for all t.
Hence $u(x) \leq \inf\{w(\lambda,t;x);\ t \in \mathbb{R}\} = \inf v < \rho_1$, a contradiction.

Remark 1. Let $f \in C^1$ satisfy (F1). The proof also shows that, if (F2) is not satisfied, there is no solution u of (P_λ) with $\max u \in (\rho_1,\rho_2)$ even if u changes sign.

Remark 2. Let $f \in C^1$ satisfy (F1) and let $\Omega \subset \mathbb{R}^N$ be an unbounded domain. Note that the same technique shows that problem

$$(P^*) \quad \begin{cases} -\Delta u = f(u) \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \\ \lim\limits_{\substack{|x|\to\infty \\ x\in\Omega}} u(x) = 0 \end{cases}$$

may have a solution u, with $\max u \in (\rho_1,\rho_2)$, only if condition (F2) is satisfied.

SUFFICIENCY : We will prove a stronger result, which will be used later on. Let $x^* \in \Omega$. Then define $\lambda^* = \mu \,.\, \alpha^2 \,.\, d(x^*,\Gamma)^{-2}$ and $z_\lambda = w(\lambda,x^*)$, where μ, α and w are defined in corollary 2.3.

Lemma 3.1. *Let* f *satisfy* (F1), (F2) *and* (F^*).
Then 1) *for* $\lambda > \lambda^*$ *problem* (P_λ) *possesses a solution* $u_\lambda \in [z_\lambda, \rho_2]$,
2) *there exists* $\lambda^{**} > \lambda^*$, $c > 0$ *and* $\tau \in (\rho_1,\rho_2)$, *such that for* $\lambda > \lambda^{**}$ *every solution* $u \in [z_\lambda, \rho_2]$ *of* (P_λ) *satisfies*

(3.4) $\quad u(x) > \min(c \,.\, \lambda^{\frac{1}{2}} \,.\, d(x,\Gamma),\tau)$ for all $x \in \Omega$.

Remark 3. It follows from (3.4) that $u_\lambda > 0$, for $\lambda > \lambda^{**}$, and that $\max u_\lambda \in (\rho_1, \rho_2)$, for λ large enough.

Remark 4. Lemma 3.1, 2), shows $\frac{\partial}{\partial n} u_\lambda < 0$ on Γ for $\lambda > \lambda^{**}$, even when $f(0) < 0$. ($\frac{\partial}{\partial n}$ denotes the outward normal derivative)

Proof of lemma 3.1. By corollary 2.3 one knows that for $\lambda > \lambda^*$, z_λ is a subsolution of (P_λ), with $z_\lambda < \rho_2$. Since ρ_2 is a supersolution of (P_λ), lemma A.1 yields a solution $u_\lambda \in [z_\lambda, \rho_2]$ of (P_λ), for $\lambda > \lambda^*$. This completes the proof of the first assertion.
Since Ω satisfies a uniform interior sphere condition, there exists $\varepsilon_0 > 0$ such that $\Omega = \cup \{B(x,\varepsilon);\ x \in \Omega_\varepsilon\}$, for $\varepsilon \in (0,\varepsilon_0]$, where $\Omega_\varepsilon = \{x \in \Omega;\ d(x,\Gamma) > \varepsilon\}$. Set

$\lambda^{**} = \max(\lambda^*, \mu \cdot \alpha^2 \cdot \varepsilon_0^{-2})$,

$c = \mu^{-\frac{1}{2}} \cdot \inf\{(\alpha-r)^{-1} \cdot v(r)\ ;\ r \in [0,\alpha)\}$ and

$\tau = v(0)$, with μ, v and α defined in corollary 2.3.

Note that $c > 0$ since $v > 0$ on $[0,\alpha)$ and $v'(\alpha) < 0$. Let (λ,u) be a solution of (P) with $\lambda > \lambda^{**}$ and $u \in [z_\lambda, \rho_2]$. Since for $\lambda > \lambda^{**}$, $\Omega_{\alpha.(\mu/\lambda)^{\frac{1}{2}}}$ is arcwise connected and since $w(\lambda,y)$ is a subsolution for $y \in \Omega_{\alpha.(\mu/\lambda)^{\frac{1}{2}}}$, with $w(\lambda,y) < 0$ on Γ, one finds by lemma A.2 that

$$u > w(\lambda,y) \text{ in } \Omega \text{ for all } y \in \Omega_{\alpha.(\mu/\lambda)^{\frac{1}{2}}}.$$

Hence

$$u(x) > c \cdot \lambda^{\frac{1}{2}} \cdot d(x,\Gamma) \text{ for all } x \in \Omega\backslash\Omega_{\alpha.(\mu/\lambda)^{\frac{1}{2}}}, \text{ and}$$

$$u(x) > \tau \text{ for all } x \in \Omega_{\alpha.(\mu/\lambda)^{\frac{1}{2}}},$$

which completes the proof.

4. PROOF OF THE SECOND THEOREM.

As mentioned in the introduction theorem 2 will be a consequence of a sharper version, theorem *2'*.

Theorem 2'. *Let* $\Gamma \in C^3$ *and let* $f \in C^{1,\gamma}$, *for some* $\gamma \in (0,1)$ *satisfy* (F1), (F2) *and* (F3). *Then for some* $\lambda_1 > 0$,

1) *there exists* $\phi \in C^1([\lambda_1,\infty);C^2(\bar{\Omega}))$, *such that* $(\lambda,\phi(\lambda))$ *is a solution of* (P) *for* $\lambda \geq \lambda_1$, *with* $\phi(\lambda) > 0$ *in* Ω, $\max \phi(\lambda) \in (\rho_1,\rho_2)$ *and* $\lim_{\lambda\to\infty} \max \phi(\lambda) = \rho_2$;
2) *if* $\mu_0(\lambda,u)$ *denotes the principal eigenvalue of*

(LP) $$\begin{cases} -\lambda^{-1} \,.\, \Delta h - f'(u) \,.\, h = \mu h \text{ in } \Omega , \\ h = 0 \text{ on } \Gamma, \end{cases}$$

then $\mu_0(\lambda,\phi(\lambda)) > 0$ *for* $\lambda > \lambda_1$;
3) *for all nonnegative* $z \in C_0^\infty(\Omega)$ *with* $\max z \in (\rho_1,\rho_2)$, *there exists* $\lambda(z) > \lambda_1$, *such that, if* (λ,u) *is a solution of* (P) *with* $\lambda > \lambda(z)$ *and* $u \in [z,\rho_2]$, *then* $u = \phi(\lambda)$.

Remark 1. Theorem 2 follows from theorem 2' by choosing a nonnegative function $z_0 \in C_0^\infty(\Omega)$ and setting $\lambda_0 = \lambda(z_0)$, as in the third assertion of theorem 2'.

Remark 2. If $\rho_1 > 0$, let C denote the component of solutions (P) in $\mathbb{R}_+ \times C^2(\bar{\Omega})$ containing $\{(\lambda,\phi(\lambda));\ \lambda \geq \lambda_1\}$. Since C is connected, one has for $(\lambda,u) \in C$ that $\max u \in (\rho_1,\rho_2)$ (see [2]) and $\lambda > 0$. By using degree arguments as in [19],[20], one can show that for λ large enough, $C \cap (\{\lambda\} \times C^2(\bar{\Omega}))$ contains at least two solutions of (P). The proof of this assertion will appear elsewhere.

For the proof of theorem 2' we need the following lemmas.

Lemma 4.1. *Let* $f \in C^1$ *satisfy* (F1), (F2) *and* (F^*). *For every* $\delta > 0$ *there is* $c(\delta) > 0$, *such that for all solutions* (λ,u) *of* (P), *with* $\lambda > \lambda^{**}$ *and* $u \in [z_\lambda, \rho_2]$, *the following holds*

$$(4.1) \qquad u(x) > \min(c(\delta) \, . \, \lambda^{\frac{1}{2}} \, . \, d(x,\Gamma) \, , \, \rho_2 - \delta) \text{ for all } x \in \Omega,$$

with λ^{**} *and* z_λ *as in lemma 3.1.*

Proof of lemma 4.1 . If $\rho_2 - \delta < \tau$, we are done with $c(\delta) = c$ as in lemma 3.1. Otherwise, by (F1) there exists $\sigma > 0$ such that

$$\sigma.(u-\tau) < f(u) \text{ for all } u \in [\tau, \rho_2-\delta].$$

Let ν denote the principal eigenvalue of

$$\begin{cases} -\Delta\psi = \nu \, . \, \psi & \text{in } B, \\ \psi = 0 & \text{on } \partial B, \end{cases}$$

where B denotes the unit ball in $\mathbb{R}^N$.

Then by using lemma A.3 with $\Omega' = \Omega_{k.\lambda^{-\frac{1}{2}}}$, $k = c^{-1} \, . \, \tau$, one finds

$$(4.2) \qquad u(x) > \rho_2 - \delta \text{ for all } x \in \Omega_{((\nu/\sigma)^{\frac{1}{2}}+k).\lambda^{-\frac{1}{2}}} ,$$

since $(\Omega')_{(\nu/\sigma\lambda)^{\frac{1}{2}}} = \Omega_{((\nu/\sigma)^{\frac{1}{2}}+k).\lambda^{-\frac{1}{2}}}$.

By (3.4) one finds

$$(4.3) \qquad u(x) > c(\delta) \, . \, \lambda^{-\frac{1}{2}} \, . \, d(x,\Gamma) \text{ for all } x \in \Omega\backslash\Omega_{((\nu/\sigma)^{\frac{1}{2}}+k).\lambda^{-\frac{1}{2}}},$$

with $c(\delta) = \tau . \, ((\nu/\sigma)^{\frac{1}{2}}+k)^{-1}$.

This completes the proof of the lemma.

Lemma 4.2. *Let* $f \in C^{1,\gamma}$ *for some* $\gamma \in (0,1)$, *satisfy* (F1), (F2), (F3) *and* (F^*). *Then there exists* $\lambda_1 > \lambda^{**}$, *such that for every solution* u *of* (P_λ), *with* $\lambda > \lambda_1$ *and* $u \in [z_\lambda, \rho_2]$, *one finds* $\mu_0(\lambda,u) > 0$.

Proof. Suppose this is not the case. Then there exists a sequence $\{(\lambda_n,u_n)\ ;\ n \in \mathbb{N}\}$ of solutions of (P), with $u_n \in [z_{\lambda_n}, \rho_2]$, $\mu_n := \mu_0(\lambda_n,u_n) \le 0$ for all n, and $\lim_{n\to\infty} \lambda_n = \infty$.

Let ε be defined by (F3). Since $\mu_n \le 0$, for all n, the associated eigenfunctions v_n, normalized by $\max v_n = 1$, satisfy

$$-\lambda_n^{-1} \cdot \Delta v_n(x) = (f'(u_n(x)) + \mu_n) \cdot v_n(x) \le 0 \text{ for } x \in \Omega_{K.\lambda_n^{-1/2}},$$

where

$$K = (c(\varepsilon))^{-1} \cdot (\rho_2 - \varepsilon)\ . \tag{4.4}$$

The constant $c(\varepsilon)$ is defined in the previous lemma.

Hence the function v_n is subharmonic in $\Omega_{K.\lambda_n^{-1/2}}$, and v_n attains its maximum outside of $\Omega_{K.\lambda_n^{-1/2}}$. Like in [3] let $y^n \in \Omega\backslash\Omega_{K.\lambda_n^{-1/2}}$ be a point where v_n attains its maximum and let $x^n \in \Gamma$ be a point which minimizes $\{d(x,y^n)\ ;\ x \in \Gamma\}$. Since x^n and μ_n are bounded, there exists a subsequence, still denoted $\{(\lambda_n,u_n)\}$, such that $\lim_{n\to\infty} x^n = \bar{x} \in \Gamma$ and $\lim_{n\to\infty} \mu_n = \bar{\mu} \le 0$. Let $\mathcal{O}$ be an open neighbourhood of $\bar{x}$ in $\mathbb{R}^N$, chosen so small that it permits C^3 local coordinates $(\xi_1,\dots,\xi_N) : \mathcal{O} \to \mathbb{R}^N$, such that $x \in \Omega \cap \mathcal{O}$ if and only if $\xi_1(x) > 0$, $\xi(\bar{x}) = 0$. In these coordinates the Laplacian is given by

$$\Delta u = \sum_{i,j} a_{ij}(\xi) \cdot \frac{\partial}{\partial \xi_i}\frac{\partial}{\partial \xi_j}\tilde{u} + \sum_j b_j(\xi) \cdot \frac{\partial}{\partial \xi_j}\tilde{u},$$

where $a_{ij} \in C^2$, $b_j \in C^1$ and $u(x) = \tilde{u}(\xi(x))$.

Moreover we choose the local coordinates such that $a_{ij}(0) = \delta_{ij}$.

Next define the functions

$$U_n(\eta) = \tilde{u}_n(\xi(x^n) + \lambda_n^{-1/2} \cdot \eta),$$
$$V_n(\eta) = \tilde{v}_n(\xi(x^n) + \lambda_n^{-1/2} \cdot \eta).$$

Since $\{U_n\}$ and $\{V_n\}$ are precompact in C^2_{loc}, there exists a convergent subsequence. Hence there are $U, V \in C^2(\bar{D})$, bounded and positive in $D = \{(x_1,x')\ ;\ x_1 > 0,\ x' \in \mathbb{R}^{N-1}\}$, satisfying respectively

$$\begin{cases} -\Delta U = f(U) \text{ in } D, \\ U = 0 \text{ on } \partial D, \end{cases}$$

$$\begin{cases} -\Delta V - f'(U) \,.\, V = \bar{\mu} \,.\, V \text{ in } D, \\ V = 0 \text{ on } \partial D. \end{cases}$$

Moreover by lemma 4.1 for every $\delta > 0$ the following inequalities,

$$(4.5) \qquad \min(c(\delta) \,.\, x_1 \,,\, \rho_2 - \delta) \le U(x_1,x') \le \rho_2 \quad \text{for all } x_1 > 0,\ x' \in \mathbb{R}^{N-1}$$

hold. From proposition 2.5 we have

$$U(x_1,x') = u_{\delta_1}(x_1) \text{ for } x_1 \ge 0,\ x' \in \mathbb{R}^{N-1}.$$

Set $S(x_1) = \sup\{V(x_1,x')\ ;\ x' \in \mathbb{R}^{N-1}\}$. Then $0 < S \le 1$ in $\mathbb{R}_+$ and we obtain by using lemma 2.6 that $S \in C[0,\infty)$, $S(0) = 0$ and

$$(4.6) \qquad \int_{\mathbb{R}_+} (S.(-\phi'') - (f'(u_{\delta_1}) + \bar{\mu}) \,.\, S \,.\, \phi)dx \le 0 \text{ for all } \phi \in \mathcal{D}^+(\mathbb{R}_+).$$

Since $u'_{\delta_1} > 0$ on $\overline{\mathbb{R}}_+$, there exists a smallest $C > 0$ such that $W := C \,.\, u'_{\delta_1} - S \ge 0$ on $[0,K+1]$, where K is defined in (4.4).

Then one finds by using (4.6) and $(-u'_{\delta_1})'' = f'(u_{\delta_1}) \,.\, u'_{\delta_1}$ in $\mathbb{R}_+$, that

$$(4.7) \qquad \int_{\mathbb{R}_+} (W.(-\phi'') - f'(u_{\delta_1}).W \,.\, \phi)dx \ge 0 \text{ for all } \phi \in \mathcal{D}^+(\mathbb{R}_+).$$

Since W is nonnegative in $[0,K+1]$, there is $\omega > 0$ such that

$$\int_{\mathbb{R}_+} (W.(-\phi'') + \omega \,.\, W \,.\, \phi)dx \ge 0 \text{ for all } \phi \in \mathcal{D}^+((0,K-1)).$$

By [5, corollary p. 581] and the fact that $W \not\equiv 0$, one obtains

$$(4.8) \qquad W \ge b \,.\, x \,.\, (K+1-x) \text{ for all } x \in [0,K+1] \text{ and some } b > 0.$$

By construction W vanishes somewhere in $[0,K+1]$. Since $W(0) > 0$, one finds $W(K+1) = 0$. Moreover $f'(u_{\delta_1}) \le 0$ on (K,∞).

Hence (4.6) yields that S is convex on (K,∞). Since W is the sum of a C^1 and a concave function on (K,∞), (4.8) shows $0 > \frac{d^-}{dx} W(K+1) \geq \frac{d^+}{dx} W(K+1)$, and therefore $W(x) < 0$ on $(K+1,K+1+c)$ for some $c > 0$. Moreover W cannot vanish on $(K+1,\infty)$. Otherwise there would be $c > 0$ such that $W < 0$ on $(K+1,K+1+c)$ and $W(K+1) = W(K+1+c) = 0$. But this cannot happen since by (4.7) W is concave as long as W is negative on (K,∞).

Hence W is concave on $(K+1,\infty)$. Since $\frac{d^+}{dx} W(K+1) < 0$, W is not bounded below, contradicting $W = C.u'_{\delta_1} - S \geq -1$ on $\mathbb{R}_+$. This completes the proof of lemma 4.2.

It follows from lemma 4.2 that for $\lambda > \lambda_1$, (P_λ) possesses at most one solution in $[z_\lambda,\rho_2]$. Indeed, choose $\omega > 0$ such that $\lambda . f'(u) + \omega > 0$ for $u \in [0,\rho_2]$, and define the mapping $K : C(\bar{\Omega}) \to C(\bar{\Omega})$ by

$$K(u) := (-\Delta + \omega)^{-1} (\lambda.f(u) + \omega . u),$$

where $(-\Delta+\omega)^{-1}$ is the inverse of $-\Delta+\omega$ with homogeneous Dirichlet boundary conditions. By our choice of ω, K maps $[z_\lambda,\rho_2]$ into itself and K has no fixed point on its boundary. Since K is compact, the Leray-Schauder degree on (z_λ,ρ_2) is well defined. Because (z_λ,ρ_2) is convex one finds

$$\text{degree } (I - K, (z_\lambda,\rho_2), 0) = 1.$$

If (λ,u) is a solution of (P), with $u \in [z_\lambda,\rho_2]$ and $\mu_0(\lambda,u) > 0$, it follows that u is an isolated fixed point of K. Moreover, the local degree of $I - K$ at u is 1. From the additivity of the degree it follows that K possesses at most one fixed point in (z_λ,ρ_2). We denote this solution by $\phi(\lambda)$. Since $\mu_0(\lambda,\phi(\lambda)) > 0$, for $\lambda > \lambda_1$, one finds by the implicit function theorem and Schauder estimates, that $\lambda \to \phi(\lambda) \in C^1([\lambda_1,\infty); C^{2,\gamma}(\bar{\Omega}))$. The estimate (4.1) implies that $\lim_{\lambda\to\infty} \max \phi(\lambda) = \rho_2$.

It remains to prove the third assertion of theorem 2'. Let $z \in \mathcal{D}^+(\Omega)$ with $\max z \in (\rho_1,\rho_2)$. It follows form the first part of the proof, that it is sufficient to show that there exists $\lambda(z) > \lambda_1$, such that any solution u of (P_λ), with $\lambda > \lambda(z)$ and $u \in [z,\rho_2]$, is larger than z_λ. This will be done in two steps.

First note that, form the definition of z,there exists an $s \in (\rho_1,\rho_2)$ and a ball $B(x_0,r) \subset \Omega$, such that $z > s$ in $B(x_0,r)$. Let $\sigma > 0$ be such that $f(u) > \sigma \cdot (u-s)$ for $u \in [s,\tau]$, where $\tau = \max z_\lambda$. For $\lambda > \lambda_1(z) := ((\nu/\sigma)^{\frac{1}{2}} + \mu^{\frac{1}{2}})^2 \cdot r^{-2}$, with μ defined in lemma 2.2, we can apply lemma A.3 in order to get

$$u(x) > \tau \quad \text{for} \quad x \in B(x_0,(\mu/\lambda)^{\frac{1}{2}}) \subset B(x_0,r-(\nu/\sigma\lambda)^{\frac{1}{2}}).$$

Observe that $w(\lambda,x_0) < u$ in Ω for $\lambda > \lambda_1(z)$. By corollary 2.3 $w(\lambda,x_0)$ is a subsolution of (P_λ) for $\lambda > \lambda_1(z)$.
Finally, like in the proof of lemma 3.1 part 2), one uses lemma A.2 to show that if $u > w(\lambda,x_0)$ in Ω and $\lambda > \lambda(z) := \max(\lambda_1(z),\lambda^{**})$ also the following estimate holds,

$$u > w(\lambda,x^*) = z_\lambda.$$

This completes the proof of theorem 2'.

5. PROOF OF THE THIRD THEOREM

Note that, if (λ,u) is a positive solution of (P), then $v := (u(0))^{-1}.u$ satisfies

$$\begin{cases} -\Delta v = (u(0))^{\alpha-1} . \lambda . v^{\alpha} . g(u(o).v) & \text{in } B \\ v = 0 \quad \text{on } \partial B. \end{cases}$$

Moreover by defining $w(r) := v(R^{-1}r)$ with $\varepsilon = u(0)$ and

(5.1) $\qquad R = u(0)^{\frac{1}{2}(\alpha-1)}.\lambda^{\frac{1}{2}}$ one gets

(5.2) $$-w'' - \frac{N-1}{r} w' = w^{\alpha} . g(\varepsilon.w)$$

(5.3) $$\begin{cases} w(0) = 1 \\ w'(0) = 0 \end{cases}$$

$$\begin{cases} w(R) = 0 \\ w > 0 \text{ on } [0,R). \end{cases}$$

Let $w(\varepsilon,\cdot)$ denote the unique solution of the initial value problem (5.2-5.3)

Lemma 5 . *There exists* $\varepsilon_1 > 0$ *such that for* ε *in* $[0,\varepsilon_1)$, $w(\varepsilon,\cdot)$ *possesses a first zero, which we denote by* $R(\varepsilon)$. *Moreover* R *as a function of* ε *is* $C^1(0,\varepsilon_1) \cap C[0,\varepsilon_1)$ *and* $\frac{d}{d\varepsilon} R$ *is bounded on* $(0,\frac{1}{2}\varepsilon_1)$

We first show that the assertion of theorem 3 is an easy consequence of this lemma. By (5.1) we have $\lambda(\varepsilon) = R(\varepsilon)^2 . \varepsilon^{1-\alpha}$, and hence $\frac{d}{d\varepsilon}\lambda(\varepsilon) = R(\varepsilon) . \varepsilon^{-\alpha}.(2\varepsilon\frac{d}{d\varepsilon} R(\varepsilon) + (1-\alpha) R(\varepsilon))$, $0 < \varepsilon < \varepsilon_1$. Since $\alpha - 1 > 0$, $R(0) > 0$ and $\frac{d}{d\varepsilon} R$ is bounded on $(0,\frac{1}{2}\varepsilon_1)$, it follows that $\frac{d}{d\varepsilon}\lambda(\varepsilon) < 0$ on some interval $(0,\varepsilon_0)$.
Then for $\lambda > \lambda(\varepsilon_0)$, $u_\lambda(r) = \varepsilon(\lambda). w(R(\varepsilon(\lambda)).r)$ is a solution of (P_λ) on the unit ball, where $\varepsilon(\lambda)$ is the inverse of the function $\lambda(\varepsilon)$. This function $\varepsilon(\lambda)$ is well defined on $(\lambda(\varepsilon_0),\infty)$, decreasing and satisfies $\lim_{\lambda\to\infty} \varepsilon(\lambda) = 0$. This completes the proof of the theorem.

Proof of lemma 5 : It is known, see [17], that (5.2-5.3) with $\varepsilon = 0$ possesses a solution w, having a first positive zero which we denoted by R(0). We want to obtain the function $w(\varepsilon,\cdot)$ by a perturbation argument.

Since we are only interested in bounded positive solutions, we modify the right-hand-side of (5.2) by setting $h(\varepsilon,w) = k(w) \, . \, g(\varepsilon w)$ where k is a C^1-function satisfying

$$k(w) = \begin{cases} 0 & \text{for } w \le 0 \\ w^{\alpha} & \text{for } 0 < w < 1 \\ 0 & \text{for } w \ge 2. \end{cases}$$

The function h is $C^1((-1,1) \times \mathbb{R})$ and has bounded derivatives. The initial value problem

$$(5.4) \qquad -w'' - \frac{N-1}{r} w' = h(\varepsilon,w) \qquad \varepsilon \in (-1,1),$$

$$(5.5) \qquad \begin{cases} w(0) = 1 \\ w'(0) = 0, \end{cases}$$

possesses a unique solution $w(\varepsilon,\cdot)$ on $[0,\infty)$.

For ε in $[0,1)$, since $w(\varepsilon,\cdot)$ is decreasing until it possibly becomes zero, this function $w(\varepsilon,\cdot)$ is identical with the one in the lemma, as long as it is positive.

We claim, for every $r > 0$, $w(\cdot,r)$ is a C^1-function of ε.

First this will be proven for $r \in (0,\delta)$, with δ small enough. Note that (5.4-5.5) can be rewritten as $w = T(\varepsilon,w)$, where

$$T(\varepsilon,z)(r) = 1 - \int_0^r t^{1-N} \int_0^t s^{N-1} \, h(\varepsilon,z(s))ds\,dt, \text{ for } z \text{ in } C[0,\delta].$$

For every $\delta > 0$, $T : (-1,1) \times C[0,\delta] \to C[0,\delta]$, where $C[0,\delta]$ is equipped with the supremum-norm, is continuously Fréchet-differentiable. For δ small enough, $T(\varepsilon,\cdot) : C[0,\delta] \to C[0,\delta]$ is a strict contraction with a unique fixed point $z(\varepsilon)$ such that $\varepsilon \to z(\varepsilon)$ is continuously differentiable. Since $w(\varepsilon,r) = z(\varepsilon)(r)$, the claim is proven for $r < \delta$. By repeating the argument it can be shown that $\varepsilon \to w(\varepsilon,r)$ is continuously differentiable for every $r > 0$.

Since $w(0,R(0)) = 0$ and $w_r(0,R(0)) < 0$ it follows from the implicit function theorem, that there exist an $\varepsilon_1 > 0$ and a continuously differentiable function $R(\cdot)$, defined on $(-\varepsilon_1,\varepsilon_1)$, such that $w(\varepsilon,R(\varepsilon)) = 0$. From (5.4) it follows that $R(\varepsilon)$ is the unique zero of $w(\varepsilon,\cdot)$ on $\mathbb{R}^+$. This completes the proof.

Proof of the corollary : Since $u(0)$ parametrizes the solutions (λ,u) of (P), $\rho^* = \inf\{\sigma > 0;$ (P) has a solution (λ,u), with $u(0) = \rho$, for all $\rho \in [\sigma,\rho_2)\}$. Suppose $\rho^* > 0$ and let v be the solution of the initial value problem

$$(5.6) \qquad -v'' - \frac{N-1}{r} \cdot v' = f(v),$$

$$(5.7) \qquad \begin{cases} v(0) = \rho^* , \\ v'(0) = 0. \end{cases}$$

Since $f(\rho) > 0$ on $(0,\rho^*]$, v is strictly decreasing while v is positive. If v has a (first) positive zero R, then $(R^2, v(R^{-1}\cdot))$ is a solution of (P), which contradicts the definition of ρ^*.
If v stays positive, then

$$(5.8) \qquad \lim_{r\to\infty} v(r) = 0.$$

Otherwise, there are $c > 0$ and $R > 0$ such that $f(v(s)) > c > 0$ for $s > R$. By integrating (5.6), one finds

$$v'(r) = (R/r)^{N-1} \cdot v'(R) - r^{1-N} \cdot \int_R^r s^{N-1} f(v(s))ds \le$$

$$\le (R/r)^{N-1} \cdot v'(R) - (c/N) \cdot (r-R.(R/r)^{N-1}) < -1,$$

for r large enough, contradicting the fact that v stays positive.
The existence of a positive function satisfying (5.6-5.8), is contradicted by theorem 2.2 of [15], if $\alpha \le N/(N-2)$, and by theorem 3.1 of [15], if the integral condition of (G2) is satisfied. Therefore $\rho^* = 0$. This completes the proof.

6. APPENDIX.

In this section we state, for the sake of completeness, a definition and some lemmas concerning sub- and supersolutions of problem

$$\text{(H)} \quad \begin{cases} -\Delta u = h(u) & \text{in } \Omega \subset \mathbb{R}^N, \\ u = g & \text{on } \Gamma, \end{cases}$$

where Ω is a bounded domain with C^3-boundary, $h \in C^1$ and $g \in C^0$.

Definition : *We call a function* v *a subsolution (supersolution) of* (H) *if* :

i) $v \in C(\bar{\Omega})$,

ii) $v \le (\ge)\ g$ *on* $\partial\Omega$, *and*

iii) $\int_\Omega (v.(-\Delta\phi) - h(v)\ .\phi)dx \le (\ge)\ 0$ *for every* $\phi \in \mathcal{D}^+(\Omega)$,

where $\mathcal{D}^+(\Omega)$ *consists of all nonnegative functions in* $C_0^\infty(\Omega)$.

Lemma A.1. *Let* v *and* w *be respectively a sub- and supersolution of* (H) *with* $g = 0$. *If* $v \le w$ *in* Ω, *then there exists a solution* $u \in C^2(\bar{\Omega})$ *of* (H) *with* $g = 0$, *which satisfies* $v \le u \le w$ *in* Ω.

Proof : We essentially follow the proof in [21] on page 24. Choose a number $\omega > 0$ such that $h'(u) + \omega \ge 0$ for $\min v \le u \le \max w$, and define the nonlinear map T by $u_1 = Tu$, where

$$\begin{cases} -\Delta u_1 + \omega\ .\ u_1 = h(u) + \omega\ .\ u \text{ in } \Omega\ , \\ u_1 = 0 \quad \text{on } \partial\Omega. \end{cases}$$

Clearly $T : C(\bar{\Omega}) \to C(\bar{\Omega})$ is compact. (Where $C(\bar{\Omega})$ is equipped with the supremum-norm.)

It is standard that T is monotone on $[v,w]$. Next we show that $v_1 := Tv \ge v$ in Ω.

By the definition of a subsolution and by the construction of v_1, we have

$$\int_\Omega (v.(-\Delta\phi) + \omega . v . \phi)\, dx \le \int_\Omega (h(v) + \omega . v)\phi \, dx =$$

$$= \int_\Omega (v_1.(-\Delta\phi) + \omega . v_1 . \phi) dx \quad \text{for every } \phi \in \mathcal{D}^+(\Omega).$$

Thus $z = v_1 - v$ satisfies $z \ge 0$ on $\partial\Omega$, and

$$\int_\Omega (z.(-\Delta\phi) + \omega . z . \phi) dx \ge 0 \quad \text{for every } \phi \in \mathcal{D}^+(\Omega).$$

We claim that z is nonnegative in Ω.
Otherwise there exists a ball $B(x_0,r) \subset \Omega$, such that z is negative in $B(x_0,r)$ and achieves its minimum in x_0.
Hence

$$\int_\Omega z.(-\Delta\phi) dx \ge 0 \quad \text{for every } \phi \in \mathcal{D}^+(B(x_0,r)).$$

This shows z is superharmonic on $B(x_0,r)$, and from the minimum principle we get $z(x) = z(x_0)$ on $B(x_0,r)$.
Then

$$\int_{B(x_0,r)} (z.(-\Delta\phi) + \omega . z . \phi) dx = \omega . z(x_0) \int_{B(x_0,r)} \phi \, dx < 0$$

for every nontrivial $\phi \in \mathcal{D}^+(B(x_0,r))$, a contradiction.
Thus $Tv = v_1 \ge v$ on $\bar{\Omega}$. Similarly, one proves $Tw \le w$ on $\bar{\Omega}$. Now it is standard, see [1], that T possesses a fixed point in [v,w], which is a solution of (H) with $g = 0$.

Next we prove an appropriate version of the sweeping principle of Serrin, [22],[21].

Let $\Gamma = \partial\Omega$ be the union of two disjoint closed subsets Γ_1 and Γ_2, where Γ_1 or Γ_2 may be empty. Let $e \in C^1(\bar{\Omega})$ satisfy $e > 0$ on $\Omega \cup \Gamma_1$ and $e = 0$, $\frac{\partial e}{\partial n} < 0$ on Γ_2. (n is the outward normal).
Set $C_e(\bar{\Omega}) = \{u \in C(\bar{\Omega});\ |u| < \alpha . e \text{ for some } \alpha > 0\}$ and for $u \in C_e(\bar{\Omega})$ define $\| u \|_e = \inf \{\alpha > 0 ;\ |u| < \alpha . e\}$.

Lemma A.2. *Let* u *be a supersolution of* (H) *and let* $A = \{v_t ; t \in [0,1]\}$ *be a family of subsolutions of* (H) *satisfying* $v_t < g$ *on* Γ_1 *and* $v_t = g$ *on* Γ_2, *for all* $t \in [0,1]$.

If 1) $t \to (v_t - v_0) \in C_e(\bar{\Omega})$ *is continuous with respect to the* $\| \cdot \|_e$-*norm*

2) $u \geq v_0$ in $\bar{\Omega}$, *and*

3) $u \not\equiv v_t$, *for all* $t \in [0,1]$,

then there exists $\alpha > 0$, *such that for all* $t \in [0,1]$ $u - v_t \geq \alpha . e$ in $\bar{\Omega}$.

Proof . Set $E = \{t \in [0,1] ; u \geq v_t \text{ in } \bar{\Omega}\}$. By 2) E is not empty. Moreover E is closed. For $t \in E$ $w_t := u - v_t$ satisfies

$$\int_\Omega (w.(-\Delta\phi) + \omega . w . \phi)\, dx \geq 0 \quad \text{for all } \phi \in \mathcal{D}^+(\Omega) \text{ and some } \omega > 0.$$

Since $w_t \not\equiv 0$ it follows from [5, corollary p. 581] that there is $\beta > 0$, such that $w_t \geq \beta . u_0$, for some $u_0 \in C^1(\bar{\Omega})$, which satisfies $u_0 > 0$ in Ω, $u_0 = 0$ and $\frac{\partial}{\partial n} u_0 < 0$ on Γ. The function w_t is positive on Γ_1, which is compact, and continuous on $\bar{\Omega}$. Hence there exists $\gamma > 0$ such that $w_t > \gamma . e$. Since $t \to (w_t - w_0)$ is continuous with respect to the $\|.\|_e$-norm, E is also open. Hence $E = [0,1]$ and there is $\alpha > 0$, such that $w_t > \alpha . e$ in $\bar{\Omega}$ for all $t \in [0,1]$.

This completes the proof of lemma A.2.

Let ψ be the principal eigenfunction, with eigenvalue ν, of

$$\begin{cases} -\Delta v = \lambda . v & \text{in } B, \\ v = 0 \text{ on } \partial B, \end{cases}$$

where B denotes the unit ball in $\mathbb{R}^N$.

Let ψ be normalized such that $\max \psi = 1$.

Lemma A.3. *Let* u *satisfy* $-\Delta u = \lambda . f(u)$ *in an open* $\Omega' \subset \Omega$, *such that* $u(x) > a$ *for* $x \in \Omega'$. *Let* $\sigma > 0$ *be such that* $f(u) > \sigma . (u-a)$ *for* $u \in [a,b]$.

If $x_0 \in (\Omega')_{(\nu/\sigma\lambda)^{\frac{1}{2}}}$, *then* $u(x_0) > b$.

Proof. Set $\theta(x_0,\lambda,t;x) = a + t\,.\,\psi((\sigma\lambda/\nu)^{\frac{1}{2}}\,.\,(x-x_0))$ for $x \in B(\ )$ and $t \in [0,b-a]$, where $B(\ ) = B(x_0,(\nu/\sigma\lambda)^{\frac{1}{2}})$. The set $\{\theta(x_0,\lambda,t); t \in [0,b-a]\}$ is the family of subsolutions of the problem

$$(Pb)\quad \begin{cases} -\Delta v = \lambda.\ f(v) & \text{in } B(\ ) \\ v = u \quad \text{on } \partial B(\ ), \end{cases}$$

and $\overline{B(\ )} \subset \Omega'$.

By using lemma A.2 one finds $u(x_0) > b$.

It remains to show that $\theta(x_0,\lambda,t)$ is a subsolution of (Pb_λ). By the assumption of the lemma $u > a = \theta(x_0,\lambda,t)$ on $\partial B(\ )$. The integral condition is also satisfied :

$$\int_{B(\ )} (\theta(-\Delta\phi) - \lambda.f(\theta).\phi)dx = \int_{B(\ )} (-\Delta\theta - \lambda.f(\theta))\phi\ dx \le$$

$$\le \int_{B(\ )} (-\Delta\theta - \lambda\sigma(\theta - a))\phi\ \ dx = 0 \text{ for all } \phi \in \mathcal{D}^+(B(\ ))$$

This completes the proof of the lemma.

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References.

[1] Amann, H.; Fixed point equations and nonlinear eigenvalue problems in ordered Banach spaces", SIAM Rev. 18 (1976), 620-709.

[2] Ambrosetti, A.- Hess, P.; "Positive solutions of asymptotically linear elliptic eigenvalue problems", J.Math.Anal.Appl. 73 (1980), 411-422.

[3] Angenent, S.B.; "Uniqueness of the solution of a semilinear boundary value problem", Math. Ann. 272 (1985), 129-138.

[4] Berestycki, H.-Lions, P.-L.;"Une méthode locale pour l'existence de solutions positives de problèmes semilinéaires elliptiques dans $\mathbb{R}^N$", J.d'Analyse Math.38 (1980), 144-187.

[5] Clément, Ph.- Peletier, L.A.; "Positive superharmonic solutions to semi-linear elliptic eigenvalue problems", J.Math. Anal. Appl. 100 (1984), 561-582.

[6] Clément, Ph.-Sweers, G.; "Existence et multiplicité des solutions d'un problème aux valeurs propres elliptique semilinéaire", C.R.Acad. Sc. Paris 302, Série I, 19 (1986), 681-683.

[7] Dancer, E.N.; "Uniqueness for elliptic equations when a parameter is large", Nonl. Anal. Theory, Methods and Appl. 8 (1984), 835-836.

[8] Dancer, E.N.; "On the number of positive solutions of weakly nonlinear elliptic equations when a parameter is large", to appear in Proc. London Math. Soc.

[9] Dancer, E.N.; "Multiple fixed points of positive mappings", preprint (1985)

[10] De Figueiredo, D.; "On the uniqueness of positive solutions of the Dirichlet problem $-\Delta u = \lambda \sin u$", Nonl. P.D.E. and Appl. Collège de France Seminar Vol. 7, Pitman (1985), 80-83.

[11] Gardner, R.A.-Peletier, L.A.; "The set of positive solutions of semilinear equations in large balls", to appear.

[12] Gidas, B.-Ni, W.M.-Nirenberg, L.; "Symmetry and related properties via the maximum principle", Comm. Math. Phys. 68 (1979), 209-243.

[13] Hess, P.; "On multiple solutions of nonlinear elliptic eigenvalue problems", Comm. Part. Diff. Eq. 6 (1981), 951-961.

[14] Lions, P.L.; "On the existence of positive solutions of semilinear elliptic equations", SIAM Rev. 24 (1982), 441-467.

[15] Ni, W.M.- Serrin, J.; "Existence and nonexistence theorems for ground states of quasilinear partial differential equations. The anomalous case.",Univ. of Minnesota Math.Report 84-150 (1984).

[16] Nirenberg. L.; "Topics in nonlinear functional analysis", Lect. Notes Courant Inst. of Math. Sc., New York Univ. (1974).

[17] Pohozaev, S.I.; "Eigenfunctions of the equations $\Delta u + \lambda f(u) = 0$", Soviet Math. Doklady 6 (1965), 1408-1411.

[18] Protter, M.H.-Weinberger, H.F.; "Maximum principles in differential equations", Prentice-Hall, Englewood Cliffs, NJ (1967).

[19] Rabinowitz, P.H.; "Pairs of positive solutions of nonlinear elliptic partial differential equations", Indiana Univ. Math. J. 23 (1973), 172-185.

[20] Rabinowitz, P.H.; "Some aspects of nonlinear eigenvalue problems", Rocky Mountain J. Math. 3 (1973), 161-202.

[21] Sattinger, D.H.; "Topics in stability and bifurction theory" Lect. Notes Math. 309, Berlin Heidelberg New York, Springer (1973).

[22] Serrin, J.; "Nonlinear equations of second order", A.M.S. Symp. in Partial Diff. Eq., Berkeley, August 1971.

[23] Trèves, F.; "Basic linear partial differential equations", Academic Press, New York (1975).

[24] Dancer,E.N.-Schmitt, K.; "On positive solutions of semilinear elliptic equations", preprint (1986).

Chapter 2

On the maximum of solutions for a semilinear elliptic problem.

ABSTRACT. In this note one studies some properties of a semilinear elliptic eigenvalue problem with nondefinite right hand side. In the first part it is shown that every solution will have its maximum in some specified interval J. If the domain is inside a cone in $\mathbb{R}^N$ with $N > 1$, then J is strictly less than in the one-dimensional case. In the second part one shows for bounded domains that if the maximum is inside some subinterval of J, then for any eigenvalue there will be at most one solution.

1. INTRODUCTION AND MAIN RESULTS.

In a previous paper,[1], existence and uniqueness results in order intervals of $C(\bar{\Omega})$ were obtained for the following eigenvalue problem

$$(1.1)\qquad \begin{cases} -\Delta u = \lambda f(u) & \text{in } \Omega \subset \mathbb{R}^N, \\ u = 0 & \partial\Omega \quad , \end{cases}$$

with $\lambda > 0$ and f changing sign. In this paper we will give some results which are related with the maximum of the solutions. The assumptions on f are basically the same as in [1]:

(1.2) $f \in C^1$ and there exist ρ_1, ρ_2 with $0 < \rho_1 < \rho_2$ such that $f(\rho_1) = f(\rho_2) = 0$ and $f > 0$ in (ρ_1, ρ_2).

And the condition orginating from [6,8]:

$$(1.3)\qquad \int_{\rho}^{\rho_2} f(s)ds > 0 \text{ for all } \rho \in [0,\rho_1].$$

For functions satisfying (1,2), (1,3) define

$$(1.4)\qquad \rho^* := \inf\{\sigma \in [\rho_1,\rho_2]; \int_{\rho}^{\sigma} f(s)ds > 0 \text{ for all } \rho \in [0,\rho_1]\}.$$

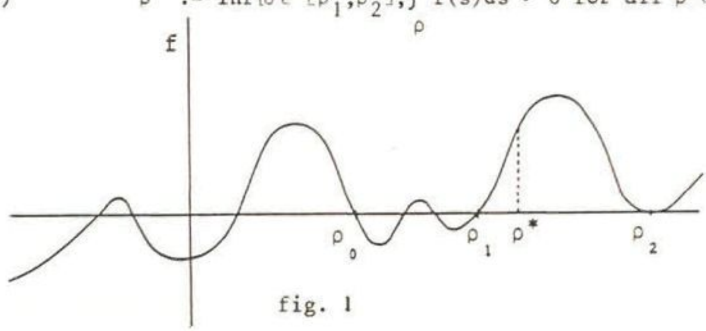

fig. 1

For any positive solution (λ,u) of (1.1) with max $u \in (\rho_1,\rho_2)$ it is proved in [3], that max $u \geq \rho^*$ if Ω satisfies some symmetry properties and $f(0) \geq 0$. For nonsymmetrical bounded domains it is shown in [1,4] that condition (1.3) is necessary for the existence of a solution with max $u \in (\rho_1,\rho_2)$. This proves again that such a solution satisfies max $u \geq \rho^*$. If not, one changes f on (max u,ρ_2), such that (1.3) is no longer satisfied, and one finds a contradiction.

By a more direct proof, which uses a one dimensional argument, one can show the strict inequality, even for some unbounded domains:

Proposition 1. Let $\Omega \subset C \subset \mathbb{R}^N$, with $N > 1$ and where C is a cone. Suppose f satisfies (1.2),(1.3) and (λ,u) satisfies (1.1), with $\sup u \in (\rho_1,\rho_2)$.
If $f(0) \geq 0$ or if $\partial\Omega \in C^1$ then $\sup u > \rho^*$.

The cone C, with vertex in x, is defined by $C := \{x+t(y-x); t > 0, y \in B\}$, where B is an open ball such that $x \notin \bar{B}$.

Remark 1. If $N = 1$, and without more assumptions, one finds $\sup u \geq \rho^*$.

2. If one keeps f and Ω fixed then ρ^* is in general not optimal. An example will be given in section 2.
3. A similar result can be found when $\rho_2 = \infty$, e.g. $f > 0$ on $[\rho_1,\infty)$.
4. For $\Omega = \mathbb{R}^N$, and hence no boundary conditions, there are related results in [9].

Now assume Ω is bounded.
Like in [12] it can be shown that if there is a solution (λ,u_1) of (1.1), with λ large and $\max u_1 \in (\rho_1,\rho_2)$, there has to be a second solution (λ,u_2) with $\max u_2 \in (\rho_1,\rho_2)$. This will be shown elsewhere.
Hence one cannot expect uniqueness. However, in [1] it is proven that there is an order interval $[v(.),\rho_2] \subset C(\bar{\Omega})$, with $v \geq 0$, in which there exists for λ large enough exactly one solution if $\partial\Omega$ is C^3 and under the condition :

$$(1.5) \qquad \begin{cases} f \in C^{1,\gamma}, \text{ for some } \gamma \in (0,1), \\ f' \leq 0 \text{ in } (\rho_2-\delta,\rho_2), \text{ for some } \delta > 0. \end{cases}$$

Dancer [5], raised the question whether is is possible to find a subinterval of (ρ_1,ρ_2), so that for λ large there is a unique positive solution (λ,u) of (1.1) with max u in this subinterval. In the following theorem we will state a slightly stronger result.

Theorem 2. Let $\tau \leq 0$ and suppose Ω is bounded and $\partial\Omega$ is C^3. If f satisfies (1.2) (!.5) and

$$(1.6) \qquad \int_{\rho}^{\rho_2} f(s)ds > 0 \text{ for all } \rho \in [\tau,\rho_1],$$

then there exists $\delta_1 > 0$ such that for all $\lambda > 0$, there is at most one $u_\lambda \in C^2(\bar{\Omega})$ with

i) (λ,u_λ) is a solution of (1.1),

ii) $\tau < u_\lambda$ in Ω,

iii) $\max u_\lambda \in (\rho_2-\delta_1,\rho_2)$.

In [1] it is proved that a curve of positive solutions $(\lambda,\phi(\lambda))$ exists with $\max \phi(\lambda) \in (\rho_1,\rho_2)$ and $\lim_{\lambda\to\infty} \max \phi(\lambda) = \rho_2$.

Combining these results one finds that for all λ large enough there is exactly one u_λ which satisfies i) ii) and iii). Moreover, since the maximal solution below ρ_2 is increasing with λ for starshaped Ω, one finds :

Corollary 3. Suppose Ω is starshaped and bounded, and $\partial\Omega$ is C^3. If f satisfies (1.2) (1.3) and (1.5) then there exists $\delta_2 > 0$, such that max u parametrizes the positive solutions (λ,u) of (1.1) with $\max u \in (\rho_2-\delta_2,\rho_2)$.

For the proof of Theorem 2 we use the Lemmas 5.1 and 5.2 of [14] and the results in [1].

We will finish this chapter by showing that ρ^* is not optimal for positive solutions in bounded Ω.

Lemma 2.2 : Let $f(u) = u - 1$ and let Ω be the unit ball in $\mathbb{R}^N$ with $N > 1$. Then there is $\rho^{**} > 2 = \rho^*$ such that every positive solution (λ, u) of (1.1) with $\max u > 1$ satisfies $\max u > \rho^{**}$.

Proof : By [7] every positive solution (λ, u) is radially symmetric and $u(0) = \max u$. Hence by [10]:

(2.8) $$u(r) = 1 + (u(0)-1) \int_0^\pi \cos(r\sqrt{\lambda} \cos \Theta)(\sin \Theta)^{N-2} d\Theta / \int_0^\pi (\sin \Theta)^{N-2} d\Theta.$$

Since $\nu_N := \inf \{\int_0^\pi \cos(z \cos \Theta)(\sin \Theta)^{N-2} d\Theta / \int_0^\pi (\sin \Theta)^{N-2} d\Theta;\ z > 0\} < 0$ satisfies $\nu_N > -1$ for $N \geq 2$, one finds independently of λ that $0 = u(1) \geq 1 + (u(0)-1).\nu_N$. Hence

(2.9) $$u(0) \geq 1 - \nu_N^{-1} > 2. \qquad \square$$

3. UNIQUENESS OF SOLUTIONS WITH MAXIMUM CLOSE TO ρ_2.

This section will be divided in four parts. For the sake of completeness we will start by recalling some results from [1]. Next we will show some technical lemmas. Thirdly we show an estimate at the boundary for the solutions close to ρ_2 defined in [1]. After these preliminaries we use results from [14] to prove Theorem 2.

3.1 EARLIER RESULTS.

<u>Theorem 3.1</u> [1, Theorem 2']. Let Ω be bounded, $\partial\Omega$ be C^3 and let f *satisfy* (1.2), (1.3) *and* (1.5). *Then for some* $\lambda_1 > 0$,

1) there exists $\phi \in C^1([\lambda_1,\infty); C^2(\bar{\Omega}))$, such that $(\lambda,\phi(\lambda))$ is a solution of (1.1) for $\lambda \geq \lambda_1$, with $\phi(\lambda) > 0$ in Ω, $\max \phi(\lambda) \in (\rho_1,\rho_2)$ and $\lim_{\lambda\to\infty} \max \phi(\lambda) = \rho_2$,
2) for all nonnegative $z \in C_0^\infty(\Omega)$ with $\max z \in (\rho_1,\rho_2)$, there exists $\lambda(z) > \lambda_1$, such that if (λ,u) is a solution of (1.1) with $\lambda > \lambda(z)$ and $u \in [z,\rho_2] \subset C(\Omega)$, then $u = \phi(\lambda)$.

Furthermore, by using [1, Proposition 2.5], one can prove a related result on the half space $D = \mathbb{R}^N_+ = \{(x_1,\ldots,x_2) \in \mathbb{R}^N;\ x_1 > 0\}$.

<u>Proposition 3.2.</u> Let f satisfy (1.2), (1.3) and (1.5). Let $u \in C^2(D) \cap C(\bar{D})$ be a solution of

$$\text{(3.1)}\qquad \begin{cases} -\Delta u = f(u) \text{ in } D, \\ u = 0 \text{ on } \partial D. \end{cases}$$

If there are $c_1 > 0$, $c_2 \in (\rho_1,\rho_2)$ such that

$$\text{(3.2)}\qquad \min(c_1.x_1,c_2) < u(x_1,x') \leq \rho_2 \text{ for } (x_1,x') \in \mathbb{R}_+ \times \mathbb{R}^{N-1},$$

then $u(x_1,x') = U(x_1)$, where U is the solution of

$$(3.3)\qquad \begin{cases} -U'' = f(U), \\ U(0) = 0, \\ U'(0) = (2\int_0^{\rho_2} f(s)ds)^{\frac{1}{2}}. \end{cases}$$

Proof. In order to apply [1, Proposition 2.5], one has to show that $u < \rho_2$ in D and that $\lim_{x_1\to\infty} u(x_1,x') = \rho_2$ uniformly for $x' \in \mathbb{R}^{N-1}$.
The inequality is straightforward since the strong maximum principle [11, Th. 2.3.6] shows that if $u \leq \rho_2$ than $u < \rho_2$ or $u \equiv \rho_2$. In order to show the uniform convergence we will prove that for any $\varepsilon > 0$ there is $c(\varepsilon) > 0$, with

$$(3.4)\qquad u(x_1,x') > \rho_2-\varepsilon \text{ for } x_1 > c(\varepsilon) \text{ and } x' \in \mathbb{R}^{N-1}.$$

Let V be the first eigenfunction, normalized such that max V = 1, with eigenvalue ν, of

$$(3.5)\qquad \begin{cases} -\Delta V = \nu \, . \, V \text{ in } B, \\ v = 0 \quad \text{on } \partial B. \end{cases}$$

B denotes the unit ball in $\mathbb{R}^N$.
Let $\sigma > 0$ be such that

$$(3.6)\qquad f(u) > \sigma(u-c_2) \text{ for } u \in [c_2,\rho_2-\varepsilon]$$

and define

$$(3.7)\qquad W(y,t;x) = c_2 + t\, V((\sigma/\nu)^{\frac{1}{2}}.(x-y)).$$

Inequality (3.2) shows that if $y_1 > c_2/c_1 + (\nu/\sigma)^{\frac{1}{2}}$ then

$$(3.8)\qquad u > c_2 = W(y,0) \text{ on } B(y,(\nu/\sigma)^{\frac{1}{2}}).$$

For $y \in \mathbb{R}^N$ and $r > 0$ we define

$$(3.9)\qquad B(y,r) = \{x \in \mathbb{R}^N; \| x-y \| < r\},$$

with $\| . \|$ the Euclidean norm.

One uses the sweeping principle of Serrin, [13,1] , to find for all $t \in [0,\rho_2-\varepsilon-c_2]$ that

(3.10) $$u > W(y,t) \text{ on } B(y,(\nu/\sigma)^{\frac{1}{2}}).$$

Hence if $x_1 > c(\varepsilon) := c_2/c_1 + (\nu/\sigma)^{\frac{1}{2}}$ then , with $x = (x_1,x')$,

(3.11) $$u(x) > W(x,\rho_2-\varepsilon-c_2;x) = \rho_2-\varepsilon \text{ for all } x' \in \mathbb{R}^{N-1}$$

which shows (3.4).

One concludes by using [1, Proposition 2.5]. □

3.2. MODIFICATION OF f AND CONSTRUCTION OF SUBSOLUTIONS.

Since we are only interested in solutions above τ, we may change f on $(-\infty,\tau)$.

Lemma 3.3. Suppose $f \in C^{1,\gamma}$, for some $\gamma \in (0,1)$, satisfies (1.1) and (1.6). Then there exists $f^* \in C^{1,\gamma}$, satisfying $f^* = f$ on $[\tau,\infty)$ and

(3.12) $$\begin{cases} f^* = 0 \text{ on} (-\infty,\tau-1], \\ \int_{\rho}^{\rho_2} f^*(s)ds > 0 \text{ for all } \rho \le \rho_1. \end{cases}$$

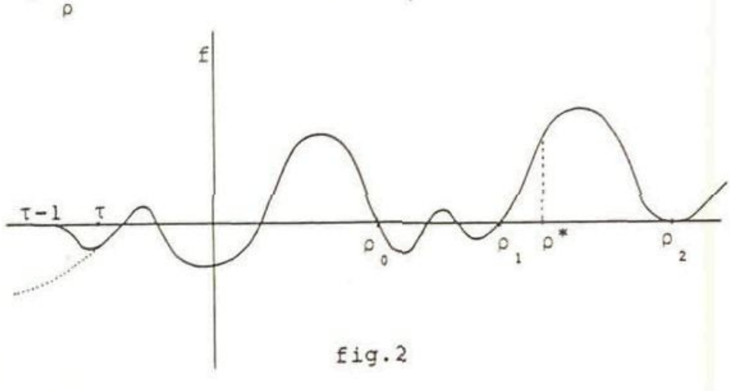

fig.2

With this modified function f we can state the equivalent of [1,lemma 2.2].

Lemma 3.4. Let $f \in C^{1,\gamma}$ satisfy (1.1) and (3.12). Then there exist $\mu > 0$ and $v \in C^2(\mathbb{R}^N)$, radially symmetric, which satisfy

$$(3.13) \qquad \begin{cases} -\Delta v = \mu . f(v) \text{ in } \mathbb{R}^N, \\ v(0) \in (\rho_1, \rho_2), \\ v(1) = \tau - 1, \\ v'(r) < 0 \text{ for } r > 0. \end{cases}$$

See [1] for a proof.
Next we define a family of subsolutions.

Corollary 3.5. Let f, μ and v be like in Lemma 3.4, and let α be the unique zero of v. Then for $y \in \Omega$, which is not necessarily bounded, and $\lambda > \mu .\ \alpha^2 .\ d(y,\partial\Omega)^{-2}$,

$$(3.14) \qquad w(\lambda,y;x) := v((\lambda/\mu)^{\frac{1}{2}}.(x-y)) \quad x \in \Omega,$$

is a subsolution of (1,1).

Proof. A direct calculation shows $-\Delta w = \lambda f(w)$, and since $\alpha < (\lambda/\mu)^{\frac{1}{2}} d(y,\partial\Omega)$ one finds that $w(\lambda,y) < 0$ on $\partial\Omega$. □

In the next lemma we will show that a solution above this subsolution $w(\lambda,y)$ is above max $w(\lambda,y)$ everywhere except near the boundary. To do so we need that Ω satisfies a uniform interior sphere condition :
There is $\varepsilon > 0$ such that

$$(3.15) \qquad \Omega = \cup\{B(y,\varepsilon);\ y \in \Omega(\varepsilon)\} ,$$

where $B(y,\varepsilon) = \{x \in \mathbb{R}^N; \|x-y\| < \varepsilon\}$ and

$$(3.16) \qquad \Omega(\varepsilon) = \{x \in \Omega;\ d(x,\partial\Omega) > \varepsilon\} .$$

Note that if Ω is bounded and has a C^3-boundary, it will satisfy a uniform interior sphere condition.

Lemma 3.6. Let f, μ, v and α be like in Corollary 3.5, and let Ω satisfy (3.15). Then there exists $c > 0$ such that every solution (λ,u) of (1.1), with $\lambda > \mu\, \alpha^2\, \varepsilon^{-2}$ and for some $y \in \Omega$

(3.17) $$\max(\tau-1,\ w(\lambda,y)) \le u \le \rho_2 \text{ in } \Omega,$$

also satisfies

(3.18) $$\min(c\ \lambda^{\frac{1}{2}} d(x;\partial\Omega),\ v(0)) < u(x) < \rho_2 \text{ for } x \in \Omega .$$

Proof. Assume without loss of generality that ε in (3.15) is so small that $\Omega(\varepsilon)$ is arcwise connected. For $\lambda > \mu\,\alpha^2\,\varepsilon^{-2}$, $\Omega(\alpha(\mu/\lambda)^{\frac{1}{2}})$ contains y and $\Omega(\varepsilon)$. Hence one can use the sweeping principle of Serrin, [13,2] , to find $w(\lambda,x) < u$ for all $x \in \Omega(\alpha(\mu/\lambda)^{\frac{1}{2}})$. Defining $c > 0$ by

(3.19) $$c = \mu^{-\frac{1}{2}} \inf \{(\alpha-r)^{-1}\ v(r)\ ;\ r \in [0,\alpha)\},$$

which is positive since $v > 0$ in $[0,\alpha)$ and $v'(\alpha) < 0$, one finds that

(3.20) $$\min(c\ \lambda^{\frac{1}{2}}\ d(x,\partial\Omega),\ v(0)) < \sup \{w(\lambda,y,x);\ y \in \Omega(\alpha(\mu/\lambda)^{\frac{1}{2}}\}.$$

This shows the first part of (3.18). The second part is a direct consequence of the strong maximum principle, [1, Th. 2.3.6]. □

Corollary 3.7. Let Ω be bounded, $\partial\Omega$ be C^3 and let f satisfy (1.2), (1.5) and (1.6). Then there is $\lambda^* > 0$ such that if (λ,u) is a solution of (1.1) with $\lambda > \lambda^*$ and

(3.21) $$\max(\tau,w(\lambda,y)) < u \le \rho_2 \text{ in } \Omega \text{ for some } y \in \Omega ,$$

then $u = \phi(\lambda)$, defined by Theorem 3.1.

Proof. Let z be a fixed nonnegative function in $C_0^\infty(\Omega)$ with $\max z \in (\rho_1,v(0))$. Then there is $\lambda^{**} > 0$ such that $z(x) < c.(\lambda^{**})^{\frac{1}{2}}.d(x,\partial\Omega)$ for $x \in \Omega$, with c defined in Lemma 3.6. Define $\lambda^* = \max(\lambda(z),\lambda^{**})$, where $\lambda(z)$ is defined in Theorem 3.1. Since $u > \tau$ we may assume f satisfies (3.12).

Then Lemma 3.6 yields (3.18). After observing that $z(x) < \min(c.\lambda^{\frac{1}{2}}.d(x,\partial\Omega),\ v(0))$ for $\lambda > \lambda^{**}$ and $x \in \Omega$, one uses Theorem 3.1 in order to find $u = \phi(\lambda)$ for $\lambda > \lambda^*$.

3.3. ESTIMATES FOR THE DERIVATIVE AT THE BOUNDARY.

Lemma 3.8. Let Ω, f and ϕ be like in Theorem 3.1. Then

$$\lim_{\lambda\to\infty} \lambda^{-\frac{1}{2}} . \frac{\partial}{\partial n}\phi(\lambda)(x) = -(2\int_0^{\rho_2} f(s)ds)^{\frac{1}{2}}, \tag{3.22}$$

uniformly for all $x \in \partial\Omega$. (n denotes the outward normal).

Proof. It is sufficient to show that any sequence $(\lambda_k,x_k) \in \mathbb{R}_+ \times \partial\Omega$ with $\lim_{k\to\infty} \lambda_k = \infty$ has a subsequence for which the following equality holds

$$\lim_{i\to\infty} \lambda_{k_i}^{-\frac{1}{2}} .\phi(\lambda_{k_i})(x_{k_i}) = -(2\int_0^{\rho_2} f(s)ds)^{\frac{1}{2}}. \tag{3.23}$$

Let $(\lambda_k,x_k) \in \mathbb{R}_+ \times \partial\Omega$ be a sequence with $\lim_{k\to\infty} \lambda_k = \infty$.
Since $\partial\Omega$ is compact, a subsequence exists, still denoted (λ_k,x_k), with x_k converging to some $\bar{x} \in \partial\Omega$. By a local change of coordinates $x \to \xi(x)$, from $O(\bar{x})$ to $\mathbb{R}^N$, see [1, Lemma 4.2], one defines the functions

$$U_k(\eta) = \phi(\lambda_k)(\xi^{-1}(\xi(x_k)+\lambda_k^{-\frac{1}{2}} .\eta)),\ \eta \in \mathbb{R}^N_+. \tag{3.24}$$

These functions U_k satisfy for k large enough

$$\begin{cases} \Delta U_k + \lambda_k^{-\frac{1}{2}} .(\sum_{ij} a_{kij} \frac{\partial}{\partial\eta_i}\frac{\partial}{\partial\eta_j} U_k + \sum_i b_{ki} \frac{\partial}{\partial\eta_i} U_k) + f(U_k) = 0 \\ \qquad\qquad \text{in } D \cap \frac{1}{2}\lambda_k^{-\frac{1}{2}}.\xi(O(\bar{x})), \\ U_k = 0 \text{ on } \partial D \cap \frac{1}{2}\lambda_k^{-\frac{1}{2}}.\xi(O(\bar{x})), \end{cases} \tag{3.25}$$

with $D = \mathbb{R}^N_+$. Since $\partial\Omega$ is C^3, the functions a_{kij} and b_{ki} exist and are uniformly bounded on $\bar{D} \cap \frac{1}{2}\lambda_k^{-\frac{1}{2}}.\xi(O(\bar{x}))$. Furthermore, because of Lemma 3.6, one finds

$$(3.26)\qquad \min(c.\lambda^{\frac12}.d(x,\partial\Omega),v(0)) < \phi(\lambda)(x) \text{ for } x \in \Omega,$$

and hence

$$(3.27)\qquad \min(c.\eta_1,v(0)) < U_k(\eta) \quad \text{for } \eta = (\eta_1,\dots,\eta_N) \in D\cap \tfrac12\lambda_k^{-\frac12}.\xi(O(\bar{x})).$$

Since $\{U_k\}$ is precompact in $C^2_{loc}(D)$, there exists a convergent subsequence, which we still denote with index k. Hence there is $U \in C^2(D) \cap C(\bar{D})$, with $\lim_{k\to\infty} U_k = U$ in $C^2_{loc}(D)$, satisfying (3.1). Moreover

$$(3.28)\qquad \min(c.\eta_1,v(0)) \le U(\eta) \le \rho_2 \quad \text{for } \eta \in D.$$

By Proposition 3.2 one finds $U(\eta) = U(\eta_1)$ and U is the solution of (3.3). Since

$$(3.29)\qquad \lim_{k\to\infty} -\lambda_k^{-\frac12}.\frac{\partial}{\partial n}\phi(\lambda_k)(x_k) = \lim_{k\to\infty}\frac{\partial}{\partial\eta_1}U_k(0) = \frac{\partial}{\partial\eta_1}U(0)$$

this last observation proves the lemma. □

If (λ,u) is a solution of (1.1), with $\lambda > \lambda_1$ and $u < \rho_2$, then one finds by Theorem 3.1 that $u \le \phi(\lambda)$. Hence $-\frac{\partial u}{\partial n} \le -\frac{\partial}{\partial n}\phi(\lambda)$ on $\partial\Omega$. If f satisfies (1.6) and $u > \tau$, then it is also possible to estimate $-\frac{\partial u}{\partial n}$ from below if λ is large enough. In order to do so we have to modify f once again. Let f satisfy (3.12). Then $C := \int_{\tau-1}^{\rho_2} f(s)ds$ is positive. Set

$$f^{**}(\rho) = \begin{cases} f(\rho) \text{ for } \rho \in \mathbb{R}/(\tau-2,\tau-1), \\ C.(\cos(2\pi.(\rho-\tau+1))-1) \text{ for } \rho \in (\tau-2,\tau-1). \end{cases}$$

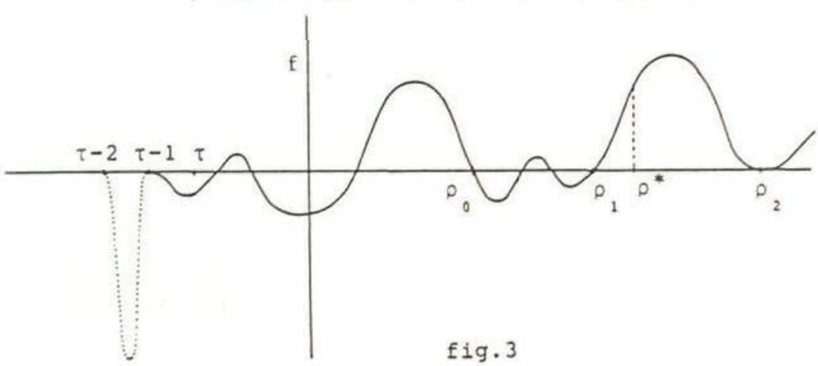

fig.3

Since $u > \tau$, (λ,u) is also a solution of

$$(3.30)\qquad \begin{cases} -\Delta u = \lambda\, f^{**}(u) \text{ in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega. \end{cases}$$

Moreover we can use Theorem 3.1 for

$$(3.31)\qquad \begin{cases} -\Delta v = -\lambda\, f^{**}(-v) \text{ in } \Omega, \\ v = 0 \quad \text{on } \partial\Omega \end{cases}$$

in order to find a minimal solution $(\lambda,\psi(\lambda))$ of (3.30) for λ large with $\min \psi \in (\tau-2,\tau-1)$.

The conditions equivalent to (1.2) and (1.5) are easily checked. The equivalent of (1.3) also holds:

$$(3.32)\qquad \int_{\rho}^{-(\tau-2)} -f^{**}(-s)ds = -\int_{\tau-2}^{-\rho} f^{**}(s)ds =$$
$$= C - \int_{\tau-1}^{-\rho} f(s)ds =$$
$$= \int_{\tau-1}^{\rho_2} f(s)ds - \int_{\tau-1}^{-\rho} f(s)ds =$$
$$= \int_{-\rho}^{\rho_2} f(s)ds > 0 \text{ for } \rho \in [0,-(\tau-1)],$$

since we may assume f satisfies (3.12).

Hence $\psi(\lambda) \le u$ in Ω for λ large enough, and $-\frac{\partial}{\partial n}\psi(\lambda) \le -\frac{\partial u}{\partial n}$ on $\partial\Omega$.

Applying Lemma 3.8 shows

$$(3.33)\qquad \lim_{\lambda\to\infty} \lambda^{-\frac{1}{2}} \cdot \frac{\partial}{\partial n}\psi(\lambda)(x) = \Big(-2\int_{\tau-2}^{0} f^{**}(s)ds\Big)^{\frac{1}{2}} = \Big(2\int_{0}^{\rho_2} f(s)ds\Big)^{\frac{1}{2}}$$

for $x \in \partial\Omega$.

3.4 PROOF OF THEOREM 2.

We will prove Theorem 2 by showing for λ large that a solution (λ,u) of (1.1), with $\tau < u < \rho_2$ and $|\max u - \rho_2|$ small enough, satisfies $u > w(\lambda,y)$ in Ω for some $y \in \Omega$. Hence one finds by Corollary 3.7 that $u = \phi(\lambda)$. After that we will show that max u is bounded away from ρ_2 for bounded λ.

<u>Lemma 3.9.</u> Let Ω be bounded, $\partial\Omega$ be C^3 and let f satisfy (1.2),(1.5) and (1.6). Then there exists a function $o(\lambda)$, with $\lim_{\lambda\to\infty} \lambda^{-1}.o(\lambda) = 0$, such that for any solution (λ,u) of (1.1) with $\tau < u < \rho_2$ one finds

$$(3.34) \qquad Q(\lambda,u)(x) := \tfrac{1}{2}\|\nabla u(x)\|^2 + \lambda . \int_0^{u(x)} f(s)ds \le \lambda . \int_0^{\rho_2} f(s)ds + o(\lambda)$$

for all $x \in \Omega$.

Proof. Let (λ,u) be a solution of (1.1) with $\tau < u < \rho_2$.
By the Lemmas 5.1 and 5.2 of [14], $Q(\lambda,u)$ assumes its maximum on $\partial\Omega$ or at a critical point of u. In the preceding section we found $-\frac{\partial}{\partial n}\psi(\lambda) \le -\frac{\partial u}{\partial n} \le -\frac{\partial}{\partial n}\phi(\lambda)$ on $\partial\Omega$. Hence by (3.22) and (3.33) a function $o(\lambda)$ exists, with $\lim_{\lambda\to\infty} \lambda.o(\lambda) = 0$, such that

$$\begin{aligned}(3.35) \qquad & \max\{Q(\lambda,u)(x) \; ; \; x \in \partial\Omega\} = \\ & = \max\{\tfrac{1}{2}\,|\tfrac{\partial u}{\partial n}(x)|^2 \; ; \; x \in \partial\Omega\} \le \\ & \le \max\{\tfrac{1}{2}\,|\tfrac{\partial}{\partial n}\psi(\lambda)(x)|^2 \; ; \; x \in \partial\Omega\} \vee \\ & \qquad \max\{\tfrac{1}{2}|\tfrac{\partial}{\partial n}\phi(\lambda)(x)|^2 \; ; \; x \in \partial\Omega\} \le \\ & \le \lambda . \int_0^{\rho_2} f(s)ds + o(\lambda).\end{aligned}$$

By condition (1.6)

$$\begin{aligned}(3.36) \qquad & \max\{Q(\lambda,u)(x) \; ; \; x \in \Omega \text{ and } \nabla u(x) = 0\} \le \\ & \le \max\{\lambda . \int_0^{u(x)} f(s)ds \; ; \; x \in \Omega\} \le \\ & \le \lambda . \int_0^{\rho_2} f(s)ds - \lambda.\min \int_{u(x)}^{\rho_2} f(s)ds \; ; \; x \in \Omega\} \le\end{aligned}$$

$$\leq \lambda.\int_0^{\rho_2} f(s)ds - \lambda.\min\{\int_u^{\rho_2} f(s)ds \; ; \; \tau < u < \rho_2\} \leq$$

$$\leq \lambda.\int_0^{\rho_2} f(s)ds.$$

Hence $Q(\lambda,u)(x) \leq \lambda . \int_0^{\rho_2} f(s)ds + o(\lambda)$ for $x \in \Omega$. □

Now we are able to prove Theorem 2 for λ large.
Since $f(\rho_2) = 0$ and $f \in C^1$, there exists $c > 0$ such that

$$2\int_u^{\rho_2} f(s)ds \leq c \, .|\rho_2-u|^2 \text{ for all } u \in [\tau,\rho_2]. \tag{3.37}$$

For any solution (λ,u) of (1.1), with $\tau < u < \rho_2$, one finds by Lemma 3.9 and (3.37) that

$$\begin{aligned} \| \nabla u(x)\|^2 &= 2Q(\lambda,u)(x) - 2\lambda\int_0^{u(x)} f(s)ds \leq \\ &\leq 2\lambda\int_{u(x)}^{\rho_2} f(s)ds + 2.o(\lambda) \leq \\ &\leq c.\lambda.|\rho_2-u(x)|^2 + 2.o(\lambda) \quad \text{for all } x \in \Omega . \end{aligned} \tag{3.38}$$

Let $x^1 \in \Omega$ and let $\varepsilon > 0$ be small enough such that $\varepsilon < (4c\lambda)^{-\frac{1}{2}}$ and $B(x^1,\varepsilon) := \{x \in \mathbb{R}^N; \| x-x^1\| < \varepsilon\} \subset \Omega$. Then for all $x^2 \in B(x^1,\varepsilon)$ there is $y \in B(x^1,\varepsilon)$ with

$$\begin{aligned} |u(x^1) - u(x^2)|^2 &\leq \| x^1-x^2\|^2 . \| \nabla u(y)\|^2 \leq \\ &\leq \varepsilon^2.(c.\lambda.|\rho_2-u(y)|^2 + 2.o(\lambda)) \leq \\ &\leq \tfrac{1}{4}|\rho_2-u(y)|^2 + \tfrac{1}{2}(c\lambda)^{-1}.o(\lambda). \end{aligned} \tag{3.39}$$

Hence, using the norm $\| v\|_* := \sup \{|v(x)| \; ; x \in B(x^1,\varepsilon)\}$, one finds

$$\begin{aligned} \|u(x^1)-u\|_*^2 &\leq \tfrac{1}{4}\| \rho_2-u\|_*^2 + \tfrac{1}{2}(c\lambda)^{-1}.o(\lambda) \leq \\ &\leq \tfrac{1}{4}(|\rho_2-u(x^1)| + \| u(x^1)-u\|_*)^2 + \tfrac{1}{2}(c\lambda)^{-1}.o(\lambda) \leq \\ &\leq \tfrac{1}{4}\Big(|\rho_2-u(x^1)| + \|u(x^1)-u\|_* + (c\lambda)^{-\frac{1}{2}}.(2.o(\lambda))^{\frac{1}{2}}\Big)^2 \end{aligned} \tag{3.40}$$

Consequently for all $x^2 \in B(x^1,\varepsilon)$:

$$(3.41)\qquad |u(x^1) - u(x^2)| \le \| u(x^1) - u\|_* \le$$

$$\le |\rho_2 - u(x^1)| + (c\lambda)^{-\frac{1}{2}}.(2.o(\lambda))^{\frac{1}{2}}.$$

Let (μ,v) be fixed by Lemma 3.4. Choose an integer $m > 2.(c\mu)^{\frac{1}{2}}$, with c from (3.37), and set $\delta = 2^{-(m+1)}.(\rho_2 - v(0))$. Finally let $\lambda_2 > \lambda_1$ be such that $\lambda^{-1}.o(\lambda) < \frac{1}{2}c.\delta^2$ for $\lambda > \lambda_2$.
Now suppose (λ,u) is a solution of (1.1) with $\tau < u < \rho_2$, $\lambda > \lambda_2$ and $u(x^0) = \max u > \rho_2 - \delta$. We will show that

$$(3.42)\qquad u > v(0) \text{ in } B(x,(\mu/\lambda)^{\frac{1}{2}}).$$

Let $\varepsilon < (4c\lambda)^{-\frac{1}{2}}$ and small enough such that $B(x^0,m\varepsilon) \subset \Omega$.
Then for any $x \in B(x^0,m\varepsilon)$ there exist $x^1,\dots,x^m = x \in B(x^0,m\varepsilon)$ such that $\| x^i - x^{i-1}\| < \varepsilon$ for $i = 1,\dots,m$.
By (3.41) and the triangle inequality one finds

$$(3.43)\qquad |u(x^0) - u(x^i)| \le |u(x^0) - u(x^{i-1})| + |u(x^i) - u(x^{i-1})| \le$$

$$\le |u(x^0) - u(x^{i-1})| + |\rho_2 - u(x^{i-1})| + (c\lambda)^{-\frac{1}{2}}.(2.o(\lambda))^{\frac{1}{2}} \le$$

$$\le 2.|u(x^0) - u(x^{i-1})| + |\rho_2 - u(x^0)| + (c\lambda)^{-\frac{1}{2}}.(2.o(\lambda))^{\frac{1}{2}}$$

$$\text{for } i = 1,\dots,m.$$

A repeated use of (3.43) shows

$$(3.44)\qquad |u(x^0) - u(x^m)| \le (2^m - 1).\left(|\rho_2 - u(x^0)| + (c\lambda)^{-\frac{1}{2}}.(2.o(\lambda))^{\frac{1}{2}}\right).$$

Because of the definition of m, δ and λ_2 one finds

$$(3.45)\qquad u(x) \ge u(x^0) - (2^m - 1).\left((\rho_2 - u(x^0) + (c\lambda)^{-\frac{1}{2}}.(2.o(\lambda))^{\frac{1}{2}}\right) >$$

$$> \rho_2 - \delta - (2^m - 1).2\delta >$$

$$> \rho_2 - 2^{m+1}\,.\,\delta = v(0) \text{ for } x \in B(x^0,m\varepsilon).$$

Since $u = 0$ on $\partial\Omega$, (3.45) shows that if $B(x^0, m\varepsilon) \subset \Omega$ also $\overline{B(x^0, m\varepsilon)} \subset \Omega$. Since this holds for any $\varepsilon < (4c\lambda)^{-\frac{1}{2}}$ it shows that $B(x^0, m.(4c\lambda)^{-\frac{1}{2}}) \subset \Omega$. Hence by the choice of m one finds

$$u > v(0) \geq w(\lambda, x^0) \text{ in } B(x^0, (\mu/\lambda)^{\frac{1}{2}}) \subset B(x^0, m.(4c\lambda)^{-\frac{1}{2}}) \subset \Omega. \tag{3.46}$$

By definition $u > \tau \geq w(\lambda, x^0)$ in $\Omega \setminus B(x^0, (\mu/\lambda)^{\frac{1}{2}})$. Then Corollary 3.7 shows $u = \phi(\lambda)$. This completes the proof of Theorem 2 if $\lambda > \lambda_2$.

What remains to be shown is that solutions (λ, u) of (1.1) with $0 < \lambda < \lambda_2$ cannot have their maximum arbitrarily close by ρ_2. The next lemma will prove this and Corollary 3.

Lemma 3.10. Let $\partial\Omega$ be C^3 and let f satisfy (1.2), (1.5) and (1.6). Let ϕ and λ_1 be as in Theorem 3.1. Then there is $c \geq 1$, which only depends on the domain Ω, such that for any solution (λ^*, u^*) of (1.1) with $u^* < \rho_2$, the following inequality holds

$$\max u^* < \max \phi(\lambda) \text{ for all } \lambda > \max(c.\lambda^*, \lambda_1). \tag{3.47}$$

Moreover, if Ω is starshaped (3.47) holds with $c = 1$.

As a direct result one finds for all solutions (λ, u) of (P), with $u \leq \rho_2$ and $\lambda \in (0, \lambda_2)$, that $\max u < \max \phi(c.\lambda_2)$.

Proof of Lemma 3.10. For any bounded domain Ω, there is $y \in \Omega$ and $r \in (0,1]$ such that

$$\theta\,.(-y+\Omega) \subset -y+\Omega \quad \text{for all } \theta \in (0,r]. \tag{3.48}$$

If Ω is starshaped with y as center, then (3.48) holds with $r = 1$. Define $c = r^{-2}$ and set

$$v(x) = \begin{cases} \max(\phi(\lambda)(x)\ ,\ u^*(y+(\lambda/\lambda^*)^{\frac{1}{2}}.(x-y)) \\ \qquad\qquad \text{for } x \in y + (\lambda^*/\lambda)^{\frac{1}{2}}.(-y+\Omega) \\ \phi(\lambda)(x) \quad \text{elsewhere in } \Omega. \end{cases} \tag{3.49}$$

Since $y + (\lambda^*/\lambda)^{\frac{1}{2}}.(-y+\Omega) \subset \Omega$ for $\lambda > c.\lambda^*$ and $\phi(\lambda) > 0$ in Ω for $\lambda > \lambda_1$, v is a subsolution of (1.1). Hence there is a solution v^* of (1.1) with $\phi(\lambda) \le v \le v^* < \rho_2$. The fact that $\phi(\lambda)$ is the maximal solution shows that $\phi(\lambda) \equiv v \equiv v^*$. Hence $u^*(y + (\lambda/\lambda^*)^{\frac{1}{2}}.(x-y)) \le \phi(\lambda)(x)$ for $x \in y + (\lambda^*/\lambda)^{\frac{1}{2}}.(-y+\Omega)$. Using the strong maximum principle yields the strict inequality and concludes the proof of Lemma 3.10. □

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REFERENCES.

[1] Ph. Clément, G. Sweers, Existence and multiplicity results for a semilinear elliptic eigenvalue problem, to appear in Annali della Scuola Normale Superiore di Pisa.

[2] Ph. Clément, G. Sweers, Existence et multiplicité des solutions d'un problème aux valeurs propres elliptique semilinéaire, C.R.Acad.Sci. Paris 302, I Ser. 19 (1986), 681-683.

[3] C. Cosner, K. Schmitt, A priori bounds for positive solutions of a semilinear elliptic equation, Proc.Am.Math.Soc. 95 (1985), 47-50.

[4] E.N. Dancer, K. Schmitt, On positive solutions of semilinear elliptic equations, preprint 1986.

[5] E.N. Dancer, private communication.

[6] P. Fife, Semilinear elliptic boundary value problems with small parameters, Arch.Ration.Mech.Anal. 52 (1973), 205-232.

[7] B. Gidas, W.M. Ni, L. Nirenberg, Symmetry and related properties via the maximum principle, Commun.Math.Phys. 68 (1979), 209-243.

[8] P. Hess, On multiple solutions of nonlinear elliptic eigenvalue problems, Commun.Partial Differ.Equations 6 (1981), 951-961.

[9] L. Modica, A gradient bound and a Liouville theorem for nonlinear Poisson equations, Commun.Pure Appl.Math. 38 (1985), 679-684.

[10] S.D. Poisson, Second mémoire sur la distribution de la chaleur dans les corps solides, J. de l'École Polytechnique 12, cahier 19 (1823), 249-403.

[11] M. Protter, H. Weinberger, Maximum principles in differential equations, Prentice Hall, Englewood Cliffs N.J., 1967.

[12] P.H. Rabinowitz, Pairs of positive solutions of nonlinear elliptic partial differential equations, Indiana Univ.Math.J. 23 (1973), 172-185.

[13] J. Serrin, *Nonlinear equations of second order*, A.M.S.*Symposium in Partial* Diff.Eq., Berkeley, August 1971.

[14] R. Sperb, Maximum principles and their applications, Math. in Science and Engineering 157, Academic Press, New York, 1981.

Chapter 3

Results for the boundary layer of solutions for a semilinear elliptic problem.

(Some results for a semilinear elliptic problem with a large parameter)

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Some Results for a Semilinear Elliptic Problem with a Large Parameter

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ABSTRACT

Consider the following eigenvalue problem

$$(P)\quad \begin{cases} -\Delta u = \lambda f(u) & \text{in } \Omega \subset \mathbb{R}^n, \text{ bounded,} \\ u = 0 & \text{on } \partial\Omega, \text{ smooth,} \end{cases}$$

where f changes sign. In this note we will show results which can be found by using the so-called sweeping principle of Serrin, 1971. Especially we will give estimates for the boundary layer of positive solutions near a zero of f. For some f a solution u will have a free boundary. We show for such f that f(u)=0 except near $\partial\Omega$. Next to this we improve a result for existence of a solution.

1. INTRODUCTION

We are interested in pairs $(\lambda,u) \in \mathbb{R}_+ \times C^2(\Omega)$ satisfying (P) and $u > 0$ in Ω. First, note that a solution satisfies $f(\max u) \geq 0$. If $f \in C^1$, the strong maximum principle even shows $f(\max u) > 0$. Secondly, if ρ is a zero of f then $u \equiv \rho$ satisfies the differential equation for all λ. So one could expect the existence of a solution (λ,u), where λ is large and u is near a zero of f (with $f(\max u) \geq 0$) except for a boundary layer. Results for this problem were presented by Fife,1973 and by Clément et al., 1986. The results here are strongly related to this last paper.
Assume that there are two numbers $0 < \rho_1 < \rho_2$ such that

(F1) $\quad f(\rho_1) = f(\rho_2) = 0$ and $f > 0$ in (ρ_1,ρ_2), (F2) $f \in C^{\gamma}(-\infty,\rho_2] \cap C^1(-\infty,\rho_2)$ and there is $\delta > 0$ such that $f' \le 0$ in $(\rho_2-\delta,\rho_2)$.

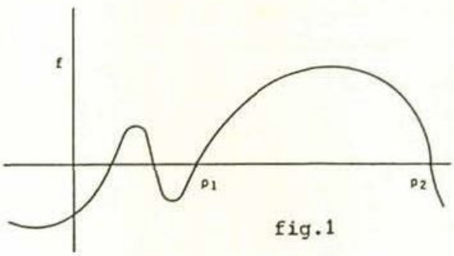


fig.1

In 1981 Hess showed, if $f(0) > 0$, that the following condition is sufficient for existence of a positive solution (λ,u) with $\max u \in (\rho_1,\rho_2)$.

(F3) $$J(\rho) := \int_{\rho}^{\rho_2} f(s)ds > 0 \text{ for every } \rho \in [0,\rho_1].$$

In the first theorem, it will be proven that this condition is sufficient and necessary when $f \in C^1[0,\max u]$, even if $f(0) < 0$. In the second theorem we will show that the solutions, which are found in this way, are near ρ_2.

2. THEOREMS AND PROOFS

Before stating the first theorem we will shortly explain the sweeping principle of Serrin, 1971. A formulation can also be found in the paper by Clément et al., 1986.

Fix λ, let u be a solution of (P) and let $\{v(t) \in C(\bar{\Omega});\ t \in [0,1]\}$ be a continuous family of subsolutions, such that $v(0) < u$ in Ω and for all t $v(t) < u$ on $\partial\Omega$ as well as $v(t) < \rho_2$ in Ω. Then $v(t) < u$ in Ω for all $t \in [0,1]$. Since, if there exists $t^* \in [0,1]$ such that $v(t^*) \le u$ and for some $x^* \in \Omega$ $v(t^*,x^*) = u(x^*)$, the strong maximum principle implies $v(t^*) \equiv u$, a contradiction.

THEOREM 1 :

Let f satisfy (F1)(F2)(F3) and let Ω satisfy a uniform interior sphere condition. Then there exists $c_1 > 0$, $c_2 \in (\rho_1,\rho_2)$ and $\lambda_0 > 0$ such that for all $\lambda > \lambda_0$ a positive solution $(\lambda,u(\lambda))$ of (P) exists with

(1) $$\min\,(c_1.d(x,\partial\Omega).\lambda^{\frac{1}{2}},\ c_2) < u(\lambda) \le \rho_2.$$

Moreover every solution (λ,u) of (P) (not necessarily positive) with $\max u \in (\rho_1,\rho_2)$ satisfies $\int_\rho^{\max u} f(s)ds > 0$ for every $\rho \in [0,\rho_1]$.

PROOF:

Replace f by f^*, where f^* satisfies (F1) and

$f^*(u) = 1$ for $u < -1$,

$f^*(u) \le f(u)$ for $0 \le u \le \rho_2$,

$f^* \in C^1(\mathbb{R})$,

$\int_\rho^{\rho_2} f^*(s)ds > 0$ for all $\rho < \rho_2$.

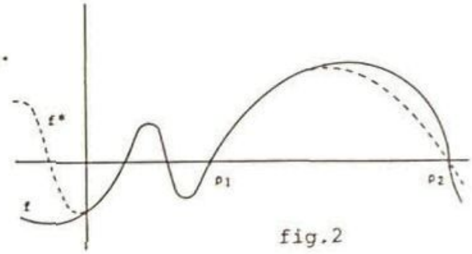


fig.2

Like Hess in 1981, one finds for μ large enough, a minimizer v of $I(v,\mu) = \frac{1}{2}\int_\Omega |\nabla v|^2dx - \mu\int_\Omega\int_0^v f^*(s)ds\,dx$ in the cone $\{v \in W^{1,2}(\Omega);\ v > -1 \text{ in } B,\ v = -1 \text{ on } \partial B\}$ with $\max v \in (\rho_1,\rho_2)$. (B denotes the unit ball). Gidas et al. showed in 1979 that v is radially symmetric and $v'(r) < 0$ for $r \in (0,1]$. Let $\theta \in (0,1)$ be the number such that $v(\theta) = 0$. Since Ω satisfies a uniform interior sphere condition, $\Omega = \cup\{B(x,\varepsilon);\ x \in \Omega(\varepsilon)\}$ for all $\varepsilon \in (0,\varepsilon_0)$, where ε_0 is some positive constant and $B(x,\varepsilon) = \{y \in \mathbb{R}^N;\ |x-y| < \varepsilon\}$, $\Omega(\varepsilon) = \{x \in \Omega;\ d(x,\partial\Omega) > \varepsilon\}$. Then $w(\lambda,x) := \sup\{v(\theta.\varepsilon^{-1}.|x-y|);\ y \in \Omega(\varepsilon)\}$, with $\lambda = \mu.(\theta/\varepsilon)^2$, is a subsolution of (P), with f replaced by f^*, for all $\lambda \ge \lambda_0 := \mu.(\theta/\varepsilon_0)^2$. Since $0 < w(\lambda) < \rho_2$ and $f^* \le f$ on $[0,\rho_2]$, $w(\lambda)$ is also a subsolution of the original (P). Note that $W(\lambda) \equiv \rho_2$ is a supersolution of (P) for all λ. By an iteration scheme one shows the existence of a solution in between. By condition (F2) there exist two strictly increasing continuous functions f_1 and f_2 such that $f = f_1 - f_2$ on $[0,\rho_2]$ and $f_2(0) = 0$. Because of (F2) one may assume $f_1 \in C^1[0,\rho_2]$. Define T by $u = T(v)$, where u is the unique solution of

$$\begin{cases} -\Delta u + \lambda f_2(u) = \lambda f_1(v) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

See the paper of Brezis et al. from 1973. Define $W_n = T^n(W(\lambda))$ and $w_n = T^n(w(\lambda))$. $\{W_n\}$ and $\{w_n\}$ are sequences of respectively decreasing supersolutions and increasing subsolutions. Since $W_n > w_n$ in Ω the sequences converge to a solution of (P). Standard regularity theory shows that these solutions, or maybe just one solution, are $C^2(\Omega)$. The estimate (1) is valid since the solutions are between $w(\lambda)$ and $W(\lambda)$.

The last part will also be proven with a sweeping argument. Suppose there is a solution of (P) with $\max u \in (\rho_1,\rho_2)$ and $\int_{\rho^*}^{\max u} f(s)ds = 0$ for some $\rho^* \in [0,\rho_1]$.

Let $\bar{u}$ be the solution of

$$-\bar{u}'' = \lambda f(\bar{u}) \qquad , t \in \mathbb{R},$$
$$\bar{u}(0) = \max u \quad ,$$
$$\bar{u}'(0) = 0.$$

Set $U(t,x_1,\dots,x_N) = \bar{u}(x_1-t)$ for $x \in \mathbb{R}^N$.

Note that $\max U = \max u$ and $\inf U \geq \rho^*$. Moreover there exists t^* and $x^* \in \overline{\Omega^*}$, with $\Omega^* = \Omega \cap \{x \in \mathbb{R}^N ; x_1 > t^*\}$, such that

$$U(t^*) \geq u \text{ in } \Omega^*,$$
$$U(t^*,x^*) = u(x^*) \text{ and } \nabla U(t^*,x^*) = \nabla u(x^*).$$

The strong maximum principle shows $U(t^*) \equiv u$, which is a contradiction. For a more detailed proof see the authors paper of 1986. □

THEOREM 2:

Let Ω satisfy an interior sphere condition and let f satisfy (F1) and (F2) with ρ_1 not necessarily positive. If $\rho_1 > 0$ then assume (F3) is also satisfied.

Suppose that $f(u) > c(\rho_2-u)^\alpha$ for $u \in (\rho_2-\delta,\rho_2)$, where $c,\alpha,\delta > 0$. Then there is $C > 0$ such that for any nonnegative $z \in C_0^\infty(\Omega)$, with $\max z \in (\rho_1,\rho_2)$, $\lambda(z) > \lambda_0$ exists for which the following holds.

Let (λ,u) be a solution of (P) with $z \leq u \leq \rho_2$ in Ω and $\lambda > \lambda(z)$.

1) If $0 < \alpha < 1$ then $u(x) \geq \min(C.\lambda^{\frac{1}{2}}.d(x,\partial\Omega),\rho_2)$.

2) If $\alpha = 1$, then $u(x) > \rho_2(1-\exp(-C.\lambda^{\frac{1}{2}}.d(x,\partial\Omega)))$, for $x \in \Omega$.

3) If $1 < \alpha$, then $u(x) > \rho_2(1-(1+C.\lambda^{\frac{1}{2}}.d(x,\partial\Omega))^{-p})$ for $x \in \Omega$, with $p = 2(\alpha-1)^{-1}$.

REMARK 1.

Case 1) shows that a solution near ρ_2 will have a free boundary within a distance of order $\lambda^{-\frac{1}{2}}$ from $\partial\Omega$.

Define $m := \frac{1}{2}(\rho_1 + \max z)$ and $M := v(0)$. Then there exists a ball $B(x^*,r)$, such that $B(x^*,r) \subset \{x \in \Omega;\ z(x) > m\}$, and a constant σ, such that $f(u) > \sigma(u-m)$ for $u \in [m,M]$. By the lemma one finds

$$u(x) > M \text{ for } x \in B(x^*, r-(\sigma.\lambda/\nu)^{-\frac{1}{2}}).$$

When $r-(\sigma.\lambda/\nu)^{-\frac{1}{2}} > \theta.(\lambda/\mu)^{-\frac{1}{2}}$ the first step is finished since

$$u(x) > M \geq v((\lambda/\mu)^{\frac{1}{2}}|x-x^*|) \text{ for } x \in B(x^*, \theta(\lambda/\mu)^{-\frac{1}{2}})$$

Hence set $\lambda(z) = \max\ (\lambda_0, r^{-2}((\nu/\sigma)^{\frac{1}{2}} + \mu^{\frac{1}{2}})^2)$.

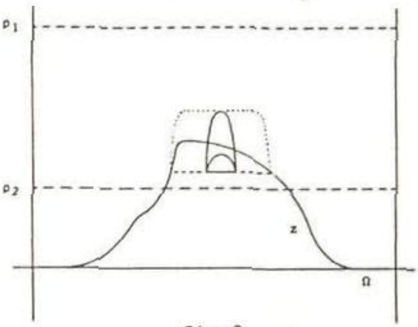


fig.3

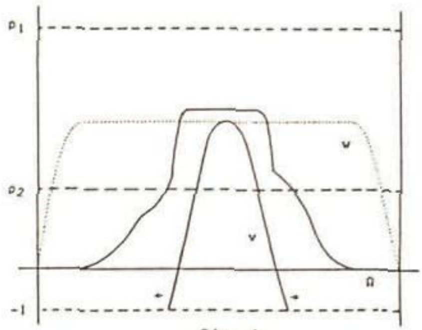


fig.4

In the second step we prove that a solution (λ,u), with $u \in (w(\lambda),\rho_2]$ and $\lambda > \lambda(z)$, satisfies the statement of the theorem. If $\rho_1 < 0$ set $M = 0$. We may assume that c is such that

$$f(u) > c(\rho_2 - u)^{\alpha} \quad \text{for } u \in [M,\rho_2] .$$

Define $M_k = \rho_2 - 2^{-k} . (\rho_2 - M)$

and $\quad \sigma_k = c . 2^{-(k+1).(\alpha-1)} . (\rho_2-M)^{\alpha-1}$.

Then $f(u) > \sigma_k . (u-M_k)$ for $u \in [M_k, M_{k+1}]$.

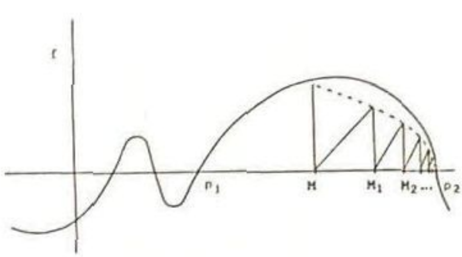


fig.5

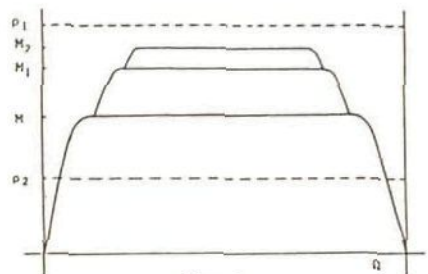


fig.6

REFERENCES

BREZIS, H. and STRAUSS, W.A., 1973, *Semi-linear second-order elliptic equations in L^1*, J.Math.Soc.Japan, 25,4, 565-590.

CLEMENT, Ph. and SWEERS, G., 1986, *Existence and multiplicity results for a semilinear elliptic eigenvalue problem*, to appear in Annali della Scuola Normale Superiore di Pisa.

FIFE, P., 1973, *Semilinear boundary value problems with small parameters*, Arch. Rat. Mech. Anal. 52, 205-232.

GIDAS, B., NI, W.M., NIRENBERG,L., 1979, *Symmetry and related properties via the maximum principle*, Comm. Math. Phys.68, 209-243.

HESS, P., 1981, *On multiple solutions of nonlinear elliptic eigenvalue problems*, Comm. Part. Diff. Eq. 6, 951 - 961.

SERRIN, J., 1971, *Nonlinear equations of second order*, A.M.S.symposium in Partial Diff. Eq., Berkeley, August 1971.

SWEERS,G., 1986, *On the maximum of solutions for a semilinear elliptic problem*, submitted.

Chapter 4

Getting a solution between sub- and supersolutions without monotone iteration.

SOMMARIO: Se esiste una sotto-sopra soluzione per un problema semilineare ellittico allora si può provare l'esistenza di una soluzione usando il metodo della iterazione monotona. Per applicare questo metodo è necessario assumere una regolarità del secondo membro più forte della continuità.
In questo nota si prova l'esistenza di una soluzione nella sola ipotesi di continuità del secondo membro usando il teorema di Schauder e una versione del principio di massimo forte assumendo l'esistenza di una sotto (sopra) soluzione debole.

SUMMARY: If there exist a sub- and a supersolution for a semilinear elliptic problem, then one can show the existence of a solution by a monotone iteration scheme. In order to do this one needs more than continuity of the right hand side. In this note the Schauder fixed point theorem and a version of the strong maximum principle is used to get existence of a solution with only continuity of the right hand side under the existence of a weak sub- and supersolution.

1. INTRODUCTION AND MAIN RESULT.

We consider the following nonlinear boundary value problem:

$$(1) \qquad \begin{cases} -\Delta u = f(x,u) & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded domain of $\mathbb{R}^N$.

For f we only assume

(H1) $\quad f\colon \bar{\Omega} \times \mathbb{R} \to \mathbb{R}$ is continuous.

We also assume that

(H2) $\quad g\colon \partial\Omega \to \mathbb{R}$ is continuous.

In this note we are interested in the existence of solutions of (1) lying between sub- and supersolutions defined in a rather weak sense. Due to the special form of the left hand side we can define

DEFINITION 1: A function u is called a sub(super)solution of (1) if

i) $u \in C(\bar{\Omega};\mathbb{R})$

ii) $\int_\Omega (u(-\Delta\varphi) - f(x,u)\varphi)\, dx \leq (\geq)\ 0$ for every $\varphi \in \mathcal{D}^+(\Omega)$

iii) $u \leq (\geq)\ g$ on $\partial\Omega$

are satisfied, where $\mathcal{D}^+(\Omega)$ consists of all nonnegative functions in $C_0^\infty(\Omega)$.

DEFINITION 2: A function u is called a solution of (1) if

i) $u \in C(\bar{\Omega};\mathbb{R})$

ii) $\int_\Omega (u(-\Delta\varphi) - f(x,u)\varphi)\, dx = 0$ for every $\varphi \in C_0^\infty(\Omega)$

iii) $u = g$ on $\partial\Omega$

are satisfied.

If f satisfies some additional assumption, like for example $u \to f(.,u) + \omega u$ is increasing for some $\omega \in \mathbb{R}$, and if $\partial\Omega$ satisfies some smoothness condition, then the following is known, see [2], [5], [6, Ch.10] and [3].

If $\underline{u}$ is a subsolution, $\bar{u}$ is a supersolution such that $\underline{u} \leq \bar{u}$, then problem (1) possesses a minimal and a maximal solution in the order interval $[\underline{u},\bar{u}]$. These solutions are obtained by the method of monotone iterations.

In [1] another method is used to prove the existence of a solution lying between a sub- and a supersolution for a very general quasilinear elliptic problem. The goal of this note is to show the existence of a solution lying between a sub- and supersolution, assuming only the continuity of f and for a much larger class of sub- and supersolutions.

We shall use the Schauder fixed point theorem and a version of the strong maximum principle.

Observe that if $f \equiv 0$, then problem (1) possesses a solution for every $g \in \partial\Omega$, if and only if all boundary points are regular, see [4, Th.2.14]. Therefore we assume

(H3) Ω is a bounded domain of $\mathbb{R}^N$ and every point of $\partial\Omega$ is regular.

Then we have

THEOREM: Assume (H1), (H2) and (H3), and let $\underline{u}$ respectively $\bar{u}$ be a sub- respectively a supersolution of problem (1), satisfying $\underline{u} \leq \bar{u}$ in $\bar{\Omega}$.

Then problem (1) possesses at least one solution u satisfying $\underline{u} \leq u \leq \bar{u}$ in $\bar{\Omega}$.

2. PROOF

We shall proceed in four steps.

STEP 1: Reduction to homogeneous boundary condition.

Let h denote the unique harmonic function on Ω, continuous on $\bar{\Omega}$, satisfying $h = g$ on $\partial\Omega$. Set $v = u - h$. Then u is a solution of problem (1) if and only if v is a solution of

$$(2) \qquad \begin{cases} -\Delta v = f(x, h(x) + v) & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases}$$

Observe that the modified right hand side again satisfies (H1). Since both $\underline{u} - h$ and $\bar{u} - h$ are sub- respectively supersolution for the modified problem and are also ordered, we may assume without loss of generality that $g \equiv 0$.

STEP 2: Modification of f.

Define

$$f^{*}(x,u) = \begin{cases} f(x,\underline{u}(x)) & \text{if } u < \underline{u}(x), \\ f(x,u) & \text{if } \underline{u}(x) \leq u \leq \bar{u}(x), \\ f(x,\bar{u}(x)) & \text{if } \bar{u}(x) < u, \end{cases} \qquad \text{and } x \in \bar{\Omega}.$$

Then $f^{*}: \bar{\Omega} \times \mathbb{R} \to \mathbb{R}$ is continuous and bounded. Note that, if u is a solution of

$$(3) \qquad \begin{cases} -\Delta u = f^{*}(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

and $\underline{u} \leq u \leq \bar{u}$ in $\bar{\Omega}$, then u is a solution of (1) with $g = 0$. In fact every solution of (3) satisfies $\underline{u} \leq u \leq \bar{u}$ in $\bar{\Omega}$. This is done in

STEP 3: Use of the maximum principle.

Let u be a solution of (3) and set $\Omega^{+} = \{ x \in \bar{\Omega} \ ; \ \bar{u}(x) < u(x) \}$

We want to prove that Ω^{+} is empty. Assume to the contrary that Ω^{+} is not empty. First, note that Ω^{+} is open, since u and $\bar{u}$ are continuous. Moreover we have

$$\int_{\Omega^+} (u-\bar{u})(-\Delta\varphi)\, dx \le \int_{\Omega^+} (f^*(x,u(x)) - f(x,\bar{u}(x)))\varphi\, dx = 0$$

for every $\varphi \in \mathcal{D}^+(\Omega^+)$.

Then $u - \bar{u} \in C(\overline{\Omega^+})$ is subharmonic and nonnegative in Ω^+. Such functions achieve its maximum at the boundary, see [4].
Since $u - \bar{u} = 0$ on $\partial\Omega^+$ it follows that $u = \bar{u}$ in Ω^+. Hence Ω^+ is empty, a contradiction. Similarly one proves that $\underline{u} \le u$ in $\bar{\Omega}$.

STEP 4: Application of the Schauder fixed point theorem.

It remains to show that problem (3) possesses a solution. Let us recall that problem (1) with f depending only on x and $g = 0$ has exactly one solution $u \in C(\bar{\Omega})$. Let $K: C(\bar{\Omega}) \to C(\bar{\Omega})$ denote the solution operator, that is $u = Kf$. Then it is known that K is a linear compact operator in $C(\bar{\Omega})$ equipped with the usual maximum norm $\|.\|$ (see also Appendix).
Let $F: C(\bar{\Omega}) \to C(\bar{\Omega})$ denote the Niemytski operator associated with f^*, that is

$$F(u)(x) = f^*(x,u(x)) \qquad \text{for } u \in C(\bar{\Omega}),\ x \in \bar{\Omega}.$$

Then F is continuous and there is $M > 0$ such that $\|F(u)\| \le M$. Finally observe that u is a solution of problem (3) if and only if u satisfies

$$u = KF(u).$$

A straightforward application of the Schauder fixed point theorem guarantees the existence of such a solution. This completes the proof of the theorem. □

REMARK: If u is a solution of (1), then it follows from standard regularity theory that $u \in W^{2,p}_{loc}(\Omega)$ for all $p \in [1,\infty)$, although $\underline{u}$ and $\bar{u}$ do not need to posses such regularity.

3. APPENDIX

PROPOSITION: Let Ω satisfy (H3) and $f \in C(\bar{\Omega})$, then there exists a unique $u \in C(\bar{\Omega})$ satisfying

i) $\int_\Omega (u(-\Delta\varphi) + f\varphi) \, dx = 0$ for every $\varphi \in C_0^\infty(\Omega)$,

ii) $u = 0$ on $\partial\Omega$.

Moreover the mapping $f \to u$ is compact in $C(\bar{\Omega})$.

PROOF: The uniqueness is a direct consequence of the maximum principle for harmonic functions. For the existence we extend f by 0 outside of $\bar{\Omega}$ and set

$$w(x) = \int_{\mathbb{R}^N} \Gamma(x-y) \, f(y) \, dy,$$

the Newtonian potential of f, see [4, p.50].

Then $w \in C^1(\bar{\Omega})$, see [4, Lemma 4.1], and the mapping $f \to w$ from $C(\bar{\Omega})$ in $C^1(\bar{\Omega})$ is continuous, where $C(\bar{\Omega})$ and $C^1(\bar{\Omega})$ are equipped with the usual norm. Since $C^1(\bar{\Omega})$ is compactly inbedded in $C(\bar{\Omega})$, the mapping $f \to w$ from $C(\bar{\Omega})$ into $C(\bar{\Omega})$ is compact.

Let $h \in C(\bar{\Omega})$ be the unique harmonic function satisfying $h = w$ on $\partial\Omega$ (here we use (H3)). Then $u = w - h$ is a solution of i), ii). Since the mapping $w \to h$ from $C(\bar{\Omega})$ into $C(\bar{\Omega})$ is continuous, we have that the mapping $f \to u$ from $C(\bar{\Omega})$ into $C(\bar{\Omega})$ is compact. □

4. REFERENCES

[1] K. Akô, On the Dirichlet problem for quasi-linear elliptic differential equations of second order, *J. Math. Soc. Japan*, Vol. 13, No.1 (1961), 45-62.

[2] H. Amann, Fixed point equations and nonlinear eigenvalue problems in ordered Banach spaces, *SIAM Rev.* 18 (1976), 620-709.

[3] Ph. Clément, G. Sweers, Existence and multiplicity results for a semilinear elliptic eigenvalue problem, to appear in *Annali della Scuola Normale Superiore di Pisa*.

[4] D. Gilbarg, N.S. Trudinger, *Elliptic partial differential equations of second order*, Springer-Verlag, Berlin/Heidelberg/New York, 1977.

[5] D.H. Sattinger, Topics in stability and bifurcation theory, *Springer Lect. Notes Math.* 309, Berlin/Heidelberg/New York, 1973.

[6] J. Smoller, *Shock waves and reaction-diffusion equations*, Springer-Verlag, Berlin/Heidelberg/New York, 1983.

Chapter 5

Semilinear elliptic problems

on

domains with corners.

SEMILINEAR ELLIPTIC PROBLEMS ON DOMAINS WITH CORNERS

ABSTRACT

In this note it is shown that there exist sign-changing stable solutions of some semilinear elliptic problems with Dirichlet boundary condition, if the (smooth) boundary is close to a cone somewhere. Moreover, for problems with these nonlinear right hand sides, there is a critical angle for corners of the domain if one wants existence of a positive solution.

1. INTRODUCTION

Consider the semilinear eigenvalue problem

$$(1.1) \qquad \begin{cases} -\Delta u = \lambda\, f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

with $f \in C(\mathbb{R})$ and $\lambda > 0$. If $f(0) \geq 0$ and $f(\rho) = 0$ for some $\rho > 0$, one can show, see e.g. [2,6], that there exists a solution u of (1.1), which satisfies

$$(1.2) \qquad 0 \leq u \leq \rho \text{ in } \Omega$$

if the boundary is just regular (see [8, page 25]).

In [4,5] it is shown for $f \in C^1(\mathbb{R})$ that there is also a positive solution for λ large when $f(0) < 0$, if the following conditions are satisfied:

(1.3) $f(\rho) = 0$ for some $\rho > 0$

(Assume $f(s) < 0$ for $s > \rho$)

(1.4) $\int_t^\rho f(s)ds > 0$ for all $t \in [0,\rho)$.

and finally

(1.5) $\Omega = \bigcup_x \{ B(x,\epsilon);\ x \in \Omega,\ d(x,\partial\Omega) > \epsilon \}$ for some $\epsilon > 0$,

where $B(x,\epsilon) = \{ y;\ d(x,y) < \epsilon \}$.

In [15] there are related results, for which proofs one needs (1.5). If Ω is like in (1.5), Ω is said to satisfy a uniform interior sphere condition. We will show that in order to find positive solutions this condition can be replaced by a cone condition. The angle of the cone depends on f.

Domains with a sharper corner will not have positive solutions. For bounded λ smooth domains close to these edgy domains will also not have a positive solution. Nevertheless there may exist a stable solution with positive maximum. Hence such a stable solution will change sign. These investigations were initiated by a question of W. Jäger. The results answer a question of Matano whether there are sign-changing stable solutions of (1.1) on convex domains. Matano himself recently found sign-changing stable solutions on convex domains with even $f(0) = 0$, [11].

A solution u of (1.1) is called stable, if for every $\epsilon > 0$ there is $\delta > 0$ such that, for every $U_0 \in L^\infty(\Omega)$ with $\| U_0 - u \|_\infty < \delta$, the solution U of the related parabolic problem:

$$(1.6)\qquad \begin{cases} U_t - \Delta U = \lambda f(U) & \text{in } \Omega \times \mathbb{R}^+, \\ U = 0 & \text{on } \partial\Omega \times \mathbb{R}^+, \end{cases}$$

with $\lim_{t \downarrow 0} \| U(t) - U_0 \|_{L_1(\Omega)} = 0$ satisfies $\| U(t) - u \|_\infty < \epsilon$ for all $t > 0$. See [3,10,13,14].

2. RESULTS

We will state and prove results for domains in $\mathbb{R}^2$. In higher dimensions similar results hold, but these will not be proved. The first lemma shows that there will not be a positive solution if for example the domain is convex and has a corner with angle less than $\frac{1}{2}\pi$ or close to $\frac{1}{2}\pi$, or if the domain is close to such a domain.

Lemma 2.1: Set $\lambda = 1$ and suppose f satisfies (1.3) (1.4) and $f(0) < 0$.

a. There is $t_1 \in (0,1)$ such that if

$$(t_1,0) \in \Omega \subset \{ (x_1,x_2);\ |x_2| < x_1 \} \tag{2.1}$$

there will be no positive solution of (1.1).

b. There is $c > 1$ such that if

$$\{ (t,0);\ 0 < t \leq 1 \} \subset \Omega \subset \{ (x_1,x_2);\ |x_2| < cx_1 \} \tag{2.2}$$

there will be no positive solution of (1.1).

Remark: By rescaling one finds that Lemma 2.1 b. holds for all $\lambda > 1$. Lemma 2.1 a. holds if one replaces t_1 by $t_\lambda = \lambda^{-\frac{1}{2}} t_1$.

In the proofs we will use a weak version of sub and supersolutions. For a definition see the Appendix.

Proof: i) *Estimating solutions from above.*

Set $f_M = \max \{ f(s) : 0 \leq s \leq \rho \}$ and $K = (2\rho f_M)^{\frac{1}{2}}$. Define $U \in C^1(\mathbb{R})$ by

$$\begin{cases} U(t) = K.t - \frac{1}{2}f_M.t^2 & \text{for } t \leq K.f_M^{-1}, \\ U(t) = \rho & \text{for } t > K.f_M^{-1}. \end{cases} \tag{2.3}$$

Let u be a solution of (1.1) and suppose (2.1) or (2.2) is satisfied. By the maximum principle one finds

(2.4) $$u(x_1,x_2) \le U(x_1) < K.x_1 \quad \text{for } (x_1,x_2) \in \Omega.$$

ii) *Taking a subdomain of* Ω.

Take $\epsilon \in (0,K)$ such that

(2.5) $$f(s) < \tfrac{1}{2}f(0) \qquad \text{for } |s| < \epsilon$$

and set

(2.6) $$\Omega^\nabla = \Omega \cap \{ (x_1,x_2) \in \mathbb{R}^2;\ x_1 < K^{-1}\epsilon \}.$$

By (2.4) one finds that

(2.7) $$-\Delta u = f(u) < \tfrac{1}{2}f(0) \qquad \text{for } x \in \Omega^\nabla.$$

iii) *Defining a superfunction on the subdomain* Ω^∇.

Define

(2.8) $$k = K\,\epsilon^{-1} \exp(-8\ f(0)^{-1}\ K^2\epsilon^{-1}) \ ,$$

(2.9) $$v(x_1,x_2) = -\ \tfrac{1}{8}f(0)\ ((x_1+\tfrac{1}{2}k^{-1})^2-x_2^2)\ \ln(kx_1+\tfrac{1}{2}).$$

Then one finds for

(2.10) $$|x_2| < x_1 + \tfrac{1}{2}k^{-1}$$

that

(2.11) $$-\Delta v(x_1,x_2) = \tfrac{1}{8}f(0)\ (3 + x_2^2(x_1+\tfrac{1}{2}k^{-1})^{-2}) > \tfrac{1}{2}f(0).$$

One also finds that v has a negative minimum for $|x_2| < x_1 + \frac{1}{2}k^{-1}$ in

(2.12) $$\bar{x} = (\ k^{-1}(e^{-1/2}-\tfrac{1}{2})\ ,\ 0\ ).$$

Define

(2.13) $$t_1 = k^{-1}(e^{-1/2}-\tfrac{1}{2})$$

(2.14) $$c = 1 + K(2\epsilon k)^{-1}.$$

Note that $t_1 \in (0,1)$ and hence $\bar{x} \in \Omega$.

By this choice of c one finds that if Ω satisfies (2.2) then every $x \in \Omega^\nabla$ satisfies (2.10).

iv) *Contradicting positivity of u by a lowest supersolution.*

Let α be the smallest number such that

(2.15) $$u \leq v + \alpha \qquad \text{in } \overline{\Omega^\nabla}.$$

If $\alpha \leq 0$ then

(2.16) $$u(\bar{x}) \leq v(\bar{x}) < 0.$$

If $\alpha > 0$, let $x^* \in \overline{\Omega^\nabla}$ be such that

(2.17) $$u(x^*) = v(x^*) + \alpha.$$

Since $v + \alpha - u$ is nonnegative and superharmonic in Ω^∇:

(2.18) $$-\Delta(v+\alpha-u) = -\Delta v + \Delta u > \tfrac{1}{2}f(0) - f(u) \geq 0$$

the minimum principle shows $x^* \in \partial\Omega^\nabla$.

First we will show that $x^* \in \partial\Omega$.

Similar to (2.4) the maximum principle yields

(2.19) $$\begin{cases} u(x_1,x_2) \leq U((c^2+1)^{-1}(cx_1-x_2)) \ . \\ u(x_1,x_2) \leq U((c^2+1)^{-1}(cx_1+x_2)) \ . \end{cases}$$

Hence if $x \in \partial\Omega^\nabla \backslash \partial\Omega$, which means that $x_1 = K^{-1}\epsilon$ and $|x_2| < cK^{-1}\epsilon$, then

(2.20) $$\begin{aligned} v(K^{-1}\epsilon,x_2) &> -\tfrac{1}{8}f(0)\ ((K^{-1}\epsilon+\tfrac{1}{2}k^{-1})^2 - x_2^2).-8f(0)^{-1}\ K^2\ \epsilon^{-1} = \\ &= \ (c^2K^{-2}\epsilon^2-x_2^2)\ K^2\ \epsilon^{-1} \quad \geq \\ &\geq \ (cK^{-1}\epsilon-|x_2|)\ c\ K \quad > \\ &> \ K(c^2+1)^{-1}(cK^{-1}\epsilon-|x_2|) \ > \ u(K^{-1}\epsilon,x_2) \ . \end{aligned}$$

Finally $x^* \in \partial\Omega$ yields $u(x^*) = 0 = v(x^*) + \alpha$ and hence

(2.21) $$u(\bar{x}) \leq v(\bar{x}) + \alpha < v(x^*) + \alpha = 0. \ \square$$

The first part of Lemma 2.1 can be used to construct sign-changing stable solutions on smooth domains.

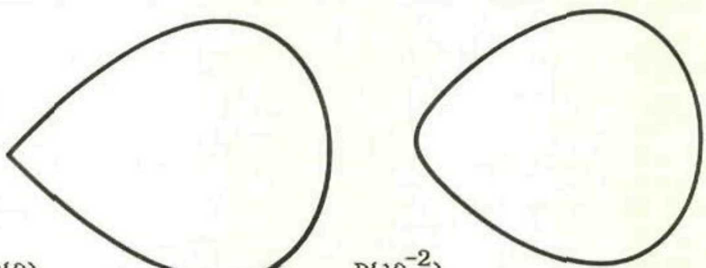
As an example:

Corollary 2.2: Set $f(u) = (u^2-1)(10-u)$ and

$$D(\epsilon) = \{ (x_1,x_2) \in \mathbb{R}^2;\ x_1 > 0,\ x_2^2 < x_1^2(1-x_1) - \epsilon \} \tag{2.22}$$

a. Then there is $\lambda_1 > 0$ such that for all $\lambda > \lambda_1$ and $\epsilon \in (0,\frac{1}{10})$ there is a stable solution $u_{\lambda,\epsilon}$ of (1.1) on $D(\epsilon)$ with $\max u_{\lambda,\epsilon} \in (1,10)$.

b. For all $\lambda > \lambda_1$ there is $\epsilon(\lambda) > 0$ such that for $\epsilon \in (0,\epsilon(\lambda)]$ $u_{\lambda,\epsilon}$ changes sign in $D(\epsilon)$.



$D(0)$ $D(10^{-2})$

Proof: First note that for $\epsilon \in (0,\frac{1}{10})$

$$(\tfrac{2}{3},0) \in D(\tfrac{1}{10}) \subset D(\epsilon) \subset D(0) \subset \{ (x_1,x_2);\ |x_2| < x_1 \}. \tag{2.23}$$

Hence there is $\delta > 0$ with $B((\frac{2}{3},0),\delta) \subset D(\epsilon)$ for all $\epsilon \in [0,\frac{1}{10}]$. By Lemma A.1 there exist $\mu > 0$ and $v \in C^2(\overline{B(0,1)})$, with v radially symmetric, which satisfy

$$\begin{cases} -\Delta v = \mu\, f(v) & \text{in } B(0,1), \\ 1 < v(0) < 10\ , & \\ v'(r) < 0 & \text{for } 0 < r \le 1, \\ v(1) = -1\ . & \end{cases} \tag{2.24}$$

Extend v by -1 outside of $B(0,1)$.

Now define $\lambda_1 = \mu\, \delta^{-2}$ and

$$V(x_1,x_2) = v((\lambda/\mu)^{1/2}(x_1-\tfrac{2}{3}),(\lambda/\mu)^{1/2}x_2) \tag{2.25}$$

which is, see Corollary A.5, a subsolution of (1.1) for all $\lambda > \lambda_1$ and $\epsilon < \frac{1}{10}$ on $D(\epsilon)$, satisfying $V = -1$ on $\partial D(\epsilon)$. The constant function $W = 10$ is a supersolution for all $\lambda > 0$.

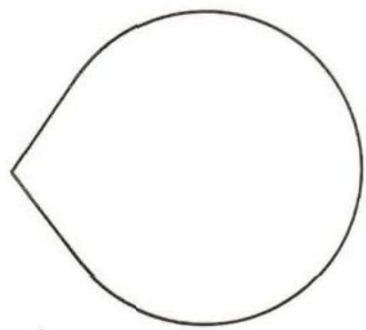
By Lemma A.6 there exists a stable solution $u_{\lambda,\epsilon}$ of (1.1) in $[V,W]$. This proves the first part. Fix $\lambda > \lambda_1$ and let t_1 be defined in Lemma 2.1. Take $\epsilon(\lambda)$ so small that $\lambda^{-1/2} t_1 \in D(\epsilon(\lambda))$. The second part of the corollary is a consequence of the remark following Lemma 2.1. □

In the next lemma we will show that for a special domain with a corner which is just below π there does exist a positive solution of (1.1). Define for $c > 0$

(2.26) $$A(c) = \{(x_1,x_2);\ |x_2| < cx_1 + (1+c^2)^{1/2},\ x_1 < -c(1+c^2)^{-1/2}\},$$

(2.27) $$S(c) = A(c) \cup B(0,1),$$

where $B(0,1)$ is the unit ball.



$S(\frac{4}{3})$

Lemma 2.3: Let f satisfy (1.3), (1.4) and suppose $f(0) < 0$. Then there are $c > 1$, λ and u, which satisfy (1.1)-(1.2) on $S(c)$.

Proof: There is a radially symmetric subsolution (λ,U) of (1.1)-(1.2) on $B(0,1)$, which satisfies $U'(r) < 0$ for $r \in (0,1]$. Since f is continuous there is $f^* \in C^1$ with $f^* \leq f$, which still satisfies (1.3) and (1.4). Lemma A.1 yields the existence of a radially symmetric solution on $B(0,1)$ with f replaced by f^*, which is a subsolution of the original problem. We will show that for some $c \in (1,\infty)$ there exists a positive solution on $S(c)$ with the same λ.

Set the negative number $f_m = \min \{ f(s) \; ; \; 0 \leq s \leq \rho \}$ and define for $c > 1$

$$V_c(x_1,x_2) = -\tfrac{1}{2}\lambda \, f_m \, (c^2-1)^{-1} \, \left(\left(cx_1 + (1+c^2)^{1/2} \right)^2 - x_2^2 \right) \quad , \tag{2.28}$$

which is positive on $S(c)$. Moreover, one finds directly that V_c satisfies:

$$-\Delta V_c = \lambda f_m \leq \lambda f(V_c) \qquad \text{if } 0 \leq V_c \leq \rho. \tag{2.29}$$

Define

$$\begin{cases} W_c(x) = V_c(x) & \text{for } x \in \overline{A(c)} \setminus B(0,1) \; , \\ W_c(x) = \max(\, V_c(x) \, , \, U(x) \,) & \text{for } x \in A(c) \cap B(0,1) \; , \\ W_c(x) = U(x) & \text{for } x \in \overline{B(0,1)} \setminus A(c) \; . \end{cases} \tag{2.30}$$

Since for some $\alpha > 0$

$$U(x) > \alpha(1-|x|^2) \qquad \text{for } x \in B(0,1) \; , \tag{2.31}$$

one finds for $c = \left(1 - \tfrac{1}{2}\lambda \, f_m \alpha^{-1} \right)^{1/2} > 1$ that

$$\begin{aligned} V_c(x_1,x_2) &= -\tfrac{1}{2}\lambda \, f_m \, (c^2-1)^{-1} \, \left((1+c^2)^{-1} - x_2^2 \right) = \\ &= \alpha \left((1+c^2)^{-1} - x_2^2 \right) < \\ &< U(x_1,x_2) \qquad \text{for } (x_1,x_2) \in \partial A(c) \cap B(0,1). \end{aligned} \tag{2.32}$$

For $x \in A(c) \cap \partial B(0,1)$ one finds

$$V_c(x) > 0 = U(x) \; . \tag{2.33}$$

Hence $W_c \in C(\overline{S(c)})$ and by Corollary A.5 one finds that W_c is a subsolution. By construction W_c is positive in $S(c)$.

Applying the results in [6] shows the existence of a solution $u \in [W_c, \rho] \subset C(\overline{S(c)})$. Hence u satisfies (1.2). □

Before we are able to state the main result for domains in $\mathbb{R}^2$ we need the following.

Definition 2.4: A domain Ω has the uniform interior cone property with constant c if $\Omega = \cup \{ \epsilon S_i ; i \in I \}$ for some $\epsilon > 0$, where every S_i is an orthonormal transformed of S(c) for a fixed c. (S(c) is defined in (2.27) ; T is an orthonormal transformation if $\|T(x)-T(y)\|_2 = \|x-y\|_2$ for all x, y $\in \mathbb{R}^2$).

Proposition 2.5: Let f satisfy (1.3), (1.4) and $f(0) < 0$. Then there is $c_0 \in (1,\infty)$ for which the following holds. Let Ω be bounded and convex.

1) If Ω has the uniform interior cone property with $c > c_0$, then λ_0 exists such that for all $\lambda > \lambda_0$ there is a solution u_λ of (1.1)-(1.2).
2) If Ω does not have the uniform interior cone property for some $c < c_0$, then there are no solutions (λ, u) of (1.1)-(1.2).

Remark: If Ω is not convex, part 1) of this proposition is still true.

Proof:

i) Let Ω_1 and Ω_2 be two bounded domains in $\mathbb{R}^2$. Suppose there exists a solution (λ_1, u_1) of (1.1)-(1.2) on Ω_1, and suppose there is $\epsilon > 0$ and a family $\{ T_i ; i \in I \}$ of orthonormal transformations in $\mathbb{R}^2$ such that $\Omega_2 = \cup \{ T_i(\epsilon\Omega_1) ; i \in I \}$. Since Ω_1 and Ω_2 are open one can assume without loss of generality that I is countable. By

Corollary A.5:

(2.34) $$v_n(x) = \sup \{ u(\epsilon^{-1}T_i^{-1}(x)) \; ; \; i \in \{i_1,\dots,i_n\} \subset I \}$$

is a subsolution on Ω_2 for $\lambda = \lambda_1\epsilon^{-2}$. Using the dominated convergence theorem one finds that

(2.35) $$v(x) = \lim_{n\to\infty} v_n(x)$$

satisfies condition ii) in Definition A.2. Since $\{v_n\}$ is an equicontinuous family, v also satisfies the conditions ii) and iii) in Definition A.2/A.3. Hence v is a subsolution on Ω_2 for $\lambda = \lambda_1\epsilon^{-2}$. By $v > 0$ in Ω_2, $\max v = \max u$ and again the supersolution $w = \rho$, one gets the existence of a solution (λ_2, u_2) of (1.1)-(1.2) on Ω_2 with $\lambda_2 = \lambda_1\epsilon^{-2}$.

ii) If Ω_1 is also convex then $\Omega_1 = \cup \{ x + \theta(\Omega_1 - x) \; ; \; x \in \Omega_1 \}$ for all $\theta \in (0,1)$. By part i) there will be a solution of (1.1)-(1.2) on Ω_2 for all $\lambda \geq \lambda_2$.

iii) Define $J = \{ c \in (0,\infty) \; ; \;$ there exists a solution (λ,u) of (1.1)-(1.2) on $S(c)$ $\}$. By Lemma 2.1 there is $c_1 > 1$ such that $c_1 \notin J$. Lemma 2.3 shows that there is $c_2 < \infty$ such that $c_2 \in J$. Part i) of this proof shows that if $c \in J$, then $[c,\infty) \subset J$. Hence

(2.36) $$c_0 = \inf \{ c \in J \} \in (1,\infty)$$

is well defined. With part ii) this proves Proposition 2.5. 1).

iv) We still have to prove the second statement.

Suppose Ω is convex but does not have the uniform interior cone condition with constant c for some $c < c_0$. Since Ω is also bounded there is $x^* \in \partial\Omega$, $\epsilon > 0$, and an orthonormal transformation T such that

(2.37) $$\begin{cases} T(\epsilon\Omega) \subset S(c), \\ T(\epsilon x^*) = \tilde{x}, \end{cases}$$

where $\tilde{x} = (-c^{-1}(1+c^2)^{1/2} , 0)$ is the vertex point of $S(c)$.

Since $\partial S(c)\backslash\{\tilde{x}\}$ is C^1 there is $\theta > 0$ and a family of orthonormal transformations $\{ T_i ; i \in I \}$ such that

(2.38) $$S(c) = \cup \{ T_i(\theta\epsilon\Omega) ; i \in I \}.$$

If there is an solution of (1.1)-(1.2) on Ω, then part i) gives the existence of a solution of (1.1)-(1.2) on $S(c)$, which is contradicted by part iii). □

For domains in higher dimensions there is no longer a unique critical cone. For example in $\mathbb{R}^3$ one may use the following superfunctions to prove nonpositivity:

(2.39) $$v(x_1,x_2,x_3) = c_1(x_1^2-\theta x_2^2-(1-\theta)x_3^2)\ \ln(c_2x_1).$$

With every $\theta \in (0,1)$ one can find a critical cone. Replace $S(c)$ in Definition 2.4 by

(2.40) $$S(\theta,c) = \{ (x_1,x_2,x_3) ; (x_1,(\theta x_2^2+(1-\theta)x_3^2)^{1/2}) \in S(c) \}$$

and one can prove the equivalent of Proposition 2.5 for every $\theta \in (0,1)$.

APPENDIX

Lemma A.1: Let $f \in C^1(\mathbb{R})$ satisfy

(a.1) $$f(\rho) = 0 \quad \text{for some} \quad \rho > 0$$

and

(a.2) $$\int_u^\rho f(s)\,ds > 0 \quad \text{for all } u \in [0,\rho).$$

Then for all $\epsilon > 0$ there is $\mu > 0$ and $v \in C^2[0,1]$ such that:

(a.3) $$\begin{cases} -\left(v'' + \frac{N-1}{r} v' \right) = \mu f(v), \\ v(0) \in (\rho-\epsilon,\rho), \\ v'(0) = v(1) = 0, \\ v'(r) < 0 \quad \text{for } r \in (0,1]. \end{cases}$$

Proof: For a proof see also [5].

Change f for negative numbers such that

(a.4) $$f(s) > |f(-s-2)| \quad \text{for } s \leq -1$$

and

(a.5) $$\int_u^\rho f(s)\,ds > 0 \qquad \text{for all } u < \rho.$$

Moreover assume

(a.6) $$f(s) < 0 \qquad \text{for } s > \rho.$$

Take a minimizing sequence $\{u_n\}$, for fixed μ, of

(a.7) $$I(u,\mu) = ½ \int_B |\nabla u|^2 dx - \mu \int_B \int_{-1}^{u} f(s)\,ds\,dx\ ,$$

for $u+1 \in W_0^{1,2}(B)$, where B denotes the unit ball in $\mathbb{R}^N$. Since $I(|u_n+1|-1,\mu) < I(u_n,\mu)$ and since $I(.,\mu)$ is sequentially weakly lower semicontinuous and coercive, $I(.,\mu)$ possesses a minimizer $u_\mu \geq -1$ in $W^{1,2}(B)$ with $u_\mu=-1$ on ∂B. Regularity theory, see [8], shows that $u_\mu \in C^2(\bar{B})$. By [7] one finds that u_μ is radially symmetric and $u_\mu'(r) < 0$ for all $r \in (0,1)$. Hence u_μ satisfies the first and fourth condition in (a.3). By the strong maximum principle one finds $u_\mu(0) < \rho$.

Suppose $u_\mu(0) \leq \rho-\epsilon$ for all $\mu > 0$. Then define

$$\text{(a.8)} \qquad \begin{cases} w_\delta(r) = \rho & \text{for } r < 1-\delta, \\ w_\delta(r) = \delta^{-1}(1-r)(1+\rho) - 1 & \text{for } 1-\delta < r < 1. \end{cases}$$

Since $I(u_\mu,\mu) > I(w_\delta,\mu)$ for μ large and δ small if $u_\mu \leq \rho-\epsilon$, this yields a contradiction. Hence for some μ_1 one finds that $\rho-\epsilon < u_{\mu_1}(0) < \rho$. Since u_{μ_1} is strictly decreasing for $r > 0$, there is a unique $r_1 \in (0,1)$ with $u_{\mu_1}(r_1) = 0$. Then v and μ defined by

$$\text{(a.9)} \qquad v(r) = u_{\mu_1}(rr_1), \quad \mu = \mu_1 r_1^2$$

satisfy (a.3). □

Definition A.2: Let Ω be an open bounded domain in $\mathbb{R}^N$, and let $f \in C(\mathbb{R})$. We call a function u a superfunction (subfunction) of

$$\text{(a.10)} \qquad -\Delta u = f(u) \qquad \text{in } \Omega,$$

if i) $u \in C(\bar{\Omega})$,

ii) $\int_\Omega (u(-\Delta\varphi) - f(u)\varphi)\, dx \geq (\leq)\ 0$ for all $\varphi \in \mathcal{D}^+(\Omega)$,

where $\mathcal{D}^+(\Omega)$ consists of all nonnegative functions in $C_0^\infty(\Omega)$.

Definition A.3: Let Ω be an open bounded domain in $\mathbb{R}^N$, let $f \in C(\mathbb{R})$ and $g \in C(\partial\Omega)$. We call a function u a supersolution (subsolution) of

$$\text{(a.11)} \qquad \begin{cases} -\Delta u = f(u) & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega, \end{cases}$$

if u satisfies i), ii) and

iii) $u \geq (\leq)\ g$ on $\partial\Omega$.

Lemma A.4: Let u_1 and u_2 be subfunctions of (a.10) on an open bounded domain Ω, with f only continuous. Then u defined by

$$u(x) = \max\,(u_1(x), u_2(x)) \quad \text{for} \quad x \in \bar{\Omega}, \tag{a.12}$$

is a subfunction of (a.10) on Ω.

Corollary A.5: Let v_i be a subfunction of (a.10) on $\Omega = \Omega_i$, where i is 1 or 2. Define v by

$$\begin{cases} v(x) = v_1(x) & \text{for} \quad x \in \bar{\Omega}_1 \setminus \Omega_2, \\ v(x) = \max\,(v_1(x), v_2(x)) & \text{for} \quad x \in \Omega_1 \cap \Omega_2, \\ v(x) = v_2(x) & \text{for} \quad x \in \bar{\Omega}_2 \setminus \Omega_1. \end{cases} \tag{a.13}$$

If

$$\begin{cases} v_1 < v_2 & \text{on } \partial\Omega_1 \cap \Omega_2, \\ v_2 < v_1 & \text{on } \partial\Omega_2 \cap \Omega_1, \end{cases} \tag{a.14}$$

then v is a subfunction of (a.10) on $\Omega = \Omega_1 \cup \Omega_2$.

Remark 1. Let v_i be a subsolution of (a.11) on Ω_i with $g = g_i$, where i=1 or 2. If v_1, v_2 and v are like in Corollary A.5, then v is a subsolution on $\Omega_1 \cup \Omega_2$ for every g with $g \geq g_1$ on $\partial\Omega_1 \setminus \Omega_2$ and $g \geq g_2$ on $\partial\Omega_2 \setminus \Omega_1$.

Remark 2. Let $\{ u_i \;;\; i=1,\ldots,k \}$ be a family of subfunctions on Ω. Then one finds that the maximum of these subfunctions is again a subfunction.

Remark 3. Similar results hold for superfunctions and supersolutions if one replaces maximum by minimum and reverses the inequality signs.

Proof of the Corollary: By construction i) in Definition A.2 is immediate; ii) remains to be proved. Let $\varphi \in \mathcal{D}^+(\Omega_1 \cup \Omega_2)$. Because of (a.14) and the continuity of v_1 and v_2, it is possible to find φ_1, φ_2 and $\varphi_3 \in \mathcal{D}^+(\Omega_1 \cup \Omega_2)$ such that $\varphi = \varphi_1+\varphi_2+\varphi_3$, $v = v_i$ on support(φ_i) for i=1,2 , and support(φ_3) $\subset \Omega_1 \cap \Omega_2$. Hence it is sufficient to prove ii) for all $\varphi \in \mathcal{D}^+(\Omega_1 \cap \Omega_2)$. This follows from Lemma A.4.

Proof of Lemma A.4: Let J_ϵ be the mollifier defined in [1, 2.17]:

With $J(x) = 0$ for $|x| \geq 1$ and $J(x) = \exp((|x|^2-1)^{-1})$ for $|x| < 1$,

$J_\epsilon(x) = J(x/\epsilon)/ (\int_{\mathbb{R}^N} J(y/\epsilon)\, dy)$.

Define for $\psi \in C(\bar{\Omega})$ the function $J_\epsilon * \psi \in C_0^\infty(\mathbb{R}^N)$ by:

(a.15) $$(J_\epsilon * \psi)(x) = \int_\Omega J_\epsilon(x-y)\, \psi(y)\, dy \quad \text{for } x \in \mathbb{R}^N.$$

Let $\eta > 0$ and define $\Omega_\eta = \{ x \in \Omega \ ; \ d(x,\partial\Omega) > \eta \}$. Suppose $\varphi \in \mathcal{D}^+(\Omega_\eta)$. By [1, Lemma 2.18] one finds that $J_\epsilon * \varphi \in \mathcal{D}^+(\Omega)$ if $\epsilon < ½\eta$, $J_\epsilon * u_i \in C_0^\infty(\mathbb{R}^N)$ and $\lim_{\epsilon \downarrow 0} J_\epsilon * u_i = u_i$ uniformly on Ω_η.

Hence for $\epsilon < ½\eta$ and $\varphi \in \mathcal{D}^+(\Omega_\eta)$:

(a.16) $$\begin{aligned} 0 &\geq \int_\Omega (-\Delta(J_\epsilon * \varphi)\, u_i - (J_\epsilon * \varphi)\, f(u_i))\, dx = \\ &= \int_{\mathbb{R}^N} ((J_\epsilon * -\Delta\varphi)\, u_i - (J_\epsilon * \varphi)\, f(u_i))\, dx = \\ &= \int_{\mathbb{R}^N} (-\Delta\varphi\, (J_\epsilon * u_i) - \varphi\, (J_\epsilon * f(u_i)))\, dx = \\ &= \int_{\Omega_\eta} (-\Delta(J_\epsilon * u_i) - (J_\epsilon * f(u_i)))\, \varphi\, dx . \end{aligned}$$

Since $-\Delta(J_\epsilon * u_i) - (J_\epsilon * f(u_i))$ is continuous one gets pointwise in Ω_η that

(a.17) $$-\Delta(J_\epsilon * u_i) - (J_\epsilon * f(u_i)) \leq 0 .$$

For $v \in C^2(\Omega)$ and $\varphi \in \mathcal{D}^+(\Omega)$ one finds by the Kato inequality

[9, Lemma A] that

$$\text{(a.18)} \qquad \int_\Omega \Delta\varphi\ |v|\ dx \ \geq \int_\Omega \text{sign}(v)\ \varphi\ \Delta v\ dx \quad .$$

Again let $\varphi \in \mathcal{D}^+(\Omega_\eta)$ and $\epsilon < ½\eta$. Write $u_{i,\epsilon} = J_\epsilon * u_i$. Using (a.17) and (a.18) one finds:

$$\text{(a.19)} \qquad \int_{\Omega_\eta} \max(u_{1,\epsilon}, u_{2,\epsilon}).-\Delta\varphi\ dx =$$

$$= \int_{\Omega_\eta} ½(u_{1,\epsilon} + u_{2,\epsilon}).-\Delta\varphi\ dx + ½\int_{\Omega_\eta} |u_{1,\epsilon} - u_{2,\epsilon}|.-\Delta\varphi\ dx \leq$$

$$\leq ½\int_{\Omega_\eta} -\Delta(u_{1,\epsilon} + u_{2,\epsilon})\varphi\ dx - ½\int_{\Omega_\eta} \text{sign}(u_{1,\epsilon} - u_{2,\epsilon})\ \varphi\ \Delta(u_{1,\epsilon} - u_{2,\epsilon})\ dx =$$

$$= \int_{\Omega_\eta} \Big(\chi_{[u_{1,\epsilon} > u_{2,\epsilon}]}.-\Delta u_{1,\epsilon} + \chi_{[u_{1,\epsilon} < u_{2,\epsilon}]}.-\Delta u_{2,\epsilon} +$$

$$\chi_{[u_{1,\epsilon} = u_{2,\epsilon}]}.-\Delta(½u_{1,\epsilon} + ½u_{2,\epsilon}) \Big)\ \varphi\ dx \qquad \leq$$

$$\leq \int_{\Omega_\eta} \Big(\chi_{[u_{1,\epsilon} > u_{2,\epsilon}]}(J_\epsilon * f(u_1)) + \chi_{[u_{1,\epsilon} < u_{2,\epsilon}]}(J_\epsilon * f(u_2)) +$$

$$\chi_{[u_{1,\epsilon} = u_{2,\epsilon}]}\big(½J_\epsilon * f(u_1) + ½J_\epsilon * f(u_2) \big) \Big)\ \varphi\ dx \quad .$$

Set $u_3 = \max(u_1, u_2)$. With the identities $J_\epsilon * f(u_1) = (J_\epsilon * f(u_1) - f(u_1)) + (f(u_1) - f(u_3)) + f(u_3)$ and $\chi_{[u_{1,\epsilon} > u_{2,\epsilon}]}(f(u_1) - f(u_3)) = \chi_{[u_{1,\epsilon} > u_{2,\epsilon}]}\ \chi_{[u_1 < u_2]}(f(u_1) - f(u_2))$, we will show that the first term in the last integral of (a.19) converges to the right expression if $\epsilon \downarrow 0$.

First, since $f(u_1)$ is continuous, $\lim_{\epsilon \downarrow 0} (J_\epsilon * f(u_1)) = f(u_1)$ uniformly on Ω_η. Secondly, $\chi_{[u_{1,\epsilon} > u_{2,\epsilon}]}.\chi_{[u_1 < u_2]}$ goes to zero pointwise on Ω since u_1 and u_2 are continuous. And thirdly, since u_1 is continuous, $\lim_{\epsilon \downarrow 0} u_{1,\epsilon} = u_1$ uniformly on Ω_η. Hence, since all the involved functions are bounded independently of ϵ:

(a.20)
$$\lim_{\epsilon\downarrow 0}\int_{\Omega_\eta} \chi_{[u_{1,\epsilon}>u_{2,\epsilon}]}(J_\epsilon * f(u_1))\,\varphi\,dx =$$

$$= \lim_{\epsilon\downarrow 0}\int_{\Omega_\eta} \chi_{[u_{1,\epsilon}>u_{2,\epsilon}]}(\,(J_\epsilon * f(u_1)) - f(u_1)\,)\,\varphi\,dx +$$

$$\lim_{\epsilon\downarrow 0}\int_{\Omega_\eta} \chi_{[u_{1,\epsilon}>u_{2,\epsilon}]}\,\chi_{[u_1<u_2]}(\,f(u_1) - f(u_2)\,)\,\varphi\,dx +$$

$$\lim_{\epsilon\downarrow 0}\int_{\Omega_\eta} \chi_{[u_{1,\epsilon}>u_{2,\epsilon}]}\,f(u_3)\,\varphi\,dx =$$

$$= \int_{\Omega_\eta} \chi_{[u_1>u_2]}\,f(u_3)\,\varphi\,dx .$$

Similarly one treats the second and third term at the end of (a.19). Since $\lim_{\epsilon\downarrow 0}\max(u_{1,\epsilon},u_{2,\epsilon}) = \max(u_1,u_2) = u_3$ one has shown:

(a.21)
$$\int_\Omega (\,u_3.-\Delta\varphi - f(u_3)\,\varphi\,)\,dx \le 0 \qquad \text{for all } \varphi \in \mathcal{D}^+(\Omega_\eta)$$

Since $\bigcup_{\eta>0}\Omega_\eta = \Omega$ (a.21) is valid for all $\varphi \in \mathcal{D}^+(\Omega)$. □

Lemma A.6: Let $f \in C^1(\mathbb{R})$, Ω be bounded and $\partial\Omega \in C^3$ and set $g=0$. If u_1, respectively u_2, with $u_1 < u_2$ in Ω, are respectively a sub and a supersolution of (a.11) with $u_1 < 0 < u_2$ on $\partial\Omega$, then there exists a stable solution $u \in [u_1,u_2] \subset C(\bar\Omega)$ of (a.11).

Proof: In order to get sub and supersolutions in $C^2(\bar\Omega)$, we will use the first two steps in a monotone iteration scheme.

Set $\epsilon = \min\{\,-u_1(x),\ u_2(x)\ ;\ x \in \partial\Omega\,\}$. Define

(a.22)
$$\omega = \max\{\,f'(u)\ ;\ \min u_1(x) \le u \le \max u_2(x)\,\} ,$$

and the operator T_σ: $C(\bar\Omega) \to C(\bar\Omega)$ by

(a.23)
$$T_\sigma(u) = (\,-\Delta + \omega\,)^{-1}(\,\omega u + \lambda f(u)\,) .$$

where $(-\Delta+\omega)^{-1}$ is the inverse of $-\Delta+\omega$ with Dirichlet boundary condition $u = \sigma\varepsilon$, $\sigma \in \{-,+\}$. The operators T_σ are order preserving. Moreover, if v is a subsolution of (a.11) with $g = -\varepsilon$, then $T_-(v) \geq v$ and $T_-(v)$ is also a subsolution, (see e.g. [13] or [5]). By regularity theory (see [8]) $T_-^2(u_1)$, $T_+^2(u_2) \in C^2(\bar{\Omega})$. Since $T_-^2(u_1) < T_+^2(u_2)$ in $\bar{\Omega}$, are respectively a sub and a supersolution, one can use [13, Th.3.6]. The unique solution U_1 of

(a.24)
$$\begin{cases} U_t - \Delta U = f(U) & \text{in } \Omega \times \mathbb{R}^+, \\ U = 0 & \text{on } \partial\Omega \times \mathbb{R}^+, \\ U(0) = \varphi & \text{on } \Omega, \end{cases}$$

with $\varphi = T_-^2(u_1)$, satisfies $U_1(x,t) \uparrow v_1(x)$ for $t \to \infty$, and v_1 is a solution of (a.11) with $g = 0$. Similar the unique solution U_2 of (a.24) with $\varphi = T_+^2(u_2)$ satisfies $U_2(x,t) \downarrow v_2(x)$ for $t \to \infty$, and $v_2 \geq v_1$ is also a solution of (a.11) with $g = 0$. By the maximum principle for elliptic problems, [12, Th.2.6], one finds:

(a.25)
$$u_1 \leq T_-^2(u_1) < v_1 \leq v_2 < T_+^2(u_2) \leq u_2 \quad \text{in } \bar{\Omega}.$$

By the maximum principle for parabolic problems, [12, Th.3.12], every solution U of (a.24) with $T_-^2(u_1) \leq \varphi \leq v_1$ converges to v_1 for $t \to \infty$. Hence v_1 is stable from below. Similarly v_2 is stable from above. By [10, Th.4.3] one finds that there is at least one stable solution $u \in [v_1,v_2] \subset [u_1,u_2] \subset C(\bar{\Omega})$. □

REFERENCES

[1] R.A. Adams, Sobolev Spaces, Academic Press, New York/San Francisco/London, 1975.

[2] K. Ako, On the Dirichlet problem for quasilinear elliptic differential equations of the second order, J. Math. Soc. Japan 13 (1961), 45-62.

[3] H. Amann, Supersolutions, monotone iterations, and stability, J. Differential Equations, 21 (1976), 363-377.

[4] Ph. Clément, G. Sweers, Existence et multiplicité des solutions d'un problème aux valeurs propres elliptique semilinéaire, C.R. Acad. Sci., Paris 302, I Ser., 19 (1986), 681-683.

[5] Ph. Clément, G. Sweers, Existence and multiplicity results for a semilinear elliptic eigenvalue problem, to appear in Annali della Scuola Normale Superiore di Pisa.

[6] Ph. Clément, G. Sweers, Getting a solution between sub- and supersolutions without monotone iteration, to appear in Rendiconti della Universitá di Trieste.

[7] B. Gidas, W.M. Ni, L. Nirenberg, Symmetry and related properties via the maximum principle, Comm. Part. Phys. 68 (1979), 209-243.

[8] D. Gilbarg, N.S. Trudinger, Elliptic partial differential equations of second order, Springer-Verlag, Berlin/Heidelberg/New York, 1977.

[9] T. Kato, Schrödinger operators with singular potentials, Israel J. Math. 13 (1972), 135-148.

[10] H. Matano, Asymptotic behaviour and stability of solutions of semilinear diffusion equations, Publ. RIMS, Kyoto Univ. 15 (1979), 401-454.

[11] H. Matano, in preparation.

[12] M. Protter, H. Weinberger, Maximum principles in differential equations, Prentice Hall, Englewood Cliffs N.J., 1967.

[13] D.H. Sattinger, Monotone methods in nonlinear elliptic and parabolic boundary value problems, Indiana Univ. Math. J. 21 (1972), 979-1000.

[14] D.H. Sattinger, Topics in stability and bifurcation theory, Springer Lect. Notes Math. 309, Berlin/Heidelberg/New York, 1973.

[15] J. Smoller, A. Wassermann, Existence of positive solutions for semilinear elliptic equations in general domains, Arch. for Rat. Mech. Anal. 98 (1987), 229-249.

Chapter 6

Existence and stability for a nonautonomous semilinear elliptic eigenvalue problem.

1. INTRODUCTION.

Consider the eigenvalue problem

$$(1.1)\quad \begin{cases} -\Delta u = \lambda F(u) & \text{in } \Omega, \\ u = \varphi & \text{on } \partial\Omega, \end{cases}$$

with Ω a bounded smooth domain in $\mathbb{R}^N$ and λ positive. In [3] one considers $\varphi = 0$ and autonomous F: $F(u)(x) = f(u(x))$, where $f \in C^1(\mathbb{R})$ is a function which changes sign. In that paper one gives necessary and sufficient conditions, which are independent of the dimension, in order to obtain existence of a positive solution near a falling zero of f except for a boundary layer. The zero ρ satisfies for some $\epsilon > 0$

$$(1.2)\quad f(\rho) = 0 \quad \text{and} \quad f(s) > 0 \quad \text{for} \quad s \in (\rho-\epsilon,\rho).$$

A basic idea to treat such a problem comes from [1]. Angenent essentially considers $f \in C^1(\bar{\Omega}\times\mathbb{R})$ which is positive untill the first zero ρ and $f_u(x,\rho(x)) < 0$ in $\bar{\Omega}$. Under additional conditions in [1] as well as in [3] one obtains a curve of stable solutions. In this paper some of these results are generalized.

First it is shown that for existence of such a solution with boundary layer, it is sufficient to assume $f \in C^0(\bar{\Omega}\times\mathbb{R})$, with $\rho \in C^0(\bar{\Omega})$ a falling zero, when the integral condition of [5] at the boundary is satisfied (f may change sign). Fife obtained results for (1.1) using asymptotic expansions for infinitely smooth functions. In this paper we approach by using sub and supersolutions. Without much more difficulties we also state an existence result for solutions with interior layers which is related to results in [6].

Secondly, in order to get strongly stable solutions it is also necessary in [5] to assume $f_u(x,\rho(x)) < 0$ for $x \in \bar{\Omega}$. The method used here allows one to weaken this condition to $f_u(x,u) \leq 0$ for $| u - \rho(x) | < \epsilon$ and $x \in \bar{\Omega}$ for some positive ϵ.

2. EXISTENCE WITH ONLY CONTINUITY

Consider the eigenvalue problem

$$(2.1)\qquad \begin{cases} -\Delta u(x) = \lambda\, f(x,u(x)) & \text{for } x \in \Omega, \\ \quad u(x) = \varphi(x) & \text{for } x \in \partial\Omega, \end{cases}$$

with Ω bounded and $\partial\Omega$ regular, see [7]. In this section we assume $f \in C(\bar{\Omega}\times\mathbb{R})$ is bounded, $\varphi \in C(\partial\Omega)$. We will show that there exist solutions which are close to zeros of f for λ large. In order to do so we need the following conditions.

C.1: There are $\alpha,\ \gamma \in C^2(\bar{\Omega})$ with

$$(2.2)\qquad \alpha(x) < \gamma(x) \qquad \text{for } x \in \bar{\Omega},$$

$$(2.3)\qquad f(x,\alpha(x)) > 0 > f(x,\gamma(x)) \qquad \text{for } x \in \bar{\Omega}.$$

C.2: There are $\beta_i \in C^2(\bar{\Omega})$, with

$$(2.4)\qquad \alpha(x) < \beta_i(x) < \gamma(x) \qquad \text{for } x \in \bar{\Omega},$$

and open $\Omega_i \subset \Omega$, $i = 1,\dots, k + 1$, such that

$$(2.5)\qquad f(x,\beta_i(x)) > 0 \qquad \text{for } x \in \bar{\Omega}_i \quad i = 1,\dots,k,$$

$$(2.6)\qquad f(x,\beta_i(x)) < 0 \qquad \text{for } x \in \bar{\Omega}_i \quad i = k{+}1,\dots,k{+}l,$$

$$(2.7)\qquad \int_{\omega}^{\beta_i(x)} f(x,s)\, ds > 0 \qquad \text{for all } \omega \in [\alpha(x),\beta_i(x)),\ x \in \bar{\Omega}_i,\ i = 1,\dots,k,$$

$$(2.8)\qquad \int_{\omega}^{\beta_i(x)} f(x,s)\, ds > 0 \qquad \text{for all } \omega \in (\beta_i(x),\gamma(x)],\ x \in \bar{\Omega}_i,\ i = k{+}1,\dots,k{+}l.$$

Remark 1. Let $\rho \in C(\bar{\Omega})$ be such that $f(x,\rho(x)) = 0$ for $x \in \bar{\Omega}$.

Call ρ a falling isolated zero if there is $\epsilon_0 > 0$ with

$$(2.9) \qquad (\omega-\rho(x))\ f(x,\omega) < 0 \qquad \text{for } 0 < |\omega-\rho(x)| < \epsilon_0,\ x \in \bar{\Omega}.$$

If ρ is a falling isolated zero of f then there exist $\beta_1, \beta_2 \in C^2(\bar{\Omega})$, with $\rho-\epsilon_0 < \beta_1 < \rho < \beta_2 < \rho+\epsilon_0$ in $\bar{\Omega}$, which satisfy respectively (2.5) and (2.6) with $\Omega_1 = \Omega_2 = \Omega$.

Remark 2: Notice that it is no loss of generality to assume $\alpha, \beta_i, \gamma \in C^2(\bar{\Omega})$ instead of $C(\bar{\Omega})$, since f is continuous.

Theorem 2.1: Let $f \in C(\bar{\Omega}\times\mathbb{R})$, $\varphi \in C(\partial\Omega)$. Suppose C.1 and C.2 are satisfied. If

$$(2.10) \qquad \alpha(x) < \varphi(x) < \gamma(x) \qquad \text{for } x \in \partial\Omega,$$

then there are $\lambda_0 > 0$ and $c > 0$ such that the following holds. For all $\lambda > \lambda_0$ there exists a solution u_λ of (2.1) which satisfies:

$$(2.11) \qquad \alpha(x) \le u_\lambda(x) \le \gamma(x) \qquad \text{for all } x \in \bar{\Omega},$$

$$(2.12) \qquad \beta_i(x) \le u_\lambda(x) \qquad \text{for all } x \in \bar{\Omega}_i \cap \Omega(c\lambda^{-1/2}),\ i = 1,\dots,k,$$

$$(2.13) \qquad u_\lambda(x) \le \beta_i(x) \qquad \text{for all } x \in \bar{\Omega}_i \cap \Omega(c\lambda^{-1/2}),\ i = k+1,\dots,k+l.$$

$(\Omega(\delta) = \{ x \in \Omega;\ d(x,\partial\Omega) > \delta \})$

In this section we will only prove two corollaries of Theorem 2.1. The theorem itself will be proven in the next section.

Corollary 2.2: Let $f \in C(\bar{\Omega}\times\mathbb{R})$ and suppose C.1 is satisfied.

Let $\varphi \in C(\partial\Omega)$ satisfy (2.10). If $\rho \in C(\bar{\Omega})$ is a falling isolated zero of f, see Remark 1, which is such that

$$\int_{\omega}^{\rho(x)} f(x,s)\,ds > 0 \quad \text{for all } \omega \in [\alpha(x),\rho(x)) \cup (\rho(x),\gamma(x)],\quad x \in \partial\Omega, \tag{2.14}$$

then for all $\epsilon > 0$ the following holds.

There are $\lambda_0 > 0$ and $c > 0$ such that for all $\lambda > \lambda_0$ there exists a solution u_λ of (2.1), which satisfies (2.11) and

$$|u_\lambda(x) - \rho(x)| < \epsilon \qquad \text{for all } x \in \Omega(c\lambda^{-1/2}). \tag{2.15}$$

This corollary is related to existence results in [5]. It is also possible to state a corollary which is related to existence results in [6] for solutions with interior transition layers. In the one-dimensional case there are related results in [2].

Corollary 2.3: Let $f \in C(\bar{\Omega}\times\mathbb{R})$ and suppose C.1 is satisfied. Let

$\varphi \in C(\partial\Omega)$ satisfy (2.10). Let $\Gamma \subset \Omega$ be a closed curve such that $\bar{\Omega} = \bar{\Omega}_1 \cup \bar{\Omega}_2$, $\partial\Omega_1 = \partial\Omega \cup \Gamma$, $\partial\Omega_2 = \Gamma$ with Ω_1 and Ω_2 open. Finally let $\rho_1, \rho_2 \in C(\bar{\Omega})$ be two falling isolated zeros of f (see (2.9)) with $\rho_1 < \rho_2$, which satisfy

$$\int_{\omega}^{\rho_1(x)} f(x,s)\,ds > 0 \quad \text{for all } \omega \in [\alpha(x),\rho_1(x)) \cup (\rho_1(x),\gamma(x)],\ x \in \partial\Omega, \tag{2.16}$$

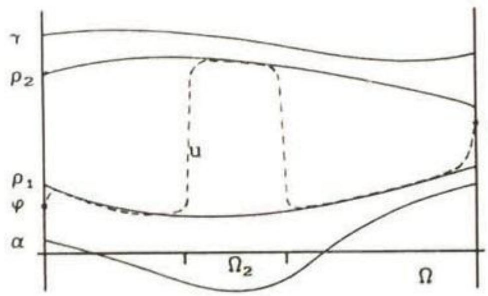
and

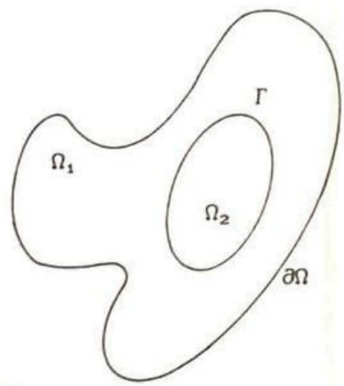
$$\int_{\omega}^{\rho_1(x)} f(x,s)\,ds > 0 \quad \text{for all } \omega \in (\rho_1(x),\rho_2(x)],\ x \in \Omega_1, \tag{2.17}$$

$$(2.18) \qquad \int_{\omega}^{\rho_2(x)} f(x,s)\, ds > 0 \quad \text{for all } \omega \in [\rho_1(x), \rho_2(x)),\ x \in \Omega_2.$$

Then for all $\epsilon > 0$ there is $\lambda_0 > 0$ such that for all $\lambda > \lambda_0$ there exists a solution u_λ of (2.1), which satisfies (2.11) and

$$(2.19) \qquad |u_\lambda(x) - \rho_i(x)| < \epsilon \qquad \text{for all } x \in \Omega_i(\epsilon) \qquad i = 1,2.$$





Remark: From (2.17) and (2.18) if follows that

$$(2.20) \qquad \int_{\rho_1(x)}^{\rho_2(x)} f(x,s)\, ds = 0 \qquad \text{for } x \in \Gamma.$$

Proof of Corollary 2.2:

Since f, ρ, α and γ are continuous, (2.14) also holds for all $x \in \bar{\Omega}\backslash\Omega(\delta)$ and for some $\delta > 0$ small enough. Take β_1, β_2 as in Remark 1 with $\epsilon = \epsilon_0$ and take $\eta \in C(\bar{\Omega}\times\mathbb{R})$ such that

$$(2.21) \qquad \begin{cases} \eta(x,\omega) \geq 0 & \text{for } \omega \leq \rho(x), \\ \eta(x,\omega) \leq 0 & \text{for } \omega \geq \rho(x), \\ \eta(x,\omega) = 0 & \text{if } \beta_1(x) \leq \omega \leq \beta_2(x) \\ & \quad \text{or } x \in \Omega\backslash\Omega(\tfrac{1}{2}\delta), \\ \eta(x,\omega) = 1 & \text{for } \omega \leq \rho(x)-\epsilon\ ,\ x \in \Omega(\delta), \\ \eta(x,\omega) = -1 & \text{for } \rho(x)+\epsilon \leq \omega\ ,\ x \in \Omega(\delta). \end{cases}$$

For c large enough $f_c \in C(\bar{\Omega}\times\mathbb{R})$, defined by

$f_c(x,\omega) = f(x,\omega) + c\eta(x,\omega)$, is such that C.1, C.2 are satisfied

Applying Theorem 2.1 shows that for $\lambda > \lambda_0$ there is a solution u_λ of (2.1), with f replaced by f_c, and

$$\beta_1(x) \leq u_\lambda(x) \leq \beta_2(x) \qquad \text{for } x \in \Omega(c\lambda^{-1/2}).$$

Hence $\eta(x,u_\lambda(x)) = 0$ for $\lambda > 2c^2\delta^{-2}$, which shows u_λ is a solution of (2.1), with the orginal f, for $\lambda > \max(2c^2\delta^{-2}, \lambda_0)$.

Since $\beta_1 > \rho-\epsilon$ and $\beta_2 < \rho+\epsilon$, (2.15) is also satisfied. □

Proof of Corollary 2.3:

Assume without loss of generality that (2.16) holds for $x \in \bar{\Omega}\backslash\Omega(\delta)$.

Let $\beta_1 < \rho_1 < \beta_3$ respectively $\beta_2 < \rho_2 < \beta_4$ as in Remark 1, with $\epsilon_0 = \epsilon$. Assume $2\epsilon < \delta$. Similar to the previous proof we take $\eta \in C(\bar{\Omega}\times\mathbb{R})$ such that

(2.22)
$$\begin{cases} \eta(x,\omega) \geq 0 & \text{if } \omega \leq \rho_1(x) \text{ and } x \in \bar{\Omega}_1 \\ & \text{or } \omega \leq \rho_2(x) \text{ and } x \in \bar{\Omega}_2, \\ \eta(x,\omega) \leq 0 & \text{if } \omega \geq \rho_1(x) \text{ and } x \in \bar{\Omega}_1 \\ & \text{or } \omega \geq \rho_2(x) \text{ and } x \in \bar{\Omega}_2, \\ \eta(x,\omega) = 1 & \text{if } \omega \leq \rho_1(x)-\epsilon \text{ and } x \in \overline{\Omega(2\epsilon)} \\ & \text{or } \omega \leq \rho_2(x)-\epsilon \text{ and } x \in \overline{\Omega_2(2\epsilon)}, \\ \eta(x,\omega) = -1 & \text{if } \omega \geq \rho_1(x)+\epsilon \text{ and } x \in \overline{\Omega_1(2\epsilon)} \\ & \text{or } \omega \geq \rho_2(x)+\epsilon \text{ and } x \in \overline{\Omega(2\epsilon)}, \\ \eta(x,\omega) = 0 & \text{if } x \in \bar{\Omega}\backslash\Omega(\epsilon) \\ & \text{or } \beta_1(x) \leq \omega \leq \beta_3(x) \text{ and } x \in \bar{\Omega}_1 \\ & \text{or } \beta_1(x) \leq \omega \leq \beta_4(x) \text{ and } x \in \Gamma+\overline{B(0,\epsilon)} \\ & \text{or } \beta_2(x) \leq \omega \leq \beta_4(x) \text{ and } x \in \bar{\Omega}_2. \end{cases}$$

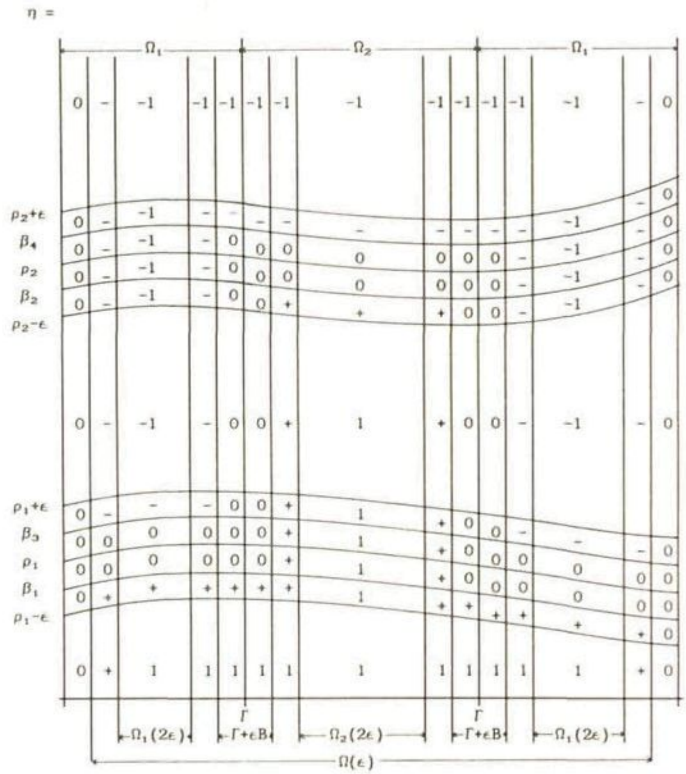


Illustration for the proof of Corollary 2.3.

Define $k = l = 2$ and

$$\left[\begin{array}{l} \Omega_1^* = \Omega, \\ \Omega_2^* = \Omega_2(\epsilon), \\ \Omega_3^* = \Omega_1(\epsilon), \\ \Omega_4^* = \Omega. \end{array}\right. \tag{2.23}$$

Then C.1 and C.2 with Ω_i^* are satisfied if f is replaced by $f_c = f + c\eta$ for c large. Application of Theorem 2.1 shows that there are c, λ_0 and u_λ for all $\lambda > \lambda_0$, with u_λ a solution of (2.1) where f is replaced by f_c.

The function u_λ satisfies

$$(2.24)\qquad \begin{cases} u_\lambda(x) \ge \beta_1(x) & \text{for } x \in \Omega(c\lambda^{-1/2}), \\ u_\lambda(x) \ge \beta_2(x) & \text{for } x \in \Omega_2(\epsilon) \cap \Omega(c\lambda^{-1/2}), \\ u_\lambda(x) \le \beta_3(x) & \text{for } x \in \Omega_1(\epsilon) \cap \Omega(c\lambda^{-1/2}), \\ u_\lambda(x) \le \beta_4(x) & \text{for } x \in \Omega(c\lambda^{-1/2}). \end{cases}$$

For $\lambda > \lambda_* = \max(\lambda_0, c^2\epsilon^{-2})$, one finds by (2.24) and (2.22) that, $\eta(x, u_\lambda(x)) = 0$. Hence u_λ is a solution of (2.1) with the original f. Moreover for $\lambda > \lambda_*$ it follows from (2.24) that (2.19) holds. □

3. PROOF OF EXISTENCE.

In this section we will construct for λ large appropriate subsolutions. We will use the definition of sub- and supersolution of [3].
The supremum of a family of these subsolutions will again be a subsolution. Similarly one constructs a supersolution. In [4] one shows that there exists a solution between these sub- and supersolutions. The sub- respectively supersolution will be constructed such that the estimates (2.12) and (2.13) are satisfied.

Lemma 3.1: Let $\alpha \in C^2(\bar{\Omega})$ be such that

$$(3.1)\qquad f(x, \alpha(x)) > 0 \qquad \text{for all } x \in \bar{\Omega},$$

and for some $x_0 \in \Omega$, $b > \alpha(x_0)$

$$(3.2)\qquad f(x_0, b) > 0.$$

If

$$(3.3)\qquad \int_\omega^b f(x_0, s)\,ds > 0 \qquad \text{for all } \omega \in [\alpha(x_0), b)$$

then for all $\epsilon > 0$ there are λ_1, $c \in \mathbb{R}^+$, $v \in C^2(\mathbb{R})$ with

$$(3.4)\qquad \begin{cases} v(0) > b , & \\ v'(r) < 0 & \text{for } r > 0, \\ v(r) < b + \epsilon - cr^2 & \text{for } r \geq 0. \end{cases}$$

such that for $\lambda > \lambda_1$, $w(\lambda)$ defined by

$$(3.5)\qquad w(\lambda,x) = \max \{ \alpha(x) , v(\lambda^{1/2}|x-y|) \}$$

is a subfunction of (2.1), for all $y \in B(x_0,\lambda_1^{-1/2})$.

Proof: Let $\sigma \in (0 , \frac{1}{3}f(x_0,\alpha(x_0)))$ be small enough such that

$$(3.6)\qquad \int_\omega^b (f(x_0,s) - \sigma)ds > 0 \qquad \text{for } \omega \in [\alpha(x_0),b)$$

and let $f^* \in C^1(\mathbb{R})$ be bounded and such that

$$(3.7)\qquad \begin{cases} f^*(s) < f(x_0,s) - \frac{1}{3}\sigma & \text{for } s \in [\alpha(x_0),\infty), \\ f^*(s) < 0 & \text{for } s \in [b+\epsilon,\infty), \\ f^*(s) > f(x_0,s) - \frac{2}{3}\sigma & \text{for } s \in [\alpha(x_0),b), \\ f^*(s) > \frac{1}{3}f(x_0,\alpha(x_0)) & \text{for } s \in (-\infty,\alpha(x_0)]. \end{cases}$$

Moreover let f^* have a unique zero b^* in $[b,b+\epsilon]$. Minimizing, see [8] or [3],

$$(3.8)\qquad I(\lambda,u) = \int_{B(0,1)} \left(\frac{1}{2}|\nabla u|^2 - \lambda \int_{\alpha(x_0)}^u f^*(s)ds \right) dx$$

in $\alpha(x_0) + W_0^{1,2}(B(0,1))$ for $\lambda = \lambda^*$ large enough, and extending by the solution of an ordinary differential equation gives a function $V \in C^2(\mathbb{R}^N)$, which satisfies

$$(3.9)\qquad \begin{cases} -\Delta V = \lambda^* f^*(V) , & \\ V(x) = V(|x|) & \text{for } x \in \mathbb{R}^N, \\ V'(r) < 0 & \text{for } r > 0, \\ b < V(0) < b^* , & \\ V(1) = \alpha(x_0) . & \end{cases}$$

Let $\delta_1 > 0$ be such that

(3.10) $$f^*(\omega) < f(x,\omega) \qquad \text{for } \omega \in [\alpha(x), b+\epsilon],\ |x-x_0| < \delta_1.$$

Define $v(r) = V((\lambda^*)^{-\frac{1}{2}} r)$ and take λ_1 large enough such that

(3.11) $$\lambda_1 > 2\,\delta_1^{-2},$$

(3.12) $$v(\tfrac{1}{2}\delta_1 \lambda_1^{\frac{1}{2}}) < \min_{\bar{\Omega}} \alpha(x)$$

and

(3.13) $$-\Delta\alpha(x) < \lambda_1 f(x,\alpha(x)) \qquad \text{for } x \in \Omega.$$

If $|x-x_0| \geq \delta_1$ and $y \in B(x_0, \lambda_1^{-\frac{1}{2}})$ then $|x-y| > \frac{1}{2}\delta_1$, and hence for $\lambda > \lambda_1$

(3.14) $$v(\lambda^{\frac{1}{2}}|x-y|) < v(\tfrac{1}{2}\lambda_1^{\frac{1}{2}}\delta_1) < \alpha(x).$$

If $|x-x_0| < \delta_1$ and $v(\lambda^{\frac{1}{2}}|x-y|) > \alpha(x)$ then (3.10) shows

(3.15) $$f^*(v(\lambda^{\frac{1}{2}}|x-y|)) < f(x, v(\lambda^{\frac{1}{2}}|x-y|)).$$

Using [9, lemma A], which shows the distributional inequality

(3.16) $$-\Delta|u| \leq -\,\text{sign}(u)\,\Delta u\ ,$$

one is able to prove for w_λ defined in (3.5)

(3.17) $$\int_\Omega -\Delta\varphi\, w_\lambda\, dx = \tfrac{1}{2}\int_\Omega -\Delta\varphi\,(\alpha + v_\lambda + |\alpha - v_\lambda|)\, dx \leq$$
$$\leq \tfrac{1}{2}\int_\Omega \varphi\,(-\Delta(\alpha+v_\lambda) - \text{sign}(\alpha-v_\lambda).\Delta(\alpha-v_\lambda))dx.$$

Define $h(x) = \begin{cases} 1 & \text{if } \alpha(x) > v_\lambda(x) \\ \frac{1}{2} & \text{if } \alpha(x) = v_\lambda(x) \\ 0 & \text{if } \alpha(x) < v_\lambda(x) \end{cases}$. Then for $y \in B(x_0, \lambda_1^{-\frac{1}{2}})$

(3.17') $$\int_\Omega -\Delta\varphi\, w_\lambda\, dx \leq \int_\Omega \varphi\,(h.-\Delta\alpha + (1-h).-\Delta v_\lambda)\, dx =$$
$$= \int_\Omega \varphi\, h.-\Delta\alpha\, dx + \int_{B(x_0,\delta)} \varphi\,(1-h).-\Delta v_\lambda\, dx =$$
$$\leq \int_\Omega \varphi\, h\, \lambda_1\, f(.,\alpha)\, dx + \int_{B(x_0,\delta)} \varphi\,(1-h)\,\lambda\, f^*(v_\lambda)\, dx \leq$$
$$\leq \int_\Omega \varphi\, h\, \lambda\, f(.,\alpha)\, dx + \int_{B(x_0,\delta)} \varphi\,(1-h)\,\lambda\, f(.,v_\lambda)\, dx =$$
$$= \int_\Omega \varphi\,\lambda\, f(.,w_\lambda)\, dx \quad \text{for all } \varphi \in \mathcal{D}^+(\Omega) = \{\varphi \in C_0^\infty(\Omega);\ \varphi \geq 0\}.$$

For a more detailed proof see [14, Lemma A.4].

The estimate (3.4) follows directly from the construction of f^* and (3.9). □

Proof of Theorem 2.1: Take $\epsilon > 0$, but small enough such that

$$\epsilon < \frac{1}{2} \min \{ \beta_j(x) - \beta_i(x) \; ; \; x \in \bar{\Omega}_i \cap \bar{\Omega}_j, \; i \in \{1,\dots,k\}, \; j \in \{k+1,\dots,k+l\} \}. \tag{3.18}$$

For every $i \in \{1,\dots,k\}$ and $z \in \bar{\Omega}_i$ let $v(i,z;x)$ and $\lambda_1(i,z)$ be defined in Lemma 3.1. Since $\bar{\Omega}_i$ is compact, there exists a finite covering

$$\bigcup_m \{ B(z_m, \lambda_1^{-1/2}(i,z_m)) \; ; \; m = 1,\dots,m_i \} \quad \text{of } \bar{\Omega}_i \text{ with } z_m \in \Omega_i. \tag{3.19}$$

Define

$$\lambda_1 = \max_{i,m} \{ \lambda_1(i,z_m) \; ; \; m = 1,\dots,m_i \}. \tag{3.20}$$

Finally, because of (3.4) there is $c > 0$ such that

$$\max_{i,m} v(i,z_m;r) < \min_{\bar{\Omega}} \alpha(x) \qquad \text{for all } r > c. \tag{3.21}$$

For $\lambda > \lambda_1$ set $w(\lambda,i,y;x) = \max(\alpha(x) \; , \; v(i,z_m;|x-y|))$. Then $W(\lambda)$ defined by

$$W(\lambda,x) = \sup\{ w(\lambda,i,y;x) \;\; ; \;\; y \in B(z_m,(\lambda_1(i,z_m))^{-1/2}) \cap \mathbb{Q}^N \cap \Omega(c\lambda^{-1/2}), \; m = 1,\dots,m_i, \; i = 1,\dots,k \} \tag{3.22}$$

is a subsolution of (2.1).

Because of (3.21)

$$w(\lambda,i,y;x) = \alpha(x) \quad \text{for } x \in \partial\Omega, \tag{3.23}$$

is satisfied, which shows the condition at the boundary. Using Kato's inequality, see [14, Lemma A.4], one shows that the maximum of a finite number of these subfunctions w is again a subfunction. Since

$$\alpha < W(\lambda) < \gamma \text{ in } \Omega, \tag{3.24}$$

one uses the dominated convergence theorem in order to show that $W(\lambda)$ satisfies, as the supremum of countable many subsolutions:

$$\int_\Omega \left(-\Delta\varphi\, W(\lambda) - \varphi\, f(.,W(\lambda)) \right) dx \le 0 \quad \text{for all } \varphi \in \mathfrak{D}^+(\Omega), \tag{3.25}$$

where $\mathfrak{D}^+(\Omega)$ consists of all nonnegative functions in $C_0^\infty(\Omega)$. Since $\{ w(\lambda,i,y) \in C(\bar{\Omega});$ with i and y as in (3.22) $\}$ is equicontinuous, $W(\lambda) \in C(\bar{\Omega})$ and $W(\lambda,x) = \alpha(x)$ for $x \in \partial\Omega$. Hence $W(\lambda)$ is a subsolution.

Similarly one constructs a supersolution $W^*(\lambda)$ for $\lambda > \lambda_1^*$. From the construction it follows that $W(\lambda) < W^*(\lambda)$. Moreover $W(\lambda)$ and $W^*(\lambda)$ satisfy (2.11) (2.12) and (2.13).

By [4] there is for $\lambda > \lambda_0 = \max(\lambda_1,\lambda_1^*)$ a solution u_λ of (2.1) with $W(\lambda) \le u_\lambda \le W^*(\lambda)$ in Ω. This completes the proof of Theorem 2.1. □

4. STABILITY

In order to tell something about stability one will need more smoothness. In this chapter we will assume that the bounded domain Ω has a C^3 boundary. Also assume $f \in C^{1,\gamma}(\bar{\Omega}\times\mathbb{R})$, for some $\gamma > 0$, and $\varphi \in C^2(\partial\Omega)$.

Suppose u is a solution of (2.1) and consider the linear eigenvalue problem:

$$\begin{cases} -\lambda^{-1}\Delta v - f_u(x,u)v = \mu v & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.1}$$

Definition 4.1: We call a solution u of (2.1) strongly stable if the first eigenvalue μ_0 of (4.1) satisfies $\mu_0 > 0$.

The notion of ordinary stability is related with the parabolic problem, see e.g. [10]:

$$(4.2) \quad \begin{cases} U_t - \Delta U = \lambda f(x,U) & \text{in } \Omega\times\mathbb{R}^+, \\ U = \varphi & \text{on } \partial\Omega\times\mathbb{R}^+. \end{cases}$$

Definition 4.2: We call a solution u of (2.1) stable if for every $\epsilon > 0$ there is $\delta > 0$ such that, for every $U_0 \in L^\infty(\Omega)$ with $\| U_0 - u \|_\infty < \delta$, the solution U of (4.2) with $\lim_{t\downarrow 0} \| U(t) - U_0 \|_{L_1(\Omega)} = 0$ satisfies $\| U(t) - u \|_\infty < \epsilon$ for all $t \geq 0$.

Remark : Strongly stable $\Rightarrow$ Stable $\Rightarrow$ $\mu_0 \geq 0$.

In general these implications cannot be reversed.

Suppose u is a strongly stable solution of (2.1) for $\lambda = \lambda_1$. Then the implicit function theorem and Schauder estimates yield the existence of a curve of solutions. That is $\Phi \in C^1((\lambda_1-\epsilon,\lambda_1+\epsilon) \; ; \; C^2(\bar{\Omega}))$ exists with $\Phi(\lambda_1) = u$ and $\Phi(\lambda)$ is a solution of (4.1) if $| \lambda - \lambda_1 | < \epsilon$, where ϵ is some positive number. However strong stability is in general harder to prove then just stability.

Combining the results of Sattinger and Matano, [12] and [10], one finds the existence of a stable solution u, if there are a subsolution u_1 and a supersolution u_2 of (2.1), with $u_1 < u_2$ in $\bar{\Omega}$ and $u_1 < \varphi < u_2$ on $\partial\Omega$. The function u satisfies $u_1 < u < u_2$ in $\bar{\Omega}$.

In the proof of Theorem 2.1, see (3.24), we constructed a sub and supersolution $W(\lambda)$, respectively $W^*(\lambda)$. Hence under the additional smoothness conditions one can assert in Theorem 2.1 that there exists a stable solution $u(\lambda)$. Under such general conditions as C.1 and C.2 it will not be possible to prove strong stability. For the case that there is only a boundary layer one can prove the following.

Theorem 4.3: Let Ω be bounded and $\partial\Omega \in C^3$. Suppose $f \in C^{1,\gamma}(\bar{\Omega}\times\mathbb{R})$ for some $\gamma > 0$, and let $\rho \in C^2(\bar{\Omega})$ be a falling isolated zero of $f(x,\rho)$:

$$(4.3)\qquad (\rho(x) - u)\, f(x,u) > 0 \quad \text{for } u \in [\rho(x)-\delta, \rho(x)+\delta]\setminus\{\rho(x)\}$$

and

$$(4.4)\qquad f_u(x,u) \le 0 \quad \text{for } u \in [\rho(x)-\delta, \rho(x)+\delta],$$

with $x \in \bar{\Omega}$. Let $\varphi \in C^2(\bar{\Omega})$.

If

$$(4.5)\qquad \int_u^{\rho(x)} f(x,s)\, ds > 0 \quad \text{for } u \in [\varphi(x), \rho(x)) \text{ and } x \in \partial\Omega,$$

then for all $\epsilon \in (0,\delta)$ there are $c > 0$, $\lambda_1 > 0$ such that for all $\lambda > \lambda_1$ a strongly stable solution $u(\lambda)$ exists with

$$(4.6)\qquad \min\{\, \varphi(x^*) + c\lambda^{1/2}|\, x-x^*| \,,\ \rho(x)-\epsilon \ :\ x^*\in \partial\Omega \,\} < u(\lambda;x) \text{ and}$$
$$u(\lambda;x) < \rho(x)+\epsilon \qquad \text{in } \Omega.$$

Moreover, $u(\lambda)$ is the only solution of (2.1) which satisfies (4.5).

Proof: In order to simplify arguments we will assume that φ is harmonic. Also we will assume that

$$(4.7)\qquad \varphi(x) < \rho(x) - \epsilon \qquad \text{for } x \in \bar{\Omega}.$$

If U is a solution of

$$(4.8)\qquad \begin{cases} -\Delta U = \lambda f(x, U+\varphi) & \text{in } \Omega, \\ U = 0 & \text{on } \partial\Omega, \end{cases}$$

then $u = U + \varphi$ is a solution of (2.1). Hence we may assume without loss of generality that $\varphi = 0$.

The proof is basically the same as in [3].

i) *Construction of sub and supersolutions.*

First change f for $u < 0$ in a neighbourhood N of $\partial\Omega$ such that

$$(4.9)\qquad f(x,u) > 0 \qquad\qquad \text{for } u < -1 \text{ and } x \in N \cap \Omega$$

and

$$(4.10)\qquad \int_u^{\rho(x)} f(x,s)\, ds > 0 \qquad\qquad \text{for } u < \rho(x) \text{ and } x \in N \cap \Omega.$$

Since we are only interested in solutions for λ large which satisfy (4.6), we may change f outside of this region and assume without loss of generality that (4.9) and (4.10) hold for all $x \in \bar{\Omega}$. Let $\{ B(y_i,\sigma) \ ; \ i = i,\dots,M \}$ be a covering of $\bar{\Omega}$. Like in the proof of Theorem 2.1 assume that σ is small enough such that $w(\lambda,i,y;x) = \max\{ -1 \ , \ v_i(\lambda^{1/2}|x-y|) \}$, with v_i defined by Lemma 3.1 (take $x_0 = y_i$), is a subfunction for all $y \in B(y_i,2\sigma)$. Define for $\theta \in [0,1]$ the subfunctions (see (3.22)):

$$(4.11)\qquad \tilde{w}(\lambda,i,\theta;x) = \sup\{ w(\lambda,i,y;x); \ y \in B(y_i,\sigma) \cap \Omega \text{ with } w(\lambda,i,y) < -\theta \text{ on } \partial\Omega \}.$$

Since $\partial\Omega$ is C^3 one finds that

$$(4.12)\qquad \tilde{w}(\lambda,i,\theta) > -\theta \qquad\qquad \text{on } B(y_i,\sigma) \cap \Omega.$$

Hence

$$(4.13)\qquad W(\lambda,\theta;x) = \max\{ \tilde{w}(\lambda,i,\theta;x) \ ; \ i = 1,\dots,M \},$$

with $\theta = 0$, is a positive subsolution and even satisfies

$$(4.14)\qquad \min\{ c\lambda^{1/2}|\, x-x^*| \ , \ \rho(x)-\epsilon \ ; \ x^* \in \partial\Omega \} < W(\lambda,0;x) < \rho(x) \quad \text{in } \Omega$$

for some $c > 0$ and all λ large enough.

Also for λ large enough, $W^*(\lambda;x) = \rho(x) + \epsilon$ is a supersolution. By sweeping with subsolutions as in [3] one can prove that every solution $u(\lambda)$ (for the changed f) with λ large enough, which satisfies

(4.15) $$\min\{ c\lambda^{1/2}| x-x^*| , \rho(x)-\epsilon ; x^*\in \partial\Omega \} < W(\lambda,0;x) < \rho(x)+\epsilon$$

for $x \in \bar{\Omega}$, or

(4.16) $$W(\lambda,-1) < u(\lambda) < \rho + \epsilon \qquad \text{in } \Omega,$$

also satisfies

(4.16) $$W(\lambda,0) < u(\lambda) < W^*(\lambda) \qquad \text{in } \Omega.$$

ii) *Using degree arguments.*

Choose ω such that $\lambda f_u(x,u) + \omega > 0$ for $x \in \bar{\Omega}$ and $u \in [-1 , \max_{\bar{\Omega}} \rho(x) + \epsilon]$. Define the continuous mappings F, K : $C(\bar{\Omega}) \to C(\bar{\Omega})$ respectively by

(4.18) $$F(u)(x) = f(x,u(x)) \qquad \text{for } x \in \bar{\Omega},$$

(4.19) $$K(u) = (-\Delta + \omega)_0^{-1} (F(u) + \omega u),$$

where $(-\Delta + \omega)_0^{-1}$ is the inverse of $-\Delta+\omega$ with homogeneous Dirichlet boundary conditions. Since K is order preserving on $[W(\lambda,-1),W^*(\lambda)] \subset C(\bar{\Omega})$,

(4.20) $$W(\lambda,-1) < K(W(\lambda,-1)) < K(W^*(\lambda)) < W^*(\lambda) \quad \text{in } \bar{\Omega},$$

and since K is compact, the Leray-Schauder degree of I-K on $(W(\lambda,-1),W^*(\lambda))$ is well defined. Since $(W(\lambda,-1),W^*(\lambda))$ is convex in $C(\bar{\Omega})$, one finds by a homotopy argument that

(4.21) $$\text{degree}(I-K , (W(\lambda,-1),W^*(\lambda)) , 0) = +1 .$$

If u is a strongly stable solution of (2.1) and

(4.22) $$u(\lambda) \in (W(\lambda,-1),W^*(\lambda)) \subset C(\bar{\Omega}),$$

then the local degree of I-K at $u(\lambda)$ is also +1. If every solution $u(\lambda)$ in (4.22) is strongly stable then the additivity of the degree shows that there is exactly one solution of (2.1) in $[W(\lambda,-1),W^*(\lambda)]$.

iii) *Showing strong stability.*

We have to show that, when λ is large, for every solution $u(\lambda)$ of (2.1), with $u(\lambda) \in [W(\lambda,0),W^*(\lambda)]$ the first eigenvalue $\mu_0(\lambda,u(\lambda))$ of the linearized problem (4.1) is positive. Suppose this is not true. Then there exists a sequence $\{ \lambda_n \}$, with $\lim_{n\to\infty} \lambda_n = \infty$, with solutions $u(\lambda_n) \in [W(\lambda_n,0),W^*(\lambda_n)]$ and with $\mu_n = \mu_0(\lambda_n,u(\lambda_n)) \leq 0$. Let v_n denote the associated eigenfunctions, which are normalized by $\max v_n = 1$. Take $y_n \in \Omega$ such that $v_n(y_n) = 1$. By (4.15) there is $c > 0$ such that

(4.24) $$| u(\lambda,x) - \rho(x) | < \epsilon \qquad \text{for } x \in \Omega(c\lambda^{-1/2}),$$

where $\Omega(\tau) = \{ x \in \Omega \ ; \ d(x,\partial\Omega) > \tau \}$.

Since

(4.25) $$-\Delta v_n(x) = \lambda_n \left(f_u(x,u_n(\lambda_n;x)) + \mu_n \right) v_n(x) \leq 0 \text{ for } x \in \Omega(c\lambda^{-1/2}),$$

by (4.4) one finds that $y_n \in \Omega\backslash\Omega(c\lambda^{-1/2})$. Similar to [1] or [3, Lemma 4.2] one constructs subsequences such that $\mu_n \to \bar{\mu} \leq 0$ and $y_n \to \bar{x} \in \partial\Omega$, uses a local change of coordinates, rescales and will find the limit problems

(4.26) $$\begin{cases} -\Delta U = f(\bar{x},U) & \text{in } \mathbb{R}^N_+ = \mathbb{R}_+ \times \mathbb{R}^{N+1}, \\ U = 0 & \text{on } \partial\mathbb{R}^N_+ = \mathbb{R}_+ \times \{0\}. \end{cases}$$

(4.27) $$\begin{cases} -\Delta V - f_u(\bar{x},U)\, V = \bar{\mu}\, V & \text{in } \mathbb{R}^N_+ , \\ V = 0 & \text{on } \partial\mathbb{R}^N_+ . \end{cases}$$

with $0 < V \leq 1$ in $\mathbb{R}^N_+$. Also U satisfies

(4.28) $$\min(cx_1 \ , \ \rho(\bar{x})-\epsilon) \leq U(x) \leq \rho(\bar{x})+\epsilon \quad \text{for } x = (x_1,\dots,x_N) \in \mathbb{R}^N_+.$$

Since

(4.29) $$(\rho(\bar{x}) - u) \, f(\bar{x},u) > 0 \quad \text{for } u \in [\rho(\bar{x})-\epsilon,\rho(\bar{x})+\epsilon]\backslash\{\rho(\bar{x})\}.$$

one uses sweeping arguments from below and from above, see [13, Prop. 3.2] to show that

$$(4.30)\qquad \lim_{x_1\to\infty} U(x_1,x') = \rho(\bar{x}) \qquad \text{for } x'\in \mathbb{R}^{N-1}.$$

Next, by the strong maximum principle one finds that $U < \rho(\bar{x})$ on $\mathbb{R}^N_+$. [3, Prop. 2.5] or [13, Prop. 3.2] shows that

$$(4.31)\qquad U(x_1,x') = U(x_1) \qquad \text{for } (x_1,x') \in \mathbb{R}_+\times \mathbb{R}^{N-1},$$

and U is a solution of

$$(4.32)\qquad \begin{cases} -U'' = f(\bar{x},U), \\ U(0) = 0\ , \\ U'(0) = \left(2 \int_0^{\rho(\bar{x})} f(\bar{x},s)\, ds \right)^{½}. \end{cases}$$

Define

$$(4.33)\qquad S(x_1) = \sup\{ V(x_1,x') \ ;\ x'\in \mathbb{R}^{N-1} \} \qquad \text{for } x_1 \in \mathbb{R}_+.$$

Then $0 < S \le 1$ in $\mathbb{R}_+$ and by [3, Lemma 2.6], $S \in C[0,\infty)$ and

$$(4.34)\qquad \int_{\mathbb{R}_+} \left(S\,(-\varphi'') - (\, f_u(\bar{x},U) + \bar{\mu}\,)\, S\, \varphi \right) dx_1 \le 0$$
$$\text{for all nonnegative } \varphi \in C_0^\infty(\mathbb{R}_+).$$

Since $f_u(\bar{x},U(x_1)) + \bar{\mu} \le 0$ for $x_1 > c$ one finds that S is convex on (c,∞). Since S is also bounded, S is constant on (c,∞). Finally, since S is a positive constant on (c,∞), (4.34) yields that $f_u(\bar{x},U(x_1)) + \bar{\mu} = 0$, and hence $f_u(\bar{x},U(x_1)) = 0$, for $x_1 >$ c. This last assertion is contradicted by the fact that neither $f(\bar{x},.)$ on $[\rho(\bar{x})-\epsilon,\rho(\bar{x})]$, nor U on (c,∞) are constant. □

REFERENCES

[1] S.B. Angenent, Uniqueness of the solution of a semilinear boundary value problem, Math. Ann. 272 (1985), 129-138.

[2] S.B. Angenent, Qualitative behaviour of solutions of elliptic and parabolic equations, thesis R.U.Leiden 1986.

[3] Ph. Clément, G. Sweers, Existence and multiplicity results for a semilinear elliptic eigenvalue problem, Annali della Scuola Normale Superiore di Pisa 14 (1987).

[3] Ph. Clément, G. Sweers, Getting a solution between sub- and supersolutions without monotone iteration, to appear in Rendiconti dell'Istituto di Matematica dell'Università di Trieste (1988).

[5] P.C. Fife, Semilinear elliptic boundary value problems with small parameters, Arch. Ration. Mech. Anal. 52 (1973), 205-232.

[6] P.C. Fife, W.M Greenlee, Interior transition layers for elliptic boundary value problems with a small parameter, Russian Math. Surveys 29:4 (1974), 103-131; from Успехи матем. наук 29:4 (1974), 103-130.

[7] D. Gilbarg, N.S. Trudinger, Elliptic partial differential equations of second order, Springer-Verlag, Berlin/Heidelberg/New York, 1977.

[8] P. Hess, On multiple solutions of nonlinear elliptic eigenvalue problems, Commun. Partial Differ. Equations 6 (1981), 951-961.

[9] T. Kato, Schrödinger operators with singular potentials, Israel J. Math. 13 (1972), 135-148.

[10] H. Matano, Asymptotic behaviour and stability of solutions of semilinear diffusion equations, Publ. RIMS, Kyoto Univ. 15 (1979), 401-454.

[11] M. Protter, H. Weinberger, Maximum principles in differential equations, Prentice Hall, Englewood Cliffs N.J., 1967.

[12] D.H. Sattinger, Monotone Methods in nonlinear elliptic and parabolic boundary value problems, Indiana Univ. Math. J. 21 (1972), 979-1000.

[13] G. Sweers, On the maximum of solutions for a semilinear elliptic problem, to appear in Proceedings of the Royal Society of Edinburgh.

[14] G. Sweers, Semilinear elliptic problems on domains with corners, report 87-83 of Department of Math. and Inf., Univ. of Techn. Delft, (1987).

B.

Chapter 7

Remarks on eigenvalues and eigenfunctions of a special elliptic system.

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Remarks on eigenvalues and eigenfunctions of a special elliptic system

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Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and $\mathbf{v} = (v^1, v^2, \ldots, v^n)$ a vector function defined on $\bar{\Omega}$. The operator defined by $\mathbf{v} \to \Delta \mathbf{v} + \alpha \operatorname{grad} \operatorname{div} \mathbf{v}$ is called *Lamé's* operator. For $n = 3$ it shows up in the theory of elasticity and then $\alpha = 1 + \frac{\lambda}{\mu} = \frac{1}{1-2\nu}$, *where* λ *and* μ *are the (positive) Lamé constants and* $\nu \in (0, \frac{1}{2})$ is the Poisson ratio, see e.g. [16, § 33]. Therefore the assumption $\alpha > 0$ is justified. For many metals such as for instance iron a realistic range of α is the interval (2, 3). The eigenvectors describe the deformation of isotropic vibrating elastic bodies with fixed boundaries. Bibliographical remarks are made at the end of this note.

Lemma 1. The first eigenvalue Λ_1 of the Lamé operator is characterized by

$$\Lambda_1(\Omega) := \min_{0 \neq \mathbf{v} \in [H_0^1(\Omega)]^n} R(\mathbf{v}), \quad \text{where}$$

$$R(\mathbf{v}) := \frac{\int_\Omega \left\{ \sum_{i=1}^n |\nabla v^i|^2 + \alpha (\operatorname{div} \mathbf{v})^2 \right\} dx}{\int_\Omega \left\{ \sum_{i=1}^n (v^i)^2 \right\} dx}.$$

For the proof we observe that if $\mathbf{u}$ is a solution to the variational problem, then the first variation of $R(\mathbf{u})$ has to vanish, i.e.

$$\int_\Omega \left\{ \sum_{i=1}^n \nabla u^i \nabla \phi^i + \alpha \operatorname{div} \mathbf{u} \operatorname{div} \phi - \Lambda_1 \mathbf{u} \phi \right\} dx = 0.$$

for all testfunctions $\phi \in [C_0^\infty(\Omega)]^n$. But this is just a weak formulation of the partial differential system for $\mathbf{u}$. The second variation of $R(\mathbf{u})$ should be nonnegative, and this is in fact the case since

$$\int_\Omega \left\{ \sum_{i=1}^{n} |\nabla \phi^i|^2 + \alpha (\operatorname{div} \phi)^2 - \Lambda_\iota |\phi|^2 \right\} \mathrm{d}x \geqq 0 .$$

A simple consequence of Lemma 1 is the following estimate. □

Corollary 1. $\Lambda_1(\Omega) > \lambda_1(\Omega)$.
In fact, certainly

$$\Lambda_1(\Omega) \geqq \min_{0 \neq \mathbf{v} \in [H_0^1(\Omega)]^n} \frac{\int_\Omega \left\{ \sum_{i=1}^{n} |\nabla v^i|^2 \right\} \mathrm{d}x}{\int_\Omega \left\{ \sum_{i=1}^{n} (v^i)^2 \right\} \mathrm{d}x} \geqq \lambda_1(\Omega)$$

holds. This proves the weak inequality $\Lambda_1(\Omega) \geqq \lambda_1(\Omega)$. To prove the strict inequality one has to observe that equality holds only if the components of $\mathbf{u}$ are multiples of the first eigenfunction u_1 of the Laplace operator, and if $\operatorname{div} \mathbf{u} = 0$ in Ω. But, as W. Velte has kindly pointed out [17], these facts are contradictory, since

$$\int_\Omega \operatorname{div} \mathbf{u} p \,\mathrm{d}x = - \int_\Omega \mathbf{u} \operatorname{grad} p \,\mathrm{d}x \quad \text{for any } p \in C^1(\Omega).$$

In fact, for $p = x_j$ we obtain $\int_\Omega u^j \mathrm{d}x = 0$, so that u^j cannot be a multiple of u_1. □

Remark 1. The well-known Krahn Faber inequality, see e.g. [13, 7], states that

$$\lambda_1(\Omega) \geqq \lambda_1(\Omega^*) = \left[\frac{\omega_n}{n\,|\Omega|}\right]^{2/n} j^2_{(n-2)/2},$$

where Ω^* denotes the n-dimensional ball of the same volume as $\Omega \subset \mathbb{R}^n$ with center in the origin, where n is the number of dimensions, ω_n the surface area of the unit sphere in $\mathbb{R}^n$, $|\Omega|$ the n-dimensional volume of Ω and $j_{(n-2)/2}$ is the first zero of the Bessel Function $J_{(n-2)/2}(x)$. In particular for $n = 3$ we get

$$\Lambda_1(\Omega) > \left(\frac{4\pi}{3}\right)^{2/3} |\Omega|^{-2/3} \pi^2 .$$

This is considerably better than the estimate of Sprössig [15]:

$$\Lambda_1(\Omega) \geqq \left(\frac{4\pi}{3}\right)^{2/3} |\Omega|^{-2/3} \frac{1}{2},$$

or of Levine and Protter [11]:

$$\Lambda_1(\Omega) \geqq \left(\frac{4\pi}{3}\right)^{2/3} |\Omega|^{-2/3} \frac{3}{5}\left(\frac{3\pi}{2}\right)^{2/3},$$

because $\pi^2 \sim 9.86$ and $3/5 \cdot (3\pi/2)^{2/3} \sim 1.44$.

Now that one has an estimate from below, let us look for an estimate from above for $\Lambda_1(\Omega)$. In view of the Lemma this is a relatively simple task. We just have to choose a suitable testfunction $\mathbf{v}$ in the Rayleigh quotient $R(\mathbf{v})$.

Corollary 2a. If $n \geqq 2$ then $\Lambda_1(\Omega) < \left(1 + \frac{\alpha}{n}\right) \lambda_1(\Omega)$.

Corollary 2b. If $n = 1$ then $\Lambda_1(\Omega) = (1 + \alpha)\,\lambda_1(\Omega)$.

Corollary 2c. If $\tilde{\Omega} \subset \Omega$ then $\Lambda_1(\Omega) \leqq \Lambda_1(\tilde{\Omega})$.

To prove Corollary 2a, we let u_1 be the first eigenfunction of the Laplace operator on Ω and suppose without loss of generality that

$$\int_\Omega \left|\frac{\partial u_1}{\partial x_1}\right|^2 \mathrm{d}x \leqq \int_\Omega \left|\frac{\partial u_1}{\partial x_j}\right|^2 \mathrm{d}x \qquad \text{for } j = 2, \ldots, n.$$

Then we choose $\mathbf{v}_1$ with components $v_1^1 = u_1$ and $v_1^2 = v_1^3 = \cdots = v_1^n \equiv 0$. Assuming that the $L^2(\Omega)$-norm of v_1^1 is one, we evaluate $R(\mathbf{v}_1)$ and obtain

$$R(\mathbf{v}_1) = \lambda_1(\Omega) + \alpha \int_\Omega \left|\frac{\partial u_1}{\partial x_1}\right|^2 \mathrm{d}x \leqq \left(1 + \frac{\alpha}{n}\right) \lambda_1(\Omega).$$

To prove the strict inequality we observe that $\Lambda_1(\Omega) = \left(1 + \frac{\alpha}{n}\right) \lambda_1(\Omega)$ if and only if

$$\int_\Omega \left|\frac{\partial u_1}{\partial x_i}\right|^2 \mathrm{d}x \leqq \int_\Omega \left|\frac{\partial u_1}{\partial x_j}\right|^2 \mathrm{d}x \qquad \text{for } i, j \in \{1, \ldots, n\},$$

and if each vector $\mathbf{v}_i$ given by the components $v_i^j = \delta_{ij} u_1$ minimizes the Rayleigh quotient. Here δ_{ij} is the Kronecker symbol. But then each vector $\mathbf{v}_i$ is an eigensolution of the Lamé operator with eigenvalue $\Lambda = \left(1 + \frac{\alpha}{n}\right) \lambda_1(\Omega)$. If we write out the systems for all $\mathbf{v}_i$ we obtain

$$\frac{\partial^2 u_1}{\partial x_i^2} + \frac{\lambda_1(\Omega)}{n} u_1 = 0 \quad \text{and} \quad \frac{\partial^2 u_1}{\partial x_i \partial x_j} = 0 \qquad \text{for all } i \neq j.$$

These are too many equations to have u_1 as a solution. In fact, if we integrate the second set of equations with respect to x_j, we obtain

$$\frac{\partial u_1}{\partial x_i} = f_i(x_i), \qquad u_1(x) = F_i(x_i) + G_i(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n).$$

Now the first set of equations implies after separation of variables, that each G_i has to be constant. Consequently u_1 must be constant, a contradiction. This completes the proof of Corollary 2a.

The proofs of Corollaries 2b and 2c are trivial. □

Remark 2. The estimates presented here are optimal for $\alpha \to 0$. For large values of α one can sometimes find a better upper bound to Λ_1 by choosing a testvector with vanishing divergence in the Rayleigh quotient. As an example, let Ω be the unit square in $\mathbb{R}^2$ and choose

$$u^1 = (\sin \pi x)^2 \sin 2\pi y \qquad u^2 = -\sin 2\pi x\,(\sin \pi y)^2.$$

Then $R(\mathbf{u}) = 5.33\,\pi^2$, see [17]. This upper bound for Λ_1 is better than the one in Corollary 2a for $\alpha > 3.78$ while our bound is better for smaller values of α. Another reasoning of this type and relations with another eigenvalue problem are exhibited in Remark 8.

For large values of α one can interpret the divergence term in the Rayleigh quotient as a penalty term which forces the first eigenfunction to have (almost) vanishing divergence. In fact there exists a constant c independent of α such that

$$\int_\Omega (\operatorname{div} \mathbf{u}_1)^2\, \mathrm{d}x \leqq \frac{c}{\alpha}.$$

Remark 3. Suppose that Ω is a ball. We call a vector function $\mathbf{v}(x)$ a *radial vector* or *rotationally invariant*, iff all components of $\mathbf{v}$ depend only on $r = |x|$. If one minimizes $R(\mathbf{v})$ over the subset of radial vectors, then the corresponding Euler equations turn out to be

$$\left(1 + \frac{\alpha}{n}\right)\left(u^j_{rr} + \frac{n-1}{r}\, u^j_r\right) + \Lambda u^j = 0\,.$$

In this case $\Lambda^r_j(\Omega)$, the minimum of $R(\mathbf{v})$ over the set of radial vectors in $[H^1_0(\Omega)]^n$, satisfies

$$\Lambda^r_1 = \left(1 + \frac{\alpha}{n}\right) \lambda_1(\Omega)\,.$$

Incidentally, this gives another proof of the weak inequality for the case that Ω is a ball. To prove the strict inequality we could also argue as follows. If $\Lambda_1(\Omega) = \Lambda^r_1(\Omega)$ then there exists a radial eigensolution to the Lamé operator. This will contradict Lemma 2. Notice that these observations show another connection between the Laplace and the Lamé operators.

Since the first eigenvalues $\lambda_1(\Omega)$ and $\Lambda_1(\Omega)$ of the Laplace and the Lamé operators are so closely related, one might wonder whether the eigenfunctions u_1 and $\mathbf{u}_1$ have something in common. If Ω is a ball, one of the features of u_1 is its radial symmetry. However, $\mathbf{u}_1$ is not radial. In fact, the Lamé operator has no radial eigensolutions whatsoever (in the sense defined in Remark 3).

Lemma 2. Let $n \geqq 2$, let Ω be a ball, and let $\mathbf{u}$ be any eigensolution associated with any eigenvalue Λ of the Lamé operator.

Then $\mathbf{u}$ is not rotationally invariant, i.e. $\mathbf{u}$ is not a function of r only.

We prove this Lemma by contradiction and suppose that $\mathbf{u} = \mathbf{u}(r)$, where $r^2 = \sum_{i=1}^{n} x_i^2$. Then the j-th component of the system can be rewritten as

$$u^j_{rr} + \frac{n-1+\alpha}{r} u^j_r + \frac{\alpha}{r^2} x_j \sum_{i=1}^{n} \left\{ x_i \left(u^i_{rr} - \frac{1}{r} u^i_r \right) \right\} + \Lambda u^j = 0 .$$

We introduce the notation

$$f^j(r) := -\frac{r^2}{\alpha} \left(u^j_{rr} + \frac{n-1+\alpha}{r} u^j_r + \Lambda u^j \right) \quad \text{and} \quad a^j(r) := u^j_{rr} - \frac{1}{r} u^j_r$$

and obtain

$$\sum_{i=1}^{n} x_i a^i(r) = \frac{f^j(r)}{x_j} \quad \text{for } j = 1, \ldots, n .$$

Consequently $x_i f^j(r) = x_j f^i(r)$ in Ω for any $i, j \in \{1, \ldots, n\}$, i.e. $f^j(r) = 0$ for $j = 1, \ldots, n$. But then $a^j(r) = 0$ for $j = 1, \ldots, n$, i.e. there are constants b_j and c_j such that $u^j(r) = b_j r^2 + c_j$. Since $f^j = 0$ we have $b_j = c_j = 0$. Therefore $\mathbf{u}$ must be zero, a contradiction. □

Another feature of the first eigenfunction u_1 of the Laplace operator is its uniqueness (modulo scaling). The first eigenfunction $\mathbf{u}_1$ of the Lamé operator is in general not unique, as can be seen from the following Lemma.

Lemma 3. Let $\Omega \in \mathbb{R}^2$ be the unit disk and let $\mathbf{u}_1 = (u(x, y), v(x, y))$ be an eigensolution associated with $\Lambda_1(\Omega)$. Suppose that $\alpha < 3.176$.

Then $\mathbf{u}_1$ is not unique modulo scaling.

Remark 4. The Lemma remains correct, if Ω is a two-dimensional domain which is invariant under reflections along the x- and y-axes, and under reflection across the diagonal $x = y$. In this case the precise upper bound for α is given by $1 + \alpha/2 < \lambda_2(\Omega)/\lambda_1(\Omega)$. Upper bounds for the ratio of the two eigenvalues can be found in [2]. The present upper bound for a general two-dimensional domain seems to be 2.586, the conjectured upper bound is the one which is obtained for the disk, namely 2.538 … .

With slightly more notational efforts one can prove Lemma 3 also for the ball in $\mathbb{R}^3$. In that case the bound for α is given by $1 + \alpha/3 < \lambda_2(\Omega)/\lambda_1(\Omega) \approx 2.04$ or $\alpha < 3.12$. This bound is in the physically realistic range, see our introduction.

Notice that even for $\alpha = 0$ there are two linearly independent eigensolutions, namely $(u_1(x, y), 0)$ and $(0, u_1(x, y))$, with $u_1(x, y)$ being the first eigenfunction of the Laplace operator.

For the **proof of Lemma 3** we assume that (u, v) is unique and use some invariance properties of the class of solutions to reach a contradiction. An easy calculation shows that the following statements i)–iii) are equivalent.

i) $(u(x, y), v(x, y))$ is an eigensolution.
ii) $(-u(-x, y), v(-x, y))$ is an eigensolution.
iii) $(v(y, x), u(y, x))$ is an eigensolution.

So there exist nonzero constants a and b such that $u(x, y) = -au(-x, y) = bv(y, x)$ and $v(x, y) = av(-x, y) = bu(y, x)$. A combination of these two equations shows that $b^2 = a^2 = 1$, so that we have to study several cases. The case $a = 1$ leads to $u(0, y) = 0$ and $v(x, 0) = 0$, while the case $a = -1$ leads to $v(0, y) = 0 = u(x, 0)$. Therefore, in both cases we may minimize the Rayleigh quotient $R(u, v)$ over the class of functions which vanish on a diameter and obtain the estimate $R(u, v) \geqq \lambda_2(\Omega)$. This contradicts Corollary 2a and the smallness assumption on α, and completes the proof of Lemma 3. □

Now that we know that the eigensolutions to the Lamé operator have features which distinguish them from the eigenfunctions of the Laplace operator, let us attempt to calculate some of the eigensolutions explicitly.

Let Ω be the ball with radius 1 in $\mathbb{R}^n$. We already know from Lemma 2, that there are no radial solutions. But maybe $\mathbf{u}_1 = \operatorname{grad} p(r)$ for some potential $p(r)$. This conjecture leads to the "Ansatz"

$$u^i(x) = x_i f(r)$$

and a straightforward calculation gives

$$f'' + \frac{n+1}{r} f' + \frac{\Lambda}{1+\alpha} f = 0 \quad \text{in } \Omega .$$

Notice that the last equation can be interpreted as

$$\Delta_{n+2} f + \frac{\Lambda}{1+\alpha} f = 0 \quad \text{in } \Omega_{n+2}$$

where Δ_{n+2} and Ω_{n+2} denote the Laplace operator and the ball of radius one in $\mathbb{R}^{n+2}$. Thus we have shown the following result:

Lemma 4.

If w_j denote the radial eigenfunctions and ϱ_j the associated eigenvalues of the Laplace operator on Ω_{n+2} under Dirichlet boundary conditions, then $\tilde{\varrho}_j := (1 + \alpha)\, \varrho_j$ and $\mathbf{w}_j = \mathbf{x}\, w_j$ are eigenvalues and eigensolutions of the Lamé operator on Ω_n. Moreover, each component $x_i w_j$ is an eigenfunction of the Laplace operator on Ω_n, with eigenvalue ϱ_j.

Remark 5. Lemma 4 shows another intrinsic relationship between the Laplace and Lamé operator.

Notice, however, that $\tilde{\varrho}_1 = (1 + \alpha)\,\lambda_1(\Omega_{n+2}) > \left(1 + \frac{\alpha}{n}\right)\lambda_1(\Omega_n) \geqq \Lambda_1(\Omega_n)$, so that $\mathbf{w}_1$ is not a first eigensolution of the Lamé operator.

We conclude our remarks with another "Ansatz" and assume that $n = 2$ and that $\mathbf{u} = ((u(x, y), 0)$. Then the system is rewritten as

$$(1 + \alpha)\,u_{xx} + u_{yy} + \Lambda u = 0 \quad \text{in } \Omega,$$
$$u_{xy} = 0 \quad \text{in } \Omega.$$

Because of the second equation there exist functions $f(x)$ and $g(y)$ such that $u(x, y) = f(x) + g(y)$. But now the first equation of the system can be solved by separation of variables and has the solution

$$u(x, y) = c_1 \sin\sqrt{\frac{\Lambda}{1+\alpha}}\,x + c_2 \cos\sqrt{\frac{\Lambda}{1+\alpha}}\,x + c_3 \sin\sqrt{\Lambda}\,y + c_4 \cos\sqrt{\Lambda}\,y.$$

This representation has to satisfy the boundary condition $u(x, y) = 0$ on $\partial\Omega$. Now it is easy to see that for many shapes of Ω, for instance for Ω a disk, we have $u \equiv 0$, so that this Ansatz is in general not successful. For special domains Ω, however, we *do* get a solution to the eigenvalue problem.

Lemma 5. Let $\Omega \in \mathbb{R}^2$ be the rhombus with corners in $(\pm\sqrt{1 + \alpha}, 0)$ and $(0, \pm 1)$. Then $\Lambda = \pi^2$ is an eigenvalue and the vector function with vanishing second component and positive first component $u(x, y) = \cos\frac{\pi x}{\sqrt{1 + \alpha}} + \cos \pi y$ is an eigensolution of the Lamé operator.

In fact, it follows from the above Ansatz and from the identity

$$u(x, y) = \cos\frac{\pi x}{\sqrt{1+\alpha}} + \cos \pi y$$
$$= 2\cos\left(\frac{\pi x}{2\sqrt{1+\alpha}} + \frac{\pi y}{2}\right) \cdot \cos\left(\frac{\pi x}{2\sqrt{1+\alpha}} - \frac{\pi y}{2}\right)$$

that $u(x, y)$ vanishes on the set $\left\{\left|\frac{x}{\sqrt{1+\alpha}}\right| + |y| = 1\right\}$. This proves the Lemma. □

Remark 6. Notice that, as $\alpha \to 0$, we approach the first eigenfunction of the Laplace operator on a square. This seems to indicate that the eigenvalue which we found in Lemma 5, is in fact the smallest eigenvalue of the Lamé operator for the rhombus. In this case one would even have an eigensolution which is (componentwise) nonnegative and thus unique (in the class of vectors with vanishing second component).

Other eigensolutions on the rhombus can be obtained by scaling the solution from Lemma 5.

Remark 7. For the sake of completeness we want to record a nodal line result. It is based on the following observation. If $\mathbf{u}_1$ is an eigensolution of the Lamé operator then the function $w = \operatorname{div} \mathbf{u}_1$ satisfies $(1 + \alpha)\, \Delta w + \Lambda w = 0$ in Ω. Furthermore, if α satisfies the smallness condition $1 + \frac{\alpha}{n} < \lambda_2(\Omega)/\lambda_1(\Omega)$ (which was also used in Lemma 3), then w cannot vanish identically in Ω. This follows from a reasoning similar to the proof of Lemma 1. If, however, w is not identical zero, then *w cannot have a closed nodal surface in Ω*. In fact, otherwise there would be a set $\tilde{\Omega} \subset \Omega$ such that the restriction of w to $\tilde{\Omega}$ is the first eigenfunction of the Laplace operator on $\tilde{\Omega}$ under Dirichlet boundary conditions. Therefore $\Lambda_1 = (1 + \alpha)\, \lambda_1(\tilde{\Omega}) > \left(1 + \frac{\alpha}{n}\right) \lambda_1(\Omega)$, a contradiction to Corollary 2a.

Remark 8. It should also be noted, as S. Heinze has kindly pointed out to us, that the components of any eigensolution of the Lamé operator satisfy

$$\begin{aligned} (1 + \alpha)\, \Delta\Delta u^i + \Lambda \Delta u^i &= 0 \quad \text{in } \Omega, \\ u^i &= 0 \quad \text{on } \partial\Omega. \end{aligned}$$

This follows from the identity $\Delta = \operatorname{grad} \operatorname{div} - \operatorname{curl} \operatorname{curl}$. The buckling of a clamped plate is modelled by this equation and boundary condition plus the condition that the normal derivative of the solution be zero on the boundary, see [13, p. 456].

Another relationship between our eigenvalue problem and the one describing the buckling of a plate is the following. Let $n = 2$ and let ϕ be any function in $H_0^2(\Omega)$. Set $\mathbf{u} = (\phi_y, -\phi_x)$. Then $\operatorname{div} \mathbf{u} = 0$ and

$$R(\mathbf{u}) = \frac{\int_\Omega (\Delta\phi)^2 \, dx}{\int_\Omega |\nabla\phi|^2 \, dx}$$

is an upper bound for Λ_1. If one minimizes this new Rayleigh quotient over all functions ϕ in $H_0^2(\Omega)$, and if ψ is the minimizer, then ψ is the first eigenfunction of the problem

$$\begin{aligned} \Delta\Delta\psi + \nu \Delta\psi &= 0 \quad \text{in } \Omega, \\ \psi &= 0 \quad \text{on } \partial\Omega, \\ \frac{\partial\psi}{\partial n} &= 0 \quad \text{on } \partial\Omega. \end{aligned}$$

The corresponding first eigenvalue $\nu_1(\Omega)$ is an upper bound for $\Lambda_1(\Omega)$, which is independent of α (and greater than or equal to $\lambda_2(\Omega)$, see [13, p. 464]).

It was kindly pointed out to us by L. Payne, that the last inequality $\Lambda_1(\Omega) \leqq \nu_1(\Omega)$ is valid in any number of dimensions.

Bibliographical remarks

The oldest reference to the Lamé operator that is known to us is the book of Poincaré [14, p. 96–99]. He studied the eigenvalue problem under the boundary condition

$$\mathbf{n} \cdot \frac{\partial \mathbf{u}}{\partial x_i} + \frac{\partial \mathbf{u}_i}{\partial \mathbf{n}} + \mathbf{n}_i \cdot \operatorname{div} \mathbf{u} = 0 \quad \text{for } i = 1, 2, 3 .$$

The oldest reference that we could find to our problem, i.e. the eigenvalue problem under Dirichlet boundary conditions, is the paper by Korn. He proved the existence of countably many eigenfunctions with positive eigenvalues (for $n = 3$). Weyl reduced the eigenvalue problem for $n = 3$ to a system of integral equations and studied the asymptotic distribution of eigenvalues. Friedrichs investigated the spectrum of the Lamé operator under Neumann boundary conditions. Under those boundary conditions and for Ω a cylinder in $\mathbb{R}^3$ Michlin and Smoliziki calculated the smallest nontrivial eigenvalue of an eigensolution with rotational symmetry, see [12, pp. 186–188]. For $n = 2$ another version of our Lemma 1 can be found in the book of Courant and Hilbert, see [3, pp. 530–582]. There is also a paper and book of Kupradze on the subject. He gave a potential theoretic approach mainly to exterior problems. He also reduced the interior problem for $n = 3$ to a solvable integral equation (see [9, §47]), provided Λ satisfies the conditions

$$\Lambda \neq \lambda_j \quad \text{and} \quad \left(\frac{2 + 5\alpha}{2 + 3\alpha}\right) \Lambda \neq \lambda_j .$$

Here λ_j are eigenvalues of the Laplace operator on Ω under Dirichlet boundary conditions. If Ω is a ball in $\mathbb{R}^3$, these assumptions can be verified for $\Lambda_1(\Omega)$ (using Lemma 1 and Remark 4 of our paper), provided $\alpha < 1.22$.

W. Velte compared in [18] the eigenvalues Λ_j with the corresponding discrete eigenvalues of a finite difference approximation. We refer to his paper for previous results by H. F. Weinberger and J. Hersch on such comparisons.

If $n = 2$ and if Ω is the square with sides of length π, one has very precise numerical bounds for Λ_j. The following list is extracted from the book of Fichera [4, p. 90–92] and is based on results by Bassotti. It shows lower and upper bounds for Λ_1. These bounds depend on α.

α	lower bound	upper bound
0.5	2.474	2.480
1	2.926	2.932
1.5	2.926	3.363
2	3.244	3.779
3	4.381	4.568

Notice that we derived the bounds $2 < \Lambda_1 < \min\{2 + \alpha, 5.33\}$ in Corollary 2a and Remark 2 for every α. For the values of α which are listed in the table, our upper estimate is fairly close to the true value, but our lower bound is definitely improvable.

Finally the papers of Sprössig and Levine and Protter seem to be the first ones to contain an explicit positive lower bound for $\Lambda_1(\Omega)$ and for general Ω. It should also be mentioned that Levine and Protter gave primarily bounds for sums of eigenvalues.

Note added in proof (September 15, 87):

After this paper had been accepted for publication, the estimate

$$\Lambda_1(\Omega) \geqq \frac{n+\alpha}{n\delta^2} \tag{$*$}$$

by G. Fichera [4, p. 128] and [20, p. 225 and 227] was kindly brought to our attention by M. A. Sneider. Here δ denotes the radius of the smallest ball containing Ω. Notice that $(*)$ contradicts Remarks 2 and 8 of our paper if α is chosen large enough. Unfortunately in [4, 20] there is no proof of the inequality of which $(*)$ is the result. We have been able to verify $(*)$ only for $\alpha \leqq 2$.

References

[1] L. Bassotti, *Calcolo numerico degli autovalori relativi al primo problema dell'elastostatica piana in un quadrato*. Riv. Mat. Univ. Parma Ser. 2, **9**, 221–245 (1968).
[2] G. Chiti, *A bound for the ratio of the first two eigenvalues of a membrane*. SIAM J. Math. Anal. *14*, 1163–1167 (1983).
[3] R. Courant, D. Hilbert, *Methoden der mathematischen Physik*. Vol. II, Springer, Heidelberg 1968.
[4] G. Fichera, *Numerical and quantitative analysis*. Pitman, London 1978.
[5] K. O. Friedrichs, *On the boundary value problems of the theory of elasticity and Korn's inequality*. Annals of Math. *48*, 441–471 (1947).
[6] S. Heinze, personal communication.
[7] B. Kawohl, *Rearrangements and convexity of level sets in PDE*. Springer Lect. Notes in Math. *1150*, Heidelberg 1985.
[8] A. Korn, *Die Eigenschwingungen eines elastischen Körpers mit ruhender Oberfläche*. Königl. Bayr. Akad. der Wiss. München, Sitzungsber. d. Math. Phys. Kl. *36*, 351–402 (1906).
[9] W. D. Kupradse, *Randwertaufgaben der Schwingungstheorie und Integralgleichungen*. VEB Deutscher Verlag der Wissenschaften, Berlin 1956.
[10] V. D. Kupradze, *Boundary value problems of steady state elastic vibrations* (In Russian). Uspekhi matem. nauk. *8* No. 3 (55), 21–74 (1953).
[11] H. Levine, M. H. Protter, *Unrestricted lower bounds for eigenvalues for classes of elliptic equations and systems of equations with applications to problems in elasticity*. Math. Methods Appl. Sci. *7*, 210–222 (1985).
[12] S. G. Michlin, C. L. Smolizki, *Näherungsmethoden zur Lösung von Differential- und Integralgleichungen*. Teubner, Leipzig 1969.
[13] L. E. Payne, *Isoperimetric inequalities and their applications*. SIAM Rev. *9*, 453–488 (1967).
[14] H. Poincaré, *Leçons sur la théorie de l'élasticité*. Georges Carré, Paris 1892.

[15] W. Sprössig, *Untere Abschätzungen des ersten Eigenwerts spezieller elliptischer Randwertaufgaben*. Wiss. Zeitschr. TH Leuna-Merseburg *27*, 556–560 (1985).
[16] H. Triebel, *Höhere Analysis*. VEB Deutscher Verlag der Wissenschaften, Berlin 1972.
[17] W. Velte, *Über ein Stabilitätsproblem der Hydrodynamik*. Arch. Ration. Mech. Anal. *9*, 9–20 (1962).
[18] W. Velte, *Untere Schranken für Eigenwerte in der linearen Elastizitätstheorie*. in: Numerical Treatment of Eigenvalue Problems. Vol. 3 Eds.: J. Albrecht, L. Collatz, W. Velte Birkhäuser Basel ISNM 69, 192–204 (1984).
[19] H. Weyl, *Das asymptotische Verteilungsgesetz der Eigenschwingungen eines beliebig gestalteten elastischen Körpers*. Rend. Circ. Mat. Palermo *39*, 1–49 (1915).
[20] G. Fichera, *Methods of functional linear analysis in mathematical physics*. In: Proc. Int. Congr. Math. (Amsterdam 1954), III, sect. V, 216–228, 1956.

Abstract

Let $\Lambda_1(\Omega)$ be the first eigenvalue of the vector-valued problem

$$\begin{aligned} \Delta \mathbf{u} + \alpha \operatorname{grad}\operatorname{div}\mathbf{u} + \Lambda\mathbf{u} &= 0 \quad \text{in } \Omega,\\ \mathbf{u} &= 0 \quad \text{on } \partial\Omega, \end{aligned}$$

with $\alpha > 0$. Let $\lambda_1(\Omega)$ be the first eigenvalue of the scalar problem

$$\begin{aligned} \Delta u + \lambda u &= 0 \quad \text{in } \Omega,\\ u &= 0 \quad \text{on } \partial\Omega. \end{aligned}$$

The paper contains a proof of the inequality $\left(1 + \frac{\alpha}{n}\right)\lambda_1(\Omega) > \Lambda_1(\Omega) > \lambda_1(\Omega)$ and improves recent estimates of Sprössig [15] and Levine and Protter [11]. Moreover we show, if Ω is a ball, that an eigensolution $\mathbf{u}_1$ associated with $\Lambda_1(\Omega)$ is not unique and that the eigensolutions for this and higher eigenvalues are never rotationally invariant. Finally we calculate some eigensolutions explicitly.

Zusammenfassung

Es sei $\Lambda_1(\Omega)$ der erste Eigenwert des vektorwertigen Problems

$$\begin{aligned} \Delta \mathbf{u} + \alpha \operatorname{grad}\operatorname{div}\mathbf{u} + \Lambda\mathbf{u} &= 0 \quad \text{in } \Omega,\\ \mathbf{u} &= 0 \quad \text{auf } \partial\Omega, \end{aligned}$$

wobei $\alpha > 0$. Es sei $\lambda_1(\Omega)$ der erste Eigenwert des skalaren Problems

$$\begin{aligned} \Delta u + \lambda u &= 0 \quad \text{in } \Omega,\\ u &= 0 \quad \text{auf } \partial\Omega. \end{aligned}$$

In der Arbeit wird ein Beweis der Ungleichung $\left(1 + \frac{\alpha}{n}\right)\lambda_1(\Omega) > \Lambda_1(\Omega) > \lambda_1(\Omega)$ gegeben. Damit werden frühere Abschätzungen von Sprössig [2] sowie Levine und Protter [1] verbessert.

Es kann auch gezeigt werden, falls Ω eine Kugel ist, daß eine Eigenlösung $\mathbf{u}_1$ nicht eindeutig ist, und daß die Eigenlösungen für diesen und höhere Eigenwerte nicht rotationsinvariant sind. Für spezielle Rhombusgebiete lassen sich Eigenlösungen explizit angeben.

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Chapter 8

A counterexample with convex domain

to

a conjecture of De Saint Venant.

A COUNTEREXAMPLE WITH CONVEX DOMAIN TO A CONJECTURE OF DE SAINT VENANT

INTRODUCTION.

The elliptic problem

$$(1) \qquad \begin{cases} -\Delta u = 1 & \text{in } \Omega \subset \mathbb{R}^2, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

with Ω which has the orthogonal axes as axes of symmetry, was first studied by B. De Saint Venant. The function ∇u contains the stress components of an elastic bar, with cross section Ω, under torsion. The points of interest for mechanical engineers are the points where the stress, $|\nabla u(x)|$, becomes maximal. It is believed that the following conjecture holds, see [1], [3].

For convex domains Ω, $|\nabla u(x)|$ attains its maximum on the intersection of $\partial\Omega$ and the largest inscribed circle.

We will give an example where this conjecture will lead to a contradiction. Hence we may state:

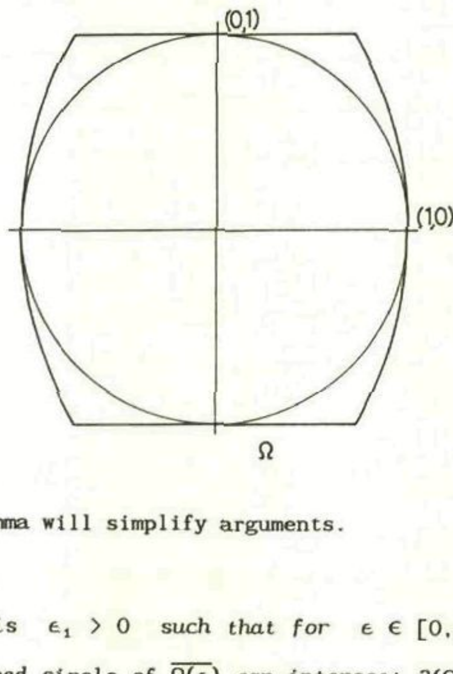
The conjecture is not true in general.

This answers a question in [4].

THE PROOF.

Set $\Omega = \{ (x_1,x_2) \in \mathbb{R}^2;\ (|x_1|+1)^2 + x_2^2 < 4\ ,\ |x_2| < 1 \}$. Note that the largest inscribed circle intersects $\partial\Omega$ precisely at the x_1 and x_2-axes. By [2, Th.6.13 and the remark following Th.6.19] there is a unique solution u of (1) and $u \in C^0(\bar{\Omega}) \cap C^\infty(\bar{\Omega}\backslash P)$, with $P = \{(\sqrt{3}-1,1), (\sqrt{3}-1,-1), (1-\sqrt{3},1), (1-\sqrt{3},-1)\}$. By the symmetry of Ω one finds that $u(x_1,x_2) = u(-x_1,x_2) = u(x_1,-x_2)$ *for all* $x \in \Omega$. *In [5] it is proven* that the level sets of u, $\Omega(\epsilon) := \{ x \in \Omega\ ;\ u(x) > \epsilon \}$, are convex and that there is only one local maximum. Note that $u_\epsilon = u - \epsilon$ is the unique solution of (1), where Ω is replaced by $\Omega(\epsilon)$.

We will show that the conjecture cannot be true for all $\Omega(\epsilon)$.



Ω

The following lemma will simplify arguments.

Lemma 1 : *There is* $\epsilon_1 > 0$ *such that for* $\epsilon \in [0,\epsilon_1]$ *the largest inscribed circle of* $\overline{\Omega(\epsilon)}$ *can intersect* $\partial(\Omega(\epsilon))$ *only at the* x_1 *and* x_2*-axes.*

Since $x = 0$ is the only critical point and since $u \in C^{\infty}(\Omega)$, the implicit function theorem shows $\partial(\Omega(\epsilon)) \in C^{\infty}$ for all $\epsilon \in (0,u(0))$. For $x \in \bar{\Omega}\setminus(P\cup\{0\})$ one can introduce a local C^{∞}-transformation of coordinates $(x_1,x_2) \to (\epsilon,\tau)$ such that $u(x(\epsilon,\tau)) = \epsilon$. Let these transformations be orientation preserving. Then the curvature K of $\partial(\Omega(\epsilon))$, defined by

$$K(x(\epsilon,\tau)) = \frac{\frac{\partial^2 x_1}{\partial\tau^2}\frac{\partial x_2}{\partial\tau} - \frac{\partial x_1}{\partial\tau}\frac{\partial^2 x_2}{\partial\tau^2}}{\left[(\frac{\partial x_1}{\partial\tau})^2 + (\frac{\partial x_2}{\partial\tau})^2 \right]^{3/2}}$$

is a nonnegative $C^{\infty}(\bar{\Omega}\setminus(P\cup\{0\}))$-function.

Set $A = \{ x \in \bar{\Omega} ; d(x,P) < d(0,P) - 1 \}$. Since $K = 0$ or $K = \frac{1}{2}$ for $x \in \partial\Omega\setminus P$, and $K \in C^{\infty}(\bar{\Omega}\setminus A)$ there is $\delta > 0$ such that $K(x) < \frac{3}{4}$ for all $x \in B := \{ x \in \bar{\Omega}\setminus A ; d(x,\partial\Omega) < \delta \}$. By the continuity of u there is $\epsilon_1 > 0$ such that $\partial(\Omega(\epsilon)) \subset A \cup B$ for all $\epsilon \in [0,\epsilon_1]$. Thus, since a circle with centre 0 and radius less than 1 cannot intersect A, we have proven the first lemma. □

The next step will be to show that if the conjecture is true, the stress will attain its maximum in every point of the intersection of $\partial\Omega$ and the largest inscribed circle.

Lemma 2 : *If the conjecture is true, then* $\frac{\partial u}{\partial n}(1,0) = \frac{\partial u}{\partial n}(0,1)$.

This lemma is proved by contradiction; we suppose $\frac{\partial u}{\partial n}(1,0) < \frac{\partial u}{\partial n}(0,1)$, where n denotes the outward normal. Since $u \in C^{\infty}(\bar{\Omega}\setminus P)$, there is a constant c_1 such that uniformly

(2) $$\left|u(x-\delta n) + \delta\frac{\partial u}{\partial n}(x)\right| \le c_1.\delta^2 \quad \text{for } x \in \{(1,0),(0,1)\} \text{ and } \delta \in [0,1].$$

Let $x_1(\epsilon)$, $x_2(\epsilon) > 0$ for $\epsilon \in [0,\epsilon_1]$ be such that $u(x_1(\epsilon),0) = \epsilon = u(0,x_2(\epsilon))$. Since $\frac{\partial u}{\partial n}(1,0) < \frac{\partial u}{\partial n}(0,1) < 0$ one finds by (2) that $x_1(\epsilon) > x_2(\epsilon)$ for $\epsilon > 0$ small.

Hence by lemma 1 the largest inscribed circle of $\overline{\Omega(\epsilon)}$ intersects $\partial(\Omega(\epsilon))$ only at $(0,\pm x_2(\epsilon))$. But for ϵ small

$$\frac{\partial u}{\partial n}(x_1(\epsilon),0) < \frac{\partial u}{\partial n}(0,x_2(\epsilon)) < 0,$$

which gives a contradiction. A similar proof holds if $\frac{\partial u}{\partial n}(1,0) > \frac{\partial u}{\partial n}(0,1)$. Hence $\frac{\partial u}{\partial n}(1,0) = \frac{\partial u}{\partial n}(0,1)$. □

Lemma 3 : *Suppose* $\frac{\partial u}{\partial n}(1,0) = \frac{\partial u}{\partial n}(0,1)$. *Then the conjecture is not true for* $\Omega(\epsilon)$ *with* $\epsilon > 0$ *and small enough.*

Set $a := -\frac{\partial u}{\partial n}(1,0) = -\frac{\partial u}{\partial n}(0,1)$. Since v and w, defined by $v(x,y) = \frac{1}{2}(1-x^2)$ and $w(x,y) = \frac{1}{4}(1-x^2-y^2)$, are respectively a super- and a subsolution of (1), one finds $w < u < v$ in Ω.

Moreover $\frac{1}{2} = -\frac{\partial w}{\partial n}(1,0) < a < -\frac{\partial v}{\partial n}(1,0) = 1$. Similar to (2) there is a constant c_2 such that uniformly

(3) $$\left| u(x-\delta n) + \delta\frac{\partial u}{\partial n}(x) - \tfrac{1}{2}\delta^2\,\frac{\partial^2 u}{\partial n^2}(x) \right| < c_2\,\delta^3$$

for $x \in \{(1,0),(0,1)\}$, $\delta \in [0,1]$.

By the differential equation one finds

$$-1 = \Delta u(x) = \frac{\partial^2 u}{\partial n^2}(x) + K(x)\,\frac{\partial u}{\partial n}(x) \qquad \text{for } x \in \partial\Omega\backslash P.$$

Hence

(4) $$\frac{\partial^2 u}{\partial n^2}(1,0) = -1 + a\,K(1,0) = -1 + \tfrac{1}{2}a$$

and

(5) $$\frac{\partial^2 u}{\partial n^2}(0,1) = -1 + a\,K(0,1) = -1.$$

There is then $c_3 > 0$ such that, for $x_1(.)$ and $x_2(.)$ defined in the proof of lemma 2,

(6) $$|\; x_1(\epsilon) - (1 - \frac{1}{a}\epsilon - \frac{2-a}{4a^3}\,\epsilon^2) \;| \le c_3\,\epsilon^3$$

and

(7) $$|\; x_2(\epsilon) - (1 - \frac{1}{a}\epsilon - \frac{2}{4a^3}\,\epsilon^2) \;| \le c_3\,\epsilon^3.$$

These inequalities show that for ϵ small enough $x_1(\epsilon) > x_2(\epsilon)$. Hence, by lemma 1, the largest inscribed circle of $\overline{\Omega(\epsilon)}$ intersects $\partial(\Omega(\epsilon))$ only at $(0,\pm x_2(\epsilon))$. But by using the Taylor expansion and (4-7) there are constants c_4 and c_5 such that uniformly

$$\frac{\partial u}{\partial n}(x_1(\epsilon),0) + \frac{\partial u}{\partial n}(0,x_2(\epsilon)) \ge$$

$$\ge (1-x_1(\epsilon))\,\frac{\partial^2 u}{\partial n^2}\,(1,0) - (1-x_2(\epsilon))\,\frac{\partial^2 u}{\partial n^2}\,(0,1) +$$

$$-c_4\Big[\,(1-x_1(\epsilon))^2 + (1-x_2(\epsilon))^2\,\Big] \ge$$

$$\ge \tfrac{1}{2}\epsilon - c_5\epsilon^2 > 0 \qquad \text{for } \epsilon \text{ small enough.}$$

Hence $|\nabla u(x)|$ is not maximal on $\overline{\Omega(\epsilon)}$ in $(0,\pm x_2(\epsilon))$ for ϵ small, which proves lemma 3. □

Acknowledgement: I would like to thank B. Kawohl for bringing this conjecture to my attention.

References

[1] L.N.G. Filon, *On the resistance to torsion of certain forms of shafting, with special reference to the effect of keyways*, Philos. Trans. R. Soc. Lond. A 193 (1900), 309-352.

[2] D. Gilbarg and N.S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Springer-Verlag, New York, 1977.

[3] B. Kawohl, *On the location of maxima of the gradient for solutions to quasilinear elliptic problems and a problem raised by Saint Venant*, Journal of Elasticity 17 (1987), 195-206.

[4] B. Kawohl, *Some qualitative properties of nonlinear partial differential equations*, to appear in M.S.R.I.-proceedings, Nonlinear diffusion equations and their equilibrium states, Eds. Ni, Peletier and Serrin, Berkeley 1986.

[5] L.G. Makar-Limanov, *Solution of Dirichlet's problem for the equation* $\Delta u = -1$ *in a convex region*, Math. Notes Acad. Sci. USSR 9 (1971), 52-53.

[6] B. de Saint Venant, *Mémoire sur la torsion des prismes, avec des considérations sur leur flexion ainsi que sur l'équilibre intérieur des solides élastiques en général, et de formules pratiques pour le calcul de leur résistance à divers efforts s'exercant simultanément*, Mémoires présentés par divers savants à l'académie des sciences de l'institut impérial de France, 2.Ser.14 (1856), 233-560.

Chapter 9

A strong maximum principle

for

a non-cooperative elliptic system.

ABSTRACT: In this note it is shown that on a ball in $\mathbb{R}^N$, with $N > 2$, a maximum principle holds for a special elliptic system. This system is such that the classical maximum principle is not applicable.

INTRODUCTION AND RESULTS.

A linear elliptic system

(1) $$-\Delta u_\alpha + \sum_{\beta=1}^{k} h_{\alpha\beta} u_\beta = f_\alpha \qquad \text{with } \alpha \in \{1,2,\dots,k\}$$

is called cooperative if

(2) $$h_{\alpha\beta} \leq 0 \qquad \text{for } \alpha \neq \beta.$$

If also

(3) $$\sum_{\beta=1}^{k} h_{\alpha\beta} \geq 0 \qquad \text{for } \alpha \in \{1,2,\dots,k\}$$

one can extend the results of the maximum principle to system (1); see [3, p.191].

In this note we consider a system, with Dirichlet boundary conditions, which is in some sense the simplest non-cooperative system:

(4) $$\begin{cases} -\Delta u = f_1 - \lambda v & \text{in } \Omega, \\ -\Delta v = f_2 & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is the unit ball in $\mathbb{R}^N$, $N > 2$. The classical maximum principle gives the following positivity result which is not uniform:

Lemma 1. Let $f_1, f_2 \in C^\gamma(\bar{\Omega})$ for some $\gamma > 0$ with $f_1 \geq 0$ and $f_1 \not\equiv 0$. Then there is $\lambda(f_1,f_2) > 0$ such that for all $\lambda \in [0,\lambda(f_1,f_2))$ the solution u_λ of (4) satisfies

(5) $$u_\lambda > 0 \qquad \text{in } \Omega.$$

The classical maximum principle doesn't show that it is possible to find a uniform result for $f_2 \leq f_1$. Nevertheless we prove for

(6) $$\begin{cases} -\Delta u = f - \lambda v & \text{in } \Omega, \\ -\Delta v = f & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases}$$

the following theorem. (The same result holds for (4) if $f_2 \le f_1$ and $f_1 = f$.)

Theorem 2 *a*) There is a largest $\lambda_0 > 0$ for which the following holds. For all $(u,v) \in C^2(\bar\Omega) \times C^2(\bar\Omega)$ satisfying (6), such that $\lambda < \lambda_0$, $f \ge 0$ and $f \ne 0$, one finds that

(7) $$u > 0 \qquad \text{in } \Omega.$$

b) $\lambda_0 \le (\lambda_1^{-1} + \lambda_2^{-1})^{-1}$ ($< \lambda_1$), where λ_1, λ_2 are respectively the first and second eigenvalue of

(8) $$\begin{cases} -\Delta\varphi = \lambda\varphi & \text{in } \Omega, \\ \varphi = 0 & \text{on } \partial\Omega. \end{cases}$$

Remark 1. If N = 1 then

(9) $$u(x) = \int_{-1}^{1} \tfrac{1}{2}\,(1-|x-y|-xy)\left[\, 1 - \frac{\lambda}{6}\,(2+2|x-y|-x^2-y^2)\,\right] f(y)\,dy.$$

Since $\max\{\, \frac{1}{6}(2+2|x-y|-x^2-y^2) \;;\; -1 \le x,y \le 1 \,\} = \frac{2}{3}$ one finds that $\lambda_0 = \frac{3}{2} \le (\lambda_1^{-1} + \lambda_2^{-1})^{-1} = \frac{1}{5}\pi^2 \approx 1.97$.

A direct calculation shows that $(\lambda_1^{-1} + \lambda_2^{-1} + \lambda_3^{-1} + \dots)^{-1} = \frac{3}{2}$.

I cannot explain this similarity.

Remark 2. Let H be a subspace of $C(\bar\Omega)$ such that the inverse B of $-\Delta$, with zero Dirichlet boundary condition, into $C^2(\bar\Omega)$ is well defined. Theorem 2a then shows that

(10) $$B\,(I - \lambda B)\,f > 0 \qquad \text{in } \Omega$$

for all $\lambda < \lambda_0$ and $f \in H$ with $f \ge 0$, $f \ne 0$.

Remark 3. The classical maximum principle [3, Th.2.2] shows that $Bf > 0$ for f as in (10). If also $f(x_0) = 0$ for some $x_0 \in \Omega$, then

(11) $$(\,(I - \lambda B)\,f\,)(x_0) < 0 \quad \text{for all } \lambda > 0.$$

Remark 4. Consider the system

$$\begin{cases} -\Delta u = f - \lambda^2 v & \text{in } \Omega, \\ -\Delta v = u & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega. \end{cases}$$

Hence for B as in Remark 2 and $\lambda < \lambda_1$

$$u = \left[\sum_{k=0}^{\infty} (\lambda^4 B^4)^k \right] (I+\lambda B)(I-\lambda B)Bf.$$

If also $\lambda < \lambda_0$ then Theorem 2a together with the classical maximum principle shows that if $f \geq 0$, $f \neq 0$ then $u > 0$ in Ω. Both this system as well as (6) cannot be uncoupled as in [1, Remark 1.7] in order to find a maximum principle. Recent results concerning [1, Remark 1.7] can be found in [4].

Remark 5. Let Ω be an arbitrary domain, and let φ_1, φ_2 be the first and second eigenfunctions of (8). Set $H = \{ c_1\varphi_1 + c_2\varphi_2;\ c_1, c_2 \in \mathbb{R}\}$. One can prove that $B(I-\lambda B)$ from H into H is positive if and only if $\lambda \leq \lambda_0 = (\lambda_1^{-1} + \lambda_2^{-1})^{-1}$.
One could also hope that in general $\lambda_0 = (\sum \lambda_i^{-1})^{-1}$, with the summation over all eigen- functions. However, direct but tedious computations show that with $H = \{ c_1\varphi_1 + c_2\varphi_2 + c_3\varphi_3 ;\ c_i \in \mathbb{R} \}$ and $\Omega = (0,1)$ the following inequality holds:

$$\lambda_0 < (\lambda_1^{-1} + \lambda_2^{-1} + \lambda_3^{-1})^{-1}.$$

PROOFS:

Lemma 1 can be proved by a straightforward application of the classical strong maximum principle. Let φ_1 be the first eigenfunction of (8) with $\varphi_1 > 0$ in Ω. Simce $v = 0$ on $\partial\Omega$ and $v \in C^2(\bar{\Omega})$ there is $c_1 > 0$ such that

(12) $$v \leq c_1\varphi_1 \qquad \text{in } \Omega.$$

Let $w \in C^2(\bar{\Omega})$ be the solution of

(13) $$\begin{cases} -\Delta w = v & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega. \end{cases}$$

The maximum principle, [3, Th.2.6], then shows that

(14) $$w \leq \frac{c_1}{\lambda_1}\varphi_1 \qquad \text{in } \Omega.$$

Since $-\Delta(u+\lambda w) = f_1 \geq 0$ and $f_1 \neq 0$ the strong maximum principle, [3, Th.2.6 and Th.2.7], implies that $u + \lambda w > c_2\varphi_1$ in Ω for some $c_2 > 0$. Hence

$$(15)\qquad u > c_2\varphi_1 - \lambda w \geq c_2\varphi_1 - \lambda \frac{c_1}{\lambda_1}\varphi_1 \geq 0 \quad \text{in } \Omega \text{ if } \lambda \leq \lambda_1 \frac{c_2}{c_1}. \qquad \square$$

Proof of Theorem 2a.

Equations (6) can be rewritten as

$$(16)\qquad \begin{aligned} u(x) &= \int_\Omega G(x,y)\ (\ f(y) - \lambda\ v(y)\ )\ dy = \\ &= \int_\Omega G(x,y) \left[f(y) - \lambda \int_\Omega G(y,z)\ f(z)\ dz \right] dy = \\ &= \int_\Omega \left[G(x,y) - \lambda \int_\Omega G(x,z)\ G(z,y)\ dz \right] f(y)\ dy, \end{aligned}$$

where, see [2, (2.12), (2.13)],

$$(17)\qquad \left[\begin{array}{l} G(x,y) = g_n \left[\ |\ x - y\ |^{2-n} - |(\ |y|x - |y|^{-1}y\ )|^{2-n} \right], \quad y \neq 0 \\ G(x,0) = g_n\ (\ |x|^{2-n} - 1\ ). \end{array} \right.$$

The Euclidean norm is denoted by $|.|$, and $g_n = (n(n-2)\omega_n)^{-1}$, where ω_n is the volume of the unit ball in $\mathbb{R}^N$.

To prove the theorem, it is sufficient to show that

$$(18)\qquad M(x,y) := (G(x,y))^{-1} \int_\Omega G(x,z)\ G(z,y)\ dz < M$$

for some $M < \infty$. One finds then $u > 0$ for all $\lambda \leq \lambda_0 = M^{-1}$.

We will prove (18) by direct computations.

In order to simplify the notations we set

$$(xy) = |x-y|\ ,\ (XY) = |(\ |y|x - |y|^{-1}y\ )|\ , \text{ etcetera.}$$

Note that $(XY) = \left[\ |y|^2|x|^2 - 2(x.y) + 1\ \right]^{1/2} = (YX)$, and hence

$$(19)\qquad \begin{aligned} (XY)^2 - (xy)^2 &= |y|^2|x|^2 + 1 - |x|^2 - |y|^2 = \\ &= (\ 1 - |x|^2)(\ 1 - |y|^2) > 0 \quad \text{for } x,y \in \Omega, \end{aligned}$$

(20) $$(xy)^{-1} - (XY)^{-1} = \frac{(1 - |x|^2)(1 - |y|^2)}{(XY) + (xy)} (xy)^{-1}(XY)^{-1}$$

for $x \neq y$.

Using (17)-(20) one finds that

(21) $$g_n^{-1} M(x,y) = \int_\Omega \frac{((xz)^{2-n} - (XZ)^{2-n})((yz)^{2-n} - (YZ)^{2-n})}{(xy)^{2-n} - (XY)^{2-n}} dz =$$

$$= \int_\Omega \frac{((xz)^{-1} - (XZ)^{-1}) \sum_{k=0}^{n-3} (xz)^{3-n+k} (XZ)^{-k}}{((xy)^{-1} - (XY)^{-1}) \sum_{k=0}^{n-3} (xy)^{3-n+k} (XY)^{-k}} \cdot$$

$$\cdot ((yz)^{-1} - (YZ))^{-1}) \sum_{k=0}^{n-3} (yz)^{3-n+k} (YZ)^{-k} dz =$$

$$= \int_\Omega \frac{(1-|x|^2)(1-|z|^2)}{1 + (xz)(XZ)^{-1}} \frac{1 + (xy)(XY)^{-1}}{(1-|x|^2)(1-|y|^2)} \frac{(1-|y|^2)(1-|z|^2)}{1 + (yz)(YZ)^{-1}} \cdot$$

$$\cdot \frac{\left[\sum_{k=0}^{n-3} (xz)^{2-n+k}(XZ)^{-2-k}\right]\left[\sum_{k=0}^{n-3} (yz)^{2-n+k}(YZ)^{-2-k}\right]}{\sum_{k=0}^{n-3} (xy)^{2-n+k}(XY)^{-2-k}} dz =$$

$$= (xy)^{n-2}(XY)^2 \left[\sum_{k=0}^{n-3} \left[\frac{(xy)}{(XY)}\right]^k\right]^{-1} \int_\Omega \frac{(1 - |z|^2)^2(1 + (xy)(XY)^{-1})}{(1 + (xz)(XZ)^{-1})(1 + (yz)(YZ)^{-1})} \cdot$$

$$\cdot \left[\sum_{k=0}^{n-3} \left[\frac{(xz)}{(XZ)}\right]^k\right]\left[\sum_{k=0}^{n-3} \left[\frac{(yz)}{(YZ)}\right]^k\right] (yz)^{2-n}(xz)^{2-n}(YZ)^{-2}(XZ)^{-2} dz \leq$$

$$\leq (xy)^{n-2}(XY)^2 \int_\Omega (1 - |z|^2)^2 \, 2(n-2)^2(yz)^{2-n}(xz)^{2-n}(YZ)^{-2}(XZ)^{-2} dz.$$

Assume $x \neq y$ and split Ω in Ω_1 and Ω_2, where

$$\Omega_1 = \{ z \in \Omega;\ |x-z| < |y-z| \},\ \Omega_2 = \{ z \in \Omega;\ |x-z| > |y-z| \}.$$

If $|x-z| < |y-z|$, then $|y-z| \geq |y-x| - |x-z| \geq |y-x| - |y-z|$ and

$$|(\ |y|x - |y|^{-1}y\ )| \leq |y||x-z| + |(\ |y|z - |y|^{-1}y\ )| \leq$$
$$\leq |y-z| + |(\ |y|z - |y|^{-1}y\ )|.$$

Hence

(22) $$(xy) \leq 2(yz) \quad \text{and}$$

(23) $$(XY) \leq 2(YZ) \quad \text{for } z \in \Omega_1.$$

By exchanging x and y, respectively X and Y one finds equivalent inequalities for $z \in \Omega_2$. Moreover

(24)
$$\begin{aligned}(XZ)^2 &= |x|^2|z|^2 - 2(x.z) + 1 \geq \\ &\geq |x|^2|z|^2 - 2|x||z| + 1 = \\ &= (\ 1 - |x||z|\ )^2 \geq \\ &\geq (\ 1 - |z|\ )^2 \geq \tfrac{1}{4}(\ 1 - |z|^2\ )^2 \quad \text{for } z \in \Omega.\end{aligned}$$

Combining (21) to (24) yields

(25)
$$\begin{aligned}M(x,y) &\leq 2(n-2)^2 g_n \left[\int_{\Omega_1} \left[\frac{1-|z|^2}{(XZ)} \right]^2 \left[\frac{(xy)}{(yz)} \right]^{n-2} \left[\frac{(XY)}{(YZ)} \right]^2 (xz)^{2-n} dz \right. + \\ &\qquad \left. + \int_{\Omega_2} \left[\frac{1-|z|^2}{(YZ)} \right]^2 \left[\frac{(xy)}{(xz)} \right]^{n-2} \left[\frac{(XY)}{(XZ)} \right]^2 (yz)^{2-n} dz \right] \leq \\ &\leq 2^{n+3}(n-2)^2 g_n \left[\int_{\Omega_1} (xz)^{2-n}\, dz + \int_{\Omega_2} (yz)^{2-n}\, dz \right] < \\ &< 2^{n+4}(n-2)^2 g_n \int_{2\Omega} |z|^{2-n}\, dz = 2^{n+5} \frac{(n-2)\ \omega_{n-1}}{n\omega_n} .\end{aligned}$$

which completes the proof of 2a.

In order to prove 2b let φ_1 and φ_2 be respectively the first and second eigenfunction of (8), with $\varphi_1 > 0$ in Ω.
First note that $\lambda_0 \leq \lambda_1$. Indeed, if $\lambda > \lambda_1$ then $u = \lambda_1^{-1}(1 - \frac{\lambda}{\lambda_1})\varphi_1$ is a solution of (6) with $f = \varphi_1 > 0$ in Ω, while $u < 0$ in Ω.
Suppose $(\lambda_1^{-1} + \lambda_2^{-1})^{-1} < \lambda \leq \lambda_1$ and let $c > 0$ be the largest constant such that

$$\varphi_1 - c\varphi_2 \geq 0 \qquad \text{in } \Omega. \tag{26}$$

Let U be the solution of (6) with $f = \varphi_1 - c\varphi_2$. Then

$$U = \lambda_1^{-2}(\lambda_1-\lambda)(\varphi_1-c\varphi_2) - c(\lambda_1^{-2}-\lambda_2^{-2})(\lambda - (\lambda_1^{-1}+\lambda_2^{-1})^{-1})\varphi_2. \tag{27}$$

If $\lambda = \lambda_1$ then U is negative somewhere in Ω since φ_2 changes sign. If $\lambda < \lambda_1$ then U is negative somewhere since c is the largest constant such that (26) holds. This is a contradiction. □

REFERENCES.

[1] D.G. De Figueiredo, E. Mitidieri, *A maximum principle for an elliptic system and applications to semilinear problems*, SIAM J.Math.Anal. 17 (1986), 836-849.

[2] D. Gilbarg, N.S. Trudinger, *Elliptic partial differential equations of second order*, Springer-Verlag, Berlin/Heidelberg/New York, 1977.

[3] M.H. Protter, H.F. Weinberger, *Maximum principles in differential equations*, Prentice-Hall, Englewood Cliffs N.J., 1967.

[4] H.F. Weinberger, *Some remarks on invariant sets for systems*, IMA Preprint Series 333, University of Minnesota, August 1987.

SAMENVATTING.

In dit proefschrift bekijken we enkele semilineaire elliptische eigenwaarde problemen. In deel A beschouwen we het probleem $-\Delta u = \lambda f(u)$, in een open en begrensd gebied Ω in $\mathbb{R}^N$, met Dirichlet randvoorwaarden op $\partial\Omega$ en $\lambda > 0$. We nemen aan dat $\partial\Omega$ voldoende glad is en dat f een differentieerbare functie is met een vallend nulpunt ρ; dat wil zeggen $f(\rho) = 0$ en $(s-\rho)f(s) < 0$ voor $0 < |s-\rho| < \delta$, met een $\delta > 0$. We laten zien dat een integraalvoorwaarde voor f, die noodzakelijk en voldoende is voor het bestaan van oplossingen (λ,u) met $\max u \in (\rho-\delta,\rho)$ in het ééndimensionale geval, ook noodzakelijk is voor gebieden in hogere dimensies. Dat deze voorwaarde voldoende is werd door Fife in 1973 bewezen met behulp van asymptotische ontwikkelingen. De voorwaarde is ook voldoende voor het bestaan van een positieve oplossing.

Voor een functie f die tussen 0 en ρ nog één of meerdere nulpunten bezit kan men geen eenduidigheid verwachten. Desalniettemin is het mogelijk, met de extra voorwaarde voor f, $f' \leq 0$ in $(\rho-\delta,\rho)$, om eenduidigheid te bewijzen in een orde-interval in $C(\bar{\Omega})$ voor λ groot genoeg. Daarnaast laten we ook zien dat er een eenduidige oplossing bestaat voor λ groot, indien men a priori eist dat $\rho-\epsilon < \max u < \rho$ voor een $\epsilon > 0$. Ook wordt stabiliteit van deze oplossingen bewezen.

Vervolgens worden de gevolgen onderzocht voor de oplossingen, indien f slechts continu is of indien $\partial\Omega$ niet langer glad is. Het is nog steeds mogelijk voor continue f om een oplossing u te vinden in $[u_1,u_2]$, als $u_1 \leq u_2$ respectivelijk een onder- en een bovenoplossing is. We laten verder zien dat er voor $\partial\Omega$ een kritieke hoek bestaat als $f(0) < 0$: indien $\partial\Omega$ een scherpere hoek bevat bestaan er geen positieve oplossingen u, met $\max u < \rho$, voor welke $\lambda > 0$ dan ook. Voor de bewijzen gebruiken we o.a. een zwakke vorm van boven- en onderoplossingen alsmede het zogeheten "Sweeping Principle" van Serrin.

In deel B worden drie onafhankelijke elliptische problemen beschouwd, waarvan de eerste twee hun oorsprong vinden in de elasticiteits-theorie. Het eerste betreft enkele resultaten voor de eigenfuncties en eigenwaarden van het Lamé-stelsel met Dirichlet randvoorwaarden. In de tweede notitie vindt men een *tegenvoorbeeld voor een vermoeden van B. De Saint Venant uit 1859.(Ofschoon* er twijfels bestonden was er zover bekend nog geen tegenvoorbeeld.) Voor het derde probleem, een lineair elliptisch stelsel, wordt een niet-standaard maximum principe bewezen.

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Stellingen behorende bij het proefschrift (¤)

Semilinear Elliptic Eigenvalue Problems

van G. Sweers.

Beschouw het stelsel van M tweede orde partiële differentiaalvergelijkingen in $\mathbb{R}^N$

$$(*) \qquad E(u) = f,$$

met $u, f: \mathbb{R}^N \to \mathbb{R}^M$ en $E_r(u) = \sum_{s=1}^{M} E_{rs}(u_s) = f_r \quad r=1,\dots,M$, waarbij

$E_{rs}(u_s) = -\sum_{h=1}^{N}\sum_{k=1}^{N} a_{hk}^{rs} \frac{\partial}{\partial x_h}\frac{\partial}{\partial x_k} u_s$. Neem aan dat $a_{hk}^{rs} = a_{kh}^{rs}$.

Het stelsel (*) heet elliptisch indien $\det(\ \alpha(\xi)\ ) \neq 0 \qquad$ voor alle $\xi \in \mathbb{R}^N \backslash \{0\}$. waarbij $(\alpha(\xi))_{rs} = \sum_{h=1}^{N}\sum_{k=1}^{N} a_{hk}^{rs}\, \xi_h\, \xi_k$.

(¤) Het onderzoek waaruit bijbehorend proefschrift is ontstaan, is ondersteund door Z.W.O., de Nederlandse organisatie voor Zuiver Wetenschappelijk Onderzoek (vanaf 1-2-88: N.W.O.).

Het stelsel (*) heet sterk elliptisch (Vishik), indien er $\rho > 0$ bestaat zodat

$$\sum_{r=1}^{M} \sum_{s=1}^{M} \sum_{h=1}^{N} \sum_{k=1}^{N} a_{hk}^{rs} \xi_h \xi_k \eta_r \eta_s \geq \rho \, |\xi|^2 \, |\eta|^2$$

voor alle $\xi \in \mathbb{R}^N$, $\eta \in \mathbb{R}^M$.

$|.|$ is de Euclidische norm.

Noem het stelsel (*) matrix-elliptisch indien

$$\sum_{r=1}^{M} \sum_{s=1}^{M} \sum_{h=1}^{N} \sum_{k=1}^{N} a_{hk}^{rs} \beta_{hr} \beta_{ks} \geq 0 \quad \text{voor alle } \beta \in (\mathbb{R}^N)^M.$$

STELLING 1 (Fichera): *Neem aan (*) is matrix-elliptisch en $a_{hk}^{rs} = a_{hk}^{sr}$. Zij Ω een begrensd gebied in $\mathbb{R}^N$ met een gladde rand. Zij $w \in C^2(\bar{\Omega})$ zodanig dat $w \leq 0$ en $\xi^T(E_{rs}(w)(x)) \xi > 0$ in $\bar{\Omega}$ voor alle $\xi \in \mathbb{R}^M \setminus \{0\}$. definieer $A(w) \in C(\bar{\Omega})$ door $A(w)(x) = \max_{1 \leq i \leq M} ((E_{rs}(w)(x))^{-1})_{ii}$, het maximum van de elementen op de hoofddiagonaal van $(E_{rs}(w)(x))^{-1}$ Dan geldt voor alle $u \in (C^2(\bar{\Omega}))^M$ met $u = 0$ op $\partial\Omega$ dat*

$$(**) \qquad \left[\int_{\Omega} |u|^2 dx \right]^{1/2} \leq 2N \max_{x \in \bar{\Omega}} \left[A(w)(x) \right] \max_{x \in \bar{\Omega}} |w(x)| \left[\int_{\Omega} |E(u)|^2 dx \right]^{1/2}$$

LEMMA: Het Lamé-stelsel, gedefinieerd voor $M = N > 1$ en $\sigma \in \mathbb{R}$ door

$$a_{hk}^{rs} = \delta_{rs} \delta_{hk} + \tfrac{1}{2}\sigma \left(\delta_{rh} \delta_{sk} + \delta_{rk} \delta_{sh} \right) \quad h,k,r,s \in \{1,\dots,N\},$$

1) is sterk elliptisch dan en slechts dan als $\sigma > -1$;
2) is matrix-elliptisch dan en slechts dan als $\sigma \in [\frac{-2}{N+1}, 2]$.

STELLING 2: *De ongelijkheid (**) is niet juist voor het Lamé-stelsel met σ voldoende groot.*

Referenties: G. Fichera, Methods of linear analysis in mathematical physics, In: Proc.Int.Congr.(Amsterdam 1954),III,sect V, 216-228,1956.
G. Fichera, Numerical and quantitative analysis, Pitman, London 1978.

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Singuliere elliptische perturbaties in $\mathbb{R}^N$.

Definieer voor $s \in \mathbb{R}^2$, $\epsilon > 0$ and $u \in H_p^{s_1+s_2}(\mathbb{R}^N)$, met $p \in (1,\infty)$ de norm

$$\|u\|_{s,p,\epsilon} = \| \mathcal{F}^{-1} \langle\xi\rangle^{s_1} \langle\epsilon\xi\rangle^{s_2} \mathcal{F} u \|_{L_p(\mathbb{R}^N)} .$$

waarbij $\langle\xi\rangle = (1 + |\xi|^2)^{½}$ voor $\xi \in \mathbb{R}^N$ en waarbij $\mathcal{F}$ de Fourier-transformatie voorstelt:

$$(\mathcal{F}u)(\xi) = (2\pi)^{-½N} \int_{\mathbb{R}^N} e^{-ix.\xi} u(x) dx.$$

Beschouw de differentiaalvergelijking

(#) $\quad (\epsilon^2\Delta^2-\Delta-1)u = f$ in $\mathbb{R}^N$ met $\epsilon \in (0,\epsilon_0]$.

STELLING 3: *Voor alle* $s \in \mathbb{R}^2$, $p \in (1,\infty)$ *en* $\epsilon_0 > 0$ *voldoende klein, is er* $C > 0$ *zodat voor alle* $\epsilon \in (0,\epsilon_0]$ *en* u,f *die voldoen aan* (#), *met* $u \in H_p^{s_1+s_2}(\mathbb{R}^N)$, *geldt:*

$$\|u\|_{s,p,\epsilon} \le C (\|f\|_{s-(2,2),p,\epsilon} + \|u\|_{s-(1,0),p,\epsilon}) .$$

Beschouw de differentiaalvergelijking

(##) $\quad (\epsilon^2\Delta^2-\Delta+1)u = f$ in $\mathbb{R}^N$ met $\epsilon \in (0,\epsilon_0]$.

STELLING 4: *Voor alle* $s \in \mathbb{R}^2$, $p \in (1,\infty)$ *en* $\epsilon_0 > 0$ *voldoende klein, is er* $C > 0$ *zodat voor alle* $\epsilon \in (0,\epsilon_0]$ *en* u,f *die voldoen aan* (##), *met* $u \in H_p^{s_1+s_2}(\mathbb{R}^N)$, *geldt:*

$$\|u\|_{s,p,\epsilon} \le C \|f\|_{s-(2,2),p,\epsilon} .$$

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Singuliere elliptische perturbaties in $\mathbb{R}^N_+ = \mathbb{R}^{N-1} \times \mathbb{R}_+$.

Definieer voor $s \in \mathbb{R}^2$, $\epsilon > 0$ and $u \in H_p^{s_1+s_2}(\mathbb{R}^N)$, met $p \in (0,\infty)$ de norm:

$$\|u\|^+_{s,p,\epsilon} = \inf \{ \|v\|_{s,p,\epsilon} \; ; \; v \in H_p^{s_1+s_2}(\mathbb{R}^N) \text{ met } v = u \text{ in } \mathbb{R}^N_+ \} .$$

Beschouw het randvoorwaardenprobleem:

$$(@) \quad \left[\begin{array}{ll} (\epsilon^2\Delta^2 - \Delta + I)u = f & \text{in } \mathbb{R}^N_+ . \\ u(x',0) = 0 & \text{voor } x' \in \mathbb{R}^{N-1} . \\ \frac{d}{dx_n} u(x',0) = 0 & \text{voor } x' \in \mathbb{R}^{N-1} . \end{array} \right.$$

met $\epsilon \in (0,\epsilon_0]$.

STELLING 5: Voor alle $p \in (1,\infty)$ *en* $\epsilon_0 > 0$ *voldoende klein, is er* $C > 0$ *zodat voor alle* $\epsilon \in (0,\epsilon_0]$ *en* u,f *die voldoen aan* (@), *met* $u \in H_p^2(\mathbb{R}^N)$, *geldt:*

$$\|u\|^+_{(1,1),p,\epsilon} \leq C \left(\|f\|^+_{(-1,-1),p,\epsilon} + \|u\|^+_{(0,1),p,\epsilon} \right) .$$

Beschouw het randvoorwaardenprobleem met $m_1, m_2 \in \mathbb{N}\backslash\{0\}$:

$$(@@) \quad \left[\begin{array}{ll} \left[(-\epsilon^2\Delta)^{m_2} + I\right](-\Delta)^{m_1} u = f & \text{in } \mathbb{R}^N_+ . \\ \left[\frac{d}{dx_n}\right]^i u(x',0) = 0 \quad \text{voor } x' \in \mathbb{R}^{N-1} & \text{en } i \in \{0,1,\ldots,m_1+m_2-1\} . \end{array} \right.$$

STELLING 6: Voor alle $p \in (1,\infty)$ *en* $\epsilon_0 > 0$ *voldoende klein, is er* $C > 0$ *zodat voor alle* $\epsilon \in (0,\epsilon_0]$ *en* u, f *die voldoen aan* (@@), *met* $u \in H_p^{m_1+m_2}(\mathbb{R}^N)$, *geldt:*

$$\|u\|^+_{(m_1,m_2),p,\epsilon} \leq C \left(\|f\|^+_{(-m_1,-m_2),p,\epsilon} + \|u\|^+_{(m_1-1,m_2),p,\epsilon} \right) .$$

Referenties: M. Gueugnon, Perturbations singulières dans des espaces L^p pour l'operateur $\epsilon\Delta^2 - \Delta + I$, C.R.Acad.Sc. Paris 293 (1981), 129-131.

M. Gueugnon, Perturbations singulières dans des espaces L^p pour des problèmes elliptiques généraux, C.R.Acad.Sc. Paris 301 (1985), 555-558.

B. Najman, Singular Perturbations in L^p, to appear in Glasnik Matematički.

G. Sweers, Estimates for elliptic singular perturbations in L^p-spaces, doctoraalscriptie K.U.Nijmegen 1984.

Om een technische stelling bij te sluiten volgt een formulering van de kettingregel.

STELLING 7: *Neem* $\alpha \in \tilde{\mathbb{N}} = \bar{\mathbb{N}}^N\setminus\{0\}$, $\eta \in C^{|\alpha|}(\mathbb{R}^N;\mathbb{R}^M)$ *en* $g \in C^{|\alpha|}(\mathbb{R}^M;\mathbb{R})$, *waarbij* $|\alpha| = \sum_{k=1}^{N} \alpha_k$. *Definieer* $D_x^\alpha = \prod_{k=1}^{N} \left[\frac{\partial}{\partial x_k}\right]^{\alpha_k}$.

Dan zijn er $C(\beta) \in \bar{\mathbb{N}}$ *zodat voor* $x \in \mathbb{R}^N$:

$$D_x^\alpha\, g(\eta(x)) = \sum_{s=1}^{|\alpha|} \left\{ \sum_{\substack{\alpha^1+\alpha^2+\ldots+\alpha^s=\alpha \\ \alpha^j\in\tilde{\mathbb{N}}}} C(\alpha^1,\alpha^2,\ldots,\alpha^s) \sum_{k_1=1}^{M}\sum_{k_2=1}^{M}\ldots\sum_{k_s=1}^{M} \left[\prod_{r=1}^{s}\left[D_x^{\alpha^r}\eta_{k_r}(x)\right]\right]\left[\prod_{r=1}^{s}\frac{\partial}{\partial\eta_{k_r}}\right] g(\eta(x)) \right\}.$$

Een spel met vlakke graphen.

Spelregels:

1) Er bestaan alleen punten en krommen die twee punten verbinden. Een kromme kan een punt in de vorm van een lus met hetzelfde punt verbinden.
2) Krommen snijden of raken elkaar of zichzelf niet, en raken punten alleen aan de einden.
3) In ieder punt komen hooguit drie krommen bijeen.
4) In den beginne zijn er alleen n punten.
5) Om beurten doen de twee deelnemers een zet.
6) Een zet bestaat uit het verbinden van twee (niet noodzakelijk verschillende) punten door een kromme, en het vervolgens in tweeën delen van deze kromme door er een nieuw punt op te plaatsen.
7) De deelnemer die geen zet meer kan doen verliest.

Zoals in ieder eindig spel zonder de mogelijkheid van remise en zonder toevalsgenerator (b.v. een dobbelsteen) bestaat er voor een van beide deelnemers een winnende strategie. Noem de deelnemers A en B en laat A beginnen. Men zou met de volgende stelling zijn voordeel kunnen doen.

STELLING 8: Als $n \in \{1,2,6\}$ *is er een winnende strategie voor* B.
Als $n \in \{3,4,5\}$ *is er een winnende strategie voor* A.

STELLING 9: Ten onrechte wordt er in het analyse-deel van de wiskunde slechts een zeer beperkt gebruik gemaakt van illustrerende figuren.

STELLING 10: Roken moet mogen. Men dient de roker zelfs te verplichten alle door hem (m/v) geproduceerde rook te inhaleren. Als men nu ook de "exhalatie" verbiedt doet men ook recht aan een ander motto: de vervuiler betaalt.

STELLING 11: Herinvoering van paard en wagen voorkomt het ontstaan van files vóór de Coen-tunnel.

STELLING 12: Er is geen laatste