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Stochastic Generalized Nash Equilibrium-Seeking in Merely Monotone Games

Barbara Franci  and Sergio Grammatico 

Abstract—We solve the stochastic generalized Nash equilibrium (SGNE) problem in merely monotone games with expected value cost functions. Specifically, we present the first distributed SGNE-seeking algorithm for monotone games that require one proximal computation (e.g., one projection step) and one pseudogradient evaluation per iteration. Our main contribution is to extend the relaxed forward–backward operator splitting by the Malitsky (Mathematical Programming, 2019) to the stochastic case and in turn to show almost sure convergence to an SGNE when the expected value of the pseudogradient is approximated by the average over a number of random samples.

Index Terms—Stochastic generalized Nash equilibrium problems, stochastic variational inequalities.

I. INTRODUCTION

IN a generalized Nash equilibrium problem (GNEP), some agents interact with the aim of minimizing their individual cost functions under some joint feasibility constraints. Due to the presence of the shared constraints, computing a GNE is usually hard. Despite this challenge, GNEPs have been studied extensively within the system and control community, for their wide applicability, e.g., in energy markets [1]–[5].

Unfortunately, the stochastic counterpart of GNEP is not studied as much [6]–[9]. A stochastic GNEP (SGNEP) is a constrained equilibrium problem where the cost functions are expected value functions. Such problems arise when there is some uncertainty expressed through a random variable with an unknown distribution. For instance, networked Cournot games with market capacity constraints and uncertainty in the demand can be modeled as SGNEPs [10], [11]. Other examples arise in transportation systems [12] and in electricity markets [13].

If the random variable is known, the expected value formulation can be solved with a standard technique for the deterministic counterpart. In fact, one possible approach for SGNEPs is to

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recast the problem as a stochastic variational inequality (SVI) through the use of the Karush–Kuhn–Tucker (KKT) conditions. Then, the problem can be written as a monotone inclusion and solved via operator-splitting techniques. To find a zero of the resulting operator, we propose a stochastic relaxed forward–backward (SRFB) algorithm. Our iterations are the stochastic counterpart of the golden ratio algorithm [14] for deterministic variational inequalities, which reduces to a stochastic relaxation of a forward–backward algorithm when applied to non-GNEPs.

Besides the shared constraints, the additional difficulty in SGNEPs is that the pseudogradient mapping is usually not directly accessible, for instance, because the expected value is hard to compute. For this reason, often the search for a solution of a SVI relies on samples of the random variable. Depending on the number of samples, there are two main methodologies available: stochastic approximation (SA) and sample average approximation (SAA). In the SA scheme [15], each agent samples one or a finite number of realizations of the random variable. While it can be computationally light, it may also require stronger assumptions on the mappings and on the parameters involved [7], [16], [17]. To weaken the assumptions, it is often used in combination with the so-called variance reduction [18], [19], taking the average over an increasing number of samples. This approach, although it may be computationally costly, is used, for instance, in machine learning problems, where there is a huge number of data available. If the average is taken over an infinite number of samples instead, we have the SAA scheme [20].

Independently of the approximation scheme, it is desirable to obtain distributed iterations, where each agent knows only its cost function and constraints [1], [21]. In a full-decision information setting, the agents have access to the decisions of the other agents that affect their cost functions [1], while in a partial-decision information setting, the shared information is even more limited [3]. In both cases, the key aspect is that the agents can communicate without the need for a central coordinator. Alternatively, a payoff-based information setup has been considered in [22], [23], where the agents have access to the values of their own cost functions.

Besides being distributed, an algorithm for SGNEPs should converge under mild monotonicity assumptions and it should be relatively fast. For SVIs, there exist several methods that may be used for SGNEPs. Among others, one can consider the stochastic preconditioned forward–backward algorithm (SpFB) [21], [24] for its convergence speed and low computational cost.

TABLE I

ALGORITHMS FOR SGNEPs THAT CONVERGE UNDER ONLY MONOTONICITY (MON.) ARE MARKED WITH \checkmark . # PROX AND # F INDICATE THE NUMBER OF PROXIMAL STEPS AND THE NUMBER OF COMPUTATIONS OF THE STOCHASTICALLY APPROXIMATED PSEUDOGRAIENT PER ITERATION, RESPECTIVELY. ALL THESE ALGORITHMS USE THE VARIANCE REDUCTION AND A FIXED STEP SIZE

	SFBF [18]	SEG [19]	SSE [29]	SPRG [29]	SpFB [24]	SRFB
MON.	\checkmark	\checkmark	\checkmark	\times	\times	\checkmark
# prox	1	2	1	1	1	1
# F	2	2	2	1	1	1

The downside of this algorithm is that the pseudogradient mapping must be, monotone and, cocoercive [24], [25], strongly monotone [21], [26], or satisfy the variational stability [27]. Similarly, one can consider the stochastic projected reflected gradient (SPRG) scheme [28], [29] that is fast but requires the weak sharpness property (implied by cocoercivity) which is, however, hard to check on the problem data. Nonetheless, weakening the assumption on the pseudogradient to mere monotonicity translates into having computationally expensive algorithms. In this case, one could apply the extragradient (EG) scheme [19], [30] with two projection steps per iteration or the forward-backward-forward (FBF) algorithm [18] that has one projection but two evaluations of the pseudogradient for each iteration. Recently, the stochastic subgradient EG (SSE) algorithm has been considered, that use one proximal step but still two computations of the approximated pseudogradient [29]. These considerations are summarized in Table I, where we consider variance-reduced schemes with *fixed step sizes* for SGNEPs in comparison with our proposed SRFB algorithm.

Essentially, for merely monotone games, our SRFB algorithm is the only one to perform one proximal step and one stochastic approximation of the pseudogradient mapping with fixed step size.

Another option that uses one projection and one computation of the pseudogradient is the iterative Tikhonov regularization [7], which, however, is not proven to converge with variance reduction in SGNEPs and uses vanishing step sizes and vanishing regularization coefficients. Other algorithms have been proposed for saddle points problems (without coupling constraints) [31], convergent in the stochastic case if the mapping is strongly monotone [32], [33].

In light of the above considerations, our main contributions in this article are summarized as follows.

- In the context of, non-strictly/strongly monotone and noncocoercive, monotone SGNEPs, we propose the first distributed algorithm with a single proximal computation (e.g., projection) and a single stochastic approximation of the pseudogradient per iteration (Section IV).
- We show that our algorithm converges almost surely (a.s.) to a stochastic generalized Nash equilibrium under monotonicity of the pseudogradient with the SA scheme and the variance reduction (Section IV-B).
- For the stochastic non-GNEP, we show convergence with and without the variance reduction and under several variants of monotonicity (Section VI).

We emphasize that, unlike [7], [8], [24], we do not assume that the pseudogradient mapping is strictly/strongly monotone nor cocoercive or similar.

II. NOTATION AND PRELIMINARIES

A. Notation

Let \mathbb{R} indicate the set of real numbers and let $\bar{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$. $\langle \cdot, \cdot \rangle : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$ denotes the standard inner product, and $\|\cdot\|$ represents the associated Euclidean norm. We indicate that a matrix A is positive definite, i.e., $x^\top A x > 0$, with $A \succ 0$. Given a symmetric $\Phi \succ 0$, denote the Φ -induced inner product as $\langle x, y \rangle_\Phi = \langle \Phi x, y \rangle$. The associated Φ -induced norm, $\|\cdot\|_\Phi$, is defined as $\|x\|_\Phi = \sqrt{\langle \Phi x, x \rangle}$. $A \otimes B$ indicates the Kronecker product between matrices A and B . $\mathbf{0}_m$ indicates the vector with m entries all equal to 0. Given N vectors $x_1, \dots, x_N \in \mathbb{R}^n$, $\mathbf{x} := \text{col}(x_1, \dots, x_N) = [x_1^\top, \dots, x_N^\top]^\top$.

$J_F = (\text{Id} + F)^{-1}$ is the resolvent of the operator $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$ and Id indicates the identity operator. The set of fixed points of the operator F is $\text{fix } F = \{x \in \mathbb{R}^n : x = F(x)\}$. For a closed set $C \subseteq \mathbb{R}^n$, the mapping $\text{proj}_C : \mathbb{R}^n \rightarrow C$ denotes the projection onto C , i.e., $\text{proj}_C(x) = \arg\min_{y \in C} \|y - x\|$. The residual mapping is, in general, defined as $\text{res}(x^k) = \|x^k - \text{proj}_C(x^k - F(x^k))\|$. Let g be a proper, lower semi-continuous, convex function. We denote the subdifferential as the maximal monotone operator $\partial g(x) = \{u \in \Omega \mid (\forall y \in \Omega) \langle y - x, u \rangle + g(x) \leq g(y)\}$. The proximal operator is defined as $\text{prox}_g(v) = \arg\min_{u \in \Omega} \{g(u) + \frac{1}{2}\|u - v\|^2\} = J_{\partial g}(v)$. ι_C is the indicator function of the set C , that is, $\iota_C(x) = 1$ if $x \in C$ and $\iota_C(x) = 0$ otherwise. The set-valued mapping $N_C : \mathbb{R}^n \rightarrow \mathbb{R}^n$ denotes the normal cone operator for the set C , i.e., $N_C(x) = \emptyset$ if $x \notin C$, $\{v \in \mathbb{R}^n \mid \sup_{z \in C} v^\top(z - x) \leq 0\}$ otherwise. Given two sets A and B , with a slight abuse of notation, we indicate with $\text{col}(A, B)$ or $\begin{bmatrix} A \\ B \end{bmatrix}$ the Cartesian product $A \times B$. This notation is common in (S)GNEPs [1].

B. Operator Theory

Let us collect some notions on properties of operators. The definitions are taken from [34]. First, we recall that F is ℓ -Lipschitz continuous if, for $\ell > 0$, $\|F(x) - F(y)\| \leq \ell\|x - y\|$, for all $x, y \in \text{dom}(F)$.

Definition 1 (Monotone Operators): Given a mapping $F : \text{dom}(F) \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^n$, we say that F is (strictly) monotone if for all $x, y \in \text{dom}(F)$ ($x \neq y$) $\langle F(x) - F(y), x - y \rangle (>) \geq 0$; F is (strictly) pseudomonotone if for all $x, y \in \text{dom}(F)$ ($x \neq y$) $\langle F(y), x - y \rangle \geq 0 \Rightarrow \langle F(x), x - y \rangle (>) \geq 0$; β -cocoercive with $\beta > 0$, if for all $x, y \in \text{dom}(F)$ $\langle F(x) - F(y), x - y \rangle \geq \beta\|F(x) - F(y)\|^2$; and F is firmly nonexpansive if for all $x, y \in \text{dom}(F)$ $\|F(x) - F(y)\|^2 \leq \|x - y\|^2 - \|(\text{Id} - F)(x) - (\text{Id} - F)(y)\|^2$.

An example of firmly nonexpansive operator is the projection operator over a nonempty, compact, and convex set [35, Proposition 4.16]. We note that a firmly nonexpansive operator is also nonexpansive and firmly quasinonexpansive [35, Definition 4.1]. We note that if a mapping is β -cocoercive it is also $1/\beta$ -Lipschitz continuous [35, Remark 4.15].

III. STOCHASTIC GENERALIZED NASH EQUILIBRIUM PROBLEMS

We consider a set of noncooperative agents $\mathcal{I} = \{1, \dots, N\}$, each of them choosing its strategy $x_i \in \mathbb{R}^{n_i}$ with the aim of minimizing its local cost function within its feasible strategy set. The local decision set of each agent is indicated with Ω_i , i.e., for all $i \in \mathcal{I}$, $x_i \in \Omega_i \subseteq \mathbb{R}^{n_i}$. Besides the local set, each agent is subject to some joint feasibility constraints, $g(x) \leq 0$. Let us set $n = \sum_{i=1}^N n_i$ and $\Omega = \prod_{i=1}^N \Omega_i$, then, the collective feasible set can be written as

$$\mathcal{X} = \Omega \cap \{y \in \mathbb{R}^n \mid g(y) \leq 0_m\} \quad (1)$$

where $g : \mathbb{R}^n \rightarrow \mathbb{R}^m$ [5]. Let us also indicate with $\mathcal{X}_i(x_{-i})$ the piece of coupling constraints corresponding to agent i , which is affected by the decision variables of the other agents $x_{-i} = \text{col}((x_j)_{j \neq i})$.

Assumption 1 (Constraint Qualification): For each $i \in \mathcal{I}$, the set Ω_i is nonempty, closed, and convex. The set \mathcal{X} satisfies Slater's constraint qualification. ■

Assumption 2 (Separable Convex Coupling Constraints): The mapping g in (1) has a separable form, $g(x) := \sum_{i=1}^N g_i(x^i)$, for some convex differentiable functions $g_i : \mathbb{R}^{n_i} \rightarrow \mathbb{R}^m$, $i \in \mathcal{I}$ and it is ℓ_g -Lipschitz continuous. Its gradient ∇g is bounded, i.e., $\sup_{x \in \mathcal{X}} \|\nabla g(x)\| \leq B_{\nabla g}$.

The local cost function of agent i is defined as

$$\mathbb{J}_i(x_i, x_{-i}) = \mathbb{E}_{\xi}[J_i(x_i, x_{-i}, \xi(\varpi))] + f_i(x_i) \quad (2)$$

for some measurable function $J_i : \mathbb{R}^{n_i} \times \mathbb{R}^d \rightarrow \mathbb{R}$. The cost function \mathbb{J}_i of agent $i \in \mathcal{I}$ depends on the local variable x_i , the decisions of the other players x_{-i} , and the random variable $\xi : \Xi \rightarrow \mathbb{R}^d$ that expresses the uncertainty. \mathbb{E}_{ξ} represents the mathematical expectation with respect to the distribution of the random variable $\xi(\varpi)^1$ in the probability space $(\Xi, \mathcal{F}, \mathbb{P})$. We assume that $\mathbb{E}[J_i(x, \xi)]$ is well defined for all feasible $x \in \mathcal{X}$ [6]. Moreover, the cost function presents the typical splitting in a smooth part and a nonsmooth part. The latter is indicated with $f_i : \mathbb{R}^{n_i} \rightarrow \bar{\mathbb{R}}$ and it can represent a local cost or local constraints via the indicator function, i.e., $f_i(x_i) = \iota_{\Omega_i}(x_i)$.

Assumption 3 (Cost Function Convexity): For each $i \in \mathcal{I}$, the function f_i in (2) is lower semicontinuous and convex and $\text{dom}(f_i) = \Omega_i$. For each $i \in \mathcal{I}$ and $x_{-i} \in \mathcal{X}_{-i}$, the function $\mathbb{J}_i(\cdot, x_{-i})$ is convex and continuously differentiable. ■

Given the decision variables of the other agents x_{-i} , each agent i aims at choosing a strategy x_i that solves its local optimization problem, i.e.,

$$\forall i \in \mathcal{I} : \begin{cases} \min_{x_i \in \Omega_i} \mathbb{J}_i(x_i, x_{-i}) \\ \text{s.t.} \quad g(x_i, x_{-i}) \leq 0. \end{cases} \quad (3)$$

From a game-theoretic perspective, the solution concept that we are seeking is that of stochastic generalized Nash equilibrium (SGNE).

Definition 2: A stochastic generalized Nash equilibrium is a collective strategy $x^* \in \mathcal{X}$, such that for all $i \in \mathcal{I}$

$$\mathbb{J}_i(x_i^*, x_{-i}^*) \leq \inf\{\mathbb{J}_i(y, x_{-i}^*) \mid y \in \mathcal{X}_i(x_{-i}^*)\}.$$

¹From now on, we use ξ instead of $\xi(\varpi)$ and \mathbb{E} instead of \mathbb{E}_{ξ} .

In other words, an SGNE is a set of strategies, where no agent can decrease its objective function by unilaterally deviating from its decision. To guarantee that an SGNE exists, we make further assumptions on the cost functions [6].

Assumption 4 (Convexity and Measurability): For each $i \in \mathcal{I}$ and for each $\xi \in \Xi$, the function $J_i(\cdot, x_{-i}, \xi)$ is convex, Lipschitz continuous, and continuously differentiable. The function $J_i(x_i, x_{-i}, \cdot)$ is measurable and for each x_{-i} , the Lipschitz constant $\ell_i(x_{-i}, \xi)$ is integrable in ξ . ■

Existence of an SGNE of the game in (3) is guaranteed, under Assumptions 1–4, by [6, Section 3.1], while uniqueness does not hold in general [6, Section 3.2]. Within all the possible Nash equilibria, we focus on those that corresponds to the solution set of an appropriate SVI. To this aim, let us denote the pseudogradient mapping as

$$\mathbb{F}(x) = \text{col}(\mathbb{E}[\nabla_{x_1} J_1(x_1, x_{-1})], \dots, \mathbb{E}[\nabla_{x_N} J_N(x_N, x_{-N})]) \quad (4)$$

and let $\partial f(x) = \text{col}(\partial f_1(x_1), \dots, \partial f_N(x_N))$. The possibility to exchange the expected value and the pseudogradient \mathbb{F} in (4) is guaranteed by Assumption 4. Then, the associated SVI reads as

$$\langle \mathbb{F}(x^*), x - x^* \rangle + \sum_{i \in \mathcal{I}} \{f_i(x_i) - f_i(x_i^*)\} \geq 0, \text{ for all } x \in \mathcal{X} \quad (5)$$

where \mathcal{X} is the intersection of the local and coupling constraints as in (1). When Assumptions 1–4 hold, any solution of SVI(\mathcal{X}, \mathbb{F}) in (5) is an SGNE of the game in (3) while vice versa does not hold in general. This is because a game may have a Nash equilibrium while the corresponding VI may have no solution [36, Proposition 12.7].

Assumption 5 (Existence of a Variational Equilibrium): The SVI in (5) has at least one solution, i.e., $\text{SOL}(\mathcal{X}, \mathbb{F}) \neq \emptyset$. ■

Remark 1: Assumption 5 is satisfied if the sets Ω_i and $i \in \mathcal{I}$ are compact [34, Corollary 2.2.5]. ■

We call variational equilibria (v-SGNE) the SGNE that are also solution of the associated SVI, namely, the solution of the SVI(\mathcal{X}, \mathbb{F}) in (5) with \mathbb{F} in (4) and \mathcal{X} in (1).

In the remaining part of this section, we recast the SGNEP as a monotone inclusion, i.e., the problem of finding a zero of a set-valued monotone operator. To this aim, we characterize the SGNE of the game in terms of the KKT conditions of the coupled optimization problems in (3). Let us define the Lagrangian function, for each $i \in \mathcal{I}$, with

$$\mathcal{L}_i(x, \lambda_i) := \mathbb{J}_i(x_i, x_{-i}) + f_i(x_i) + \lambda_i^\top g(x_i^*, x_{-i}^*)$$

where $\lambda_i \in \mathbb{R}_{\geq 0}^m$ is the Lagrangian dual variable associated with the coupling constraints. Then, a set of strategies x^* is an SGNE if and only if the KKT conditions are satisfied [37, Theorem 4.6].

Moreover, according to [38, Theorem 3.1] and [39, Theorem 3.1], the v-SGNE are those equilibria such that the shared constraints have the same dual variable for all the agents, i.e., $\lambda_i = \lambda$ for all $i \in \mathcal{I}$, and solve the SVI(\mathcal{X}, \mathbb{F}) in (5). Thus, x^* is a v-SGNE if the following KKT inclusions, for all $i \in \mathcal{I}$, are

Algorithm 1. Stochastic Relaxed Forward–Backward (SRFB).

Initialization: $x_i^0 \in \Omega_i, \lambda_i^0 \in \mathbb{R}_{\geq 0}^m$, and $z_i^0 \in \mathbb{R}^m$.

Iteration k : Agent i

(1) Updates the variables

$$\begin{aligned}\bar{x}_i^k &= (1 - \delta)x_i^k + \delta\bar{x}_i^{k-1} \\ \bar{z}_i^k &= (1 - \delta)z_i^k + \delta\bar{z}_i^{k-1} \\ \bar{\lambda}_i^k &= (1 - \delta)\lambda_i^k + \delta\bar{\lambda}_i^{k-1}\end{aligned}\quad (7)$$

(2) Receives x_j^k for all $j \in \mathcal{N}_i^J$ and z_j^k, λ_j^k for $j \in \mathcal{N}_i^\lambda$, then updates

$$\begin{aligned}x_i^{k+1} &= \text{prox}_{f_i}[\bar{x}_i^k - \alpha_i(\hat{F}_i(x_i^k, \mathbf{x}_{-i}^k, \xi_i^k) + \nabla g_i(x_i)^T \lambda_i^k)] \\ z_i^{k+1} &= \bar{z}_i^k - \nu_i \sum_{j \in \mathcal{N}_i^\lambda} w_{i,j}(\lambda_i^k - \lambda_j^k) \\ \lambda_i^{k+1} &= \text{proj}_{\mathbb{R}_{\geq 0}^m} \{ \bar{\lambda}_i^k + \tau_i g_i(x_i^k) \\ &\quad - \tau \sum_{j \in \mathcal{N}_i^\lambda} w_{i,j}[(z_i^k - z_j^k) - (\lambda_i^k - \lambda_j^k)] \}\end{aligned}\quad (8)$$

satisfied for some $\lambda \in \mathbb{R}_{\geq 0}^m$

$$\begin{cases} 0 \in \mathbb{E}[\nabla_{x_i} J_i(x_i^*, \mathbf{x}_{-i}^*, \xi)] + \partial f_i(x_i^*) + \nabla g(x_i^*, \mathbf{x}_{-i}^*)^T \lambda \\ 0 \in -g(\mathbf{x}^*) + N_{\mathbb{R}_{\geq 0}^m}(\lambda). \end{cases}\quad (6)$$

IV. DISTRIBUTED STOCHASTIC RELAXED FORWARD–BACKWARD ALGORITHM

In this section, we describe the details that lead to the distributed iterations in Algorithm 1 which include an averaging step (7) and a proximal step (8). The averaging step induces some inertia but it allows us to prove convergence under mild monotonicity assumptions. Moreover, for the decision variable x_i , the proximal update guarantees that the local constraints are always satisfied while the coupling constraints are enforced asymptotically through the dual variable λ_i , which should be nonnegative. The variable z_i is an auxiliary variable to force consensus on the dual variables. We note that to update the primal variable, we use an approximation \hat{F} of the pseudogradient mapping \mathbb{F} , characterized in Section IV-A.

We suppose that each player i knows its local data Ω_i and their part $\mathcal{X}_i(\mathbf{x}_{-i})$. We also suppose that the agents have access to a pool of samples of the random variable and are able to compute, given the actions of the other players \mathbf{x}_{-i} , the pseudogradient \mathbb{F} of their own cost functions (or an approximation \hat{F}). The set of agents j , whose decisions affect the cost function of agent i , are denoted by \mathcal{N}_i^J . Specifically, $j \in \mathcal{N}_i^J$ if the function $J_i(x_i, \mathbf{x}_{-i})$ explicitly depends on x_j . Under these premises, Algorithm 1 is distributed in the sense that each agent knows its own problem data and variables and communicates with

the other agents only to access the information to compute \mathbb{F} (full-decision information setup [1]).

Let us also introduce the graph $\mathcal{G}^\lambda = (\mathcal{I}, \mathcal{E}^\lambda)$ through which a local copy of the dual variable is shared. According to [38, Theorem 3.1] and [39, Theorem 3.1], we seek for a v-SGNE with the consensus of the dual variables. Therefore, along with the dual variable, agents share through \mathcal{G}^λ a copy of an auxiliary variable $z_i \in \mathbb{R}^m$ whose role is to force consensus. A deeper insight on this variable is given later in this section. The set of edges \mathcal{E}^λ of the multiplier graph \mathcal{G}^λ is given by $(i, j) \in \mathcal{E}^\lambda$ if player j shares its $\{\lambda_j, z_j\}$ with player i . For all $i \in \mathcal{I}$, the neighboring agents in \mathcal{G}^λ form the set $\mathcal{N}_i^\lambda = \{j \in \mathcal{I} : (i, j) \in \mathcal{E}^\lambda\}$. In this way, each agent controls its own decision variable and a local copy of the dual variable λ_i and of the auxiliary variable z_i and, through the graphs, it obtains the other agents variables.

Assumption 6 (Graph Connectivity): The multiplier graph \mathcal{G}^λ is undirected and connected. ■

The weighted adjacency matrix associated to \mathcal{G}^λ is denoted with $W \in \mathbb{R}^{N \times N}$. Then, letting $D = \text{diag}\{d_1, \dots, d_N\}$, where $d_i = \sum_{j=1}^N w_{i,j}$ is the degree of agent i , the associated Laplacian is given by $L = D - W \in \mathbb{R}^{N \times N}$. It follows from Assumption 6 that $L = L^T$.

Let us now rewrite the KKT conditions in (6) in compact form as

$$0 \in \mathcal{T}(\mathbf{x}, \boldsymbol{\lambda}) = \begin{bmatrix} \mathbb{F}(\mathbf{x}) + \partial f(\mathbf{x}) + \nabla g(\mathbf{x})^T \boldsymbol{\lambda} \\ N_{\mathbb{R}_{\geq 0}^m}(\boldsymbol{\lambda}) - g(\mathbf{x}) \end{bmatrix}. \quad (9)$$

$\mathcal{T} : \mathcal{X} \times \mathbb{R}_{\geq 0}^m \Rightarrow \mathbb{R}^n \times \mathbb{R}^m$ is a set-valued mapping and it follows that the v-SGNE of the game in (3) correspond to the zeros of the mapping \mathcal{T} which can be split as a summation of two operators, $\mathcal{T} = \mathcal{A} + \mathcal{B}$, where,

$$\begin{aligned}\mathcal{A} : \begin{bmatrix} \mathbf{x} \\ \boldsymbol{\lambda} \end{bmatrix} &\mapsto \begin{bmatrix} \mathbb{F}(\mathbf{x}) \\ 0 \end{bmatrix} + \begin{bmatrix} \nabla g(\mathbf{x})^T \boldsymbol{\lambda} \\ -g(\mathbf{x}) \end{bmatrix} \\ \mathcal{B} : \begin{bmatrix} \mathbf{x} \\ \boldsymbol{\lambda} \end{bmatrix} &\mapsto \begin{bmatrix} \partial f(\mathbf{x}) \\ N_{\mathbb{R}_{\geq 0}^m}(\boldsymbol{\lambda}) \end{bmatrix}.\end{aligned}\quad (10)$$

To force consensus on the dual variables, Yi and Pavel [1] proposed the Laplacian constraint $\mathbf{L}\boldsymbol{\lambda} = 0$. This is why, to preserve monotonicity, we expand the two operators \mathcal{A} and \mathcal{B} in (10) and introduce the auxiliary variable $\mathbf{z} = \text{col}(z_1, \dots, z_N) \in \mathbb{R}^{Nm}$. Let us first define $\mathbf{L} = L \otimes \text{Id}_m \in \mathbb{R}^{Nm \times Nm}$, where L is the Laplacian of \mathcal{G}^λ and $\boldsymbol{\lambda} = \text{col}(\lambda_1, \dots, \lambda_N) \in \mathbb{R}^{Nm}$. Then, the two operators \mathcal{A} and \mathcal{B} in (10) can be rewritten as

$$\begin{aligned}\bar{\mathcal{A}} : \begin{bmatrix} \mathbf{x} \\ \mathbf{z} \\ \boldsymbol{\lambda} \end{bmatrix} &\mapsto \begin{bmatrix} \mathbb{F}(\mathbf{x}) \\ 0 \\ \mathbf{L}\boldsymbol{\lambda} \end{bmatrix} + \begin{bmatrix} \nabla g(\mathbf{x})^T \boldsymbol{\lambda} \\ \mathbf{L}\mathbf{z} \\ -g(\mathbf{x}) - \mathbf{L}\mathbf{z} \end{bmatrix} \\ \bar{\mathcal{B}} : \begin{bmatrix} \mathbf{x} \\ \mathbf{z} \\ \boldsymbol{\lambda} \end{bmatrix} &\mapsto \begin{bmatrix} \partial f(\mathbf{x}) \\ \mathbf{0} \\ N_{\mathbb{R}_{\geq 0}^m}(\boldsymbol{\lambda}) \end{bmatrix}.\end{aligned}\quad (11)$$

From now on, we indicate the state variable as $\boldsymbol{\omega} = \text{col}(\mathbf{x}, \mathbf{z}, \boldsymbol{\lambda})$. The properties of the operators in (11) depends on the properties of \mathbb{F} and are described in the next section. We here show that the zeros of $\bar{\mathcal{A}} + \bar{\mathcal{B}}$ are the same as the zeros of \mathcal{T} in (9).

Lemma 1: Let Assumptions 1–6 hold and consider the operators \mathcal{A} and \mathcal{B} in (10), and the operators $\bar{\mathcal{A}}$ and $\bar{\mathcal{B}}$ in (11). Then, the following statements hold.

- (i) Given any $\omega^* \in \text{zer}(\bar{\mathcal{A}} + \bar{\mathcal{B}})$, \mathbf{x}^* is a v-SGNE of game in (3), i.e., \mathbf{x}^* solves the SVI(\mathcal{X}, \mathbb{F}) in (5). Moreover, $\lambda^* = \mathbf{1}_N \otimes \lambda^*$ and $(\mathbf{x}^*, \lambda^*)$ satisfy the KKT condition in (6), i.e., $\text{col}(\mathbf{x}^*, \lambda^*) \in \text{zer}(\mathcal{A} + \mathcal{B})$.
- (ii) $\text{zer}(\mathcal{A} + \mathcal{B}) \neq \emptyset$ and $\text{zer}(\bar{\mathcal{A}} + \bar{\mathcal{B}}) \neq \emptyset$.

Proof: See Appendix I. ■

Since the distribution of the random variable is unknown, the expected value mapping can be hard to compute. Therefore, we take an approximation of the pseudogradient mapping, properly defined in Section IV-A. Therefore, in what follows, we replace $\bar{\mathcal{A}}$ with

$$\hat{\mathcal{A}} : \begin{bmatrix} (\mathbf{x}, \xi) \\ \mathbf{z} \\ \lambda \end{bmatrix} \mapsto \begin{bmatrix} \hat{F}(\mathbf{x}, \xi) \\ 0 \\ \mathbf{L}\lambda \end{bmatrix} + \begin{bmatrix} \nabla g(\mathbf{x})^\top \lambda \\ \mathbf{L}\mathbf{z} \\ -g(\mathbf{x}) - \mathbf{L}\mathbf{z} \end{bmatrix} \quad (12)$$

where \hat{F} is an approximation of the expected value mapping \mathbb{F} in (4), given some realizations of the random vector ξ .

Then, Algorithm 1 can be written in compact form as [14]

$$\begin{aligned} \bar{\omega}^k &= (1 - \delta)\omega^k + \delta\bar{\omega}^{k-1} \\ \omega^{k+1} &= (\text{Id} + \Phi^{-1}\bar{\mathcal{B}})^{-1}(\bar{\omega}^k - \Phi^{-1}\hat{\mathcal{A}}(\omega^k)) \end{aligned} \quad (13)$$

where $\Phi \succ 0$ contains the inverse of step size sequences

$$\Phi = \text{diag}(\alpha^{-1}, \nu^{-1}, \sigma^{-1}) \quad (14)$$

and $\alpha^{-1}, \nu^{-1}, \sigma^{-1}$ are diagonal matrices.

A. Approximation Scheme

We now enter the details of the approximation introduced in Algorithm 1. We use an SA scheme with variance reduction; hence, we suppose that, at each iteration k , the agents have access to a pool of samples of the random variable and are able to compute an approximation of $\mathbb{F}(\mathbf{x})$ of the form

$$\begin{aligned} \hat{F}(\mathbf{x}, \xi) &= \text{col}(\hat{F}_i(\mathbf{x}, \xi)) \\ &= \text{col} \left(\frac{1}{S_k} \sum_{t=1}^{S_k} \nabla_{x_1} J_1(\mathbf{x}, \xi_1^{(t)}), \dots, \frac{1}{S_k} \sum_{t=1}^{S_k} \nabla_{x_N} J_N(\mathbf{x}, \xi_N^{(t)}) \right) \end{aligned} \quad (15)$$

where $\xi = \text{col}(\bar{\xi}_1, \dots, \bar{\xi}_N)$, for all $i \in \mathcal{I}$, $\bar{\xi}_i = \text{col}(\xi_i^{(1)}, \dots, \xi_i^{(S_k)})$ and ξ is an independent identically distributed (i.i.d.) sequence of random variables drawn from \mathbb{P} . Approximations of the form (15) are very common in Monte–Carlo simulation approaches and machine learning [18]; they are easy to obtain in case we are able to sample from the measure \mathbb{P} . Typical assumptions when using an approximation as in (15) are related to the choice of a proper batch size sequence S_k [18], [19].

Assumption 7 (Increasing Batch Size): The batch size sequence $(S_k)_{k \geq 1}$ is such that, for some $c, k_0, a > 0$,

$$S_k \geq c(k + k_0)^{a+1}.$$

Form Assumption 7, it follows that $1/S_k$ is summable, which is a standard assumption when used in combination with the forthcoming variance reduction assumption to control the stochastic error [18], [19]. For $k \geq 0$, the approximation error is defined as

$$\epsilon_k = \hat{F}(\mathbf{x}^k, \xi^k) - \mathbb{F}(\mathbf{x}^k).$$

Remark 2: Since there is no uncertainty in the constraints, we indicate with $\varepsilon_k = \text{col}(\epsilon_k, 0, 0)$ the error on the extended operator, i.e., $\hat{\mathcal{A}}(\omega^k, \xi^k) - \mathcal{A}(\omega^k) = \varepsilon_k$. $\hat{\mathcal{A}}$ is the operator in (12) with approximation \hat{F} in (15). ■

In the stochastic framework, there are usually assumptions on the expected value and variance of the stochastic error ϵ_k [7], [19], [30]. Let us define the filtration $\mathcal{F} = \{\mathcal{F}_k\}$, that is, a family of σ -algebras such that $\mathcal{F}_0 = \sigma(X_0)$ and $\mathcal{F}_k = \sigma(X_0, \xi_1, \xi_2, \dots, \xi_k) \quad \forall k \geq 1$, such that $\mathcal{F}_k \subseteq \mathcal{F}_{k+1}$ for all $k \in \mathbb{N}$. In other words, \mathcal{F}_k contains the information up to time k .

Assumption 8 (Zero mean error): The stochastic error is such that, for all $k \in \mathbb{N}$, a.s., $\mathbb{E}[\epsilon^k | \mathcal{F}_k] = 0$. ■

Assumption 9 (Variance control): There exist $p \geq 2, \sigma_0 \geq 0$ and a measurable locally bounded function $\sigma : \text{SOL}(\mathcal{X}, \mathbb{F}) \rightarrow \mathbb{R}$ such that for all $(\mathbf{x}, \mathbf{x}^*) \in \mathcal{X} \times \text{SOL}(\mathcal{X}, \mathbb{F})$ ■

$$\mathbb{E}[\|\hat{F}(\mathbf{x}, \xi) - \mathbb{F}(\mathbf{x})\|^p]^{1/p} \leq \sigma(\mathbf{x}^*) + \sigma_0 \|\mathbf{x} - \mathbf{x}^*\|.$$

Remark 3: When the feasible set is compact (as in Remark 1) a uniform bounded variance, i.e., for some $\sigma > 0$

$$\sup_{\mathbf{x} \in \mathcal{X}} \mathbb{E}[\|\hat{F}(\mathbf{x}, \xi) - \mathbb{F}(\mathbf{x})\|^2] \leq \sigma^2$$

can be considered instead of Assumption 9. ■

B. Convergence Analysis

Now, we study the convergence of the algorithm. First, to ensure that $\bar{\mathcal{A}}$ and $\bar{\mathcal{B}}$ have the properties that we use for the analysis, we make the following assumption.

Assumption 10 (Monotonicity): \mathbb{F} in (4) is monotone and $\ell_{\mathbb{F}}$ -Lipschitz continuous for some $\ell_{\mathbb{F}} > 0$. ■

Lemma 2: Let Assumptions 6 and 10 hold and let $\Phi \succ 0$. Then, the operators $\bar{\mathcal{A}}$ and $\bar{\mathcal{B}}$ in (11) have the following properties.

- 1) $\bar{\mathcal{A}}$ is monotone and $\ell_{\bar{\mathcal{A}}}$ -Lipschitz continuous.
- 2) The operator $\bar{\mathcal{B}}$ is maximally monotone.
- 3) $\Phi^{-1}\bar{\mathcal{A}}$ is monotone and ℓ_{Φ} -Lipschitz continuous.
- 4) $\Phi^{-1}\bar{\mathcal{B}}$ is maximally monotone.

Proof: See Appendix I. ■

Finally, we indicate how to choose the parameters of the algorithm. This is fundamental for the convergence analysis and, in practice, for the convergence speed.

Assumption 11 (Averaging Parameter): The averaging parameter δ in (13) is such that

$$\frac{1}{\varphi} \leq \delta \leq 1$$

where $\varphi = \frac{1+\sqrt{5}}{2}$ is the golden ratio. ■

Assumption 12 (Step Size Bound): The steps size is such that

$$0 < \|\Phi^{-1}\| \leq \frac{1}{2\delta(2\ell_{\bar{\mathcal{A}}}+1)}$$

where $\ell_{\bar{\mathcal{A}}}$ is the Lipschitz constant of $\bar{\mathcal{A}}$ as in Lemma 2. ■

We are now ready to state our convergence result.

Theorem 1: Let Assumptions 1–12 hold. Then, the sequence $(\mathbf{x}^k)_{k \in \mathbb{N}}$ generated by Algorithm 1 with \hat{F} as in (15) converges a.s. to a v-SGNE of the game in (3).

Proof: See Appendix III. ■

In the following, let us consider the case where the local nonsmooth cost is determined by the local constraints, i.e., $f_i(x_i) = \iota_{\Omega_i}(x_i)$. Then, the problem is slightly different and we can show that the algorithm converges under a weaker assumption than monotonicity.

The first difference is that the operator $\bar{\mathcal{B}}$ is now given by

$$\bar{\mathcal{B}} : \begin{bmatrix} \mathbf{x} \\ \mathbf{z} \\ \boldsymbol{\lambda} \end{bmatrix} \mapsto \begin{bmatrix} \mathbf{N}_{\Omega}(\mathbf{x}) \\ \mathbf{0} \\ \mathbf{N}_{\mathbb{R}_{\geq 0}^m}(\boldsymbol{\lambda}) \end{bmatrix}$$

hence, we have a projection instead of the proximal operator

$$\begin{aligned} \bar{\omega}^k &= (1 - \delta)\omega^k + \delta\bar{\omega}^{k-1} \\ \omega^{k+1} &= \text{proj}_{\mathcal{Z}'} [\bar{\omega}^k - \Phi^{-1}\hat{\mathcal{A}}(\omega^k)] \end{aligned} \quad (16)$$

where $\mathcal{Z}' = \Omega \times \mathbb{R}^{mN} \times \mathbb{R}_{\geq 0}^{mN}$. We also call $\mathcal{Z} = \mathcal{X} \times \mathbb{R}^{mN} \times \mathbb{R}_{\geq 0}^{mN}$ and \mathcal{Z}^* the set of v-SGNE, i.e., $\mathcal{Z}^* = \text{zer}(\bar{\mathcal{A}} + \bar{\mathcal{B}})$. To show convergence, let the mapping $\bar{\mathcal{A}}$ satisfy the following assumption.

Assumption 13 (Almost Restricted Pseudomonotonicity): The operator $\bar{\mathcal{A}}$ in (11) is such that for all $(\omega, \omega^*) \in \mathcal{Z} \times \mathcal{Z}^*$,

$$\langle \bar{\mathcal{A}}(\omega), \omega - \omega^* \rangle \geq 0.$$

Remark 4: The property in Assumption 13 is implied by both monotonicity and pseudomonotonicity, but it does not necessarily hold for $\bar{\mathcal{A}}$ if we assume it directly on \mathbb{F} . It corresponds to the concept of weak solution of a VI, compared to that of strong solution as in (5) [40]. This assumption is also used in [18] and [41]; an example of a mapping that satisfies (13) is in [40, Equation (2.4)]. ■

We can now state the corresponding convergence result.

Corollary 1: Let Assumptions 1–9 and 11–13 hold. Then, the sequence generated by Algorithm 1 in (16) with $\hat{\mathcal{A}}$ as in Remark 2 converges a.s. to a v-SGNE of the game in (3).

Proof: See Appendix III. ■

V. CONVERGENCE UNDER COCOERCIVITY

Having a mild monotonicity condition on the pseudogradient implies taking a small, although constant, step size sequence. However, if the pseudogradient mapping satisfies a stronger monotonicity assumption, a larger step size can be chosen. One possibility is that the pseudogradient satisfies the cut property (described in detail in Remark 6), which is hard to check on the problem data but it follows directly from cocoercivity.

Assumption 14 (Cocoercivity): \mathbb{F} in (4) is β -cocoercive for some $\beta > 0$. ■

Algorithm 2: Stochastic Relaxed Preconditioned Forward–Backward (SRpFB).

Initialization: $x_i^0 \in \Omega_i, \lambda_i^0 \in \mathbb{R}_{\geq 0}^m$, and $z_i^0 \in \mathbb{R}^m$. Iteration k : Agent i

1) Updates the variables

$$\bar{x}_i^k = (1 - \delta)x_i^k + \delta\bar{x}_i^{k-1}$$

$$\bar{z}_i^k = (1 - \delta)z_i^k + \delta\bar{z}_i^{k-1}$$

$$\bar{\lambda}_i^k = (1 - \delta)\lambda_i^k + \delta\bar{\lambda}_i^{k-1}$$

2) Receives x_j^k for all $j \in \mathcal{N}_i^J, \lambda_j^k$ for $j \in \mathcal{N}_i^\lambda$, then updates

$$x_i^{k+1} = \text{proj}_{\Omega_i} [\bar{x}_i^k - \alpha_i (\hat{F}_i(x_i^k, \mathbf{x}_{-i}^k, \xi_i^k) - A_i^\top \lambda_i^k)]$$

$$z_i^{k+1} = \bar{z}_i^k - \nu_i \sum_{j \in \mathcal{N}_i^\lambda} w_{i,j} (\lambda_i^k - \lambda_j^k)$$

3) Receives $z_{j,k+1}$ for all $j \in \mathcal{N}_i^\lambda$, then updates

$$\begin{aligned} \lambda_i^{k+1} &= \text{proj}_{\mathbb{R}_{\geq 0}^m} [\bar{\lambda}_i^k + \sigma_i (A_i(2x_i^{k+1} - x_i^k) - b_i) \\ &\quad + \sigma_i \sum_{j \in \mathcal{N}_i^\lambda} w_{i,j} (2(z_i^{k+1} - z_j^{k+1}) - (z_i^k - z_j^k)) \\ &\quad - \sigma_i \sum_{j \in \mathcal{N}_i^\lambda} w_{i,j} (\lambda_i^k - \lambda_j^k)] \end{aligned}$$

For instance, every symmetric, affine, and monotone mapping is cocoercive (see also [34, Example 2.9.25]).

The operator splitting that we used in Section IV is not cocoercive, even when the mapping \mathbb{F} is. For this reason, here we have to consider a different splitting. Moreover, to obtain distributed iterations, we consider affine coupling constraints.

Assumption 15 (Affine Coupling Constraints): $g(\mathbf{x}) = \mathbf{A}\mathbf{x} - \mathbf{b}$, where $\mathbf{A} = [A_1, \dots, A_N] \in \mathbb{R}^{m \times n}$ and $\mathbf{b} \in \mathbb{R}^m$. ■

[38, Theorem 3.1] and [39, Theorem 3.1] hold also in this case. To obtain the extended operator as in (11), let $\mathbf{A} = \text{diag}\{A_1, \dots, A_N\} \in \mathbb{R}^{Nm \times n}$, where A_i represents the individual coupling constraints and let $\mathbf{b} \in \mathbb{R}^{Nm \times n}$. Then, the operator \mathcal{T} in (9) can be split as

$$\begin{aligned} \bar{\mathcal{C}} : \begin{bmatrix} \mathbf{x} \\ \mathbf{z} \\ \boldsymbol{\lambda} \end{bmatrix} &\mapsto \begin{bmatrix} \mathbb{F}(\mathbf{x}) \\ \mathbf{0} \\ \mathbf{L}\boldsymbol{\lambda} + \mathbf{b} \end{bmatrix} \\ \bar{\mathcal{D}} : \begin{bmatrix} \mathbf{x} \\ \mathbf{z} \\ \boldsymbol{\lambda} \end{bmatrix} &\mapsto \begin{bmatrix} \mathbf{N}_{\Omega}(\mathbf{x}) \\ \mathbf{0} \\ \mathbf{N}_{\mathbb{R}_{\geq 0}^m}(\boldsymbol{\lambda}) \end{bmatrix} + \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{A}^\top \\ \mathbf{0} & \mathbf{0} & \mathbf{L} \\ -\mathbf{A} & -\mathbf{L} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{z} \\ \boldsymbol{\lambda} \end{bmatrix} \end{aligned} \quad (17)$$

where $\bar{\mathcal{C}}$ is the first part of the operator $\bar{\mathcal{A}}$ in (11) and the second part is in $\bar{\mathcal{D}}$, including the linear constraints. Lemma 1 guarantees that a zero of $\bar{\mathcal{C}} + \bar{\mathcal{D}}$ exists. Moreover, $\bar{\mathcal{C}}$ and $\bar{\mathcal{D}}$ in (17) have the following properties.

Lemma 3: Let Assumption 14 hold and let $\Phi > 0$. The operators $\bar{\mathcal{C}}$ and $\bar{\mathcal{D}}$ in (17) have the following properties.

- (i) $\bar{\mathcal{C}}$ is θ -cocoercive, where $0 < \theta \leq \min\{\frac{1}{2d^*}, \beta\}$ and d^* is the maximum weighted degree of \mathcal{G}^λ .
- (ii) The operator $\bar{\mathcal{D}}$ is maximally monotone.
- (iii) $\Phi^{-1}\bar{\mathcal{C}}$ is $\theta\gamma$ -cocoercive, where $\gamma = \frac{1}{|\Phi^{-1}|}$.
- (iv) $\Phi^{-1}\bar{\mathcal{D}}$ is maximally monotone.

Proof: See Appendix I. ■

Also in this case, we use an approximation to compute the expected value, therefore, similarly to (12),

$$\hat{\mathcal{C}} : \begin{bmatrix} (\mathbf{x}, \xi) \\ \mathbf{z} \\ \boldsymbol{\lambda} \end{bmatrix} \mapsto \begin{bmatrix} \hat{F}(\mathbf{x}, \xi) \\ 0 \\ \mathbf{L}\boldsymbol{\lambda} + \mathbf{b} \end{bmatrix}. \quad (18)$$

In this case, the SRFB algorithm is given by

$$\begin{aligned} \bar{\omega}^k &= (1 - \delta)\omega^k + \delta\bar{\omega}^{k-1} \\ \omega^{k+1} &= (\text{Id} + \Psi^{-1}\bar{\mathcal{D}})^{-1}(\bar{\omega}^k - \Psi^{-1}\hat{\mathcal{C}}(\omega^k)) \end{aligned} \quad (19)$$

where the preconditioning matrix Ψ is defined as

$$\Psi = \begin{bmatrix} \alpha^{-1} & 0 & -\mathbf{A}^\top \\ 0 & \nu^{-1} & -\mathbf{L} \\ -\mathbf{A} & -\mathbf{L} & \sigma^{-1} \end{bmatrix} \quad (20)$$

with α^{-1} , ν^{-1} , σ^{-1} defined as in (14) and \mathbf{A} and \mathbf{L} are, respectively, the extended constraints and Laplacian matrix. We note that this type of preconditioning cannot be used for nonlinear coupling constraints $g(\mathbf{x}) \leq 0$ as in (1).

The distributed SRFB iterations read as in Algorithm 2. We note that Algorithm 2 differs from Algorithm 1 in the computation of the dual variable $\boldsymbol{\lambda}^{k+1}$ which, in this case, depends also on the variables \mathbf{x}^{k+1} and \mathbf{z}^{k+1} .

A. Convergence Analysis

Since the matrix Ψ must be positive definite [42], we postulate the following assumption.

Assumption 16 (Step Size Sequence): The step size sequence is such that, given $\gamma > 0$, for every agent $i \in \mathcal{I}$

$$\begin{aligned} 0 < \alpha_i &\leq \left(\gamma + \max_{j \in \{1, \dots, n_i\}} \sum_{k=1}^m |[A_i^\top]_{jk}| \right)^{-1} \\ 0 < \nu_i &\leq (\gamma + 2d_i)^{-1} \\ 0 < \sigma_i &\leq \left(\gamma + 2d_i + \max_{j \in \{1, \dots, m\}} \sum_{k=1}^{n_i} |[A_i]_{jk}| \right)^{-1} \end{aligned}$$

where $[A_i^\top]_{jk}$ indicates the entry (j, k) of the matrix A_i^\top . Moreover,

$$\|\Psi^{-1}\| \leq \frac{1}{\delta(2\ell_{\bar{\mathcal{C}}} - 1)}$$

where δ is the averaging parameter and $\ell_{\bar{\mathcal{C}}}$ is the Lipschitz constant of $\bar{\mathcal{C}}$. ■

We are now ready to state the convergence result.

Theorem 2: Let Assumptions 1–11, 14 and 16 hold. Then, the sequence $(\mathbf{x}_k)_{k \in \mathbb{N}}$ generated by Algorithm 2 with \hat{F} as in (15) converges a.s. to a v-SGNE of the game in (3).

Proof: See Appendix IV. ■

Algorithm 3: Stochastic Relaxed Forward–Backward.

Initialization: $\mathbf{x}_i^0 \in \Omega_i$

Iteration k : Agent i receives \mathbf{x}_j^k for all $j \in \mathcal{N}_i^J$, then updates

$$\begin{aligned} \bar{\mathbf{x}}_i^k &= (1 - \delta)\mathbf{x}_i^k + \delta\bar{\mathbf{x}}_i^{k-1} \\ \mathbf{x}_i^{k+1} &= \text{proj}_{\Omega_i}[\bar{\mathbf{x}}_i^k - \alpha_i \hat{F}_i(\mathbf{x}_i^k, \mathbf{x}_{-i}^k, \xi_i^k)] \end{aligned}$$

VI. STOCHASTIC NASH EQUILIBRIUM PROBLEMS

In this section, we consider a nongeneralized SNEP, namely, an SGNEP without shared constraints; see [9], [26] for recently proposed algorithms. We consider that the local cost function of agent i is defined as in (2) with $f_i(\mathbf{x}_i) = \iota_{\Omega_i}(\mathbf{x}_i)$. Assumptions 1–5 hold also in this case.

The aim of each agent i , given the decision variables of the other agents \mathbf{x}_{-i} , is to choose a strategy \mathbf{x}_i that solves its local optimization problem, i.e.,

$$\forall i \in \mathcal{I} : \min_{\mathbf{x}_i \in \Omega_i} \mathbb{J}_i(\mathbf{x}_i, \mathbf{x}_{-i}). \quad (21)$$

As a solution, we aim to compute a stochastic Nash equilibrium (SNE), that is, a collective strategy $\mathbf{x}^* \in \Omega$ such that for all $i \in \mathcal{I}$,

$$\mathbb{J}_i(\mathbf{x}_i^*, \mathbf{x}_{-i}^*) \leq \inf\{\mathbb{J}_i(y, \mathbf{x}_{-i}^*) \mid y \in \Omega_i\}.$$

We note that, compared to Definition 2, here we consider only local constraints. Also in this case, we study the associated SVI given by

$$\langle \mathbb{F}(\mathbf{x}^*), \mathbf{x} - \mathbf{x}^* \rangle \geq 0 \text{ for all } \mathbf{x} \in \Omega \quad (22)$$

where \mathbb{F} is the pseudogradient mapping as in (4). The stochastic variational equilibria (v-SNE) of the game in (21) are defined as the solutions of the SVI(Ω, \mathbb{F}) in (22).

Remark 5: A collective strategy $\mathbf{x}^* \in \mathcal{X}$ is a Nash equilibrium of the game in (21) if and only if \mathbf{x}^* is a solution of the SVI in (22) [34, Proposition 1.4.2], [6, Lemma 3.3]. ■

The SRFB iterations for SNEPs are shown in Algorithm 3.

A. Convergence Analysis

In the SNEP case, we can consider sampling only one realization of the random variable, i.e., $S_k = 1$

$$\begin{aligned} \hat{F}(\mathbf{x}^k, \boldsymbol{\xi}^k) &= \text{col}(\hat{F}_i(\mathbf{x}^k, \boldsymbol{\xi}^k)) \\ &= \text{col}(\nabla_{\mathbf{x}_1} J_1(\mathbf{x}^k, \xi_1^k), \dots, \nabla_{\mathbf{x}_N} J_N(\mathbf{x}^k, \xi_N^k)) \end{aligned} \quad (23)$$

where $\boldsymbol{\xi}^k = \text{col}(\xi_1^k, \dots, \xi_N^k) \in \mathbb{R}^N$ is a collection of i.i.d. random variables drawn from \mathbb{P} . Taking fewer samples is less computationally expensive, but we have to make some further assumptions on the pseudogradient mapping.

Assumption 17 (Cut Property): \mathbb{F} in (4) is such that

$$\left. \begin{aligned} \mathbf{x}^* &\in \text{SOL}(\mathbb{F}, \Omega) \\ \bar{\mathbf{x}} &\in C \\ \bar{\mathbf{x}} &\neq \mathbf{x}^* \\ \langle \mathbb{F}(\bar{\mathbf{x}}), \bar{\mathbf{x}} - \mathbf{x}^* \rangle &= 0 \end{aligned} \right\} \implies \bar{\mathbf{x}} \in \text{SOL}(\mathbb{F}, \Omega). \quad (24)$$

Remark 6: The cut property means that, given a solution \mathbf{x}^* , it can be verified if another point $\bar{\mathbf{x}}$ is also a solution by looking only at $\bar{\mathbf{x}}$ and \mathbf{x}^* instead of comparing $\bar{\mathbf{x}}$ with all the points in \mathcal{X} . A very intuitive example is the search for a minimum of a single-valued function [43].

The class of mappings that satisfy this assumption is that of paramonotone (or monotone⁺) operators. A paramonotone operator is a monotone operator such that for all $\mathbf{x}, \mathbf{y} \in \mathcal{X}$

$$\langle \mathbb{F}(\mathbf{x}) - \mathbb{F}(\mathbf{y}), \mathbf{x} - \mathbf{y} \rangle = 0 \Rightarrow \mathbb{F}(\mathbf{x}) = \mathbb{F}(\mathbf{y}).$$

This property does not hold in general for monotone operators. It holds for strongly and strictly monotone operators, because, in these cases, there is only one solution [34, Theorem 2.3.3], and for cocoercive operators. In fact, strict monotonicity implies paramonotonicity that in turn implies monotonicity [43, Definition 2.1]. The same holds for cocoercive operators that are also paramonotone and consequently monotone [34, Definition 2.3.9]. We refer to [43], [44] for a deeper insight on this class of operators. ■

Assumption 18 (Bounded Pseudogradient): \mathbb{F} is bounded, i.e., there exists $B > 0$ such that for all $\mathbf{x} \in \mathcal{X}$ $\|\mathbb{F}(\mathbf{x})\|^2 \leq B_{\mathbb{F}}$.

Even if this assumption is quite strong, it is reasonable in our game theoretic framework. On the other hand, we do not require \mathbb{F} to be Lipschitz continuous, which is practical since computing the Lipschitz constant is difficult in general.

With a little abuse of notation, we denote the approximation error again with $\epsilon_k = \hat{F}(\mathbf{x}^k, \xi_k) - \mathbb{F}(\mathbf{x}^k)$.

Concerning the assumptions on the stochastic error, we still suppose that it has zero expected value (Assumption 8) but we do not need an explicit bound on the variance.

Assumption 19 (Parameter and Step Sizes): The averaging parameter is such that $\delta \in (0, 1)$. The step size is square summable and such that

$$\sum_{k=0}^{\infty} \gamma_k = \infty, \quad \sum_{k=0}^{\infty} \gamma_k^2 < \infty \quad \text{and} \quad \sum_{k=0}^{\infty} \gamma_k^2 \mathbb{E} [\|\epsilon_k\|^2 | \mathcal{F}_k] < \infty. \quad \blacksquare$$

It follows from Assumption 19 that we can take a larger bound on the averaging parameter δ . Since δ is no longer related to the golden ratio, the algorithm reduces to a relaxed FB iteration. Moreover, we note that in this case, we must take a vanishing step size sequence to control the stochastic error.

We now state the main convergence result of this section.

Theorem 3: Let Assumptions 1–6, 8, 13, and 17–19 hold. Then, the sequence $(\mathbf{x}_k)_{k \in \mathbb{N}}$ generated by Algorithm 3 with \hat{F} as in (23) converges to a solution of the game in (21).

Proof: See Appendix V. ■

Remark 7: We note that Theorem 3 holds also in the deterministic case, under the same assumptions with the exception of those on the stochastic error (that is not present).

Formally, under Assumptions 1–6 and 17–19, Algorithm 3 converges to a v-NE of the game in (21). Equivalently, one can use [14, Algorithm 1] to find a deterministic NE. ■

B. Discussion on Further Monotonicity Assumptions

In this section, we discuss some consequences of Theorems 1 and 3. In particular, we discuss different monotonicity notions that can be used to find an SNE in relation with the two possible approximation schemes.

First of all, Algorithm 1 with the approximation as in (15) can be used also for SNEPs.

Corollary 2: If Assumptions 1–5 and 10–12 hold, then, the sequence $(\mathbf{x}^k)_{k \in \mathbb{N}}$ generated by Algorithm 1 with \hat{F} as in (15) converges to a v-SNE of the game in (21).

Proof: Set $g \equiv 0$ and apply Theorem 1. ■

Remark 8: In Corollary 2 as well, the condition presented in Remark 4 can be used instead of monotonicity. ■

The same result holds in the case of cocoercive mappings, but in this case, Assumption 14 can be reduced to the cut property for the pseudogradient (Assumption 17).

Corollary 3: Let Assumptions 1–5, 7–11, 14, and 16 hold. Then, the sequence $(\mathbf{x}_k)_{k \in \mathbb{N}}$ generated by Algorithm 2 with \hat{F} as in (15) converges a.s. to a v-SNE for the game in (21).

Proof: Set $A = 0$ and $b = 0$ and apply Theorem 2. ■

Besides the cut property, there are other assumptions that can be considered.

Assumption 20 (Weak Sharpness): \mathbb{F} satisfies the weak sharpness property, i.e., for all $\mathbf{x} \in \mathcal{X}$, $\mathbf{x}^* \in \text{SOL}(\mathbb{F}, \Omega)$, and for some $c > 0$,

$$\langle \mathbb{F}(\mathbf{x}^*), \mathbf{x} - \mathbf{x}^* \rangle \geq c \min_{\mathbf{x}^* \in \text{SOL}(\mathbb{F}, \Omega)} \|\mathbf{x} - \mathbf{x}^*\|.$$

Remark 9: Assumption 20 is stronger than that in Remark 4 and it is often used in addition to monotonicity [29], [30]. It is sometimes considered a property of the solution set and it is implied by paramonotonicity [45, Th. 4.1]. ■

Corollary 4: Let Assumptions 1–5, 8, 19, and 20 hold. Then, the sequence $(\mathbf{x}_k)_{k \in \mathbb{N}}$ generated by Algorithm 3 with \hat{F} as in (23) converges to a v-SNE of the game in (21).

Proof: See Appendix VI. ■

Remark 10: As a technical assumption in addition to monotonicity, we can also consider the acute angle relation, i.e., $\langle \mathbb{F}(\mathbf{x}), \mathbf{x} - \mathbf{x}^* \rangle > 0$ for all $\mathbf{x} \in \mathcal{X}$ and $\mathbf{x}^* \in \text{SOL}(\mathbb{F}, \Omega)$, $\mathbf{x} \neq \mathbf{x}^*$, also known as variational stability [27]. It is implied by strict pseudomonotonicity, which in turn is implied by strict monotonicity [30, Definition 2] and [27, Corollary 2.4]. It is stronger than the assumption in Remark 4, since the condition is satisfied with the strict inequality.

Formally, if Assumptions 1–5, 8, 19, and the acute angle relation hold, then, the sequence $(\mathbf{x}_k)_{k \in \mathbb{N}}$ generated by Algorithm 3 with \hat{F} as in (23) converges a.s. to a v-SNE of the game in (21). We propose a proof in Appendix VI. The same result for $\delta = 0$ is established in [27, Theorem 4.7].

VII. NUMERICAL SIMULATIONS

Let us now propose some numerical simulations to corroborate the theoretical analysis. We compare our algorithm with the SpFB [24], stochastic forward–backward–forward (SFBF) [18],

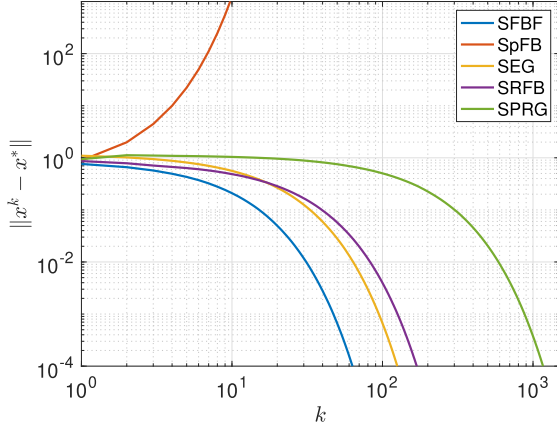


Fig. 1. Distance of the primal variable from the solution.

[46], stochastic extragradient (SEG) [19], and SPRG [28], [29] algorithms, using variance reduction.

We present two sets of simulations: a Cournot game and an academic example. While the first is a realistic application to an electricity market with market capacity constraints, the second is built to show the advantages of the SRFB algorithm.

All the simulations are performed on Matlab R2019a with a 2.3 GHz Intel Core i5 and 8 GB LPDDR3 RAM.

A. Illustrative Example

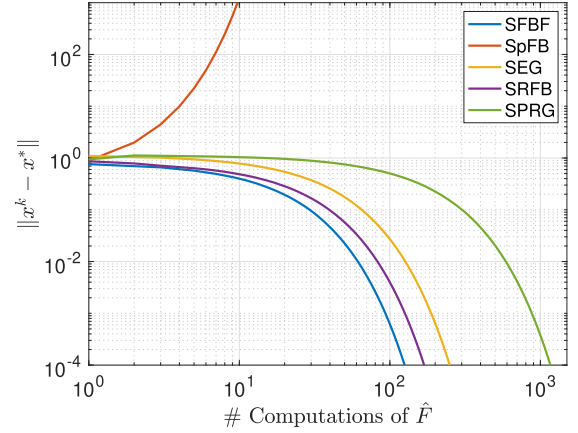
We start with the built up example, that is, a monotone (non-coercive) SNE problem with two players with strategies x_1 and x_2 , respectively, and pseudogradient mapping $\mathbb{F}(x_1, x_2) = (R_1(\xi)x_2, -R_2(\xi)x_1)^\top$.

The mapping is monotone and the random variables are sampled from a normal distribution with mean 1 and finite variance. The problem is unconstrained and the optimal solution is $(0,0)$. The step sizes are taken to be the highest possible. As one can see from Fig. 1, the SpFB does not converge in this case, because stronger monotonicity properties on the mapping should be taken. Moreover, we note that the SPRG is not guaranteed to converge under mere monotonicity. From Fig. 2 instead, we note that the SRFB algorithm is less computationally expensive than the EG.

B. Case Study: Network Cournot Game

We consider the network Cournot game proposed in [14] with the addition of markets capacity constraints [1], [8], which may model the electricity market, the gas market, or the transportation system [12], [13]. Let us consider a set of N companies that sell a commodity in a set of m markets. Each company decides the quantity x_i of product to be delivered in the n_i markets it is connected with. Each company has a local cost function $c_i(x_i)$ related to the production of the commodity. We assume that, the cost function is not uncertain as the companies should know their own cost of production.

Since the markets have a bounded capacity $b = [b_1, \dots, b_m]$, the collective constraints are given by $Ax \leq b$, where $A =$

Fig. 2. Distance of the primal variable from the solution versus number of evaluations of \hat{F} in (15).

$[A_1, \dots, A_N]$. Each A_i indicates in which markets each company participates. The prices are collected in a mapping $P : \mathbb{R}^m \times \Xi \rightarrow \mathbb{R}$ that denotes the inverse demand curve. The random variable $\xi \in \Xi$ represents the demand uncertainty. The cost function of each agent is therefore given by

$$\mathbb{J}_i(x_i, x_{-i}, \xi) = c_i(x_i) - \mathbb{E}[P(x, \xi) \sum_{i \in \mathcal{I}} x_i]. \quad (25)$$

As a numerical setting, we consider a set of 20 companies and 7 markets, connected as in [1, Fig. 1]. Following [1], we suppose that the dual variables graph is a cycle graph with the addition of the edges $(2,15)$ and $(6,13)$. Each company i has local constraints of the form $0 < x_i \leq \theta_i$, where each component of θ_i is randomly drawn from $[1, 1.5]$. The maximal capacity b_j of a market j is randomly drawn from $[0.5, 1]$. The local cost function of company i is

$$c_i(x_i) = q_i^\top x_i + \frac{\beta_i}{\beta_i + 1} \pi_i^{\frac{1}{\beta_i}} \sum_{j=1}^{n_i} ([x_i]_j)^{\frac{\beta_i + 1}{\beta_i}}$$

where $[x_i]_j$ indicates the j component of x_i . π_i is randomly drawn from $[0.5, 5]$, and each component of q_i is randomly drawn from $[1, 100]$. Similarly to [14], we assume that the inverse demand function is of the form

$$P(x, \xi) = \Lambda(\xi)^{\frac{1}{\gamma}} \left(\sum_{i \in \mathcal{I}} x_i \right)^{-\frac{1}{\gamma}}$$

where $\gamma = 1.1$ and $\Lambda(\xi)$ is drawn following a normal distribution with mean 5000 and finite variance. We note that the mapping in (25) is monotone but it may be not Lipschitz continuous depending on β and γ .

We simulate the SpFB, SFBF, SEG, and SRFB to make a comparison using the SA scheme with variance reduction. Since the mapping is not Lipschitz continuous, we tune the step sizes to be half of the minimum step that causes instability. The plots in Fig. 3 and 4 show, respectively, the residual of x_k ($\text{res}(x_k)$) that measures the distance from x_k being a solution and the number of computations of the approximations \hat{F} in (15) of the pseudogradient needed to reach a solution. As one can see, our

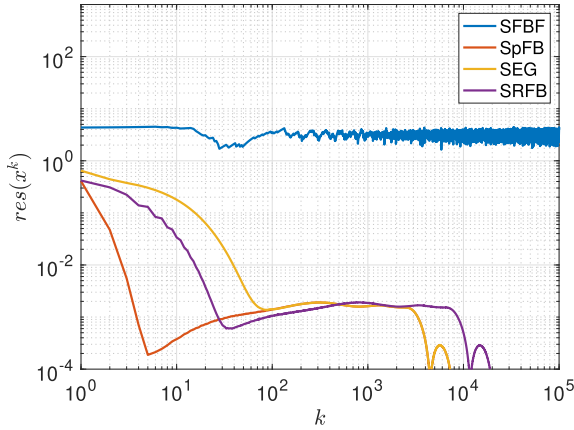


Fig. 3. Residual distance of the primal variable from the solution.

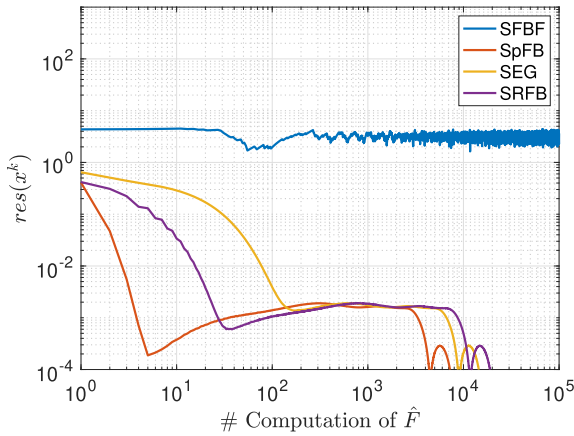


Fig. 4. Residual distance of the primal variable from the solution versus number of evaluations of \hat{F} in (15).

algorithm is slower than the SpFB, as ours involves the averaging step, but it is faster than the EG scheme. Remarkably, the fact that the mapping is only monotone and not Lipschitz continuous prevents the SFBF from converging but it does not affect the other algorithms.

VIII. CONCLUSION

The SRFB algorithm is applicable to stochastic (generalized) Nash equilibrium seeking in merely monotone games. To approximate the expected valued pseudogradient, the stochastic approximation scheme (with or without variance reduction) can be used to guarantee almost sure convergence to an equilibrium.

Our SRFB algorithm is the first distributed algorithm with single proximal computation and single approximated pseudogradient computation per iteration for merely monotone stochastic games.

It remains an open question whether for monotone SGNEPs the stochastic approximation with only one random sample per iteration can guarantee almost sure convergence to an equilibrium, instead of the variance reduced approach. We also leave for future work a comprehensive comparison between

the SRFB algorithm and the most popular fixed-step algorithms for SVIs and SGNEPs, especially, SEG and SFBF, in terms of computational complexity and convergence speed.

APPENDIX I

PROPERTIES OF THE EXTENDED OPERATORS

Proof of Lemma 1: The proof of (i) can be obtained similarly to [1, Theorem 2]. Concerning (ii), given Assumptions 1–5, the game in (3) has at least one solution x^* ; therefore, there exists a $\lambda^* \in \mathbb{R}_{\geq 0}^m$, such that the KKT conditions in (6) are satisfied [39, Th. 3.1]. It follows that $\text{zer}(\mathcal{A} + \mathcal{B}) \neq \emptyset$. The existence of z^* such that $\text{col}(x^*, z^*, \lambda^*) \in \text{zer}(\bar{\mathcal{A}} + \bar{\mathcal{B}})$ follows using some properties of the normal cone and of the Laplacian matrix as a consequence of Assumption 6 [1, Theorem 2].

Proof of Lemma 2: $\bar{\mathcal{A}} = \mathcal{A}_1 + \mathcal{A}_2$ is given by a sum; therefore, it is monotone if both the addends are [35, Proposition 20.10]. \mathcal{A}_2 is monotone because of [47, Theorem 1] and monotonicity of \mathcal{A}_1 follows from

$$\begin{aligned} \langle \mathcal{A}_1(\omega_1) - \mathcal{A}_1(\omega_2), \omega_1 - \omega_2 \rangle &= \langle \mathbb{F}(x_1) - \mathbb{F}(x_2), x_1 - x_2 \rangle \\ &\quad + \langle \mathbf{L}\lambda_1 - \mathbf{L}\lambda_2, \lambda_1 - \lambda_2 \rangle \geq 0 \end{aligned}$$

since Assumption 10 holds and \mathbf{L} is cocoercive by the Baillon–Haddad theorem and, therefore, monotone [35, Example 20.5]. To show that $\bar{\mathcal{A}}$ is Lipschitz continuous, we use the fact that \mathbb{F} is ℓ_F -Lipschitz and \mathbf{L} is ℓ_L -Lipschitz continuous

$$\begin{aligned} \|\bar{\mathcal{A}}_1(\omega_1) - \bar{\mathcal{A}}_1(\omega_2)\| &\leq \|\mathbb{F}(x_1) - \mathbb{F}(x_2)\| + \|\mathbf{L}\lambda_1 - \mathbf{L}\lambda_2\| \\ &\leq (\ell_F + \ell_L)(\|x_1 - x_2\| + \|\lambda_1 - \lambda_2\|). \end{aligned}$$

Similarly, we can prove that the skew symmetric part is $\ell_{\mathcal{A}_2} = \ell_L + \ell_g + B_{\nabla g}$ -Lipschitz continuous (with constant that depends on A and L) from which it follows that $\bar{\mathcal{A}}$ is $\ell_{\bar{\mathcal{A}}} = \ell_{\mathcal{A}_1} + \ell_{\mathcal{A}_2}$ -Lipschitz continuous.

$\bar{\mathcal{B}}$ is maximally monotone by [35, Proposition 20.23] because ∂f is maximally monotone by Assumption 3 and Moreau theorem [35, Theorem 20.25] and the normal cone is maximally monotone [35, Example 20.26].

The fact that $\Phi^{-1}\bar{\mathcal{A}}$ is monotone follows from the fact that $\bar{\mathcal{A}}$ is monotone: $\langle \Phi^{-1}(\bar{\mathcal{A}}(\omega_1) - \bar{\mathcal{A}}(\omega_2)), \omega_1 - \omega_2 \rangle_{\Phi} = \langle \bar{\mathcal{A}}(\omega_1) - \bar{\mathcal{A}}(\omega_2), \omega_1 - \omega_2 \rangle \geq 0$. Similarly, it holds that $\Phi^{-1}\bar{\mathcal{A}}$ is Lipschitz continuous and that $\Phi^{-1}\bar{\mathcal{B}}$ is maximally monotone. ■

Proof of Lemma 3: First, we notice that $\|L\| \geq 2d^*$ and that by the Baillon–Haddad theorem the Laplacian operator is $\frac{1}{2d^*}$ -cocoercive. Then, Statement 1) follows the below computation

$$\begin{aligned} \langle \bar{\mathcal{C}}(\omega_1) - \bar{\mathcal{C}}(\omega_2), \omega_1 - \omega_2 \rangle &= \langle \mathbb{F}(x_1) - \mathbb{F}(x_2), x_1 - x_2 \rangle + \langle \mathbf{L}\lambda_1 - \mathbf{L}\lambda_2, \lambda_1 - \lambda_2 \rangle \\ &\geq \beta \|\mathbb{F}(x_1) - \mathbb{F}(x_2)t\|^2 + \frac{1}{2d^*} \|\mathbf{L}\lambda_1 - \mathbf{L}\lambda_2t\|^2 \\ &\geq \min \left\{ \beta, \frac{1}{2d^*} \right\} (\|\mathbb{F}(x_1) - \mathbb{F}(x_2)t\|^2 + \|\mathbf{L}\lambda_1 - \mathbf{L}\lambda_2t\|^2) \\ &\geq \theta \|\bar{\mathcal{C}}(\omega_1) - \bar{\mathcal{C}}(\omega_2)t\|^2. \end{aligned}$$

The operator $\bar{\mathcal{D}}$ is given by a sum; therefore, it is maximally monotone if both the addends are [35, Proposition 20.23].

The first part is maximally monotone because the normal cone is maximally monotone and the second part is a skew symmetric matrix [35, Corollary 20.28]. Statement 3) follows from Statement 1), and 4) follows from 2) [1, Lemma 7]. ■

APPENDIX II USEFUL LEMMAS

We here recall some known facts about norms and sequence of random variables. Moreover, we include two preliminary results that are useful for the forthcoming convergence proofs.

a) Norm Properties: Now we recall some properties of the norms that we will use in the proofs. We use the cosine rule (or Pythagorean identity)

$$\langle x, y \rangle = \frac{1}{2} (\langle x, x \rangle + \langle y, y \rangle - \|x - y\|^2) \quad (26)$$

and the following two properties of the norm [35, Corollary 2.15], $\forall a, b \in \mathcal{X}$, $\forall \alpha \in \mathbb{R}$:

$$\begin{aligned} \|\alpha a + (1 - \alpha)b\|^2 &= \alpha\|a\|^2 + (1 - \alpha)\|b\|^2 \\ &\quad - \alpha(1 - \alpha)\|a - b\|^2 \end{aligned} \quad (27)$$

$$\|a + b\|^2 \leq 2\|a\|^2 + 2\|b\|^2. \quad (28)$$

b) Property of the Projection and Proximal Operator: By [35, Proposition 12.26], the projection operator and the proximity operators satisfy, respectively, the following inequalities. Let C be a nonempty closed-convex set and let g be a proper lower semicontinuous function, then, for all $x, y \in C$

$$\bar{x} = \text{proj}_C(x) \Leftrightarrow \langle \bar{x} - x, y - \bar{x} \rangle \geq 0 \quad (29)$$

$$\bar{x} = \text{prox}_f(x) \Leftrightarrow \langle \bar{x} - x, y - \bar{x} \rangle \geq f(\bar{x}) - f(y). \quad (30)$$

Moreover, by [35, Proposition 16.44], it holds that

$$\text{prox}_f = (\text{Id} + \partial f)^{-1}. \quad (31)$$

c) Sequence of Random Variables: We now recall some results concerning sequences of random variables, given the probability space $(\Xi, \mathcal{F}, \mathbb{P})$. The Robbins–Siegmund lemma is widely used in the literature to prove a.s. convergence of sequences of random variables. It first appeared in [48].

Lemma 4 (Robbins–Siegmund Lemma, [48]): Let $\mathcal{F} = (\mathcal{F}_k)_{k \in \mathbb{N}}$ be a filtration. Let $\{\alpha_k\}_{k \in \mathbb{N}}$, $\{\theta_k\}_{k \in \mathbb{N}}$, $\{\eta_k\}_{k \in \mathbb{N}}$, and $\{\chi_k\}_{k \in \mathbb{N}}$ be nonnegative sequences, such that $\sum_k \eta_k < \infty$, $\sum_k \chi_k < \infty$, and let

$$\forall k \in \mathbb{N}, \quad \mathbb{E}[\alpha_{k+1} | \mathcal{F}_k] + \theta_k \leq (1 + \chi_k)\alpha_k + \eta_k \quad \text{a.s.}$$

Then, $\sum_k \theta_k < \infty$ and $\{\alpha_k\}_{k \in \mathbb{N}}$ converges a.s. to a nonnegative random variable.

We also need this result for L_p norms, known as Burkholder–Davis–Gundy inequality [49].

Lemma 5 (Burkholder–Davis–Gundy inequality): Let $\{\mathcal{F}_k\}$ be a filtration and $\{U_k\}_{k \in \mathbb{N}}$ a vector-valued martingale relative to this filtration. Then, for all $p \in [1, \infty)$, there exists a universal

constant $c_p > 0$ such that for every $k \geq 1$

$$\mathbb{E} \left[\left(\sup_{0 \leq i \leq N} \|U_i\| \right)^p \right]^{\frac{1}{p}} \leq c_p \mathbb{E} \left[\left(\sum_{i=1}^N \|U_i - U_{i-1}\|^2 \right)^{\frac{p}{2}} \right]^{\frac{1}{p}}.$$

When combined with Minkowski inequality, we obtain for all $p \geq 2$ a constant $C_p > 0$, such that for every $k \geq 1$,

$$\mathbb{E} \left[\left(\sup_{0 \leq i \leq N} \|U_i\| \right)^p \right]^{\frac{1}{p}} \leq C_p \sqrt{\sum_{k=1}^N \mathbb{E} (\|U_i - U_{i-1}\|^p)^{\frac{2}{p}}}.$$

d) Preliminary Results. Given Lemma 5, we prove a preliminary result on the variance of the stochastic error.

Proposition 1: For all $k \in \mathbb{N}$, if Assumption 9 holds, we have

$$\mathbb{E} [\|\epsilon_k\|^2 | \mathcal{F}_k] \leq \frac{C(\sigma(\mathbf{x}^*)^2 + \sigma_0^2 \|\mathbf{x} - \mathbf{x}^*\|^2)}{S_k} \quad \text{a.s.}$$

Proof: We first prove that, for $p \geq 2$ and $1 \leq q \leq p$,

$$\mathbb{E} [\|\epsilon_k\|^q | \mathcal{F}_k]^{\frac{1}{q}} \leq \frac{c_q(\sigma(\mathbf{x}^*) + \sigma_0 \|\mathbf{x} - \mathbf{x}^*\|)}{\sqrt{S_k}}.$$

Let us define the process $\{M_i^S(\mathbf{x})\}_{i=0}^S$ as $M_0(\mathbf{x}) = 0$ and for $1 \leq i \leq S$

$$M_i^S(\mathbf{x}) = \frac{1}{S} \sum_{j=1}^i \hat{F} \text{maximally monotone}(\mathbf{x}, \xi_j) - \mathbb{F}(\mathbf{x}).$$

Let $\mathcal{F}_i = \sigma(\xi_1, \dots, \xi_i)$. Then, $\{M_i^S(\mathbf{x}), \mathcal{F}_i\}_{i=1}^S$ is a martingale starting at 0. Let

$$\begin{aligned} \Delta M_{i-1}^S(\mathbf{x}) &= M_i^S(\mathbf{x}) - M_{i-1}^S(\mathbf{x}) \\ &= \hat{F}(\mathbf{x}, \xi_i) - \mathbb{F}(\mathbf{x}). \end{aligned}$$

Then, by (9), we have

$$\begin{aligned} \mathbb{E} [\|\Delta M_{i-1}^S(\mathbf{x})\|^q]^{\frac{1}{q}} &= \frac{1}{S} \mathbb{E} [\|\hat{F}(\mathbf{x}, \xi_i) - \mathbb{F}(\mathbf{x})\|^q]^{\frac{1}{q}} \\ &\leq \frac{\sigma(\mathbf{x}) + \sigma_0 \|\mathbf{x} - \mathbf{x}^*\|}{S}. \end{aligned}$$

By applying Lemma 5, we have

$$\begin{aligned} \mathbb{E} [\|M_S^S(\mathbf{x})\|^q]^{\frac{1}{q}} &\leq c_q \sqrt{\sum_{i=1}^N \mathbb{E} \left[\left\| \frac{\hat{F}(\mathbf{x}, \xi_i) - \mathbb{F}(\mathbf{x})}{S} \right\|^q \right]^{\frac{2}{q}}} \\ &\leq c_q \sqrt{\frac{1}{S^2} \sum_{i=1}^N \mathbb{E} [\|\hat{F}(\mathbf{x}, \xi_i) - \mathbb{F}(\mathbf{x})\|^q]^{\frac{1}{q}}} \\ &\leq \frac{c_q(\sigma(\mathbf{x}) + \sigma_0 \|\mathbf{x} - \mathbf{x}^*\|)}{\sqrt{S}}. \end{aligned}$$

We note that $M_{S_k}^S(\mathbf{x}^k) = \epsilon_k$, hence,

$$\mathbb{E} [\|\epsilon_k\|^q | \mathcal{F}_k]^{\frac{1}{q}} \leq \frac{c_q(\sigma(\mathbf{x}) + \sigma_0 \|\mathbf{x} - \mathbf{x}^*\|)}{\sqrt{S_k}}.$$

Then, the claim follows by noting that $\mathbb{E} [\|\epsilon\|^{2q} | \mathcal{F}_k]^{\frac{1}{q}} = \mathbb{E} [\|\epsilon\|^q | \mathcal{F}_k]^{\frac{2}{q}}$ and $C = 2c_q^2$. ■

Remark 11: If Proposition 1 holds, then it follows that ■

$$\mathbb{E} [\|\Phi^{-1} \varepsilon_k\|_{\Phi}^2 | \mathcal{F}_k] \leq \frac{C \|\Phi^{-1}\| (\sigma(x^*)^2 + \sigma_0^2 \|x - x^*\|^2)}{S_k}.$$

In the next Lemma, we collect some inequalities that follow from the definition of the algorithm in (13).

Lemma 6: Let $(\omega_k, \bar{\omega}_k)_{k \in \mathbb{N}}$ be generated by Algorithm 1 defined as in (13). Then, the following equations hold:

- (i) $\omega^k - \bar{\omega}^{k-1} = \frac{1}{\delta}(\omega^k - \bar{\omega}^k)$;
- (ii) $\omega^{k+1} - \omega^* = \frac{1}{1-\delta}(\bar{\omega}^{k+1} - \omega^*) - \frac{\delta}{1-\delta}(\bar{\omega}^k - \omega^*)$;
- (iii) $\frac{\delta}{(1-\delta)^2} \|\bar{\omega}^{k+1} - \omega^k\|^2 = \delta \|\omega^{k+1} - \omega^k\|^2$.

Proof: It follows immediately from (13). ■

APPENDIX III PROOFS OF SECTION IV-B

The proof uses the Φ -induced norm and inner product and finds its inspiration in [14], [18], [19].

Proof of Theorem 1: First, let us define $H(x, \lambda) = \sum_{i \in \mathcal{I}} f_i(x_i) + \iota_{\mathbb{R}_{\geq 0}}(\lambda_i)$ and note that $\Phi^{-1} \bar{\mathcal{B}} = \alpha \partial [\sum_{i \in \mathcal{I}} f_i(x_i) + \iota_{\mathbb{R}_{\geq 0}}(\lambda_i)] = \alpha \partial H$. By (31), $(\text{Id} + \Phi^{-1} \bar{\mathcal{B}})^{-1} = (\text{Id} + \alpha \partial H)^{-1} = \text{prox}_{\alpha H}$. Therefore, by using the property of proximal operators in (30), we have that

$$\begin{aligned} \langle \omega^{k+1} - \bar{\omega}^k + \Phi^{-1} \hat{\mathcal{A}}(\omega^k, \xi^k), \omega^* - \omega^{k+1} \rangle &\geq \\ &\geq \alpha(H(\omega^{k+1}) - H(\omega^*)) \end{aligned} \quad (32)$$

$$\begin{aligned} \langle \omega^k - \bar{\omega}^{k-1} + \Phi^{-1} \hat{\mathcal{A}}(\omega^{k-1}, \xi^{k-1}), \omega^{k+1} - \omega^k \rangle &\geq \\ &\geq \alpha(H(\omega^k) - H(\omega^{k+1})). \end{aligned} \quad (33)$$

By using Lemma 6(i), (33) becomes

$$\begin{aligned} \langle \frac{1}{\delta}(\omega^k - \bar{\omega}^k) + \Phi^{-1} \hat{\mathcal{A}}(\omega^{k-1}, \xi^{k-1}), \omega^{k+1} - \omega^k \rangle &\geq \\ &\geq \alpha(H(\omega^k) - H(\omega^{k+1})). \end{aligned} \quad (34)$$

Then, by adding (32) and (34), we obtain

$$\begin{aligned} \langle \omega^{k+1} - \bar{\omega}^k + \Phi^{-1} \hat{\mathcal{A}}(\omega^k, \xi^k), \omega^* - \omega^{k+1} \rangle &+ \\ + \langle \frac{1}{\delta}(\omega^k - \bar{\omega}^k) + \Phi^{-1} \hat{\mathcal{A}}(\omega^{k-1}, \xi^{k-1}), \omega^{k+1} - \omega^k \rangle &\geq \\ &\geq \alpha(H(\omega^k) - H(\omega^*)). \end{aligned} \quad (35)$$

Now, we use the cosine rule in (26)

$$\begin{aligned} \langle \omega^{k+1} - \bar{\omega}^k, \omega^* - \omega^{k+1} \rangle &= \\ -\frac{1}{2} (\|\omega^{k+1} - \bar{\omega}^k\|^2 + \|\omega^{k+1} - \omega^*\|^2 - \|\omega^* - \bar{\omega}^k\|^2) & \\ \langle \frac{1}{\delta}(\omega^k - \bar{\omega}^k), \omega^{k+1} - \omega^k \rangle &= \\ -\frac{1}{2\delta} (\|\omega^k - \bar{\omega}^k\|^2 + \|\omega^k - \omega^{k+1}\|^2 - \|\omega^{k+1} - \bar{\omega}^k\|^2) & \end{aligned}$$

and we note that

$$\begin{aligned} \langle \Phi^{-1} \hat{\mathcal{A}}(\omega^k, \xi^k), \omega^* - \omega^{k+1} \rangle &= \\ = -\langle \Phi^{-1} \bar{\mathcal{A}}(\omega^k), \omega^k - \omega^* \rangle + \langle \varepsilon^k, \omega^* - \omega^k \rangle &+ \\ + \langle \Phi^{-1} \bar{\mathcal{A}}(\omega^k), \omega^k - \omega^{k+1} \rangle + \langle \varepsilon^k, \omega^k - \omega^{k+1} \rangle. & \end{aligned}$$

Then, by reordering and substituting in (35), we obtain

$$\begin{aligned} -\|\omega^{k+1} - \bar{\omega}^k\|^2 - \|\omega^{k+1} - \omega^*\|^2 + \|\omega^* - \bar{\omega}^k\|^2 &+ \\ -\frac{1}{\delta} \|\omega^k - \bar{\omega}^k\|^2 - \frac{1}{\delta} \|\omega^k - \omega^{k+1}\|^2 + \frac{1}{\delta} \|\omega^{k+1} - \bar{\omega}^k\|^2 &+ \\ -2\langle \Phi^{-1} \bar{\mathcal{A}}(\omega^k), \omega^k - \omega^* \rangle + 2\langle \Phi^{-1} \varepsilon^k, \omega^* - \omega^k \rangle &+ \\ + 2\langle \Phi^{-1} (\bar{\mathcal{A}}(\omega^k) - \bar{\mathcal{A}}(\omega^{k-1})), \omega^k - \omega^{k+1} \rangle &+ \\ + 2\langle \Phi^{-1} (\varepsilon^k - \varepsilon^{k-1}), \omega^k - \omega^{k+1} \rangle \geq \alpha(H(\omega^k) - H(\omega^*)). & \end{aligned} \quad (36)$$

Since $\bar{\mathcal{A}}$ is monotone, it holds that

$$\begin{aligned} \langle \Phi^{-1} \bar{\mathcal{A}}(\omega^k), \omega^k - \omega^* \rangle + \alpha(H(\omega^k) - H(\omega^*)) &\geq \\ \geq \langle \Phi^{-1} \bar{\mathcal{A}}(\omega^*), \omega^k - \omega^* \rangle + \alpha(H(\omega^k) - H(\omega^*)) &\geq 0. \end{aligned} \quad (37)$$

Now we apply Lemma 6(ii) and Lemma 6(iii) to $\|\omega^{k+1} - \omega^*\|$

$$\begin{aligned} \|\omega^{k+1} - \omega^*\|^2 &= \frac{1}{1-\delta} \|\bar{\omega}^{k+1} - \omega^*\|^2 - \frac{\delta}{1-\delta} \|\bar{\omega}^k - \omega^*\|^2 + \\ &+ \delta \|\omega^{k+1} - \omega^k\|^2. \end{aligned} \quad (38)$$

By substituting in (36), grouping and reordering, we have

$$\begin{aligned} \frac{1}{1-\delta} \|\bar{\omega}^{k+1} - \omega^*\|^2 + \frac{1}{\delta} \|\omega^k - \omega^{k+1}\|^2 &\leq \\ \leq \left(\frac{\delta}{1-\delta} + 1 \right) \|\omega^* - \bar{\omega}^k\|^2 - \frac{1}{\delta} \|\omega^k - \bar{\omega}^k\|^2 &+ \\ + 2\langle \Phi^{-1} (\bar{\mathcal{A}}(\omega^k) - \bar{\mathcal{A}}(\omega^{k-1})), \omega^k - \omega^{k+1} \rangle &+ \\ + 2\langle \Phi^{-1} \varepsilon^k, \omega^* - \omega^k \rangle + 2\langle \Phi^{-1} (\varepsilon^k - \varepsilon^{k-1}), \omega^k - \omega^{k+1} \rangle & \end{aligned} \quad (39)$$

where we used Assumption 11. Moreover, by using Lipschitz continuity of \mathbb{F} and Cauchy–Schwartz and Young’s inequality, we obtain that

$$\begin{aligned} \langle \Phi^{-1} (\bar{\mathcal{A}}(\omega^k) - \bar{\mathcal{A}}(\omega^{k-1})), \omega^k - \omega^{k+1} \rangle &\leq \\ \leq \frac{\ell_{\bar{\mathcal{A}}} \|\Phi^{-1}\|}{2} (\|\omega^k - \omega^{k-1} t\|^2 + \|\omega^k - \omega^{k+1} t\|^2). & \end{aligned}$$

Similarly, we can bound the term with the stochastic errors

$$\begin{aligned} 2\langle \Phi^{-1} (\varepsilon^k - \varepsilon^{k-1}), \omega^k - \omega^{k+1} \rangle & \\ \leq 2\|\Phi^{-1}\| \|\varepsilon^k - \varepsilon^{k-1}\| \|\omega^k - \omega^{k+1}\| & \\ \leq \|\Phi^{-1}\| \|\varepsilon^k - \varepsilon^{k-1} t\|^2 + \|\Phi^{-1}\| \|\omega^k - \omega^{k+1} t\|^2. & \end{aligned}$$

Substituting in (39), it yields

$$\begin{aligned} \frac{1}{1-\delta} \|\bar{\omega}^{k+1} - \omega^*\|^2 + \frac{1}{\delta} \|\omega^k - \omega^{k+1}\|^2 &\leq \\ \leq \frac{1}{1-\delta} \|\omega^* - \bar{\omega}^k\|^2 - \frac{1}{\delta} \|\omega^k - \bar{\omega}^k\|^2 &+ \\ + \ell_{\bar{\mathcal{A}}} \|\Phi^{-1}\| (\|\omega^k - \omega^{k-1} t\|^2 + \|\omega^k - \omega^{k+1} t\|^2) &+ \\ + \|\Phi^{-1}\| \|\varepsilon^k - \varepsilon^{k-1} t\|^2 + \|\Phi^{-1}\| \|\omega^k - \omega^{k+1} t\|^2 &+ \\ + 2\langle \Phi^{-1} \varepsilon^k, \omega^* - \omega^k \rangle & \end{aligned} \quad (40)$$

Now consider the residual function of ω^k

$$\begin{aligned} \text{res}(\omega^k)^2 &= \|\omega_k - (\text{Id} + \Phi^{-1}\bar{\mathcal{B}})^{-1}(\omega_k - \Phi^{-1}\bar{\mathcal{A}}(\omega_k)t)\|^2 \\ &\leq 2\|\omega_k - \omega^{k+1}t\|^2 + 2\|\bar{\omega}_k - \omega_k + \Phi^{-1}\varepsilon_k t\|^2 \\ &\leq 2\|\omega_k - \omega^{k+1}t\|^2 + 4\|\bar{\omega}_k - \omega_k t\|^2 + \|\Phi^{-1}\varepsilon_k t\|^2 \end{aligned}$$

where we added and subtracted $\omega^{k+1} = \text{proj}(\bar{\omega}_k - \Phi^{-1}\bar{\mathcal{A}}(\omega_k))$ in the first inequality and used the firmly nonexpansiveness of the projection and (28). It follows that

$$\|\bar{\omega}_k - \omega_k t\|^2 \geq \frac{1}{4} \text{res}(\omega^k)^2 - \frac{1}{2} \|\omega_k - \omega^{k+1}t\|^2 - 4\|\Phi^{-1}\varepsilon_k t\|^2$$

Substituting in (40)

$$\begin{aligned} \frac{1}{1-\delta} \|\bar{\omega}^{k+1} - \omega^*\|^2 + \frac{1}{\delta} \|\omega^k - \omega^{k+1}\|^2 &\leq \frac{1}{1-\delta} \|\omega^* - \bar{\omega}^k\|^2 \\ &\quad - \frac{1}{\delta} \left(\frac{1}{4} \text{res}(\omega^k)^2 - \frac{1}{2} \|\omega_k - \omega^{k+1}t\|^2 - \|\Phi^{-1}\varepsilon_k t\|^2 \right) + \\ &\quad + \ell_{\bar{\mathcal{A}}} \|\Phi^{-1}\| (\|\omega^k - \omega^{k-1}t\|^2 + \|\omega^k - \omega^{k+1}t\|^2) + \\ &\quad + \|\Phi^{-1}\| \|\varepsilon^k - \varepsilon^{k-1}t\|^2 + \|\Phi^{-1}\| \|\omega^k - \omega^{k+1}t\|^2 + \\ &\quad + 2\langle \Phi^{-1}\varepsilon^k, \omega^* - \omega^k \rangle. \end{aligned}$$

By taking the expected value, grouping and using Proposition 1 and Assumptions 8 and 12, we have

$$\begin{aligned} &\mathbb{E} \left[\frac{1}{1-\delta} \|\bar{\omega}^{k+1} - \omega^*\|^2 | \mathcal{F}_k \right] + \\ &+ \mathbb{E} \left[\left(\frac{1}{2\delta} - \ell_{\bar{\mathcal{A}}} \|\Phi^{-1}\| - \|\Phi^{-1}\| \right) \|\omega^k - \omega^{k+1}\|^2 | \mathcal{F}_k \right] \leq \\ &\leq \frac{1}{1-\delta} \|\omega^* - \bar{\omega}^k\|^2 + \ell_{\bar{\mathcal{A}}} \|\Phi^{-1}\| \|\omega^k - \omega^{k-1}\|^2 + \\ &\quad + \|\Phi^{-1}\| C \left(\frac{2}{S_k} + \frac{2}{S_{k-1}} + \frac{1}{\delta} \frac{1}{S_k} \right) \\ &\quad (\sigma(x^*)^2 + \sigma_0^2 \|x - x^*t\|^2) \\ &\quad - \frac{1}{\delta} \|\omega^k - \bar{\omega}^k\|^2 - \frac{1}{4\delta} \text{res}(\omega^k)^2. \end{aligned}$$

To use Lemma 4, let

$$\begin{aligned} \alpha_k &= \frac{1}{1-\delta} \|\omega^* - \bar{\omega}^k\|^2 + \ell_{\bar{\mathcal{A}}} \|\Phi^{-1}\| \|\omega^k - \omega^{k-1}\|^2, \\ \theta_k &= \frac{1}{\delta} \|\omega^k - \bar{\omega}^k\|^2 + \frac{1}{4\delta} \text{res}(\omega^k)^2 \\ \eta_k &= \|\Phi^{-1}\| C \left(\frac{2}{S_k} + \frac{2}{S_{k-1}} + \frac{1}{\delta} \frac{1}{S_k} \right) \\ &\quad (\sigma(x^*)^2 + \sigma_0^2 \|x - x^*t\|^2). \end{aligned}$$

By applying the Robbins Siegmund lemma, we conclude that, α_k converges and that $\sum_k \theta_k$ is summable. This implies that the sequence $(\bar{\omega}^k)_{k \in \mathbb{N}}$ is bounded and that $\|\omega^k - \bar{\omega}^k\| \rightarrow 0$ (otherwise $\sum \frac{1}{\delta} \|\omega^k - \bar{\omega}^k\|^2 = \infty$). Therefore, $(\omega^k)_{k \in \mathbb{N}}$ has at least one cluster point $\tilde{\omega}$. Moreover, since $\sum \theta_k < \infty$, $\text{res}(\omega^k)^2 \rightarrow 0$ and $\text{res}(\tilde{\omega}^k)^2 = 0$. ■

Proof of Corollary 1: The proof is similar to Theorem 1 but we do not use monotonicity. Hence, the steps are the same,

except for (37). Indeed, the terms in H are not present since the projection satisfies (29) and $\langle \bar{\mathcal{A}}(\omega^k), \omega^k - \omega^* \rangle \geq 0$ by Assumption 13. The conclusion follows as in Theorem 1. ■

APPENDIX IV PROOF OF THEOREM 2

Proof of Theorem 2: The first part of the proof is the same as Theorem 1 since the resolvent is firmly nonexpansive but we do not use the residual or monotonicity. Then, taking the expected value and grouping in (40), we have

$$\begin{aligned} &\mathbb{E} \left[\frac{1}{1-\delta} \|\bar{\omega}^{k+1} - \omega^*\|^2 | \mathcal{F}_k \right] + \\ &+ \mathbb{E} \left[\left(\frac{1}{\delta} - \ell_{\bar{\mathcal{C}}} \|\Phi^{-1}\| - \|\Phi^{-1}\| \right) \|\omega^k - \omega^{k+1}\|^2 | \mathcal{F}_k \right] \leq \\ &\leq \frac{1}{1-\delta} \|\omega^* - \bar{\omega}^k\|^2 + \ell_{\bar{\mathcal{C}}} \|\Phi^{-1}\| \|\omega^k - \omega^{k-1}\|^2 + \\ &\quad - 2\langle \Phi^{-1}\bar{\mathcal{C}}(\omega^k), \omega^k - \omega^* \rangle - \frac{1}{\delta} \|\omega^k - \bar{\omega}^k\|^2 + \\ &\quad + \left(\frac{2\|\Phi^{-1}\|C}{S_k} + \frac{2\|\Phi^{-1}\|C}{S_{k-1}} \right) (\sigma(x^*)^2 + \sigma_0^2 \|x - x^*t\|^2). \end{aligned}$$

where the inequality follows by Proposition 1 and Assumptions 8 and 16. To use Lemma 4, let

$$\begin{aligned} \alpha_k &= \frac{1}{1-\delta} \|\omega^* - \bar{\omega}^k\|^2 + \ell_{\bar{\mathcal{C}}} \|\Phi^{-1}\| \|\omega^k - \omega^{k-1}\|^2 \\ \theta_k &= \frac{1}{\delta} \|\omega^k - \bar{\omega}^k\|^2 + 2\langle \Phi^{-1}\bar{\mathcal{C}}(\omega^k), \omega^k - \omega^* \rangle \end{aligned}$$

$$\eta_k = 2\|\Phi^{-1}\|C \left(\frac{1}{S_k} + \frac{1}{S_{k-1}} \right) (\sigma(x^*)^2 + \sigma_0^2 \|x - x^*t\|^2).$$

Then, α_k converges and θ_k is summable. This implies that $\{\bar{\omega}^k\}_{k \in \mathbb{N}}$ is bounded and that $\|\omega^k - \bar{\omega}^k\| \rightarrow 0$. Therefore, $\{\omega^k\}_{k \in \mathbb{N}}$ has at least one cluster point $\tilde{\omega}$. Moreover, $\langle \Phi^{-1}\bar{\mathcal{C}}(\omega^k), \omega^k - \omega^* \rangle \rightarrow 0$ and $\langle \bar{\mathcal{C}}(\tilde{\omega}), \tilde{\omega} - \omega^* \rangle = 0$. Since $\bar{\mathcal{C}}$ is cocoercive, it also satisfies the cut property; therefore, $\tilde{\omega}$ is a solution. ■

APPENDIX V PROOF OF THEOREM 3

Proof of Theorem 3: We start by using the fact that the projection is firmly quasinonexpansive.

$$\begin{aligned} &\|\mathbf{x}^{k+1} - \mathbf{x}^*t\|^2 \leq \\ &\leq \|\mathbf{x}^* - \bar{\mathbf{x}}^k + \lambda_k \hat{F}(\mathbf{x}^k, \xi^k)t\|^2 - \|\bar{\mathbf{x}}^k - \lambda_k \hat{F}(\mathbf{x}^k, \xi^k) - \mathbf{x}^{k+1}t\|^2 \\ &\leq \|\mathbf{x} - \bar{\mathbf{x}}^k t\|^2 - \|\bar{\mathbf{x}}^k - \mathbf{x}^{k+1}t\|^2 + 2\lambda_k \langle \hat{F}(\mathbf{x}^k, \xi^k), \mathbf{x}^* - \bar{\mathbf{x}}^k \rangle + \\ &\quad + 2\lambda_k \langle \hat{F}(\mathbf{x}^k, \xi^k), \bar{\mathbf{x}}^k - \mathbf{x}^{k+1} \rangle \\ &= \|\mathbf{x}^* - \bar{\mathbf{x}}^k t\|^2 - \|\bar{\mathbf{x}}^k - \mathbf{x}^{k+1}t\|^2 + 2\lambda_k \langle \hat{F}(\mathbf{x}^k, \xi^k), \bar{\mathbf{x}}^k - \mathbf{x}^{k+1} \rangle \\ &\quad + 2\lambda_k \langle \hat{F}(\mathbf{x}^k, \xi^k), \mathbf{x}^* - \bar{\mathbf{x}}^k \rangle + 2\lambda_k \langle \hat{F}(\mathbf{x}^k, \xi^k), \mathbf{x}^k - \bar{\mathbf{x}}^k \rangle \end{aligned}$$

In view of Lemmas 6.2 and 6.3 as in (38), we can rewrite the inequality as

$$\begin{aligned} \frac{1}{1-\delta} \|\bar{x}^{k+1} - x^*t\|^2 &\leq \frac{1}{1-\delta} \|\bar{x}^k - x^*t\|^2 + \\ &+ 2\lambda_k \langle \hat{F}(x^k, \xi^k), x^* - x^k \rangle + 2\lambda_k \langle \hat{F}(x^k, \xi^k), x^k - \bar{x}^k \rangle + \\ &+ 2\lambda_k \langle \hat{F}(x^k, \xi^k), \bar{x}^k - x^{k+1} \rangle - (\delta + 1) \|x^{k+1} - \bar{x}^k t\|^2. \end{aligned} \quad (41)$$

By applying the Young's inequality to the inner products, we obtain

$$\begin{aligned} 2\lambda_k \langle \hat{F}(x^k, \xi^k), x^k - \bar{x}^k \rangle &\leq \lambda_k^2 \|\hat{F}(x^k, \xi^k)t\|^2 \\ &+ \|x^k - \bar{x}^k t\|^2 \\ 2\lambda_k \langle \hat{F}(x^k, \xi^k), \bar{x}^k - x^{k+1} \rangle &\leq \lambda_k^2 \|\hat{F}(x^k, \xi^k)t\|^2 \\ &+ \|\bar{x}^k - x^{k+1}t\|^2. \end{aligned}$$

Then, (41) becomes

$$\begin{aligned} \frac{1}{1-\delta} \|\bar{x}^{k+1} - x^*t\|^2 &\leq \frac{1}{1-\delta} \|\bar{x}^k - x^*t\|^2 + \\ &+ 2\lambda_k \langle \mathbb{F}(x^k), x^* - x^k \rangle + 2\lambda_k \langle \epsilon^k, x^* - x^k \rangle + \\ &+ 4\lambda_k^2 \|\mathbb{F}(x^k)t\|^2 + 4\lambda_k^2 \|\epsilon^k t\|^2 - (\delta + 1) \|x^{k+1} - \bar{x}^k t\|^2 + \\ &+ \|x^k - \bar{x}^k t\|^2 + \|\bar{x}^k - x^{k+1}t\|^2. \end{aligned} \quad (42)$$

By using Lemma 6.1 and Assumption 8, reordering and taking the expected value, we have

$$\begin{aligned} \mathbb{E} \left[\frac{1}{1-\delta} \|\bar{x}^{k+1} - x^*t\|^2 | \mathcal{F}_k \right] &+ \mathbb{E} [\delta \|x^{k+1} - \bar{x}^k t\|^2 | \mathcal{F}_k] \leq \\ &\leq \frac{1}{1-\delta} \|\bar{x}^k - x^*t\|^2 + \delta^2 \|x^k - \bar{x}^{k-1}t\|^2 + \\ &+ 2\lambda_k \langle \mathbb{F}(x^k), x^* - x^k \rangle + 4\lambda_k^2 \|\mathbb{F}(x^k)t\|^2 \\ &+ 4\lambda_k^2 \mathbb{E} [\|\epsilon^k t\|^2 | \mathcal{F}_k]. \end{aligned} \quad (43)$$

Thanks to Lemma 4, we conclude that $(\bar{x}^k)_{k \in \mathbb{N}}$ and $(x^k)_{k \in \mathbb{N}}$ are bounded sequence and that they have a cluster point, that is, $\bar{x}^k \rightarrow \bar{x}$ and $x^k \rightarrow x$. Since $\bar{x}^k = (1-\delta)x^k + \delta\bar{x}^{k-1}$, taking the limit, we obtain that $\bar{x} = x$. Moreover, since $\langle \mathbb{F}(x^k), x^* - x^k \rangle \leq 0$, again by Lemma 4, we obtain that $\langle \mathbb{F}(x), x - x^* \rangle = 0$ which, for the cut property, implies that x is a solution. ■

APPENDIX VI PROOFS OF SECTION VI-B

Proof of Corollary 4: We use the weak sharpness property in (43) to obtain

$$\begin{aligned} \mathbb{E} \left[\frac{1}{1-\delta} \|\bar{x}^{k+1} - x^*t\|^2 | \mathcal{F}_k \right] &+ \mathbb{E} [\delta \|x^{k+1} - \bar{x}^k t\|^2 | \mathcal{F}_k] \leq \\ &\leq \frac{1}{1-\delta} \|\bar{x}^k - x^*t\|^2 + \delta^2 \|x^k - \bar{x}^{k-1}t\|^2 + 4\lambda_k^2 \mathbb{E} [\|\epsilon^k t\|^2 | \mathcal{F}_k] \\ &- 2\lambda_k c \min_{x^* \in \text{SOL}(\mathbb{F}, \Omega)} \|x - x^*\| + 4\lambda_k^2 \|\mathbb{F}(x^k)t\|^2. \end{aligned}$$

Applying Lemma 4, $(\bar{x}^k)_{k \in \mathbb{N}}$ and $(x^k)_{k \in \mathbb{N}}$ are bounded sequences and they have a cluster point \bar{x} . Moreover, $\min_{x^* \in \text{SOL}(\mathbb{F}, \Omega)} \|x - x^*\| \rightarrow 0$ and $\|\bar{x} - x^*\| = 0$, that is, \bar{x} is a solution. ■

Proof of the Statement in Remark 10: We apply Lemma 4 to (43). Therefore, $(\bar{x}^k)_{k \in \mathbb{N}}$ and $(x^k)_{k \in \mathbb{N}}$ are bounded sequences and that they have a cluster point \bar{x} . Moreover, $\langle \mathbb{F}(x^k), x^* - x^k \rangle \rightarrow 0$ and $\langle \mathbb{F}(x), x - x^* \rangle = 0$ but this contradicts the acute angle property. Therefore, \bar{x} must be a solution. ■

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