Master Thesis

A Parametric Structural Design Tool (Grasshopper Interface) for Plate Structures

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Contents

Abstract: The thesis presents a parametric design tool for plate structural analysis. The goal of the thesis is to establish a real-time visualized program for structural calculation and to make it parameterized. The tool is based on a visualized drawing program Rhino with Grasshopper plug-in to generate the parametric environment for the plate structural analysis. The solution of plate analysis is computed by membrane analogy. For out-of-plane behavior, such analogy generates the solution of sum of bending moment. Followed by rain-flow analysis, the relation between shear force flows and the structural geometry is presented. And for in-plane behavior, the solution is so-called stress function. With such solution other structural behavior results can be calculated by applied finite difference method.

Keywords: Design Tools, Plates Structure, Membrane Analogy, Rain-flow Analysis, Boundary Conditions, Stress Function.

The basic outline of the thesis:

Chapter 1: Background Introduction
Chapter 2: Theoretical Framework
Chapter3: Out – of – Plane Parametric Design Tool
Chapter4: Out – of – Plane Result Verification
Chapter5: In – Plane Parametric Design Tool
Chapter6: In – Plane Result Verification
Chapter7: Reinforcement Calculation
Chapter8: Conclusion & Recommendation

1. Background Introduction

1.1. Background

In the recent years, accompanied by the development of construction and advanced computer technology, numerous complicate designs in the past now become possible. So far, it is like an explosion of construction with complex geometry. Lots of buildings with fantastic form are coming out. With aid of computer, more insightful analysis can be achieved, and therefore to enhance the quality of the design.

For shell structure, the shape is critical and plays an important role in structure. However, due to the form complexity, the load path and structural behavior is quite inconvenient to understand. Respected to the reason above, in conceptual design stage when determining the shell shape, an insightful visualization of the shell's structural analysis will be beneficial for generating a qualitative design.

A computational tool for analysis is needed. Now the Finite Element Method (FEM) is widely used among the building field. Accurate analysis result can be gained but changing the model, especially the structural shape. However, such model modifying is time consuming and also not insightful. Each time people need to regenerate a new model to compute the result, which is just repeating the work that was done before. Next problem is the results comparison is not that easy. To contrast the value, people need to first subtract the number from the tabulated result which will definitely cost lots of human labor to finish, if the compared data are quite huge.

To overcome these obstacles, a suitable tool needs to apply in the conceptual design phrase. Introducing real-time visualized figure can be easier for intuitive view of comparison. To modify the shape easily, and to become perceptive, parametric method is a solution. By changing the parameters, the relation between different parameters and the structural behavior will be unlocked. Therefore structure evaluation is much more effective. Not only parametric design way helps designer to change the model faster and to understand the structural behavior easily, but also it can be combine with some other optimization method, for example "Genetic Algorithm" and "Evolutionary Optimization". With these helps the whole design will be promoted into a better level. By these reasons, parametric method will become one of the most important ways for building design in the future. And therefore, implementing the parametric design method into plate analysis is a good starting point.

Another consideration is that, in plate or shell structure analysis, people will assume the solution first. Normally, such solutions consist of different forms. The widely used forms are polynomial form and Fourier series. Such formulas of the solutions describe most parts of the structural behavior. But still the solutions that have been found so for can not 100% fulfill the entire structural phenomenon. Searching for the correct solution will be time – consuming. And, of course, each situation has its unique solution, which means that, in each case, people need to repeat the whole process to find the answers.

Fortunately, accompanied with technology improvement, the real – time visualized computational program, like Rhino, occurred. With assist of these programs to generate the NURBS surface, the solution assumption is no longer needed. By the "membrane analogy" the solution for the structural analysis is computed and visualized by the computational programs. Saving time on looking for the solution and the solution that can satisfy all the structure behavior are the incentives for me to step into this field.

This thesis is to develop such parametric structural design tool for the plate design first. The tool can be developed on the visualization program. Hence Rhino and Grasshopper plug-in are chosen as the program environment for the tool. The plate structural calculation plate's out-of-plane behavior with simple support boundary has been defined by Mr. M. Oosterhuis. The main task of this thesis is to develop the Grasshopper program with different boundary conditions, and upgrade the program for in-plane analysis.

1.2. Objective

As what is mentioned above, the former program is only valid in plate supported with edge simple support. Still there are some other kinds of boundary conditions need to be satisfied. This thesis will search a way to introduce different types of boundary conditions into the program.

Since the tool is now merely for plate with out-of-plane load, to develop the tool into another level, the in-plane analysis should be added to the program. Under this circumstance, the tool can reach the goal to analyze plate structure.

Objectives summary:

- Out-of-plane program:

1. Define the theoretical framework in calculating the sum of bending moments with the membrane analogy.
2. In the membrane analogy to compute the sum of bending moment, program will be extended to satisfy different boundary conditions.
3. In the structural behavior calculation component, since the program now is only valid for simple supported edge, the program will be upgraded to fit different boundary condition.
4. Verify the result of produced calculation in a qualitative and quantitative manner (compared with FEM program).

- In-plane program:

1. Define the theoretical framework of in-plane structural behavior, respected with the stress function, and combine with membrane analogy.
2. Under the theory of membrane analogy, define the correct boundary for the membrane simulation.
3. Concerning the usability and functionality, define the parameters that should be introduced into the tool and implementing the theory into computational program (Grasshopper).
4. Generate the calculation component to compute the structure behavior based on the

membrane solution.

5. Verify the result of produced calculation in a qualitative and quantitative manner (compared with FEM program).

In case that the theory and computational program stated above has finished and pass the verification of the result with the FEM computational program, the main goals of the thesis are considered to be reach.

To level up the model application, the reinforcement calculation will be introduced. This goal makes the program more practical. Therefore the program can be used for plate reinforcement design. Then the objective statement is:

- Generate the reinforcement calculation component for plate design.

1.3. Methodology

The research in this field is not very deep. The theory of membrane analogy is still not clear. To discover the relation behind the theory is one of the main tackles. For this purpose, some ways to find the answers are needed. One is to look through the differential equation of plate and slab and compared that with membrane equation to find whether there are connections. Another is based on the previous model, to make some trials and errors to discover the relation.

In former tool the software framework is developed in Rhino and Grasshopper. The thesis will follow this way to improve the program. Inside the Grasshopper, there are some predefined components for generating model. However, Grasshopper is not sophisticated software, the components is not enough. So making own components become necessary. Grasshopper provides two code languages for user to script. One is VB, the other is C#. In the former model, the code language is VB, I will follow this selection.

For the comparison with finite element method calculation, the FEM program I used is TNO Diana. The reason to introduce such comparison into the report is to check the validation of the tool and also the accuracy.

1.4. Scope

The structural type that is considered in this thesis is rectangular plate structure. The reason to choose such type of structure is stated below:

- Rectangle is the basic shape for a wild range of plate structural calculation. Most of the plate theories are based on this shape. By implementing rectangle into the program, the coordinate system does not need to be changed.
- Plate structure is the basic theory for shell structure. Only after the plate theory is defined the theory of shell structure can be generated. Therefore, at the initial stage, plate is the

optimal structure choice for the program.

- The analytical result of rectangular plate structural behavior has been published in numerous articles. Such analytical result is widely agreed, and can be used for validation.

The next announcement of the scope is loading type. Since the program is separated into two parts. One is for the out – of – plane analysis. Another is in – plane analysis.

- For out – of – plane program, the load is added perpendicular to the plane.
- For in – plane program, load will be presented parallel to the plane and acting on the boundary.

Considering the boundary constraints, following statements are made.

- For out – of – plane program, the types of boundary constraints are simple supported, free edges with corner supported and fixed edges.
- For in – plane program, the analysis case is confined to two parallel loaded edges and two parallel edges fixed.

2. Theoretical Framework

This chapter will present the basic theories of the whole thesis. The theoretical framework is the basic of the parametric design program. They describe the relations in between the different parameters of plate structure, like geometry, boundary conditions, load cases, etc. Only by implementing these theories into the design tool, the program is valid. Therefore, to emphasize the framework at the first beginning is at the utmost position.

To define which theories should be used in the tool, the basic analysis processes are stated followed:

- **Basic process for out – of – plane analysis program:**

a. Implement the boundary component to generate the correct boundary value of the membrane for the analogy
b. Combine membrane analogy to indicate the value for sum of bending moment
c. Introduce force density method to form the membrane shape
d. Apply rain flow analogy to give image of principal shear trajectories
e. Together with finite difference method to calculate structural behavior

- **Basic process for in – plane analysis program:**

a. Implement the boundary component to generate the correct boundary value of the membrane for the analogy
b. Combine membrane analogy to indicate the value for stress function
c. Introduce force density method to form the membrane shape
d. Together with finite difference method to calculate structural behavior

According to the basic analysis processed stated above, the following theories should be introduced:

1) Differential equations for thin plates
2) Membrane analogy
3) Force density method
4) Rain flow analogy
5) Finite difference method

2.1. Differential Equations for Thin Plates

The term “plate” is described as a flat structural member for systems in which there is a transfer of forces. Plenty of researches and equations derivations have been present to specify the thin plate mechanic behavior.

There are two main categories for thin plate classification which are plates that are loaded in – plane and out – of – plane. For both cases, differential equations are given initially to state the relations between displacements, strains, stress and loads. These equations are based on mathematical theory.

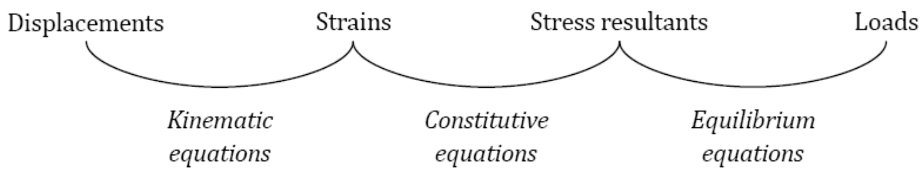


Figure 2.1: Relations of different structural behavior (Picture from J. Blaauwendraad [1])

The relations are present respectively as Kinematic, Constitutive and Equilibrium equation. In this thesis, the scope has been announced previously that the plate is homogeneous isotropic plates. The chapter will be separated into two parts. One is about the theory of in – plane structural behavior, while another is for out – of – plane mechanics.

Before deriving the differential equations of structural mechanics, some assumptions need to be stated before.

1. The material is elastic, homogeneous and isotropic.
2. The Poisson’s ratio is zero.
3. The shape is initially flat.
4. The deflection is small compared with the thickness of the plate.
5. The straight lines which is initially normal to the mid - plane remain straight and normal to the middle surface.
6. The stress normal to the middle plane n_{zz} is small compare with other stress components and therefore it is neglected.

Comprehensive researches give the following equations for thin plate mechanics

2.1.1. In – Plane Mechanics

In flat plate that is loaded in – plane, the state of stress is called plane stress. The stress are parallel the mid – plane. The meanings of stress components and strain components are showed below.

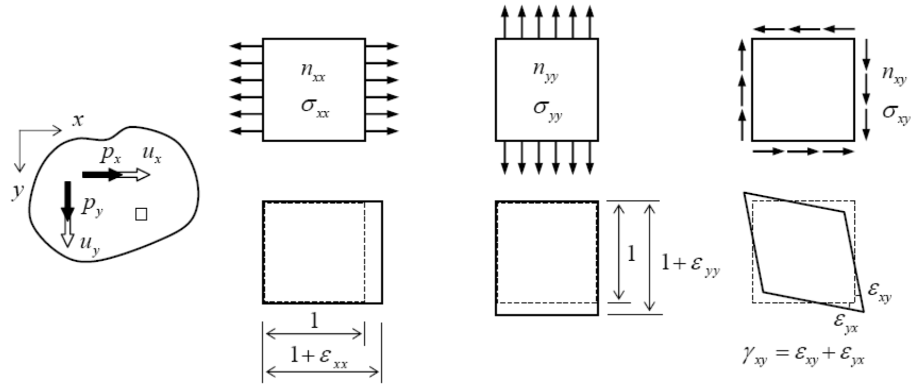


Figure 2.2: Quantities of a plate loaded in – plane (Picture from J. Blaauwendraad [1])

For deriving the basic differential equations, the elementary rectangular unit is set with infinitesimal small dimensions d_x and d_y. The thickness is t.

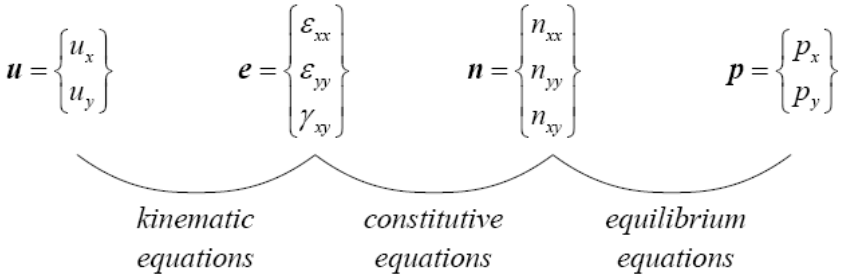


Figure 2.3: Relations between the quantities (Picture from J. Blaauwendraad [1])

There are three series of basic equations to be presented.

Kinematic Equations

The elementary plate will deform after applying a load. The new state can be described by three rigid body displacements. The rigid body displacements are strainless movement and therefore the relation between displacements and deformations are stated by Kinematic Equations

$$\varepsilon_{xx} = \frac{\partial u_x}{\partial x}$$

$$\varepsilon_{yy} = \frac{\partial u_y}{\partial y}$$

$$\gamma_{xy} = \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}$$

There is a relation equation represent the strain compatibility.

$$\frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y} \tag{2.1}$$

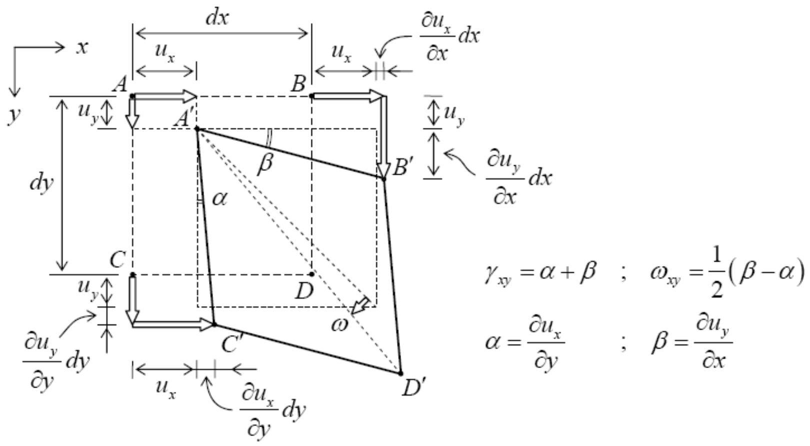


Figure 2.4: Displaced and deformed state of an elementary plate part (Picture from J. Blaauwendraad [1])

Constitutive Equations

The equations declare the Hooke's law in plate mechanics behavior. The relation between the stresses and the strains is provided.

$$\varepsilon_{xx} = \frac{1}{E}\left(\sigma_{xx} - \upsilon\sigma_{yy}\right) = \frac{1}{Et}\left(n_{xx} - \upsilon n_{yy}\right)$$

$$\varepsilon_{yy} = \frac{1}{E}\left(\sigma_{yy} - \upsilon\sigma_{xx}\right) = \frac{1}{Et}\left(n_{yy} - \upsilon n_{xx}\right)$$

$$\gamma_{xy} = \frac{2(1+\upsilon)}{E}\sigma_{xy} = \frac{n_{xy}}{Gt}$$

There is another interesting situation that, by substituted the above equations into strain compatibility equation (2.1) will yields:

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\left(n_{xx} + n_{yy}\right) = 0 \qquad (2.2)$$

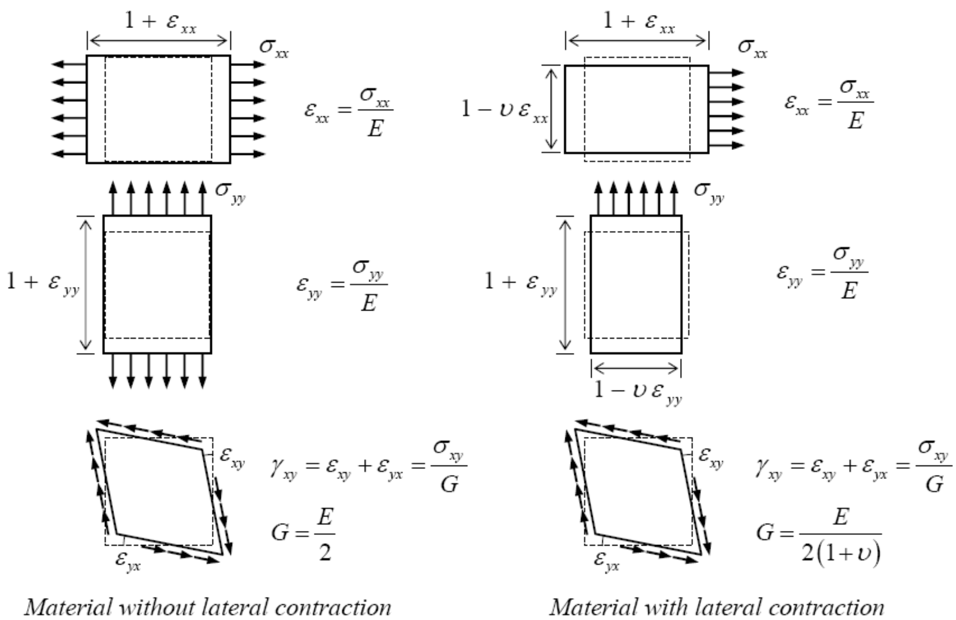


Figure 2.5: Stress and strain relations (Picture from J. Blaauwendraad [1])

Equilibrium Equations

The last equations give the relations between the loads and the stress resultants. The equilibrium equations are expressed as follows:

$$\frac{\partial n_{xx}}{\partial x} + \frac{\partial n_{xy}}{\partial y} = -p_x$$

$$\frac{\partial n_{yy}}{\partial y} + \frac{\partial n_{xy}}{\partial x} = -p_y$$

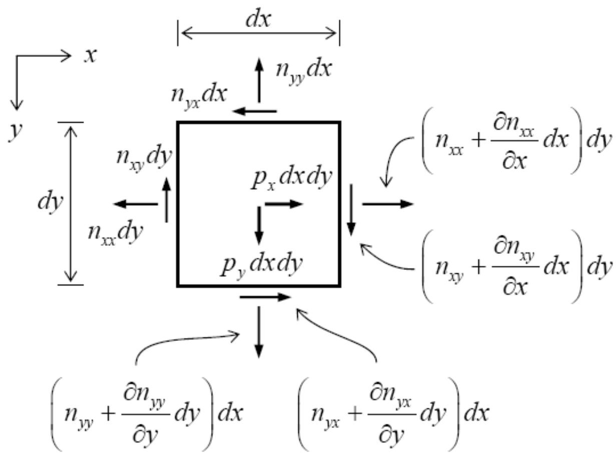


Figure 2.6: Equilibrium of an elementary plate part (Picture from J. Blaauwendraad [1])

Normally the surface load p_x and p_y do not exist, the equilibrium equation will become:

$$\frac{\partial n_{xx}}{\partial x} + \frac{\partial n_{xy}}{\partial y} = 0 \tag{2.3}$$

$$\frac{\partial n_{yy}}{\partial y} + \frac{\partial n_{xy}}{\partial x} = 0 \tag{2.4}$$

The invention of Airy Stress Function is based on this formula. Assume:

$$n_{xx} = \frac{\partial^2 \emptyset}{\partial y^2}$$

$$n_{yy} = \frac{\partial^2 \emptyset}{\partial x^2}$$

$$n_{xy} = -\frac{\partial^2 \emptyset}{\partial x \partial y}$$

The applied stress function automatically satisfies the force equilibrium equations (2.3) and (2.4). By replacing the stress function into strain compatibility equation:

$$\frac{\partial^4 \emptyset}{\partial x^4} + 2\frac{\partial^4 \emptyset}{\partial x^2 \partial y^2} + \frac{\partial^4 \emptyset}{\partial x^4} = \nabla^2\nabla^2\emptyset = 0 \tag{2.5}$$

All the basic equations for in – plane behavior have been determined now.

Another concept is the principal stress:

$$n_1 = \frac{n_{xx} + n_{yy}}{2} + \sqrt{(\frac{n_{xx} - n_{yy}}{2})^2 + {n_{xy}}^2}$$

$$n_2 = \frac{n_{xx} + n_{yy}}{2} - \sqrt{(\frac{n_{xx} - n_{yy}}{2})^2 + {n_{xy}}^2}$$

2.1.2. Out – of – Plane Mechanics

For this part, the derivation of the basic equations for out – of – plane mechanics of plate structure will be stated. The considered situation is the same as the previous chapter, a rectangular plate with thickness t.

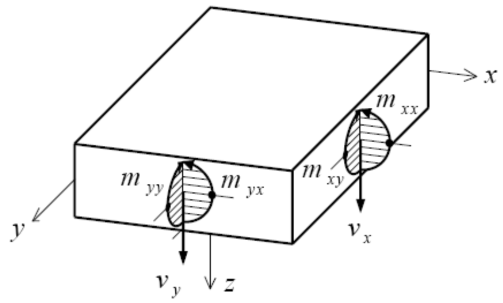


Figure 2.7: Stress quantities in the plate (Picture from J. Blaauwendraad [1])

To define the connectivity equation between different components, the concept of in – plane behavior will be continued. Three equations will be applied to describe the relation.

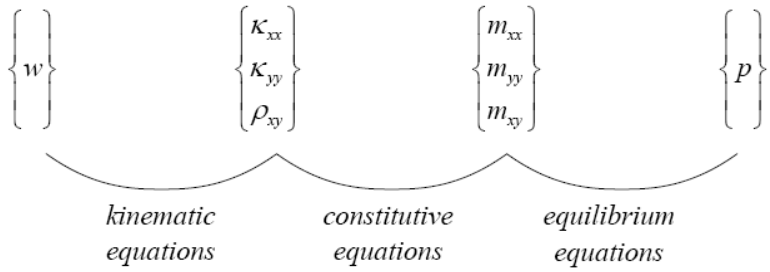


Figure 2.8: Relation scheme (Picture from J. Blaauwendraad [1])

There are three series of basic equations to be presented.

Kinematic Equations

The definition of Kinematic Equations for thin plate that is subjected to the loads acting normal to the mid – plane is as follows:

$$\kappa_{xx} = \frac{\partial \varphi_x}{\partial x} = -\frac{\partial^2 w}{\partial x^2}$$

$$\kappa_{yy} = \frac{\partial \varphi_y}{\partial y} = -\frac{\partial^2 w}{\partial y^2}$$

$$\rho_{xy} = 2\kappa_{xy} = -2\frac{\partial^2 w}{\partial x \partial y}$$

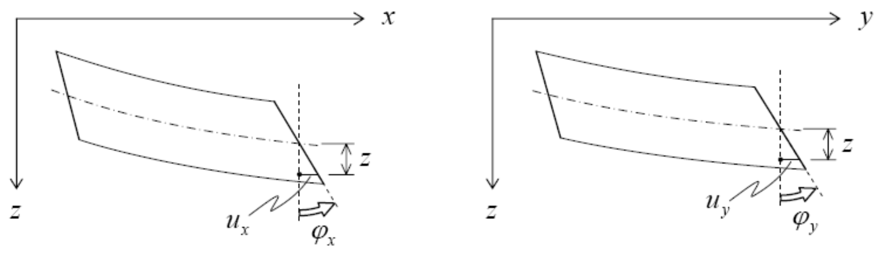


Figure 2.9: Determination of the displacements from the mid – plane (Picture from J. Blaauwendraad [1])

Constitutive Equations

In the theory of thick plate mechanics, the shear strain has to be taken into account. However in thin plate theory, the shear deformation is relatively small. Therefore, in constitutive relations, the shear strain is meaningless and can be neglect.

Then the equations that declared the Hooke's law in plate mechanics behavior are simplified. The relation between the stresses and the strains is provided.

$$m_{xx} = \frac{Et^3}{12(1-\upsilon^2)}\left(\kappa_{xx} + \upsilon\kappa_{yy}\right) = D\left(\kappa_{xx} + \upsilon\kappa_{yy}\right)$$

$$m_{yy} = \frac{Et^3}{12(1-\upsilon^2)}\left(\kappa_{yy} + \upsilon\kappa_{xx}\right) = D\left(\kappa_{yy} + \upsilon\kappa_{xx}\right)$$

$$m_{xy} = \frac{Et^3}{12(1+\upsilon)}\kappa_{xy} = (1-\upsilon)D\kappa_{xy}$$

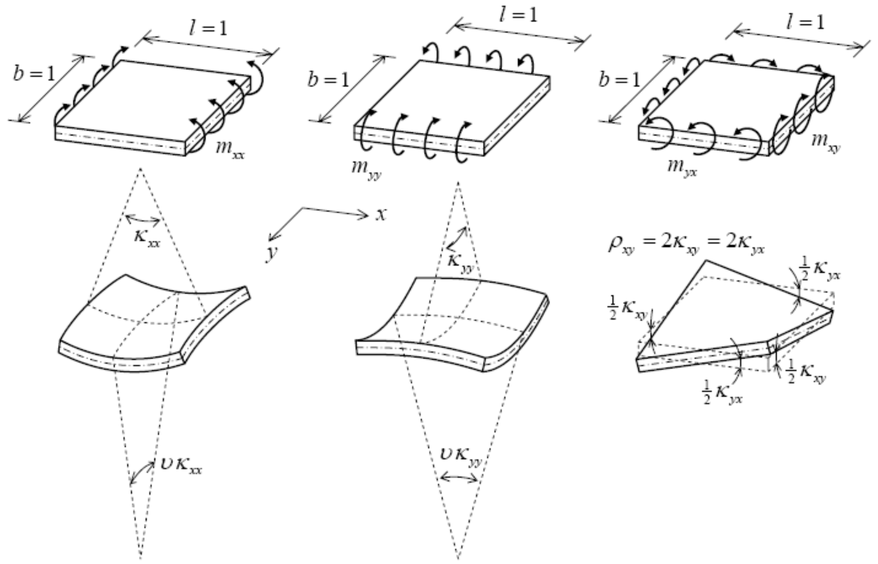


Figure 2.10: Stress resultants and deformation (Picture from J. Blaauwendraad [1])

Equilibrium Equations

In the preceding sections, the kinematic and constitutive relations are found. For equilibrium in w – direction of infinitesimal small unit the load will be stood by shear forces. And not only the equilibrium in vertical direction should be achieved, the moment equilibrium also.

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = -p \tag{2.6}$$

$$\frac{\partial m_{xx}}{\partial x} + \frac{\partial m_{xy}}{\partial y} = v_x \tag{2.7}$$

$$\frac{\partial m_{yy}}{\partial y} + \frac{\partial m_{xy}}{\partial x} = v_y \tag{2.8}$$

Substitution of the equations (2.7) and (2.8) in the first one (2.6) yields:

$$\frac{\partial^2 m_{xx}}{\partial x^2} + 2\frac{\partial^2 m_{xy}}{\partial x \partial y} + \frac{\partial^2 m_{yy}}{\partial y^2} = -p \tag{2.9}$$

By replacing the moment with displacement will lead to:

$$\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial x^4} = \nabla^2 \nabla^2 w = -\frac{p}{D} \tag{2.10}$$

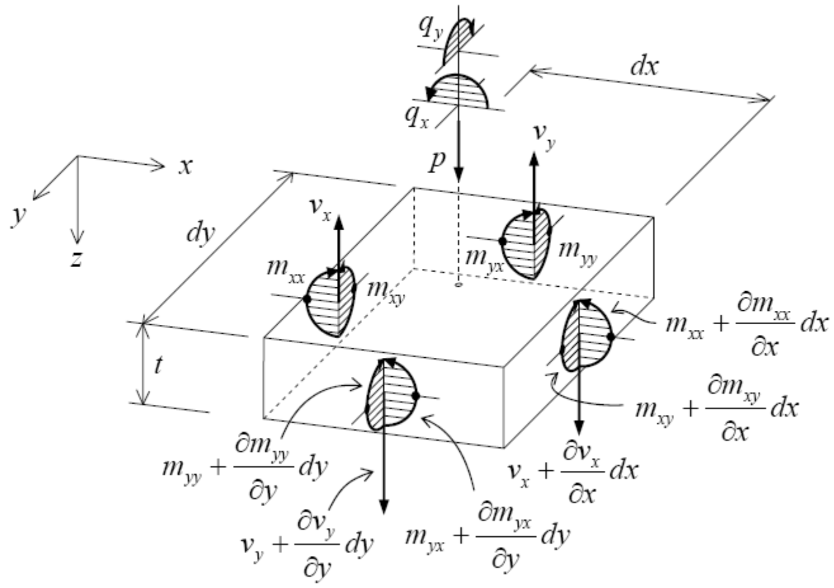


Figure 2.11: Equilibrium of an elementary plate part (Picture from J. Blaauwendraad [1])

All the basic equations for out – of – plane behavior have been determined now.

The principle moments equations are:

$$m_1 = \frac{m_{xx} + m_{yy}}{2} + \sqrt{(\frac{m_{xx} - n_{yy}}{2})^2 + {m_{xy}}^2}$$

$$m_2 = \frac{m_{xx} + m_{yy}}{2} - \sqrt{(\frac{m_{xx} - m_{yy}}{2})^2 + {m_{xy}}^2}$$

2.2. Membrane Analogy

In previous plate structure research, people are focusing on looking for the solution for the analysis. Lots of trials and errors are made. During the solution searching, the polynomial form and Fourier series are used quite frequently. However, no matter polynomial form or Fourier series, still no solution is found that can fully satisfy all the structural conditions.

In this thesis, another way is utilized which is call membrane analogy. Such method is invented by pioneering aerodynamicist *L. Prandtl* in 1903, also known as the soap-film analogy. By integrating membrane analogy to plate structure, the solution that can fulfill all the structural conditions will be generated.

In introducing the membrane analogy, this part will be divided into two parts, because in different type of structural mechanics, the metaphor has different meaning. Then, in the following, this part will be separated into out – of – plane and in – plane behavior.

2.2.1. Membrane Mechanics

To facilitate the understanding of the logic behind the membrane analogy, the knowledge behind the membrane structure need to be introduced.

Before showing the equations of membrane structure, some assumptions have to be announced.

- The deflection of elastic membrane structure is assumed to be very small.
- Due to the stiffness of membrane is very small, therefore, there is no out – of – plane moments and shear occurred. Either the in – plane shear N_{xy}.

Following figure shows the mechanics system of in – plane structure. According to the assumptions, $N_{xy} = 0$.

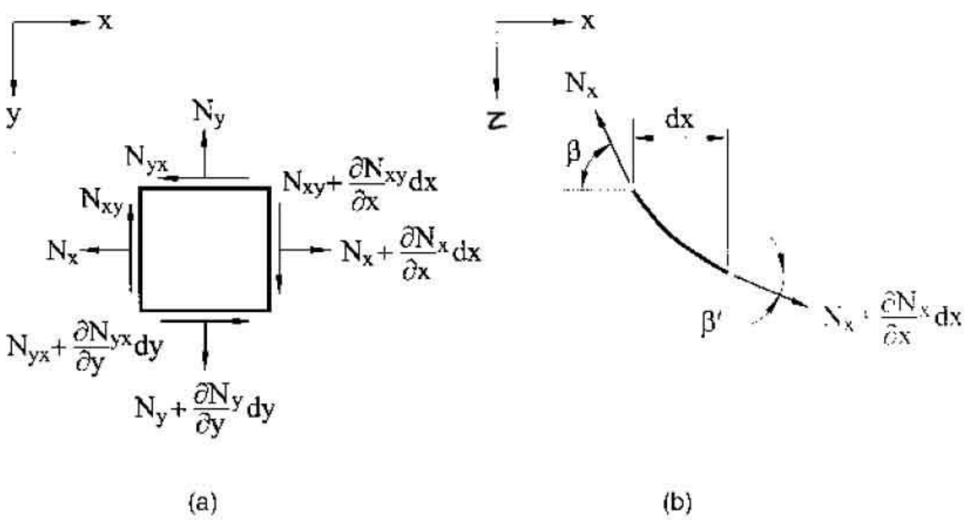


Figure 2.12: Equilibrium of an elementary membrane unit (Picture from E. Ventsel [4])

Considering the projection of x – direction forces on z axis, the z component of the force in x – direction is:

$$-N_x \cdot dy \cdot sin\beta + \left(N_x + \frac{\partial N_x}{\partial x} \cdot dx\right) \cdot dy \cdot sin\beta' \tag{2.11}$$

Due to the deflection is assumed to be very small.

$$sin\beta \approx \beta = \frac{\partial w_m}{\partial x}$$

And

$$sin\beta' \approx \beta' = \beta + \frac{\partial \beta}{\partial x} dx = \frac{\partial w_m}{\partial x} + \frac{\partial^2 w_m}{\partial x^2} dx \tag{2.12}$$

Substituting the equation (2.12) into the x – direction component (2.11):

$$-N_x \cdot dy \cdot \frac{\partial w_m}{\partial x} + \left(N_x + \frac{\partial N_x}{\partial x} \cdot dx\right) \cdot dy \cdot \left(\frac{\partial w_m}{\partial x} + \frac{\partial^2 w_m}{\partial x^2} dx\right)$$

Neglecting the higher order terms leads to:

$$N_x \frac{\partial^2 w_m}{\partial x^2} dxdy + \frac{\partial N_x}{\partial x} \frac{\partial w_m}{\partial x} dxdy \tag{2.13}$$

Since in the x – direction the equilibrium equation leads to:

$$\frac{\partial N_x}{\partial x} = 0$$

Then the above component (2.13) is:

$$N_x \frac{\partial^2 w_m}{\partial x^2} dxdy$$

In y – direction the formula form is the same only by replacing the x with y.

$$N_y \frac{\partial^2 w_m}{\partial y^2} dxdy$$

Then in the vertical direction the equilibrium equation is:

$$N_x \frac{\partial^2 w_m}{\partial x^2} dxdy + N_y \frac{\partial^2 w_m}{\partial y^2} dxdy + pdxdy = 0$$

If assume $N_x = N_y = T$, then:

$$-p = T \frac{\partial^2 w_m}{\partial x^2} + T \frac{\partial^2 w_m}{\partial y^2}$$

$$-\frac{p}{T} = \frac{\partial^2 w_m}{\partial x^2} + \frac{\partial^2 w_m}{\partial y^2} \tag{2.14}$$

In the soap – film theory, *T* is equal to 1.

2.2.2. Out – of – Plane Behavior

The out – of – plane tool is based on this membrane analogy method to indicate the sum of bending moment (the summation of m_{xx} and m_{yy}) in the plate structure.

To fully describe the structural behavior of plate or shell, the normal force, shear and moment should be computed. The moment is expressed by second order differential equation of plate deflection and the shear is by third order differential equation. Hence the plate deflection should be known beforehand. However to determine the solution for deflection is difficult, therefore the introduction of membrane analogy become necessary. Applying this analogy, the solution of sum of bending moment is obtained, further by using finite - difference method the shear and deflection can be derived. That is the reason why the membrane analogy is presented.

Here, it comes to the theory of plate structure. To assume the function *m* (sum of bending moment) as followed:

$$m = \frac{m_{xx} + m_{yy}}{1 + \upsilon}$$

With

$$m_{xx} = -D(\frac{\partial^2 w}{\partial x^2} + \upsilon \frac{\partial^2 w}{\partial y^2})$$

$$m_{yy} = -D(\frac{\partial^2 w}{\partial y^2} + \upsilon \frac{\partial^2 w}{\partial x^2})$$

Therefore the sum of bending moment *m* is:

$$m = \frac{m_{xx} + m_{yy}}{1 + \upsilon} = -D(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2})$$

The equilibrium equation of plate (2.10) is:

$$p = -D\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right) = \frac{\partial^2 m}{\partial x^2} + \frac{\partial^2 m}{\partial y^2}$$

Compared to the equilibrium equation of membrane (2.14):

$$p = -T(\frac{\partial^2 w_m}{\partial x^2} + \frac{\partial^2 w_m}{\partial y^2})$$

Here, the w_m is the deflection of membrane. And with the hypothesis that $T = 1$ N/m^2, the formula can be rewritten as:

$$-p = \frac{\partial^2 w_m}{\partial x^2} + \frac{\partial^2 w_m}{\partial y^2} \tag{2.15}$$

The expressions of the equilibrium formulas for plate and membrane have the same form and same general solution.

$$-p = \frac{\partial^2 w_m}{\partial x^2} + \frac{\partial^2 w_m}{\partial y^2} = -(\frac{\partial^2 m}{\partial x^2} + \frac{\partial^2 m}{\partial y^2})$$

$$w_m \sim -m$$

For this reason, the value of sum of bending moment in plate can be achieved by directly determining the displacement of an elastic membrane with a support layout similar to that of the original plate.

2.2.3. In – Plane Behavior

Compared to the out – of – plane tool, the in – plane program apply membrane analogy method to indicate the sum of normal force (the summation of n_{xx} and n_{yy}). In the out – of – plane mechanics, the normal force, shear and moment are presented by the plate displacement *w*. But in the in – plane behavior, features are linked to stress function *Φ*.

Here, the article assume n as sum of normal force:

$$n = n_{xx} + n_{yy}$$

With

$$n_{xx} = \frac{\partial^2 \emptyset}{\partial y^2}$$

$$n_{yy} = \frac{\partial^2 \emptyset}{\partial x^2}$$

Then the sum of normal force becomes:

$$n = n_{xx} + n_{yy} = \frac{\partial^2 \emptyset}{\partial x^2} + \frac{\partial^2 \emptyset}{\partial y^2}$$

Because the stress function is computed base on equilibrium function, so at this place, the compatibility equation (2.1) for strain is applied.

The equation is:

$$\frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2} - \frac{\partial^2 \gamma_{xy}}{\partial x \partial y} = 0$$

Rewrite the above equation in terms of the stress components:

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\left(n_{xx} + n_{yy}\right) = \frac{\partial^2 n}{\partial x^2} + \frac{\partial^2 n}{\partial y^2} = 0$$

Then again, compared with the equilibrium equation of membrane (2.15) (assume *T = 1* N/m^2):

$$-p = \frac{\partial^2 w_m}{\partial x^2} + \frac{\partial^2 w_m}{\partial y^2}$$

For this analogy, the membrane load *p* is set to be 0. It means that the membrane has no load adding on it.

$$\frac{\partial^2 w_m}{\partial x^2} + \frac{\partial^2 w_m}{\partial y^2} \sim \frac{\partial^2 n}{\partial x^2} + \frac{\partial^2 n}{\partial y^2}$$

$$w_m \sim n$$

After generating the correct boundary value, the membrane can simulate the figure of “n – hill”. Then the stress function can be achieved by finite difference method base on this equation:

$$n = \frac{\partial^2 \emptyset}{\partial x^2} + \frac{\partial^2 \emptyset}{\partial y^2}$$

At this point the solution for in – plane structural behavior is obtained.

2.3. Force Density Method

In the visualization environment, form finding is the main problem. Rhino is a computational graphic program. The combination of drawing with structure calculation is the critical problem to solve. If the calculation result cannot be presented by the drawing program, it will be meaningless for the integration of drawing tool and mechanic calculation.

In the research of membrane structure, form finding is always the major topic. From the physical models like soap bubbles, hanging fabric and air inflated membranes to energy method such as the least complementary energy and variation principle; lots of approaches have been developed. The studies and experiments of these approaches give a broader understanding about membrane form finding. However there is no method perfect. Each one has its own advantages and limitation.

The background of the thesis is based on a membrane will small deflection. The equilibrium of the structure will form the geometry. Furthermore, because of programming environment, linear calculation is a better choice. By compared different form finding methods, the energy ways need complex process of iterative. Such energy methods require a number of calculation steps until the equilibrium shape is found, which will occupy plenty of CPU capacity. It is not an efficient way.

The Force Density Method is the decision. In the field of network computation, Force Density Method is a solution to determine the shape. The concept is based upon the "force – length ratios" which is also name as "force density". To description of equilibrium, this concept is quite suitable. Only by transforming the membrane into equivalent network system, the force density method can solve the equilibrium equation of membrane.

Arguments have been draw that Force Density Method is a linear analysis approach. Linear computation process has been proved that it is fast and sometimes the process can be reversed. Comprehensive sources show that the theory of FDM fits the practical result of membrane shape with small deflection assumption. And the most important issue is FDM can be easily to be adapted to the Grasshopper parametric environment.

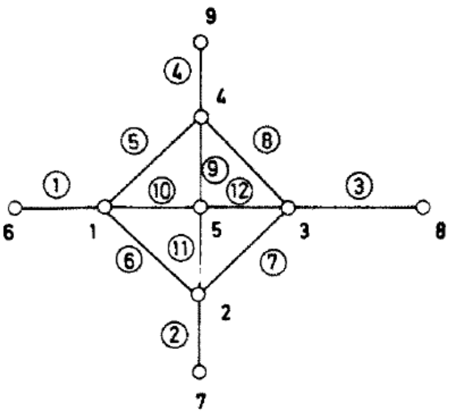
Next part will introduce the mathematical theory behind Force Density Method. The content is based on the article of H. J. Schek in 1973. It is proposed to use the method for Olympic Stadium in Munich for determining the membrane form.

First of all the membrane is transformed into a discrete cable network system which is consist of a series of nodes and lines.
The shape of the network is described by the node and the connected branches. Therefore, a matrix to for this topological description is necessary. It starts with the graph of mesh. This matrix presents the connectivity between the nodes and branches. The nodes are separated into free nodes and fixed nodes. The number of free points is n, and for fixed points it is n_f. So in total the

number of all the nodes is $n_s = n + n_f$. The usual branch – node matrix $\boldsymbol{C_s}$ is defined by

$$\boldsymbol{C_s} = \begin{cases} +1 & for\ i(j) = 1 \\ -1 & for\ k(j) = 1 \\ 0 & other\ cases \end{cases}$$

It has been classified into free and fixed nodes by matrices **C** and $\mathbf{C_f}$.



	1	2	3	4	5	6	7	8	9
1	1	.	.	.	.	−1	.	.	.
2	.	1	.	.	.	.	−1	.	.
3	.	.	1	.	.	.	.	−1	.
4	.	.	.	1	.	.	.	.	−1
5	1	.	.	−1	.	.	.	.	.
6	1	−1	.	.	.	.	.	.	.
7	.	1	−1	.	.	.	.	.	.
8	.	.	1	−1	.	.	.	.	.
9	.	.	.	1	−1	.	.	.	.
10	1	.	.	.	−1	.	.	.	.
11	.	1	.	.	−1	.	.	.	.
12	.	.	1	.	−1	.	.	.	.
	C					C_f			

Figure 2.13: Graph and branch – node matrix (Picture from H. Schek [9])

The next step is to state the structure of equilibrium formula with Force Density Method.
The free nodes are interpreted as points $\boldsymbol{P_i}$ with coordinates (x_i, y_i, z_i), $I = 1, \ldots, n$, and the boundary nodes are as $\boldsymbol{P_{fi}}$ with coordinates (x_{fi}, y_{fi}, z_{fi}), $I = 1, \ldots, n$.
The coordinates of all the free nodes form the $\boldsymbol{n}$ – vector $\boldsymbol{x, y, z}$ and the $\boldsymbol{n_{fi}}$ – vector $\boldsymbol{x_{fi}, y_{fi}, z_{fi}}$ for all the boundary nodes.

The coordinated difference $\boldsymbol{u, v, w}$ of the mesh node is

$$\boldsymbol{u = C \cdot x + C_s \cdot x_f}$$
$$\boldsymbol{v = C \cdot y + C_s \cdot y_f}$$
$$\boldsymbol{w = C \cdot z + C_s \cdot z_f}$$

With the diagonal transformation, $\boldsymbol{U, V, W}$ and **L** referring to $\boldsymbol{u, v, w}$ and $\boldsymbol{l}$, the equilibrium follow

$$\boldsymbol{C^t U L^{-1} s = p_x}$$
$$\boldsymbol{C^t V L^{-1} s = p_y}$$
$$\boldsymbol{C^t W L^{-1} s = p_z}$$

Here we have used the obvious representation for the Jacobian matrices

$$\frac{\partial \boldsymbol{l}}{\partial \boldsymbol{x}} = \boldsymbol{C^t U L^{-1}}$$
$$\frac{\partial \boldsymbol{l}}{\partial \boldsymbol{y}} = \boldsymbol{C^t V L^{-1}}$$
$$\frac{\partial \boldsymbol{l}}{\partial \boldsymbol{z}} = \boldsymbol{C^t W L^{-1}}$$

Then the definition of force density is

$$\boldsymbol{q = L^{-1} s}$$

With the rewritten symbols,

$$\boldsymbol{C^t U q = p_x}$$

$$\boldsymbol{C^tVq = p_y}$$
$$\boldsymbol{C^tWq = p_z}$$

With the identities

$$\boldsymbol{Uq = Qu}$$
$$\boldsymbol{Vq = Qv}$$
$$\boldsymbol{Wq = Qw}$$

And the equilibrium equations is

$$\boldsymbol{C^tQCx + C^tQC_fx_f = p_x}$$
$$\boldsymbol{C^tQCy + C^tQC_fy_f = p_y}$$
$$\boldsymbol{C^tQCz + C^tQC_fz_f = p_z}$$

For simplicity

$$\boldsymbol{D = C^tQC}$$
$$\boldsymbol{D_f = C^tQC_f}$$

The equilibrium equations will have the form

$$\boldsymbol{Dx = p_x - D_fx_f}$$
$$\boldsymbol{Dy = p_y - D_fy_f}$$
$$\boldsymbol{Dz = p_z - D_fz_f}$$

The purpose of making use of Force Density Method is to get the new coordinates of nodes under load equilibrium. Then

$$\boldsymbol{x = D^{-1}(p_x - D_fx_f)}$$
$$\boldsymbol{y = D^{-1}(p_y - D_fy_f)}$$
$$\boldsymbol{z = D^{-1}(p_z - D_fz_f)}$$

In the parametric program the linear equilibrium equations will be solved and therefore the new coordinates of free points are achieved. In the article of *H. Scheck* (1973), there are two examples are showed by means of Force Density Method.

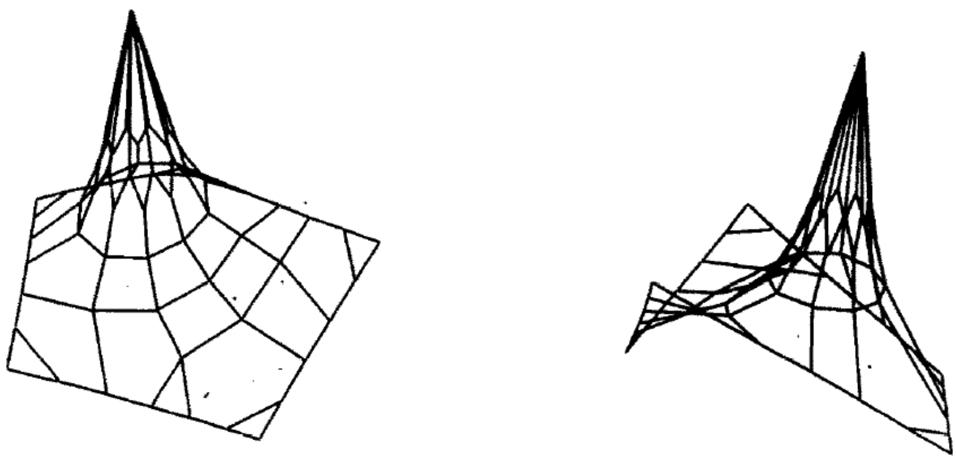


Figure 2.14: Examples views (Picture from H. Schek [9])

2.4. Rain - Flow Analogy

The idea of Rain – Flow Analogy is inspired by *Beranek* (1976). The analytical method is making use of water stream lines which fall on a curved surface. The Rain - Flow Analogy is also named as "Rain Shower Analogy", which is a method of simulation for principal shear trajectories. The phenomenon of water stream lines indicates the load path for out – of – plane structural mechanic behavior. To assume that function of m can be regarded as a "hill". The gradient of the "hill" represent the value of shear.

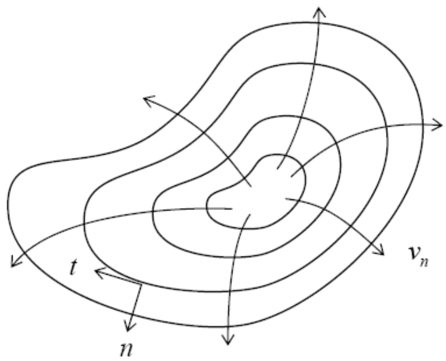


Figure 2.15: Rain – flow analogy (Picture from J. Blaauwendraad [1])

- Trajectories of Principle Shear Force

In determining the trajectories, one can visualize the uniform load of the structure as water drops falling down on the "m – hill" surface. Such shape is defined by the value of bending moment summation. The highnesses of the hill represent the value of corresponded points. Following in the shape of the "hill" the drops will flow downward and generate stream lines. These lines follow the steepest decent direction and run to the edge. Such streams' directions coincide to the principal shear orientation. Using the rain flow analogy, the indication of principal shear trajectories can be obtained. Also from the shear trajectories, the load path of the plate is known. There are some characters of the

- Magnitude of Principle Shear Force

The magnitude of principle shear forces is another topic. Not only the directions of the principle shear are important, but also the value of those. The magnitude can be achieved by integrating the associated load flows between the stream lines. But according to the theory behind Rain – Flow Analogy, the calculation is much simpler, since the gradient of the "hill" equal to the value of principle shear forces.

For example the formula of sum of bending moment is:

$$m = \frac{m_{xx} + m_{yy}}{1 + v} = -D(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2})w$$

The shear in x – direction is:

$$v_x = \frac{\partial m_{xx}}{\partial x} + \frac{\partial m_{xy}}{\partial y} = -D\frac{\partial}{\partial x}\left(\frac{\partial^2 w}{\partial x^2} + \upsilon\frac{\partial^2 w}{\partial y^2}\right) - D(1-\upsilon)\frac{\partial}{\partial y}\frac{\partial^2 w}{\partial x \partial y} = -D\frac{\partial}{\partial x}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)w$$
$$= \frac{\partial}{\partial x}(m)$$

The shear in y – direction is:

$$v_y = \frac{\partial m_{yy}}{\partial y} + \frac{\partial m_{xy}}{\partial x} = -D\frac{\partial}{\partial y}\left(\frac{\partial^2 w}{\partial y^2} + \upsilon\frac{\partial^2 w}{\partial x^2}\right) - D(1-\upsilon)\frac{\partial}{\partial x}\frac{\partial^2 w}{\partial x \partial y} = -D\frac{\partial}{\partial y}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)w$$
$$= \frac{\partial}{\partial y}(m)$$

The equations above show that the gradient of "m – hill" equal to the shear value in that direction. From figure below, it shows two triangular plate parts with shear forces acting on the edges. Base on the plate equilibrium, the v_n and v_t follows these formulas.

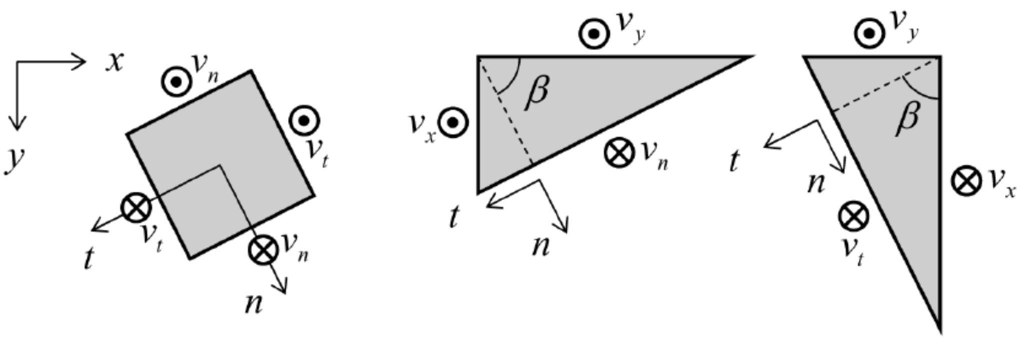


Figure 2.16: Shear of an elementary plate (Picture from J. Blaauwendraad [1])

$$v_n = v_x \cdot cos\beta + v_y \cdot sin\beta$$
$$v_t = -v_x \cdot sin\beta + v_y \cdot cos\beta$$

To determine the maximal value of v_n, it leads to

$$\frac{\partial v_n}{\partial \beta} = 0$$

Then it requires:

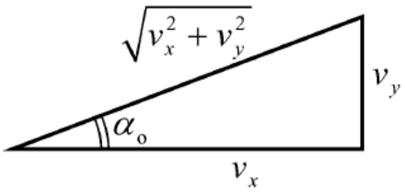
$$-v_x \cdot sin\beta + v_y \cdot cos\beta = 0$$

Therefore:

$$tan\beta = \frac{v_y}{v_x}$$

Following the graph below, it set that the angle *β* is α_0. Therefore the maximal shear force becomes

$$v_0 = \sqrt{v_x^2 + v_y^2}$$



$$\tan\alpha_o = \frac{v_y}{v_x}, \qquad \sin\alpha_o = \frac{v_y}{\sqrt{v_x^2 + v_y^2}}, \qquad \cos\alpha_o = \frac{v_x}{\sqrt{v_x^2 + v_y^2}}$$

Figure 2.17: Relations of shear forces (Picture from J. Blaauwendraad [1])

According to the derivation, the maximal shear force is perpendicular to the minimal one. And the minimal shear force is equal to 0.

It has been stated that the gradient value of the surface is equal to the shear in corresponding direction. In the curved surface geometry, the slope of the contour lines of surface is zero, and the steepest direction is perpendicular to the contour lines. Such phenomenon aligns with the principal shear forces. In structural mechanics the principal shear trajectories is perpendicular to the minimal one, and the value of minimal one is zero.

Therefore, when the theory comes to the Rain – Flow Analogy, the principal shear trajectories are in the direction of the slope, and perpendicular to the contour lines of the "m – hill". In these directions of streams the shears reach to their maximum value. According to the theory of principal shear, in perpendicular to the minimal shear orientation which is also the contour lines' direction, the shear is equal to zero.

The principal shear expression is:

$$v_n = \frac{\partial}{\partial n}(m)$$

Here n is the direction of principal shear.

2.5. Finite Difference Method

The process to calculate the structural behavior of plate is base on partial differential equation. To achieve this goal in the visualization environment, the Finite Difference Method is a solution. In mathematics, the Finite Difference Method (FDM) is numerical method to approximate derivatives in the partial differential equation by linear combinations of function values at the grid points. People can find that such method is widely used in Finite Element Method (FEM).

In both methods, FDM and FEM, a series of equations generate the matrixes for calculation. FEM will assemble the stiffness matrix which is quite huge and highly complicated. Unlike FEM, the FDM skip this step and can be used for calculation effectively. This is the main reason that this thesis presents FDM instead of FEM, where FDM provide a more straightforward formulation of the answer. It helps to make the program easier for user to understand the parametric application and get familiar with the program.

To use Finite Difference Method to attempt to solve the partial differential equation, the domain needs to be discretized by dividing into grid. If the step size of the grid is chosen appropriately, the error by applied FDM to approximate the exact analytical solution will become small and acceptable. Within the FDM, a grid is implemented over the plate. The analysis process is based on the value that is distributed to each grid point. By using those values and the means of finite difference operators, the derivatives of the partial differential equations will be replaced. Due to the convenience of this computation process with FDM, the visualized environment computation program is realized.

The next paragraphs will introduce the Finite Difference Operators and the boundary conditions of plate structure.

2.5.1. Finite Difference Operators

The case to be discussed is a square mesh grid with equal space of h_x and h_y. The grid is based on Cartesian coordinates.

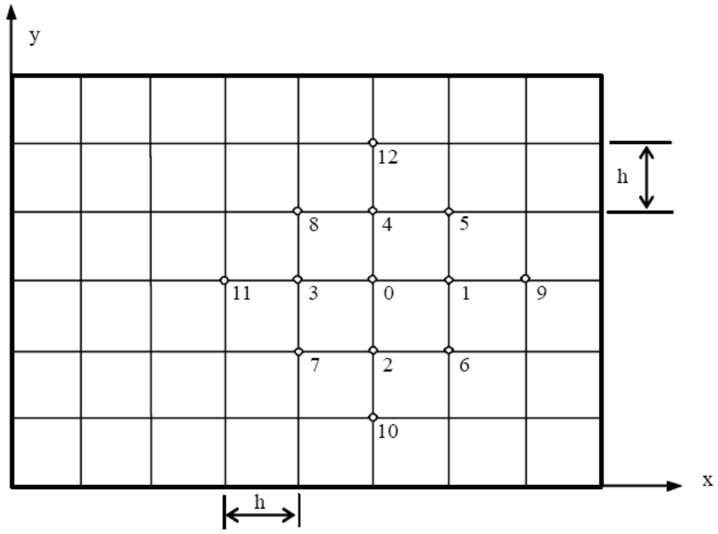


Figure 2.18: Grid for finite difference method

Considered a continuous function *f(x)*, and it is known that Taylor series expansion of the function at point x_0 is defined as follow.

$$f = f_0 + \left(\frac{\partial f}{\partial x}\right)(x - x_0) + \frac{1}{2!}\left(\frac{\partial^2 f}{\partial x^2}\right)(x - x_0)^2 + \frac{1}{3!}\left(\frac{\partial^3 f}{\partial x^3}\right)(x - x_0)^3 + \cdots \qquad (2.16)$$

Base on above equation (2.16), the value of point x_3 and x_1 can be computed.

$$f_3 = f_0 - \left(\frac{\partial f}{\partial x}\right)h + \frac{1}{2!}\left(\frac{\partial^2 f}{\partial x^2}\right)h^2 - \frac{1}{3!}\left(\frac{\partial^3 f}{\partial x^3}\right)h^3 + \cdots$$

$$f_1 = f_0 + \left(\frac{\partial f}{\partial x}\right)h + \frac{1}{2!}\left(\frac{\partial^2 f}{\partial x^2}\right)h^2 + \frac{1}{3!}\left(\frac{\partial^3 f}{\partial x^3}\right)h^3 + \cdots$$

Assume that the distance increment *h* is small enough. The third order differential terms can be neglected. It leads to:

$$f_3 = f_0 - \left(\frac{\partial f}{\partial x}\right)h + \frac{1}{2!}\left(\frac{\partial^2 f}{\partial x^2}\right)h^2 \qquad (2.17)$$

$$f_1 = f_0 + \left(\frac{\partial f}{\partial x}\right)h + \frac{1}{2!}\left(\frac{\partial^2 f}{\partial x^2}\right)h^2 \qquad (2.18)$$

By solving the equations above (2.17) and (2.18):

$$\frac{\partial f}{\partial x} = \frac{f_1 - f_3}{2h}$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{f_1 - 2f_0 + f_3}{h^2}$$

With same method:

$$\frac{\partial f}{\partial y} = \frac{f_2 - f_4}{2h}$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{f_2 - 2f_0 + f_4}{h^2}$$

And:

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{(f_6 + f_8) - (f_5 + f_7)}{4h^2}$$

The expressions are also referred to Finite Difference operators. And by implementing these operators, the third and forth order differential derivatives of the function *f(x, y)* can be derived.

$$\left(\frac{\partial}{\partial x}\right)_0 = \frac{1}{2h}\cdot \begin{bmatrix} 1 & \boxed{0} & -1 \end{bmatrix} \qquad \left(\frac{\partial^2}{\partial x^2}\right)_0 = \frac{1}{h^2}\cdot \begin{bmatrix} 1 & \boxed{-2} & 1 \end{bmatrix}$$

$$\left(\frac{\partial}{\partial y}\right)_0 = \frac{1}{2h}\cdot \begin{bmatrix} 1 \\ \boxed{0} \\ -1 \end{bmatrix} \qquad \left(\frac{\partial^2}{\partial y^2}\right)_0 = \frac{1}{h^2}\cdot \begin{bmatrix} 1 \\ \boxed{-2} \\ 1 \end{bmatrix}$$

$$\left(\frac{\partial^2}{\partial x \partial y}\right)_0 = \frac{1}{4h^2}\cdot \begin{bmatrix} -1 & 0 & 1 \\ 0 & \boxed{0} & 0 \\ 1 & 0 & -1 \end{bmatrix}$$

Figure 2.19: Finite difference operators

Of course the FDM is not only valid in Cartesian coordinates. Polar or other types of coordinates can be conveniently applied by the transformation of the corresponding equations relating the x and y coordinates to the set of coordinates and coefficient patterns.

2.5.2. Boundary Condition

For those points line on the edge or next to it, part of the operator points fall outside the grid of the plate mesh. Some ways have to be introduced in the boundary point calculation.

For out – of – plane program:

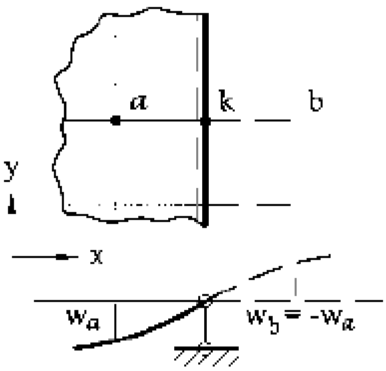


Figure 2.20: Simple support (Picture from E. Ventsel [4])

(a) The simply supported edge

According to the boundary conditions of simply supported edge:

$$w_k = 0$$

$$M_{xx} = \frac{\partial^2 w}{\partial x^2} = \frac{w_a - 2w_k + w_b}{h^2} = 0$$

This will lead to

$$w_k = 0$$
$$w_b = -w_a$$

(b) The fixed edge

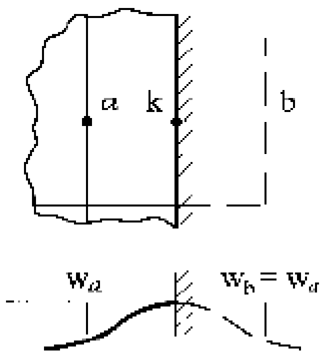


Figure 2.21: Fixed edge (Picture from E. Ventsel [4])

According to the boundary conditions of fixed edge:

$$w_k = 0$$

$$\frac{\partial w}{\partial x} = \frac{w_b - w_a}{2h} = 0$$

This will lead to

$$w_k = 0$$
$$w_b = w_a$$

(c) The free edge

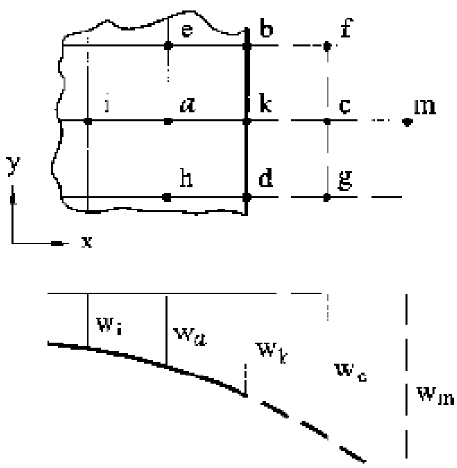


Figure 2.22: Free edge (Picture from E. Ventsel [4])

According to the boundary conditions of fixed edge:

$$M_{xx} = \frac{\partial^2 w}{\partial x^2} = \frac{w_a - 2w_k + w_c}{h^2} = 0$$

$$V_x = 0$$

For in – plane program:

The situation is more complex than the out – of – plane program. For the previous program the function is vertical displacement. The physical meaning of that is quite clear. Therefore the boundary conditions are easier to be adapted to the program. However, in the in – plane tool, the function is stress function. Still in recent research, the physical meaning is in the mist. Not like the out – of – plane program, some of the boundary value cannot be set directly. The stress function cannot be introduced in the same way. So in the calculation process, the outer points are set.

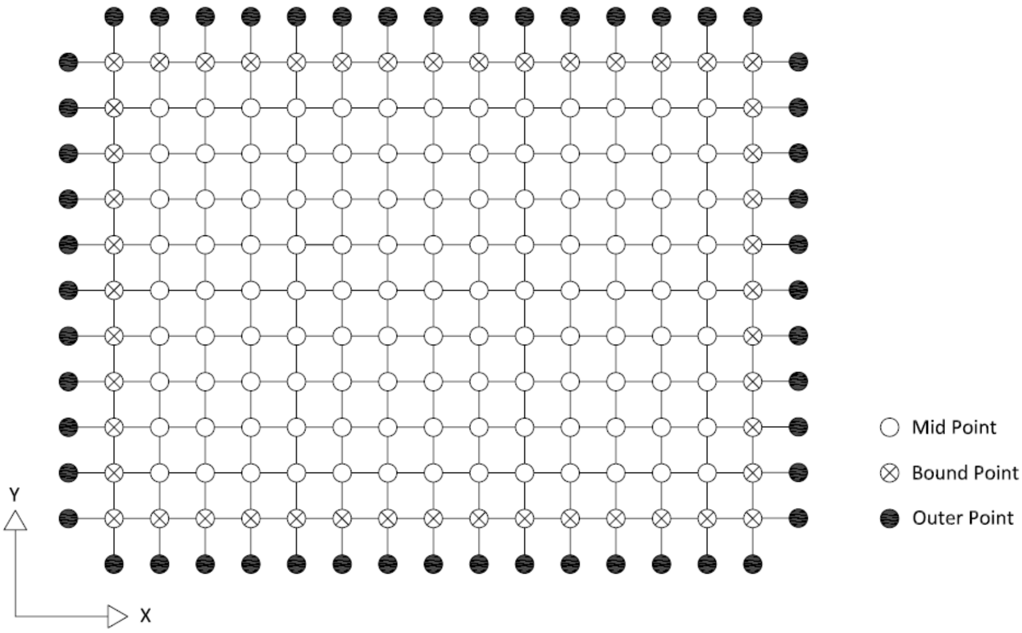


Figure 2.23: Points classification

2.5.3. Linear Equations Calculation

The partial differential equation is computed by finite difference method and transformed into linear equations. Under this circumstance, some calculation processes can be reversed.

For example with membrane analogy, the equilibrium equation is:

$$-p = \frac{\partial^2 w_m}{\partial x^2} + \frac{\partial^2 w_m}{\partial y^2}$$

By rewrite the above formula with FDM manner:

$$\boldsymbol{P} = \boldsymbol{A} \times \boldsymbol{W}$$

Since the boundary points of the membrane are set beforehand, the displacement of those points are pre – defined. Only the middle free points are unknown. Therefore the membrane displacements of free points share the same dimension of membrane load. Then the FDM matrix ***A*** is a square matrix. The formula can be rewrite as:

$$\boldsymbol{W} = \boldsymbol{A}^{-1} \times \boldsymbol{P}$$

It means that the FDM is not a one – way computational process. The method can be reversed. Of course the square matrix should not be singular and with full rank. Otherwise the equations are not linear. The process presented above cannot be achieved.

2.6. Theory Application

2.6.1. Introduction

The description of the theory framework is the main content of the previous chapters. The theories are presented separately. To realize the goals of parametric program, the theories will be combined. These methods will build up the whole structural calculation process. For this reason, the relation between the theories should be announced, and also the structure of these combinations.

The upcoming part will express the application of theory for different purposes. Since the thesis is separated by two parts, the out – of – plane program and the in – plane program, this section will elaborate the application with the same order.

2.6.2. Out – of – Plane Program

The results for structural evaluation of the program are as follows:
- Shear force
- Principle shear trajectories
- Deformation
- Bending moment
- Torsion
- Boundary reaction

To realize all these goals the combinations will be:
(1) Membrane Analogy + Finite Difference Method + Force Density + Rain Flow Analogy
(2) Membrane Analogy + Finite Difference Method + Force Density + Finite Difference Method

The combinations share the membrane analogy, the force density method and finite difference method. These three methods are used for generating the calculation solution. The solution in out – of – plane program is the sum of bending moment. The membrane analogy is to simulate the equilibrium equation of thin plate subjected to loads acting perpendicular on its surface. The finite difference method is applied for satisfy the boundary condition. After these two theories, the equilibrium equation and boundary condition will be defined. Then in the visualized computational program, the force density method is for form shaping. With help of force density method, the solution for calculation is found.

Then the combination of rain flow analogy will lead to the result of shear force and principle shear trajectories. The finite difference method based on the found solution will come to the result of deformation, bending moment, torsion and boundary reaction. At this stage, the structure evaluation process is done.

2.6.3. In – Plane Program

The results for structural evaluation of the program are as follows:

- Normal stress
- Shear stress
- Deformation

To realize all these goals the combination will be:

(1) Membrane Analogy + Finite Difference Method + Force Density + Finite Difference Method

The application purpose is the same as the out – of – plane program. The membrane analogy, force density method and finite difference method are used for generating the calculation solution. According to this solution, the result can be derived by finite difference method. Then the result of normal and shear stresses are computed, so as to deformation.

3. Out – of – Plane Parametric Design Tool

3.1. Introduction

First of all, for the out – of – plane parametric design tool, most of the work was done by *Michiel Oosterhuis* (2010). The previous model is a structural analysis tool for rectangular plate supported by four edges. The boundary conditions are all simple supported. The task for me in this model is to extend the boundary conditions to fixed and free edges.
The new model will line with the old model. Therefore, at this part, the previous model structure will be briefly showed. And next section is the method to realize different boundary conditions in the program. The theories and the deriving procedure are the main story. The basic idea and the equations will be within this chapter.

3.2. Usability & Functionality

For freeform structures like shell, the basic form is the determinant factor. It will determine the structural behavior. Whether the structure is optimal, in this type of freeform building, mainly depends on the original shape. However, in freeform structure, such mechanic behavior is not that easy to figure out. Therefore the program is developed to solve this problem, in the conceptual design phase.

Since the model is for the conceptual design phase, the expected users for the out – of – plane model are architects and structural engineers. It is upmost task is to give structural evaluation with easy model modification. Based on this goal the usability is confined.

- Provide real – time results during the design process.
- Able to change the design parameters, such as geometry, load case and support conditions.
- Able to modify the program for different users and further developments.

The demands for qualitative and quantitative insight of thin plate structure for plate out – of – plane mechanics determine the functionality of the program. Below results are the evaluation criteria that other structural calculation program used.

- Bending moment
- Torsion moment
- Shear force
- Principal moment
- Principal shear
- Displacement

The out – of – plane parametric tool uses the same criteria. The result of these will be computed by the program in Grasshopper environment.

To provide straightforward view of the result is another objective. Instead of just presenting the number, showing the value shape of the result helps people to have a complete idea of the structural behavior. Then in the terms of functionality, the drawing shape of the results is one of the outputs.

3.3. Previous Model

Due to the thesis is based on previous model which is done by *M. Oosterhuis* (*A Parametric Structural Design Tool for Plate Structures.* 2010); the introduction of this model should be described beforehand.

The basic outline of the previous model is described below. Each component will be explained separately.

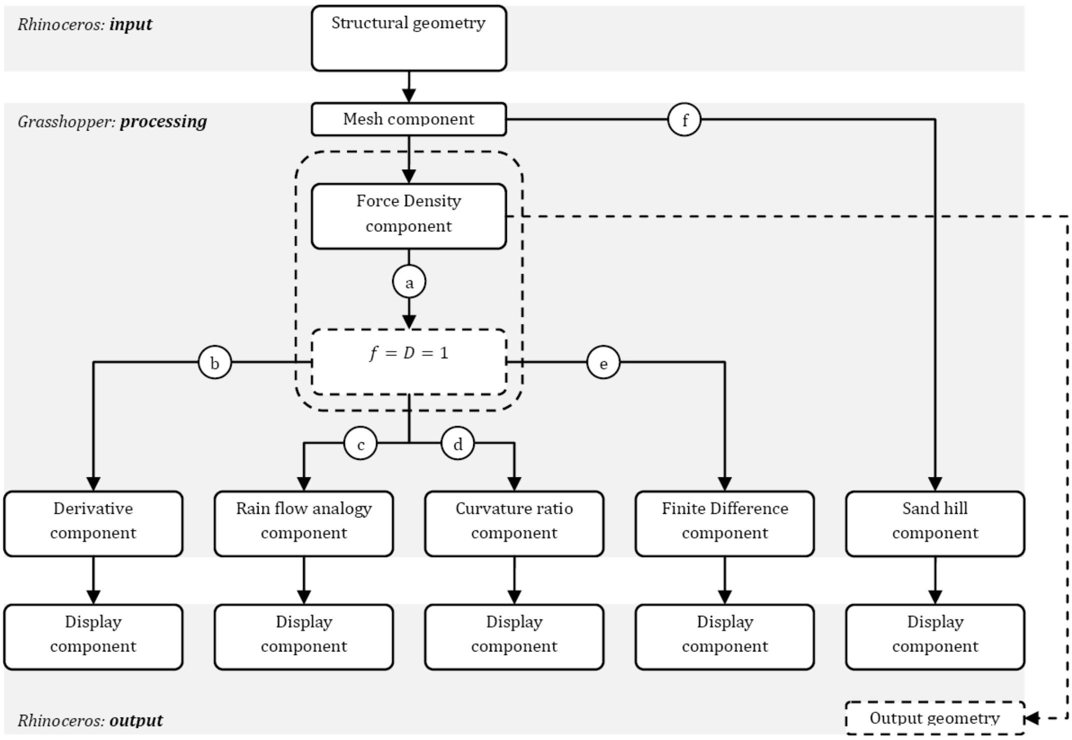


Figure 3.1: Previous model outline (Picture from M. Oosterhuis [14])

- **Structural Geometry**

The input component of structural geometry define the rectangle plate dimension and the mesh width

- **Meshing Component**

For calculation purposes, meshing component generate the grid of the plates. After considered the convenience of implementing finite difference method, the square mesh is decided. Also in this component, the nodes are sorted for different application.

- **Force Density Component**

Here is the place that membrane analogy is utilized. The component is used for form finding. By integrating the equilibrium equations with force density method, the membrane is shaped by Rhino.

- **Derivative Component**

The meshing of plate structure generates some marked points. This component shows the magnitude and the direction of principal shear in these marked points.

- **Rain Flow Analogy**

To determine the principal shear trajectories, this component make use of rain flow analogy. The analogy is based on gradient descent algorithm.

- **Finite Difference Component**

According to the plate theory, the mechanics behaviors are computed by differential equations. However in Rhino program, such application to calculate the differential equations does not exist. For this purpose, finite difference method component is applied to achieve the result for deflections, shears and moments.

- **Curvature Ratio & Sand Hill Component**

In *M. Oosterhuis* (2010) thesis, these two components are described. But in the model I gained from him such components were missing. And, on the other hand, this thesis does not need these components. They are outside of the thesis scope.

At last the outlines of previous model the thesis will follow is

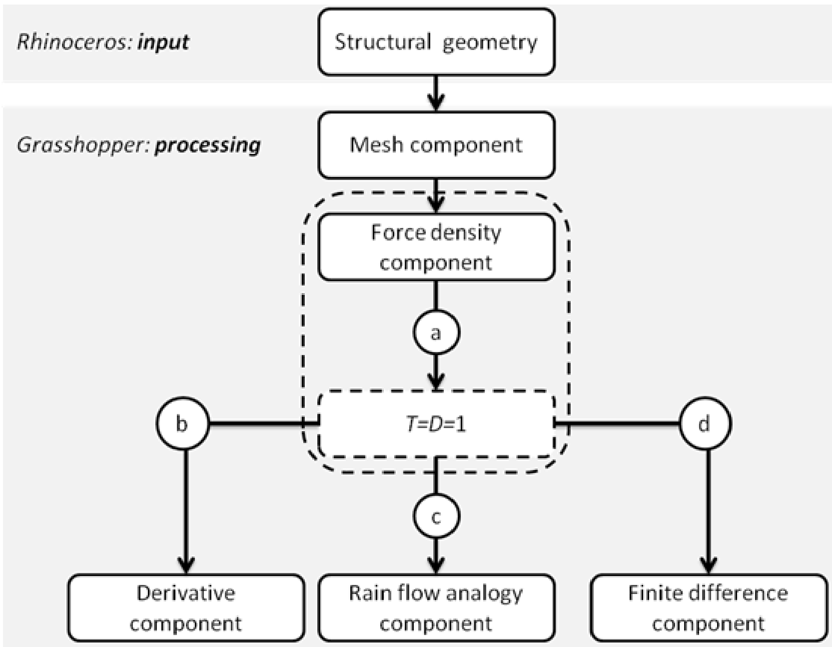


Figure 3.2: Reduced outline of previous model

3.4. Analysis Case

The scope of the out – of – plane program is confined to certain boundary conditions. The objective is to extend the program to satisfy different boundary conditions. Therefore, the analysis case will align with previous model. The load is uniform distributed load, acting perpendicular to the plate mid – plane. The plate is rectangular and constrained at the boundaries. The types of constraints are variable, with simple supported, free edges and fixed edges.

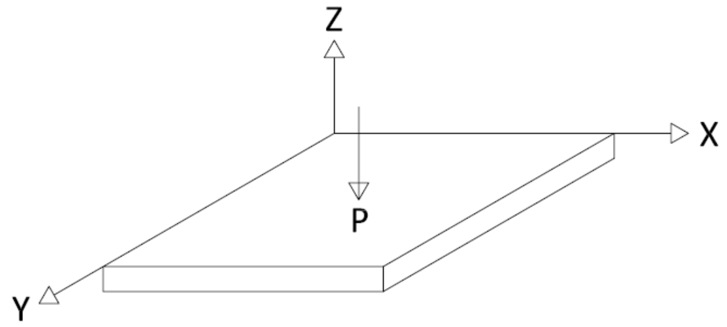


Figure 3.3: Analysis case

P is the distributed load. For further development, the program should be satisfied into a variety of analysis demands.

3.5. Boundary Condition Component

In previous article (*M. Oosterhuis.* 2010), the plate analysis tool had been developed. However the tool is only valid in plate and the boundary condition is merely restricted to four edge simple supported. In this thesis, such tool will be extended to be applicable with different boundary conditions like free edge and fixed edge support.

In the membrane analogy, people can define different membrane boundary conditions according to the force density methods in the program. Coincidently, in plate with simple support, the moments in x - and y – direction are both equal to zero. Since the membrane nodes coordinate value in z – axis present the value of bending moment. It implies that setting the coordinate value in z – axis of the membrane boundary nodes to zero conform the simple supported edge behavior. Unlike simple supported edge, other types of boundary conditions do not only refer to the bending moment. In fixed edge, the rotation is constrained; and in free edge, the Kirchhoff shear appears. It means further research in the relation with membrane analogy and the boundary conditions is critical.

In the beginning stage of the thesis research, I have tried to unlock the relation between the membrane boundary feature and the corresponding plate boundary conditions. Unfortunately the analogy between these two conditions is still in mystery. In line with this reason, generating the correct membrane boundary which is corresponding to the sum of bending moment should be combined with other method.

Based on this idea, another calculation component is added to generate the boundary. Due to only after the boundary value is defined, the correct membrane form – finding is achieved, such component is placed before the force density component.

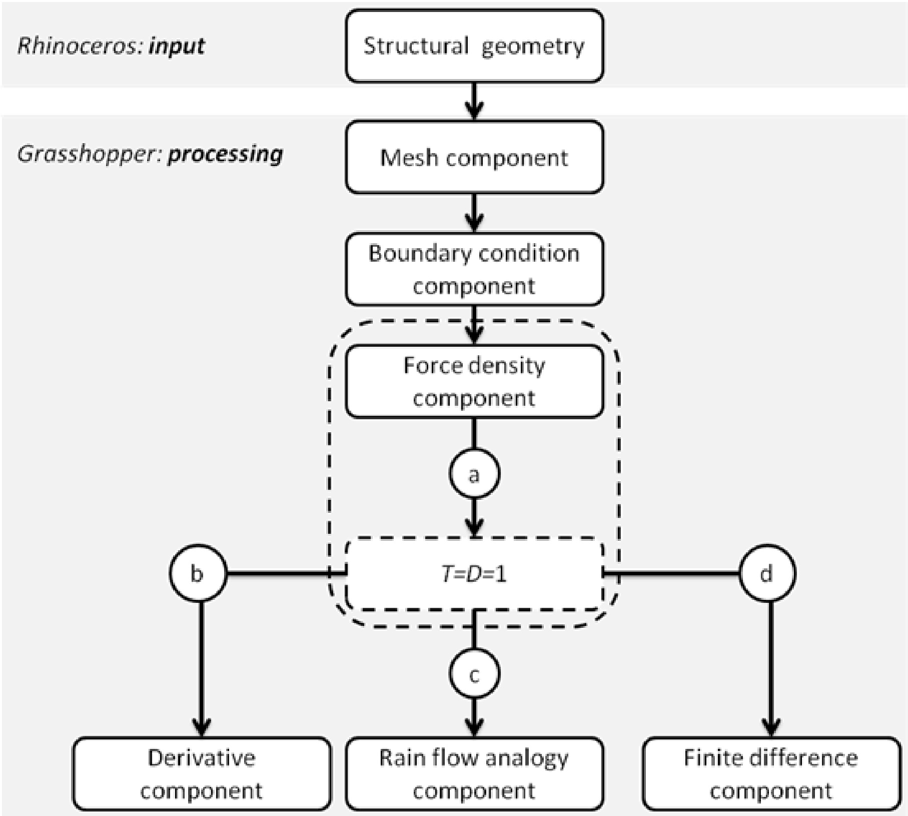


Figure 3.4: Outline of out – of – plane model

3.5.1. Basic Idea

The basic idea of this method is:

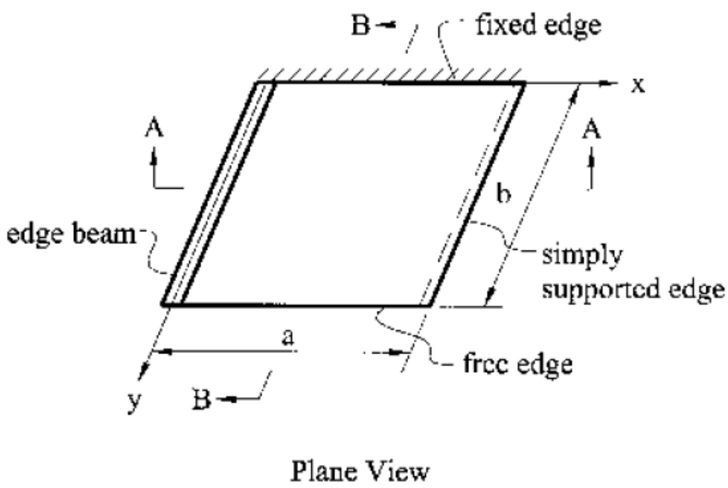
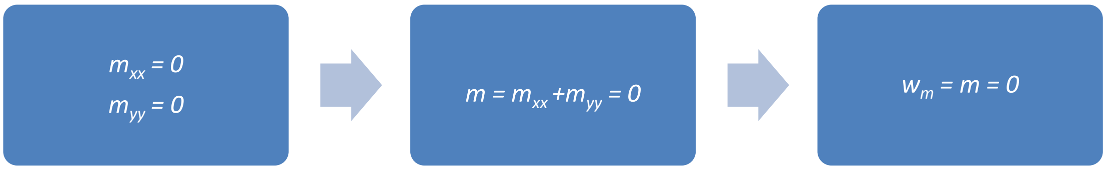
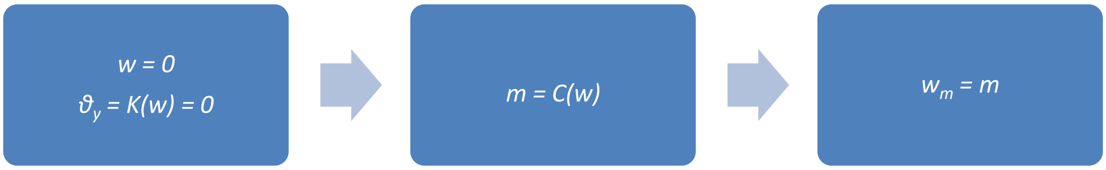


Figure 3.5: Edges description (Picture from E. Ventsel [4])

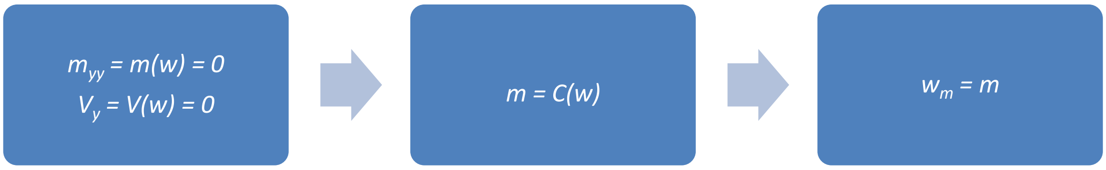
- Simple supported edge ($x = a$)



- fixed edge ($y = 0$)



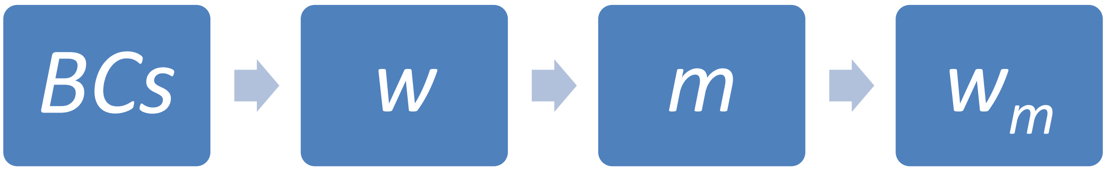
- free edge ($y = b$)



As the figure shows that in fixed edge and free edge, the boundary condition should be translated into plate deflections first. After that the sum of bending moment can be expressed by the deflections with assist of finite difference method. Then the membrane boundary is determined.

- Previous Model:

- New Model



The main function of this component is to generate the equations that represent the relation between the plate deflection and membrane shape. At the input component, the boundary condition will be defined. Based on the selected conditions, the component first translated the boundary condition with the language of deflection *w*. According to the relation equations and the known deflection w, the membrane boundary value can be computed.

The next part is to state the equations of relation and explain the meanings.

3.5.2. Boundary Equations

The formula below is derived from the equation in the thesis of *H. Schek* (1973).

The free nodes are interpreted as points $P_{i\text{-}free}$ with coordinates $(x_{i\text{-}free}, y_{i\text{-}free}, z_{i\text{-}free})$, *I = 1, …, n*, and the boundary nodes are as $P_{i\text{-}fixed}$ with coordinates $(x_{i\text{-}fixed}, y_{i\text{-}fixed}, z_{i\text{-}fixed})$, *I = 1, …, n.*

The coordinates of all the free nodes form the n_{free} – vector $x_{free}, y_{free}, z_{free}$ and the n_{fixed} – vector $x_{fixed}, y_{fixed}, z_{fixed}$ for all the boundary nodes.

- **The coordinate differences f of connected points**:

$$f = C_{free} \cdot z_{free} + C_{fixed} \cdot z_{fixed}$$

C is the branch-node matrix.

$$C = [C_{free} \quad C_{fixed}]$$

- **The equilibrium equation is**:

$$C^t_{free} F q = p_z$$

P_z is the matrix of load in z direction.
The equation can be rewritten as:

$$C^t_{free} Q f = p_z$$

With the identity:

$$Fq = Qf$$

(**F** and **Q** are the diagonal matrices belonging to **f** and **q**)
Then the equilibrium formula can be extended:

$$C^t_{free} Q C_{free} \cdot z_{free} + C^t_{free} Q C_{fixed} \cdot z_{fixed} = p_z$$

And **Q** is the force density matrix. In minimal surface all the force density should be equal to one. Therefore, the matrix **Q** equal to unit matrix **E**.
And rephrase the formula:

$$C^t_{free} C_{free} \cdot z_{free} + C^t_{free} C_{fixed} \cdot z_{fixed} = p_z$$

For simplicity, set:

$$D_{free} = C^t_{free} C_{free}$$
$$D_{fixed} = C^t_{free} C_{fixed}$$

Then:

$$D_{free} \cdot z_{free} + D_{fixed} \cdot z_{fixed} = p_z$$

With given loads and giving boundary value, the shape is stated by the equilibrium equation.

$$z_{free} = D^{-1}_{free} \cdot (p_z - D_{fixed} \cdot z_{fixed})$$

With

$$z = \begin{bmatrix} z_{fixed} \\ z_{free} \end{bmatrix}$$

In this method the boundary value z_{fixed} is unknown. The displacement of the membrane can be presented by the plate boundary conditions.

- **The finite difference method**:

According to membrane analogy (2.15):

$$z = w_m = m$$

W is the displacement of plate. The deflection f of the plate can be computed by the finite difference method. The calculation is as followed:

$$\boldsymbol{C} \cdot \boldsymbol{w} = \boldsymbol{m} \tag{3.1}$$

1) ***w*** is the deflection matrix.
2) ***C*** is the FDM computed matrix for sum of bending moment. But here the matrix ***C*** is different with different boundary conditions.
3) ***m*** is the matrix of sum of bending moment. And ***m*** can be represented by the membrane displacement.

$$\boldsymbol{m} = \boldsymbol{z} = \begin{bmatrix} \boldsymbol{z_{fixed}} \\ \boldsymbol{D_{free}^{-1}} \cdot (\boldsymbol{p_z} - \boldsymbol{D_{fixed}} \cdot \boldsymbol{z_{fixed}}) \end{bmatrix}$$

Therefore equation (3.1) is rewritten:

$$\boldsymbol{C} \cdot \boldsymbol{w} = \begin{bmatrix} \boldsymbol{z_{fixed}} \\ \boldsymbol{D_{free}^{-1}} \cdot (\boldsymbol{p_z} - \boldsymbol{D_{fixed}} \cdot \boldsymbol{z_{fixed}}) \end{bmatrix}$$

Then the deflection of the plate is:

$$\boldsymbol{w} = \boldsymbol{C^{-1}} \begin{bmatrix} \boldsymbol{z_{fixed}} \\ \boldsymbol{D_{free}^{-1}} \cdot (\boldsymbol{p_z} - \boldsymbol{D_{fixed}} \cdot \boldsymbol{z_{fixed}}) \end{bmatrix}$$

Above equation shows the relation between membrane boundary value $\boldsymbol{z_{fixed}}$ and the plate deflection ***w***. Only after these two connections are unlocked, the whole boundary condition component to can be realized. Next step is to describe the relation between plate boundary condition and the plate deflection.

- **The all fixed boundary**:

In fixed boundary, two conditions are defined. One is the deflection w is equal to zero, which is quite easy to modify. The other is the curvature is set to be zero. Since the curvature is the first order differential equation of deflection *w*. That means this boundary condition should be integrated with finite difference method.

For fixed boundary:

$$K_{fixed} = \frac{\partial w}{\partial x} = 0$$

And the curvature is computed as followed:

$$K_{fixed} = \frac{\partial w}{\partial x} = \boldsymbol{K} \cdot \boldsymbol{w} = \boldsymbol{K} \cdot \boldsymbol{C^{-1}} \begin{bmatrix} \boldsymbol{z_{fixed}} \\ \boldsymbol{D_{free}^{-1}} \cdot (\boldsymbol{p_z} - \boldsymbol{D_{fixed}} \cdot \boldsymbol{z_{fixed}}) \end{bmatrix} \tag{3.2}$$

For simplicity assume:

$$\boldsymbol{K} \cdot \boldsymbol{C^{-1}} = \boldsymbol{R} = [\boldsymbol{R_{fixed}} \quad \boldsymbol{R_{free}}]$$

Then equation (3.2) is:

$$\boldsymbol{K_{fixed}} = [\boldsymbol{R_{fixed}} \quad \boldsymbol{R_{free}}] \begin{bmatrix} \boldsymbol{z_{fixed}} \\ \boldsymbol{D_{free}^{-1}} \cdot (\boldsymbol{p_z} - \boldsymbol{D_{fixed}} \cdot \boldsymbol{z_{fixed}}) \end{bmatrix} = \boldsymbol{0}$$

Extend the equation:

$$\boldsymbol{R_{fixed}} \cdot \boldsymbol{z_{fixed}} + \boldsymbol{R_{free}} \cdot \boldsymbol{D_{free}^{-1}} \cdot (\boldsymbol{p_z} - \boldsymbol{D_{fixed}} \cdot \boldsymbol{z_{fixed}}) = \boldsymbol{0}$$

By rephrasing the order of each term:

$$\boldsymbol{R_{free}} \cdot \boldsymbol{D_{free}^{-1}} \cdot \boldsymbol{p_z} = (\boldsymbol{R_{free}} \cdot \boldsymbol{D_{free}^{-1}} \cdot \boldsymbol{D_{fixed}} - \boldsymbol{R_{fixed}}) \cdot \boldsymbol{z_{fixed}}$$

Then the relation between fixed edge conditions and the membrane boundary value is found. The component calculates the value base on the followed equation:

$$\boldsymbol{z_{fixed}} = (\boldsymbol{R_{free}} \cdot \boldsymbol{D_{free}^{-1}} \cdot \boldsymbol{D_{fixed}} - \boldsymbol{R_{fixed}})^{-1} \cdot \boldsymbol{R_{free}} \cdot \boldsymbol{D_{free}^{-1}} \cdot \boldsymbol{p_z} \tag{3.3}$$

- **The all free boundary with four corners supported**:

In free edge boundary condition, the Kirchhoff's shear stress is equal to zero.

$$V_x = Q_x + \frac{\partial m_{xy}}{\partial y} = \frac{\partial m}{\partial x} + \frac{\partial m_{xy}}{\partial y} = 0$$

In finite difference method the first term and the second term of the equation above can be computed. The first term represent the shear stress is:

$$\frac{\partial m}{\partial x} = \boldsymbol{M} \cdot \boldsymbol{m} = \boldsymbol{M} \cdot \boldsymbol{z} = \boldsymbol{M_{fixed}} \cdot \boldsymbol{z_{fixed}} + \boldsymbol{M_{free}} \cdot \boldsymbol{z_{free}} \tag{3.4}$$

With identity:

$$\boldsymbol{M} = [\boldsymbol{M_{fixed}} \quad \boldsymbol{M_{free}}]$$

The second term is calculated:

$$\frac{\partial m_{xy}}{\partial y} = \boldsymbol{T} \cdot \boldsymbol{w} = \boldsymbol{T} \cdot \boldsymbol{C^{-1}} \begin{bmatrix} \boldsymbol{z_{fixed}} \\ \boldsymbol{D_{free}^{-1}} \cdot (\boldsymbol{p_z} - \boldsymbol{D_{fixed}} \cdot \boldsymbol{z_{fixed}}) \end{bmatrix} \tag{3.5}$$

For simplicity assume:

$$\boldsymbol{T} \cdot \boldsymbol{C^{-1}} = \boldsymbol{N} = [\boldsymbol{N_{fixed}} \quad \boldsymbol{N_{free}}]$$

Then Kirchhoff's shear stress should be (with (3.4) and (3.5)):

$$V_x = \frac{\partial m}{\partial x} + \frac{\partial m_{xy}}{\partial y} = \boldsymbol{M_{fixed}} \cdot \boldsymbol{z_{fixed}} + \boldsymbol{M_{free}} \cdot \boldsymbol{z_{free}} + \boldsymbol{N_{fixed}} \cdot \boldsymbol{z_{fixed}} + \boldsymbol{N_{free}} \cdot \boldsymbol{z_{free}} = \boldsymbol{0}$$

By replacing:

$$\boldsymbol{z_{free}} = \boldsymbol{D_{free}^{-1}} \cdot (\boldsymbol{p_z} - \boldsymbol{D_{fixed}} \cdot \boldsymbol{z_{fixed}})$$

The equation can be extended:

$$(\boldsymbol{M_{fixed}} + \boldsymbol{N_{fixed}}) \cdot \boldsymbol{z_{fixed}} + (\boldsymbol{M_{free}} + \boldsymbol{N_{free}}) \cdot \boldsymbol{D_{free}^{-1}} \cdot (\boldsymbol{p_z} - \boldsymbol{D_{fixed}} \cdot \boldsymbol{z_{fixed}}) = \boldsymbol{0}$$

Again, for simplicity, assume:

$$\boldsymbol{S_{free}} = \boldsymbol{M_{free}} + \boldsymbol{N_{free}}$$
$$\boldsymbol{S_{fixed}} = \boldsymbol{M_{fixed}} + \boldsymbol{N_{fixed}}$$

Then:

$$\boldsymbol{S_{fixed}} \cdot \boldsymbol{z_{fixed}} + \boldsymbol{S_{free}} \cdot \boldsymbol{D_{free}^{-1}} \cdot (\boldsymbol{p_z} - \boldsymbol{D_{fixed}} \cdot \boldsymbol{z_{fixed}}) = \boldsymbol{0}$$

Rephrasing the orders:

$$\boldsymbol{S_{free}} \cdot \boldsymbol{D_{free}^{-1}} \cdot \boldsymbol{p_z} = (\boldsymbol{S_{free}} \cdot \boldsymbol{D_{free}^{-1}} \cdot \boldsymbol{D_{fixed}} - \boldsymbol{S_{fixed}}) \cdot \boldsymbol{z_{fixed}}$$

Then the relation between free edge conditions and the membrane boundary value is found. The component calculates the value base on the followed equation:

$$\boldsymbol{z_{fixed}} = (\boldsymbol{S_{free}} \cdot \boldsymbol{D_{free}^{-1}} \cdot \boldsymbol{D_{fixed}} - \boldsymbol{S_{fixed}})^{-1} \cdot \boldsymbol{S_{free}} \cdot \boldsymbol{D_{free}^{-1}} \cdot \boldsymbol{p_z} \tag{3.6}$$

At this stage, the connections between plate boundary conditions with membrane shape have unlocked.

- **The all simple supported boundary**:

In simple supported boundary, two conditions are defined. One is the deflection *w* is equal to zero, which is quite easy to modify. The other is the bending moment is set to be zero. This boundary condition should be integrated with finite difference method.

For simple supported boundary:

$$B_{fixed} = \frac{\partial^2 w}{\partial x^2} = 0$$

And the moment is computed as followed:

$$B_{fixed} = \frac{\partial^2 w}{\partial x^2} = \boldsymbol{B} \cdot \boldsymbol{w} = \boldsymbol{B} \cdot \boldsymbol{C}^{-1} \begin{bmatrix} \boldsymbol{z_{fixed}} \\ \boldsymbol{D_{free}^{-1}} \cdot (\boldsymbol{p_z} - \boldsymbol{D_{fixed}} \cdot \boldsymbol{z_{fixed}}) \end{bmatrix} \tag{3.7}$$

For simplicity assume:

$$\boldsymbol{B} \cdot \boldsymbol{C}^{-1} = \boldsymbol{H} = [\boldsymbol{H_{fixed}} \quad \boldsymbol{H_{free}}]$$

Then equation (3.7) is:

$$\boldsymbol{B_{fixed}} = [\boldsymbol{H_{fixed}} \quad \boldsymbol{H_{free}}] \begin{bmatrix} \boldsymbol{z_{fixed}} \\ \boldsymbol{D_{free}^{-1}} \cdot (\boldsymbol{p_z} - \boldsymbol{D_{fixed}} \cdot \boldsymbol{z_{fixed}}) \end{bmatrix} = \boldsymbol{0}$$

Extend the equation:

$$\boldsymbol{H_{fixed}} \cdot \boldsymbol{z_{fixed}} + \boldsymbol{H_{free}} \cdot \boldsymbol{D_{free}^{-1}} \cdot (\boldsymbol{p_z} - \boldsymbol{D_{fixed}} \cdot \boldsymbol{z_{fixed}}) = \boldsymbol{0}$$

By rephrasing the order of each term:

$$\boldsymbol{H_{free}} \cdot \boldsymbol{D_{free}^{-1}} \cdot \boldsymbol{p_z} = (\boldsymbol{H_{free}} \cdot \boldsymbol{D_{free}^{-1}} \cdot \boldsymbol{D_{fixed}} - \boldsymbol{H_{fixed}}) \cdot \boldsymbol{z_{fixed}}$$

Then the relation between hinged edge conditions and the membrane boundary value is found. The component calculates the value base on the followed equation:

$$\boldsymbol{z_{fixed}} = (\boldsymbol{H_{free}} \cdot \boldsymbol{D_{free}^{-1}} \cdot \boldsymbol{D_{fixed}} - \boldsymbol{H_{fixed}})^{-1} \cdot \boldsymbol{H_{free}} \cdot \boldsymbol{D_{free}^{-1}} \cdot \boldsymbol{p_z} \tag{3.8}$$

3.5.3. Equations Combination

For further consideration, the plate structure may have different type of boundary conditions in each edge. To make this thinking functional, the equations stated in above chapter will be combined.

If people look at the equations for fixed edge and free edge:

- Simple Supported Edge equation (3.8):

  $$\boldsymbol{z_{fixed}} = (\boldsymbol{H_{free}} \cdot \boldsymbol{D_{free}^{-1}} \cdot \boldsymbol{D_{fixed}} - \boldsymbol{H_{fixed}})^{-1} \cdot \boldsymbol{H_{free}} \cdot \boldsymbol{D_{free}^{-1}} \cdot \boldsymbol{p_z}$$

  (Based on the same theory the equation for the simple supported edge can be obtained by the same way. The matrix ***H*** represents the relation between simple supported conditions with membrane boundary)
- Fixed Edge equation (3.3):

  $$\boldsymbol{z_{fixed}} = (\boldsymbol{R_{free}} \cdot \boldsymbol{D_{free}^{-1}} \cdot \boldsymbol{D_{fixed}} - \boldsymbol{R_{fixed}})^{-1} \cdot \boldsymbol{R_{free}} \cdot \boldsymbol{D_{free}^{-1}} \cdot \boldsymbol{p_z}$$
- Free Edge equation (3.6)

  $$\boldsymbol{z_{fixed}} = (\boldsymbol{S_{free}} \cdot \boldsymbol{D_{free}^{-1}} \cdot \boldsymbol{D_{fixed}} - \boldsymbol{S_{fixed}})^{-1} \cdot \boldsymbol{S_{free}} \cdot \boldsymbol{D_{free}^{-1}} \cdot \boldsymbol{p_z}$$

The equations share the same form.

According to this phenomenon, the following presumption is made. Rectangular plate has four edges. Each edge has its boundary condition. Therefore, the edges are named by A, B, C and D for the equation.

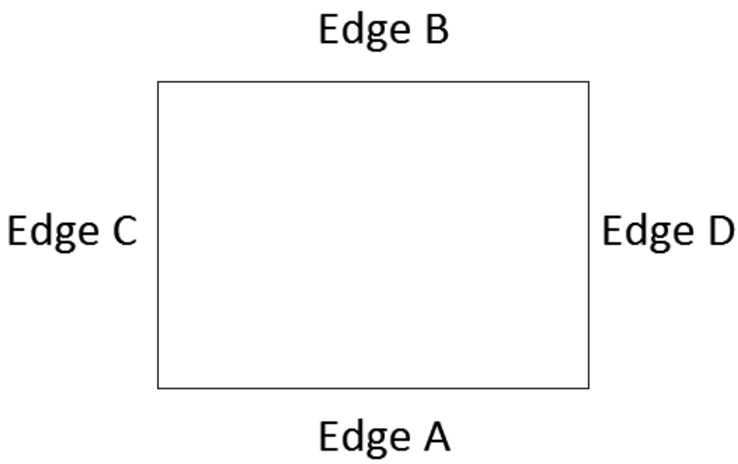


Figure 3.6: Edges classification

The equation is rewritten into following shape:

$$\boldsymbol{z}_{fixed} = \begin{bmatrix} \boldsymbol{z}_{A-fixed} \\ \boldsymbol{z}_{B-fixed} \\ \boldsymbol{z}_{C-fixed} \\ \boldsymbol{z}_{D-fixed} \end{bmatrix} = (\begin{bmatrix} \boldsymbol{E}_{A-free} \\ \boldsymbol{E}_{B-free} \\ \boldsymbol{E}_{C-free} \\ \boldsymbol{E}_{D-free} \end{bmatrix} \cdot \boldsymbol{D}_{free}^{-1} \cdot \boldsymbol{D}_{fixed} - \begin{bmatrix} \boldsymbol{E}_{A-fixed} \\ \boldsymbol{E}_{B-fixed} \\ \boldsymbol{E}_{C-fixed} \\ \boldsymbol{E}_{D-fixed} \end{bmatrix})^{-1} \cdot \begin{bmatrix} \boldsymbol{E}_{A-free} \\ \boldsymbol{E}_{B-free} \\ \boldsymbol{E}_{C-free} \\ \boldsymbol{E}_{D-free} \end{bmatrix} \cdot \boldsymbol{D}_{free}^{-1} \cdot \boldsymbol{p}_z$$

With identities:

$$\boldsymbol{E}_{i-free} = \begin{cases} \boldsymbol{R}_{i-free} & (fixed\ edge) \\ \boldsymbol{S}_{i-free} & (free\ edge) \\ \boldsymbol{H}_{i-free} & (simple\ supported\ edge) \end{cases} \quad With\ i = A, B, C, D$$

$$\boldsymbol{E}_{i-fixed} = \begin{cases} \boldsymbol{R}_{i-fixed} & (fixed\ edge) \\ \boldsymbol{S}_{i-fixed} & (free\ edge) \\ \boldsymbol{H}_{i-fixed} & (simple\ supported\ edge) \end{cases} \quad With\ i = A, B, C, D$$

In the boundary component, such equation is applied.

4. Out – of – Plane Result Verification

4.1. Introduction

In this chapter, the result comparison is presented. In qualitative comparison the generated outputs from the parametric model are compared to the general results produced by *ir. W. J. Beranek* (1976). The produced results from the parametric tool are compared with FEM (finite element method) program TNO Diana. The quantitative comparison evaluates the accuracy of the numerical result generated by Grasshopper model.

In the comparison, rectangular plate subjected to a uniform distributed load p is set. In line with Grasshopper model, the grid I used in Diana model is square with dimension of 1m x 1m. Such meshing is the same as the parametric tool. The mesh type in Diana model is CQ24P.

The properties of the analysis model are:

1) $L = 14.0\ m$
2) $W = 10.0\ m$
3) $p = 0.1\ N/m^2$
4) $\upsilon = 0$

The geometry of the plate and the definition of nodes location are showed below. The long edge points are defined as Bound X; and short edge points are Bound Y.

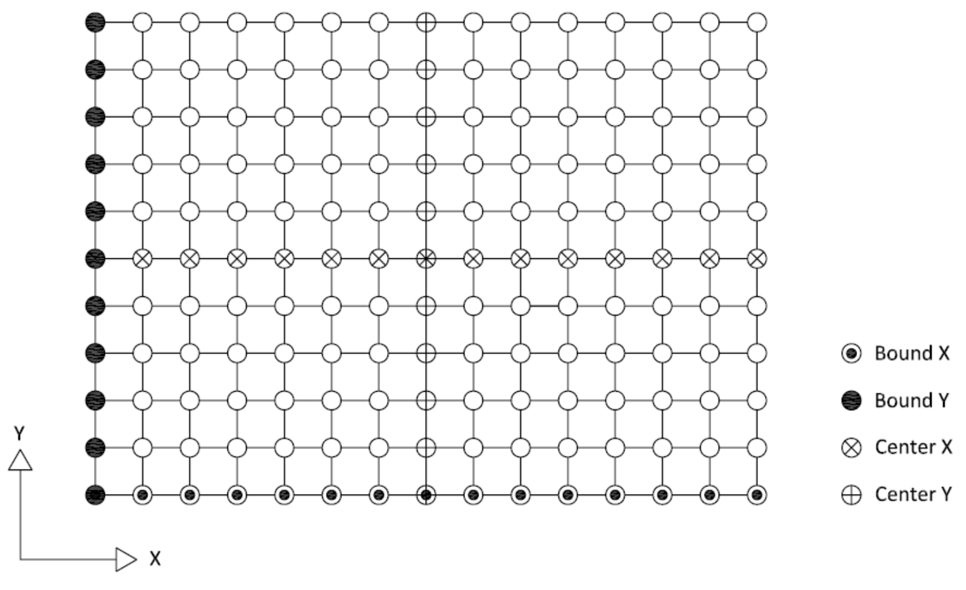


Figure 4.1: Nodes classification

Two boundary cases are taken into account.

- Thin plate with 4 edges fixed
- Thin plate with 4 corner support

In plate structural theory of out – of – plane mechanics, the in – plane strain is ignored. However, the in – plane mechanics still exist. It is just the matter of how much such behavior influence the out – of – plane behavior. Under the consideration of this question, the thickness of the plate is taken into account, because the span/thickness ratio plays an important role in the mechanics. That is why, in the following comparison, two different types of thickness are applied, with 0.1m and 1.0m thickness.

The total cases I present in this chapter are 2 x 2 = 4.

- Four edges fixed with *t = 0.1m*
- Four edges fixed with *t = 1.0m*
- Four corner support with *t = 0.1m*
- Four corner support with *t = 1.0m*

4.2. Fixed Boundary Case

4.2.1. Qualitative Verification

In qualitative manner, the form – finding result will be compared.

- Result of membrane shape:

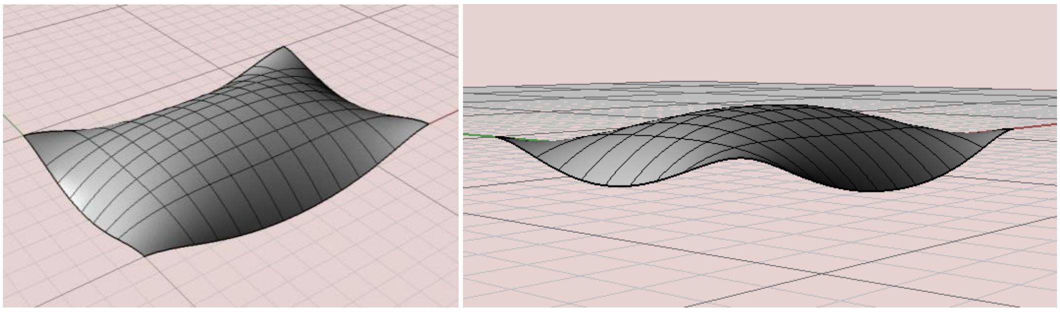


Figure 4.2: Membrane analogy of bending moment summation

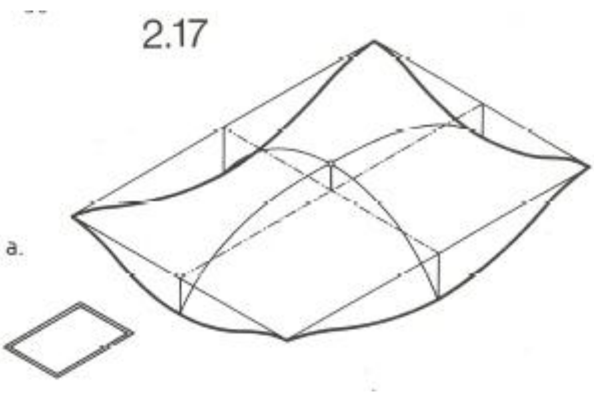


Figure 4.3: General bending moment summation (Picture from W. J. Beranek [6])

As the figures show that, the result from Grasshopper model is quite similar to the general result. The moments in the edges are negative, which means that the bending moments are negative. This conforms to the fact. And in the middle of plate, positive moments occur.

- Result of principal shear trajectories:

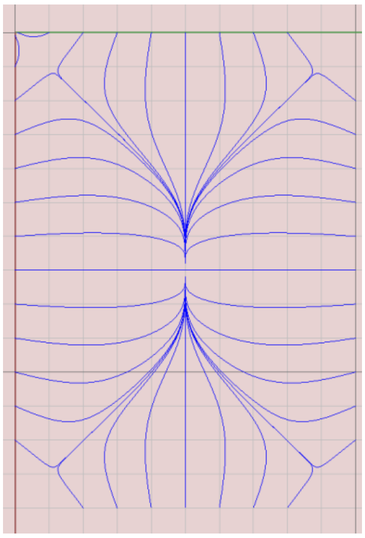


Figure 4.4: Rain – flow analogy in GH

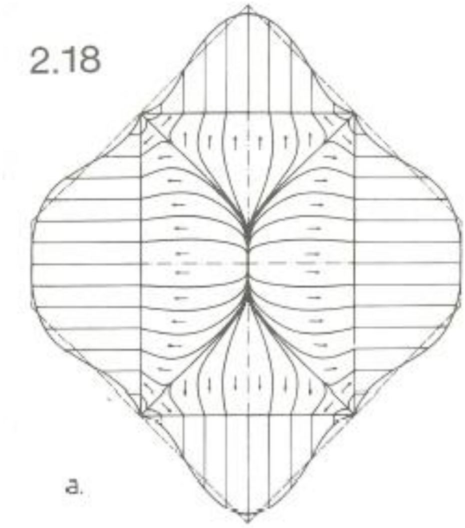


Figure 4.5: General Rain – flow analogy

It can be seen again that the result from Grasshopper aligned with the general result *(Picture from W. J. Beranek [6])* quite well. Especially at the corners, the trajectories are perpendicular to the angular bisectors.

The verification presents a good calculation performance of the Grasshopper model, in qualitative manner.

4.2.2. Quantitative Verification

In the table the M (Diana) is the sum of bending moment which is calculated in TNO Diana FEM program. And M (GH) is the value of Rhino Grasshopper model. The ratio is the number of M (Diana) divided by M (GH). The figure shows the shape of the moment line.

In the comparison of Bound Y results, the analytical results will be used. The results are from *Tafeln fur Gleichmassig Vollbelastete Rechteckplatten. (Bautechnik – Archiv Heft 11)* written by *F. Czerny.*

- Bound X (100mm Thickness)

The unit of Node X is m, and M is N.

Node X	0	1	2	3	4	5
M(Diana)	0.01749	-0.0684995	-0.24984	-0.42333	-0.5513	-0.64818
M(GH)	0	-0.096	-0.258	-0.417	-0.545	-0.635
Ratio X	0	0.713536458	0.96839	1.015168	1.01155	1.020756

Node X	6	7	8	9	10	11
M(Diana)	-0.70124	-0.7181	-0.70124	-0.64818	-0.5513	-0.42333
M(GH)	-0.687	-0.704	-0.687	-0.635	-0.545	-0.417
Ratio X	1.020721	1.020028	1.020721	1.020756	1.01155	1.015168

Node X	12	13	14
M(Diana)	-0.24984	-0.0685	0.01749
M(GH)	-0.258	-0.096	0
Ratio X	0.96839	0.713536	0

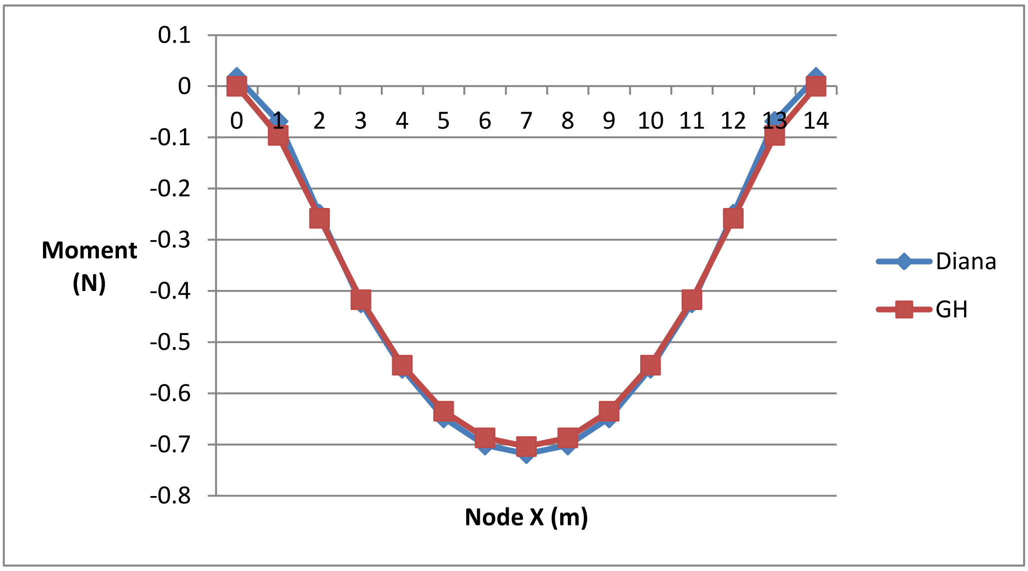


Figure 4.6: Sum of Bending Moment in Bound X (t = 100mm)

It can be seen that the results from Diana model and Grasshopper model almost coincide. Only slight differences occur.

- Bound X (1000mm Thickness)

The unit of Node X is m, and M is N.

Node X	0	1	2	3	4	5
M(Diana)	0.001155	-0.1079635	-0.26415	-0.42193	-0.55018	-0.64037
M(GH)	0	-0.096	-0.258	-0.417	-0.545	-0.635
Ratio X	0	1.124619792	1.023849	1.011819	1.009511	1.008456

Node X	6	7	8	9	10	11
M(Diana)	-0.69245	-0.70948	-0.69245	-0.64037	-0.55018	-0.42193
M(GH)	-0.687	-0.704	-0.687	-0.635	-0.545	-0.417
Ratio X	1.007937	1.00778	1.007937	1.008456	1.009511	1.011819

Node X	12	13	14
M(Diana)	-0.26415	-0.10796	0.001155
M(GH)	-0.258	-0.096	0
Ratio X	1.023849	1.12462	0

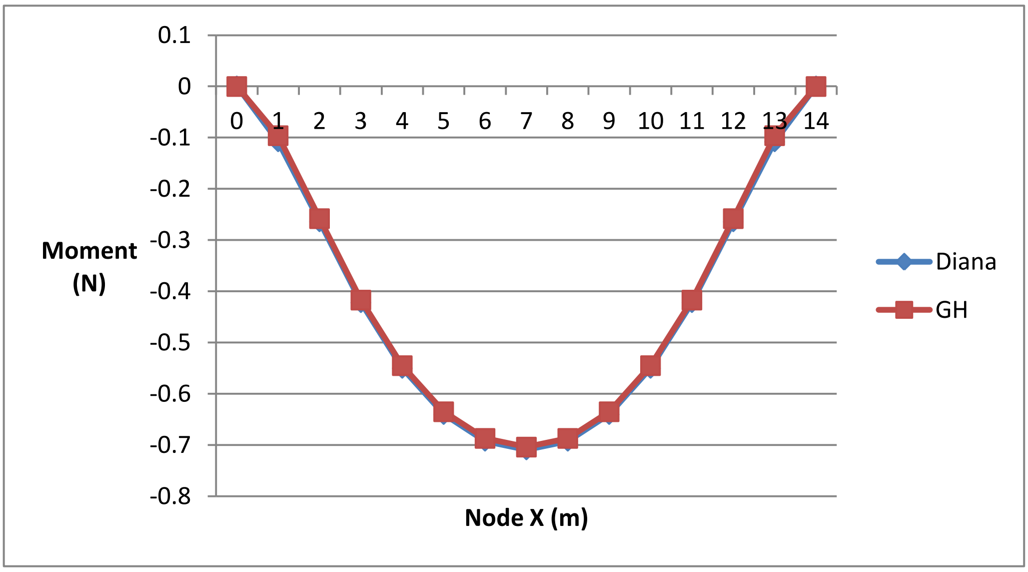


Figure 4.7: Sum of Bending Moment in Bound X (t = 1000mm)

Compared the results with different thickness, the Grasshopper model is close to the thicker case better. To interpret the phenomenon, the model thickness has to be taken into account. 100mm thickness is relative small to the dimensions of plate structure. Therefore, there may be large deflection which will cause in – plane mechanic behavior. With 1000mm, the ratio of thickness and plate dimension is logical to be considered as pure out – of – plane behavior. Base on this reason, the 1000mm thickness result is close to the Grasshopper model.

- Bound Y (100mm Thickness)

The unit of Node Y is m, and M is N.

Node Y	0	1	2	3	4	5
M(Analy)	0	-0.074	-0.259	-0.425	-0.526	-0.572
M(Diana)	0.01749	-0.0686355	-0.24937	-0.41644	-0.52351	-0.56106
M(GH)	0	-0.096	-0.257	-0.408	-0.509	-0.544
RatioA/G	0	0.770833333	1.007782	1.041667	1.033399	1.051471
RatioD/G	0	0.714953125	0.970327	1.020683	1.028497	1.03136

Node Y	6	7	8	9	10
M(Analy)	-0.526	-0.425	-0.259	-0.074	0
M(Diana)	-0.52351	-0.41644	-0.24937	-0.06864	0.01749
M(GH)	-0.509	-0.408	-0.257	-0.096	0
RatioA/G	1.033399	1.041667	1.007782	0.770833	0
RatioD/G	1.028497	1.020683	0.970327	0.714953	0

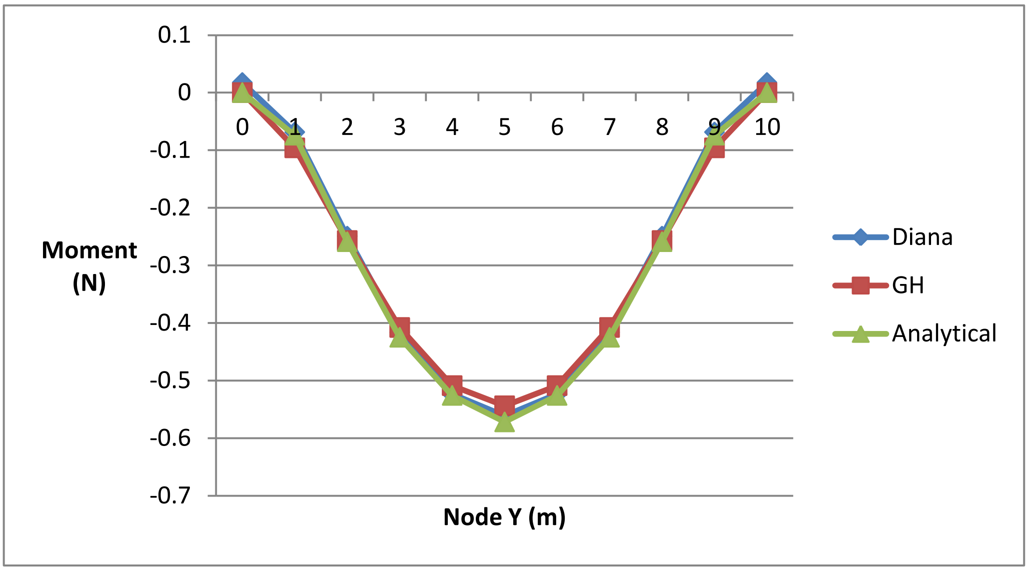


Figure 4.8: Sum of Bending Moment in Bound Y (t = 100mm)

- Bound Y (1000mm Thickness)

The unit of Node Y is m, and M is N.

Node Y	0	1	2	3	4	5
M(Analy)	0	-0.074	-0.259	-0.425	-0.526	-0.572
M(Diana)	0.001155	-0.1079985	-0.26229	-0.41069	-0.51099	-0.54563
M(GH)	0	-0.096	-0.257	-0.408	-0.509	-0.544
RatioA/G	0	0.770833333	1.007782	1.041667	1.033399	1.051471
RatioD/G	0	1.124984375	1.020597	1.006588	1.003918	1.002991

Node Y	6	7	8	9	10
M(Analy)	-0.526	-0.425	-0.259	-0.074	0
M(Diana)	-0.51099	-0.41069	-0.26229	-0.108	0.001155
M(GH)	-0.509	-0.408	-0.257	-0.096	0
RatioA/G	1.033399	1.041667	1.007782	0.770833	0
RatioD/G	1.003918	1.006588	1.020597	1.124984	0

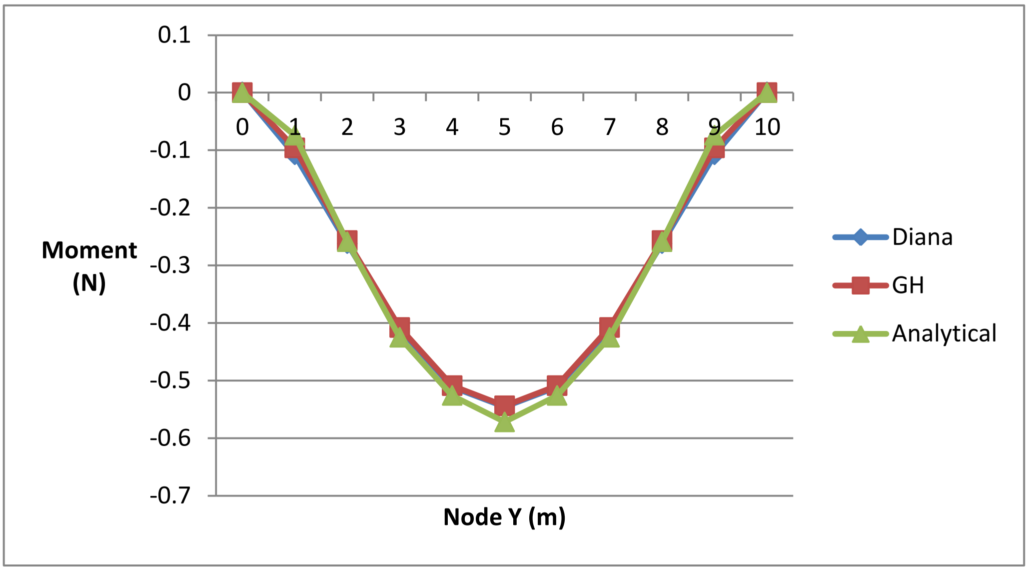


Figure 4.9: Sum of Bending Moment in Bound Y (t = 1000mm)

With results verification of Bound Y and also by introduced the analytical results; it is convincing that the Grasshopper model can generate precise structural analysis. And again, with the reason that has been declared above, the GH results fit the 1000mm Diana better.

4.3. Free Boundary Case

4.3.1. Qualitative Verification

In qualitative manner, the form – finding result will be compared.

- Result of membrane shape from grasshopper model:

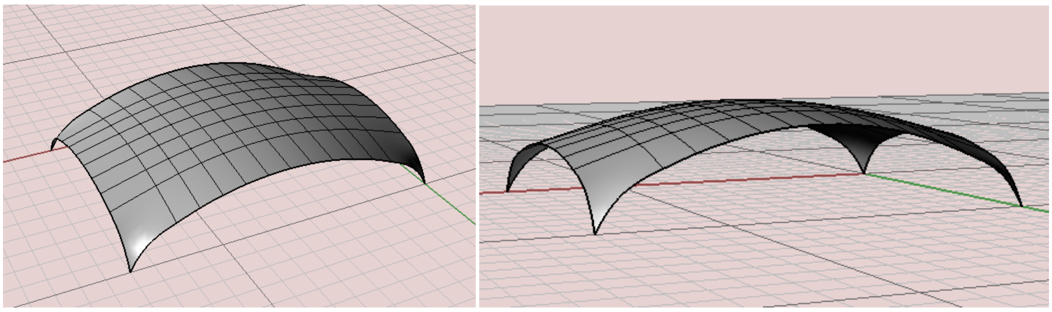


Figure 4.10: Membrane analogy of bending moment summation

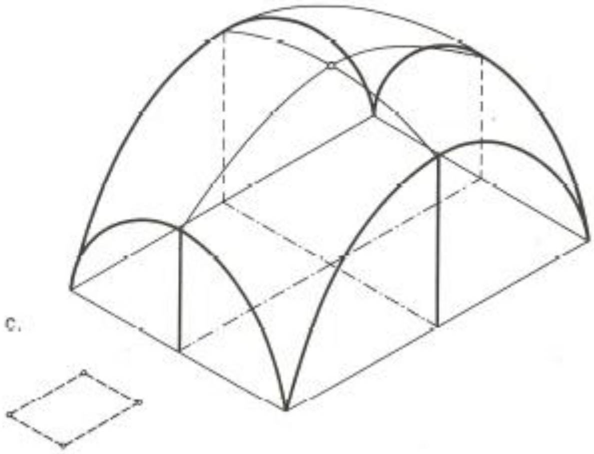


Figure 4.11: Analytical bending moment summation (Picture from W. J. Beranek [6])

The scales of the results are a bit different. But in qualitative manner, the shapes of the membrane are close to each other.

- Result of principal shear trajectories:

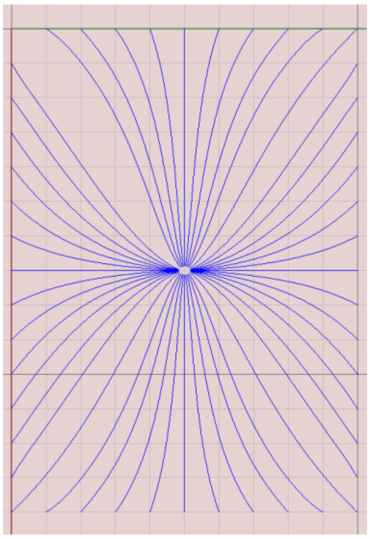


Figure 4.12: Rain – flow analogy in GH

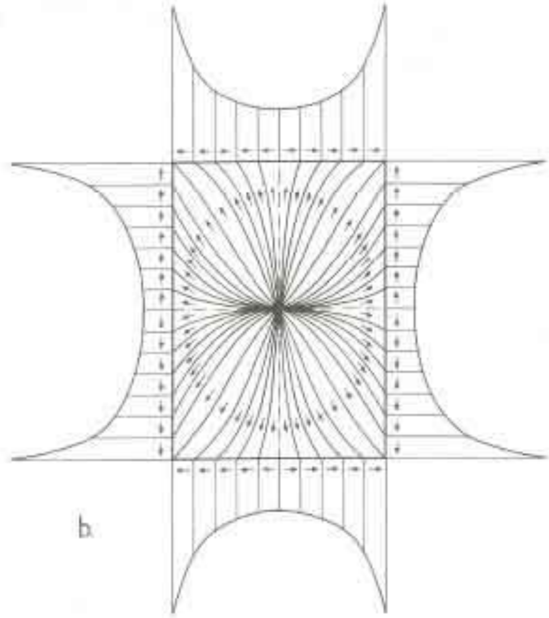


Figure 4.13: Analytical Rain – flow analogy

The verification presents a good calculation performance of the Grasshopper model, in qualitative manner.

4.3.2. Quantitative Verification

- Bound X (100mm Thickness)

The unit of Node X is m, and M is N.

Node X	0	1	2	3	4	5
M(Diana)	0.649	1.03225	1.646495	2.03595	2.353865	2.572335
M(GH)	0	1.219	1.714	2.105	2.404	2.616
Ratio X	0	0.846800656	0.960616	0.967197	0.979145	0.983308

Node X	6	7	8	9	10	11
M(Diana)	2.704205	2.74355	2.704205	2.572335	2.353865	2.03595
M(GH)	2.743	2.786	2.743	2.616	2.404	2.105
Ratio X	0.985857	0.984763	0.985857	0.983308	0.979145	0.967197

Node X	12	13	14
M(Diana)	1.646495	1.03225	0.649
M(GH)	1.714	1.219	0
Ratio X	0.960616	0.846801	0

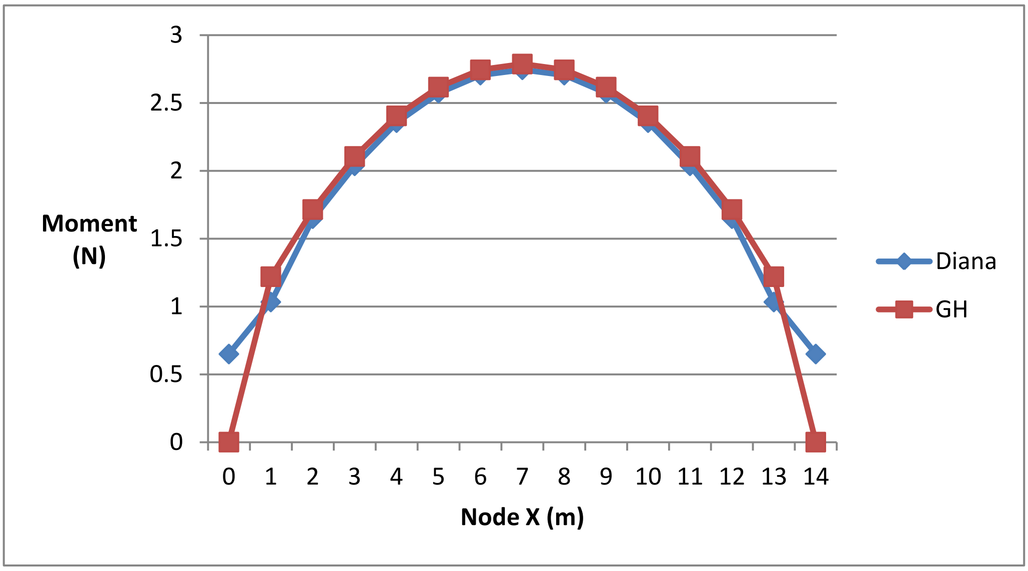


Figure 4.14: Sum of Bending Moment in Bound X (t = 100mm)

According to the mechanics theory, the moment in the corner support should be equal to zero. But based on the *Kirchhoff* theory, the existing of concentrated shear force leads to unusual stress distribution in the boundary edges. The Diana program is not precise enough to come out with the right results of such special mechanic phenomenon. Therefore, in Diana model, when it comes to the corner point, the results are not fully correct.

The above theory is presented in the book of *Plates and FEM* which is written by Professor *Blaauwendraad*.

- Bound X (1000mm Thickness)

The unit of Node X is m, and M is N.

Node X	0	1	2	3	4	5
M(Diana)	1.753	1.435	1.62175	1.98955	2.2887	2.5059
M(GH)	0	1.219	1.714	2.105	2.404	2.616
Ratio X	0	1.177194422	0.946179	0.945154	0.952038	0.957913

Node X	6	7	8	9	10	11
M(Diana)	2.63705	2.6882	2.63705	2.5059	2.2887	1.98955
M(GH)	2.743	2.786	2.743	2.616	2.404	2.105
Ratio X	0.961374	0.964896	0.961374	0.957913	0.952038	0.945154

Node X	12	13	14
M(Diana)	1.62175	1.435	1.753
M(GH)	1.714	1.219	0
Ratio X	0.946179	1.177194	0

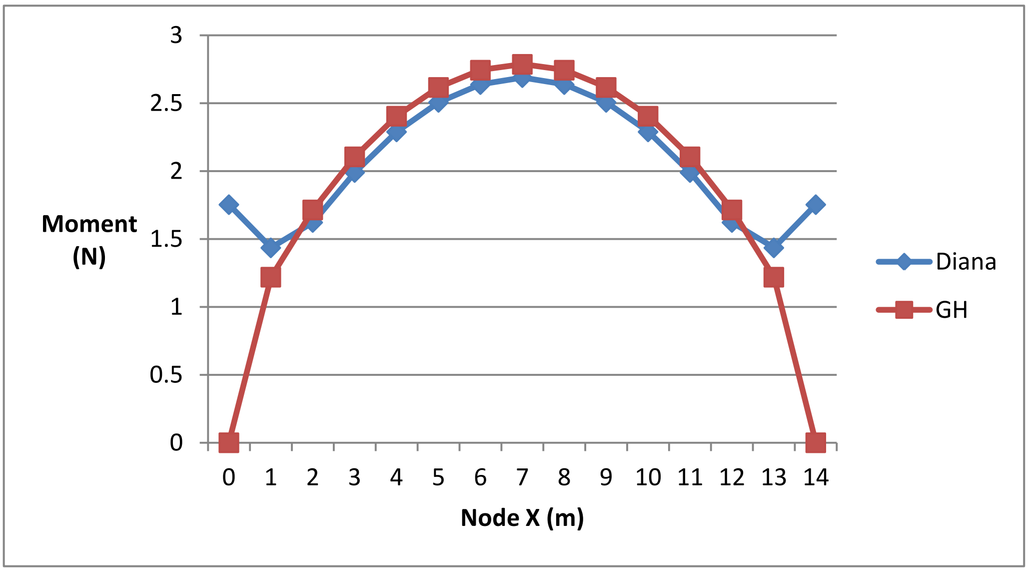


Figure 4.15: Sum of Bending Moment in Bound X (t = 1000mm)

In 1000mm thickness model the deviation of stress distribution in the corner points become larger.

- Bound Y (100mm Thickness)

The unit of Node Y is m, and M is N.

Node Y	0	1	2	3	4	5
M(Diana)	0.649	0.97965	1.487014	1.7455	1.90895	1.9552
M(GH)	0	1.182	1.573	1.831	1.979	2.028
Ratio Y	0	0.828807107	0.945336	0.953304	0.964603	0.964103

Node Y	6	7	8	9	10
M(Diana)	1.90895	1.7455	1.487014	0.97965	0.649
M(GH)	1.979	1.831	1.573	1.182	0
Ratio Y	0.964603	0.953304	0.945336	0.828807	0

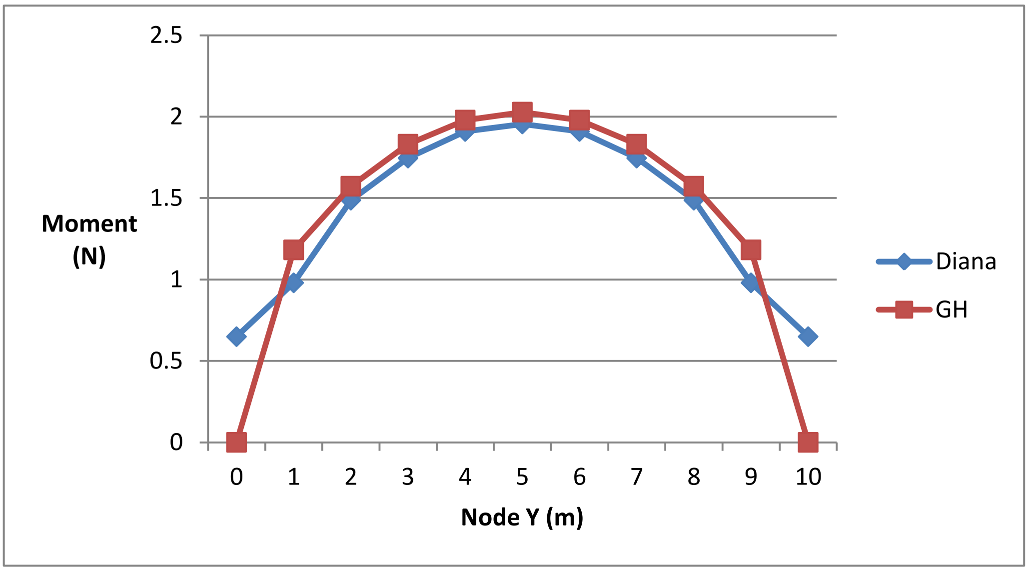


Figure 4.16: Sum of Bending Moment in Bound Y (t = 100mm)

- Bound Y (1000mm Thickness)

The unit of Node Y is m, and M is N.

Node Y	0	1	2	3	4	5
M(Diana)	1.753	1.3855	1.45415	1.6865	1.8218	1.8774
M(GH)	0	1.182	1.573	1.831	1.979	2.028
Ratio Y	0	1.172165821	0.924444	0.921081	0.920566	0.92574

Node Y	6	7	8	9	10
M(Diana)	1.8218	1.6865	1.45415	1.3855	1.753
M(GH)	1.979	1.831	1.573	1.182	0
Ratio Y	0.920566	0.921081	0.924444	1.172166	0

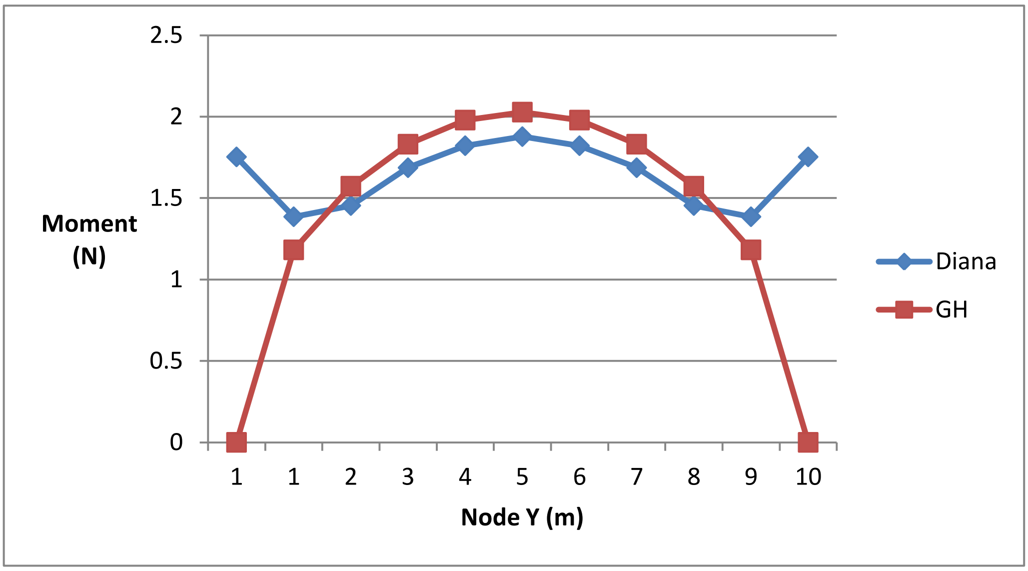


Figure 4.17: Sum of Bending Moment in Bound Y (t = 1000mm)

As it can be seen that in Diana program, for free edges, the results is not fulfill the plate theory. The sum of bending moment in the corner should be zero. It means that the Diana results of the boundaries are not that reliable for free edge. However, at the rest part of the produced results still can be used for validation. So the points at the central line are used for comparison.

- Center X (1000mm Thickness)

The unit of Node X is m, and M is N.

Node X	0	1	2	3	4	5
M(Diana)	1.8774	2.1685	2.443	2.68	2.875	3.012
M(GH)	2.0275	2.2987	2.5556	2.7853	2.9759	3.1183
Ratio X	0.925968	0.94335929	0.95594	0.962194	0.966094	0.965911

Node X	6	7	8	9	10	11
M(Diana)	3.108	3.133	3.108	3.012	2.875	2.68
M(GH)	3.2062	3.2359	3.2062	3.1183	2.9759	2.7853
Ratio X	0.969372	0.968201	0.969372	0.965911	0.966094	0.962194

Node X	12	13	14
M(Diana)	2.443	2.1685	1.8774
M(GH)	2.5556	2.2987	2.0275
Ratio X	0.95594	0.943359	0.925968

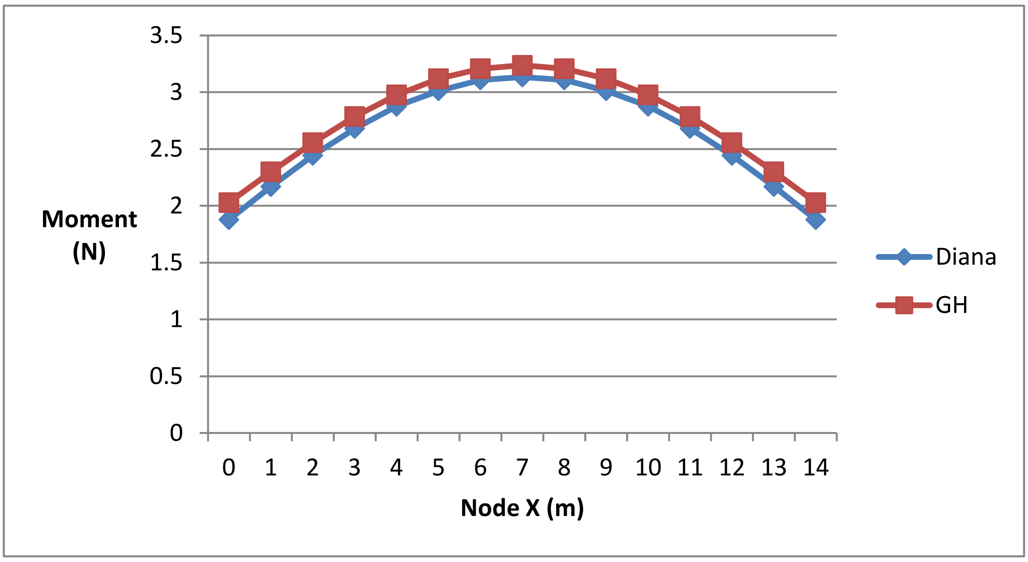


Figure 4.18: Sum of Bending Moment in Center X (t = 1000mm)

- Center Y (1000mm Thickness)

The unit of Node Y is m, and M is N.

Node Y	0	1	2	3	4	5
M(Diana)	2.6882	2.829	2.961	3.0535	3.1085	3.133
M(GH)	2.7855	2.9345	3.0604	3.1559	3.2156	3.2359
Ratio Y	0.965069	0.96404839	0.967521	0.967553	0.966694	0.968201

Node Y	6	7	8	9	10
M(Diana)	3.1085	3.0535	2.961	2.829	2.6882
M(GH)	3.2156	3.1559	3.0604	2.9345	2.7855
Ratio Y	0.966694	0.967553	0.967521	0.964048	0.965069

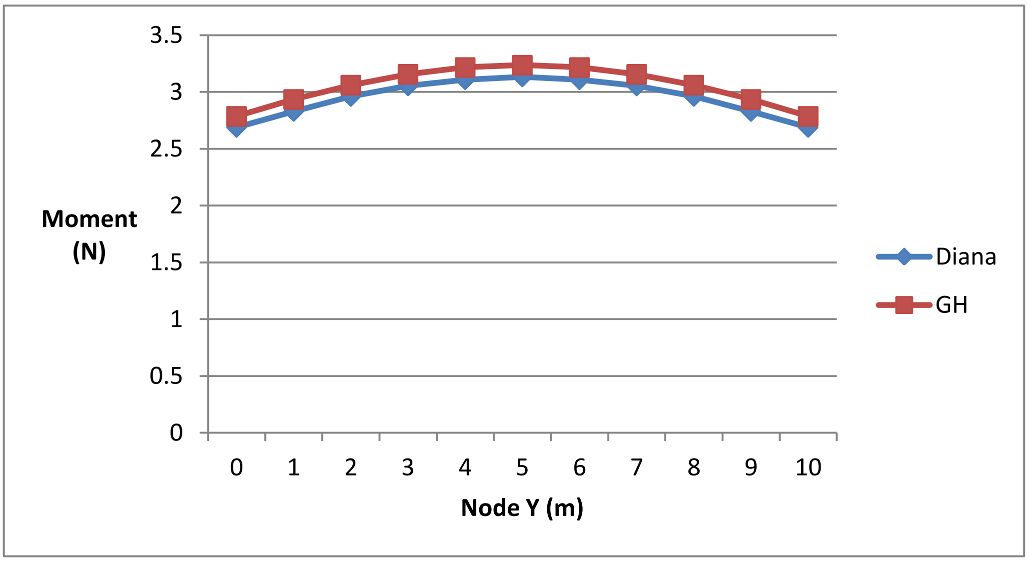


Figure 4.19: Sum of Bending Moment in Center Y (t = 1000mm)

4.4. Conclusion

Since the main task of this part is to generate the correct membrane boundary that represented the sum of bending moment, the verification is focus on the membrane shape. Other structural calculation result like displacement and shear will not be used for checking the validation. These results are computed by finite difference component which is produced by previous model, and the validation has been testified before.

According to the evaluation of different points with results comparison, the Grasshopper model shows good calculation performance of out – of – plane mechanic analysis.

5. In– Plane Parametric Design Tool

5.1. Introduction

In this chapter, the development of in – plane parametric design tool for thin plate will be present. The model will follow the structure of out – of – plane model. The reason of establishing a new model on previous one is that they share the same theory of plate analysis. Another consideration is that in the final stage, the out – of – plane model and the in – plane program will be combined as one integrated tool for structure calculation. Models within similar structure will dramatically reduce the complexity during the combination.

In the following section, the program components will be introduced. By giving the description of different components, the theoretical framework and the working process will be presented. And one will understand how the theories are implemented into the conceived model.

5.2. Usability & Functionality

The targeted users of the program are the same as the out – of – plane model which are architects and structural engineers. The goals are providing insightful structural evaluation in the conceptual design stage for freeform structure.

The usability and functionality aligns with the out – of – plane program. Therefore the usability is as follows:

- Provide real – time results during the design process.
- Able to change the design parameters, such as geometry, load case and support conditions.
- Able to modify the program for different users and further developments.

When it comes to functionality, it demands the program to convey the qualitative and quantitative insight of thin plate structure for plate in – plane mechanics. The chosen result to present should be directly linked to the criteria of structural analysis. The following numerical result will be computed and showed in the program.

- Stress function
- Normal and shear stress
- Principle stress
- Displacement

In comprehensive FEM computational program the results are presented by number. However in this thesis the model will come out with the shape of the value, which has a more straightforward view of the result.

5.3. General Outlines

As the thesis mentioned before, the structure of in – plane model will be similar with out – of – plane program. The basic outlines are as follows:

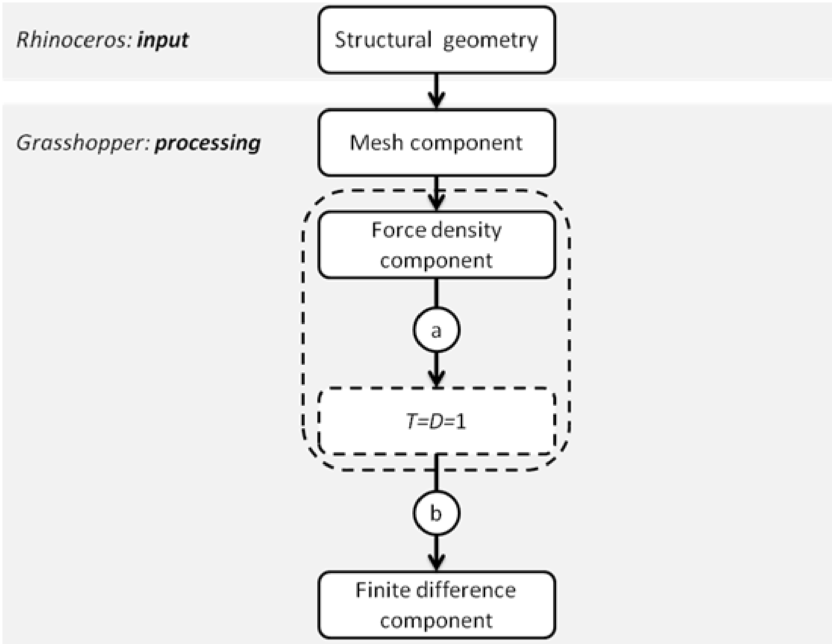


Figure 5.1: Outline of in – plane model

- **Structural Geometry**

The input component of structural geometry define the rectangle plate dimension and the mesh width

- **Meshing Component**

For calculation purposes, meshing component generate the grid of the plates. After considered the convenience of implementing finite difference method, the square mesh is decided. Also in this component, the nodes are sorted for different application.

- **Force Density Component**

Here is the place that membrane analogy is utilized. The component is used for form finding. By integrating the equilibrium equations with force density method, the membrane is shaped by Rhino.

- **Finite Difference Component**

According to the plate theory, the mechanics behaviors are computed by differential equations. However in Rhino program, such application to calculate the differential equations does not exist. For this purpose, finite difference method component is applied to achieve the result for deflections, shears and moments.

Although the models share similar outlines, there are several modifications for in – plane tool.

One is that the rain - flow analogy and derivation components are deleted. The reason of this is the theory of implementing the analogy into in – plane behavior still unclear. The meaning of the stream lines of the membrane in the in – plane behavior has not be defined yet. Quite little researches pay attention to this topic, therefore, less material show the answer of this analogy. Since the model uses the same structure as the previous one, most of the component will be the same. There is no changes in the geometry defined component and the meshing component. For finite difference component, the concept is the same. The only difference is the differential formula.

5.4. Analysis Case

The scope of the in – plane program is confined to certain boundary conditions. The main task of the thesis for in – plane behavior analysis is to set the foundation of the program. It is the first try of in – plane tool realization. Based on this purpose the analysis situation is confined as a rectangular plate under loads acting on the edges that are parallel to x – axis. The edges parallel to y – axis is the fixed edges, which the vertical displacement u_y is restricted. There is no vertical displacement at those edges. But the horizontal displacement u_x is free. And there is no external load that will acting on these edges.

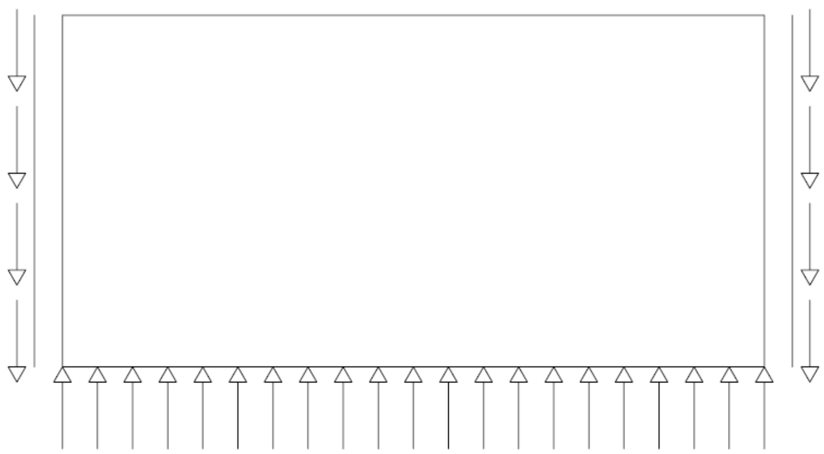


Figure 5.2: Analysis case

And of course, in the future, the sophisticated parametric program should fulfill the demands for different boundary conditions calculation. But at this thesis, the analysis case is confined for the above mentioned situation.

5.5. Force Density Component

The main difference between out – of – plane program and in – plane program is the force density component for membrane analogy. In out – of – plane program the membrane is used to simulate the shape of bending moment summation. But for in – plane program, the different type of mechanics will lead to different meaning of this analogy. This time the membrane shape is linked to the stress function. The theory has been spoken out in the previous chapter.

By using the stress function as the solution for plate analysis, the other structural behavior can be derived from this answer. After applying finite difference method, the structural result like normal stresses, shear stresses and displacement will be showed based on the membrane shape.

5.5.1. Basic Idea

The membrane analogy is base on a second order differential equation (2.15).

$$-p = \frac{\partial^2 w_m}{\partial x^2} + \frac{\partial^2 w_m}{\partial y^2} = \nabla^2 w_m \tag{5.1}$$

However, according to the theory the stress function should be computed by a forth order equation (2.5).

$$0 = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\left(\frac{\partial^2 \emptyset}{\partial x^2} + \frac{\partial^2 \emptyset}{\partial y^2}\right) = \nabla^2 \nabla^2 \emptyset \tag{5.2}$$

Then the problem now comes to how to extend the membrane analogy second order differential equation to forth order equation. As the theory said that, the second order differential equation of stress function will lead to sum of normal stress.

$$n = \frac{\partial^2 \emptyset}{\partial x^2} + \frac{\partial^2 \emptyset}{\partial y^2} = \nabla^2 \emptyset \tag{5.3}$$

The membrane analogy can be used here by regarding the sum of normal stress *n* is a known parameter. But in fact, the sum of normal stress cannot be calculated beforehand. To solve this question another second order differential equation need to be used.

$$0 = \frac{\partial^2 n}{\partial x^2} + \frac{\partial^2 n}{\partial y^2} = \nabla^2 n \tag{5.4}$$

This time the symbol at the left side of the equation, which is equal to zero, is a known parameter in the plate mechanics theory.

Then the following idea shows the process to solve the biharmonic equation. In equation (5.1) and (5.4):

$$-p = \frac{\partial^2 w_m}{\partial x^2} + \frac{\partial^2 w_m}{\partial y^2} = \nabla^2 w_m$$

$$0 = \frac{\partial^2 n}{\partial x^2} + \frac{\partial^2 n}{\partial y^2} = \nabla^2 n$$

In this equation series, the membrane analogy can be applied by regarding the membrane with no loads. It means

$$-p = 0$$

Then:

$$\nabla^2 w_m = \frac{\partial^2 w_m}{\partial x^2} + \frac{\partial^2 w_m}{\partial y^2} = 0$$

$$\nabla^2 n = \frac{\partial^2 n}{\partial x^2} + \frac{\partial^2 n}{\partial y^2} = 0$$

$$w_m \sim n$$

Under the given boundaries of the membrane, the shape can be generated. The membrane form represents the value of sum of bending moment *n*. Now the values of *n* are obtained. And again the membrane analogy can be used. In equation (5.1) and (5.3):

$$-\frac{p}{T} = \frac{\partial^2 w_m}{\partial x^2} + \frac{\partial^2 w_m}{\partial y^2} = \nabla^2 w_m$$

$$n = \frac{\partial^2 \emptyset}{\partial x^2} + \frac{\partial^2 \emptyset}{\partial y^2} = \nabla^2 \emptyset$$

This time the values of *n* can be regarded as the loads on the membrane.

$$-\frac{p}{T} = n$$

Then:

$$\nabla^2 w_m = \frac{\partial^2 w_m}{\partial x^2} + \frac{\partial^2 w_m}{\partial y^2} = -\frac{p}{T} = n$$

$$\nabla^2 \emptyset = \frac{\partial^2 \emptyset}{\partial x^2} + \frac{\partial^2 \emptyset}{\partial y^2} = n$$

$$w_m \sim \emptyset$$

At this stage, the results of stress function are achieved. The computation procedure above shows that the membrane analogy is implemented twice. Therefore in this thesis such process is call “double membrane analogy”. Then it leads to the conclusion that by applying the membrane analogy of the equation (5.4), the answer of the sum of normal stress will become a known factor. And by using the membrane analogy again in the equation (5.3), the answer of stress function will be found.

It may come to a question why not use the sum of normal stress as the solution to compute the structural behavior. Like the out – of – plane program, the solution for calculation is the sum of bending moment which is a second order differential equation and computed by using the membrane analogy once.

The reasons are several. One is that in out – of – plane program the membrane of bending moment summation will be used for the rain – flow analysis. The gradient of the bending moment summation has meaning as shear force direction and magnitude. But for in – plane program, the gradient of sum of normal stress means nothing. So far there is no information and research showing what the definition is. And also in the in – plane program, the rain – flow component and derivative component are moved out. There is no need to stick to use sum of normal stress as solution.

Another consideration indicated the reason about choosing stress function instead of sum of normal stress as solution is the boundary condition. For in – plane program, the boundary conditions are a bit more complex than the out – of – plane one. In out – of – plane program, the simple supported edge can be easily realized by just setting the bending moment summation to be zero. The other boundary conditions like fixed edge and free edge need another approach. The method is to translate the bending moment into plate displacement first. Based on that, and combining the finite difference method, the boundary conditions are defined. The in – plane mechanics do not has this simple situation like simple supported edge. All the boundary condition should be stated with the manner of stress function. Then why not use the stress function as the solution at first. Furthermore, the physical meanings of stress function is not quite clear, the translation from sum of normal stress to stress function is not that simple as the out – of – plane program. To reduce the complexity of the tool, the stress function is a better choice.

Based on the preceding reason the force density component is used to generate the form of stress function *Φ*.

The basic boundary conditions for in – plane behavior is as follows:

- Load edge

Figure of boundary conditions

$$p = n_{yy} = \frac{\partial^2 \phi}{\partial x^2} \tag{5.5}$$

$$0 = n_{xy} = \frac{\partial^2 \phi}{\partial x \partial y} \tag{5.6}$$

- fixed edge

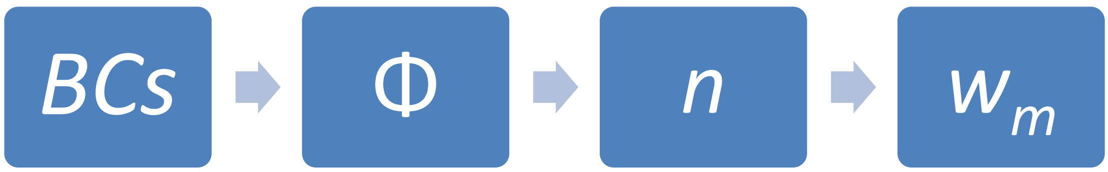
Figure of boundary conditions

$$0 = n_{xx} = \frac{\partial^2 \phi}{\partial y^2} \tag{5.7}$$

$$0 = u_y \tag{5.8}$$

Since the analysis case is confined, only the above boundary cases will be introduced in the program. As people can see that the conditions should be translated into stress function first. Then regarding the edges situation, the finite difference method helps to express the boundary shape of normal stress summation. After that the membrane analogy is completed. Therefore the in – plane model will follow the below computation procedure.

- In – Plane Model



Since the boundary conditions have been set, the edges parallel to x – axis are loaded boundary, others are fixed edges. The only input of the component is the magnitude of the acting loads. Next chapter is to present the calculation theory of the force density component.

5.5.2. Basic Equations

The formula below is derived from the equation in the thesis of *H. Schek* (1973).

The free nodes are interpreted as points $P_{i\text{-}free}$ with coordinates ($x_{i\text{-}free}$, $y_{i\text{-}free}$, $z_{i\text{-}free}$), *I = 1, …, n*, and the boundary nodes are as $P_{i\text{-}fixed}$ with coordinates ($x_{i\text{-}fixed}$, $y_{i\text{-}fixed}$, $z_{i\text{-}fixed}$), *I = 1, …, n*.

The coordinates of all the free nodes form the $\boldsymbol{n_{free}}$ – vector $\boldsymbol{x_{free}}$, $\boldsymbol{y_{free}}$, $\boldsymbol{z_{free}}$ and the $\boldsymbol{n_{fixed}}$ – vector $\boldsymbol{x_{fixed}}$, $\boldsymbol{y_{fixed}}$, $\boldsymbol{z_{fixed}}$ for all the boundary nodes.

Since the previous chapter has stated the theory of force density method, the later part will skip the introduction of this method.

Force Density:

- Equilibrium Formula

$$\boldsymbol{C_{free}^{t} Q C_{free} \cdot z_{free} + C_{free}^{t} Q C_{fixed} \cdot z_{fixed} = p_{free}}$$

The $\boldsymbol{C}$ is the branch – node matrix indicates the connectivity of network system.
And $\boldsymbol{Q}$ is the force density matrix. In minimal surface all the force density should be equal to one. Therefore, the matrix $\boldsymbol{Q}$ equal to unit matrix $\boldsymbol{E}$.
And rephrase the formula:

$$\boldsymbol{C_{free}^{t} C_{free} \cdot z_{free} + C_{free}^{t} C_{fixed} \cdot z_{fixed} = p_{free}}$$

For simplicity, set:

$$\boldsymbol{D_{free} = C_{free}^{t} C_{free}}$$
$$\boldsymbol{D_{fixed} = C_{free}^{t} C_{fixed}}$$

Then:

$$\boldsymbol{D_{free} \cdot z_{free} + D_{fixed} \cdot z_{fixed} = p_{free}}$$

With

$$\boldsymbol{z} = \begin{bmatrix} \boldsymbol{z_{fixed}} \\ \boldsymbol{z_{free}} \end{bmatrix}$$

For further explanation the formula is rewritten into other symbols:

$$\boldsymbol{D_{mid} \cdot z_{mid} + D_{bound} \cdot z_{bound} = p_{mid}}$$

The theories that are applied to the force density component are as follows.

Plate Theory:

- Stress Function

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = n$$

- Stress Summation

$$\frac{\partial^2 n}{\partial x^2} + \frac{\partial^2 n}{\partial y^2} = 0$$

- Loaded Edge

$$\frac{\partial^2 \phi}{\partial x^2} = n_{yy}$$
$$\frac{\partial^2 \phi}{\partial x \partial y} = n_{xy}$$

- Boundary Edge

$$u_y = 0$$

Lead to

$$\frac{\partial^2 \phi}{\partial x^2} = n_{yy} = \frac{\partial u_y}{\partial y} = 0$$

In here the double membrane analogy is presented. The equations below are the theories that combined with finite difference method. Since the expression of boundary conditions is complicate, the following nodes distributions will be used. The free nodes are called Mid Point. The edges nodes are distributed to Bound Point. According to finite difference method, there are hidden nodes. Those nodes are called Outer Point. The Bound Points are exactly lined on the edges of plate. The Outer Points are the imaged points located outside the plate's boundaries. The existing of the Outer Points is only used for boundary condition calculation.

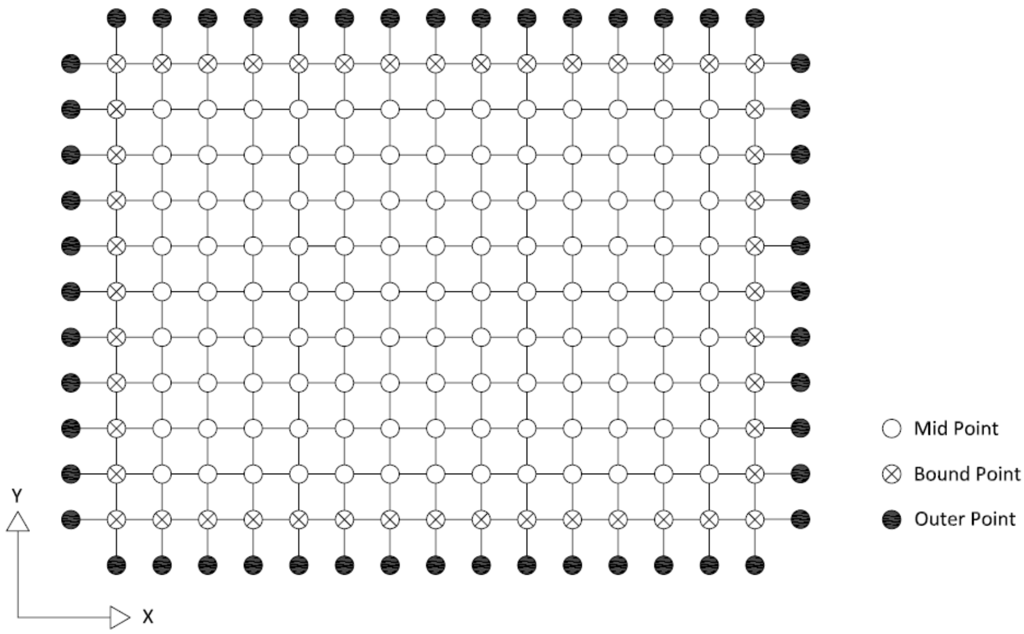


Figure 5.3: Points classification

Finite Difference Method:

- Stress Function

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}=n$$

Rewrite into finite difference method manner:

$$\boldsymbol{D_{mid}}\cdot\boldsymbol{\phi_{mid}}+\boldsymbol{D_{bound}}\cdot\boldsymbol{\phi_{bound}}=\boldsymbol{n_{mid}} \tag{5.9}$$

- Stress Summation

$$\frac{\partial^2 n}{\partial x^2}+\frac{\partial^2 n}{\partial y^2}=0$$

Rewrite into finite difference method manner:

$$\boldsymbol{D_{mid}}\cdot\boldsymbol{n_{mid}}+\boldsymbol{D_{bound}}\cdot\boldsymbol{n_{bound}}=\boldsymbol{0} \tag{5.10}$$

Lead to

$$\boldsymbol{n_{mid}}=-\boldsymbol{D_{mid}^{-1}}\cdot\boldsymbol{D_{bound}}\cdot\boldsymbol{n_{bound}}$$

Then equation (5.9) is:

$$\boldsymbol{D_{mid}}\cdot\boldsymbol{\phi_{mid}}+\boldsymbol{D_{bound}}\cdot\boldsymbol{\phi_{bound}}=-\boldsymbol{D_{mid}^{-1}}\cdot\boldsymbol{D_{bound}}\cdot\boldsymbol{n_{bound}} \tag{5.11}$$

The above equation is the double membrane analogy. By using this equation the membrane shape will be generated. The next part is implementing the boundary for membrane form finding.

- Loaded Boundary Condition (5.5)

$$\frac{\partial^2\phi}{\partial x^2} = n_{yy} = p$$

Rewrite into finite difference method manner:

$$\boldsymbol{I_{bound}} \cdot \boldsymbol{\phi_{bound}} = \boldsymbol{N}$$

Then

$$\boldsymbol{\phi_{bound}} = \boldsymbol{I_{bound}^{-1}} \cdot \boldsymbol{N} \quad (5.12)$$

The ***N*** is the applied load matrix. And $\boldsymbol{I_{bound}}$ is the difference finite operator matrix for second order differential in one direction.

Another loaded boundary condition (5.6) is the shear stress is zero.

$$\frac{\partial^2\phi}{\partial x \partial y} = n_{xy} = 0$$

Rewrite into finite difference method manner:

$$\boldsymbol{G_{mid}} \cdot \boldsymbol{\phi_{mid}} + \boldsymbol{G_{bound}} \cdot \boldsymbol{\phi_{bound}} + \boldsymbol{G_{out,x}} \cdot \boldsymbol{\phi_{out,x}} = \boldsymbol{0} \quad (5.13)$$

By replacing equation (5.12) into equation (5.11) the formula will be:

$$\boldsymbol{D_{mid}} \cdot \boldsymbol{\phi_{mid}} + \boldsymbol{D_{bound}} \cdot \boldsymbol{I_{bound}^{-1}} \cdot \boldsymbol{N} = -\boldsymbol{D_{mid}^{-1}} \cdot \boldsymbol{D_{bound}} \cdot \boldsymbol{n_{bound}}$$

And equation (5.13) is:

$$\boldsymbol{G_{mid}} \cdot \boldsymbol{\phi_{mid}} + \boldsymbol{G_{bound}} \cdot \boldsymbol{I_{bound}^{-1}} \cdot \boldsymbol{N} + \boldsymbol{G_{out,x}} \cdot \boldsymbol{\phi_{out,x}} = \boldsymbol{0}$$

Then

$$\boldsymbol{\phi_{out.x}} = -\boldsymbol{G_{out,x}^{-1}} \cdot \left(\boldsymbol{G_{mid}} \cdot \boldsymbol{\phi_{mid}} + \boldsymbol{G_{bound}} \cdot \boldsymbol{I_{bound}^{-1}} \cdot \boldsymbol{N}\right) \quad (5.14)$$

- Fixed Boundary Condition (5.8)

$$\frac{\partial^2\phi}{\partial x^2} = n_{yy} = \frac{\partial u_y}{\partial y} = 0$$

By using finite difference method:

$$\boldsymbol{J_{mid}} \cdot \boldsymbol{\phi_{mid}} + \boldsymbol{J_{bound}} \cdot \boldsymbol{\phi_{bound}} + \boldsymbol{J_{out,y}} \cdot \boldsymbol{\phi_{out,y}} = \boldsymbol{0}$$

Using the same equation (5.12) to replace $\boldsymbol{\Phi_{bound}}$:

$$\boldsymbol{J_{mid}} \cdot \boldsymbol{\phi_{mid}} + \boldsymbol{J_{bound}} \cdot \boldsymbol{I_{bound}^{-1}} \cdot \boldsymbol{N} + \boldsymbol{J_{out,y}} \cdot \boldsymbol{\phi_{out,y}} = \boldsymbol{0}$$

Lead to

$$\boldsymbol{\phi_{out.y}} = -\boldsymbol{J_{out,y}^{-1}} \cdot \left(\boldsymbol{J_{mid}} \cdot \boldsymbol{\phi_{mid}} + \boldsymbol{J_{bound}} \cdot \boldsymbol{I_{bound}^{-1}} \cdot \boldsymbol{N}\right) \quad (5.15)$$

- Boundary Condition Combination

Here the different types of boundary conditions (5.14) and (5.15) are combined into a single formula.

$$\boldsymbol{\phi_{out.x}} = -\boldsymbol{G_{out,x}^{-1}} \cdot (\boldsymbol{G_{mid}} \cdot \boldsymbol{\phi_{mid}} + \boldsymbol{G_{bound}} \cdot \boldsymbol{I_{bound}^{-1}} \cdot \boldsymbol{N})$$
$$\boldsymbol{\phi_{out.y}} = -\boldsymbol{J_{out,y}^{-1}} \cdot (\boldsymbol{J_{mid}} \cdot \boldsymbol{\phi_{mid}} + \boldsymbol{J_{bound}} \cdot \boldsymbol{I_{bound}^{-1}} \cdot \boldsymbol{N})$$

Combine the above two equations:

$$\boldsymbol{\phi_{out}} = \begin{bmatrix} -\boldsymbol{G_{out,x}^{-1}} \cdot (\boldsymbol{G_{mid}} \cdot \boldsymbol{\phi_{mid}} + \boldsymbol{G_{bound}} \cdot \boldsymbol{I_{bound}^{-1}} \cdot \boldsymbol{N}) \\ -\boldsymbol{J_{out,y}^{-1}} \cdot (\boldsymbol{J_{mid}} \cdot \boldsymbol{\phi_{mid}} + \boldsymbol{J_{bound}} \cdot \boldsymbol{I_{bound}^{-1}} \cdot \boldsymbol{N}) \end{bmatrix}$$

For simplicity assume:

$$\boldsymbol{F_{mid}} = \begin{bmatrix} \boldsymbol{G_{out,x}^{-1}} \cdot \boldsymbol{G_{mid}} \\ \boldsymbol{J_{out,y}^{-1}} \cdot \boldsymbol{J_{mid}} \end{bmatrix}$$

$$\boldsymbol{F_{bound}} = \begin{bmatrix} \boldsymbol{G_{out,x}^{-1}} \cdot \boldsymbol{G_{bound}} \\ \boldsymbol{J_{out,y}^{-1}} \cdot \boldsymbol{J_{bound}} \end{bmatrix}$$

Then

$$\boldsymbol{\phi}_{out} = -\boldsymbol{F}_{mid} \cdot \boldsymbol{\phi}_{mid} - \boldsymbol{F}_{bound} \cdot \boldsymbol{I}_{bound}^{-1} \cdot \boldsymbol{N} \tag{5.16}$$

Now the equation to calculate $\boldsymbol{\Phi}_{out}$ has been defined. Here the $\boldsymbol{n}_{bound}$ also align with membrane analogy.

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = n$$

This equation is translated in the language of finite difference method.

$$\boldsymbol{n}_{bound} = \boldsymbol{K}_{mid} \cdot \boldsymbol{\phi}_{mid} + \boldsymbol{K}_{bound} \cdot \boldsymbol{I}_{bound}^{-1} \cdot \boldsymbol{N} + \boldsymbol{K}_{out} \cdot \boldsymbol{\phi}_{out}$$

The matrices $\boldsymbol{K}$ are the finite difference operator matrices. Then by applied equation (5.16):

$$\boldsymbol{n}_{bound} = \boldsymbol{K}_{mid} \cdot \boldsymbol{\phi}_{mid} + \boldsymbol{K}_{bound} \cdot \boldsymbol{I}_{bound}^{-1} \cdot \boldsymbol{N} - \boldsymbol{K}_{out} \cdot (\boldsymbol{F}_{mid} \cdot \boldsymbol{\phi}_{mid} + \boldsymbol{F}_{bound} \cdot \boldsymbol{I}_{bound}^{-1} \cdot \boldsymbol{N})$$

$$\boldsymbol{n}_{bound} = (\boldsymbol{K}_{mid} - \boldsymbol{K}_{out} \cdot \boldsymbol{F}_{mid}) \cdot \boldsymbol{\phi}_{mid} + (\boldsymbol{K}_{bound} - \boldsymbol{K}_{out} \cdot \boldsymbol{F}_{bound}) \cdot \boldsymbol{I}_{bound}^{-1} \cdot \boldsymbol{N}$$

The component calculates the value based on the following equation (applied equation (5.11)):

$$\left[\boldsymbol{D}_{mid} + \boldsymbol{D}_{mid}^{-1} \cdot \boldsymbol{D}_{bound} \cdot (\boldsymbol{K}_{mid} - \boldsymbol{K}_{out} \cdot \boldsymbol{F}_{mid})\right] \cdot \boldsymbol{\phi}_{mid} = -[\boldsymbol{D}_{bound} + \boldsymbol{D}_{mid}^{-1} \cdot \boldsymbol{D}_{bound} \cdot (\boldsymbol{K}_{bound} - \boldsymbol{K}_{out} \cdot \boldsymbol{F}_{bound})] \cdot \boldsymbol{I}_{bound}^{-1} \cdot \boldsymbol{N}$$

$$\boldsymbol{\phi}_{mid} = -\left[\boldsymbol{D}_{mid} + \boldsymbol{D}_{mid}^{-1} \cdot \boldsymbol{D}_{bound} \cdot (\boldsymbol{K}_{mid} - \boldsymbol{K}_{out} \cdot \boldsymbol{F}_{mid})\right]^{-1} \cdot [\boldsymbol{D}_{bound} + \boldsymbol{D}_{mid}^{-1} \cdot \boldsymbol{D}_{bound} \cdot (\boldsymbol{K}_{bound} - \boldsymbol{K}_{out} \cdot \boldsymbol{F}_{bound})] \cdot \boldsymbol{I}_{bound}^{-1} \cdot \boldsymbol{N}$$

The above equation is applied to the component to calculate the stress function.

5.6. Finite Difference Component (Displacement)

In this component the computation procedure of normal stresses and shear forces is similar to the out – of – plane program. But for the outputs of displacement, the process to achieve the results needs to be introduced.

According to the plate theory (with zero Poisson' ratio assumption) the constitutive equations:

$$\varepsilon_{xx} = \frac{\partial u_x}{\partial x} = \frac{1}{Et} n_{xx}$$

$$\varepsilon_{yy} = \frac{\partial u_y}{\partial y} = \frac{1}{Et} n_{yy}$$

$$\gamma_{xy} = \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} = \frac{1}{Gt} n_{xy} = \frac{2}{Et} n_{xy}$$

Transfer the equations with finite difference method manner:

$$\boldsymbol{C}_x \cdot \boldsymbol{u}_x = \boldsymbol{n}_{xx}$$

$$\boldsymbol{C}_y \cdot \boldsymbol{u}_y = \boldsymbol{n}_{yy}$$

$$\boldsymbol{C}_x \cdot \boldsymbol{u}_y + \boldsymbol{C}_y \cdot \boldsymbol{u}_x = 2\boldsymbol{n}_{xy}$$

The stiffness *1 / Et* is a parameter that can be imported later.

By applied the finite difference method, the in – plane stresses $\boldsymbol{n}_{xx}$, $\boldsymbol{n}_{yy}$ and $\boldsymbol{n}_{xy}$ are obtained. With known values of $\boldsymbol{n}_{xx}$, $\boldsymbol{n}_{yy}$ and $\boldsymbol{n}_{xy}$, using the equations above the displacements $\boldsymbol{u}_x$ and $\boldsymbol{u}_y$ can be computed. However based on the degree of freedom, there are only two independent variables which are $\boldsymbol{u_x}$ and $\boldsymbol{u_y}$. Therefore, the stresses resultants are not fully independent. An inner relation occurs to describe the connection of the stresses, and it is the equation of strain compatibility. By looking a way to reduce three equations into two, the calculation process become linear, and then the displacements value will be generated.

Now the question goes to how to determine these two equations. Implementing the compatibility formula into constitutive equations is not easy. By taking into account the structure of the constitutive equations, one can find out that, $\boldsymbol{n_{xx}}$ can only be used to determine $\boldsymbol{u_x}$; and $\boldsymbol{n_{yy}}$ is for $\boldsymbol{u_y}$. To fully describe the plate displacement, the $\boldsymbol{n_{xy}}$ needs to be applied to determine the relation between $\boldsymbol{u_x}$ and $\boldsymbol{u_y}$.

But due to the nature of the membrane analogy, at the boundaries, the value did not fit the strain compatibility very well. Errors happen along the boundaries nodes. Therefore, introducing the $\boldsymbol{n_{xy}}$ to determine the displacement at the boundaries will lead to inaccurate results. In computation process, it should try to avoid using $\boldsymbol{n_{xy}}$ in the boundaries nodes. Then the calculation procedure is as follows:

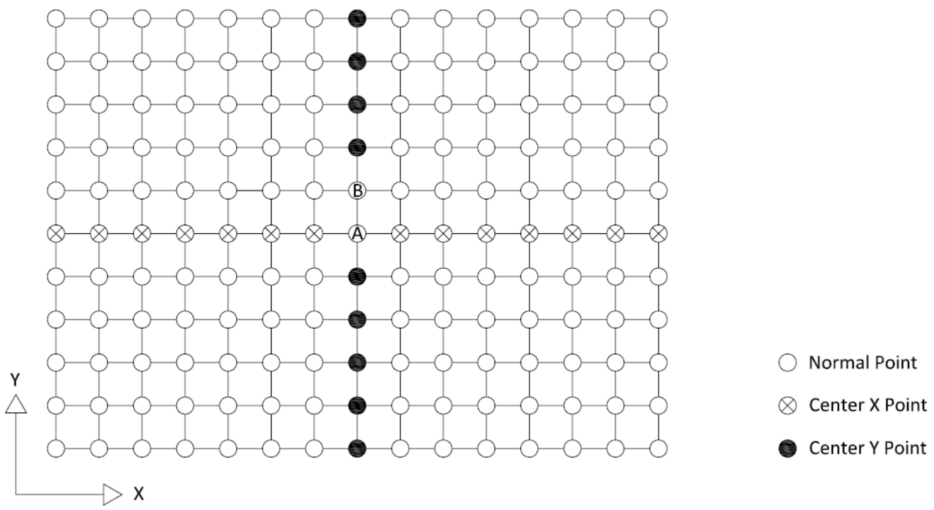


Figure 5.4: Nodes classification for displacements

The nodes distribution is as the figure shows. In fact the chosen Center X and Center Y points are not necessary to be exactly at the center lines. The only consideration is to avoid the chosen points closing to the boundaries. Otherwise the errors of the calculation become obvious. And these chosen points are used as datum marks for computation.

Only in the Center X and Center Y points the $\boldsymbol{n_{xy}}$ will be used at the rest Normal points, the displacements are only computed by $\boldsymbol{n_{xx}}$ and $\boldsymbol{n_{yy}}$.

In Center X Points:

$$\begin{bmatrix} \boldsymbol{C_x} & \boldsymbol{0} \\ \boldsymbol{C_y} & \boldsymbol{C_x} \end{bmatrix} \cdot \begin{bmatrix} \boldsymbol{u_x} \\ \boldsymbol{u_y} \end{bmatrix} = \begin{bmatrix} \boldsymbol{n_{xx}} \\ \boldsymbol{n_{xy}} \end{bmatrix}$$

In Center Y Points:

$$\begin{bmatrix} \boldsymbol{C_y} & \boldsymbol{C_x} \\ \boldsymbol{C_y} & \mathbf{0} \end{bmatrix} \cdot \begin{bmatrix} \boldsymbol{u_x} \\ \boldsymbol{u_y} \end{bmatrix} = \begin{bmatrix} \boldsymbol{n_{xy}} \\ \boldsymbol{n_{yy}} \end{bmatrix}$$

In Normal Points:

$$\begin{bmatrix} \boldsymbol{C_x} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{C_y} \end{bmatrix} \cdot \begin{bmatrix} \boldsymbol{u_x} \\ \boldsymbol{u_y} \end{bmatrix} = \begin{bmatrix} \boldsymbol{n_{xx}} \\ \boldsymbol{n_{yy}} \end{bmatrix}$$

In calculating the displacement another aspect needs to be considered that is the rigid body movement. During the process, the hypothesis is to assume the displacements u_x and u_y at point A are zero, and the horizontal displacement u_x at point B is also zero. After the computation, the whole deformed plate can be modified according to the boundaries conditions with correct rigid body movement to get the correct deformation.

6. In – Plane Result Verification

6.1. Introduction

After building up the program, the next step is to check the validation of the tool. As the same for out – of – plane verification, the produced values for comparison are come out with FEM (finite element method) program TNO Diana.

The analysis case is one edge loading. On the other loaded edges, there is no load acting. It is free edge. In line with Grasshopper model, the mesh space is square with dimension of 1m x 1m. The mesh type for structure calculation in Diana model is CQ16M.

The properties of the analysis model are:

1) $L = 20.0\ m$
2) $W = 10.0\ m$
3) $q = 0.1\ N/m$
4) $\upsilon = 0$
5) $E = 30000\ N/m^2$

The results for some special points are subtracted to check whether the program calculation is valid. The long edge points are defined as Bound X; and short edge points are Bound Y. For better verification, another two series of nodes are also used for result comparison which is Center X and Center Y. The geometry of the plate and the definition of nodes location are showed below.

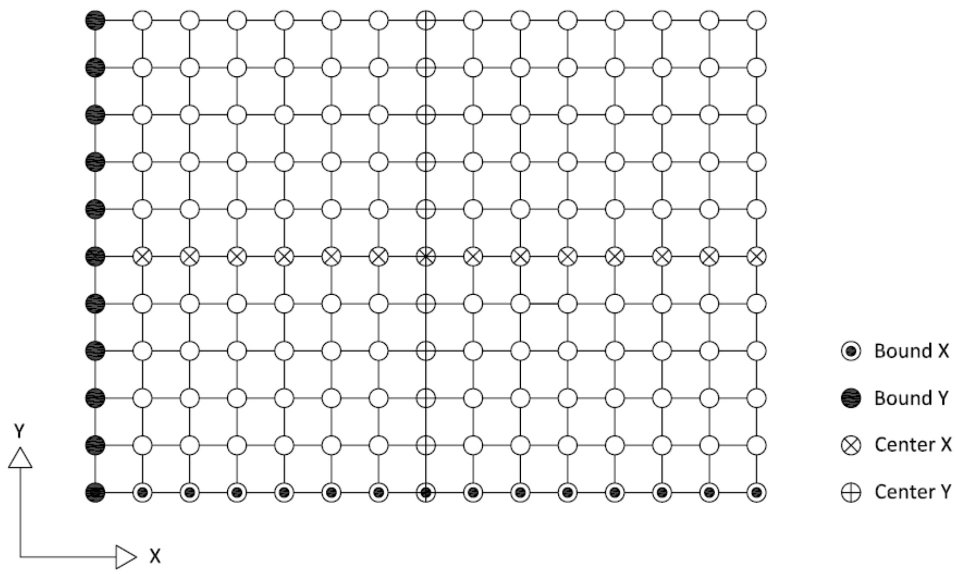


Figure 6.1: Nodes classification

Not only the Diana results will be provided to valid the model, there are two other research materials are presented. One is from the book that is written by *ir. W. J. Beranek* (*Toegepaste Mechanica K-3*). The stress function and stress results are recorded in the book. However, the results are not fully correct. The stress function does not fulfill the strain compatibility equation, and also the fixed boundary results do not align with the real condition. Another is from *J. Barber* (*Elasticity)*. These results are widely used in plate mechanics, although they are not 100% correct. In this verification section, the comparison for these results is also made to find the differences.

- The formulas from *Toegepaste Mechanica K-3*:

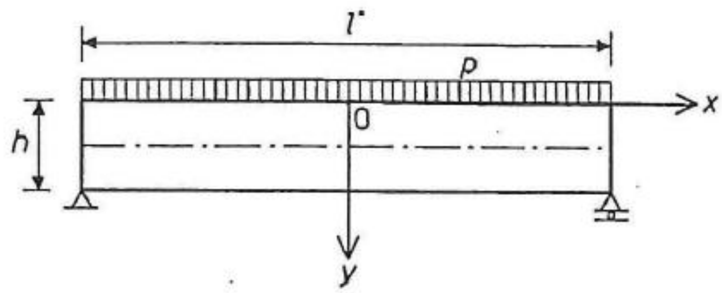


Figure 6.2: Analysis case in Toegepaste Mechanica K-3 (Picture from W. J. Beranek [5])

$$\phi = \frac{p}{8h^3}(l^2 - 4x^2)(2y^3 - 3hy^2 + h^3)$$
$$n_{xx} = \frac{\partial^2\phi}{\partial y^2} = \frac{3p}{4h^3}(l^2 - 4x^2)(2y - h)$$
$$n_{yy} = \frac{\partial^2\phi}{\partial x^2} = -\frac{p}{h^3}(2y^3 - 3hy^2 + h^3)$$
$$n_{xy} = \frac{\partial^2\phi}{\partial x\partial y} = -\frac{6p}{h^3}xy(h - y)$$

- The formulas from *Elasticity:*

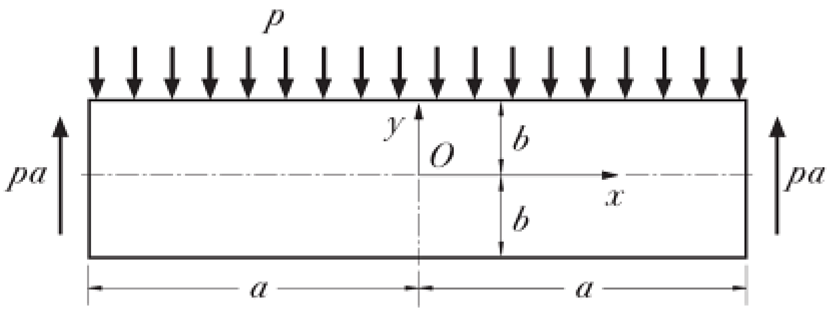


Figure 6.3: Analysis case in Elasticity (Picture from J. Barber [7])

$$\phi = \frac{p}{40b^3}(5x^2y^3 - y^5 - 15b^2x^2y - 5a^2y^3 + 2b^2y^3 - 10b^3x^2)$$
$$\sigma_{xx} = \frac{p}{20b^3}(15x^2y - 10y^3 - 15a^2y + 6b^2y)$$
$$\sigma_{xy} = \frac{3px}{4b^3}(b^2 - y^2)$$
$$\sigma_{yy} = \frac{p}{4b^3}(y^3 - 3b^2y - 2b^3)\ .$$

The formula in this book, the boundary conditions at the left and right side are so called "weak boundary condition".

$$F_x(a) = \int_{-b}^{b} \sigma_{xx}(a,y)dy = 0$$
$$F_y(a) = \int_{-b}^{b} \sigma_{xy}(a,y)dy = pa$$
$$M(a) = \int_{-b}^{b} \sigma_{xx}(a,y)ydy = 0$$

This boundary conditions only require the summation of the value to satisfy the stated requirements. Like the situation in this load case, in the fixed end the stress n_{xx} should be zero at each point. But the weak condition only demands the summation of n_{xx} is equal to zero. It does not matter whether all the point is equal to zero.

There is only one load case for the next chapter of verification.

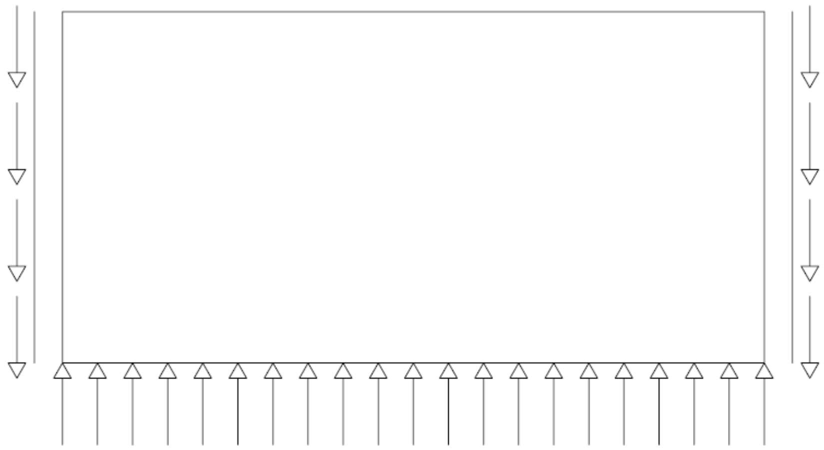


Figure 6.4: Analysis case

6.2. Verification

6.2.1. Qualitative Verification

In qualitative manner, the form finding result of stress function will be presented, and also the results of structural evaluation. The figures of normal stress, shear stress and plate displacement are the content.

- Result of stress function:

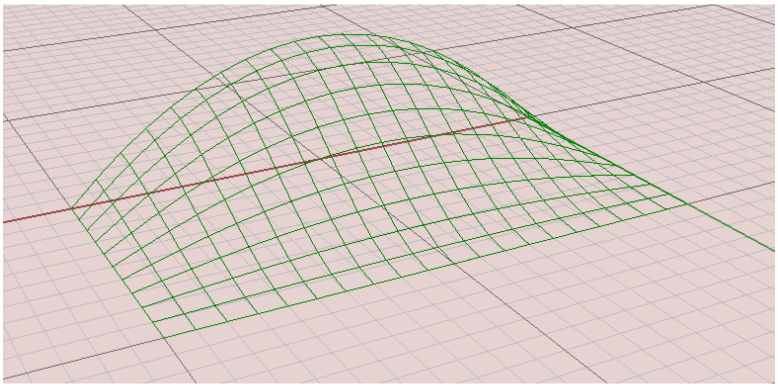


Figure 6.5: Stress function from Grasshopper model

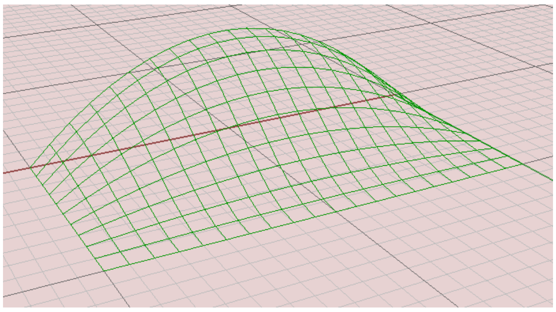


Figure 6.6: Beranek's stress function result

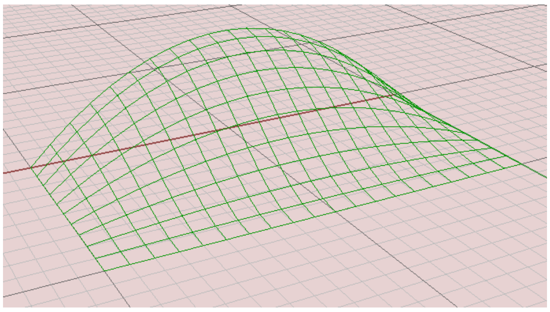


Figure 6.7: Barber's stress function result

- Result of normal stress n_{xx}:

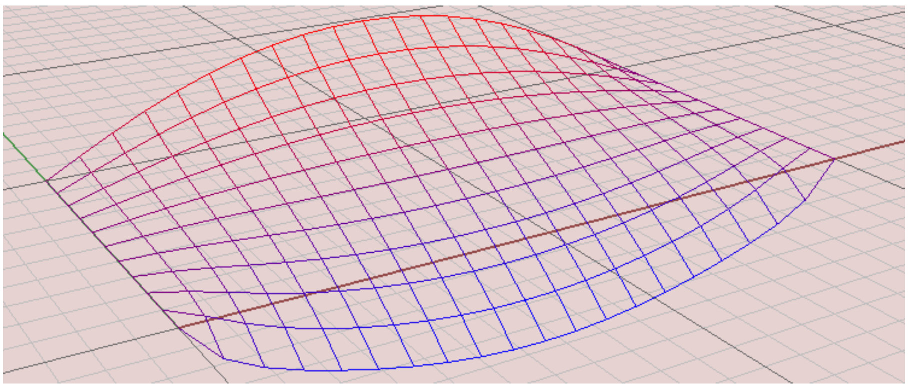


Figure 6.8: Nxx Stress from Grasshopper model

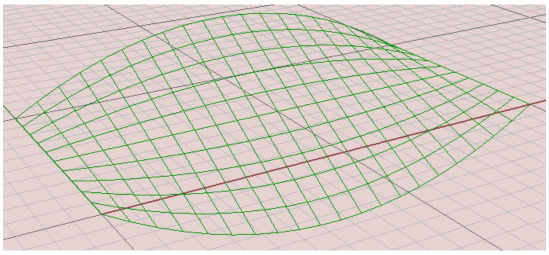


Figure 6.9: Beranek's Nxx stress result

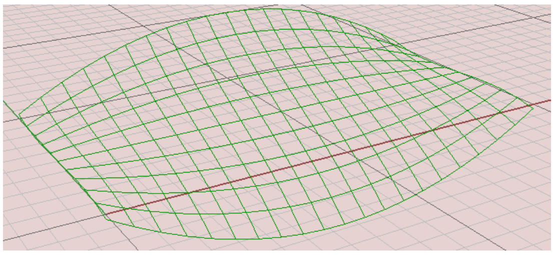


Figure 6.10: Barber's Nxx stress result

- Result of normal stress n_{yy}:

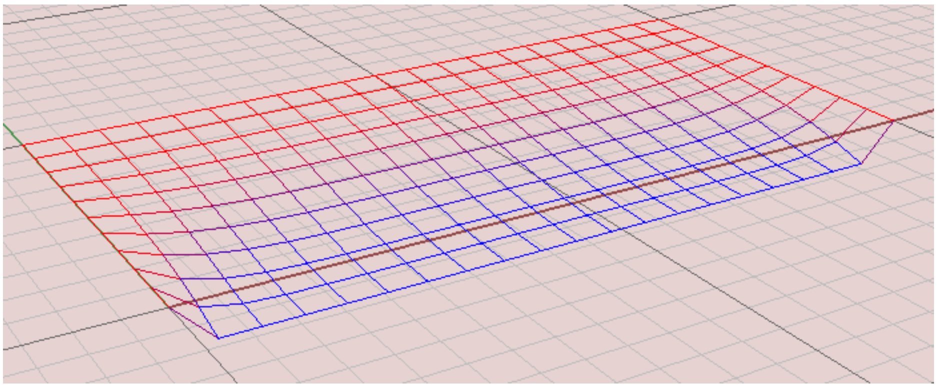


Figure 6.11: Nyy Stress from Grasshopper model

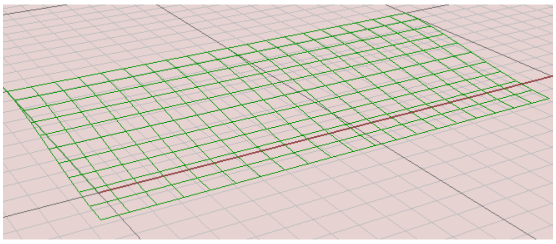


Figure 6.12: Beranek's Nyy stress result

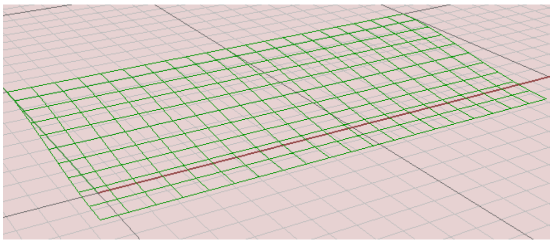


Figure 6.13: Barber's Nyy stress result

- Result of normal stress n_{xy}:

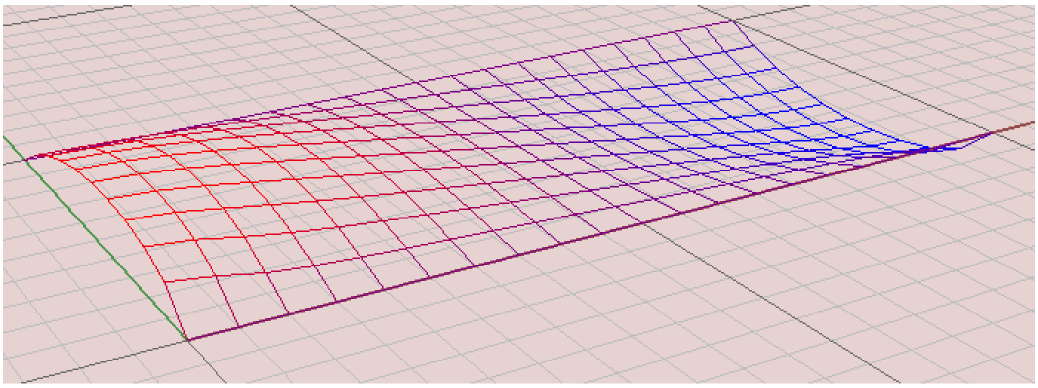
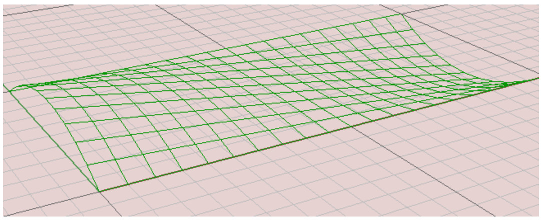


Figure 6.14: Nxy Stress from Grasshopper model



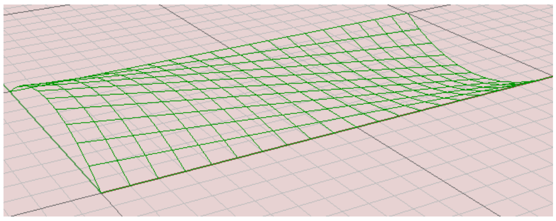


Figure 6.15: Beranek's Nxy stress result

Figure 6.16: Barber's Nxy stress result

- Result of displacement:

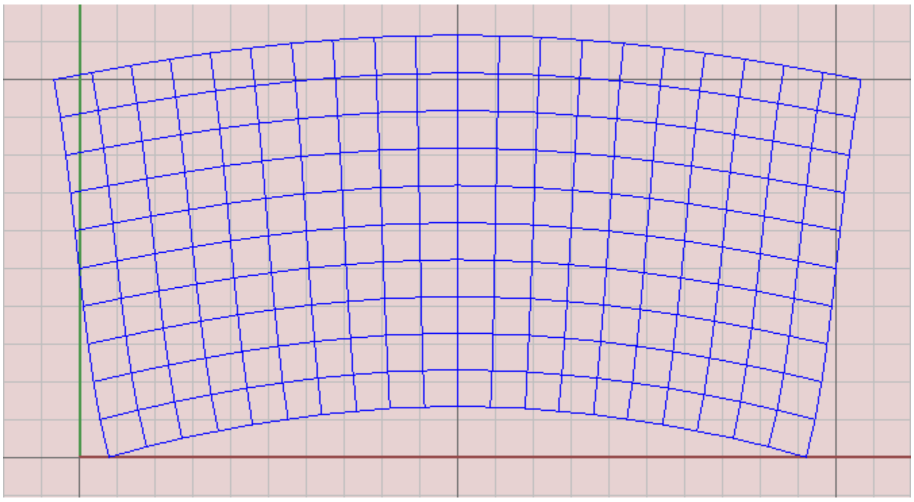


Figure 6.17: Plate deformation from Grasshopper model

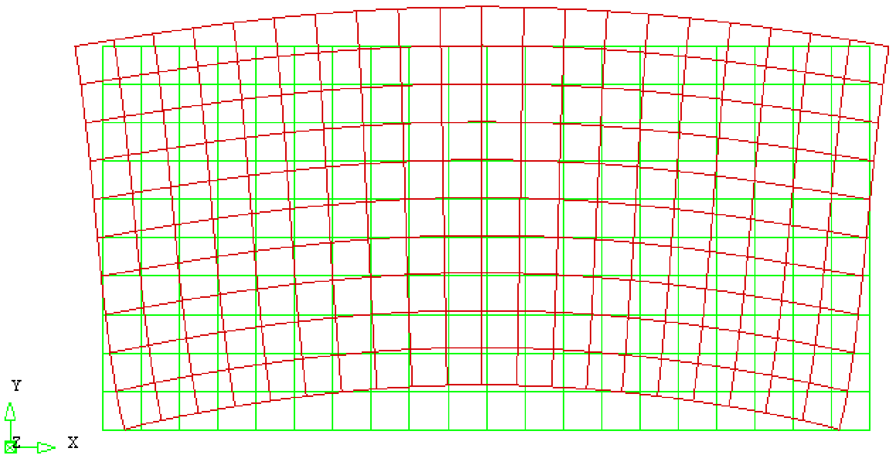


Figure 6.18: Plate deformation from Diana model

6.2.2. Quantitative Verification

In the table the (Diana) is the results which are calculated in TNO Diana FEM program. And (GH) is the value of Rhino Grasshopper model. The ratio is the number of (Diana) divided by those with (GH). The figure shows the shape of the results.

The following results are chosen to compare: $N_{xx}, N_{yy}, N_{xy}, u_x, u_y$

- Bound X (The unit of Node X is m, and Nxx, Nyy and Nxy is N/m)

Diana							
Node X	0	1	2	3	4	5	6
Nxx	-0.0257	-0.1365	-0.1675	-0.201	-0.231	-0.257	-0.2795
Nyy	-0.0254	-0.1235	-0.09785	-0.1015	-0.1	-0.1	-0.1
Nxy	0	0	0	0	0	0	0
GH							
Nxx	0	-0.1071	-0.1597	-0.1962	-0.2263	-0.2522	-0.274
Nyy	0	-0.1	-0.1	-0.1	-0.1	-0.1	-0.1
Nxy	0	0	0	0	0	0	0
Ratio							
Nxx	1	1.27451	1.048842	1.024465	1.020769	1.019033	1.020073
Nyy	1	1.235	0.9785	1.015	1	1	1
Nxy	1	1	1	1	1	1	1

Diana							
Node X	7	8	9	10	11	12	13
Nxx	-0.297	-0.311	-0.318	-0.321	-0.318	-0.311	-0.297
Nyy	-0.1	-0.1	-0.1	-0.1	-0.1	-0.1	-0.1
Nxy	0	0	0	0	0	0	0
GH							
Nxx	-0.2914	-0.304	-0.3118	-0.3143	-0.3118	-0.304	-0.2914
Nyy	-0.1	-0.1	-0.1	-0.1	-0.1	-0.1	-0.1
Nxy	0	0	0	0	0	0	0
Ratio							
Nxx	1.019218	1.023026	1.019885	1.021317	1.019885	1.023026	1.019218
Nyy	1	1	1	1	1	1	1
Nxy	1	1	1	1	1	1	1

Diana							
Node X	14	15	16	17	18	19	20
Nxx	-0.2795	-0.257	-0.231	-0.201	-0.1675	-0.1365	-0.0257
Nyy	-0.1	-0.1	-0.1	-0.1015	-0.09785	-0.1235	-0.0254
Nxy	0	0	0	0	0	0	0
GH							
Nxx	-0.274	-0.2522	-0.2263	-0.1962	-0.1597	-0.1071	0
Nyy	-0.1	-0.1	-0.1	-0.1	-0.1	-0.1	0
Nxy	0	0	0	0	0	0	0
Ratio							
Nxx	1.020073	1.019033	1.020769	1.024465	1.048842	1.27451	1
Nyy	1	1	1	1.015	0.9785	1.235	1
Nxy	1	1	1	1	1	1	1

- Bound X

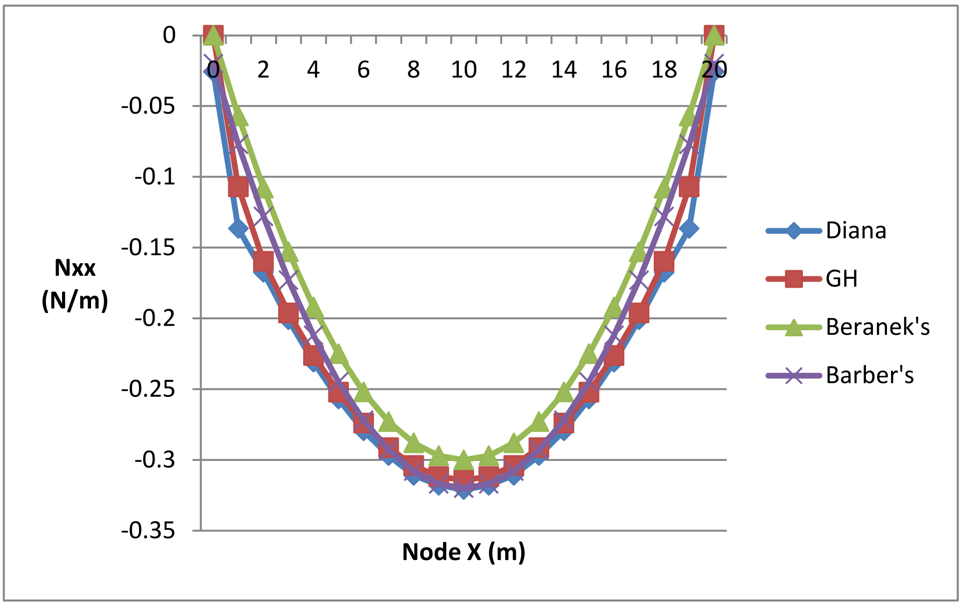


Figure 6.19: Normal Stress in X – Direction

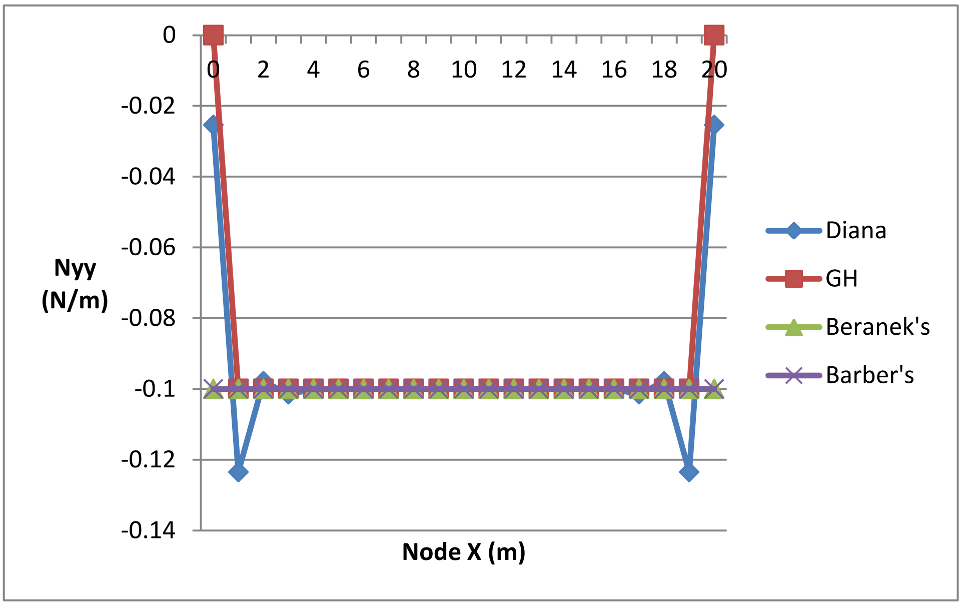


Figure 6.20: Normal Stress in Y – Direction

From the figure of normal stress in x – direction, the results from Grasshopper model fits the results from Diana better than the other two. It indicates that the model has high accuracy. Then in figure of normal stress in y – direction, the reason of why the end point has this huge difference is those end points are the intersection points of loaded edge and fixed edges. In discrete model these corner points cannot fulfill the conditions of both edges. So in here, I only applied the fixed boundary condition to the corner points. That is why in GH, the value is zero.

- Center X (The unit of Node X is m, and Nxx, Nyy and Nxy is N/m)

Diana							
Node X	0	1	2	3	4	5	6
Nxx	0	0.004775	0.00785	0.00875	0.00805	0.006575	0.004925
Nyy	0	-0.01785	-0.03338	-0.04263	-0.04848	-0.05138	-0.0525
Nxy	0.133	0.129	0.12	0.10675	0.091675	0.07635	0.0609
GH							
Nxx	0	0.00453	0.00738	0.00823	0.00765	0.00637	0.00492
Nyy	0	-0.0176	-0.0319	-0.0418	-0.0477	-0.0508	-0.0522
Nxy	0.1279	0.1245	0.1155	0.1031	0.089	0.0742	0.0593
Ratio							
Nxx	1	1.054084	1.063686	1.063183	1.052288	1.032182	1.001016
Nyy	1	1.014205	1.046238	1.019737	1.016247	1.011319	1.005747
Nxy	1.039875	1.036145	1.038961	1.035403	1.030056	1.028976	1.026981

Diana							
Node X	7	8	9	10	11	12	13
Nxx	0.00355	0.0025	0.0019	0.0017	0.0019	0.0025	0.00355
Nyy	-0.05275	-0.0526	-0.05245	-0.05235	-0.05245	-0.0526	-0.05275
Nxy	0.04555	0.0303	0.01515	0	-0.01515	-0.0303	-0.04555
GH							
Nxx	0.00364	0.00268	0.00209	0.0019	0.00209	0.00268	0.00364
Nyy	-0.0526	-0.0526	-0.0525	-0.0525	-0.0525	-0.0526	-0.0526
Nxy	0.0444	0.0296	0.0148	0	-0.0148	-0.0296	-0.0444
Ratio							
Nxx	0.975275	0.932836	0.909091	0.894737	0.909091	0.932836	0.975275
Nyy	1.002852	1	0.999048	0.997143	0.999048	1	1.002852
Nxy	1.025901	1.023649	1.023649	1	1.023649	1.023649	1.025901

Diana							
Node X	14	15	16	17	18	19	20
Nxx	0.004925	0.006575	0.00805	0.00875	0.00785	0.004775	0
Nyy	-0.0525	-0.05138	-0.04848	-0.04263	-0.03338	-0.01785	0
Nxy	-0.0609	-0.07635	-0.09168	-0.10675	-0.12	-0.129	-0.133
GH							
Nxx	0.00492	0.00637	0.00765	0.00823	0.00738	0.00453	0
Nyy	-0.0522	-0.0508	-0.0477	-0.0418	-0.0319	-0.0176	0
Nxy	-0.0593	-0.0742	-0.089	-0.1031	-0.1155	-0.1245	-0.1279
Ratio							
Nxx	1.001016	1.032182	1.052288	1.063183	1.063686	1.054084	1
Nyy	1.005747	1.011319	1.016247	1.019737	1.046238	1.014205	1
Nxy	1.026981	1.028976	1.030056	1.035403	1.038961	1.036145	1.039875

- Center X

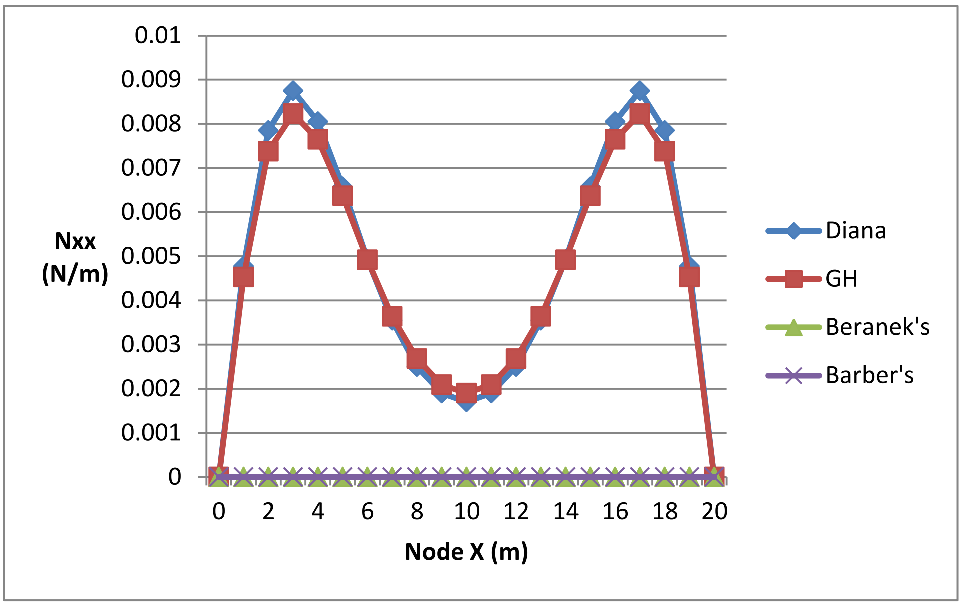


Figure 6.21: Normal Stress in X – Direction

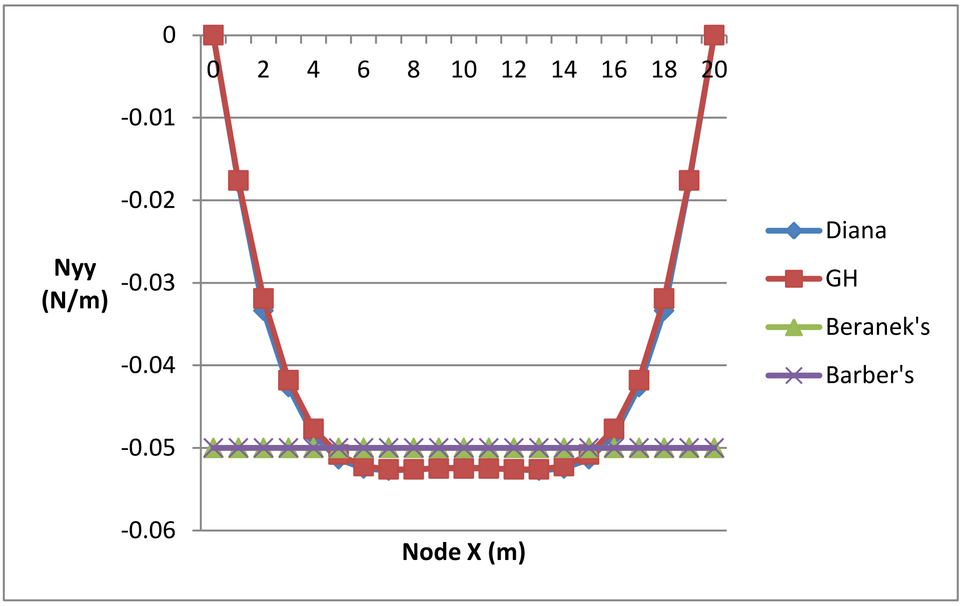


Figure 6.22: Normal Stress in Y – Direction

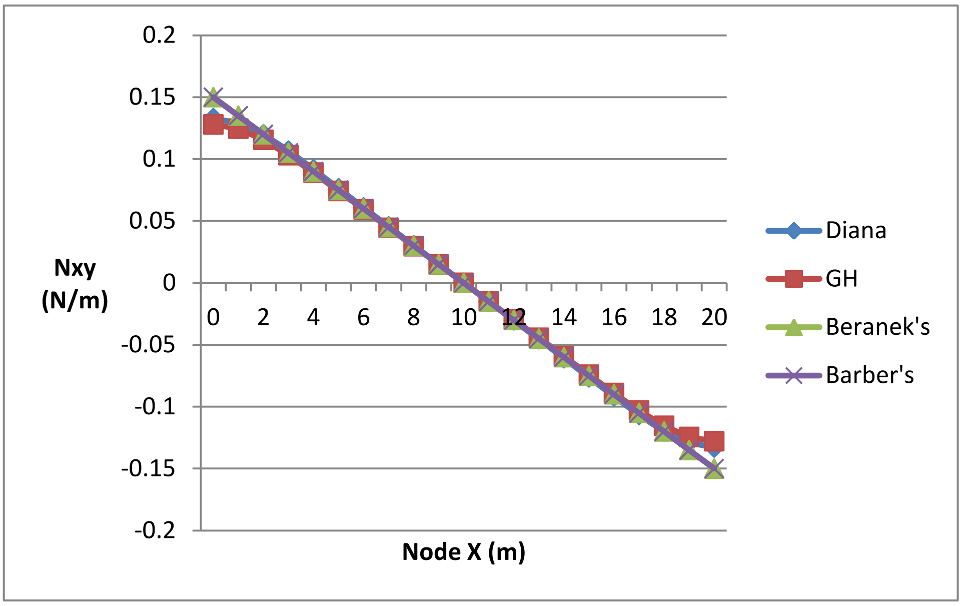


Figure 6.23: Shear stress

There is further explanation of the n_{xx} figure. Although the number is quiet small, the GH results still line with the Diana model very well. This is a convincing figure to prove the parametric tool is in high accuracy. For the result from W. J. Beranek and J. Barber, because the basic assumption for the formula is symmetric and according to this assumption the Center X is where the neutral line is. Therefore there is no stress at those points.

- Bound Y (The unit of Node Y is m, and Nxx, Nyy and Nxy is N/m)

Diana							
Node Y	0	1	2	3	4	5	6
Nxx	0	0	0	0	0	0	0
Nyy	0	0	0	0	0	0	0
Nxy	0.0341	0.1037	0.118	0.129	0.134	0.133	0.124
GH							
Nxx	0	0	0	0	0	0	0
Nyy	0	0	0	0	0	0	0
Nxy	0	0.0746	0.1064	0.1222	0.1288	0.1279	0.1198
Ratio							
Nxx	1	1	1	1	1	1	1
Nyy	1	1	1	1	1	1	1
Nxy	1	1.39008	1.109023	1.055646	1.040373	1.039875	1.035058

Diana				
Node Y	7	8	9	10
Nxx	0	0	0	0
Nyy	0	0	0	0
Nxy	0.108	0.0834	0.04865	0.0012
GH				
Nxx	0	0	0	0
Nyy	0	0	0	0
Nxy	0.1041	0.0801	0.0461	0
Ratio				
Nxx	1	1	1	1
Nyy	1	1	1	1
Nxy	1.037464	1.041199	1.055315	1

- Bound Y

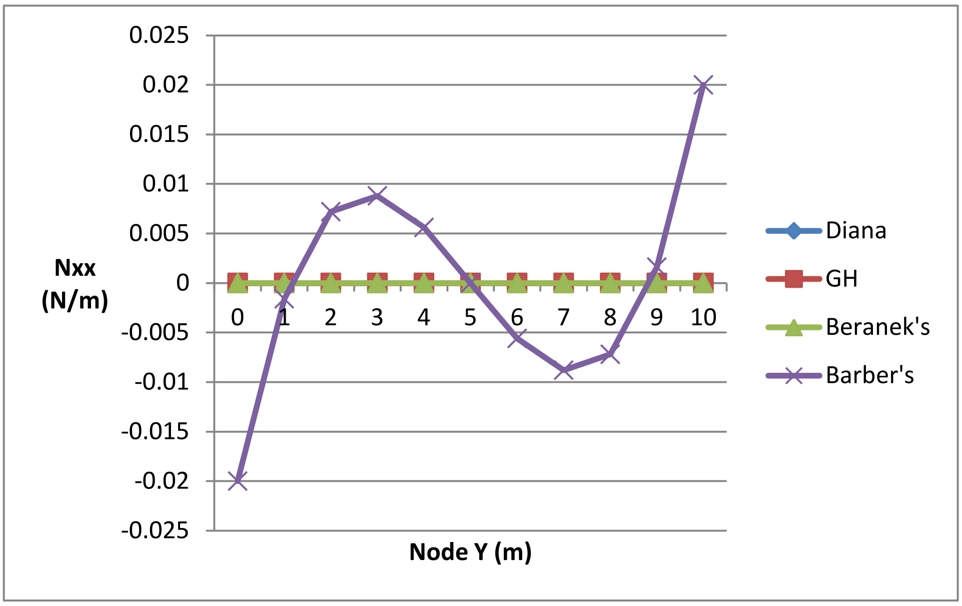


Figure 6.24: Normal Stress in X – Direction

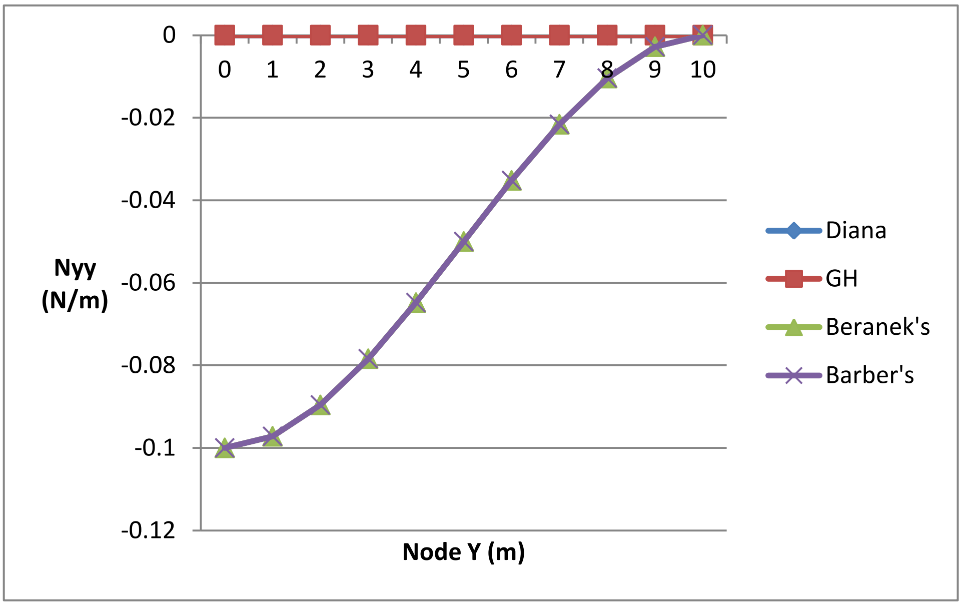


Figure 6.25: Normal Stress in Y – Direction

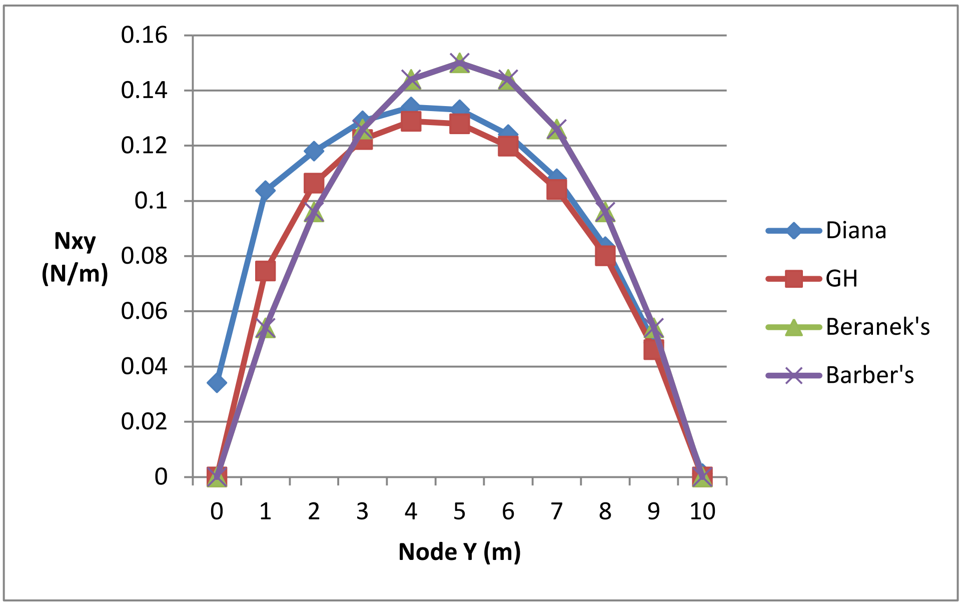


Figure 6.26: Shear stress

As the article mentioned before, the solution from J. Barber is not fully correct. The first figure shows the problem of weak boundary condition. In weak boundary condition, only the summation of the whole edge fulfills the condition requirement. It does not matter whether all the points are satisfied. From the chart, one can see that in reality, it should be no stress at each point. But from the results from book, only the summation is equal to zero.

- Center Y (The unit of Node Y is m, and Nxx, Nyy and Nxy is N/m)

Diana							
Node Y	0	1	2	3	4	5	6
Nxx	-0.321	-0.243	-0.173	-0.1105	-0.0529	0.0017	0.05565
Nyy	-0.1	-0.09785	-0.09085	-0.08025	-0.0671	-0.05235	-0.03725
Nxy	0	0	0	0	0	0	0
GH							
Nxx	-0.3143	-0.2374	-0.1692	-0.1078	-0.0515	0.0019	0.0546
Nyy	-0.1	-0.0974	-0.0905	-0.0801	-0.0671	-0.0525	-0.0375
Nxy	0	0	0	0	0	0	0
Ratio							
Nxx	1.021317	1.023589	1.022459	1.025046	1.027184	0.894737	1.019231
Nyy	1	1.00462	1.003867	1.001873	1	0.997143	0.993333
Nxy	1	1	1	1	1	1	1

Diana				
Node Y	7	8	9	10
Nxx	0.1115	0.172	0.24	0.318
Nyy	-0.02295	-0.01095	-0.00265	0
Nxy	0	0	0	0
GH				
Nxx	0.1091	0.168	0.2342	0.3105
Nyy	-0.0234	-0.0115	-0.0032	0
Nxy	0	0	0	0
Ratio				
Nxx	1.021998	1.02381	1.024765	1.024155
Nyy	0.980769	0.952174	0.828125	1
Nxy	1	1	1	1

- Center Y

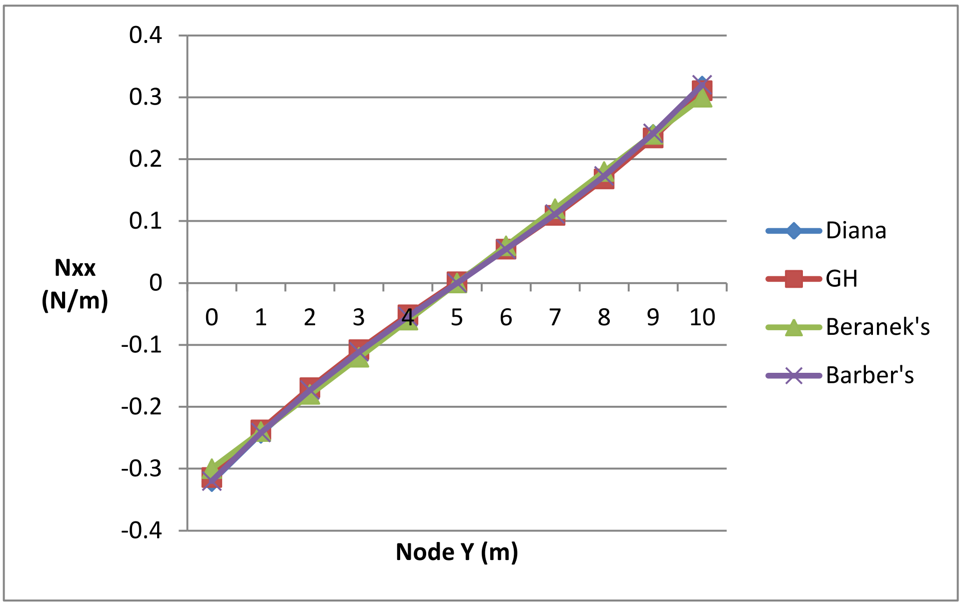


Figure 6.27: Normal Stress in X – Direction

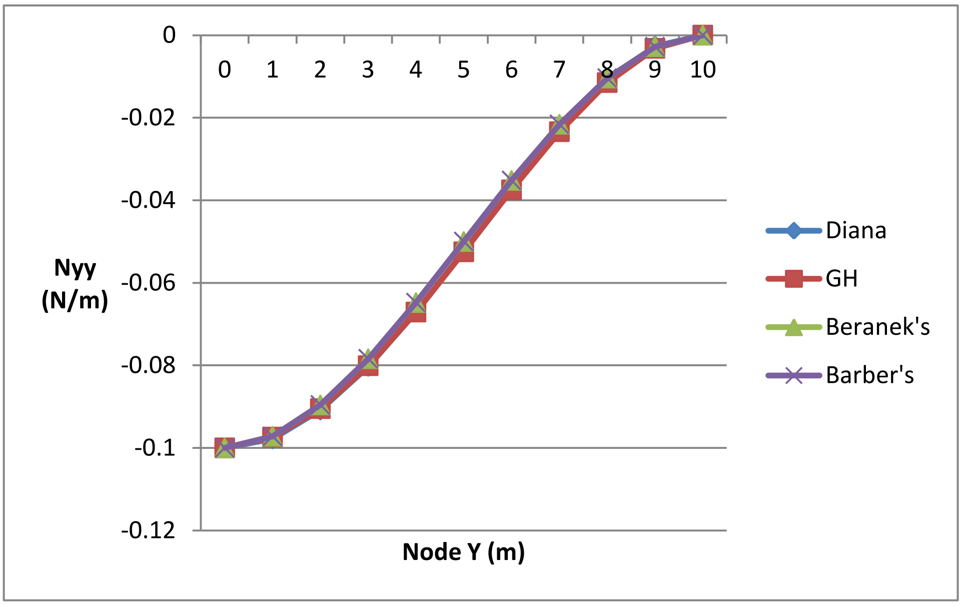


Figure 6.28: Normal Stress in Y – Direction

- Bound X (The unit of Node X is m, and DispX and DispY is mm)

Diana							
Node X	0	1	2	3	4	5	6
DispX	0.665	0.628	0.579	0.517	0.444	0.363	0.273
DispY	0	0.286	0.509	0.703	0.872	1.02	1.14
GH							
DispX	0.635	0.635	0.564	0.529	0.433	0.378	0.265
DispY	0	0.281	0.503	0.694	0.867	1.01	1.132
Ratio							
DispX	1.047244	0.988976	1.026596	0.977316	1.025404	0.960317	1.030189
DispY	1	1.017794	1.011928	1.012968	1.005767	1.009901	1.007067

Diana							
Node X	7	8	9	10	11	12	13
DispX	0.177	0.0752	-0.0299	-0.137	-0.244	-0.349	-0.45
DispY	1.23	1.3	1.34	1.36	1.34	1.3	1.23
GH							
DispX	0.195	0.071	-0.007	-0.137	-0.267	-0.345	-0.469
DispY	1.219	1.296	1.328	1.353	1.328	1.296	1.219
Ratio							
DispX	0.907692	1.059155	4.271429	1	0.913858	1.011594	0.959488
DispY	1.009024	1.003086	1.009036	1.005174	1.009036	1.003086	1.009024

Diana							
Node X	14	15	16	17	18	19	20
DispX	-0.547	-0.636	-0.718	-0.79	-0.852	-0.901	-0.939
DispY	1.14	1.02	0.872	0.703	0.509	0.286	0
GH							
DispX	-0.539	-0.652	-0.707	-0.803	-0.838	-0.909	-0.909
DispY	1.132	1.01	0.867	0.694	0.503	0.281	0
Ratio							
DispX	1.014842	0.97546	1.015559	0.983811	1.016706	0.991199	1.033003
DispY	1.007067	1.009901	1.005767	1.012968	1.011928	1.017794	1

- Bound X

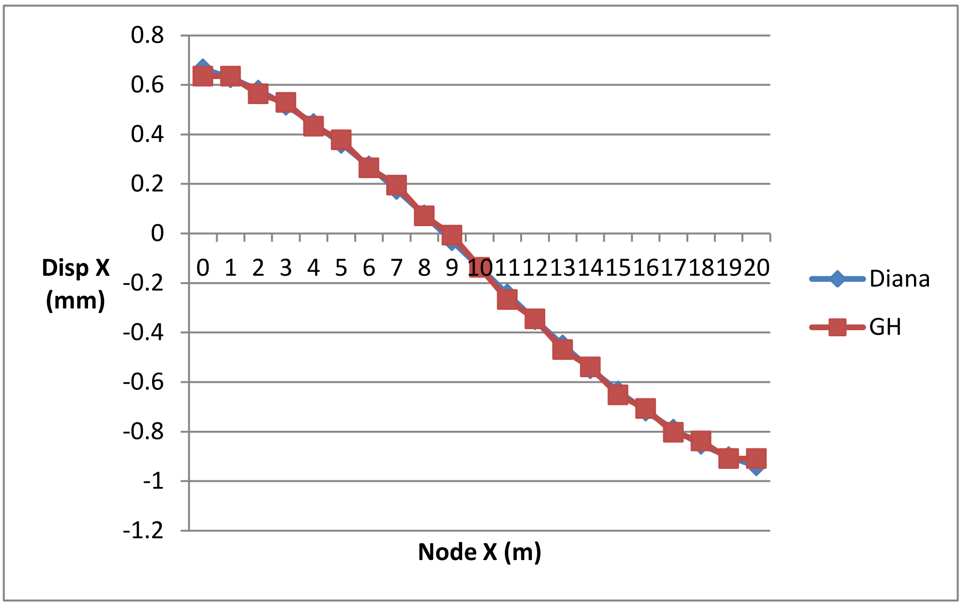


Figure 6.29: Displacement X-Direction

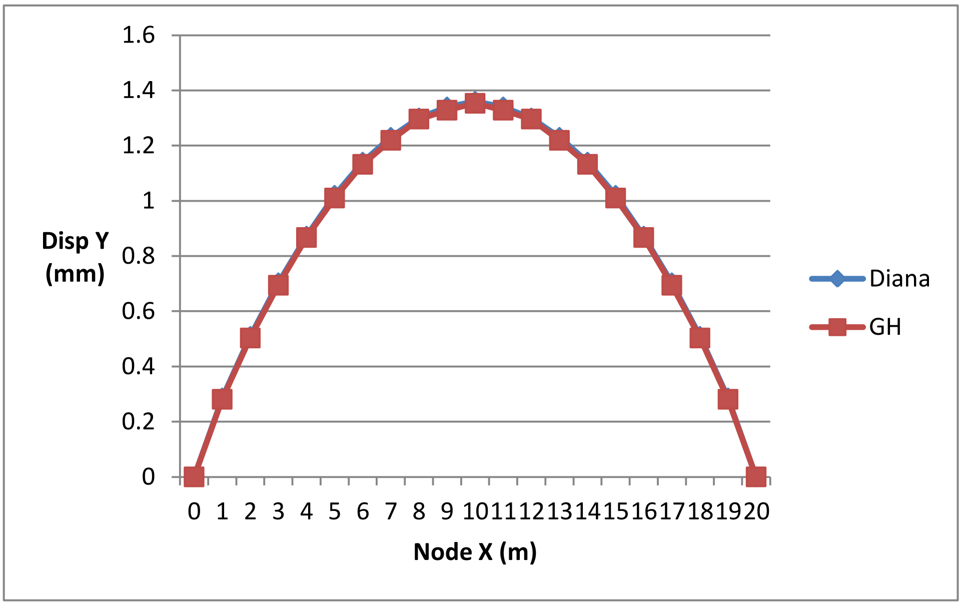


Figure 6.30: Displacement Y-Direction

- Center X (The unit of Node X is m, and DispX and DispY is mm)

Diana							
Node X	0	1	2	3	4	5	6
DispX	-0.153	-0.152	-0.15	-0.147	-0.145	-0.142	-0.14
DispY	0	0.2	0.393	0.574	0.737	0.88	1
GH							
DispX	-0.1535	-0.154	-0.151	-0.149	-0.145	-0.144	-0.141
DispY	0	0.197	0.39	0.566	0.732	0.868	0.993
Ratio							
DispX	0.996743	0.987013	0.993377	0.986577	1	0.986111	0.992908
DispY	1	1.015228	1.007692	1.014134	1.006831	1.013825	1.007049

Diana							
Node X	7	8	9	10	11	12	13
DispX	-0.139	-0.138	-0.137	-0.137	-0.136	-0.135	-0.134
DispY	1.1	1.16	1.21	1.22	1.21	1.16	1.1
GH							
DispX	-0.14	-0.138	-0.138	-0.137	-0.136	-0.136	-0.134
DispY	1.081	1.158	1.19	1.215	1.19	1.158	1.081
Ratio							
DispX	0.992857	1	0.992754	1	1	0.992647	1
DispY	1.017576	1.001727	1.016807	1.004115	1.016807	1.001727	1.017576

Diana							
Node X	14	15	16	17	18	19	20
DispX	-0.133	-0.131	-0.129	-0.126	-0.123	-0.121	-0.12
DispY	1	0.88	0.737	0.574	0.393	0.2	0
GH							
DispX	-0.133	-0.13	-0.129	-0.1253	-0.1234	-0.1204	-0.1204
DispY	0.993	0.868	0.732	0.566	0.39	0.197	0
Ratio							
DispX	1	1.007692	1	1.005587	0.996759	1.004983	0.996678
DispY	1.007049	1.013825	1.006831	1.014134	1.007692	1.015228	1

- Center X

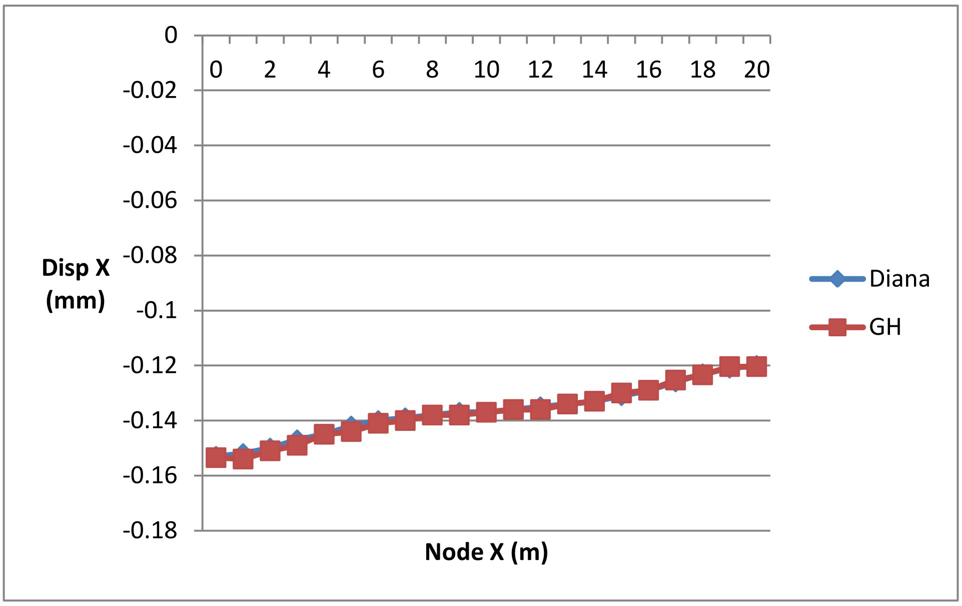


Figure 6.31: Displacement X-Direction

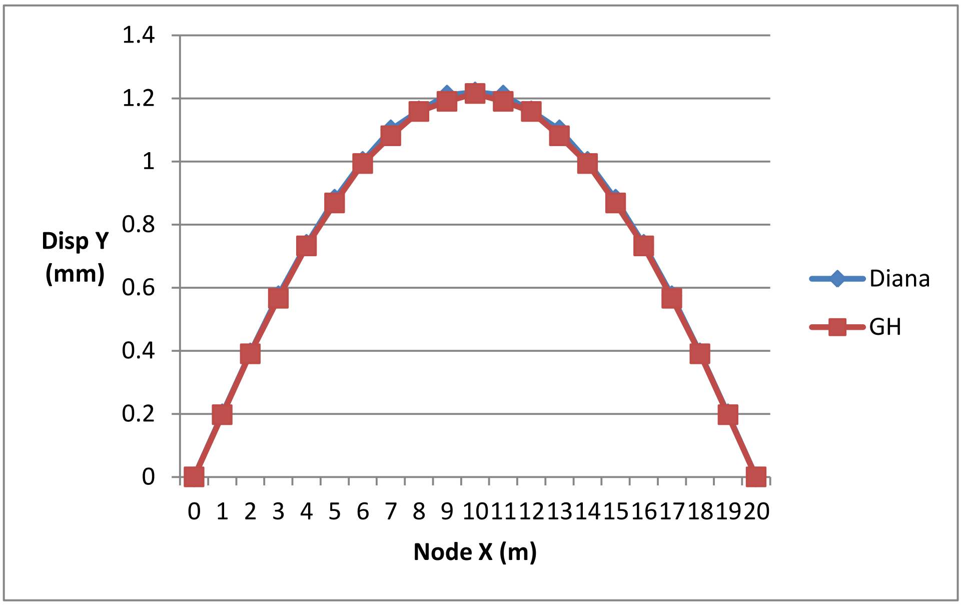


Figure 6.32: Displacement Y-Direction

- Bound Y (The unit of Node Y is m, and DispX and DispY is mm)

Diana							
Node Y	0	1	2	3	4	5	6
DispX	0.665	0.415	0.235	0.0893	-0.0371	-0.153	-0.266
DispY	0	0	0	0	0	0	0
GH							
DispX	0.635	0.401	0.226	0.084	-0.04	-0.154	-0.264
DispY	0	0	0	0	0	0	0
Ratio							
DispX	1.047244	1.034913	1.039823	1.063095	0.9275	0.993506	1.007576
DispY	1	1	1	1	1	1	1

Diana				
Node Y	7	8	9	10
DispX	-0.384	-0.512	-0.659	-0.833
DispY	0	0	0	0
GH				
DispX	-0.379	-0.504	-0.648	-0.818
DispY	0	0	0	0
Ratio				
DispX	1.013193	1.015873	1.016975	1.018337
DispY	1	1	1	1

- Bound Y

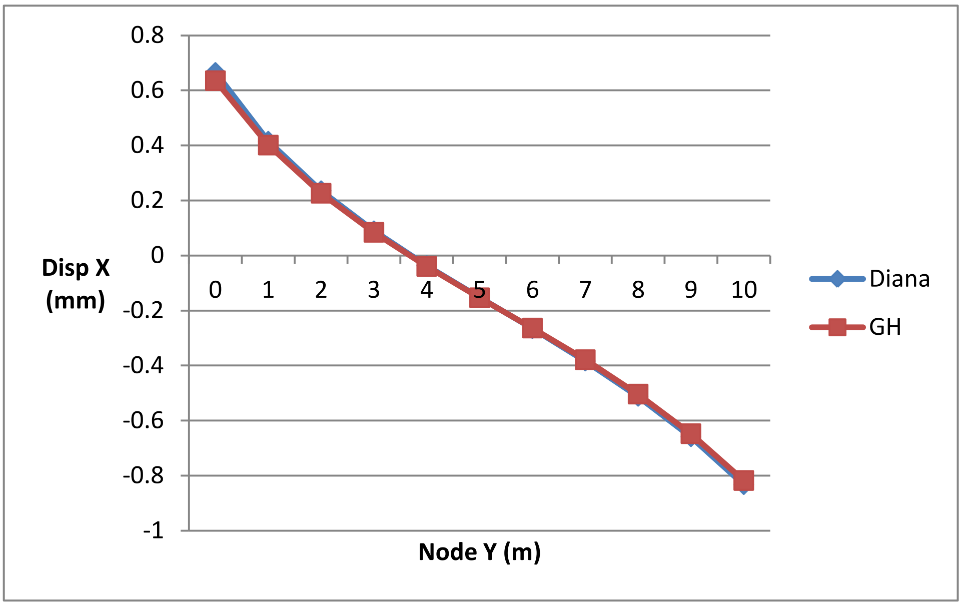


Figure 6.33: Displacement X-Direction

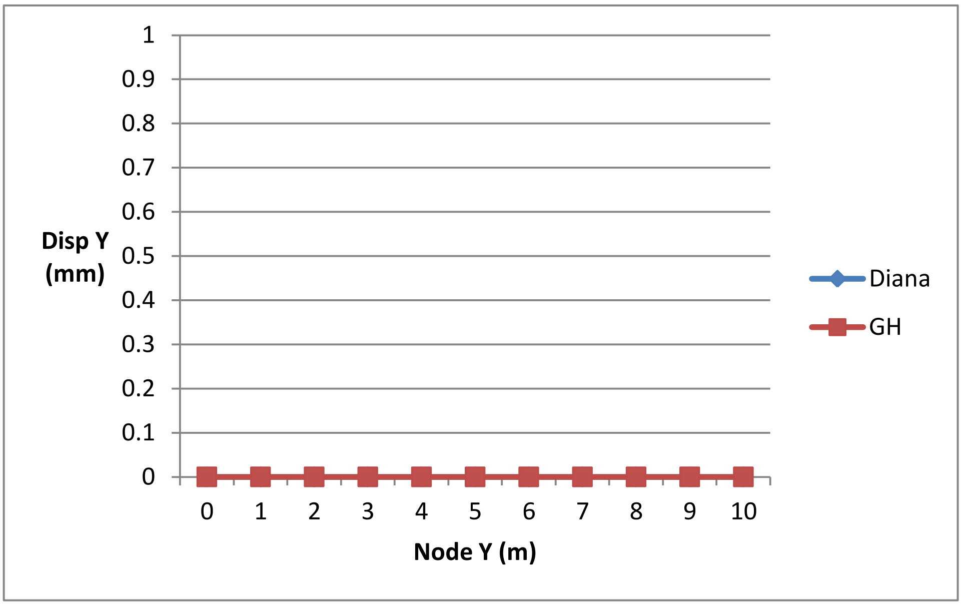


Figure 6.34: Displacement Y-Direction

- Center Y (The unit of Node Y is m, and DispX and DispY is mm)

Diana							
Node Y	0	1	2	3	4	5	6
DispX	-0.137	-0.137	-0.137	-0.137	-0.137	-0.137	-0.137
DispY	1.36	1.32	1.29	1.26	1.24	1.22	1.2
GH							
DispX	-0.137	-0.137	-0.137	-0.137	-0.137	-0.137	-0.137
DispY	1.353	1.32	1.288	1.26	1.235	1.215	1.2
Ratio							
DispX	1	1	1	1	1	1	1
DispY	1.005174	1	1.001553	1	1.004049	1.004115	1

Diana				
Node Y	7	8	9	10
DispX	-0.137	-0.137	-0.137	-0.137
DispY	1.19	1.19	1.19	1.19
GH				
DispX	-0.137	-0.137	-0.137	-0.137
DispY	1.19	1.184	1.182	1.182
Ratio				
DispX	1	1	1	1
DispY	1	1.005068	1.006768	1.006768

- Center Y

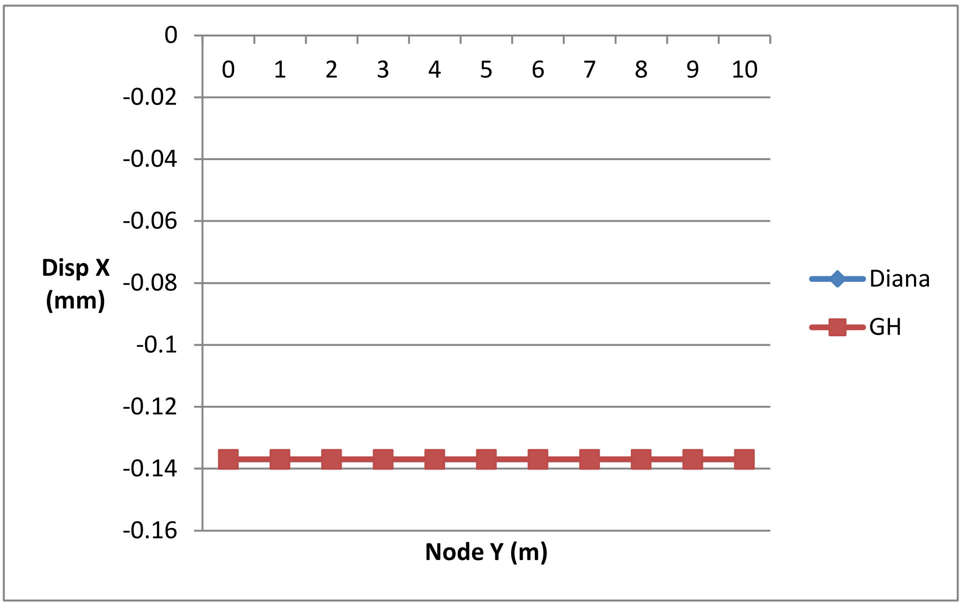


Figure 6.35: Displacement X-Direction

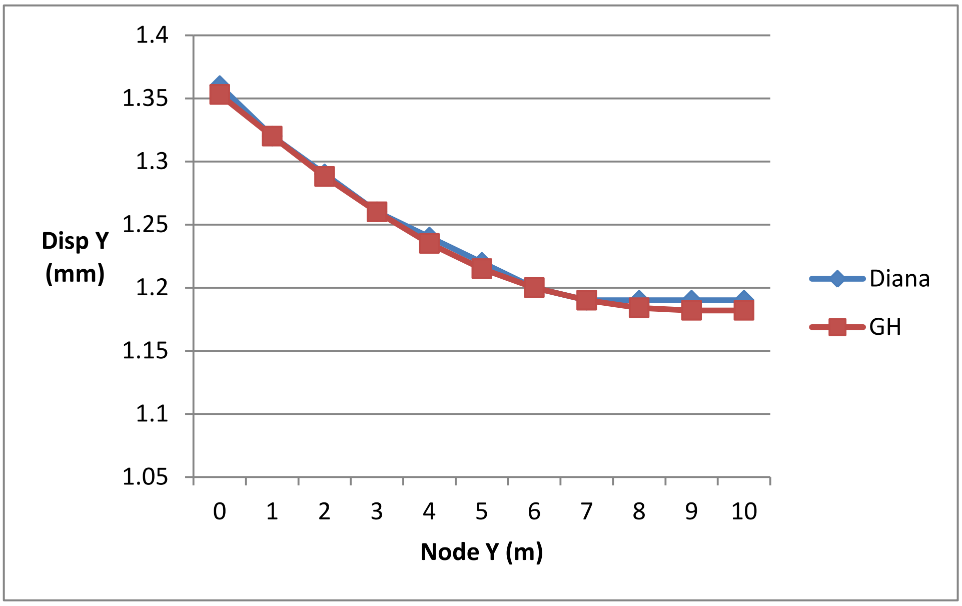


Figure 6.36: Displacement Y-Direction

6.3. Conclusion

According to the evaluation of different points with results comparison, the Grasshopper model shows good calculation performance of in – plane mechanic analysis.

7. Reinforcement Calculation

7.1. Introduction

The foremost purpose of the thesis program is for plate structural analysis. The outputs are mainly focus on mechanic behavior descriptions like normal stresses, bending moments, shear and deflections. Since these results have been generated by the parametric tool, the program can incorporate with them for further applications. Base on this, the program can be upgraded to be more practical.

Then the first idea is the reinforcement calculation. Plate structures are normally made of concrete, and reinforcements should be introduced to satisfy the safety requirement. To obtain the reinforcement bars layout in plate structures, the mechanic analysis results need to be known first. One of the advantages of the program is that the tool is real-time. It means the structural analysis can be achieved immediately. Utilized the results, the reinforcement design application is easier to be realized.

The inputs of the program are plate thickness and the design steel strength. The outputs are reinforcement area ratios in the corresponding points.

7.2. Methods

The design solution of reinforcement calculation is the *Wood & Armer* method, which is also recommended in Dutch code *NEN 6720*. The concept is to determine equivalent moments and normal stresses for reinforcement calculation. And the formulas are presented below.

- Plate subjected to simple bending:

For a selected directions x and y, two types of design moments are computed:
The lower moments M_{xd} and M_{yd} (positive, causing tension in the bottom parts of plate)

$$M_{xd} = M_{xx} + |M_{xy}|$$
$$M_{yd} = M_{yy} + |M_{xy}|$$

The upper moments M_{xg} and M_{yg} (negative, causing tension in the upper parts of plate)

$$M_{xg} = M_{xx} - |M_{xy}|$$
$$M_{yg} = M_{yy} - |M_{xy}|$$

Then the ratio can be computed.
The lower ratios:

$$\rho_{xd} = \frac{M_{xd}}{z \cdot f_{yd} \cdot d} = \frac{M_{xd}}{0.9d \cdot f_{yd} \cdot d}$$
$$\rho_{yd} = \frac{M_{yd}}{z \cdot f_{yd} \cdot d} = \frac{M_{yd}}{0.9d \cdot f_{yd} \cdot d}$$

The upper ratios:

$$\rho_{xg} = \frac{M_{xg}}{z \cdot f_{yd} \cdot d} = \frac{M_{xg}}{0.9d \cdot f_{yd} \cdot d}$$

$$\rho_{yg} = \frac{M_{yg}}{z \cdot f_{yd} \cdot d} = \frac{M_{yg}}{0.9d \cdot f_{yd} \cdot d}$$

The internal lever arm *z* is assumed to be *z = 0.9d* (*d* is the thickness of plate). The exact ratio between *z* and *d* depends on the reinforcement ratio. However, the ratio of 0.9 is most often accurate.

- Plate subjected to simple in – plane stresses:

For a selected directions x and y, the design normal stresses are computed:

The normal stress N_{xr} and N_{yr} (positive, causing tension in the plate)

$$N_{xr} = N_{xx} + |N_{xy}|$$

$$N_{yr} = N_{yy} + |N_{xy}|$$

Then the ratio can be computed.

The reinforcement ratios:

$$\rho_{xr} = \frac{N_{xr}}{f_{yd} \cdot d}$$

$$\rho_{yr} = \frac{N_{yr}}{f_{yd} \cdot d}$$

According to the equations above, the ratios of reinforcement are obtained.

7.3. Verification

Imaging a one direction spanned concrete slab subjected uniformed distributed load of 10kN/m^2 perpendicular to mid – plane. The boundary is simple supported, and it is spanned in y – direction. Thickness of plate is 200mm and the steel design strength is 435N/mm^2. The dimensions of the plate are showed below.

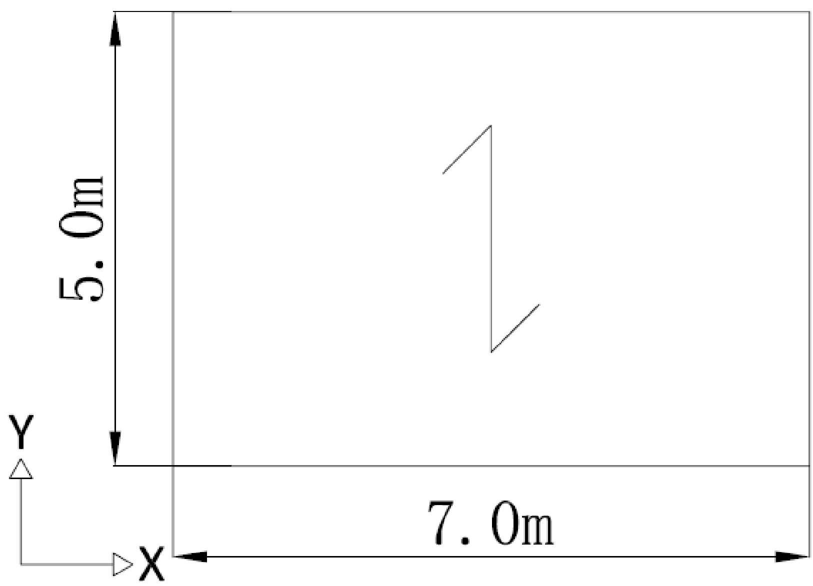


Figure 7.1: Dimensions of the plate

Calculation result of the reinforcement ratio in mid - span:

$$\rho_{mid} = \frac{M}{z \cdot f_{yd} \cdot d} = \frac{\frac{1}{8} \cdot q \cdot l^2}{0.9d^2 \cdot f_{yd}} = \frac{\frac{1}{8} \cdot 10 \cdot 10^{-3} \cdot 5000^2}{0.9 \cdot 200^2 \cdot 435} = 0.19955 \cdot 10^{-2} = 0.19955\%$$

Calculation result of the reinforcement ratio in Grasshopper program:

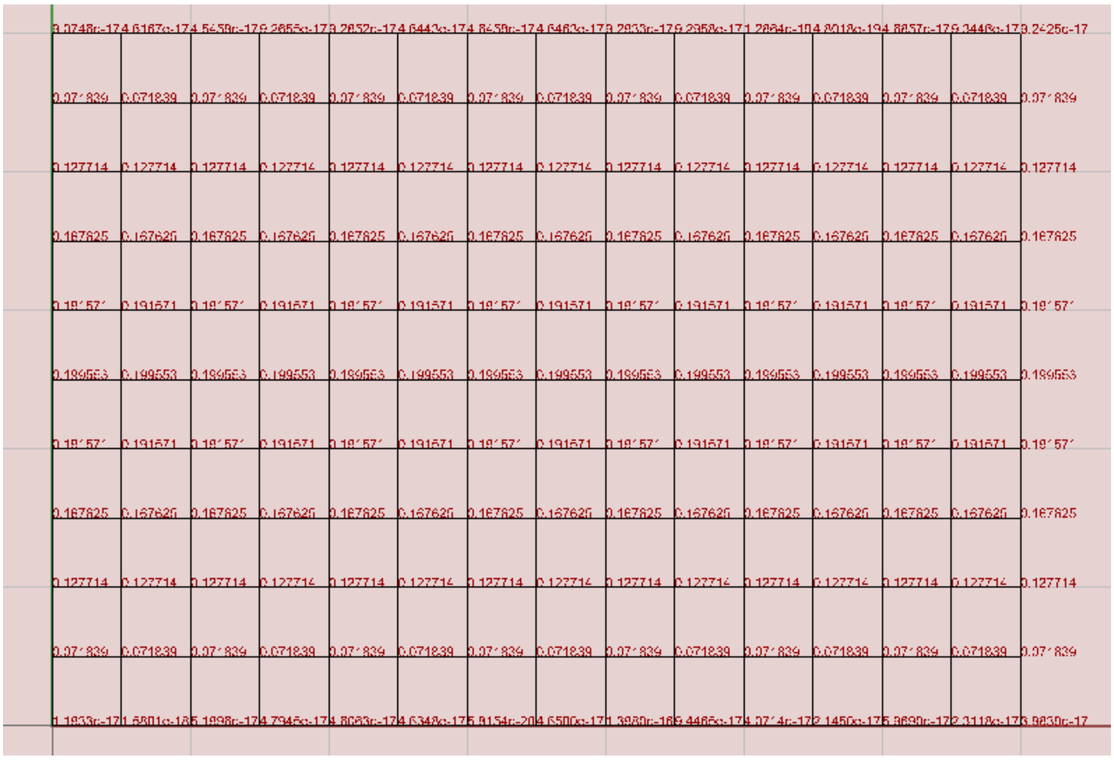


Figure 7.2: Overall value of plate reinforcement ratios

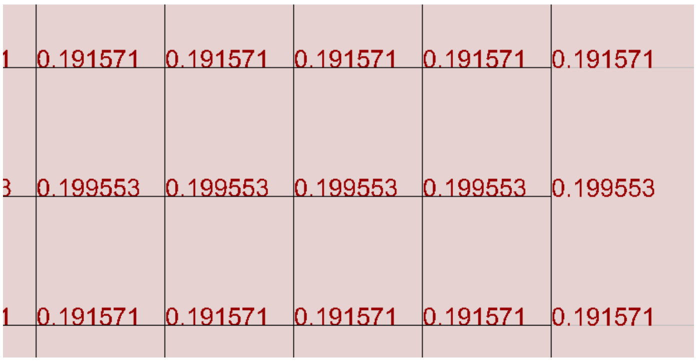


Figure 7.3: The ratios in mid – span

It can be seen that the ratio value is the same. In both hand calculation and Grasshopper model, the result is 0.19955%. It means that the reinforcement calculation component is also with high accuracy.

8. Conclusion & Recommendation

8.1. Introduction

The master thesis is an extension of a recently developed parametric design tool for thin plates loaded out – of – plane, and establishment of program for thin plates loaded in – plane. One of the goals is implementing the theories into visualization computer program to gain visual representation and real – time result. The statement of this objective is as:

"Develop a structural design tool for architects and engineers, based on simple analytical structural analysis methods, which gives both quantitative and qualitative (real time) insight in the flow and magnitude of forces within a specific structure during a conceptual design stage."

The out – of – plane and in – plane calculation models are based on preceding developed program (*M. Oosterhuis, 2010*). The final goal is to generate a sophisticate computational program for freeform structural computation. This project requires a long period to complete. This thesis is just one step for the structural calculation program.

The scope of the thesis has been confined.

- For out – of – plane program, the load is added perpendicular to the plane. The boundary conditions of each edge can be simple supported, fixed edge and free edge.
- For in – plane program, load will be presented parallel to the plane and acting on the boundary. The boundary conditions are two parallel sides loaded and two parallel sides fixed in vertical direction.
- For practical demand, the program will be able to calculate the reinforcement for plate structure.

8.2. Conclusion

The following conclusions can be drawn with respect to the analysis sequences.

Out – of – plane program:

a) The boundary component can generate the boundary shape for membrane analogy. The types of the conditions can be simple supported, fixed edge and free edge. Four edges do not need to be the same type of boundary condition. All the mentioned supported edges can be combined randomly for the edges.
b) Elastic membrane analogy integrated with force density method can determine the shape of bending moment summation, based on the boundary which is set by the boundary component.
c) The numerical finite difference method can be used in calculating the displacements. And by combining finite difference method, the displacement shape can be used to determine the stress results like bending moment, torsions and boundary shear forces, based on different

type of boundary conditions.

In – plane program:

a) The stress function shape can be computed by double elastic membrane analogy with assist of force density method. The determined shape is used for further structural evaluation.
b) The stress results can be computed by numerical finite difference method. Also the displacement of the plate.

Reinforcement component:

a) The reinforcement component fulfills the demand to compute the reinforcement ratio for steel bar design in plate structure.

Regarding the performance of the models, the defined theories provide a strong foundation of structural calculation. The results are compatible with those values from FEM program. It means the models are with high accuracy. It can be said that the parametric design tool, at this stage, can deliver the highly qualitative and quantitative insight for plate structural mechanic analysis.

8.3. Recommendation

Since the thesis is just a small step for generating freeform structural analysis program, there are some recommendations are defined for further research.

a) The program now is only valid for rectangular shape. It needs to be developed to satisfy different structure geometry. Also the program should be developed from 2D structure to 3D (from plate to shell).
b) Further research is needed to extend the program into more different types of boundary condition. For out – of – plane behavior, like elastic bearing, for in – plane, the fully fixed boundary can be developed. Another is different types of loading.
c) To fulfill the analysis of freeform, the out – of – plane and in – plane programs need to be combined.

8.4. Summary

Reflecting on the main objective for the thesis, the goals are successfully achieved. The goal to make a parametric structural design tool of plates is gain by implementing the theories into Grasshopper model. With different structural calculation component, the insightful results with high quality are obtained.

List of Symbols

L	*Length*
W	*Width*
t	*Thickness*
α, β	*Angle*
x, y, z	*Coordinates*
υ	*Poisson's ratio*
E	*Young's modulus*
D	*Stiffness*
u_x	*Displacement in X - direction*
u_y	*Displacement in Y – direction*
w	*Displacement in Z - direction*
ε	*Normal strain*
γ	*Shear strain*
κ	*Extrinsic curvature*
σ	*In – plane stress*
n	*In – plane force*
m	*Out – of – plane Moment*
v	*Out – of – plane shear*
p, q	*Distributed load*
$\emptyset$	*Stress function*
N	*Membrane force*
$\boldsymbol{C}, \boldsymbol{C_s}$	*Branch – node matrix*
$\boldsymbol{Q}$	*Force density matrix*
$\boldsymbol{K}$	*Finite difference matrix for curvature*
$\boldsymbol{M}$	*Finite difference matrix for concentrated shear force (bending moment terms)*
$\boldsymbol{T}$	*Finite difference matrix for concentrated shear force (torsion terms)*
$\boldsymbol{B}$	*Finite difference matrix for bending moment*
$\boldsymbol{I_{bound}}$	*Finite difference matrix for edge loads*
$\boldsymbol{N}$	*Edge loads matrix*
$\boldsymbol{G}$	*Finite difference matrix for external shear loads*
$\boldsymbol{J}$	*Finite difference matrix for fixed boundary*

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