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DOI

[10.1214/16-EJP7](https://doi.org/10.1214/16-EJP7)

Publication date

2016

Document Version

Final published version

Published in

Electronic Journal of Probability

Citation (APA)

Veraar, M., & Yaroslavtsev, I. (2016). Cylindrical continuous martingales and stochastic integration in infinite dimensions. *Electronic Journal of Probability*, 21(59), 1-53. <https://doi.org/10.1214/16-EJP7>

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Cylindrical continuous martingales and stochastic integration in infinite dimensions ^{*}

Mark Veraar[†] Ivan Yaroslavtsev[‡]

Abstract

In this paper we define a new type of quadratic variation for cylindrical continuous local martingales on an infinite dimensional spaces. It is shown that a large class of cylindrical continuous local martingales has such a quadratic variation. For this new class of cylindrical continuous local martingales we develop a stochastic integration theory for operator valued processes under the condition that the range space is a UMD Banach space. We obtain two-sided estimates for the stochastic integral in terms of the γ -norm. In the scalar or Hilbert case this reduces to the Burkholder-Davis-Gundy inequalities. An application to a class of stochastic evolution equations is given at the end of the paper.

Keywords: cylindrical martingale; quadratic variation; continuous local martingale; stochastic integration in Banach spaces; UMD Banach spaces; Burkholder-Davis-Gundy; random time change; γ -radonifying operators; inequalities; Itô formula; stochastic evolution equation; stochastic convolution; functional calculus.

AMS MSC 2010: Primary 60H05, Secondary 60B11; 60G44; 47D06.

Submitted to EJP on February 12, 2016, final version accepted on September 17, 2016.

Supersedes arXiv:1602.03996.

1 Introduction

Cylindrical local martingales play an important role in the theory of stochastic PDEs. For example the classical cylindrical Brownian motion W_H on a Hilbert space $H = L^2(D)$ can be used to give a functional analytic framework to model a space-time white noise on $\mathbb{R}_+ \times D$. A cylindrical (local) martingale M on a Banach space X is such that for every $x^* \in X^*$ (the dual space of X) one has that Mx^* is a (local) martingale and the mapping $x^* \rightarrow Mx^*$ is linear and satisfies appropriate continuity conditions (see Section 3.1).

Cylindrical (local) martingales have extensively studied in the literature (see [34, 60, 61, 49, 50, 73, 74]). In this paper we introduce a new type of *quadratic variation* $[[M]]$

^{*}The first named author is supported by the Vidi subsidy 639.032.427 of the Netherlands Organisation for Scientific Research (NWO).

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for a cylindrical continuous local martingale M on a Banach space X (see Definition 3.4). Moreover, we develop a stochastic calculus for those M which admit such a quadratic variation. The process $[[M]]$ is continuous and increasing and it is given by

$$[[M]]_t := \lim_{\text{mesh} \rightarrow 0} \sum_{n=1}^N \sup_{x^* \in X^*, \|x^*\|=1} ([M(t_n)x^*] - [M(t_{n-1})x^*]), \quad (1.1)$$

where the a.s. limit is taken over partitions $0 = t_0 < \dots < t_N = t$. The definition (1.1) can be given for any Banach space X , but for technical reasons we will assume that X^* is separable. The definition (1.1) is new even in the Hilbert space setting. Our motivation for introducing this class comes from stochastic integration theory and in this case M is a cylindrical continuous local martingale on a Hilbert space. A more detailed discussion on stochastic integration theory will be given in the second half of the introduction.

In many cases $[[M]]$ is simple to calculate. For instance for a cylindrical Brownian motion one has $[[W_H]]_t = t$. More generally, if $M = \int_0^\cdot \Phi dW_H$ where Φ is an $\mathcal{L}(H, X)$ -valued adapted process, then one has

$$[[M]]_t = \int_0^t \|\Phi(s)\Phi(s)^*\|_{\mathcal{L}(X^*, X)} ds.$$

These examples illustrate that Definition (1.1) is a natural object. However, one has to be careful, as there are *cylindrical* continuous martingales (even on Hilbert spaces) which do not have a quadratic variation $[[M]]$. From now on let us write $M_{\text{var}}^{\text{loc}}(X)$ for the class of cylindrical continuous local martingales which admit a quadratic variation.

If M is a continuous local martingale with values in a Hilbert space, then it is well known that it has a classical quadratic variation $[M]$ in the sense that there exists an a.s. unique increasing continuous process $[M]$ starting at zero such that $\|M\|^2 - [M]$ is a continuous local martingale again. It is simple to check that in this case $[[M]]$ exists and a.s. for all $t \geq 0$, $[[M]]_t \leq [M]_t$. Clearly, $[M]$ does not exist in the cylindrical case, but as we will see, $[[M]]$ gives a good alternative for it.

Previous attempts to define quadratic variation are usually given in the case M is actually a martingale (instead of a cylindrical martingale) and in the case X is a Hilbert space (see [14, 60, 49, 50]). We will give a detailed comparison with the earlier attempts to define the quadratic variation in Section 3.

To study SPDEs with a space-time noise one often models the noise as a cylindrical local martingale on an infinite dimensional space. We refer the reader to [13] for the case of cylindrical Brownian motion. In order to study SPDEs one uses a theory of stochastic integration for operator-valued processes $\Phi : \mathbb{R}_+ \times \Omega \rightarrow \mathcal{L}(H, X)$. Our aim is to develop a stochastic integration theory where the integrator M is from $M_{\text{var}}^{\text{loc}}(H)$ and the integrand takes values in $\mathcal{L}(H, X)$, where X is a Banach space which has the UMD property.

The history of stochastic integration in Banach spaces has an interesting history which goes back 40 years. Important contributions have been given in several papers and monographs (see [4, 7, 9, 31, 57, 58, 59, 70] and references therein). We refer to [56] for a detailed discussion on the history of the subject. Influenced by results from Garling [24] and McConell [48], a stochastic integration theory for $\Phi : [0, T] \times \Omega \rightarrow \mathcal{L}(H, X)$ with integrator W_H was developed in [53] by van Neerven and Weis and the first author. The theory is naturally restricted to the class of Banach spaces X with the UMD property (e.g. L^q with $q \in (1, \infty)$). The main result is that Φ is stochastically integrable with respect to an H -cylindrical Brownian motion if and only if $\Phi \in \gamma(0, T; H, X)$ a.s. Here $\gamma(0, T; H, X)$ is a generalized space of square functions as introduced in the influential paper [36] (see Subsection 4.1 for the definition). Furthermore, it was shown that for

any $p \in (0, \infty)$ the following two-sided estimate holds

$$c\|\Phi\|_{L^p(\Omega;\gamma(0,T;H,X))} \leq \left\| \sup_{t \in [0,T]} \left\| \int_0^t \Phi dW_H \right\|_X \right\|_{L^p(\Omega)} \leq C\|\Phi\|_{L^p(\Omega;\gamma(0,T;H,X))},$$

which can be seen as an analogue of the classical Burkholder-Davis-Gundy inequalities. This estimate is strong enough to obtain sharp regularity results for stochastic PDEs (see [55]) which can be used for instance to extend some of the sharp estimates of Krylov [40] to an $L^p(L^q)$ -setting.

The aim of our paper is to build a calculus for the newly introduced class of cylindrical continuous local martingales which admit a quadratic variation. Moreover, if M is a cylindrical continuous local martingale on a Hilbert space H , we show that there is a natural analogue of the stochastic integration theory of [53] where the integrator W_H is replaced by M . At first sight it is quite surprising that the γ -norms again play a fundamental role in this theory although the cylindrical martingales do not have a Gaussian distribution. Our theory is even new in the Hilbert space setting. The proof of the main result Theorem 4.1 is based on a sophisticated combination of time change arguments and Brownian representation results for martingales with absolutely continuous quadratic variations from [60, Theorem 2]. Theorem 4.1 gives that Φ is stochastically integrable with respect to M if and only if $\Phi Q_M^{1/2} \in \gamma(0, T, [[M]]; H, X)$ a.s. Here Q_M is a predictable operator with norm $\|Q_M\| = 1$. Moreover, two-sided Burkholder-Davis-Gundy inequalities hold again. We will derive several consequence of the integration theory among which a version Itô's formula.

We finish this introduction with some related contributions to the theory of stochastic integration in an infinite dimensional setting. In [49] Métivier and Pellaumail developed an integration theory for cylindrical martingales which are not necessarily continuous and two-sided estimates are derived in a Hilbert space setting. A theory for SDEs and SPDEs with semimartingales in Hilbert spaces is developed by Gyöngy and Krylov in [26, 27, 25]. The integration theory with respect to cylindrical Lévy processes in Hilbert cases and its application to SPDEs is developed in the monograph by Peszat and Zabczyk [64]. Some extensions in the Banach space setting have been considered and we refer to [1, 2, 45, 69, 68] and references therein. In [16] Dirksen has found an analogue of the two-sided Burkholder-Davis-Gundy inequalities in the case the integrator is a Poisson process and $X = L^q$ (also see [17, 46, 47]). By the results of our paper and the previously mentioned results, it is a natural question what structure of a cylindrical *noncontinuous* local martingales is required to build a theory which allows to have two-sided estimates for stochastic integrals.

Outline:

- In Section 2 some preliminaries are discussed.
- In Section 3 the quadratic variation of a cylindrical continuous local martingale is introduced.
- In Section 4 the stochastic integrable Φ are characterized.
- In Section 5 the results are applied to study a class of stochastic evolution equations.

2 Preliminaries

Let $F : \mathbb{R}_+ \rightarrow \mathbb{R}$ be a right-continuous function of bounded variation (e.g. nondecreasing càdlàg). Then we define μ_F on subintervals of \mathbb{R}_+ as follows:

$$\mu_F((a, b]) = F(b) - F(a), \quad 0 \leq a < b < \infty,$$

$$\mu_F(\{0\}) = 0.$$

By the Carathéodory extension theorem, μ_F extends to a measure, which we will call *the Lebesgue-Stieltjes measure* associated to F . Conversely, if μ is a measure such that $\mu((a, b]) = F(b) - F(a)$ for a given function F , then F has to be right-continuous.

Let (S, Σ) be separable measurable space and let $(\Omega, \mathbb{F}, \mathbb{P})$ be a probability space. A mapping $\nu : \Sigma \times \Omega \rightarrow [0, \infty]$ will be called a *random measure* if for all $A \in \Sigma$, $\omega \mapsto \nu(A, \omega)$ is measurable and for almost all $\omega \in \Omega$, $\nu(\cdot, \omega)$ is a measure on (S, Σ) and $(S, \Sigma, \nu(\cdot, \omega))$ is separable (i.e. such that the corresponding L^2 -space is separable).

Example 2.1. Let $F : \mathbb{R}_+ \times \Omega \rightarrow \mathbb{R}$ be a càdlàg process which is of bounded variation a.s. Then one can define a random measure $\mu_F : \mathcal{B}(\mathbb{R}_+) \times \Omega \rightarrow [0, \infty]$ such that $\mu_F(A, \omega) = \mu_{F(\omega)}(A)$.

Random measures arise naturally when working with continuous local martingales M . Indeed, for almost all $\omega \in \Omega$, the quadratic variation process $[M](\cdot, \omega)$ is continuous and increasing (see [35, 49, 66]), so as in Example 2.1 we can associate a Lebesgue-Stieltjes measure with it. Often we will denote this measure again by $[M](\cdot, \omega)$ for convenience.

Let (S, Σ, μ) be a measure space. Let X and Y be Banach spaces. An operator valued function $f : S \rightarrow \mathcal{L}(X, Y)$ is called *X -strongly measurable* if for all $x \in X$, the function $s \mapsto f(s)x$ is strongly measurable. It is called *scalarly measurable* if for all $y^* \in Y^*$, $f(s)^*y^*$ is strongly measurable. If Y is separable both measurability notions coincide.

Often we will use the notation $A \lesssim_Q B$ to indicate that there exists a constant C which depends on the parameter(s) Q such that $A \leq CB$.

2.1 Positive operators and self-adjoint operators on Banach spaces

Let X, Y be Banach spaces. We will denote the space of all bilinear operators from $X \times Y$ to \mathbb{R} as $\mathbf{B}(X, Y)$. Notice, that for each continuous $b \in \mathbf{B}(X, Y)$ there exists an operator $B \in \mathcal{L}(X, Y^*)$ such that

$$b(x, y) = \langle Bx, y \rangle, \quad x \in X, y \in Y. \tag{2.1}$$

We will call an operator $B : X \rightarrow X^*$ *self-adjoint*, if for each $x, y \in X$

$$\langle Bx, y \rangle = \langle Bx, y \rangle.$$

A self-adjoint operator B is called *positive*, if $\langle Bx, x \rangle \geq 0$ for all $x \in X$.

Remark 2.2. Notice, that if $B : X \rightarrow X^*$ is a positive self-adjoint operator, then the Cauchy-Schwartz inequality holds for the bilinear form $(x, y) := \langle Bx, y \rangle$ (see [72, 4.2]). From the latter one deduces that

$$\|B\| = \sup_{x \in X, \|x\|=1} |\langle Bx, x \rangle| \tag{2.2}$$

Moreover, if X is a Hilbert space, then (2.2) holds for any self-adjoint operator.

Further we will need the following lemma proved in [60, Proposition 32]:

Lemma 2.3. *Let (S, Σ) be a measurable space, H be a separable Hilbert space, $f : S \rightarrow \mathcal{L}(H)$ be a scalarly measurable self-adjoint operator-valued function. Let $F : \mathbb{R} \rightarrow \mathbb{R}$ be locally bounded measurable. Then $F(f) : S \rightarrow \mathcal{L}(H)$ is a scalarly measurable self-adjoint operator-valued function.*

The next lemma allows us to define a square root of a positive operator in case of a reflexive separable Banach space:

Lemma 2.4. *Let X be a reflexive separable Banach space, $B : X \rightarrow X^*$ be a positive operator. Then there exists a separable Hilbert space H and an operator $B^{1/2} : X \rightarrow H$ such that $B = B^{1/2*}B^{1/2}$.*

Proof. Since X is reflexive separable, X^* is also separable. We will use the space H , constructed in [42, p.154] (see also [9, p.15] and [61, Part 3.3]). Briefly speaking, one can find such a separable Hilbert space H that there exists a continuous dense embedding $j: X^* \hookrightarrow H$. Because of the reflexivity, $j^*: H \hookrightarrow X^{**} = X$ is a continuous dense embedding and as an embedding it has a trivial kernel.

Consider the operator $jBj^*: H \rightarrow H$. Obviously this operator is positive, so one can define a positive square root $\sqrt{jBj^*}: H \rightarrow H$ (see [22, Chapter 6.6]). Now define

$$B^{1/2} = \sqrt{jBj^*}j^{*-1}: \text{ran } j^* \rightarrow H.$$

This operator is bounded, because for each $x \in \text{ran } j^*$

$$\begin{aligned} \|\sqrt{jBj^*}j^{*-1}x\|_H^2 &= \langle \sqrt{jBj^*}j^{*-1}x, \sqrt{jBj^*}j^{*-1}x \rangle = \langle jBj^*j^{*-1}x, j^{*-1}x \rangle \\ &= \langle Bx, j^*j^{*-1}x \rangle = \langle Bx, x \rangle \leq \|B\| \|x\|^2, \end{aligned}$$

therefore it can be extended to the whole X . Moreover, for all $x, y \in \text{ran } j^*$

$$\langle B^{1/2*}B^{1/2}x, y \rangle = \langle B^{1/2}x, B^{1/2}y \rangle = \langle \sqrt{jBj^*}j^{*-1}x, \sqrt{jBj^*}j^{*-1}y \rangle = \langle Bx, y \rangle,$$

Thus $B^{1/2*}B^{1/2} = B$ on $\text{ran } j^*$, and hence on X by density and continuity. □

Remark 2.5. The square root obtained in such a way is not determined uniquely, since the operator j can be defined in different ways. The following measurability property holds: if there exists a measurable space (S, Σ) and a scalarly measurable function $f: S \rightarrow \mathcal{L}(X, X^*)$ with values in positive operators, then defined in such a polysemantic way $f^{1/2}$ will be also scalarly measurable. Indeed, since f is scalarly measurable, then jfj^* and, consequently by Lemma 2.3 and the fact that jfj^* is positive operator-valued, the square root $\sqrt{jfj^*}$ is scalarly measurable. So, $f^{1/2} = \sqrt{jfj^*}j^{*-1}x$ is measurable for all $x \in \text{ran } j^*$, and because of the boundedness of an operator $\sqrt{jfj^*}j^{*-1}$ and the density of $\text{ran } j^*$ in X one has that $f^{1/2}x$ is measurable for all $x \in X$.

2.2 Supremum of measures

In the main text we will often need explicit descriptions of the supremum of measures. The results are elementary, but we could not find suitable references in the literature. All positive measures are assumed to take values in $[0, \infty]$ (see [5, Definition 1.6.1]). In other words, a positive measure of a set could be infinite.

Lemma 2.6. *Let $(\mu_\alpha)_{\alpha \in \Lambda}$ be positive measures on a measurable space (S, Σ) . Then there exists the smallest measure $\check{\mu}$ s.t. $\check{\mu} \geq \mu_\alpha \forall \alpha \in \Lambda$. Moreover,*

$$\check{\mu}(A) = \sup_{n=1}^N \sum_{\alpha} \sup \mu_\alpha(A_n), \quad A \in \Sigma, \tag{2.3}$$

where the first supremum is taken over all the partitions $A = \bigcup_{n=1}^N A_n$ of A .

From now on we will write $\sup_{\alpha \in \Lambda} \mu_\alpha = \check{\mu}$, where $\check{\mu}$ is as in the above lemma. A similar formula as (2.3) can be found in [21, Exercise 213Y(d)] for finitely many measures.

Proof. The existence of the measure $\check{\mu}$ is well-known (see e.g. [35, Exercise 2.2] [21, Exercise 213Y(e)]), but it also follows from the proof below. To prove (2.3) let ν denote the right-hand side of (2.3).

We first show that ν is a measure. It suffices to show that ν is additive and σ -subadditive. To prove the σ -subadditivity, let $(B_k)_{k \geq 1}$ be sets in Σ and let $B = \bigcup_{k \geq 1} B_k$.

Let $A_1, \dots, A_N \in \Sigma$ be disjoint and such that $B = \bigcup_{n=1}^N A_n$. Let $B_{nk} = A_n \cap B_k$. Then by the σ -subadditive of the μ_α , we find

$$\sum_{n=1}^N \sup_{\alpha} \mu_\alpha(A_n) = \sum_{n=1}^N \sup_{\alpha} \sum_{k \geq 1} \mu_\alpha(B_{nk}) \leq \sum_{k \geq 1} \sum_{n=1}^N \sup_{\alpha} \mu_\alpha(B_{nk}) \leq \sum_{k \geq 1} \nu(B_k).$$

Taking the supremum over all A_n , we find $\nu(B) \leq \sum_{k \geq 1} \nu(B_k)$.

To prove the additivity let $B, C \in \Sigma$ be disjoint. By the previous step it remains to show that $\nu(B) + \nu(C) \leq \nu(B \cup C)$. Fix $\varepsilon > 0$ and choose $A_1, \dots, A_N \in \Sigma$ disjoint, $\alpha_1, \dots, \alpha_N \in \Lambda$ and $1 \leq M < N$ such that $\bigcup_{n=1}^M A_n = B$, $\bigcup_{n=M+1}^N A_n = C$ and

$$\nu(B) \leq \sum_{n=1}^M \mu_{\alpha_n}(A_n) + \varepsilon \quad \text{and} \quad \nu(C) \leq \sum_{n=M+1}^N \mu_{\alpha_n}(A_n) + \varepsilon.$$

Then we find that

$$\nu(B) + \nu(C) \leq \sum_{n=1}^N \mu_{\alpha_n}(A_n) + 2\varepsilon \leq \nu(B \cup C) + 2\varepsilon,$$

and the additivity follows.

Finally, we check that $\nu = \check{\mu}$. In order to check this let $\tilde{\nu}$ be a measure such that $\mu_\alpha \leq \tilde{\nu}$ for all α . Then for each $A \in \Sigma$ we find

$$\nu(A) = \sup \sum_{n=1}^N \sup_{\alpha} \mu_\alpha(A_n) \leq \sup \sum_{n=1}^N \tilde{\nu}(A_n) = \tilde{\nu}(A)$$

and hence $\nu \leq \tilde{\nu}$. Thus we may conclude that $\nu = \check{\mu}$. □

Remark 2.7. If the conditions of Lemma 2.6 are satisfied and there exists a measure μ such that $\mu_\alpha \leq \mu$, then $\check{\mu} \leq \mu$. In particular if μ is finite, then $\check{\mu}$ is finite as well.

Lemma 2.8. Let (S, Σ, ν) be a measure space. Let F be a set of measurable functions from S into $[0, \infty]$. Let $\{f_j\}_{j=1}^\infty$ be a sequence in F . Let $\bar{f} = \sup_{j \geq 1} f_j$ and assume $\sup_{f \in F} f = \bar{f}$. For each $f \in F$ let μ_f be the measure given by

$$\mu_f(A) = \int_A f \, d\nu.$$

Let $\check{\mu} = \sup_{f \in F} \mu_f$. Then $\check{\mu} = \sup_{j \geq 1} \mu_{f_j}$ and

$$\check{\mu}(A) = \int_A \bar{f} \, d\nu, \quad A \in \Sigma. \tag{2.4}$$

Proof. Since \bar{f} is the supremum of countably many measurable functions, it is measurable. Since $A \mapsto \int_A \bar{f} \, d\nu$ defines a measure which dominates all measures μ_f , the estimate “ \leq ” in (2.4) follows.

To prove the converse estimate, let $A \in \Sigma$, $\varepsilon > 0$ and $n \in \mathbb{N}$. Let $A_1 = \{s \in A : f_1(s) > (1 - \varepsilon)(\bar{f}(s) \wedge n)\}$ and let

$$A_{j+1} = \{s \in A : f_{j+1}(s) > (1 - \varepsilon)(\bar{f}(s) \wedge n)\} \setminus \bigcup_{k=1}^j A_k, \quad j \geq 1.$$

Then the $(A_j)_{j \geq 1}$ are pairwise disjoint and $\bigcup_{j \geq 1} A_j = A$, and therefore,

$$\check{\mu}(A) = \sum_{j \geq 1} \check{\mu}(A_j) \geq \sum_{j \geq 1} \mu_{f_j}(A_j) = \sum_{j \geq 1} \int_{A_j} f_j \, d\nu$$

$$\geq (1 - \varepsilon) \sum_{j \geq 1} \int_{A_j} \bar{f}(s) \wedge n \, d\nu = (1 - \varepsilon) \int_A \bar{f}(s) \wedge n \, d\nu.$$

Since $\varepsilon > 0$ and $n \in \mathbb{N}$ were arbitrary the required estimate follows. The identity $\check{\mu} = \sup_{j \geq 1} \mu_{f_j}$ follows if we replace F by $\{f_j : j \geq 1\}$ and apply (2.4) in this situation. \square

Lemma 2.9. *Let $(\mu_n)_{n \geq 1}$ be a sequence of measures on a measurable space (S, Σ) . Let $\check{\mu} = \sup_{n \geq 1} \mu_n$. Then for each $A \in \Sigma$,*

$$\left(\sup_{1 \leq n \leq N} \mu_n \right)(A) \rightarrow \check{\mu}(A), \quad \text{as } N \rightarrow \infty.$$

Proof. Let $A \in \Sigma$. Without loss of generality suppose that $\check{\mu}(A) < \infty$. Fix $\varepsilon > 0$. According to (2.3) there exists $K > 0$, a partition $A = \bigcup_{k=1}^K A_k$ of A into pairwise disjoint sets and an increasing sequence $(n_k)_{1 \leq k \leq K} \subseteq \mathbb{N}$ such that $\sum_{k=1}^K \mu_{n_k}(A_k) > \check{\mu}(A) - \varepsilon$. Hence $(\sup_{1 \leq n \leq n_K} \mu_n)(A) \geq \check{\mu}(A) - \varepsilon$, which finishes the proof. \square

Remark 2.10. Assume that in the situation above $S = \mathbb{R}$, Σ is a Borel σ -algebra. Define $\bar{\mu}$ on segments as follows:

$$\bar{\mu}(a, b] = \sup_{n=1}^N \sup_{\alpha} \mu_{\alpha}(A_n), \tag{2.5}$$

where the first supremum is taken over all the partitions $(a, b] = \bigcup_{n=1}^N A_n$ of the segments $(a, b]$ into pairwise disjoint segments. Then by Carathéodory's extension theorem $\bar{\mu}$ extends to a measure on the Borel σ -algebra. Obviously $\bar{\mu} \geq \mu_{\alpha}$ for each $\alpha \in \Lambda$ (because $(\bar{\mu} - \mu_{\alpha})((a, b]) \geq 0$ for every segment $(a, b]$, and so, by [5, Corollaries 1.5.8 and 1.5.9] for every Borel set) and $\bar{\mu} \leq \check{\mu}$. Consequently, $\bar{\mu} = \check{\mu}$. Notice that the segments in the partition $(a, b] = \bigcup_{n=1}^N A_n$ can be chosen with rational endpoints (of course except a and b). Then the supremum obtained in (2.5) will be the same.

3 Cylindrical continuous martingales and quadratic variation

In this section we assume that X is a Banach space with a separable dual space X^* . Let $(\Omega, \mathbb{F}, \mathbb{P})$ be a complete probability space with filtration $\mathbb{F} := (\mathcal{F}_t)_{t \in \mathbb{R}_+}$ that satisfies the usual conditions, and let $\mathcal{F} := \sigma(\bigcup_{t \geq 0} \mathcal{F}_t)$. We denote the predictable σ -algebra by \mathcal{P} .

In this section we introduce a class of cylindrical continuous local martingales on a Banach space X which have a certain quadratic variation. We will show that it extends several previous definitions from the literature even in the Hilbert space setting.

3.1 Definitions

A scalar-valued process M is called a continuous local martingale if there exists a sequence of stopping times $(\tau_n)_{n \geq 1}$ such that $\tau_n \uparrow \infty$ almost surely as $n \rightarrow \infty$ and $\mathbf{1}_{\tau_n > 0} M^{\tau_n}$ is a continuous martingale.

Let \mathcal{M} (resp. \mathcal{M}^{loc}) be the class of continuous (local) martingales. On \mathcal{M}^{loc} define the translation invariant metric given by

$$\|M\|_{\mathcal{M}^{\text{loc}}} = \sum_{n=1}^{\infty} 2^{-n} \mathbb{E}[1 \wedge \sup_{t \in [0, n]} |M_t|]. \tag{3.1}$$

Here and in the sequel we identify indistinguishable processes. One may check that this coincides with the ucp topology (uniform convergence compact sets in probability). The following characterization will be used frequently.

Remark 3.1. For a sequence of continuous local martingales one has $M^n \rightarrow 0$ in \mathcal{M}^{loc} if and only if for every $T \geq 0$, $[M^n]_T \rightarrow 0$ in probability and $M_0^n \rightarrow 0$ in probability (see [35, Proposition 17.6]).

The space \mathcal{M}^{loc} is a complete metric space. This is folklore, but we include a proof for convenience of the reader. Let $(M^n)_{n \geq 1}$ be a Cauchy sequence in \mathcal{M}^{loc} with respect to the ucp topology. By completeness of the ucp topology we obtain an adapted limit M with continuous paths. It remains to show that M is a continuous local martingale. By taking an appropriate subsequence without loss of generality we can suppose that $M^n \rightarrow M$ a.s. uniformly on compact sets. Define a sequence of stopping times $(\tau_k)_{k=1}^\infty$ as follows:

$$\tau_k = \begin{cases} \inf\{t \geq 0 : \sup_n \|M^n(t)\| > k\}, & \text{if } \sup_{t \in \mathbb{R}_+} \sup_n \|M^n(t)\| > k; \\ \infty, & \text{otherwise.} \end{cases}$$

Since each $(M^n)^{\tau_k}$ is a bounded local martingale with continuous paths, $(M^n)^{\tau_k}$ is a martingale as well by the dominated convergence theorem. Letting $n \rightarrow \infty$, it follows again by dominated convergence theorem that M^{τ_k} is a martingale. Therefore, M is a continuous local martingale with a localizing sequence $(\tau_k)_{k=1}^\infty$.

Let X be a Banach space. A continuous linear mapping $M : X^* \rightarrow \mathcal{M}^{\text{loc}}$ is called a *cylindrical continuous local martingale*. (Details on cylindrical martingales can be found in [3, 34]). For a cylindrical continuous local martingale M and a stopping time τ we can define $M^\tau : X^* \rightarrow \mathcal{M}^{\text{loc}}$ by $M^\tau x^*(t) = Mx^*(t \wedge \tau)$. In this way M^τ is a cylindrical continuous (local) martingale again. Two cylindrical continuous local martingales M and N are called *indistinguishable* if $\forall x^* \in X^*$ the local martingales Mx^* and Nx^* are indistinguishable.

Remark 3.2. On \mathcal{M}^{loc} it is also natural to consider the Emery topology, see [18] and also [38, 3, 34]. Because of the continuity of the local martingales we consider, this turns out to be equivalent. We find it therefore more convenient to use the ucp topology instead.

Remark 3.3. Since X^* is separable, we can find linearly independent vectors $(e_n)_{n \geq 1} \subseteq X^*$ with linear span F which is dense in X^* . For fixed $t \geq 0$ and almost all $\omega \in \Omega$ one can define $B_t : \Omega \rightarrow \mathbf{B}(F, F)$ such that $B_t(x^*, y^*) = [Mx^*, My^*]_t$ for all $x^*, y^* \in F$. Unfortunately, one can not guarantee, that $t \mapsto B_t$ is continuous a.s. Moreover, as we will see in Example 3.26 for X a Hilbert space, it may already happen that for a.a. $\omega \in \Omega$, for some $t > 0$, $B_t \notin \mathbf{B}(X^*, X^*)$.

In the following definition we introduce a new set of cylindrical martingales for which the above phenomenon does not occur.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, (S, Σ) be a measure space and let $\mathcal{M}_+(S, \Sigma)$ be a set of all positive measures on (S, Σ) . For $f, g : \Omega \rightarrow \mathcal{M}_+(S, \Sigma)$ we say that $f \geq g$ if $f(\omega) \geq g(\omega)$ for \mathbb{P} -a.a. $\omega \in \Omega$.

Definition 3.4. Let $M : X^* \rightarrow \mathcal{M}^{\text{loc}}$ be a linear mapping. Then M is said to have a quadratic variation if

- (1) There exists a smallest $f : \Omega \rightarrow \mathcal{M}_+(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+))$ such that $f \geq \mu_{[Mx^*]}$ for each $x^* \in X^*$, $\|x^*\| = 1$,
- (2) $f(\omega)[0, t]$ is finite for a.e. $\omega \in \Omega$ for all $t \geq 0$.

Let $[[M]] : \mathbb{R}_+ \times \Omega \rightarrow \mathbb{R}_+$ be such that

$$[[M]]_t(\omega) = \mathbf{1}_{f(\omega)[0, t] < \infty} f(\omega)[0, t].$$

Then $[[M]]$ is called the quadratic variation of M and we write $M \in \mathcal{M}_{\text{var}}^{\text{loc}}(X)$.

If additionally, for each $x^* \in X^*$, Mx^* is a martingale, we write $M \in \mathcal{M}_{\text{var}}(X)$.

Notice that in the definition above $f = \mu_{[[M]]}$ a.s. In the next proposition we collect some basic properties of $[[M]]$.

Proposition 3.5. *Assume $M \in \mathcal{M}_{\text{var}}^{\text{loc}}(X)$. Then M is a cylindrical continuous local martingale and the following properties hold:*

1. $[[M]]$ has a continuous version.
2. $[[M]]$ is predictable.
3. $[[M]]_0 = 0$ a.s.
4. $[[M]]$ is increasing.
5. For all $x^* \in X^*$ a.s. for all $s \leq t$,

$$[Mx^*]_t - [Mx^*]_s \leq ([[M]]_t - [[M]]_s) \|x^*\|^2.$$

In Example 3.26 we will see that not every cylindrical continuous local martingale is in $\mathcal{M}_{\text{var}}^{\text{loc}}(X)$.

Proof. Properties (3), (4) and (5) are immediate from the definitions. Properties (1) and (2) will be proved in Proposition 3.7 below.

To prove that M is a cylindrical continuous local martingale, fix $t \geq 0$ and a sequence $(x_n^*)_{n \geq 1}$ such that $x_n^* \rightarrow 0$. Then by (5), $[Mx_n^*]_t \rightarrow 0$ a.s., so by Remark 3.1 M is a continuous linear mapping. \square

Remark 3.6. Let $M \in \mathcal{M}_{\text{var}}^{\text{loc}}(X)$. Then M is a cylindrical continuous local martingale.

Proposition 3.7. *Let $M : X^* \rightarrow \mathcal{M}^{\text{loc}}$ be a cylindrical continuous local martingale. Then the following assertions are equivalent:*

1. $M \in \mathcal{M}_{\text{var}}^{\text{loc}}(X)$;
2. For any dense subset $(x_n^*)_{n \geq 1}$ of the unit ball in X^* there exists a nondecreasing right-continuous process $F : \mathbb{R}_+ \times \Omega \rightarrow \mathbb{R}_+$ such that for a.a. $\omega \in \Omega$ we have that $\mu_{F(\omega)} = \sup_n \mu_{[Mx_n^*](\omega)}$;
3. For any dense subset $(x_n^*)_{n \geq 1}$ of the unit ball in X^* there exists a nondecreasing right-continuous process $G : \mathbb{R}_+ \times \Omega \rightarrow \mathbb{R}_+$ such that for a.a. $\omega \in \Omega$ we have that $\mu_{[Mx_n^*](\omega)} \leq \mu_{G(\omega)}$.

Moreover, in this case F is a.s. continuous, predictable and $F = [[M]]$ a.s.

Proof. (1) \Rightarrow (2): Since $\mu_{[[M]]} \geq \mu_{[Mx_n^*]}$ a.s. for each $n \geq 1$, it follows that a.s. there exists $\check{\mu} := \sup_n \mu_{[Mx_n^*]} \leq \mu_{[[M]]}$ by the definition of a supremum of measures given in Lemma 2.6. By Remark 2.10 one can write $\check{\mu} = \mu_F$ where the process F is given by

$$F(t) = \sup_{j=1}^J \sup_{n \geq 1} \left([Mx_n^*]_{t_j} - [Mx_n^*]_{t_{j-1}} \right), \tag{3.2}$$

where the outer supremum is taken over all $0 = t_0 < t_1 < \dots < t_J < t$ with $t_j \in \mathbb{Q}$ for $j \in \{0, \dots, J\}$. The fact that F is increasing is clear from (3.2). The right-continuity of F follows from the fact that $\check{\mu}$ is a measure.

(2) \Rightarrow (3): This is trivial.

(3) \Rightarrow (2): Since each of the measures $\mu_{[Mx_n^*]}$ is nonatomic a.s., by (2.3) μ_F is nonatomic a.s. and finite by Remark 2.7 and hence F is finite and a.s. continuous.

(2) \Rightarrow (1): We claim that for each $x^* \in X^*$ with $\|x^*\| = 1$ a.s. $\mu_F \geq \mu_{[Mx^*]}$. Fix $x^* \in X^*$ of norm 1. Since M is a cylindrical continuous local martingale we can find $(n_k)_{k \geq 1}$ such that $x_{n_k}^* \rightarrow x^*$ and a.s. $[Mx_{n_k}^*] \rightarrow [Mx^*]$ uniformly on compact sets

as $k \rightarrow \infty$ (see [35, Exercise 17.8]). Then a.s. for all $0 \leq s < t < \infty$ one has that $[Mx_{n_k}^*]_t - [Mx_{n_k}^*]_s \leq F(t) - F(s)$ for each $k \geq 1$, so a.s.

$$[Mx^*]_t - [Mx^*]_s = \lim_{k \rightarrow \infty} [Mx_{n_k}^*]_t - [Mx_{n_k}^*]_s \leq \lim_{k \rightarrow \infty} F(t) - F(s) = F(t) - F(s),$$

and therefore $\mu_F \geq \mu_{[Mx^*]}$ a.s. We claim that F is a.s. the least function with this property. Let $\phi : \Omega \rightarrow \mathcal{M}_+(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+))$ be such that for all $x^* \in X^*$ with $\|x^*\| = 1$, $\phi \geq \mu_{[Mx^*]}$ a.s. Then $\phi \geq \sup_n \mu_{[Mx_n^*]} = \mu_F$ a.s. and hence μ_F is the smallest measure with the required property. By the definition of a quadratic variation we find that $F = [[M]]$ a.s.

Finally, note that by (3.2), F is adapted and therefore F is predictable by the a.s. pathwise continuity of F . □

Remark 3.8. Notice that by the above proposition the quadratic variation of $M \in \mathcal{M}_{\text{var}}^{\text{loc}}(X)$ has the following form a.s.

$$[[M]]_t = \sup_{n=1}^N \sup_m ([Mx_m^*]_{t_{i+1}} - [Mx_m^*]_{t_i}), \quad t \geq 0, \tag{3.3}$$

where the limit is taken over all rational partitions $0 = t_0 < \dots < t_N = t$ and $(x_m^*)_{m \geq 1}$ is a dense subset of the unit ball in X^* .

Next we give another characterization of M being in $\mathcal{M}_{\text{var}}^{\text{loc}}(X)$.

Theorem 3.9. *Let $M : X^* \rightarrow \mathcal{M}^{\text{loc}}$. Then $M \in \mathcal{M}_{\text{var}}^{\text{loc}}(X)$ if and only if there exists a mapping $a_M : \mathbb{R}_+ \times \Omega \rightarrow \mathbf{B}(X^*, X^*)$ such that for every $x^*, y^* \in X^*$, a.s. for all $t \geq 0$, $a_M(t)(x^*, y^*) = [Mx^*, My^*]_t$ and a.s. for all $t \geq 0$, $(x^*, y^*) \mapsto a_M(t)(x^*, y^*)$ is bilinear and continuous, and for all $t \geq 0$ the following limit exists*

$$G(t) := \lim_{\text{mesh} \rightarrow 0} \sum_{n=1}^N \sup_{\|x^*\|=1} (a_M(t_n)(x^*, x^*) - a_M(t_{n-1})(x^*, x^*)). \tag{3.4}$$

Here the limit is taken over partitions $0 = t_0 < \dots < t_N = t$.

If this is the case then $G(t) = [[M]]_t$ a.s. for all $t \geq 0$.

Proof. Let $M \in \mathcal{M}_{\text{var}}^{\text{loc}}(X)$. Fix a set $(x_m^*)_{m \geq 1} \subset X^*$ of linearly independent vectors such that $\text{span}(x_m^*)_{m \geq 1}$ is dense in X^* . Let $F = (y_n^*) \subset X^*$ be the \mathbb{Q} -span of $(x_m^*)_{m \geq 1}$. Then there exists $a_M : \mathbb{R}_+ \times \Omega \rightarrow \mathbf{B}(F, F)$ such that for each $n, k \geq 1$ $a_M(y_n^*, y_k^*)$ is a version of $[My_n^*, My_k^*]$ such that $\mu_{a_M(y_n^*, y_k^*)} \ll \mu_{[[M]]} \|y_n^*\| \|y_k^*\|$. Since by the last inequality a_M is bounded on $F \times F$, it can be extended to $X^* \times X^*$, and by the continuity of M , $a_M(x^*, y^*)$ is a version of $[Mx^*, My^*]$. To prove (3.4) notice that because of the boundedness of a_M and a density argument one replace the supremum over the unit sphere by the supremum over $x^* \in \{y_n^* : \|y_n^*\| \leq 1\}$. Then this formula coincides with (3.3), therefore a.s. $G(t) = [[M]]_t$ for all $t \geq 0$.

To prove the converse first note that for all $x^* \in X^*$, $\mu_{[Mx^*]} \leq \mu_G \|x^*\|^2$ a.s. and hence M is a cylindrical continuous local martingale by Remark 3.1. Since a_M is continuous one can replace the supremum by the supremum over a countable dense subset of the unit ball again. Now one can apply Proposition 3.7 and use (2.5). □

Definition 3.10. Given $M \in \mathcal{M}_{\text{var}}^{\text{loc}}(X)$ we define its cylindrical Doléans measure μ_M on the predictable σ -algebra \mathcal{P} as follows:

$$\mu_M(C) = \mathbb{E} \int_0^\infty \mathbf{1}_C d[[M]], \quad C \in \mathcal{P}.$$

Lemma 3.11. *Let $M \in \mathcal{M}_{\text{var}}^{\text{loc}}(X)$ and let τ be a stopping time and define a sequence of stopping times by*

$$\tau_n = \inf\{t \geq 0 : [[M]]_t \geq n\}, \text{ for } n \geq 1.$$

Then the following assertions hold:

1. $M^\tau \in \mathcal{M}_{\text{var}}^{\text{loc}}(X)$, $[[M^\tau]] = [[M]]^\tau$.
2. For each $n \geq 1$, $M^{\tau_n} \in \mathcal{M}_{\text{var}}(X)$.

Proof. (1): It is obvious from the scalar theory that for every $x^* \in X^*$ with $\|x^*\| \leq 1$, $M^\tau x^*$ is a continuous local martingale. Moreover,

$$d\mu_{[M^\tau x^*]} = \mathbf{1}_{[0, \tau]} d\mu_{[Mx^*]} \leq \mathbf{1}_{[0, \tau]} d\mu_{[[M]]}.$$

Since $\mu_{[[M]]}$ is the least measure which majorizes $\mu_{[Mx^*]}$ for $\|x^*\| = 1$, it follows that $\mathbf{1}_{[0, \tau]} d\mu_{[[M]]}$ is the least measure which majorizes $\mu_{[M^\tau x^*]}$ for $\|x^*\| = 1$.

(2): To check that $M^{\tau_n} \in \mathcal{M}_{\text{var}}(X)$ it remains to show that $\mathbf{1}_{\tau_n > 0} M^{\tau_n} x^*$ is a martingale. By the Burkholder-Davis-Gundy inequality [35, Theorem 26.12] and the continuity of $[[M]]$ we have for all $x^* \in X^*$

$$\mathbb{E} \sup_{s \leq t} |\mathbf{1}_{\tau_n > 0} M_s^{\tau_n} x^*| \leq C \mathbb{E} [M^{\tau_n} x^*]_t^{1/2} = C \mathbb{E} [[M]]_{t \wedge \tau_n}^{1/2} \|x^*\| \leq Cn^{1/2} \|x^*\|.$$

Therefore, the martingale property follows from the dominated convergence theorem and the fact that $\mathbf{1}_{\tau_n > 0} M^{\tau_n} x^*$ is a local martingale. □

We end this subsection with a simple but important example.

Example 3.12 (Cylindrical Brownian motion). Let X be a Banach space and $Q \in \mathcal{L}(X^*, X)$ be a positive self-adjoint operator. Let $W^Q : \mathbb{R}_+ \times X^* \rightarrow L^2(\Omega)$ be a cylindrical Q -Brownian motion (see [13, Chapter 4.1]), i.e.

- $W^Q(\cdot)x^*$ is a Brownian motion for all $x^* \in X$,
- $\mathbb{E} W^Q(t)x^* W^Q(s)y^* = \langle Qx^*, y^* \rangle \min\{t, s\} \forall x^*, y^* \in X^*, t, s \geq 0$.

The operator Q is called the *covariance operator* of W^Q . Then $W^Q \in \mathcal{M}_{\text{var}}(X)$. Indeed, since $a_{W^Q}(t)(x^*, x^*) = t\langle Qx^*, x^* \rangle$ we have $[[M]]_t = t\|Q\|$.

In the case $Q = I$ is the identity operator on a Hilbert space H , we will call $W_H = W^I$ an *H -cylindrical Brownian motion*. In this case $[[M]]_t = t$.

3.2 Quadratic variation operator

Let $M \in \mathcal{M}_{\text{var}}^{\text{loc}}(X)$. From Example 3.12 one sees that essential information about the cylindrical martingale is lost when one only considers $[[M]]$. For this reason we introduce the quadratic variation operator A_M and its $[[M]]$ -derivative Q_M .

Let $\Omega_0 \subset \Omega$ be a set of a full measure such that $G(t)$ from (3.4) is finite for all $t \geq 0$ in Ω_0 . Note that pointwise in Ω_0 for all $t \geq 0$,

$$|a_M(t)(x^*, y^*)| \leq [[M]]_t \|x^*\| \|y^*\| \quad \forall x^*, y^* \in F.$$

It follows that for all $\omega \in \Omega_0$ for all $t \geq 0$ and all $x^* \in X^*$, the bilinear map $(x^*, y^*) \mapsto a_M(t, \omega)(x^*, y^*)$ is bounded by $[[M]]_t(\omega)$ in norm, and therefore it defines a mapping $A_M(t, \omega) \in \mathcal{L}(X^*, X^{**})$. For $\omega \notin \Omega_0$ we set $A_M = 0$. Note that for each $x^*, y^* \in X^*$, for almost all $\omega \in \Omega$, and for all $t \geq 0$, $\langle A_M(t)x^*, y^* \rangle$ is a version of $[Mx^*, My^*]_t$. The function A_M is called *the quadratic variation operator of M* . By construction, for every $x^*, y^* \in X^*$, $(t, \omega) \mapsto \langle A_M(t, \omega)x^*, y^* \rangle$ is predictable. Moreover, one can check that for each $t \geq 0$ and $\omega \in \Omega$, and $x^*, y^* \in X^*$,

$$\langle A_M(t, \omega)x^*, x^* \rangle \geq 0, \quad \text{and} \quad \langle A_M(t, \omega)x^*, y^* \rangle = \langle A_M(t, \omega)y^*, x^* \rangle.$$

Proposition 3.13 (Polar decomposition). *For each $M \in \mathcal{M}_{\text{var}}^{\text{loc}}(X)$ there exists a process $Q_M : \mathbb{R}_+ \times \Omega \rightarrow \mathcal{L}(X^*, X^{**})$ such that almost surely for all $t > 0$*

$$\langle A_M(t)x^*, y^* \rangle = \int_0^t \langle Q_M(s)x^*, y^* \rangle d[[M]]_s, \quad x^*, y^* \in X^*. \tag{3.5}$$

Moreover, the following properties hold:

1. For all $x^*, y^* \in X^*$, $(t, \omega) \mapsto \langle Q_M(t, \omega)x^*, y^* \rangle$ is predictable.
2. Q is self-adjoint and positive μ_M -a.e.
3. $\|Q_M(t)\| = 1$ for $\mu_{[[M]]}$ -a.e. t on \mathbb{R}_+ . In particular, $\|Q_M(t, \omega)\| = 1$, μ_M -a.s. on $\mathbb{R}_+ \times \Omega$.

In (3.5) the Lebesgue-Stieltjes integral is considered. In the proof we use the following fact which is closely related to [5, Theorem 5.8.8] and [20, Theorem 3.21]. In the statement and its proof we use the convention that $\frac{0}{0} = 0$.

Lemma 3.14. *Let μ be a positive non-atomic σ -finite measure on \mathbb{R}_+ and let $f \in L^1_{\text{loc}}(\mathbb{R}_+, \mu)$. Define the measure ν by $d\nu = f d\mu$. Then for μ -almost all $t > 0$,*

$$\lim_{\varepsilon \downarrow 0} \frac{\nu((t - \varepsilon, t])}{\mu((t - \varepsilon, t])} = f(t).$$

Proof. It is enough to show this lemma given $\mu \geq \lambda$. If it is shown for $\mu \geq \lambda$, then in general situation one can use $\mu + \lambda$: due to the fact that $\mu \ll \mu + \lambda$ one has that there exists $g : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that $d\mu = g d(\mu + \lambda)$ and $d\nu = fg d(\mu + \lambda)$, so for μ -a.a. $t \geq 0$

$$\begin{aligned} \lim_{\varepsilon \downarrow 0} \frac{\nu((t - \varepsilon, t])}{\mu((t - \varepsilon, t])} &= \lim_{\varepsilon \downarrow 0} \frac{\nu((t - \varepsilon, t])}{(\mu + \lambda)((t - \varepsilon, t])} \bigg/ \lim_{\varepsilon \downarrow 0} \frac{\mu((t - \varepsilon, t])}{(\mu + \lambda)((t - \varepsilon, t])} \\ &= \frac{(fg)(t)}{g(t)} = f(t). \end{aligned}$$

Now let $\mu \geq \lambda$, and define $\tau : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ by $\tau(t) = \inf\{s : \mu([0, s]) > t\}$. Then $\mu \circ \tau = \lambda$ is the Lebesgue measure on \mathbb{R}_+ , $d(\nu \circ \tau) = f \circ \tau d\lambda$. By the Lebesgue differentiation theorem (see [20, Theorem 3.21]) one has

$$\lim_{\varepsilon \downarrow 0} \frac{\nu \circ \tau((t - \varepsilon, t])}{\mu \circ \tau((t - \varepsilon, t])} = f \circ \tau(t), \tag{3.6}$$

for λ -almost all t . Define $F : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ by $F(s) = \mu([0, s])$. Then F is strictly increasing and continuous since μ is nonatomic. Therefore $\tau \circ F(s) = s$ for all $s \in \mathbb{R}_+$, and it follows from (3.6) that for μ -a.a. $t \in \mathbb{R}_+$

$$\begin{aligned} \lim_{\varepsilon \downarrow 0} \frac{\nu((t - \varepsilon, t])}{\mu((t - \varepsilon, t])} &= \lim_{\varepsilon \downarrow 0} \frac{\nu \circ \tau \circ F((t - \varepsilon, t])}{\mu \circ \tau \circ F((t - \varepsilon, t])} = \lim_{\varepsilon \downarrow 0} \frac{\nu \circ \tau((F(t - \varepsilon), F(t)))}{\mu \circ \tau((F(t - \varepsilon), F(t)))} \\ &= f \circ \tau(F(t)) = f(t). \quad \square \end{aligned}$$

Proof of Proposition 3.13. Let $\Omega_0 \subset \Omega$ be a set of a full measure such that $G(t)$ from (3.4) is finite for all $t \geq 0$ in Ω_0 . Then pointwise on Ω_0 , for all $x^*, y^* \in X^*$, $\langle A_M x^*, y^* \rangle$ is absolutely continuous with respect to $[[M]]$. Let $(e_n^*)_{n \geq 1} \subseteq X^*$ be a set of linearly independent vectors, such that its linear span F is dense in X^* . Then there exists a process $Q_M : \Omega \times \mathbb{R}_+ \rightarrow \mathcal{L}(F, X^{**})$ such that $\langle Q_M e_n^*, e_m^* \rangle$ is predictable for each $n, m \geq 1$ and $\int_0^t \langle Q_M(s)e_n^*, e_m^* \rangle d[[M]]_s = \langle A_M(t)e_n^*, e_m^* \rangle$. To check the predictability, note that by Lemma 3.14 a.s. for $\mu_{[[M]]}$ -a.a. $t \geq 0$,

$$\frac{\langle A_M(t)e_n^*, e_m^* \rangle - \langle A_M(t - 1/k)e_n^*, e_m^* \rangle}{[[M]]_t - [[M]]_{t - \frac{1}{k}}} \rightarrow \langle Q_M(t)e_n^*, e_m^* \rangle$$

as $k \rightarrow \infty$. Since the left-hand side is predictable, the right-hand side has a predictable version.

Let $(f_n^*)_{n \geq 1}$ in F of length one be dense in $\{x^* \in X^* : \|x^*\| = 1\}$. Then by the definition of $[[M]]$, on Ω_0 it holds that $|\mu_{a_M(\cdot)}(x^*, x^*)| \leq \mu_{[[M]]}$ for all $m, n \geq 1$. Therefore on Ω_0 , we find that for all $m, n \geq 1$, $|\langle Q_M(s)f_n^*, f_m^* \rangle| \leq 1$ for $\mu_{[[M]]}$ -a.a. $t \geq 0$. Let $S \subseteq \mathbb{R}_+ \times \Omega_0$ be the set where $|\langle Q_M(s)f_n^*, f_m^* \rangle| \leq 1$ for all $m, n \geq 1$. Then S is predictable and for each $\omega \in \Omega_0$, $\mu_{[[M]]}(\mathbb{R} \setminus S_\omega) = 0$, where S_ω denotes its section. Taking the supremum over all $n, m \geq 1$, it follows that $\|Q_M\| \leq 1$ on S . On the complement of S we let $Q_M = 0$. Since F is dense in X^* , Q_M has a unique continuous extension to a mapping in $\mathcal{L}(X^*, X^{**})$.

Fix $t > 0$. Let $x^*, y^* \in X^*$, $(x_n^*)_{n \geq 1}, (y_n^*)_{n \geq 1} \subseteq F$ be such that $x^* = \lim_{n \rightarrow \infty} x_n^*$, $y^* = \lim_{n \rightarrow \infty} y_n^*$. Since on Ω_0 for all $t \geq 0$,

$$\langle A_M(t)x_n^*, y_m^* \rangle = \int_0^t \langle Q_M(s)x_n^*, y_m^* \rangle d[[M]]_s,$$

letting as $m, n \rightarrow \infty$, (3.5) follows by the dominated convergence theorem.

We claim that for all $\omega \in \Omega_0$, $\|Q_M\| = 1$, $\mu_{[[M]]}$ -a.e. on \mathbb{R}_+ . Since $\mu_{[[M]]}$ is a maximum for the measures $\mu_{a_M(\cdot)}(f_n^*, f_n^*)$ (where the f_n^* are as before) it follows that $\|Q_M\| = \sup_n \langle Q_M f_n^*, f_n^* \rangle = 1$, $\mu_{[[M]]}$ -a.e. on \mathbb{R}_+ . Indeed, otherwise there exists an $\alpha \in (0, 1)$ such that $C = \{t \in \mathbb{R}_+ : \|Q(t)\| < \alpha\}$ satisfies $\mu_{[[M]]}(C) > 0$. Then it follows that for the maximal measure and all measurable $B \subseteq C$

$$\mu_{a_M(\cdot)}(f_n^*, f_n^*)(B) = \int_B \langle Q_M(s)f_n^*, f_n^* \rangle d[[M]]_s \leq \alpha \mu_{[[M]]}(B).$$

This contradicts the fact that the supremum measure on the left equals $\mu_{[[M]]}$ as well. Thus $\mu_{[[M]]}(C) = 0$ and hence the claim follows.

It follows from the construction that Q_M is self-adjoint and positive $\mu_{[[M]]}$ -a.s. □

Remark 3.15. Assume that X^{**} is also separable (e.g. X is reflexive). In this case it follows from the Pettis measurability theorem that the functions $A_M x^*$ and $Q_M x^*$ are strongly progressively measurable for each $x^* \in X^*$ (see e.g. [32]). Moreover, if $\phi : \mathbb{R}_+ \times \Omega \rightarrow X^*$ is strongly progressively measurable, then $A_M \phi$ and $Q_M \phi$ are strongly progressively measurable as well.

Remark 3.16. Let H be a separable Hilbert space and X be a separable Banach space. In [50, 60, 61] cylindrical continuous martingales are considered for which the quadratic variation operator has the form

$$\langle A_M(t)x^*, y^* \rangle = \int_0^t (g^* x^*, g^* y^*)_H ds,$$

where $g : \mathbb{R}_+ \times \Omega \rightarrow \mathcal{L}(H, X)$ is such that for all $x^* \in X^*$, $g^* x^* \in L^2_{loc}(\mathbb{R}_+; H)$. In this case $[[M]]_t = \int_0^t \|g g^*\| ds$. Indeed,

$$a_M(b)(x^*, x^*) - a_M(a)(x^*, x^*) = \int_{(a,b)} \|g(s)^* x^*\|_H^2 ds$$

and hence the identity follows from Lemma 2.8, Remark 2.10, Theorem 3.9 and the separability of X^* .

3.3 Quadratic variation for local martingales

In this section we will study the case where the cylindrical local martingale actually comes from a local martingale on X . We discuss several examples and compare our definition quadratic variation from Definition 3.4 to other definitions. In order to distinguish

between martingales and cylindrical martingales we use the notation \widetilde{M} for an X -valued martingale.

For a continuous local martingale $\widetilde{M}: \mathbb{R}_+ \times \Omega \rightarrow X$ we define the associated cylindrical continuous martingale $M: \mathbb{R}_+ \times X^* \rightarrow L^0(\Omega)$ by

$$Mx^* = \langle \widetilde{M}, x^* \rangle, \quad x^* \in X^*.$$

It is a cylindrical continuous local martingale since if $(x_n^*)_{n \geq 1} \subseteq X^*$ vanishes as $n \rightarrow \infty$, then for all $t \geq 0$ almost all ω

$$\sup_{0 \leq s \leq t} |\langle \widetilde{M}_s(\omega), x_n^* \rangle| \leq \|x_n^*\| \sup_{0 \leq s \leq t} \|\widetilde{M}_s(\omega)\| \rightarrow 0 \quad n \rightarrow \infty,$$

so $\langle \widetilde{M}, x_n^* \rangle \rightarrow 0$ in the ucp topology.

Below we explain several situations where one can check that the associated cylindrical continuous local martingale M is an element of $\mathcal{M}_{\text{var}}^{\text{loc}}(X)$. In general this is not true (see Example 3.25).

First we recall some standard notation in the case H is a separable Hilbert space. Let $\widetilde{M}: \mathbb{R}_+ \times \Omega \rightarrow H$ be a continuous local martingale. Then the quadratic variation is defined by

$$[\widetilde{M}]_t = \mathbb{P} - \lim_{\text{mesh} \rightarrow 0} \sum_{n=1}^N \|M_{t_n} - M_{t_{n-1}}\|^2. \tag{3.7}$$

where $0 = t_0 < t_1 < \dots < t_N = t$. It is well known that this limit exists in the ucp sense (see [49, 2.6 and 3.2]) and the limit coincides with the unique increasing and continuous process starting at zero such that $\|\widetilde{M}\|^2 - [\widetilde{M}]$ is a continuous local martingale. Moreover, one can always choose a sequence of partitions with $\text{mesh} \rightarrow 0$ for which a.s. uniform convergence on compact intervals holds.

Observe that for an orthonormal basis $(h_n)_{n \geq 1}$, letting $M_t^n = (\widetilde{M}_t, h_n)_H$ we find that almost surely for all $t \geq 0$

$$\widetilde{M}_t = \sum_{n \geq 1} M_t^n h_n$$

with convergence in H . Moreover, the following identity for the quadratic variation holds (see [49, Chapter 14.3]): a.s.

$$[\widetilde{M}]_t = \sum_{n \geq 1} [M^n]_t, \quad \text{for all } t \geq 0. \tag{3.8}$$

Next we first consider two finite dimensional examples before returning to the infinite dimensional setting.

Example 3.17. Let $\widetilde{M} \in \mathcal{M}_{\text{var}}^{\text{loc}}(\mathbb{R})$. Then $\widetilde{M} = M1$ is a continuous real-valued local martingale, $[[M]] = [\widetilde{M}]$ and $Q_M = 1$ (where Q_M is as in Proposition 3.13).

Example 3.18. Let $d \geq 1$ and $H = \mathbb{R}^d$. Again let $M \in \mathcal{M}_{\text{var}}^{\text{loc}}(H)$. Let h_1, \dots, h_d be an orthonormal basis in H . Then $\widetilde{M} = \sum_{n=1}^d Mh_n \otimes h_n$ defines a continuous H -valued local martingale. Moreover, its quadratic variation satisfies

$$[\widetilde{M}]_t = \sum_{n=1}^d [Mh_n]_t, \quad t \geq 0.$$

and in particular the right-hand side does not depend on the choice of the orthonormal basis. It follows that

$$\frac{[\widetilde{M}]_t - [\widetilde{M}]_s}{d} \leq \sup_{\|h\|=1} ([Mh]_t - [Mh]_s) \leq [\widetilde{M}]_t - [\widetilde{M}]_s, \quad t > s \geq 0,$$

and hence from the definition of $[[M]]$ we see that

$$\frac{[\widetilde{M}]_t - [\widetilde{M}]_s}{d} \leq [[M]]_t - [[M]]_s \leq [\widetilde{M}]_t - [\widetilde{M}]_s, \quad t > s \geq 0,$$

which means that the Lebesgue-Stieltjes measures $\mu_{[\widetilde{M}]}$ and $\mu_{[[M]]}$ are equivalent a.s.

Example 3.19. Let H be a separable Hilbert space again and let \widetilde{M} be an H -valued continuous local martingale. The quadratic variation operator (see [49, Chapter 14.3]) $\langle \widetilde{M} \rangle : \mathbb{R}_+ \times \Omega \rightarrow \mathcal{L}(H)$ is defined by

$$\langle \widetilde{M} \rangle_t h, g = [\langle \widetilde{M}, h \rangle, \langle \widetilde{M}, g \rangle]_t, \quad \omega \in \Omega, t \geq 0.$$

To see that this is well-defined and bounded of norm at most $[M]_t$, choose partitions with decreasing mesh sizes such that the convergence in (3.7) holds on a set of full measure Ω_0 . Then a polarization argument shows that pointwise on Ω_0 ,

$$\begin{aligned} |[\langle \widetilde{M}, h \rangle, \langle \widetilde{M}, g \rangle]_t| &= \lim \left| \sum_{n=1}^N \langle \widetilde{\Delta} M_{t_n}, h \rangle \langle \widetilde{\Delta} M_{t_n}, g \rangle \right| \\ &\leq \lim \sum_{n=1}^N \|\Delta M_{t_n}\|^2 \|g\| \|h\| = [M]_t \|g\| \|h\|. \end{aligned}$$

The operator $\langle \widetilde{M} \rangle_t$ is positive and it follows from (3.8) that for any orthonormal basis $(h_n)_{n \geq 1}$ of H , pointwise on Ω_0 for all $t \geq 0$,

$$\sum_{n \geq 1} \langle \langle \widetilde{M} \rangle_t h_n, h_n \rangle = \sum_{n \geq 1} [\langle \widetilde{M}, h_n \rangle]_t = [M]_t.$$

Hence a.s. for all $t \geq 0$, $\langle \widetilde{M} \rangle_t$ a trace class operator and $\text{Tr} \langle \widetilde{M} \rangle_t = [M]_t$.

As in Proposition 3.13 one sees that there is a $q_{\widetilde{M}} : \mathbb{R}_+ \times \Omega \rightarrow \mathcal{L}(H)$ such that for all $g, h \in H$, $\langle q_{\widetilde{M}} g, h \rangle$ is predictable and a.s.

$$\langle \widetilde{M} \rangle_t = \int_0^t q_{\widetilde{M}}(s) d[\widetilde{M}]_s, \quad t > 0.$$

Moreover, a.s. $q_{\widetilde{M}}$ is a trace class operator $\mu_{[\widetilde{M}]}$ -a.a., and $\text{Tr}(q_{\widetilde{M}}(t)) = 1$ a.s. for all $t \geq 0$.

Define $M : \mathbb{R}_+ \times H \rightarrow L^0(\Omega)$ by the formula

$$Mh := \langle \widetilde{M}, h \rangle, \quad h \in H.$$

We claim that $M \in M_{\text{var}}^{\text{loc}}(H)$. As before a.s. for $\forall t > s > 0$, $\sup_{\|h\|=1} [Mh]_t - [Mh]_s \leq [\widetilde{M}]_t - [\widetilde{M}]_s$, so $[[M]]_t - [[M]]_s \leq [\widetilde{M}]_t - [\widetilde{M}]_s$, which means that a.s. $[[M]]_t$ is continuous in t . Such M is called the associated local H -cylindrical martingale.

Now we find that almost surely, for all $h, g \in H$ and $t \geq 0$

$$\int_0^t \langle Q_M(s)g, h \rangle d[[M]]_s = [Mh, Mg]_t = [\langle \widetilde{M}, h \rangle, \langle \widetilde{M}, g \rangle]_t = \int_0^t \langle q_{\widetilde{M}}(s)g, h \rangle d[\widetilde{M}]_s.$$

Moreover, an approximation argument yields that for all elementary progressive processes $\phi, \psi : \mathbb{R}_+ \times \Omega \rightarrow H$

$$\int_0^\infty \langle Q_M(s)\phi(s), \psi(s) \rangle d[[M]]_s = \int_0^\infty \langle q_{\widetilde{M}}(s)\phi(s), \psi(s) \rangle d[\widetilde{M}]_s. \quad (3.9)$$

Remark 3.20. Example 3.19 illustrates some of the advantages using $[[M]]$ instead of $[M]$. Indeed, $[M]$ is rather large and in order to compensate for this q_M has to be small (of trace class). On the other hand $[[M]]$ is so small that only the boundedness of Q_M is needed. The above becomes even more clear in the cylindrical case, where $[M]$ and q_M are not defined at all.

Let X be a Banach space, $\widetilde{M} : \mathbb{R}_+ \times \Omega \rightarrow X$ be a continuous local martingale. Then we say that \widetilde{M} has a *scalar quadratic variation* (see [14, Definition 4.1]), if for any $t > 0$

$$[[\widetilde{M}]_t]^\varepsilon := \int_0^t \frac{\|\widetilde{M}_{s+\varepsilon} - \widetilde{M}_s\|^2}{\varepsilon} ds \tag{3.10}$$

has a ucp limit as $\varepsilon \rightarrow 0$. In this case the limit will be denoted by $[\widetilde{M}]_t := \mathbb{P} - \lim_{\varepsilon \rightarrow 0} [[\widetilde{M}]_t]^\varepsilon$. Since in the Hilbert space case the above limit coincides with the previously defined quadratic variation, there is no risk of confusion here (see [14, Remark 4.3.3-4.3.4]).

Outside the Hilbert space setting it is not so simple to determine whether the scalar quadratic variation exists. Also note that the definition can not be extended to cylindrical (local) martingales. In the next example we show that the existence of $[M]$ implies the existence of $[[M]]$.

Example 3.21. Let \widetilde{M} be an X -valued continuous local martingale with a scalar quadratic variation. Then the associated cylindrical continuous local martingale $Mx^* := \langle \widetilde{M}, x^* \rangle$ for $x^* \in X^*$ is in $\mathcal{M}_{\text{var}}^{\text{loc}}(X)$. Indeed, choose a sequence $\varepsilon_n \rightarrow 0$ such that the limit in (3.10) converges uniformly on compact intervals on a set of full measure Ω_0 . Then for every $\omega \in \Omega_0, t > s \geq 0, x^* \in X^*$,

$$[Mx^*]_t - [Mx^*]_s = \lim_{n \rightarrow \infty} \int_s^t \frac{|(\widetilde{M}x^*)_{r+\varepsilon_n} - (\widetilde{M}x^*)_r|^2}{\varepsilon_n} dr \leq ([\widetilde{M}]_t - [\widetilde{M}]_s) \|x^*\|.$$

Therefore, $[[M]]$ exists and for all $\omega_0 \in \Omega, t > s \geq 0, [[M]]_t - [[M]]_s \leq [\widetilde{M}]_t - [\widetilde{M}]_s$. With a similar argument one sees that the existence of the tensor quadratic variations of [14] implies the existence of $[[M]]$.

It follows from Example 3.25 that there are martingales which do not admit a scalar (or tensor) quadratic variation. We do not know if the existence of $[[M]]$ implies that $[M]$ (or its tensor quadratic variation) exists in general.

3.4 Cylindrical martingales and stochastic integrals

Let X, Y be two Banach spaces, $x^* \in X^*, y \in Y$. We denote by $x^* \otimes y \in \mathcal{L}(X, Y)$ the following linear operator: $x^* \otimes y : x \mapsto \langle x^*, x \rangle y$.

Let X be a Banach space. The process $\Phi : \mathbb{R}_+ \times \Omega \rightarrow \mathcal{L}(H, X)$ is called *elementary progressive* with respect to the filtration $\mathbb{F} = (\mathcal{F}_t)_{t \in \mathbb{R}_+}$ if it is of the form

$$\Phi(t, \omega) = \sum_{n=1}^N \sum_{m=1}^M \mathbf{1}_{(t_{n-1}, t_n] \times B_{mn}}(t, \omega) \sum_{k=1}^K h_k \otimes x_{kmn},$$

where $0 \leq t_0 < \dots < t_n < \infty$, for each $n = 1, \dots, N$ the sets $B_{1n}, \dots, B_{Mn} \in \mathcal{F}_{t_{n-1}}$ and vectors h_1, \dots, h_K are orthogonal. For each elementary progressive Φ we define the stochastic integral with respect to $M \in \mathcal{M}_{\text{var}}^{\text{loc}}(H)$ as an element of $L^0(\Omega; C_b(\mathbb{R}_+; X))$ as

$$\int_0^t \Phi(s) dM(s) = \sum_{n=1}^N \sum_{m=1}^M \mathbf{1}_{B_{mn}} \sum_{k=1}^K (M(t_n \wedge t) h_k - M(t_{n-1} \wedge t) h_k) x_{kmn}. \tag{3.11}$$

Often we will write $\Phi \cdot M$ for the process $\int_0^\cdot \Phi(s) dM(s)$.

Remark 3.22. For all progressively measurable processes $\phi : \mathbb{R}_+ \times \Omega \rightarrow \mathcal{L}(H, \mathbb{R})$ with $\phi Q_M^{1/2} \in L^2(\mathbb{R}_+, [[M]]; \mathcal{L}(H, \mathbb{R}))$ one has

$$\left[\int_0^\cdot \phi \, dM \right]_t = \int_0^t \phi(s) Q_M(s) \phi^*(s) \, d[[M]]_s. \tag{3.12}$$

This can be proved analogously to [49, (14.7.4)].

One can also prove that in the situation above for each stopping time $\tau : \Omega \rightarrow \mathbb{R}_+$ a.s. for all $t \geq 0$

$$\left(\int_0^\cdot \phi \, dM \right)^{t \wedge \tau} = \int_0^t \phi(s) \mathbf{1}_{s \leq \tau} \, dM_s = \int_0^t \phi \, dM^\tau. \tag{3.13}$$

If the domain of ϕ is in a fixed finite dimensional subspace $H_0 \subseteq H$, then (3.13) is an obvious multidimensional corollary of [35, Proposition 17.15]. For general ϕ it follows from an approximation argument. Indeed, let $\phi_n : \mathbb{R}_+ \times \Omega \rightarrow \mathcal{L}(H_n, \mathbb{R})$, where $H_n \subseteq H$ is fixed finite dimensional for each $n \geq 0$, be such that $\phi_n Q_M^{1/2} \rightarrow \phi Q_M^{1/2}$ in $L^2(\mathbb{R}_+, [[M]]; \mathcal{L}(H, \mathbb{R}))$ a.s. Then thanks to Lemma 3.11 $\phi_n Q_M^{1/2} \rightarrow \phi Q_M^{1/2}$ in $L^2(\mathbb{R}_+, [[M^\tau]]; \mathcal{L}(H, \mathbb{R}))$ a.s. and $\phi_n \mathbf{1}_{\cdot \leq \tau} Q_M^{1/2} \rightarrow \phi \mathbf{1}_{\cdot \leq \tau} Q_M^{1/2}$ in $L^2(\mathbb{R}_+, [[M]]; \mathcal{L}(H, \mathbb{R}))$ a.s. So, using (3.13) for ϕ_n , (3.12) and Remark 3.1 one obtains (3.13) for general ϕ .

Remark 3.23. It follows from Remark 3.1 that for each finite dimensional subspaces $X_0 \subseteq X$ the definition of the stochastic integral can be extended to all strongly progressively measurable processes $\Phi : \mathbb{R}_+ \times \Omega \rightarrow \mathcal{L}(H, X)$ that take values in $\mathcal{L}(H, X_0)$, and satisfy $\Phi Q_M^{1/2} \in L^2(\mathbb{R}_+, [[M]]; \mathcal{L}(H, X))$ a.s. (or equivalently $\Phi Q_M^{1/2}$ is scalarly in $L^2(\mathbb{R}_+, [[M]]; H)$ a.s.). In order to deduce this result from the one-dimensional case one can approximate Φ by a process which is supported on a finite dimensional subspace of H and use Remark 3.1 together with (3.12) and the fact that X_0 is isomorphic to \mathbb{R}^d for some $d \geq 1$ since it is finite dimensional. The space of stochastic integrable Φ will be characterized in Theorem 4.1.

Proposition 3.24. Let H be a Hilbert space. Let $N \in \mathcal{M}_{\text{var}}^{\text{loc}}(H)$. Let $\Phi : \mathbb{R}_+ \times \Omega \rightarrow \mathcal{L}(H, X)$ be such that for each $x^* \in X^*$, $\Phi^* x^*$ is progressively measurable and assume that for all $x^* \in X^*$, $\omega \in \Omega_0$, $\langle \Phi(\omega) Q_N(\omega) \Phi^*(\omega) x^*, x^* \rangle \in L^1_{\text{loc}}(\mathbb{R}_+, [[N]](\omega))$. Define a cylindrical continuous local martingale $M := \int \Phi \, dN$ by

$$Mx^*(t) := \int_0^t \Phi^* x^* \, dN, \quad x^* \in X^*. \tag{3.14}$$

Then $M \in \mathcal{M}_{\text{var}}^{\text{loc}}(X)$ if and only if $\|\Phi Q_N \Phi^*\| \in L^1_{\text{loc}}(\mathbb{R}_+, [[N]])$ a.s. In this case,

$$[[M]]_t = \int_0^t \|\Phi(s) Q_N \Phi^*(s)\| \, d[[N]]_s, \quad t \geq 0, \tag{3.15}$$

$$\langle A_M(t)x^*, y^* \rangle = \int_0^t \langle \Phi(s) Q_N \Phi^*(s) x^*, y^* \rangle \, d[[N]]_s, \quad t \geq 0, x^*, y^* \in X^*,$$

$$Q_M(s) = \frac{\Phi(s) Q_N(s) \Phi^*(s)}{\|\Phi(s) Q_N(s) \Phi^*(s)\|}, \quad \text{for } \mu_{[[N]]}\text{-almost all } s \in \mathbb{R}_+.$$

In this section there are two definitions of a stochastic integral (see (3.11) and (3.14)). One can check that both integrals coincide in the sense that (3.14) would be the cylindrical continuous martingale associated to the one given in (3.11).

Proof. We first show that M is a cylindrical continuous local martingale. Clearly, each Mx^* is a continuous local martingale. It remains to prove the continuity of $x^* \mapsto Mx^*$

in the ucp topology. Fix $T > 0$. Let Ω_0 be a set of full measure such that for $\omega \in \Omega_0$, $t \mapsto \langle \Phi(t, \omega)Q_N(t, \omega)^* \Phi(t, \omega)^* x^*, x^* \rangle \in L^1(0, T)$. By the closed graph theorem for each $\omega \in \Omega_0$ there is a constant $C_T(\omega)$ such that

$$\|\langle \Phi(\cdot, \omega)Q_N(\cdot, \omega)^* \Phi(\cdot, \omega)^* x^*, y^* \rangle\|_{L^1(0, T, [[N]](\omega))} \leq C_T(\omega) \|x^*\| \|y^*\|.$$

Also note that $[Mx^*]_t = \int_0^t \langle \Phi(s)Q_N \Phi^*(s)x^*, x^* \rangle d[[N]]$ for all $x^* \in X^*$. Now if $x_n^* \rightarrow x^*$ as $n \rightarrow \infty$, it follows from the above estimate and identity that $[Mx_n^*]_T \rightarrow [Mx^*]_T$ on Ω_0 , and hence by the remarks in Subsection 3.1 also $Mx_n^* \rightarrow Mx^*$ uniformly on $[0, T]$ in probability. Since $T > 0$ was arbitrary, we find that M is a cylindrical continuous local martingale.

To prove the equivalence it suffices to observe that

$$\begin{aligned} [[M]]_t &= \lim_{\text{mesh} \rightarrow 0} \sum_{j=1}^J \sup_{x^* \in X^*, \|x^*\|=1} ([Nx^*]_{t_j} - [Nx^*]_{t_{j-1}}) \\ &= \lim_{\text{mesh} \rightarrow 0} \sum_{j=1}^J \sup_{x^* \in X^*, \|x^*\|=1} \int_{t_{j-1}}^{t_j} \langle \Phi(s)Q_N(s)\Phi^*(s)x^*, x^* \rangle d[[N]]_s \\ &= \int_0^t \|\Phi(s)Q_N(s)\Phi^*(s)\| d[[N]]_s, \end{aligned} \tag{3.16}$$

where the last equality holds true thanks to Lemma 2.8, Remark 2.10 and the separability of X^* . At the same time this proves the required formula for $[[M]]_t$. In order to find A_M it suffices to note that for all $x^*, y^* \in X^*$:

$$\begin{aligned} \langle A_M(t)x^*, y^* \rangle_H &= [Mx^*, My^*]_t = \int_0^t \langle Q_N(s)\Phi^*(s)x^*, \Phi^*(s)y^* \rangle d[[N]]_s \\ &= \int_0^t \langle \Phi(s)Q_N(s)\Phi^*(s)x^*, y^* \rangle d[[N]]_s. \end{aligned}$$

Since $d[[M]]_s = \|\Phi(s)Q_N(s)\Phi^*(s)\| d[[N]]_s$ the required identity for Q_M follows from Proposition 3.13. □

Next we present an example of a situation where \widetilde{M} is a continuous martingale which associated cylindrical continuous local martingale M is not in $M_{\text{var}}^{\text{loc}}(X)$.

Example 3.25. Let $X = \ell^p$ with $p \in (2, \infty)$ and let W be a one-dimensional Brownian motion. It follows from [70, Example 3.4] that there exists a continuous martingale $\widetilde{M} : \mathbb{R}_+ \times \Omega \rightarrow X$ such that

$$\langle \widetilde{M}_t, x^* \rangle = \int_0^t \langle \phi(s), x^* \rangle dW(s),$$

where $\phi : \mathbb{R}_+ \rightarrow X$ is such that $\langle \phi, x^* \rangle \in L^2(\mathbb{R}_+)$ for all $x^* \in X^*$, but on the other hand $\|\phi\|_{L^2(0,1;X)} = \infty$. Therefore, by Proposition 3.24 the associated cylindrical martingale satisfies $[[M]]_1 = \infty$ a.s., and hence $M \notin \mathcal{M}_{\text{var}}^{\text{loc}}(X)$.

The same construction can be done for any Banach space X which does not have cotype 2 (see [70, Proposition 6.2] and [51, Theorem 11.6]).

In the next example we construct a cylindrical continuous martingale in a Hilbert space which is not in $\mathcal{M}_{\text{var}}^{\text{loc}}(H)$.

Example 3.26. Let H be a separable Hilbert space with an orthonormal basis $(h_n)_{n \geq 1}$ and W be an one-dimensional Brownian motion. Let $[0, 1] = \cup_{n=1}^\infty A_n$ be a partition of

$[0, 1]$ into pairwise disjoint sets. Let $\psi : \mathbb{R}_+ \times \Omega \rightarrow H$ be a deterministic function such that $\psi(t) = \sum_{n=1}^{\infty} |A_n|^{-1/2} \mathbf{1}_{A_n}(t) h_n$. For each $h \in H$ one has that

$$\int_{\mathbb{R}_+} \langle \psi(s), h \rangle^2 ds = \int_{\mathbb{R}_+} \sum_{n=1}^{\infty} |A_n|^{-1} \mathbf{1}_{A_n}(s) \langle h_n, h \rangle^2 ds = \sum_{n=1}^{\infty} \langle h_n, h \rangle^2 = \|h\|^2, \quad (3.17)$$

therefore $\langle \psi, h \rangle$ is stochastically integrable with respect to W and one can define $M : H \rightarrow \mathcal{M}^{\text{loc}}$ by $Mh = \langle \psi, h \rangle \cdot W$. Obviously M is linear. Moreover, Mh is an L^2 -martingale for each $h \in H$ and thanks to (3.17) and the Itô isometry, $\|(Mh)_\infty\|_{L^2(\Omega)} = \left\| \int_0^1 \langle \psi, h \rangle dW \right\|_{L^2(\Omega)} = \|h\|$. So $Mh \rightarrow 0$ as $h \rightarrow 0$ in the ucp topology by Remark 3.1, hence M is a cylindrical continuous local martingale. On the other hand due to (3.15) one concludes that

$$[[M]]_1 = \int_0^1 \|\psi(s)\|^2 ds = \int_0^1 \sum_{n=1}^{\infty} |A_n|^{-1} \mathbf{1}_{A_n}(s) \|h_n\|^2 ds = \sum_{n=1}^{\infty} \|h_n\|^2 = \infty.$$

Consequently, $M \notin \mathcal{M}_{\text{var}}^{\text{loc}}(H)$.

3.5 Quadratic Doléans measure

Recall from Definition 3.10 that μ_M is the cylindrical Doléans measure associated with M . Since it only depends on $[[M]]$ sometimes the information get lost. In the next definition we define a bilinear-valued measure associated to M (see [49, Section 15.3]).

Definition 3.27. Let M be a cylindrical continuous martingale such that $M(t)x^* \in L^2(\Omega)$ for all $t \geq 0$. Define the quadratic Doléans measure $\bar{\mu}_M : \mathcal{P} \rightarrow \mathbf{B}(X^*, X^*)$ by

$$\langle \bar{\mu}_M(F \times (s, t]), x^* \otimes y^* \rangle = \mathbb{E}[\mathbf{1}_F([Mx^*, My^*]_t - [Mx^*, My^*]_s)]$$

for every predictable rectangle $F \times (s, t]$ and for every $x^*, y^* \in X^*$.

A disadvantage of the quadratic Doléans measure is that it can only be considered if $\langle M, x^* \rangle_t \in L^2(\Omega)$. Such a problem does not occur for $\mu_{[[M]]}$, A_M and Q_M as in Proposition 3.13.

Note that $\bar{\mu}_M$ defines a vector measure with variation (see [15, 76]) given by

$$|\bar{\mu}_M|(A) = \sup \sum_{n=1}^N \|\bar{\mu}_M(A_n)\|, \quad (3.18)$$

where the supremum is taken over all the partitions $A = \bigcup_{n=1}^N A_n$. If $|\bar{\mu}_M|([0, \infty) \times \Omega) < \infty$, then it is a standard fact that the variation $|\bar{\mu}_M|$ defines a measure again and $|\bar{\mu}_M| \ll \mu_M$ (see [15]). Under the assumption that $\bar{\mu}_M$ has bounded variation a stochastic integration theory was developed in [49, Chapter 16]. The next result connects the measure μ_M from Definition 3.10 the operator Q_M from Proposition 3.13 and the above vector measure $\bar{\mu}_M$. It provides a bridge between the theory in [49, Chapter 16] and our setting.

Proposition 3.28. Assume M is a cylindrical continuous martingale such that $\langle M, x^* \rangle_t \in L^2(\Omega)$ for all $t \geq 0$. Then the following assertions are equivalent

1. $M \in \mathcal{M}_{\text{var}}^{\text{loc}}(X)$ and $\mu_M([0, \infty) \times \Omega) < \infty$
2. $\bar{\mu}_M$ has bounded variation.

In that case $d\bar{\mu}_M = Q_M d\mu_M$ in a weak sense, namely

$$\langle \bar{\mu}_M(A), x^* \otimes y^* \rangle = \int_A \langle Q_M x^*, y^* \rangle d\mu_M, \quad x^*, y^* \in X^*, A \in \mathcal{P}. \quad (3.19)$$

Moreover, $|\bar{\mu}_M| = \mu_M$.

The identity (3.19) coincides with [49, (16.1.1)]. To prove the above result we will need a technical lemma. Let $f : \mathbb{R}_+ \times \Omega \rightarrow [0, \infty]$ be an a.s. continuous increasing predictable process. With slight abuse of terminology we say that the Doléans measure of f exists if $C \mapsto \mathbb{E} \int_0^\infty \mathbf{1}_C df$ defines a finite measure on \mathcal{P} .

Lemma 3.29. *Let $(f^n)_{n \geq 1}$ be a sequence of continuous predictable increasing processes on \mathbb{R}_+ . Suppose that for all $n \geq 1$ the corresponding Doléans measure μ_n of f^n exists. Assume also that $\mu = \sup_{n \geq 1} \mu_n$ is of bounded variation. Then $F : \mathbb{R}_+ \times \Omega \rightarrow \mathbb{R}_+$ defined by*

$$F(t) = \lim_{\text{mesh} \rightarrow 0} \sum_{k=1}^K \sup_n (f^n(t_k) - f^n(t_{k-1})), \tag{3.20}$$

where the limit is taken over all partitions $0 = t_0 < \dots < t_K = t$, is a predictable continuous increasing process and its Doléans measure exists and equals μ .

Proof. For each $N \geq 1$ define $F^N : \mathbb{R}_+ \times \Omega \rightarrow \mathbb{R}_+$ as

$$F^N(t) = \lim_{\text{mesh} \rightarrow 0} \sum_{k=1}^K \sup_{1 \leq n \leq N} (f^n(t_k) - f^n(t_{k-1})), \quad t \geq 0,$$

where the limit is taken over all partitions $0 = t_0 < \dots < t_K = t$. Then F^N is a predictable process by Remark 2.10. Moreover, it is continuous since the corresponding Lebesgue-Stieltjes measure is nonatomic by (2.3). Let us consider the corresponding Doléans measure ν_N of F^N . We claim that

$$\nu_N = \sup_{1 \leq n \leq N} \mu_n. \tag{3.21}$$

Since $\nu_N \geq \mu_n$ for each given $n \leq N$, we have $\nu_N \geq \sup_{1 \leq n \leq N} \mu_n$. Also notice that $\nu_N \leq \sum_{1 \leq n \leq N} \mu_n$.

It remains to show " \leq " in (3.21). First of all by Remark 2.10 a.s. $\mu_{F^N}(\omega) = \sup_{1 \leq n \leq N} \mu_{f^n}(\omega)$. By Lemma 2.8 a.s. the maximum of the Radon-Nikodym derivatives satisfies $\max_{1 \leq n \leq N} \frac{d\mu_{f^n}}{d\mu_{F^N}}(t) = 1$ for μ_{F^N} -a.a. $t \in \mathbb{R}_+$. So by Lemma 3.14 a.s. for μ_{F^N} -a.a. $t > 0$

$$1 = \max_{1 \leq n \leq N} \frac{d\mu_{f^n}}{d\mu_{F^N}}(t) = \max_{1 \leq n \leq N} \lim_{\varepsilon \rightarrow 0} \frac{f^n(t) - f^n(t - \varepsilon \wedge t)}{F^N(t) - F^N(t - \varepsilon \wedge t)}. \tag{3.22}$$

Notice, that for each $n \leq N$ the processes $t \mapsto f^n(t) - f^n(t - \varepsilon \wedge t)$ and $t \mapsto F^N(t) - F^N(t - \varepsilon \wedge t)$ are predictable and continuous. Therefore, the sets

$$A_n := \{(t, \omega) \in \mathbb{R}_+ \times \Omega : \lim_{\varepsilon \rightarrow 0} \frac{f^n(t) - f^n(t - \varepsilon \wedge t)}{F^N(t) - F^N(t - \varepsilon \wedge t)} = 1\}, \quad 1 \leq n \leq N,$$

are in the predictable σ -algebra \mathcal{P} . Redefine these sets to make them disjoint: $A_n := A_n \setminus (\cup_{1 \leq k < n} A_k)$. Then by (3.22) for each predictable rectangle $B \in \mathcal{P}$ we have that $\nu_N(A_n \cap B) = \mu_n(A_n \cap B)$. Clearly this extends to all $B \in \mathcal{P}$. Now it follows that for all $B \in \mathcal{P}$

$$\nu_N(B) = \sum_{1 \leq n \leq N} \nu_N(B \cap A_n) = \sum_{1 \leq n \leq N} \mu_n(B \cap A_n) \leq (\sup_{1 \leq n \leq N} \mu_n)(B),$$

and hence (3.21) holds. Letting $N \rightarrow \infty$ in (3.21) by Lemma 2.9 we obtain

$$\lim_{N \rightarrow \infty} \nu_N(A) = \lim_{N \rightarrow \infty} (\sup_{1 \leq n \leq N} \mu_n)(A) = \mu(A), \quad A \in \mathcal{P}. \tag{3.23}$$

By Lemma 2.9, pointwise on $\mathbb{R}_+ \times \Omega$, $F^N \rightarrow F$, where F is as in (3.20). Notice that $\mathbb{E}F^N(t) = \nu^N(\Omega \times [0, t]) \nearrow \mu(\Omega \times [0, t]) < \infty$, and since $F^N(t) \nearrow F(t)$ we have that

$\mu(\Omega \times [0, t]) = \mathbb{E}F(t)$, so $F(t)$ finite a.s. Moreover, F is predictable as it is the pointwise limit of the predictable processes F^N . By the monotone convergence theorem and (3.23) we find that for all $0 \leq s < t$ and $A \in \mathcal{F}_s$,

$$\mathbb{E}\mathbf{1}_A(F(t) - F(s)) = \lim_{N \rightarrow \infty} \mathbb{E}\mathbf{1}_A(F^N(t) - F^N(s)) = \lim_{N \rightarrow \infty} \nu_N((s, t] \times A) = \mu((s, t] \times A),$$

which completes the proof. □

Proof of Proposition 3.28. (1) \Rightarrow (2): Assume (1). Let $x^*, y^* \in X^*$. Then for $A = (a, b] \times F$ with $b > a \geq 0$ and $F \in \mathcal{F}_a$, it follows from Proposition 3.13 that

$$\begin{aligned} \langle \bar{\mu}_M(A), x^* \otimes y^* \rangle &= \mathbb{E}\mathbf{1}_F([Mx^*, My^*]_b - [Mx^*, My^*]_a) \\ &= \int_F \int_a^b d\langle A_M(s)x^*, y^* \rangle d\mathbb{P} \\ &= \int_F \int_a^b \langle Q_M x^*, y^* \rangle d[[M]]_s d\mathbb{P} = \int_A \langle Q_M x^*, y^* \rangle d\mu_M. \end{aligned}$$

As in [49, Chapter 16.1] this extends to each $A \in \mathcal{P}$. This proves (3.19) and since $\|Q_M\| = 1$ μ_M -a.e. it follows that

$$|\bar{\mu}_M|([0, \infty) \times \Omega) \leq \int_{[0, \infty) \times \Omega} d\mu_M = \mu_M([0, \infty) \times \Omega) < \infty.$$

(2) \Rightarrow (1): Assume (2). Let $(x_n^*)_{n \geq 1}$ be such that its \mathbb{Q} -linear span E is dense in X^* and (x_1^*, \dots, x_n^*) are linear independent for any $n \geq 1$. By a standard argument one can construct a \mathbb{Q} -bilinear mapping $a_M : \Omega \times [0, \infty) \rightarrow \mathbf{B}_{\mathbb{Q}}(E, E)$ such for all $x^*, y^* \in E$ and all $t \geq 0$, a.s. $a_M(t, \omega)(x^*, y^*) = [\langle M, x^* \rangle, \langle M, y^* \rangle]_t$.

Let $(y_n^*)_{n \geq 1} \subseteq X^*$ be equal to the intersection of E and the unit ball in X^* . Then by Definition 3.27 and (3.18) $|\bar{\mu}_M| = \sup_n \mu_{My_n^*}$, where μ_{Mx^*} is the Doléans measure of Mx^* for a given $x^* \in X^*$. Now by Lemma 3.29 one derives that there exists a predictable continuous increasing process $F : \mathbb{R}_+ \times \Omega \rightarrow \mathbb{R}$ such that a.s.

$$F(t) = \lim_{\text{mesh} \rightarrow 0} \sum_{k=1}^K \sup_n (a_M(t_k)(y_n^*, y_n^*) - a_M(t_{k-1})(y_n^*, y_n^*)),$$

where the limit is taken over all partitions $0 = t_0 < \dots < t_K = t$. In particular, $a_M(t)(y_n^*, y_n^*) \leq F(t)$ a.s. and hence as in the first part of the proof of Theorem 3.9 one sees that $a_M(t)$ extends to a bounded bilinear form on $X^* \times X^*$ a.s. and thanks to Remark 3.1 and the fact that M is a cylindrical continuous local martingale one obtains that for each $x^*, y^* \in X^*$, $a_M(x^*, y^*)$ and $[Mx^*, My^*]$ are indistinguishable. Then

$$F(t) = \lim_{\text{mesh} \rightarrow 0} \sum_{k=1}^K \sup_{x^* \in X^*, \|x^*\|=1} (a_M(t_k)(x^*, x^*) - a_M(t_{k-1})(x^*, x^*)),$$

and thanks to Theorem 3.9 we conclude that the quadratic variation of M exists.

The final identity $|\bar{\mu}_M| = \mu_M$ follows from Lemma 2.8, (3.19) and the fact that

$$\sup_{\|x^*\|=\|y^*\|=1} \langle Q_M x^*, y^* \rangle = \|Q_M\| = 1.$$

which was proved in Proposition 3.13. □

3.6 Covariation operators

In this subsection we assume that both X and Y have a separable dual space. In this section we introduce a covariation operator for $M_1 \in \mathcal{M}_{\text{var}}^{\text{loc}}(X)$, $M_2 \in \mathcal{M}_{\text{var}}^{\text{loc}}(Y)$ and develop some calculus results for them.

Proposition 3.30. *Let $M_1 \in \mathcal{M}_{\text{var}}^{\text{loc}}(X)$, $M_2 \in \mathcal{M}_{\text{var}}^{\text{loc}}(Y)$ be defined on the same probability space. Then there exists a covariation operator $A_{M_1, M_2} : \mathbb{R}_+ \times \Omega \rightarrow \mathcal{L}(X^*, Y^{**})$ such that for each $x^* \in X^*$, $y^* \in Y^*$ a.s.*

$$\langle A_{M_1, M_2}(t)x^*, y^* \rangle = [M_1x^*, M_2y^*]_t, \quad t \geq 0.$$

Proof. Let $a_{M_1, M_2} : \mathbb{R}_+ \times \Omega \rightarrow \mathbf{B}(X^*, Y^*)$ be defined as a version of $(t, \omega)(x^*, y^*) \mapsto [M_1(\omega)x^*, M_2(\omega)y^*]_t$ such that a.s. for each $t \in \mathbb{R}_+$

$$\begin{aligned} |a_{M_1, M_2}(t)(x^*, y^*)| &\leq \sqrt{a_{M_1}(t)(x^*, x^*)a_{M_2}(t)(y^*, y^*)} \\ &\leq \sqrt{[[M_1]]_t [[M_2]]_t} \|x^*\| \|y^*\| \quad \forall x^* \in X^*, y^* \in Y^*, \end{aligned} \tag{3.24}$$

To construct such a version we can argue as in the first part of the proof of Theorem 3.9. □

Proposition 3.31. *The space $\mathcal{M}_{\text{var}}^{\text{loc}}(X)$ is a vector space and equipped with the (metric) topology of ucp convergence of the quadratic variation $[[\cdot]]$ it becomes a complete metric space with the translation invariant metric given by*

$$\|M\|_{\mathcal{M}_{\text{var}}^{\text{loc}}(X)} := \sum_{n=1}^{\infty} 2^{-n} \mathbb{E}[1 \wedge [[M]]_n^{1/2}] + \sup_{\|x^*\| \leq 1} \mathbb{E}[1 \wedge |(Mx^*)_0|].$$

Moreover, for $M_1, M_2 \in \mathcal{M}_{\text{var}}^{\text{loc}}(X)$ a.s. for all $t \geq 0$ the triangle inequality holds:

$$[[M_1 + M_2]]_t^{1/2} \leq [[M_1]]_t^{1/2} + [[M_2]]_t^{1/2}. \tag{3.25}$$

The above metric does not necessarily turn $\mathcal{M}_{\text{var}}^{\text{loc}}(X)$ into a topological vector space in the case X is infinite dimensional. However, if the martingales are assumed to start at zero then it becomes a topological vector space.

Proof. Now for $M_1, M_2 \in \mathcal{M}_{\text{var}}^{\text{loc}}(X)$ one can easily prove, that $M_1 + M_2 \in \mathcal{M}_{\text{var}}^{\text{loc}}(X)$. Indeed, by the definition of the quadratic (co)variation operator and linearity for all $x^*, y^* \in X^*, t \geq 0$, a.s.

$$[(M_1 + M_2)x^*, (M_1 + M_2)y^*]_t = \langle (A_{M_1}(t) + A_{M_1, M_2}(t) + A_{M_2, M_1}(t) + A_{M_2}(t))x^*, y^* \rangle,$$

and so by (3.24) and Definition 3.4 $[[M_1 + M_2]]$ exists and a.s.

$$[[M_1 + M_2]]_t \leq [[M_1]]_t + [[M_2]]_t + 2\sqrt{[[M_1]]_t [[M_2]]_t}, \quad t \geq 0,$$

which proves (3.25). Since it is clear that $\mathcal{M}_{\text{var}}^{\text{loc}}(X)$ is closed under multiplication by scalars, it follows that $\mathcal{M}_{\text{var}}^{\text{loc}}(X)$ is a vector space.

To prove the completeness let $(M^n)_{n \geq 1} \subseteq \mathcal{M}_{\text{var}}^{\text{loc}}(X)$ be a Cauchy sequence, then $(M^n x^*)_{n \geq 1}$ is a Cauchy sequence in \mathcal{M}^{loc} for all $x^* \in X^*$, and so by Remark 3.1 and completeness it converges to a continuous local martingale Mx^* in the ucp topology. Let $(x_m^*)_{m=1}^{\infty} \subset X^*$ be a dense subset of X^* . Then due to a diagonalization argument there exists a subsequence $(n_k)_{k \geq 1}$ such that $[M^{n_k} x_m^*]_t$ converges a.s. for any $t \geq 0$ and $m \geq 1$, and $[[M^{n_k}]]_t$ has an a.s. limit for all $t \geq 0$ (recall that due to (3.25), $[[\cdot]]_t^{1/2}$ obeys a triangle inequality for each $t \geq 0$). Then a.s. for all $t \geq s \geq 0, m \geq 1$,

$$[Mx_m^*]_t - [Mx_m^*]_s = \lim_{k \rightarrow \infty} ([M^{n_k} x_m^*]_t - [M^{n_k} x_m^*]_s) \leq \liminf_{k \rightarrow \infty} ([M^{n_k}]_t - [M^{n_k}]_s) \|x_m^*\|^2.$$

By Proposition 3.7 we find $M \in \mathcal{M}_{\text{var}}^{\text{loc}}(X)$ and $[[M]] \leq \lim_{n \rightarrow \infty} [[M^n]]$, where the last limit is taken in the ucp topology. Now fix $t > 0$. To prove that a.s. $\lim_{k \rightarrow \infty} [[M - M^{n_k}]]_t = 0$ one has firstly to consider a sequence $(c_k^D)_{k,D=1}^\infty$, such that for all $k, D > 0$

$$c_k^D = \left(\lim_{\text{mesh} \rightarrow 0} \sum_{l=1}^L \sup_{1 \leq d \leq D} [(M^{n_k} - M)x_d^*]_{t_l} - [(M^{n_k} - M)x_d^*]_{t_{l-1}} \right)^{\frac{1}{2}},$$

where the limit is taken over all partitions $0 = t_0 < \dots < t_L = t$. Then by Lemma 2.9 a.s. $c_k^D \rightarrow [[M^{n_k} - M]]_t^{\frac{1}{2}}$ as $D \rightarrow \infty$, and consequently $c_k := (c_k^D)_{D=1}^\infty \in \ell^\infty$ for all $k \geq 1$, where ℓ^∞ is the space of bounded sequences. Then obviously by (3.25) a.s.

$$\sup_{D \geq 1} |c_k^D - c_l^D| \leq [[M^k - M^l]]_t^{\frac{1}{2}}, \quad k, l \geq 1,$$

which yields that $(c_k)_{k=1}^\infty$ is a Cauchy sequence in ℓ^∞ . Now one can easily show that $c_k^D \rightarrow 0$ as $k \rightarrow \infty$, so $c_k \rightarrow 0$, and a.s. $\sup_D (c_k^D)^2 = [[M - M^{n_k}]]_t \rightarrow 0$. \square

As a positive definite bilinear form the covariation operator has the following properties a.s. $\forall t > s \geq 0, x^* \in X^*$:

$$A_{M_1, M_2}(t, \omega) = \frac{A_{M_1+M_2}(t, \omega) - A_{M_1-M_2}(t, \omega)}{4},$$

$$\begin{aligned} & \langle (A_{M_1, M_2}(t) - A_{M_1, M_2}(s))x^*, x^* \rangle \\ & \leq \sqrt{\langle (A_{M_1}(t) - A_{M_1}(s))x^*, x^* \rangle \langle (A_{M_2}(t) - A_{M_2}(s))x^*, x^* \rangle}. \end{aligned} \quad (3.26)$$

Remark 3.32. One can also define a *covariation process* $[[M_1, M_2]]$ by the formula

$$[[M_1, M_2]]_t := \lim_{\text{mesh} \rightarrow 0} \sum_{n=1}^N \|A_{M_1, M_2}(t_n) - A_{M_1, M_2}(t_{n-1})\|.$$

The limit exists a.s. thanks to the Cauchy-Schwartz inequality and the fact that a.s. for each $0 \leq s < t$

$$\|A_{M_1, M_2}(t) - A_{M_1, M_2}(s)\| \leq \sqrt{\|A_{M_1}(t) - A_{M_1}(s)\|} \sqrt{\|A_{M_2}(t) - A_{M_2}(s)\|},$$

where the last is an easy consequence of (3.26).

The process $[[M_1, M_2]]$ is continuous a.s. and has some properties of a covariation process of real-valued martingales. For instance, one can prove by the formula (3.26) that for all $t > s \geq 0$

$$|[[M_1, M_2]]_t - [[M_1, M_2]]_s| \leq \sqrt{([M_1]]_t - [M_1]]_s)([M_2]]_t - [M_2]]_s) \quad a.s. \quad (3.27)$$

Unfortunately, in general $[[\cdot]]_t$ is not a quadratic form (except in the one-dimensional case).

Thanks to the continuity of covariation process one can consider the Lebesgue-Stieltjes measure $\mu_{[[M_1, M_2]]}$ for a.a. ω . By the same technique as it was mentioned before one can also construct $Q_{M_1, M_2} : \mathbb{R}_+ \times \Omega \rightarrow \mathcal{L}(X^*, Y^{**})$:

$$\langle A_{M_1, M_2}(t)x^*, y^* \rangle = \int_0^t \langle Q_{M_1, M_2}(s)x^*, y^* \rangle d[[M_1, M_2]], \quad t \geq 0, \omega \in \Omega.$$

Note, that $\|Q_{M_1, M_2}(t)\| \leq 1$ a.s. and for $\mu_{[[M_1, M_2]]}$ -a.a. $t > 0$ by the same argument, as in Proposition 3.13. Also evidently $Q_{M_1, M_2} = Q_{M_2, M_1}^*$. One can derive the following result:

Proposition 3.33 (Kunita-Watanabe inequality, cylindrical case). *Let $M_1 \in \mathcal{M}_{\text{var}}^{\text{loc}}(X)$, $M_2 \in \mathcal{M}_{\text{var}}^{\text{loc}}(Y)$ defined on the same probability space $(\Omega, \mathbb{F}, \mathbb{P})$, $f: \mathbb{R}_+ \times \Omega \rightarrow X^*$, $g: \mathbb{R}_+ \times \Omega \rightarrow Y^*$ be two strongly $\mathcal{B}_{\mathbb{R}_+} \otimes \mathcal{F}$ -measurable bounded functions. Then for all $t > 0$ and for almost all $\omega \in \Omega$*

$$\begin{aligned} & \left| \int_0^t \langle Q_{M_1, M_2}(s) f(s, \omega), g(s, \omega) \rangle d[[M_1, M_2]]_s(\omega) \right|^2 \\ & \leq \int_0^t \langle Q_{M_1}(s) f(s, \omega), f(s, \omega) \rangle d[[M_1]]_s(\omega) \int_0^t \langle Q_{M_2} g(s, \omega), g(s, \omega) \rangle d[[M_2]]_s(\omega). \end{aligned}$$

The proof is analogous to the proof of [66, Theorem II.25], for which one has to apply inequalities of the form (3.26).

Recall from (3.14) that for suitable Φ and $M \in M_{\text{var}}^{\text{loc}}(H)$, $(\Phi \cdot M) \in M_{\text{var}}^{\text{loc}}(X)$ given by

$$(\Phi \cdot M)x^* := \int_0^\cdot \Phi^* x^* dM, \quad x^* \in X^*$$

is well-defined.

Theorem 3.34. *Let H be a separable Hilbert space, $M_1 \in \mathcal{M}_{\text{var}}^{\text{loc}}(H)$, $M_2 \in \mathcal{M}_{\text{var}}^{\text{loc}}(Y)$, $\Phi: \mathbb{R}_+ \times \Omega \rightarrow \mathcal{L}(H, X)$ be such that $\Phi^* x^*$ is a strongly progressively measurable process for each $x^* \in X^*$ and let $\|\Phi Q_{M_1} \Phi^*\| \in L^1_{\text{loc}}(\mathbb{R}_+, [[M_1]])$ a.s. Then for all $t \geq 0$ and for all $x^* \in X^*$, $y^* \in Y^*$ one has*

$$\langle A_{\Phi \cdot M_1, M_2}(t)x^*, y^* \rangle = \int_0^t \langle Q_{M_1, M_2} \Phi^* x^*, y^* \rangle d[[M_1, M_2]] \quad \text{a.s.}$$

Proof. Fix $t \geq 0$ and $x^* \in X^*$, $y^* \in Y^*$. Put $\phi = \Phi^* x^*$. Firstly suppose that there exists $n > 0$ such that ϕ takes its values in a finite-dimensional subspace $\text{span}(h_1, \dots, h_n) \subseteq H$. Then by bilinearity of covariation process, the definition of Q_{M_1, M_2} , and thanks to [35, Theorem 17.11]

$$\begin{aligned} \langle A_{\Phi \cdot M_1, M_2}(t)x^*, y^* \rangle &= \left[\int_0^\cdot \phi dM_1, M_2 y^* \right]_t = \sum_{i=1}^n \left[\int_0^\cdot \langle \phi, h_i \rangle d(M_1 h_i), M_2 y^* \right]_t \\ &= \sum_{i=1}^n \int_0^t \langle \phi, h_i \rangle d[M_1 h_i, M_2 y^*]_t \\ &= \sum_{i=1}^n \int_0^t \langle \phi, h_i \rangle \langle h_i, Q_{M_2, M_1} y^* \rangle d[[M_1, M_2]]_t \\ &= \int_0^t \langle \phi, Q_{M_2, M_1} y^* \rangle d[[M_1, M_2]]_t \\ &= \int_0^t \langle Q_{M_1, M_2} \phi, y^* \rangle d[[M_1, M_2]]_t. \end{aligned}$$

In the general case one can approximate ϕ by $P_n \phi$, where $P_n \in \mathcal{L}(H)$ is an orthogonal projection on $\text{span}(h_1, \dots, h_n)$, and derive the desired by using (3.12) and inequalities of the type (3.26)-(3.27). □

One can prove the full analogues of [35, Lemma 17.10] and [35, Theorem 17.11] using the same methods as in the proof above:

Theorem 3.35 (Covariation of integrals). *Let H be a separable Hilbert space, $M_1, M_2 \in \mathcal{M}_{\text{var}}^{\text{loc}}(H)$, $\Phi_1: \mathbb{R}_+ \times \Omega \rightarrow \mathcal{L}(H, X)$, $\Phi_2: \mathbb{R}_+ \times \Omega \rightarrow \mathcal{L}(H, Y)$ be such that $\Phi_1^* x^*$, $\Phi_2^* y^*$ are strongly progressively measurable processes for each $x^* \in X^*$, $y^* \in Y^*$ and assume that*

for $j \in \{1, 2\}$, $\|\Phi_j Q_{M_j} \Phi_j^*\| \in L^1_{\text{loc}}(\mathbb{R}_+, [[M_j]])$ a.s. Then for all $t \geq 0$ and for all $x^* \in X^*$, $y^* \in Y^*$ one has

$$\langle A_{\Phi_1 \cdot M_1, \Phi_2 \cdot M_2}(t)x^*, y^* \rangle = \int_0^t \langle Q_{M_1, M_2} \Phi_1^* x^*, \Phi_2^* y^* \rangle d[[M_1, M_2]] \quad \text{a.s.}$$

Remark 3.36. To construct the analogy one has to see due to the equation above that in a weak sense

$$A_{\Phi_1 \cdot M_1, \Phi_2 \cdot M_2}(t) = \int_0^t \Phi_2 Q_{M_1, M_2} \Phi_2^* d[[M_1, M_2]]_s = \int_0^t \Phi_2 dA_{M_1, M_2}(s) \Phi_2^*,$$

which extends the scalar case.

4 Stochastic integration with respect to cylindrical continuous local martingales

Let H be a separable Hilbert space and let X be a separable Banach space with a separable dual space. In the previous section we have introduced stochastic integrals as cylindrical continuous local martingales. Often one wants the stochastic integral to be an actual local martingale instead of a cylindrical one. In this section we will characterize when this is the case we prove two-sided estimates for the stochastic integral

$$(\Phi \cdot M)_t = \int_0^t \Phi(s) dM(s),$$

where Φ is an $\mathcal{L}(H, X)$ -valued H -strongly progressively measurable processes. Here $M \in \mathcal{M}_{\text{var}}^{\text{loc}}(H)$ (see Definition 3.4).

For this characterization we need the language of γ -radonifying operators and the geometric condition UMD on the Banach space X . Both will be introduced in the next two subsection.

4.1 γ -radonifying operators

We refer to [33], [51] and [36] and references therein for further details. Let $(\gamma'_n)_{n \geq 1}$ be a sequence of independent standard Gaussian random variables on a probability space $(\Omega', \mathcal{F}', \mathbb{P}')$ (we reserve the notation $(\Omega, \mathcal{F}, \mathbb{P})$ for the probability space on which our processes live) and let H be a separable real Hilbert space. A bounded operator $R \in \mathcal{L}(H, X)$ is said to be γ -radonifying if for some (or equivalently for each) orthonormal basis $(h_n)_{n \geq 1}$ of H the Gaussian series $\sum_{n \geq 1} \gamma'_n R h_n$ converges in $L^2(\Omega'; X)$. We then define

$$\|R\|_{\gamma(H, X)} := \left(\mathbb{E}' \left\| \sum_{n \geq 1} \gamma'_n R h_n \right\|^2 \right)^{\frac{1}{2}}.$$

This number does not depend on the sequence $(\gamma'_n)_{n \geq 1}$ and the basis $(h_n)_{n \geq 1}$, and defines a norm on the space $\gamma(H, X)$ of all γ -radonifying operators from H into X . Endowed with this norm, $\gamma(H, X)$ is a Banach space, which is separable if X is separable. For later reference we note that the convergence of $\sum_{n \geq 1} \gamma'_n R h_n$ in $L^p(\Omega'; X)$ with $p \in (0, \infty)$, in probability and a.s. can all be shown to be equivalent.

If $R \in \gamma(H, X)$, then $\|R\| \leq \|R\|_{\gamma(H, X)}$. If X is a Hilbert space, then $\gamma(H, X) = \mathcal{L}^2(H, X)$ isometrically. Let G be another Hilbert space, X be another Banach space. Then by the so-called *ideal property* (see [33]) the following holds true: for all $S \in \mathcal{L}(G, H)$ and all $T \in \mathcal{L}(X, Y)$ we have $TRS \in \gamma(G, Y)$ and

$$\|TRS\|_{\gamma(G, Y)} \leq \|T\| \|R\|_{\gamma(H, X)} \|S\|. \tag{4.1}$$

Let μ be a measure on a Borel set $J \subseteq \mathbb{R}_+$ with a σ -field \mathcal{A} such that $L^2(J, \mu)$ is separable and $p \in [1, \infty)$. We say that a function $\Phi : J \rightarrow \mathcal{L}(H, X)$ belongs to $L^p(J, \mu; H)$ scalarly if for all $x^* \in X^*$, $\Phi^* x^* \in L^p(J, \mu; H)$. A function $\Phi : J \rightarrow \mathcal{L}(H, X)$ is said to represent an operator $R \in \gamma(L^2(J, \mu; H), X)$ if Φ belongs to $L^2(J, \mu; H)$ scalarly and for all $x^* \in X^*$ and $f \in L^2(J, \mu; H)$ we have

$$\langle Rf, x^* \rangle = \int_J f(s) \Phi(s)^* x^* \, d\mu(s).$$

The above notion will be abbreviated by $\Phi \in \gamma(J, \mu; H, X)$. In the case X is a Hilbert space, one has $\gamma(J, \mu; H, X) = L^2(J, \mu; \mathcal{L}_2(H, X))$ isometrically, where $\mathcal{L}_2(H, X)$ denotes the Hilbert-Schmidt operators from H to X .

If μ is the Lebesgue measure we will also write $\gamma(L^2(J; H), X)$ and $\gamma(J; H, X)$ for $\gamma(L^2(J, \mu; H), X)$ and $\gamma(J, \mu; H, X)$ respectively.

Let $\nu : \mathcal{A} \times \Omega \rightarrow [0, \infty]$ be a random measure. Typically, ν will be the Lebesgue-Stieltjes measure associated $[[M]]$ for $M \in M_{\text{var}}^{\text{loc}}(H)$. In this case we will also identify $[[M]]$ and ν . We say that $\Phi : J \times \Omega \rightarrow \mathcal{L}(H, X)$ is scalarly in $L^2(J, \nu; H)$ a.s. if for all $x^* \in X^*$, for almost all $\omega \in \Omega$, $\Phi(\cdot, \omega)^* x^* \in L^2(J, \nu(\cdot, \omega); H)$.

For such a process Φ and a family $R = (R(\omega) : \omega \in \Omega)$ with $R(\omega) \in \gamma(L^2(J, \nu(\cdot, \omega); X))$ for almost all $\omega \in \Omega$, we say that Φ represents R if for all $x^* \in X^*$, for almost all $\omega \in \Omega$, $\Phi(\cdot, \omega)^* x^* = R^*(\omega) x^*$ in $L^2(J, \nu(\cdot, \omega); H)$. As before this will be abbreviated by $\Phi \in \gamma(J, \nu; H, X)$ a.s.

In the case that ν is the Lebesgue measure the above notion of representability reduces to the one given in [53].

4.2 The UMD property

The results will be stated for the important class of UMD Banach spaces and we refer to [11], [32], [71] for details. A Banach space X is called a UMD space if for some (or equivalently, for all) $p \in (1, \infty)$ there exists a constant $\beta > 0$ such that for every $n \geq 1$, every martingale difference sequence $(d_j)_{j=1}^n$ in $L^p(\Omega; X)$, and every $\{-1, 1\}$ -valued sequence $(\varepsilon_j)_{j=1}^n$ we have

$$\left(\mathbb{E} \left\| \sum_{j=1}^n \varepsilon_j d_j \right\|^p \right)^{\frac{1}{p}} \leq \beta \left(\mathbb{E} \left\| \sum_{j=1}^n d_j \right\|^p \right)^{\frac{1}{p}}.$$

The infimum over all admissible constants β is denoted by $\beta_{p,X}$.

UMD spaces are always reflexive. Examples of UMD space include, the reflexive range of L^q -spaces, Besov spaces, Sobolev spaces. Example of spaces without the UMD property include all nonreflexive spaces, e.g. $L^1(0, 1)$ and $C([0, 1])$.

4.3 Characterization of stochastic integrability

The next result is the main result of this section.

Theorem 4.1. *Let X be a UMD space, $M \in \mathcal{M}_{\text{var}}^{\text{loc}}(H)$. For a strongly progressively measurable process $\Phi : \mathbb{R}_+ \times \Omega \rightarrow \mathcal{L}(H, X)$ such that $\Phi Q_M^{1/2}$ is scalarly in $L^2(\mathbb{R}_+, [[M]]; H)$ a.s. the following assertions are equivalent:*

(1) *There exists elementary progressive processes $(\Phi_n)_{n \geq 1}$ such that:*

- (i) *for all $x^* \in X^*$, $\lim_{n \rightarrow \infty} Q_M^{1/2} \Phi_n^* x^* = Q_M^{1/2} \Phi^* x^*$ in $L^0(\Omega; L^2(\mathbb{R}_+, [[M]]; H))$;*
- (ii) *there exists a process $\zeta \in L^0(\Omega; C_b(\mathbb{R}_+; X))$ such that*

$$\zeta = \lim_{n \rightarrow \infty} \int_0^\cdot \Phi_n(t) \, dM(t) \quad \text{in } L^0(\Omega; C_b(\mathbb{R}_+; X)).$$

(2) There exists an a.s. bounded process $\zeta : \mathbb{R}_+ \times \Omega \rightarrow X$ such that for all $x^* \in X^*$ we have

$$\langle \zeta, x^* \rangle = \int_0^\cdot \Phi^*(t)x^* \, dM(t) \quad \text{in } L^0(\Omega; C_b(\mathbb{R}_+)).$$

(3) $\Phi Q_M^{1/2} \in \gamma(L^2(\mathbb{R}_+, [[M]]); H), X)$ almost surely;

In this case ζ in (1) and (2) coincide and for all $p \in (0, \infty)$ we have

$$\mathbb{E} \sup_{t \in \mathbb{R}_+} \|\zeta(t)\|^p \sim_{p, X} \mathbb{E} \|\Phi Q_M^{1/2}\|_{\gamma(L^2(\mathbb{R}_+, [[M]]); H), X}^p.$$

A process $\Phi : \mathbb{R}_+ \times \Omega \rightarrow \mathcal{L}(H, X)$ which satisfies the above conditions and the assertions (1)–(3) will be called *stochastically integrable with respect to M* .

Remark 4.2. The case of scalar-valued continuous local martingales of Theorem 4.1 was considered in [77], where the Dambis, Dubins-Schwarz result is applied to write the continuous local martingale as a time changed Brownian motion. Unfortunately, in the vector-valued setting, this technique breaks down as one cannot do a different time change in infinitely many direction. The proof of Theorem 4.1 will be given in Subsection 4.6 after we have introduced some techniques we will use.

4.4 Time transformations

A nondecreasing, right-continuous family of stopping times $\tau_s : \Omega \rightarrow [0, \infty]$, $s \geq 0$, will be called a *random time-change*. If additionally $\tau_s : \Omega \rightarrow [0, \infty)$ then $\tau_s, s \geq 0$, will be called a *finite random time-change*. If \mathbb{F} is right-continuous, then according to [35, Lemma 7.3] the same holds true for the *induced filtration* $\mathbb{G} = (\mathcal{G}_s)_{s \geq 0} = (\mathcal{F}_{\tau_s})_{s \geq 0}$ (see [35, Chapter 7]). An $M \in \mathcal{M}_{\text{var}}^{\text{loc}}(X)$ is said to be τ -continuous if for each $x^* \in X^*$, Mx^* is an a.s. constant on every interval $[\tau_{s-}, \tau_s]$, $s \geq 0$, where we let $\tau_{0-} = 0$. Notice that if M is τ -continuous, then $[[M]]$ is τ -continuous as well by [35, Exercise 17.3] and by using Proposition 3.7. A vector-valued process F is τ -continuous if F is an a.s. constant on every interval $[\tau_{s-}, \tau_s]$, $s \geq 0$.

Proposition 4.3 (Kazamaki). *Let τ be a finite random time-change and let $M \in \mathcal{M}_{\text{var}}^{\text{loc}}(H)$ with respect to \mathbb{F} . Let X_0 be a finite dimensional Banach space. Assume also that M is τ -continuous. Let $\Phi : \mathbb{R}_+ \times \Omega \rightarrow \mathcal{L}(H, X_0)$ be \mathbb{F} -progressively measurable and assume*

$$\int_0^\infty \|\Phi Q_M^{1/2}\|_{\mathcal{L}(H, X_0)}^2 \, d[[M]] < \infty \text{ a.s.}$$

Define the process $\Psi : \mathbb{R}_+ \times \Omega \rightarrow \mathcal{L}(H, X_0)$ by $\Psi(s) = \Phi(\tau_s)$. Then following assertions hold:

1. $N = M \circ \tau : H \rightarrow \mathcal{M}^{\text{loc}}$ given by

$$Nh := (Mh) \circ \tau, \quad h \in H.$$

is in $\mathcal{M}_{\text{var}}^{\text{loc}}(H)$ with respect to \mathbb{G} ;

2. $[[N]] = [[M \circ \tau]] = [[M]] \circ \tau$ a.s.;
3. $Q_N = Q_M \circ \tau$;
4. Ψ is \mathbb{G} -progressively measurable and

$$\int_0^\infty \|\Psi Q_N^{1/2}\|_{\mathcal{L}(H, X_0)}^2 \, d[[N]] < \infty \text{ a.s.}, \tag{4.2}$$

$$(\Phi \circ \tau) \cdot (M \circ \tau) = (\Phi \cdot M) \circ \tau \text{ a.s.} \tag{4.3}$$

Note that the stochastic integrals are well-defined by Remark 3.23.

Proof. (1): By [35, Theorem 17.24] for each $h \in H$ the process $Nh = (Mh) \circ \tau$ is a continuous \mathbb{G} -local martingale and $[Mh \circ \tau] = [Mh] \circ \tau$. Thus by Proposition 3.7 $M \circ \tau : H \rightarrow \mathcal{M}^{\text{loc}}$ given by

$$(M \circ \tau)h := (Mh) \circ \tau, \quad h \in H.$$

is in $\mathcal{M}_{\text{var}}^{\text{loc}}(H)$, since for any $h \in H$ one has that $\mu_{[Mh] \circ \tau} \leq \mu_{[[M]] \circ \tau} \|h\|^2$ a.s. (notice that thanks to τ -continuity both $[Mh] \circ \tau$ and $[[M]] \circ \tau$ are a.s. continuous).

(2): Let $(x_m^*)_{m \geq 1}$ be a dense subset of the unit ball in X^* . Since M is τ -continuous, one has that a.s. $[[M]]$ and $[Mx_m^*]$ are τ -continuous for each $m \geq 1$. Now by Proposition 3.7 we find that a.s.

$$\mu_{[[N]]} = \sup_{m \geq 1} \mu_{[Nx_m^*]} = \sup_{m \geq 1} \mu_{[Mx_m^*] \circ \tau} = \mu_{[[M]] \circ \tau},$$

and therefore, $[[M]] \circ \tau$ is a version of $[[N]]$.

(3): This follows from a substitution argument:

$$\langle Q_N h_1, h_2 \rangle = \frac{d(\langle A_M \circ \tau h_1, h_2 \rangle)}{d([[M]] \circ \tau)} = \frac{d\langle A_M h_1, h_2 \rangle}{d[[M]]} \circ \tau = \langle Q_M h_1, h_2 \rangle \circ \tau, \quad h_1, h_2 \in H.$$

(4): The \mathbb{G} -progressive measurability of Ψ can be proven in the same way as in the proof of [39, Proposition 2]. Assertion (4.2) can be obtained by (2), (3) and the general version of the substitution rule (4.5).

The existence of the left hand side of (4.3) can be proved via (4.2) and Remark 3.23. The equation (4.3) is obvious for elementary progressively measurable Φ and follows by an approximation argument as in Remark 3.23. \square

We now prove a version of Proposition 4.3 for a special class of random time changes which are not necessarily finite.

Corollary 4.4. *Let $M \in \mathcal{M}_{\text{var}}^{\text{loc}}(H)$. Suppose that $(\tau_s)_{s \geq 0}$ has the following form:*

$$\tau_s = \begin{cases} \inf\{t \geq 0 : [[M]]_t > s\}, & \text{if } 0 \leq s < S; \\ \infty, & \text{otherwise,} \end{cases}$$

where $S = \sup_{t \geq 0} [[M]]_t$. Then for each $h \in H$, $M_\infty h = \lim_{t \rightarrow \infty} M_t h$ a.s. exists if $S < \infty$ and Proposition 4.3 holds true for $N := M \circ \tau$ defined as follows

$$N_s = \begin{cases} M_{\tau_s}, & \text{if } 0 \leq s < S; \\ M_\infty, & \text{otherwise,} \end{cases}$$

Moreover, if $\Psi : \mathbb{R}_+ \times \Omega \rightarrow \mathcal{L}(H, X_0)$ is stochastically integrable with respect to N , $\Phi := \Psi \circ [[M]]$, then also $\Phi \circ \tau$ is stochastically integrable with respect to N and a.s.

$$\int_0^{[[M]]_t} \Psi \, dN = \int_0^{[[M]]_t} \Phi \circ \tau \, dN = \int_0^t \Phi \, dM, \quad t \geq 0. \tag{4.4}$$

Recall the substitution rule: for a strongly measurable $f : \mathbb{R}_+ \rightarrow X$ we have $f \in L^1(\mathbb{R}_+, \mu_{[[M]]}; X)$ if and only if $f \circ \tau \in L^1(0, S; X)$, and in that case

$$\int_{\mathbb{R}_+} f(t) \, d[[M]] = \int_{[0, S)} f(\tau(s)) \, ds. \tag{4.5}$$

Proof of Corollary 4.4. According to [67, Proposition IV.1.26] and the fact that for each $h \in H$, $[[M]]\|h\| \geq [Mh]$ a.s., one can define $M_\infty h$ if $S < \infty$, so N is well-defined. Now we prove that $N \in \mathcal{M}_{\text{var}}^{\text{loc}}(H)$.

Define $\tau_s^n := \inf\{t \geq 0 : [[M]]_t > s\} \wedge n$ for each $n \geq 1$. Then τ_s^n is a finite random time change. Let $N^n := M \circ \tau^n$. Then by Proposition 4.3 $(N^n)_{n \geq 1} \subset \mathcal{M}_{\text{var}}^{\text{loc}}(H)$ and $[[N^n]]_t = t \wedge [[M]]_n$ for all $n \geq 0$ a.s. for all $t \geq 0$. Also notice that $[[N^n]]_t \rightarrow t \wedge [[M]]_\infty$ as $n \rightarrow \infty$. Therefore N^n is a Cauchy sequence in the ucp topology, and thanks to Proposition 3.31 there exists a limit $\tilde{N} \in \mathcal{M}_{\text{var}}^{\text{loc}}(H)$. Obviously $\tilde{N}_s = N_s$ a.s. for all $s < S$. If $s \geq S$, then $\tilde{N}_s h = \lim_{t \rightarrow \infty} M_t h = M_\infty h = N h$ a.s. for each $h \in H$. So, $N = \tilde{N} \in \mathcal{M}_{\text{var}}^{\text{loc}}(H)$.

By the same argument

$$[[N]]_t = \lim_{n \rightarrow \infty} [[N^n]]_t = \lim_{n \rightarrow \infty} t \wedge [[M]]_n = t \wedge [[M]]_\infty = [[M]]_{\tau_t},$$

which proves Proposition 4.3(2). To prove Proposition 4.3(3) note that since the measure $d[[N]]$ vanishes on $[S, \infty)$, one can put $Q_N(s) = 0$ if $\tau_s = \infty$, and for $\tau_s < \infty$ one has that

$$Q_N(s) = \lim_{n \rightarrow \infty} Q_{N^n}(s) = \lim_{n \rightarrow \infty} Q_M(\tau_s^n) = Q_M(\tau_s).$$

The proof of Proposition 4.3(4) is analogous to one in the main proof.

Now let us prove the last statement of the corollary.

Since a.s. $\tau \circ [[M]](s) = s$ for $\mu_{[[M]]}$ -a.a. s , we find that a.s.

$$(\Phi \circ \tau - \Psi) \circ [[M]] = \Phi \circ \tau \circ [[M]] - \Psi \circ [[M]] = 0$$

$\mu_{[[M]]}$ -a.e. Therefore according to (4.5), Proposition 4.3(2) a.s.

$$(\Phi \circ \tau - \Psi) \circ [[M]] \circ \tau = \Phi \circ \tau - \Psi = 0$$

$\mu_{[[N]]}$ -a.e., which means that a.s. $\int_0^\infty \|(\Phi \circ \tau - \Psi)Q_N^{1/2}\|^2 d[[N]] = 0$, which yields stochastic integrability of $\Phi \circ \tau$ and the first equality of (4.4) thanks to [35, Exercise 17.3]. The last equality of (4.4) is nothing more than formula (4.3). \square

The next lemma is a γ -version of this substitution result and can be proved as in [77, Lemma 3.5] where the case $H = \mathbb{R}$ was considered.

Lemma 4.5. *Let X be a Banach space, H be a separable Hilbert space. Let $F : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be increasing and continuous with $F(0) = 0$ and let μ be the Lebesgue-Stieltjes measure corresponding to F . Let $S := \lim_{t \rightarrow \infty} F(t) \leq \infty$ and define $\tau : \mathbb{R}_+ \rightarrow [0, \infty]$ as*

$$\tau(s) = \begin{cases} \inf\{t \geq 0 : F(t) > s\}, & \text{for } 0 \leq s < S; \\ \infty, & \text{for } s \geq S. \end{cases}$$

Let $\Phi : \mathbb{R}_+ \rightarrow \mathcal{L}(H, X)$ be strongly measurable and define $\Psi : \mathbb{R}_+ \rightarrow \mathcal{L}(H, X)$ by

$$\Psi(s) = \begin{cases} \Phi(\tau_s), & \text{for } 0 \leq s < S; \\ 0, & \text{for } s \geq S. \end{cases}$$

Then $\Phi \in \gamma(L^2(\mathbb{R}_+, \mu; H), X)$ if and only if $\Psi \in \gamma(L^2(\mathbb{R}_+; H), X)$. In that case

$$\|\Phi\|_{\gamma(L^2(\mathbb{R}_+, \mu; H), X)} = \|\Psi\|_{\gamma(L^2(\mathbb{R}_+; H), X)}. \tag{4.6}$$

4.5 Representation and cylindrical Brownian motion

The next theorem is an infinite time interval version of [53, Theorem 3.6], while the second part is modified thanks to [62, Theorem 5.1] and the last part modified by [53, Theorem 4.4] and [12, Theorem 5.4]. It will play an important role in the proof of Theorem 4.1. It might be instructive for the reader to check that it is exactly Theorem 4.1 in the special case that M is a cylindrical Brownian motion.

Theorem 4.6. *Let X be a UMD space. For a strongly measurable and adapted process $\Phi: \mathbb{R}_+ \times \Omega \rightarrow \mathcal{L}(H, X)$ which is scalarly in $L^2(\mathbb{R}_+; H)$ a.s. the following assertions are equivalent:*

(1) *There exists a sequence $(\Phi_n)_{n \geq 1}$ of elementary progressive processes such that:*

- (i) *for all $x^* \in X^*$ we have $\lim_{n \rightarrow \infty} \Phi_n^* x^* = \Phi^* x^*$ in $L^0(\Omega; L^2(\mathbb{R}_+, H))$,*
- (ii) *there exists a process $\zeta \in L^0(\Omega; C_b(\mathbb{R}_+; X))$ such that*

$$\zeta = \lim_{n \rightarrow \infty} \int_0^\cdot \Phi_n(t) dW_H(t) \quad \text{in } L^0(\Omega; C_b(\mathbb{R}_+; X)).$$

(2) *There exists an a.s. bounded process $\zeta: \mathbb{R}_+ \times \Omega \rightarrow X$ such that for all $x^* \in X^*$ we have*

$$\langle \zeta, x^* \rangle = \int_0^\cdot \langle \Phi(t), x^* \rangle dW_H(t) \quad \text{in } L^0(\Omega; C_b(\mathbb{R}_+)).$$

(3) $\Phi \in \gamma(L^2(\mathbb{R}_+; H), X)$ *almost surely;*

In this case ζ in (1) and (2) coincide and is in $\mathcal{M}_{\text{var}}^{\text{loc}}(X)$. Furthermore, for all $p \in (0, \infty)$ we have

$$\mathbb{E} \sup_{t \in \mathbb{R}_+} \|\zeta(t)\|^p \approx_{p, X} \mathbb{E} \|\Phi\|_{\gamma(L^2(\mathbb{R}_+; H), X)}^p. \tag{4.7}$$

For the proof of Theorem 4.1 we will also need the following result which is a simple consequence of [60, Theorem 2].

Proposition 4.7. *Let X be a reflexive separable Banach space and let $M \in \mathcal{M}_{\text{var}}^{\text{loc}}(X)$. If $[[M]]$ is absolutely continuous with respect to the Lebesgue measure, then there exists a separable Hilbert space H , an H -cylindrical Brownian motion W_H on an enlarged probability space $(\bar{\Omega}, \bar{\mathbb{F}}, \bar{\mathbb{P}})$, a progressively measurable process $z: \mathbb{R}_+ \times \Omega \rightarrow \mathbb{R}_+$ and a scalarly progressively measurable process $Q_M^{1/2}: \mathbb{R}_+ \times \Omega \rightarrow \mathcal{L}(X^*, H)$ which satisfies $Q_M^{1/2*} Q_M^{1/2} = Q_M$ a.s. and $z^{1/2} Q_M^{1/2} \in L^0(\Omega; L_{\text{loc}}^2(\mathbb{R}_+; \mathcal{L}(X^*, H)))$ such that a.s.*

$$M_t x^* = \int_0^t z^{1/2}(s) (Q_M^{1/2}(s) x^*)^* dW_H(s), \quad t \in \mathbb{R}_+, x^* \in X^*.$$

Moreover, if X is a Hilbert space, then for each progressively strongly measurable $\Phi: \mathbb{R}_+ \times \Omega \rightarrow X^$ such that $\int_0^\infty \langle Q_M \Phi, \Phi \rangle d[[M]] < \infty$ a.s. one has*

$$\int_0^t \Phi(s) dM(s) = \int_0^t z^{1/2}(s) Q_M^{1/2}(s) \Phi(s) dW_H(s), \quad t \in \mathbb{R}_+. \tag{4.8}$$

Remark 4.8. The integral in the left hand side of (4.8) exists for the special M with absolutely continuous quadratic variation thanks to the isometry given in [60, Remark 30] and the construction given in [60, p.1022].

Proof. Since $[[M]]$ is absolutely continuous with respect to the Lebesgue measure, one can find $z: \mathbb{R}_+ \times \Omega \rightarrow \mathbb{R}_+$ such that $[[M]]_t = \int_0^t z(s)ds$ for each $t \in \mathbb{R}_+$ a.s. Define H and $Q_M^{1/2}: \mathbb{R}_+ \times \Omega \rightarrow \mathcal{L}(X^*, H)$ as in Lemma 2.4. By Remark 2.5 the process $Q_M^{1/2}$ is scalarly progressively measurable. Then for all $x^*, y^* \in X^*$

$$\begin{aligned} [Mx^*, My^*]_t &= \int_0^t d[Mx^*, My^*]_s = \int_0^t \langle Q_M x^*, y^* \rangle d[[M]]_s \\ &= \int_0^t \langle z(s)Q_M(s)x^*, y^* \rangle ds \\ &= \int_0^t \langle (z(s)^{1/2}Q_M(s)^{1/2})x^*, (z(s)^{1/2}Q_M(s)^{1/2})y^* \rangle ds, \end{aligned}$$

and the rest follows from [60, Theorem 2].

The last equation is evident for elementary functions, and the general case follows from a density argument, Remark 4.8 and the isometry, mentioned in [60, Remark 30]. \square

4.6 Proof of the main characterization Theorem 4.1

To prove the result we will reduce to Theorem 4.6 by using the time transformation from Corollary 4.4 and the representation of Proposition 4.7.

Proof of Theorem 4.1. Define $\tau: \Omega \times \mathbb{R}_+ \rightarrow [0, \infty]$ as follows:

$$\tau_s = \begin{cases} \inf\{t \geq 0 : [[M]]_t > s\}, & \text{for } 0 \leq s < [[M]]_\infty; \\ \infty, & \text{for } s \geq [[M]]_\infty. \end{cases} \tag{4.9}$$

Put

$$\Psi(s) = \begin{cases} \Phi(\tau_s), & \text{for } 0 \leq s < [[M]]_\infty; \\ 0, & \text{for } s \geq [[M]]_\infty. \end{cases} \tag{4.10}$$

For each $s \geq 0$ it holds true that $[[M]]_{\tau_s} - [[M]]_{\tau_{s-}} = 0$ a.s. So, since for fixed $h \in H$, $\mu_{[[M]]} \geq \mu_{[Mh]}$, then also $[Mh]_{\tau_s} - [Mh]_{\tau_{s-}} = 0$ a.s. Therefore thanks to the fact that τ_{s-} is a stopping time, so $(Mh)^{\tau_s} - (Mh)^{\tau_{s-}}$ is a continuous local martingale with zero quadratic variation (see [66, Theorem I.18]), and by Remark 3.1 and [37, Problem 1.5.12] one concludes that Mh is τ -continuous.

It also follows that $([[M]] \circ \tau)_s = s$ for $s < [[M]]_\infty$. Let \mathbb{G} be as in Corollary 4.4. By Corollary 4.4 one can define a local H -cylindrical continuous \mathbb{G} -martingale $N: \mathbb{R}_+ \times H \rightarrow L^0(\Omega)$ such that $N = M \circ \tau$, $[[N]]_s = s$ for $s < [[M]]_\infty$, and $Q_N = Q_M \circ \tau$.

Let W_H and $(\bar{\Omega}, \bar{\mathbb{F}}, \bar{\mathbb{P}})$ be as in Proposition 4.7. We will prove the result by showing that (1), (2) and (3) for Φ are equivalent with (1), (2) and (3) in Theorem 4.6 for $\Psi Q_N^{1/2}$. (Notation $(k, \Phi) \Leftrightarrow (k, \Psi)$ for $k = 1, 2, 3$).

(1, Φ) \Rightarrow (1, Ψ): Assume (1) holds for a sequence of elementary progressive processes $(\Phi_n)_{n \geq 1}$. For all $n \geq 1$ define $\Psi_n: \mathbb{R}_+ \times \Omega \rightarrow \mathcal{L}(H, X)$ as

$$\Psi_n(s) = \begin{cases} \Phi_n(\tau_s), & \text{for } 0 \leq s < [[M]]_\infty, \\ 0, & \text{for } s \geq [[M]]_\infty. \end{cases}$$

Then it follows from the Pettis measurability theorem and Corollary 4.4 that each Ψ_n is strongly progressively measurable with respect to the time transformed filtration, and the same holds true for each $\Psi_n Q_N^{1/2}$, because Φ_n takes their values in finite dimensional subspace of X . So since Φ_n is elementary progressive it follows from (4.3), Corollary 4.4, Proposition 4.7, Remark 3.23 that for all $n \geq 1$ for all $s \in \mathbb{R}_+$ we have a.s.

$$\zeta_{\Psi_n Q_N^{1/2}}(s) = \int_0^s \Psi_n(r) Q_N^{1/2}(r) dW_H(r) = \int_0^s \Psi_n(r) dN(r) = \int_0^{\tau_s} \Phi_n(r) dM(r)$$

(recall that $z(s) = [[N]]'_s = 1$ for $s < [[M]]_\infty$).

Therefore, it follows that $(\zeta_{\Psi_n Q_N^{1/2}})_{n \geq 1}$ is a Cauchy sequence in $L^0(\Omega; C_b(\mathbb{R}_+; X))$, and hence it converges to some $\zeta_\chi \in L^0(\Omega; C_b(\mathbb{R}_+; X))$. By (4.5), Theorem 4.1 (1) (i), by the special choice of $\bar{\Omega}$ and by Fubini's theorem it follows that for every $x^* \in X^*$ we have $\lim_{n \rightarrow \infty} Q_N^{1/2} \Psi_n^* x^* = Q_N^{1/2} \Psi^* x^*$ in $L^0(\bar{\Omega}; L^2(\mathbb{R}_+; H))$. Since $\Psi_n Q_N^{1/2} h$ take values in finite dimensional subspace of X for each $h \in H$, one can approximate $(\Psi_n Q_N^{1/2})_{n \geq 1}$ to obtain a sequence of elementary progressive processes $(\hat{\chi}_n)_{n \geq 1}$ that satisfies Theorem 4.6 (1) (i) and (ii).

(1, Ψ) \Rightarrow (1, Φ): Let Theorem 4.6 (1) be satisfied for $\Psi Q_N^{1/2}$ on the enlarged probability space $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}})$. Then it follows from Theorem 4.6 that $\Psi Q_N^{1/2} \in \gamma(L^2(\mathbb{R}_+; H), X)$ $\bar{\mathbb{P}}$ -a.s. By special choice of $\bar{\Omega}$ and by Fubini's theorem we may conclude that $\Psi Q_N^{1/2} \in \gamma(L^2(\mathbb{R}_+; H), X)$ \mathbb{P} -a.s. By [53, Remark 2.8] $\Psi Q_N^{1/2} \in L^0(\Omega; \gamma(L^2(\mathbb{R}_+; H), X))$. Then by [77, Lemma 3.2], [53, Proposition 2.10] and [53, Proposition 2.12] there exist elementary progressive processes $(\chi_n)_{n \geq 1}$ in $L^0(\Omega; \gamma(L^2(\mathbb{R}_+; H), X))$ such that $\Psi Q_N^{1/2} = \lim_{n \rightarrow \infty} \chi_n$ in $L^0(\Omega; \gamma(L^2(\mathbb{R}_+; H), X))$.

Let n be fixed. Without loss of generality one can suppose that χ_n has the following form:

$$\chi_n = \sum_{i=1}^I \sum_{j=1}^J \mathbf{1}_{(t_{i-1}, t_i] \times B_{ij}} \sum_{k=1}^K h_k \otimes x_{ijk}.$$

Fix $\omega \in \Omega$. Let $P_0 : \mathbb{R}_+ \times \Omega \rightarrow \mathcal{L}(H)$ be the projection onto $\overline{\text{ran } Q_N^{1/2}(t, \omega)}$. It is easy to check that P_0 is scalarly progressively measurable and $\|P_0\| \leq 1$. By the ideal property (4.1) one has \mathbb{P} -a.e.

$$\begin{aligned} \|\Psi Q_N^{1/2} - \chi_n P_0\|_{\gamma(L^2(\mathbb{R}_+; H), X)} &= \|\Psi Q_N^{1/2} P_0 - \chi_n P_0\|_{\gamma(L^2(\mathbb{R}_+; H), X)} \\ &\leq \|\Psi Q_N^{1/2} - \chi_n\|_{\gamma(L^2(\mathbb{R}_+; H), X)}, \end{aligned}$$

thanks to $P_0 Q_N^{1/2} = Q_N^{1/2} P_0 = Q_N^{1/2}$.

Now for each $k \geq 1$ define $P_k \in \mathcal{L}(H)$ in the same way as P_0 , but by taking projections onto $Q_N^{1/2}(\text{span}(h_1, \dots, h_k))$. Note that P_k is a scalarly measurable operator. By [53, Proposition 2.4], pointwise on Ω we have $\|\chi_n P_k - \chi_n P_0\|_{\gamma(L^2(\mathbb{R}_+; H), X)} \rightarrow 0$ as $k \rightarrow \infty$.

Fix $k \geq 1$. By Lemma A.1 (applied with $F = Q_N^{1/2}$) we can find H -strongly progressive $\tilde{P}_k, L_k : \mathbb{R}_+ \times \Omega \rightarrow \mathcal{L}(H)$ such that

$$\tilde{P}_k Q_N^{1/2} = Q_N^{1/2} P_k \quad \text{and} \quad L_k Q_N^{1/2} = P_k. \tag{4.11}$$

For each $n, k \geq 1$ one let $\Psi_{nk} = \chi_n L_k \in L^0(\Omega; \gamma(L^2(\mathbb{R}_+; H), X))$. Then by (4.11) $\Psi_{nk} Q_N^{1/2} = \chi_n P_k$. Since $\Psi_{nk} Q_N^{1/2} \rightarrow \chi_n P_0$ as $k \rightarrow \infty$, we can choose a subsequence $(k_n)_{n \geq 0}$ and define $\Psi_n := \Psi_{nk_n}$ such that $\Psi Q_N^{1/2} = \lim_{n \rightarrow \infty} \Psi_n Q_N^{1/2}$ in $L^0(\Omega; \gamma(L^2(\mathbb{R}_+; H), X))$.

Without loss of generality assume that $\Psi_n Q_N^{1/2}(s) = 0$ for $s \geq [[M]]_\infty$. For each $n \geq 1$ define $\Phi_n : \mathbb{R}_+ \times \Omega \rightarrow \mathcal{L}(H, X)$ as $\Phi_n = \Psi_n \circ [[M]]$. It is easy to see that $\Phi_n Q_M^{1/2} = (\Psi_n Q_N^{1/2}) \circ [[M]]$ for each $n > 0$. Then $\Phi_n Q_M^{1/2}$ is a sequence of strongly progressively measurable processes, and $(\Phi_n Q_M^{1/2}) \circ \tau = \Psi_n Q_N^{1/2}$.

By the substitution rule (4.5) for all $x^* \in X^*$ one has

$$\left\| Q_M^{1/2} \Phi_n^* x^* - Q_M^{1/2} \Phi_n^* x^* \right\|_{L^2(\mathbb{R}_+, [[M]]; H)} = \left\| Q_N^{1/2} \Psi_n^* x^* - Q_N^{1/2} \Psi_n^* x^* \right\|_{L^2(\mathbb{R}_+; H)},$$

and we derive (1) (i) because the last expression converges to 0 in probability. By the Itô homeomorphism [53, Theorem 5.5] and the fact that $\Psi_n Q_N^{1/2} \rightarrow \Psi Q_N^{1/2}$ in

$L^0(\Omega; \gamma(L^2(\mathbb{R}_+; H), X))$ one obtains

$$\int_0^\cdot \Psi(t) Q_N^{1/2}(t) dW_H(t) = \lim_{n \rightarrow \infty} \int_0^\cdot \Psi_n(t) Q_N^{1/2}(t) dW_H(t) \quad \text{in } L^0(\Omega; C_b(\mathbb{R}_+; X)).$$

Since $\Psi_n Q_N^{1/2}$ are progressively strongly measurable processes one concludes from Proposition 4.7 and the fact that $[[N]]_s = s$ for $s \leq [[M]]_\infty$ (and so $z(s) = [[N]]'_s = 1$) that, almost surely for all $t \in \mathbb{R}_+$ and for all $n \geq 1$

$$\int_0^{[[M]]_t} \Psi_n(s) Q_N^{1/2}(s) dW_H(s) = \int_0^{[[M]]_t} \Psi_n(s) dN(s) = \int_0^t \Phi_n(s) dM(s). \quad (4.12)$$

Here the second identity follows from Corollary 4.4.

It follows that $(\int_0^t \Phi_n(s) dM(s))_{n \geq 1}$ is a Cauchy sequence in $L^0(\Omega; C_b(\mathbb{R}_+; X))$. Now as in the proof of the previous step one may conclude (1) (ii) via an approximation argument.

(2, Φ) \Rightarrow (2, Ψ): Let $\zeta : \mathbb{R}_+ \times \Omega \rightarrow X$ be the given stochastic integral process. Let $\zeta_\Psi : \mathbb{R}_+ \times \Omega \rightarrow X$ be defined as

$$\zeta_\Psi(s) = \begin{cases} \zeta(\tau_s), & \text{for } 0 \leq s < [[M]]_\infty, \\ \text{weak} - \lim_{t \rightarrow \infty} \zeta(t), & \text{for } s \geq [[M]]_\infty. \end{cases}$$

The weak limit exists a.e. and it is strongly measurable by [77, Lemma 3.8].

Moreover, by Corollary 4.4 and Proposition 4.7

$$\langle \zeta_\Psi, x^* \rangle = \int_0^\cdot \Psi(t)^* x^* dN(t) = \int_0^\cdot \langle Q_N^{1/2} \Psi(t)^*(t) x^* dW_H(t) \quad \text{in } L^0(\Omega; C_b(\mathbb{R}_+)).$$

On the other hand since ζ is a.s. bounded, the same holds for ζ_Ψ . Therefore, Theorem 4.6 (2) holds for $\Psi Q_N^{1/2}$ and ζ_Ψ .

(2, Ψ) \Rightarrow (2, Φ): Let ζ_Ψ be the stochastic integral process of $\Psi Q_N^{1/2}$ with respect to W_H . Let $\zeta : \mathbb{R}_+ \times \Omega \rightarrow X$ be defined as $\zeta = \zeta_\Psi \circ [[M]]$. Then $\zeta \in L^0(\Omega; C_b(\mathbb{R}_+; X))$ and it follows from Proposition 4.7 that for all $x^* \in X^*$, for all $t \in \mathbb{R}_+$ a.s. we have

$$\begin{aligned} \langle \zeta(t), x^* \rangle &= \langle \zeta_\Psi([[M]]_t), x^* \rangle = (\langle \zeta_\Psi, x^* \rangle)([[M]]_t) = \int_0^{[[M]]_t} Q_N^{1/2} \Psi^* x^* dW_H(r) \\ &= \int_0^{[[M]]_t} \Psi^* x^* dN(r) = \int_0^t \Phi^* x^* dM(r). \end{aligned}$$

Here the last identity follows from Corollary 4.4.

(3, Φ) \Leftrightarrow (3, Ψ): This statement is obvious by Lemma 4.5. Furthermore, from (4.6) it follows that \mathbb{P} -a.s. we have

$$\|\Phi Q_M^{1/2}\|_{\gamma(L^2(\mathbb{R}_+, [[M]]); H), X} = \|\Psi Q_N^{1/2}\|_{\gamma(L^2(\mathbb{R}_+; H), X)}. \quad (4.13)$$

Therefore $\|\Phi Q_M^{1/2}\|_{\gamma(L^2(\mathbb{R}_+, [[M]]); H), X}$ is a measurable function on Ω . Since $\zeta(t) = \zeta_\Psi([[M]]_t)$ and by using Proposition 4.7, (4.7) and (4.13) one derives for $p \in (0, \infty)$

$$\begin{aligned} \mathbb{E} \sup_{t \in \mathbb{R}_+} \|\zeta(t)\|^p &= \mathbb{E} \sup_{t \in \mathbb{R}_+} \left\| \int_0^t \Phi dM \right\|^p = \mathbb{E} \sup_{t \in \mathbb{R}_+} \left\| \int_0^t \Psi dN \right\|^p \\ &= \mathbb{E} \sup_{t \in \mathbb{R}_+} \left\| \int_0^t \Psi Q_N^{1/2} dW_H \right\|^p \\ &\approx_{p, X} \mathbb{E} \|\Psi Q_N^{1/2}\|_{\gamma(L^2(\mathbb{R}_+; H), X)}^p = \mathbb{E} \|\Phi Q_M^{1/2}\|_{\gamma(L^2(\mathbb{R}_+, [[M]]); H), X}^p, \end{aligned}$$

which proves the last part of Theorem 4.1. □

By the above proof and a limiting argument in $L_0(\Omega; C_b(\mathbb{R}_+; X))$ one obtains the following theorem, which can be seen as a vector-valued generalization of the famous Dambis-Dubins-Schwarz theorem (see [35, Theorem 18.4] for the isotropic case in finite dimensions).

Theorem 4.9. *Let H be a Hilbert space, X be a UMD Banach space, $M \in \mathcal{M}_{\text{var}}^{\text{loc}}(H)$, $(\tau_s)_{s \geq 0}$ be the time change defined as in (4.9). Then we have that there exists an H -cylindrical Brownian motion W_H that does not depend on X such that for any $\Phi : \mathbb{R}_+ \times \Omega \rightarrow \mathcal{L}(H, X)$ which is stochastically integrable with respect to M , one has a.s.*

$$\int_0^t \Phi(s) dM = \int_0^{[[M]]_t} (\Phi(s)Q_M(s)) \circ \tau dW_H, \quad t \geq 0.$$

4.7 Further consequences

During the proof of Theorem 4.1 we have obtained the following corollary, which is absolutely analogous to [77, Corollary 3.9]:

Corollary 4.10 (Kazamaki, infinite dimensional case). *Assume the conditions of Theorem 4.1 hold and formula (4.9). If $\Phi : \mathbb{R}_+ \times \Omega \rightarrow \mathcal{L}(H, X)$ is scalarly \mathbb{F} -measurable and satisfies $\Phi Q_M^{1/2} \in \gamma(L^2(\mathbb{R}_+, [[M]]; H), X)$ a.s., then the process $\Psi : \mathbb{R}_+ \times \Omega \rightarrow \mathcal{L}(H, X)$ defined as in (4.10) is \mathbb{G} -adapted and satisfies $\Psi Q_N^{1/2} \in \gamma(L^2(\mathbb{R}_+; H), X)$ a.s., and the X -valued version of (4.3) holds.*

Using this corollary one can prove the following analogue of [77, Corollary 3.10]:

Corollary 4.11. *Let X be a UMD space. For each $n \geq 1$ let $\Phi_n : \mathbb{R}_+ \times \Omega \rightarrow \mathcal{L}(H, X)$ be stochastically integrable and let $\zeta_n \in L^0(\Omega, C_b(\mathbb{R}_+, X))$ denote its stochastic integral. Then we have $\Phi_n Q_M^{1/2} \rightarrow 0$ in $L^0(\Omega; \gamma(L^2(\mathbb{R}_+, [[M]]; H), X))$ if and only if $\zeta_n \rightarrow 0$ in $L^0(\Omega; C_b(\mathbb{R}_+; X))$.*

Corollary 4.12 (Local property). *Let X be a UMD space, $\Phi : \mathbb{R}_+ \times \Omega \rightarrow \mathcal{L}(H, X)$ be stochastically integrable. Suppose that there exists $A \in \mathcal{F}$ such that for all $x^* \in X^*$ a.s. for all $t \geq 0$, $\Phi^*(t)x^* = 0$. Then a.s. in A for all $t \geq 0$*

$$\int_0^t \Phi dM = 0.$$

Proof. By Hahn-Banach and strong measurability, it is enough to show that for each $x^* \in X^*$ a.s. in A for all $t \geq 0$

$$N_t := \int_0^t \Phi^* x^* dM = 0.$$

But we know that by Remark 3.22 a.s. on A

$$[N]_\infty = \int_0^\infty (\Phi^* x^*) Q_M (\Phi^* x^*)^* d[[M]] = 0,$$

what yields the desired by [35, Exercise 17.3]. □

Remark 4.13. Due to [53, Proposition 3.2] the implication (1) \Rightarrow (2) can be proven for any Banach space X , because in the proofs of (1, Φ) \Rightarrow (1, Ψ) \Rightarrow (2, Ψ) \Rightarrow (2, Φ) one does not need the UMD property. The same holds true for (3, Φ) \Leftrightarrow (3, Ψ) because there is no restriction on X in Lemma 4.5.

The next corollary is a generalization of both [77, Corollary 4.1] and [52, Proposition 6.1]. Let \mathcal{P} denote the progressive measurable σ -algebra in the result below.

Corollary 4.14. *Let X be a UMD Banach function space over a σ -finite measure space (S, Σ, μ) and let $p \in (0, \infty)$. Let $\Phi : \mathbb{R}_+ \times \Omega \rightarrow \mathcal{L}(H, X)$ be scalarly progressive and assume that there exists a $\mathcal{P} \times \Sigma$ -measurable process $\phi : \mathbb{R}_+ \times \Omega \times S \rightarrow H$ such that for all $h \in H$ and $t \geq 0$*

$$(\Phi(t)h)(\cdot) = \langle \phi(t, \cdot), h \rangle,$$

where the equality holds in X . Then Φ is stochastically integrable with respect to M if and only if almost surely

$$\left\| \left(\int_{\mathbb{R}_+} \|Q_M^{1/2}(t)\phi(t, \cdot)\|_H d[[M]]_t \right)^{\frac{1}{2}} \right\|_X < \infty.$$

In this case

$$\mathbb{E} \sup_{t \geq 0} \left\| \int_{\mathbb{R}_+} \Phi(t) dM(t) \right\|_X^p \approx_{p, X} \mathbb{E} \left\| \left(\int_{\mathbb{R}_+} \|Q_M^{1/2}(t)\phi(t, \cdot)\|_H d[[M]]_t \right)^{\frac{1}{2}} \right\|_X^p.$$

Proof. To prove this statement note that as in [52, Proposition 6.1]

$$\left\| \left(\int_{\mathbb{R}_+} \|Q_M^{1/2}(t)\phi(t, \cdot)\|_H d[[M]]_t \right)^{\frac{1}{2}} \right\|_X \approx \|\Phi Q_M^{1/2}\|_{\gamma(L^2(\mathbb{R}_+, [[M]]); H, X)}$$

and hence the results follows from Theorem 4.1. □

Due to the canonical embedding $L^2(\mathbb{R}_+, \mu; \gamma(H, X)) \hookrightarrow \gamma(L^2(\mathbb{R}_+, \mu; H), X)$ for a measure μ for type 2 spaces, and the reversed embedding for cotype 2 spaces, stated in [51, Theorem 11.6], one obtains the full analogue of [77, Corollary 4.2]:

Corollary 4.15. *Let X be a UMD space, $p \in (0, \infty)$ and $M \in \mathcal{M}_{\text{var}}^{\text{loc}}(H)$.*

(1) *If X has type 2, then every scalarly progressively measurable process $\Phi : \mathbb{R}_+ \times \Omega \rightarrow \mathcal{L}(H, X)$ such that $\Phi Q_M^{1/2} \in L^2(\mathbb{R}_+, [[M]]; \gamma(H, X))$ almost surely is stochastically integrable with respect to M and we have*

$$\mathbb{E} \sup_{t \geq 0} \left\| \int_{\mathbb{R}_+} \Phi(t) dM(t) \right\|_X^p \lesssim_{p, X} \mathbb{E} \|\Phi Q_M^{1/2}\|_{L^2(\mathbb{R}_+, [[M]]; \gamma(H, X))}^p.$$

(2) *If X has cotype 2, then every scalarly progressively measurable process Φ which is integrable with respect to M satisfies $\Phi Q_M^{1/2} \in L^2(\mathbb{R}_+, [[M]]; \gamma(H, X))$ almost surely and we have*

$$\mathbb{E} \|\Phi Q_M^{1/2}\|_{L^2(\mathbb{R}_+, [[M]]; \gamma(H, X))}^p \lesssim_{p, X} \mathbb{E} \sup_{t \geq 0} \left\| \int_{\mathbb{R}_+} \Phi(t) dM(t) \right\|_X^p.$$

(3) *If X is a Hilbert space, then Φ is integrable with respect to M if and only if $\Phi Q_M^{1/2} \in L^2(\mathbb{R}_+, [[M]]; \mathcal{L}_2(H, X))$ almost surely, and we have*

$$\mathbb{E} \sup_{t \geq 0} \left\| \int_{\mathbb{R}_+} \Phi(t) dM(t) \right\|_X^p \approx_p \mathbb{E} \|\Phi Q_M^{1/2}\|_{L^2(\mathbb{R}_+, [[M]]; \mathcal{L}_2(H, X))}^p.$$

4.8 Itô's formula

We will say that $\Phi \in \gamma_{\text{loc}}(L^2(\mathbb{R}_+, [[M]]; H), X)$ a.s. if for every $T > 0$, $\Phi \mathbf{1}_{[0, T]} \in \gamma(L^2(\mathbb{R}_+, [[M]]; H), X)$ a.s. It is an easy consequence of Theorem 4.1 that Φ is locally stochastically integrable if and only if $\Phi Q_M^{1/2} \in \gamma_{\text{loc}}(L^2(\mathbb{R}_+, [[M]]; H), X)$.

A function $f : \mathbb{R}_+ \times X \rightarrow Y$ is said to be of class $C^{1,2}$ if it is differentiable in the first variable and twice Fréchet differentiable in the second variable and the functions f, D_1f, D_2f and D_2^2f are continuous on $\mathbb{R}_+ \times X$

For $R \in \gamma(H, X)$ and $T \in \mathcal{L}(X, X^*) = \mathbf{B}(X, X)$,

$$\text{Tr}_R(T) = \sum_{n \geq 1} T(Rh_n, Rh_n),$$

where $(h_n)_{n \geq 1}$ is any orthonormal basis for H (see [10, Lemma 2.3] for details). The following version of Itô's formula holds

Theorem 4.16. *Let H be a Hilbert space, X and Y be UMD Banach spaces, $M \in \mathcal{M}_{\text{var}}^{\text{loc}}(H)$ and let $A : \mathbb{R}_+ \times \Omega \rightarrow \mathbb{R}$ be adapted, a.s. continuous and locally of finite variation. Assume that $f : \mathbb{R}_+ \times X \rightarrow Y$ is of class $C^{1,2}$. Let $\Phi : \mathbb{R}_+ \times \Omega \rightarrow \mathcal{L}(H, X)$ be an H -strongly progressively measurable which is stochastically integrable with respect to M and assume that $\Phi Q_M^{1/2}$ belongs to $L_{\text{loc}}^2(\mathbb{R}_+, [[M]]; \gamma(H, X))$. Let $\psi : \mathbb{R}_+ \times \Omega \rightarrow X$ be strongly progressively measurable with paths in $L_{\text{loc}}^1(\mathbb{R}_+, A; X)$ a.s. Let $\xi : \Omega \rightarrow X$ be strongly \mathcal{F}_0 -measurable. Define $\zeta : \mathbb{R}_+ \times \Omega \rightarrow X$ as*

$$\zeta = \xi + \int_0^\cdot \psi(s) dA(s) + \int_0^\cdot \Phi(s) dM(s).$$

Then $s \mapsto D_2f(s, \zeta(s))\Phi(s)$ is locally stochastically integrable with respect to M and almost sure we have for all $t \geq 0$

$$\begin{aligned} f(t, \zeta(t)) - f(0, \zeta(0)) &= \int_0^t D_1f(s, \zeta(s)) ds + \int_0^t D_2f(s, \zeta(s))\psi dA(s) \\ &+ \int_0^t D_2f(s, \zeta(s))\Phi(s) dM(s) \\ &+ \frac{1}{2} \int_0^t \text{Tr}_{\Phi(s)Q_M^{1/2}(s)}(D_2^2f(s, \zeta(s))) d[[M]]_s. \end{aligned} \tag{4.14}$$

A typical application of this formula are the case where $f : X \rightarrow \mathbb{R}$ is given by $f(x) = \|x\|^p$ whenever this two time Fréchet differentiable and satisfies appropriate estimates (e.g. $X = L^p$ with $p \geq 2$). Another application is $f : X \times X^* \rightarrow \mathbb{R}$ given by $f(x, x^*) = \langle x, x^* \rangle$.

To prove this result we can reduce to the case $Y = \mathbb{R}$ in a similar way as in [10, Theorem 2.4] step 1. Indeed, if the formula holds true for $F = \mathbb{R}$, then we can apply the result to $\langle f, y^* \rangle$ for each $y^* \in Y$. After that we can apply Theorem 4.1 (2) to derive the stochastic integrability of $s \mapsto D_2f(s, \zeta(s))\Phi(s)$. The identity (4.14) then follows from the Hahn-Banach theorem.

The next step is to reduce the proof to the case where ξ is simple and both ψ and Φ have finite dimensional range (see [10, Theorem 2.4] step 2). As soon as we have this reduction, then there exists a fixed finite dimensional subspace $H_0 \subset H$ such that $H = H_0 \oplus \ker \Phi$. Then one can restrict M onto this subspace, and thanks to Example 3.18 one can use the usual finite-dimensional Itô formula to derive the required result (see e.g. [49, Section 3.3]).

Lemma 4.17. *Let X be a UMD Banach space, H be a Hilbert space, $M \in \mathcal{M}_{\text{var}}^{\text{loc}}(H)$. Let $\Phi : \mathbb{R}_+ \times \Omega \rightarrow \mathcal{L}(H, X)$ be stochastically integrable with respect to M . Assume that its paths are in $L^2(\mathbb{R}_+, [[M]]; \gamma(H, X))$ almost surely. Then there exists a sequence of progressive processes $(\Phi_n)_{n \geq 1}$ such that each Φ_n takes values in a finite dimensional subspace of X and is supported on a finite dimensional subspace of H and*

$$\Phi_n Q_M^{1/2} \rightarrow \Phi Q_M^{1/2} \text{ in } L^2(\mathbb{R}_+, [[M]]; \gamma(H, X)) \cap \gamma(L^2(\mathbb{R}_+, [[M]]; H), X) \text{ in probability.}$$

Proof. Let $(\tilde{\Phi}_n)_{n \geq 1}$ be constructed as in (4.11). Then $(\tilde{\Phi}_n Q_M^{1/2})_{n \geq 1}$ is an approximation of $\Phi Q_M^{1/2}$ in $L^2(\mathbb{R}_+, [[M]]; \gamma(H, X)) \cap \gamma(L^2(\mathbb{R}_+, [[M]]; H), X)$ in probability. By [53, Proposition 2.4] $(\tilde{\Phi}_n P_k Q_M^{1/2})_{k \geq 1}$ approximates $\tilde{\Phi}_n Q_M^{1/2}$ for each $n \geq 1$, where P_k is an orthogonal projection onto $\text{span}(h_1, \dots, h_k)$. So, choosing a subsequence $\Phi_n := \tilde{\Phi}_n P_{k_n}$ one derives the desired. \square

The next lemmas is taken from [10, Lemma 2.8]:

Lemma 4.18. *Let X be a Banach space, $A : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be an increasing continuous function, and $\psi \in L^0(\Omega; L^1(\mathbb{R}_+, A; X))$ be a progressively measurable process. Then there exists a sequence of elementary progressive processes $(\psi_n)_{n \geq 1}$ such that $\psi = \lim_{n \rightarrow \infty} \psi_n$ in $L^0(\Omega; L^1(\mathbb{R}_+, A; X))$.*

5 Stochastic evolution equations and cylindrical noise

In this section we study existence and uniqueness of solutions to the stochastic evolution equation on a UMD space X :

$$du = (Au(t) + F(t, u)) dt + G(t, u) dM, \quad t \in [0, T],$$

where $u(0) = u_0$. Here A is the generator of an analytic semigroup on X , F and G are nonlinearities and M is a cylindrical continuous local martingale on a Hilbert space H which admits a quadratic variations as introduced in Definition 3.4. We will treat the above problem by semigroup methods. The case $M = W_H$ has been extensively studied in the literature (see [8, 13, 54]). Before we start we need some preliminaries from analysis.

5.1 Analytic preliminaries

Let X and Y be two Banach spaces, $(r_n)_{n \geq 1}$ be a Rademacher sequence, i.e. a sequence of independent random variables satisfying $\mathbb{P}(r_n = 1) = \mathbb{P}(r_n = -1) = \frac{1}{2}$. A family $\mathcal{T} \subseteq \mathcal{L}(X, Y)$ is called R -bounded if there exists a constant C such that for each $N > 0$, $(x_n)_{n=1}^N \subseteq X$ and $(T_n)_{n=1}^N \subseteq \mathcal{T}$ one has

$$\left(\mathbb{E} \left\| \sum_{n=1}^N r_n T_n x_n \right\|^2 \right)^{\frac{1}{2}} \leq C \left(\mathbb{E} \left\| \sum_{n=1}^N r_n x_n \right\|^2 \right)^{\frac{1}{2}}.$$

The least such C is called R -bound of \mathcal{T} , notation $R(\mathcal{T})$.

If one replaces the Rademacher sequence by a sequence of independent Gaussian variables in the definition above, then one obtains the notion of γ -bounded family of operators, whose γ -bound is denoted by $\gamma(\mathcal{T})$. A simple randomization argument shows that R -boundedness implies γ -boundedness, and in this case $\gamma(\mathcal{T}) \leq R(\mathcal{T})$ and the converse fails in general (see [43]).

A set (Λ, \leq) with an order \leq is called a *set with a total order* if for any $x, y \in \Lambda$ it holds true that $x \leq y$ or $y \leq x$. The next result is due to [6] (for a proof see [32]):

Lemma 5.1 (Vector-valued Stein's inequality). *Let (S, \mathcal{A}, μ) be a probability space, X be a UMD space. Let Λ be a set with a total order. Then for all $1 < p < \infty$ and every increasing set $\{\mathcal{A}_\alpha\}_{\alpha \in \Lambda}$ of sub- σ -algebras of \mathcal{A} one has that the family of conditional expectations*

$$\mathcal{E}_p = \{\mathbb{E}(\cdot | \mathcal{A}_\alpha), \alpha \in \Lambda\} \subseteq \mathcal{L}(L^p(\Omega; X))$$

is R -bounded as a set of operators with an R -bound depending only on p and X .

We will need the following technical lemma about γ -spaces:

Lemma 5.2. *Let X be a Banach space, $T > 0$. Let $\psi : (0, T) \rightarrow X$ be strongly measurable and let μ be a finite positive Borel measure on $[0, T]$. Suppose that $\langle \psi, x^* \rangle \in L^2(0, T; X)$ for each $x^* \in X^*$. Then $\int_0^T \psi dt \in \gamma(0, T, \mu; X)$ and*

$$\left\| \int_0^T \psi dt \right\|_{\gamma(0, T, \mu; X)} \leq \sup_{\|x^*\| \leq 1} \|\langle \psi, x^* \rangle\|_{L^2(0, T)} \left(\int_0^T t d\mu(t) \right)^{\frac{1}{2}}$$

The integral $\int_0^T \psi dt$ is defined as a Pettis integral (see [32]). Note that the above supremum is finite by the closed graph theorem.

Proof. Let $\Psi(t) = \int_0^t \psi(s) ds$ for $t \in [0, T]$. Let $(\gamma'_n)_{n \geq 1}$ be a sequence of standard independent Gaussian random variables on a probability space $(\Omega', \mathbb{F}', \mathbb{P}')$. Let $(\phi_n)_{n \geq 1}$ be an orthonormal basis for $L^2(0, T; \mu)$. Then for a fixed $\phi \in L^2(0, T; \mu)$ and $n \geq 1$ we can write

$$\int_0^T \Psi(t) \phi(t) d\mu(t) = \int_0^T \psi(s) \int_0^T \mathbf{1}_{(s, T)}(t) \phi(t) d\mu(t) ds = \int_0^T \psi(s) \langle \mathbf{1}_{(s, T)}, \phi \rangle_{L^2(\mu)} ds,$$

where the latter is defined as a Pettis integral. By Parseval's identity we have

$$\sum_{n \geq 1} |\langle \mathbf{1}_{(s, T)}, \phi_n \rangle_{L^2(\mu)}|^2 = \int_0^T \mathbf{1}_{(s, T)} d\mu.$$

Therefore, defining $\xi : \Omega \rightarrow L^2(0, T)$ by

$$\xi(s) = \sum_{n \geq 1} \gamma'_n \langle \mathbf{1}_{(s, T)}, \phi_n \rangle_{L^2(\mu)},$$

by the previous estimate, the orthogonality of the γ'_n and the three series theorem (see [76, p. 289]) we find

$$\mathbb{E} \|\xi\|_{L^2(0, T)}^2 = \int_0^T \int_0^T \mathbf{1}_{(s, T)} d\mu ds = \int_0^T t d\mu =: C_T.$$

and the series defining ξ converges a.s. in $L^2(0, T)$. It follows that

$$\sum_{n \geq 1} \gamma'_n \int_0^T \Psi(t) \phi_n(t) d\mu(t) = \int_0^T \psi(s) \xi(s) ds,$$

converges a.s. in X and

$$\left\| \sum_{n \geq 1} \gamma'_n \int_0^T \Psi(t) \phi_n(t) d\mu(t) \right\|_X = \left\| \int_0^T \psi(s) \xi(s) ds \right\| \leq C_\psi \|\xi\|_{L^2(0, T)},$$

where $C_\psi = \sup_{\|x^*\| \leq 1} \|\langle \psi, x^* \rangle\|_{L^2(0, T)}$. Taking $L^2(\Omega')$ norms it follows from the definition of the γ -norm (note that a.s. convergence and convergence in $L^p(\Omega'; X)$ are equivalent in this setting) that

$$\|\Psi\|_{\gamma(0, T, \mu; X)} \leq C_\psi \|\xi\|_{L^2(\Omega'; L^2(0, T))} \leq C_\psi C_T. \quad \square$$

Let X be a Banach space, (S, \mathcal{A}, μ) be a σ -finite measurable space, $1 \leq p < \infty$ and H be Hilbert space. Then one can prove that

$$L^p(S; \gamma(H, X)) \simeq \gamma(H, L^p(S; X)). \tag{5.1}$$

This relation is called γ -Fubini isomorphism (for more information see [33]).

A Banach space X has *property* (α) if for all $N \in \mathbb{N}$ and all sequences $(x_{mn})_{m,n=1}^N \subseteq X$ it holds true that

$$\mathbb{E} \left\| \sum_{m,n=1}^N r_{mn} x_{mn} \right\|^2 \approx \mathbb{E}' \mathbb{E}'' \left\| \sum_{m,n=1}^N r'_m r''_n x_{mn} \right\|^2,$$

where $(r_{mn})_{m,n \geq 1}$, $(r'_m)_{m \geq 1}$ and $(r''_n)_{n \geq 1}$ are independent Rademacher sequences. This property was introduced in a slightly different manner in [65] (see [33] for the proof of the equivalence).

Sectorial operators and H^∞ -calculus For each $\phi \in (0, \pi)$ let

$$S_\phi := \{\lambda \in \mathbb{C} \setminus \{0\} : \arg(\lambda) < \phi\}$$

be an open sector of angle ϕ in the complex plane. A closed and densely defined operator A on X is *sectorial of type* $\phi \in [0, \pi)$ (see [29]) if A is bijective with dense range, $\sigma(A) \subseteq \overline{S_\phi}$ and for all $\omega \in (\phi, \pi)$

$$\sup_{\lambda \notin \overline{S_\omega}} \|\lambda R(\lambda, A)\| < \infty.$$

For details on H^∞ -calculus for sectorial operators we refer the reader to [29, 41].

5.2 Hypotheses and problem formulation

Consider the following hypothesis.

(A0) H is a separable Hilbert space. X is a separable Banach space which has UMD and satisfies property (α) . $M \in \mathcal{M}_{\text{var}}^{\text{loc}}(H)$. The operator A has a bounded H^∞ -calculus of angle $< \pi/2$.

Consider the following stochastic evolution equation:

$$\begin{cases} du = (Au(t) + F(t, u)) dt + G(t, u) dM, \\ u(0) = u_0, \end{cases} \tag{5.2}$$

where A is the generator of an analytic C_0 -semigroup $(S(t))_{t \geq 0}$ on X (see [19, 63] for details).

We make the following assumption on F and G :

(A1) The function $F : \mathbb{R}_+ \times \Omega \times X \rightarrow X$ is Lipschitz of linear growth uniformly in $\mathbb{R}_+ \times \Omega$, i.e., there are constants L_F and C_F such that for all $t \in \mathbb{R}_+$, $\omega \in \Omega$ and $x, y \in X$

$$\begin{aligned} \|F(t, \omega, x) - F(t, \omega, y)\|_X &\leq L_F \|x - y\|_X, \\ \|F(t, \omega, x)\|_X &\leq C_F (1 + \|x\|_X). \end{aligned}$$

Moreover, for all $x \in X$, $(t, \omega) \mapsto F(t, \omega, x)$ is strongly measurable and adapted in X .

(A2) The function $G : \mathbb{R}_+ \times \Omega \times X \rightarrow \mathcal{L}(H, X)$ is Lipschitz of linear growth in a γ -sense uniformly in Ω and T , i.e., there are constants L_G^γ and C_G^γ s.t. for all $b \geq a \geq 0$ and for all $\phi_1, \phi_2 : \mathbb{R}_+ \rightarrow X$ which are in $L^2(\mathbb{R}_+; X) \cap \gamma(\mathbb{R}_+, [[M]]; X)$, a.s.

$$\begin{aligned} &\|(G(\cdot, \phi_1) - G(\cdot, \phi_2)) Q_M^{1/2}\|_{\gamma(L^2(a,b,[[M]]; H), X)} \\ &\leq L_G^\gamma (\|\phi_1 - \phi_2\|_{L^2(a,b; X)} + \|\phi_1 - \phi_2\|_{\gamma(a,b,[[M]]; X)}), \\ &\|G(\cdot, \omega, \phi_1) Q_M^{1/2}\|_{\gamma(L^2(a,b,[[M]]; H), X)} \\ &\leq C_G^\gamma (1 + \|\phi_1\|_{L^2(a,b; X)} + \|\phi_1\|_{\gamma(a,b,[[M]]; X)}). \end{aligned}$$

Moreover, for all $x \in X$, $(t, \omega) \mapsto G(t, \omega, x)$ is H -strongly progressively measurable.

(A3) The initial value $u_0 : \Omega \rightarrow X$ is strongly \mathcal{F}_0 -measurable.

In the case $M = W_H$, the above Lipschitz assumptions reduce to the assumptions in [54]. A key difference with [54] is that the nonlinearities can be defined on interpolation space between X and $D(A)$, but this cannot be done for general martingales except under additional assumptions on $[[M]]$.

5.3 Existence and uniqueness result

For deterministic and stochastic convolutions we will use the following notations (see [54, 78]):

$$S * F(t) := \int_0^t S(t-s)F(s) \, ds,$$

$$S \diamond G(t) := \int_0^t S(t-s)G(s) \, dM_s.$$

We call a process $(u(t))_{t \in \mathbb{R}_+}$ a *mild solution* of (5.2) if

- (i) $u : \mathbb{R}_+ \times \Omega \rightarrow X$ is strongly measurable and adapted,
- (ii) for all $t \in \mathbb{R}_+$, $s \mapsto S(t-s)F(s, u(s))$ is in $L^1(0, t; X)$ a.s.,
- (iii) for all $t \in \mathbb{R}_+$, $s \mapsto S(t-s)G(s, u(s))$ is H -strongly progressively measurable and $GQ_M^{1/2}$ is in $\gamma(L^2(0, t, [[M]]); H), X$ a.s.,
- (iv) for all $t \in \mathbb{R}_+$, almost surely

$$u(t) = S(t)u_0 + S * F(\cdot, u)(t) + S \diamond G(\cdot, u)(t). \tag{5.3}$$

Definition 5.3. Fix $b \geq a \geq 0$ and $p \in (1, \infty)$.

1. We define $V^p(a, b, M; X)$ as the space of all strongly progressively measurable processes $\phi : \mathbb{R}_+ \times \Omega \rightarrow X$ for which

$$\|\phi\|_{V^p(a,b,M;X)} := (\mathbb{E}\|\phi\|_{L^2((a,b);X)}^p)^{\frac{1}{p}} + (\mathbb{E}\|\phi\|_{\gamma(a,b,[[M]];X)}^p)^{\frac{1}{p}} < \infty$$

2. We define $V(a, b, M; X)$ as the space of all progressively measurable processes $\phi : \mathbb{R}_+ \times \Omega \rightarrow X$ for which almost surely

$$\|\phi\|_{L^2((a,b);X)} + \|\phi\|_{\gamma(L^2(a,b,[[M]]),X)} < \infty.$$

Remark 5.4. Due to the ideal property (4.1) one can show, that if τ is a stopping time and $\phi \in V^p(a, b, M; X)$, then $\phi \in V^p(a, b, M^\tau; X)$ as well and $\|\phi\|_{V^p(a,b,M^\tau;X)} \leq \|\phi\|_{V^p(a,b,M;X)}$.

The following result is the main existence and uniqueness result:

Theorem 5.5 (Existence and uniqueness). *Suppose that (A0)–(A3) are satisfied. Then there exists a unique solution U in $V(0, T, M; X)$ of (5.2).*

Moreover, if the unbounded operator A is omitted, then property (α) is not needed in the above result.

Proposition 5.6. *Let H be a Hilbert space, $M \in \mathcal{M}_{\text{var}}^{\text{loc}}(H)$, X be a UMD space. Consider the equation:*

$$\begin{cases} du = F(t, u) \, dt + G(t, u) \, dM, \\ u(0) = u_0, \end{cases} \tag{5.4}$$

Suppose that (A1)–(A3) are satisfied. Then there exists a unique solution U in $V(0, T, M; X)$ of (5.2).

Unlike in the Brownian case one cannot ensure $L^p(\Omega)$ -integrability of the solution even if the initial value is constant in Ω .

5.4 The fix point argument

Consider the fixed point operator

$$L_T(\phi) = [t \mapsto S(t)u_0 + S * F(\cdot, \phi)(t) + S \diamond G(\cdot, \phi)(t)].$$

Proposition 5.7. *Suppose that (A0)–(A3) are satisfied. If $u_0 \in L^p(\Omega, \mathcal{F}_0; X)$, $[[M]]_T \in L^\infty(\Omega)$, then the operator L_T is bounded and well-defined on $V^p(0, T, M; X)$ and there exists a constant $C_{T,M}$, with $\lim C_{T,M} = 0$ as $T \rightarrow 0$ and $T_{M,T} := \|[[M]]_T\|_{L^\infty(\Omega)} \rightarrow 0$, such that for all $\phi_1, \phi_2 \in V^p(0, T, M; X)$,*

$$\|L_T(\phi_1) - L_T(\phi_2)\|_{V^p(0,T,M;X)} \leq C_{T,M} \|\phi_1 - \phi_2\|_{V^p(0,T,M;X)}. \tag{5.5}$$

Moreover, for $T \in (0, 1]$, there exists a constant \tilde{C} independent of T and M such that

$$C_{T,M} \leq \tilde{C} \max\{T^{\frac{1}{2}}, T_{M,T}^{\frac{1}{2}}\}. \tag{5.6}$$

Furthermore, there is a constant $C \geq 0$, independent of u_0 , such that for all $\phi \in V^p(0, T, M; X)$

$$\|L_T(\phi)\|_{V^p(0,T,M;X)} \leq C(1 + \|u_0\|_{L^p(\Omega;X)}) + C_{T,M} \|\phi\|_{V^p(0,T,M;X)}, \tag{5.7}$$

and $L_T(\phi)$ has a continuous version and

$$\|L_T(\phi)\|_{L^p(\Omega;C([0,T];X))} \leq C(1 + \|u_0\|_{L^p(\Omega;X)}) + C_2 \|\phi\|_{V^p(0,T,M;X)}. \tag{5.8}$$

Proof. Actually the assumptions even yields that $\{S(t) : t \geq 0\}$ is R -bounded and hence γ -bounded by some constant N (see [41, Theorem 2.20 and 12.8]). In particular $\|S(t)\| \leq N$ for all $t \geq 0$.

Let $Y = \gamma(\mathbb{R}; X)$. For the proof we use the following dilation result for the semigroup S from [23]. By the boundedness of the H^∞ -calculus with angle $< \frac{\pi}{2}$ yields that there exist $J \in \mathcal{L}(X, Y)$, $P \in \mathcal{L}(Y)$ and $(\tilde{S}(t))_{t \in \mathbb{R}} \subseteq \mathcal{L}(Y)$ such that

- (i) There are $c_J, C_J > 0$ such that for all $x \in X$, one has $c_J \|x\| \leq \|Jx\| \leq C_J \|x\|$.
- (ii) P is a projection onto $\text{ran } J$.
- (iii) $(\tilde{S}(t))_{t \in \mathbb{R}}$ is a strongly continuous group on Y with $\|\tilde{S}(t)y\| = \|y\|$ for all $y \in Y$.
- (iv) For all $t \geq 0$ it holds true that $JS(t) = P\tilde{S}(t)J$.

This dilation will be used to derive continuity of the stochastic convolution in a similar way as in [30]. Moreover, we use it to obtain estimates in the γ -norm.

Notice that by [78, Lemma 2.3] Y is a UMD space. Also notice that since X has property (α) then according to [28, Theorem 3.18] family $(\tilde{S}(t))_{t \in \mathbb{R}}$ is γ -bounded by some constant α_X . Now we will proceed prove in 4 steps. Fix $T \geq 0$. Let $C_P = \|P\|$.

Step 1: Estimating the initial value part. By the strong continuity and uniform boundedness of S we derive:

$$\|s \mapsto S(s)u_0\|_{L^2(0,T;X)} \leq T^{\frac{1}{2}} \|s \mapsto S(s)u_0\|_{C([0,T];X)} \leq NT^{\frac{1}{2}} \|u_0\| \tag{5.9}$$

By the γ -boundedness of $(S(t))_{t \in \mathbb{R}}$ and [36, Proposition 4.11]:

$$\|s \mapsto S(s)u_0\|_{\gamma(0,T,[[M]];X)} \leq N \|s \mapsto u_0\|_{\gamma(0,T,[[M]];X)} = [[M]]_T^{1/2} \|u_0\|.$$

Step 2. Estimating the deterministic part. We proceed in two steps.

(a): For fixed $\omega \in \Omega$ and $\psi \in L^2(0, T; X)$ we estimate the $L^p(0, T; X)$ - and $\gamma(0, T, [[M]]; X)$ -norms of $S * \psi$. One has

$$\|S * \psi\|_{L^2(0,T;X)} = T^{\frac{1}{2}} \|S * \psi\|_{C([0,T];X)} \leq TN \|\psi\|_{L^2(0,T;X)}, \tag{5.10}$$

where the continuity of $S * \psi$ is simple to check (see [44, Corollary 4.2.4]).

By the representation of S as a group, the ideal property (4.1) and [36, Proposition 4.11] we find that

$$\begin{aligned} \|S * \psi\|_{\gamma(0,T,[[M]];X)} &\leq \frac{1}{c_J} \|(JS) * \psi\|_{\gamma(0,T,[[M]];X)} \\ &= \frac{1}{c_J} \left\| P\tilde{S}(\cdot) \int_0^\cdot \tilde{S}(-s)J\psi(s) \, ds \right\|_{\gamma(0,T,[[M]];X)} \\ &\leq \frac{C_P\alpha_X}{c_J} \left\| \int_0^\cdot \tilde{S}(-s)J\psi(s) \, ds \right\|_{\gamma(0,T,[[M]];X)} \\ &\leq \frac{C_J C_P\alpha_X N}{c_J} T^{\frac{1}{2}} [[M]]_T^{\frac{1}{2}} \|\psi\|_{L^2(0,T;X)}, \end{aligned}$$

where in the last step we used Lemma 5.2 and $\int_0^T t \, d[[M]]_t \leq T[[M]]_T$.

Now let $\Psi \in V^p(0, T, M; X)$. Then by applying the inequalities above to the paths $\Psi(\cdot, \omega)$ one easily obtains that $S * \Psi \in V^p(0, T, M; X)$ and

$$\|S * \Psi\|_{V^p(0,T,M;X)} \leq C_T^1 \|\Psi\|_{V^p(0,T,M;X)},$$

where

$$C_T^1 = TN + \frac{C_J C_P\alpha_X N}{c_J} T^{\frac{1}{2}} T_{M,T}^{\frac{1}{2}} \text{ and } T_{M,T} := \|[[M]]_T\|_{L^\infty(\Omega)}.$$

(b): Let $\phi_1, \phi_2 \in V^p(0, T, M; X)$. Since F is of linear growth, $F(\cdot, \phi_1)$ and $F(\cdot, \phi_2)$ have a continuous version and belong to $V^p(0, T, M; X)$. Since F is Lipschitz in its X -variable, we deduce that $S * F(\cdot, \phi_1)$ and $S * F(s, \phi_2)$ are in $V^p(0, T, M; X)$ and

$$\begin{aligned} \|S * F(s, \phi_1) - S * F(s, \phi_2)\|_{V^p(0,T,M;X)} &\leq C_T^1 \|F(\cdot, \phi_1) - F(\cdot, \phi_2)\|_{V^p(0,T,M;X)} \\ &\leq C_T^1 L_F \|\phi_1 - \phi_2\|_{V^p(0,T,M;X)}. \end{aligned}$$

Step 3. Estimating the stochastic part.

(a): Let $\Psi : [0, T] \times \Omega \rightarrow \mathcal{L}(H, X)$ be scalarly strongly progressively measurable and suppose that $\Psi Q_M^{1/2}$ is in $L^p(\Omega; \gamma(L^2(0, T, [[M]]; H), X))$. Then by [36, Proposition 4.11] and Theorem 4.1 for each $t \in [0, T]$,

$$\zeta_M(t) := \int_0^t S(t-s)\Psi(s) \, dM(s)$$

is well-defined. Now we estimate ζ_M pathwise in the space of continuous functions. As before one sees that $\int_0^\cdot \tilde{S}^{-1}J\Psi \, dM$ is well-defined and is a.s. continuous (here we use the fact that Y is a UMD space and \tilde{S} is γ -bounded). Therefore, by the representation of S as a group it follows that we can write

$$J\zeta_M(t) = P\tilde{S}(t) \int_0^t \tilde{S}(-s)J\Psi(s) \, dM \tag{5.11}$$

Since $P\tilde{S}(t)$ is strongly continuous, the continuity follows since J is an isomorphic embedding. Moreover, by Theorem 4.1 and [36, Proposition 4.11],

$$\begin{aligned} \|\zeta_M\|_{L^p(\Omega; L^2(0,T;X))} &\leq T^{\frac{1}{2}} \|\zeta_M\|_{L^p(\Omega; C([0,T];X))} \\ &\leq \frac{T^{\frac{1}{2}}}{c_J} \left\| t \mapsto P\tilde{S}(t) \int_0^t \tilde{S}^{-1}(s)J\Psi(s) \, dM(s) \right\|_{L^p(\Omega; C([0,T];X))} \\ &\leq \frac{T^{\frac{1}{2}} C_P N}{c_J} \left\| t \mapsto \int_0^t \tilde{S}^{-1}(s)J\Psi(s) \, dM(s) \right\|_{L^p(\Omega; C([0,T];X))} \end{aligned}$$

$$\begin{aligned} &\leq \frac{T^{\frac{1}{2}}C_P N C_{p,X}}{c_J} \|\tilde{S}^{-1} J\Psi Q_M^{\frac{1}{2}}\|_{L^p(\Omega; \gamma(0,T,[[M]]; H, Y))} \\ &\leq \frac{T^{\frac{1}{2}}C_P N^2 C_{p,X} C_J}{c_J} \|\Psi Q_M^{\frac{1}{2}}\|_{L^p(\Omega; \gamma(0,T,[[M]]; H, X))}. \end{aligned}$$

To estimate the $\gamma(L^2(0, T, [[M]]), X)$ -norm of ζ_M we can again use the representation (5.11) and use [36, Proposition 4.11] and the ideal property to estimate

$$\begin{aligned} \|\zeta_M\|_{L^p(\Omega; \gamma(0,T,[[M]]; X))} &\leq \frac{1}{c_J} \left\| t \mapsto P\tilde{S}(t) \int_0^t \tilde{S}^{-1}(s) J\Psi(s) dM(s) \right\|_{L^p(\Omega; \gamma(0,T,[[M]]; Y))} \\ &\leq \frac{C_P N}{c_J} \left\| t \mapsto \int_0^t \tilde{S}^{-1}(s) J\Psi(s) dM(s) \right\|_{L^p(\Omega; \gamma(0,T,[[M]]; Y))} \end{aligned}$$

To estimate the last term recall from Theorem 4.9

$$\tilde{\zeta}_M(t) := \int_0^t \tilde{S}^{-1} J\Psi dM = \int_0^{[[M]]_t} (\mathbf{1}_{(0,T)} \tilde{S}^{-1} J\Psi Q_M^{\frac{1}{2}}) \circ \tau dW_H =: \tilde{\zeta}_{W_H}([[M]]_t).$$

Using this representation, we find

$$\begin{aligned} \|\tilde{\zeta}_M\|_{L^p(\Omega; \gamma(0,T,[[M]]; Y))} &= \|\tilde{\zeta}_{W_H} \circ [[M]]\|_{L^p(\bar{\Omega}; \gamma(0,T,[[M]]; Y))} \\ &\stackrel{(*)}{=} \|\tilde{\zeta}_{W_H}\|_{L^p(\bar{\Omega}; \gamma(0,[[M]]_T; Y))} \\ &\leq \|\mathbb{E}(\tilde{\zeta}_{W_H}(T_{M,T}) | \mathcal{G}_t)\|_{L^p(\bar{\Omega}; \gamma(0,T_{M,T}; Y))} \\ &\stackrel{(i)}{\leq} C_p \|\mathbb{E}(\tilde{\zeta}_{W_H}(T_{M,T}) | \mathcal{G}_t)\|_{\gamma(0,T_{M,T}; L^p(\bar{\Omega}; Y))} \\ &\stackrel{(ii)}{\leq} C_p \gamma(\bar{\mathcal{E}}_{p,T}) \|\tilde{\zeta}_{W_H}(T_{M,T})\|_{\gamma(0,T_{M,T}; L^p(\bar{\Omega}; Y))} \\ &= C_p \gamma(\bar{\mathcal{E}}_{p,T}) T_{M,T}^{\frac{1}{2}} \|\tilde{\zeta}_{W_H}(T_{M,T})\|_{L^p(\bar{\Omega}; Y)} \\ &\stackrel{(iii)}{\leq} C_p \gamma(\bar{\mathcal{E}}_{p,T}) T_{M,T}^{\frac{1}{2}} C_{p,X} \|(\mathbf{1}_{(0,T)} \tilde{S}^{-1} J\Psi Q_M^{\frac{1}{2}}) \circ \tau\|_{L^p(\Omega; \gamma(\mathbb{R}_+; Y))} \\ &\stackrel{(*)}{\leq} C_p \gamma(\bar{\mathcal{E}}_{p,T}) T_{M,T}^{\frac{1}{2}} C_{p,X} \|\tilde{S}^{-1} J\Psi Q_M^{\frac{1}{2}}\|_{L^p(\Omega; \gamma(0,T,[[M]]; Y))} \end{aligned}$$

In (*) we used Lemma 4.5 and $[[M]]_{\tau_s} = s$. In (i) we used (5.1). In (ii) we used Lemma 5.1 for conditional expectations on $L^p(\bar{\Omega}; Y)$ and [36, Proposition 4.11]. In (iii) we used (4.7). Therefore, combining both estimates it follows that

$$\begin{aligned} \|\zeta_M\|_{L^p(\Omega; \gamma(0,T,[[M]]; X))} &\leq \frac{C_P N}{c_J} \|\tilde{\zeta}_M\|_{L^p(\Omega; \gamma(0,T,[[M]]; Y))} \\ &\leq \frac{C_P N^2 C_J}{c_J} C_p \gamma(\bar{\mathcal{E}}_{p,T}) T_{M,T}^{\frac{1}{2}} C_{p,X} \|\Psi Q_M^{\frac{1}{2}}\|_{L^p(\Omega; \gamma(0,T,[[M]]; X))} \end{aligned}$$

where in the last step we argue as below (5.11).

Combining these estimates we conclude that

$$\|\zeta_M\|_{V^p(0,T,M;Y)} \leq C_T^2 \|\Psi Q_M^{1/2}\|_{L^p(\Omega; \gamma(L^2(0,T,[[M]]; H), X))}, \tag{5.12}$$

where

$$C_T^2 := \frac{T^{\frac{1}{2}}C_P N^2 C_{p,X} C_J}{c_J} + \frac{C_P N^2 C_J}{c_J} C_p \gamma(\bar{\mathcal{E}}_{p,T}) T_{M,T}^{\frac{1}{2}} C_{p,X}$$

(b): Let $\phi_1, \phi_2 \in V^p(0, T, M; X)$. It follows from the assumption on G that $S \diamond G(\cdot, \phi_1), S \diamond G(\cdot, \phi_2)$ have a continuous version and are $V^p(0, T, M; X)$ and

$$\left\| S \diamond G(\cdot, \phi_1) - S \diamond G(\cdot, \phi_2) \right\|_{V^p(0,T,M;X)}$$

$$\begin{aligned} &\leq C_T^2 \|(G(\cdot, \phi_1) - G(\cdot, \phi_2))Q_M^{1/2}\|_{L^p(\Omega; \gamma(L^2(0, T, [[M]]); H), X)} \\ &\leq L_G^\gamma C_T^2 \|\phi_1 - \phi_2\|_{V^p(0, T, M; X)}. \end{aligned}$$

Step 4: Collecting the estimates. It follows from the previous steps that L_T is well-defined on $V^p(0, T, M; X)$ and

$$\|L_t(\phi_1) - L_t(\phi_2)\|_{V^p(0, T, M; X)} \leq C_{T, M} \|\phi_1 - \phi_2\|_{V^p(0, T, M; X)}, \tag{5.13}$$

where $C_{T, M} = L_F C_T^1 + L_G C_T^2$ and one can check that (5.6) holds.

To prove (5.7) one has to apply (5.13) and the fact that for some positive constant C it holds true that

$$\|L_T(0)\|_{V^p(0, T, M; X)} \leq C(1 + (\mathbb{E}\|u_0\|_X^p)^{\frac{1}{p}}).$$

The final continuity statement and (5.8) follows from the previous steps. □

5.5 Existence and uniqueness when the variation is small

Theorem 5.8 (Existence and uniqueness). *Suppose that (A0)–(A3) are satisfied and $\|[[M]]_T\|_{L^\infty(\Omega)} \leq (2\tilde{C})^{-2}$, where \tilde{C} is as in (5.6). If $u_0 \in L^p(\Omega, \mathcal{F}_0; X)$, then there exists a unique solution U in $V^p(0, T, M; X)$ of (5.2). Moreover, there exists a nonnegative constant C , independent of u_0 but depending on $T \vee 1$ and \tilde{C} such that*

$$\|U\|_{V^p(0, T, M; X)} \leq C(1 + (\mathbb{E}\|u_0\|_X^p)^{\frac{1}{p}}). \tag{5.14}$$

Furthermore, U has a continuous version and there exists a constant D independent of u_0 but depending on $T \vee 1$ and \tilde{C} such that

$$\|U\|_{L^p(\Omega; C([0, T]; X))} \leq C(1 + (\mathbb{E}\|u_0\|_X^p)^{\frac{1}{p}}) \tag{5.15}$$

Proof. By Proposition 5.7 one can find $t \in [0, T \wedge 1]$, independent of u_0 , such that $t \leq \|[[M]]_T\|_{L^\infty(\Omega)}$, so $C_{t, M} \leq \frac{1}{2}$. It follows from (5.5) and the Banach fixed point argument that L_t has a unique fixed point $U \in V^p(0, t, M; X)$. This gives us a continuous progressively measurable process $U : [0, t] \times \Omega \rightarrow X$ such that a.s. for all $s \in [0, t]$,

$$U(s) = S(s)u_0 + S * F(\cdot, U)(s) + S \diamond G(\cdot, U)(s).$$

Note that (5.7) implies that

$$\|U\|_{V^p(0, t, M; X)} \leq C(1 + (\mathbb{E}\|u_0\|_X^p)^{\frac{1}{p}}) + C_{t, M} \|U\|_{V^p(0, t, M; X)},$$

and since $C_{t, M} \leq 1/2$

$$\|U\|_{V^p(0, t, M; X)} \leq 2C(1 + (\mathbb{E}\|u_0\|_X^p)^{\frac{1}{p}}). \tag{5.16}$$

and by $U = L_t(U)$, (5.8) and (5.16) we find

$$\|U\|_{L^p(\Omega; C([0, t]; X))} \leq C_2(1 + \|u_0\|_{L^p(\Omega; X)}). \tag{5.17}$$

Thanks to a standard induction argument one easily constructs a solution on each of intervals $[t, 2t], \dots, [nt, T]$, where $n = \lceil \frac{T}{t} \rceil$. This solution U on $[0, T]$ is the solution of (5.2). Moreover, according to (5.16), (5.17) and the induction one deduces (5.14) and (5.15).

For small $t \in [0, T]$ uniqueness on $[0, t]$ follows from the uniqueness of the fixed point of L_t in $V^p(0, t, M; X)$, and uniqueness on $[0, T]$ follows from the induction argument. □

Lemma 5.9. *Suppose that (A0)–(A3) are satisfied both for M and N and*

$$\|[[M]]_T\|_{L^\infty(\Omega)}, \|[[N]]_T\|_{L^\infty(\Omega)} < \frac{1}{4\tilde{C}^2}.$$

Let $U_1 \in V^p(0, T, M; X)$, $U_2 \in V^p(0, T, N; X)$ be the solutions of (5.2) with initial values $u_1, u_2 \in L^p(\Omega, \mathcal{F}_0; X)$ and cylindrical martingales M, N respectively. Finally suppose that $M \equiv N$ a.s. on the set $\{u_1 = u_2\}$. Then a.s. $U_1 \equiv U_2$ on $\{u_1 = u_2\}$.

Proof. Let $\Gamma = \{u_1 = u_2\}$. Since $U_2 \in V^p(0, T, N; X)$, then $U_2 \mathbf{1}_\Gamma \in V^p(0, T, M; X)$, because M and N coincides on Γ . Consider small t as in the beginning of the proof of Theorem 5.8. Since Γ is \mathcal{F}_0 -measurable

$$\begin{aligned} \|U_1 \mathbf{1}_\Gamma - U_2 \mathbf{1}_\Gamma\|_{V^p(0,t,M;X)} &= \|L_t(U_1) \mathbf{1}_\Gamma - L_t(U_2) \mathbf{1}_\Gamma\|_{V^p(0,t,M;X)} \\ &= \|L_t(U_1 \mathbf{1}_\Gamma) \mathbf{1}_\Gamma - L_t(U_2 \mathbf{1}_\Gamma) \mathbf{1}_\Gamma\|_{V^p(0,t,M;X)} \\ &\leq C_{t,M} \|U_1 \mathbf{1}_\Gamma - U_2 \mathbf{1}_\Gamma\|_{V^p(0,t,M;X)}, \end{aligned}$$

therefore almost surely $U_1|_{[0,t] \times \Gamma} \equiv U_2|_{[0,t] \times \Gamma}$.

To extend this result to the whole interval $[0, T]$ one has to apply the same induction argument as in the end of the proof of Theorem 5.8. \square

Let $b \geq a \geq 0$. We say that ϕ is *locally* in $V^p(a, b, M; X)$ (or simply $\phi \in V_{\text{loc}}^p(a, b, M; X)$) if there exists a sequence of increasing stopping times $(\tau_n)_{n \geq 1}$ such that $\tau_n \nearrow \infty$ a.s. and $\phi \in V^p(a, b, M^{\tau_n}; X)$ for each $n > 0$. It is evident by Remark 5.4 that $\phi \in V^p(a, b, M; X)$ implies $\phi \in V_{\text{loc}}^p(a, b, M; X)$. Obviously $V_{\text{loc}}^p(a, b, M; X) \subseteq V(a, b, M; X)$.

Lemma 5.10. *Suppose that (A0)–(A3) are satisfied. Let τ be a stopping time such that $\|[[M^\tau]]_T\|_{L^\infty(\Omega)} < \frac{1}{4C^2}$, $u_0 \in L^p(\Omega, \mathcal{F}_0; X)$, $v_0 \in L^0(\Omega, \mathcal{F}_0; X)$, $U_M \in V_{\text{loc}}^p(0, T, M; X)$, $U_{M^\tau} \in V^p(0, T, M^\tau; X)$ be solutions of (5.2) with cylindrical martingales M, M^τ and initial values u_0, v_0 respectively. Then on the set $\{u_0 = v_0\}$ and on the interval $[0, \tau \wedge T]$, one has $U_M \equiv U_{M^\tau}$ a.s.*

Proof. Let $\Gamma = \{u_0 = v_0\}$. Consider the localizing sequence $\{\tau_n\}_{n \geq 1}$ of stopping times for U_M , so $U_M \in V^p(0, T, M^{\tau_n}; X)$ for each $n \geq 1$. Then by (4.1) both $U_M \mathbf{1}_{[0, \tau \wedge \tau_n]} \mathbf{1}_\Gamma$ and $U_{M^\tau} \mathbf{1}_{[0, \tau \wedge \tau_n]} \mathbf{1}_\Gamma$ are in $V^p(0, T, M^{\tau \wedge \tau_n}; X)$ for each fixed $n \geq 1$. Let t be such that $t^\beta < \|[[M^\tau]]_T\|_{L^\infty(\Omega)}^{\frac{1}{2}}$. Then for each fixed $n \geq 1$

$$\begin{aligned} &\|U_M \mathbf{1}_{[0, \tau \wedge \tau_n]} \mathbf{1}_\Gamma - U_{M^\tau} \mathbf{1}_{[0, \tau \wedge \tau_n]} \mathbf{1}_\Gamma\|_{V^p(0,t,M^{\tau \wedge \tau_n};X)} \\ &= \|L_t(U_M) \mathbf{1}_{[0, \tau \wedge \tau_n]} \mathbf{1}_\Gamma - L_t(U_{M^\tau}) \mathbf{1}_{[0, \tau \wedge \tau_n]} \mathbf{1}_\Gamma\|_{V^p(0,t,M^{\tau \wedge \tau_n};X)} \\ &= \|L_t(U_M \mathbf{1}_{[0, \tau \wedge \tau_n]} \mathbf{1}_\Gamma) \mathbf{1}_{[0, \tau \wedge \tau_n]} \mathbf{1}_\Gamma - L_t(U_{M^\tau} \mathbf{1}_{[0, \tau \wedge \tau_n]} \mathbf{1}_\Gamma) \mathbf{1}_{[0, \tau \wedge \tau_n]} \mathbf{1}_\Gamma\|_{V^p(0,t,M^{\tau \wedge \tau_n};X)} \\ &\leq C_{t,M^{\tau \wedge \tau_n}} \|U_M \mathbf{1}_{[0, \tau \wedge \tau_n]} \mathbf{1}_\Gamma - U_{M^\tau} \mathbf{1}_{[0, \tau \wedge \tau_n]} \mathbf{1}_\Gamma\|_{V^p(0,t,M^{\tau \wedge \tau_n};X)}, \end{aligned}$$

hence $U_M \mathbf{1}_{[0, \tau \wedge \tau_n \wedge t]} \mathbf{1}_\Gamma \equiv U_{M^\tau} \mathbf{1}_{[0, \tau \wedge \tau_n \wedge t]} \mathbf{1}_\Gamma$ a.s. Letting n to infinity yields $U_M \equiv U_{M^\tau}$ on $[0, t \wedge \tau]$ for a.a. $\omega \in \Gamma$. Now by induction and the same technique as in Lemma 5.9 one obtains the required result. \square

5.6 Proof of the main existence and uniqueness result

We first proof Theorem 5.5 under additional integrability assumptions on the initial value.

Theorem 5.11 (Existence and uniqueness for integrable initial values). *Suppose that (A0)–(A3) are satisfied. If $u_0 \in L^p(\Omega, \mathcal{F}_0; X)$, then there exists a unique solution U in $V_{\text{loc}}^p(0, T, M; X)$ of (5.2).*

Proof. By Proposition 5.7 one can find $n \in \mathbb{N}$ large enough so that $\frac{T}{2^n} \leq \frac{1}{4C^2}$ and $T \leq 2^n$.

Let $\rho = \tau_{\frac{1}{4C^2}}$, where τ_s is a stopping time introduced in (4.9). Consider equation (5.2) with the cylindrical martingale M^ρ instead of M . It follows from (5.13) that $C_{\frac{T}{2^n}, M^\rho} < \frac{1}{2}$. Using the Banach fixed point argument one derives that $L_{\frac{T}{2^n}}$ has a unique fixed point $U_n \in V^p(0, \frac{T}{2^n}, M^\rho; X)$. This gives us a continuous progressive measurable process $U_n : [0, \frac{T}{2^n}] \times \Omega \rightarrow X$ such that for almost all $\omega \in \Omega$ for all $s \in [0, \frac{T}{2^n}]$,

$$U_n(t) = S(s)u_0 + S * F(\cdot, U_n) + \int_0^t S(t-s)G(s, U_n) dM_s^\rho.$$

Note that (5.7) implies that

$$\|U_n\|_{V^p(0, \frac{T}{2^n}, M^\rho; X)} \leq C(1 + (\mathbb{E}\|u_0\|_X^p)^{\frac{1}{p}}) + C_{\frac{T}{2^n}, M^\rho} \|U_n\|_{V^p(0, \frac{T}{2^n}, M^\rho; X)},$$

and since $C_{\frac{T}{2^n}, M^\rho} < \frac{1}{2}$

$$\|U_n\|_{V^p(0, \frac{T}{2^n}, M^\rho; X)} \leq 2C(1 + (\mathbb{E}\|u_0\|_X^p)^{\frac{1}{p}}). \tag{5.18}$$

To go on with a standard induction argument on each of intervals $[\frac{(k-1)T}{2^n}, \frac{kT}{2^n}]$ for $k \in \{2, \dots, 2^n\}$ we introduce the following stopping times for $k \in \{1, \dots, 2^n\}$

$$\rho_{nk} = \begin{cases} \frac{(k-1)T}{2^n} + \inf\{t \geq 0 : [[M]]_{t+\frac{(k-1)T}{2^n}} - [[M]]_{\frac{(k-1)T}{2^n}} > \frac{T}{2^n}\}, & \text{on the set } A; \\ \infty, & \text{on the set } \Omega \setminus A. \end{cases} \tag{5.19}$$

Here $A = \{0 \leq \frac{T}{2^n} < [[M]]_\infty - [[M]]_{\frac{(k-1)T}{2^n}}\}$. As one can notice, $\rho_{n1} = \rho$. By [66, Theorem I.18] and since the minimum of stopping times is a stopping time, $M^{\rho_{n1} \wedge \dots \wedge \rho_{nk}} \in \mathcal{M}_{\text{var}}(H)$. Fix $k > 1$. Then one can construct solution of equation (5.2) on the interval $[\frac{(k-1)t}{2^n}, \frac{kt}{2^n}]$ with the cylindrical martingale $M^{\rho_{n1} \wedge \dots \wedge \rho_{nk}}$ instead of M and with the initial value, obtained on the previous interval $[\frac{(k-2)T}{2^n}, \frac{(k-1)T}{2^n}]$.

Thanks to (5.18), (5.8) and a standard induction argument one may construct a solution on each of intervals $[\frac{T}{2^n}, 2\frac{T}{2^n}], \dots, [(2^n - 1)\frac{T}{2^n}, T]$. This solution U_n on $[0, T]$ is the solution of (5.2) with M replaced by M^{ρ_n} .

Define $\rho_n := \rho_{n1} \wedge \dots \wedge \rho_{n2^n}$ for each $n \in \mathbb{N}$. Then by the fixed point argument, the induction argument and Lemma 5.10, $U_n = U_m$ on $[0, \rho_n \wedge \rho_m \wedge T]$ for all $m, n \in \mathbb{N}$. Consequently, since $\rho_n \nearrow \infty$ a.s. there exists $U : [0, T] \times \Omega \rightarrow X$ such that $U = U_n$ on $[0, \rho_n \wedge T]$ for each $n \geq 1$.

Now one has to show that U is a solution of (5.2). First of all notice that for each fixed $t \geq 0$ we know that $(U - U_n)\mathbf{1}_{t \leq \rho_n} = 0$. Consequently $(S(t-s)G(s, U_n) - S(t-s)G(s, U))\mathbf{1}_{t \leq \rho_n} = 0$. Then for each fixed $t \geq 0$ according to Corollary 4.12 one has that a.s. on $\{t \leq \rho_n\}$

$$\begin{aligned} U(t) = U_n(t) &= S(t)u_0 + \int_0^t S(t-s)F(s, U_n) ds + \int_0^t S(t-s)G(s, U_n) dM_s \\ &= S(t)u_0 + \int_0^t S(t-s)F(s, U) ds + \int_0^t S(t-s)G(s, U) dM_s. \end{aligned}$$

So, letting n to infinity one can show that for each fixed $t \geq 0$ a.s.

$$U(t) = S(t)u_0 + \int_0^t S(t-s)F(s, U) ds + \int_0^t S(t-s)G(s, U) dM_s.$$

Now assume that $V \in V_{\text{loc}}^p(0, T, M; X)$ is another solution of (5.2). Then by Lemma 5.10, $V = U_n$ on $[0, \rho_{n1} \wedge \frac{T}{2^n}]$ for all $n \geq 1$. According to (5.15) $U_n(\frac{T}{2^n}) \in L^p(\Omega, \mathcal{F}_{\frac{T}{2^n}}; X)$, so again by Lemma 5.10 on the set $\{\rho_{n1} \geq \frac{T}{2^n}\}$ $V = U_n$ on $[\frac{T}{2^n}, \rho_{n2} \wedge \frac{2T}{2^n}]$ for all $n \geq 1$ (here we start our solutions from the point $\frac{T}{2^n}$). Continuing this procedure for $k = 3, \dots, 2^n$ we have that $V = U_n$ on $[0, \rho_n \wedge T]$ for all positive n . But since $U = U_n$ on $[0, \rho_n \wedge T]$ for all $n \geq 1$, $V = U$ on $[0, \rho_n \wedge T]$, therefore on whole $[0, T]$. \square

Finally we can prove Theorem 5.5 for general initial values.

Proof of Theorem 5.5. The structure of the proof is the same as in [54, Theorem 7.1]. To prove existence define a sequence $(u_n)_{n \geq 1}$ in $L^p(\Omega, \mathcal{F}_0; X)$ in the following way:

$$u_n = \mathbf{1}_{\|u_0\| \leq n} u_0.$$

Then by Theorem 5.11 for each $n \geq 1$ there exists a unique solution $U_n \in V_{\text{loc}}^p(0, T, M; X)$ of (5.2) with initial value u_n . By Lemma 5.10 one can define $U : [0, T] \times \Omega \rightarrow X$ as $U(t) = \lim_{n \rightarrow \infty} U_n(t)$ if this limit exists and 0 otherwise. Then U is strongly progressive measurable, and almost surely on $\{\|u_0\| \leq n\}$ for all $t \in [0, T]$ we have that $U(t) = U_n(t)$. Consequently, $U \in V(0, T, M; X)$ and one can check it is a solution of (5.2).

For uniqueness of the solution we will need the stopping times constructed in the proof of Theorem 5.11. Let $U, V \in V(0, T, M; X)$ be two solutions of (5.2). First of all fix $n \geq 1$ and prove that $U \mathbf{1}_{\|u_0\| \leq n} = V \mathbf{1}_{\|u_0\| \leq n}$. Let $U_n = U \mathbf{1}_{\|u_0\| \leq n}$, $V_n = V \mathbf{1}_{\|u_0\| \leq n}$. Obviously U_n and V_n are solutions of (5.2) with initial value $u_0 \mathbf{1}_{\|u_0\| \leq n}$.

Let k be large enough such that $\frac{T}{2^k} < \frac{1}{2C}$. For each $l \in \mathbb{N}$ define a stopping time σ_{nl} as follows:

$$\sigma_{nl} = \inf\{s \in [0, T] : \|U_n\|_{L^2((0,s);X)} + \|U_n\|_{\gamma(L^2(0,s,[[M]]),X)} + \|V_n\|_{L^2((0,s);X)} + \|V_n\|_{\gamma(L^2(0,s,[[M]]),X)} \geq l\}.$$

Then $U_n \mathbf{1}_{[0, \sigma_{nl}]}, V_n \mathbf{1}_{[0, \sigma_{nl}]} \in V^p(0, \frac{T}{2^k}, M; X)$. Define $(\rho_{km})_{1 \leq m \leq 2^k}$ in the same way as in (5.19). For fixed k one has the following

$$\begin{aligned} & \|U_n \mathbf{1}_{[0, \sigma_{nl} \wedge \rho_{k1}]} - V_n \mathbf{1}_{[0, \sigma_{nl} \wedge \rho_{k1}]} \|_{V^p(0, \frac{T}{2^k}, M; X)} \\ &= \|U_n \mathbf{1}_{[0, \sigma_{nl} \wedge \rho_{k1}]} - V_n \mathbf{1}_{[0, \sigma_{nl} \wedge \rho_{k1}]} \|_{V^p(0, \frac{T}{2^k}, M^{\rho_{k1}}; X)} \\ &= \|L_t(U_n) \mathbf{1}_{[0, \sigma_{nl} \wedge \rho_{k1}]} - L_t(V_n) \mathbf{1}_{[0, \sigma_{nl} \wedge \rho_{k1}]} \|_{V^p(0, \frac{T}{2^k}, M^{\rho_{k1}}; X)} \\ &= \|L_t(U_n \mathbf{1}_{[0, \sigma_{nl} \wedge \rho_{k1}]}) \mathbf{1}_{[0, \sigma_{nl} \wedge \rho_{k1}]} - L_t(V_n \mathbf{1}_{[0, \sigma_{nl} \wedge \rho_{k1}]}) \mathbf{1}_{[0, \sigma_{nl} \wedge \rho_{k1}]} \|_{V^p(0, \frac{T}{2^k}, M^{\rho_{k1}}; X)} \\ &\leq C_{\frac{T}{2^k}, M^{\rho_{k1}}} \|U_n \mathbf{1}_{[0, \sigma_{nl} \wedge \rho_{k1}]} - V_n \mathbf{1}_{[0, \sigma_{nl} \wedge \rho_{k1}]} \|_{V^p(0, \frac{T}{2^k}, M^{\rho_{k1}}; X)} \\ &\leq \frac{1}{2} \|U_n \mathbf{1}_{[0, \sigma_{nl} \wedge \rho_{k1}]} - V_n \mathbf{1}_{[0, \sigma_{nl} \wedge \rho_{k1}]} \|_{V^p(0, \frac{T}{2^k}, M; X)}, \end{aligned}$$

so a.s. $U_n \mathbf{1}_{[0, \sigma_{nl} \wedge \rho_{k1}]}(s) = V_n \mathbf{1}_{[0, \sigma_{nl} \wedge \rho_{k1}]}(s)$ for all $s \in (0, \frac{T}{2^k})$. Define again

$$\rho_k := \rho_{k1} \wedge \dots \wedge \rho_{k2^k}, \quad k \in \mathbb{N}.$$

By the standard induction argument one derives that a.s. $U_n \mathbf{1}_{[0, \sigma_{nl} \wedge \rho_k]} \equiv V_n \mathbf{1}_{[0, \sigma_{nl} \wedge \rho_k]}$ on $[0, T]$. Now taking k and l to infinity gives us the desired.

Since $U = \lim_{n \rightarrow \infty} U_n$ and $V = \lim_{n \rightarrow \infty} V_n$, then $U = V$ a.s. and uniqueness is proved. \square

Proof of Proposition 5.6. This result follows with the same method as for Theorem 5.5. Note that property (α) can be avoided since $A = 0$ and hence we can take $S(t) = \tilde{S}(t) = I$ and the γ -boundedness is clear in this case. \square

Remark 5.12. Using the time change result of Theorem 4.9 one can turn the noise part of the problem (5.2) into a cylindrical Brownian motion. Unfortunately, by using this technique the term $Au(t) dt$ becomes more involved. In particular, one has to use evolution families instead of semigroups, which complicates matters.

A A technical lemma on measurable selections

In the next lemma we show that a certain projection valued function can be chosen in a measurable way. Moreover, we give a representation formula for its inverse which is used in the proof of Theorem 4.1. In [64, Lemma 8.9] a similar measurability result was proved by applying a selector theorem by Kuratowski and Ryll-Nardzewski.

Recall from before that a function $F : S \rightarrow \mathcal{L}(H)$ is called H -strongly measurable if for all $h \in H$, $s \mapsto F(s)h$ is strongly measurable.

Lemma A.1. Let (S, Σ) be a measurable space and let H be a separable Hilbert space. Let $H_0 \subseteq H$ be a finite dimensional subspace. Let $F : S \rightarrow \mathcal{L}(H)$ be a function such that:

1. F is H -strongly measurable;
2. for all $s \in S$ and $h \in H$, $F(s)^* = F(s)$ and $\langle F(s)h, h \rangle \geq 0$.

For each $s \in S$, let $P(s) \in \mathcal{L}(H)$ be the orthogonal projection onto $F(s)H_0$. Then there exist H -strongly measurable functions $\tilde{P}, L : S \rightarrow \mathcal{L}(H)$ such that

$$\tilde{P}F = FP \text{ and } LF = P, \tag{A.1}$$

pointwise in S . Moreover, \tilde{P} is a projection.

The operator \tilde{P} will not be an orthogonal projection in general.

Proof. Let P_0 be the orthogonal projection onto H_0 . For each $s \in S$ define $\tilde{P}(s) \in \mathcal{L}(H)$ as follows:

$$\tilde{P}(s)P_0F(s)^2P_0h = F(s)^2P_0h, \text{ for } h \in H,$$

and set $\tilde{P}(s) = 0$ on $\ker P_0F(s)^2P_0$. Notice, that there is no contradiction, since if $P_0F(s)^2P_0h = 0$ for some $h \in H$ and $s \in S$, then

$$0 = \langle P_0F(s)^2P_0h, h \rangle = \|F(s)P_0h\|^2$$

and hence $h \in \ker F(s)P_0 \subseteq \ker F(s)^2P_0$. Since $P_0F(s)^2P_0$ is a finite-rank self-adjoint operator for each $s \in S$, we have $H = \ker P_0F(s)^2P_0 \oplus \text{ran } P_0F(s)^2P_0$, and thus $P_0F(s)^2P_0$ is a bounded linear operator (see [75, Theorem 6.2-G]).

In the sequel we suppress the $s \in S$ from the formulas. We claim that

- (i) $\tilde{P}h = 0$ for each $h \in H_0^\perp$;
- (ii) $\tilde{P}F^2h = F^2h$ for $h \in H_0$.
- (iii) $\tilde{P}F = FP$

Property (i) is clear from $H_0^\perp \subseteq \ker P_0F^2P_0$. For (ii) note that for every $h \in H_0$, we can write $F^2h = P_0F^2h + (1 - P_0)F^2h$. Since for all $g \in H_0$,

$$\langle (1 - P_0)F^2h, g \rangle = \langle (1 - P_0)F^2h, P_0g \rangle = \langle P_0(1 - P_0)F^2h, g \rangle = 0,$$

we find that $(1 - P_0)F^2h \in H_0^\perp$. Thus by (i) and the definition of \tilde{P} ,

$$\tilde{P}F^2h = \tilde{P}P_0F^2h + \tilde{P}(1 - P_0)F^2h = F^2h$$

and (ii) follows. To prove (iii) let $g \in \text{ran } P$. Choosing $h \in H_0$ s.t. $g = Fh$ we find

$$FPg = Fg = F^2h \stackrel{(ii)}{=} \tilde{P}F^2h = \tilde{P}Fg.$$

On the other hand, FP vanishes on the space

$$\ker P_0F = (\text{ran } FP_0)^\perp = (\text{ran } P)^\perp = \ker P.$$

The same holds true for $\tilde{P}F$. Indeed, since $(1 - P_0)Fh \in \ker P_0F^2P_0$, it follows that for $h \in \ker P_0F$

$$\tilde{P}Fh = \tilde{P}(1 - P_0)Fh = 0$$

and this gives (iii).

Next we claim that $\tilde{P}^2 = \tilde{P}$. Indeed, for each $h \in H$, $\tilde{P}h \in \text{ran } F^2P_0$ by the definition of \tilde{P} . Thus by (ii) $\tilde{P}F^2P_0 = F^2P_0$ and therefore $\tilde{P}^2h = \tilde{P}h$.

To prove H -strong measurability fix an orthonormal basis $(h_i)_{i=1}^k$ for H_0 . For each subset $\alpha \subseteq \{1, \dots, k\}$ there exists a measurable $S_\alpha \subseteq S$ such that $(Fh_i)_{i \in \alpha}$ is a basis of $\text{span}(Fh_i)_{1 \leq i \leq k}$ (because of the strong measurability of Fh for each $h \in H$ and using the Gramian matrix technique). Notice that if $(Fh_i)_{i \in \alpha}$ is a basis of $\text{span}(Fh_i)_{1 \leq i \leq k}$, then $(F^2h_i)_{i \in \alpha}$ is a basis of $\text{span}(F^2h_i)_{1 \leq i \leq k}$. Indeed, let $g = \sum_{i \in \alpha} c_i Fh_i$ be a combination of $(Fh_i)_{i \in \alpha}$ with some scalars $(c_i)_{i \in \alpha}$. If $Fg = 0$, then $g \in \ker F = (\text{ran } F)^\perp$, so $g = 0$. Let $\alpha, \beta \subseteq \{1, \dots, k\}$. We will say that $\alpha < \beta$ if $\sum_{i \in \alpha} 2^i < \sum_{i \in \beta} 2^i$. If $\alpha < \beta$, one has to redefine $S_\alpha := S_\alpha \setminus S_\beta$. After the iterations of this procedure for all pairs $\alpha, \beta \subseteq \{1, \dots, k\}$ the sets $(S_\alpha)_{\alpha \subseteq \{1, \dots, k\}}$ will be pairwise disjoint.

Now fix $\alpha \subseteq \{1, \dots, k\}$. Let $(g_i)_{i \in \alpha}$ be obtained from $(P_0 F^2 h_i)_{i \in \alpha}$ by the Gram-Schmidt process. These vectors are orthonormal and measurable because $(\langle P_0 F^2 h_i, P_0 F^2 h_j \rangle)_{i, j \in \alpha}$ are measurable. Moreover, the transformation matrix $C = (c_{ij})_{i, j \in \alpha}$ such that

$$g_i = \sum_{j \in \alpha} c_{ij} P_0 F^2 h_j, \quad i \in \alpha,$$

has measurable elements. So, $\tilde{P}g_i = \tilde{P} \sum_{j \in \alpha} c_{ij} P_0 F^2 h_j = \sum_{j \in \alpha} c_{ij} F^2 h_j$. This means that for each $h \in H$ the following hold true:

$$\tilde{P}h = \sum_{i \in \alpha} \langle h, g_i \rangle \tilde{P}g_i = \sum_{i \in \alpha} \langle h, g_i \rangle \sum_{j \in \alpha} c_{ij} F^2 h_j,$$

which is obviously measurable.

Now define L as an operator with values in $F(H_0) = P(H_0)$ as follows:

$$\begin{aligned} L(F^2 h) &= PFh, \quad h \in H, \\ Lh &= 0, \quad h \in \ker F. \end{aligned}$$

Then L is well-defined since $\ker F = \ker F^2$. Also for each $1 \leq i \leq k$ and $h \in H$

$$|\langle L(F^2 h), Fh_i \rangle| = |\langle PFh, Fh_i \rangle| = |\langle Fh, PFh_i \rangle| = |\langle F^2 h, h_i \rangle| \leq \|F^2 h\|.$$

Since the range of L is finite dimensional and equal to FH_0 , the operator L is bounded. Since $H = \overline{\text{ran } F} \oplus \ker F$ and $\ker F = \ker P$ we find $LF = P$.

As before one can show that L is H -strongly measurable, This time fixing $\alpha \subseteq \{1, \dots, k\}$ one considering the orthogonal basis $(g_i)_{i \in \alpha}$ for $\text{span}(Fh_i)_{i \in \alpha}$. \square

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Acknowledgments. The authors would like to thank Jan van Neerven for helpful comments. We are indebted to the anonymous referee for his/her valuable comments to improve the paper.

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