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DOI
10.1002/mma.10936

Publication date
2025

Document Version
Final published version

Published in
Mathematical Methods in the Applied Sciences

Mathematical Methods in the Applied Sciences

WILEY

RESEARCH ARTICLE

Global Dissipativity for Quaternion Valued Inertial Neural Networks With Unbounded Time-Varying Delays

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Received: 31 May 2024 | **Revised:** 7 March 2025 | **Accepted:** 12 March 2025

Funding: This study was supported by the Indian Institute of Technology (BHU).

Keywords: global dissipativity | global exponential dissipativity | inertial neural network | Lyapunov function | quaternion valued neural network | unbounded time varying delays

ABSTRACT

In this article, the authors investigate the global and exponential dissipativity of quaternion-valued inertial neural networks (QVINNs) with mixed time-varying delays, without utilizing order reduction of inertial neural networks (INNs) and quaternion separation methods. Using innovative Lyapunov functional and inequality techniques, several fruitful sufficient criteria with multi-parameters are derived for QVINNs to ensure global dissipativity (GD), which generalizes and refines existing results. This article estimates the attractive sets and exponentially attractive sets globally. Unlike previous studies in which quaternion-valued neural networks (QVNNs) are separated into real-valued neural networks (RVNNs) and INNs are reduced into first-order systems, the foundation of this article rests upon approaches that diverge from the traditional methods of separation and order reduction. Unlike existing results on the GD of traditional neural networks (NNs) with bounded discrete time delays, this article focuses on INNs with unbounded discrete time-varying delays, which is more realistic because neurons consider their entire past rather than partial history within bounded time delays. In general, in exponential stability, synchronization, and dissipativity results, researchers typically impose an upper bound on the rate of convergence λ, but in the present article, the authors investigate dissipativity criteria without such a restriction on the convergence rate in global exponential dissipativity (GED). Finally, to demonstrate the efficiency of our theoretical work, three consecutive examples are proposed to illustrate the effectiveness of the obtained results. The first two examples verify the proposed results, and the third one, related to QVNNs, redemonstrates the efficiency of storing true color image patterns.

1 | Introduction

In 1853, W.R. Hamilton first introduced quaternions [1], a subset of Clifford algebra different from the real and complex domains. One of its remarkable properties is that the law of commutativity does not apply to quaternion multiplication. Due to the non-commutativity inherent in quaternion algebra, progress in research within the quaternion domain has experienced a prolonged deceleration when contrasted with advancements in the real and complex fields. This has been a primary factor in the slow development of QVNNs. Fortunately, as modern mathematics has advanced and expanded, the applications of quaternions in various fields have been discovered. Due to their great potential, research on quaternions has gained considerable attention. For example, they have promising applications in three-dimensional wind forecasting [2], color face recognition [3], image compression [4], and quantum mechanics [5]. Quaternions are particularly useful in 3D and 4D applications. One such example is their utilization in succinctly expressing spatial rotations within the context of 3D geometrical affine transformations and translations [6].

Keeping these aspects in mind, researchers have been exploring the dynamic behavior of QVNNs. Initially, QVNNs were separated into four equivalent real-valued neural networks (RVNNs) or two complex-valued neural networks (CVNNs) to facilitate their study, leading to significant findings [7–11], among others. However, in the separation method, authors need to handle four RVNNs instead of a single quaternion-valued system (QVS), making the results more complex and computationally intensive. Additionally, this approach increases the original system's dimensions, complicating mathematical analysis. In practical applications, multi-dimensional data are frequently addressed in their original forms, and this issue cannot be efficiently resolved by processing multi-dimensional input through multiple real-valued neurons.

Nowadays, the separation technique is considered outdated. A complex-valued neuron can process two-dimensional data as a whole, and similarly, researchers have proposed using quaternion numbers in neural networks to describe multi-dimensional information, such as color and 3D coordinates, through quaternion neurons. The direct quaternion-based mathematical approach for QVNNs has gained increasing popularity. It is essential to recognize that utilizing the direct quaternion approach provides clear advantages over the decomposition approach when handling QVNNs. This is primarily due to its superior capability to retain the inherent properties of neural networks. A growing trend among researchers involves applying the direct quaternion approach, leading to significant advancements in addressing QVNNs and yielding remarkable results [12–15]. Motivated by this perspective, the authors of the present article discuss the dynamical behavior of QVNNs using the fully quaternion-based approach.

The introduction of inductors into neural circuits by Babcock and Westervelt in 1986 [16] led to the proposal of a class of NNs called INNs. These INNs exhibit a second-order derivative of state variables. Furthermore, an INN with time delay was introduced by Babcock and Westervelt, considering that neurons have a finite frequency response or a transfer characteristic with a time delay [17]. The authors observed that inertial components in electronic NNs might result in complex phenomena such as instability, spontaneous oscillations, ringing around fixed positions, and chaotic behavior. It is evident that INN systems differ from conventional NN systems, which are defined by first-order differential equations, making the analysis of their dynamical behaviors more complicated and challenging.

The inertial term can be introduced into neural systems with a wide range of biological backgrounds. For instance, by adding inductance, the membrane can exhibit electrical tuning and filtering activities, enabling the quasi-active membrane activity of neurons to be modelled [18]. The axon of the squid demonstrates phenomenological inductance [19], and circuits incorporating inductance can replicate key features of neurons, such as the quasi-active membrane and the axon-like behavior observed in squids [20]. Compared to first-order NNs, the dynamical behavior of INNs is generally more complex and challenging to analyze. In some cases, the presence of the inertial term can even lead to instability [21, 22]. Early research revealed that inertial terms could introduce chaos and bifurcation, highlighting the importance of considering their incorporation into artificial NNs.

It is essential to note that utilizing the direct quaternion approach provides significant advantages when handling QVNNs compared to the decomposition approach, primarily due to its superior ability to retain the inherent properties of NNs. A growing trend among researchers involves applying the direct quaternion approach, leading to significant advancements in addressing QVNNs and achieving remarkable results [12–15]. Motivated by these findings, this study investigates the dynamical behavior of QVINNs using the fully quaternion approach.

This version improves clarity, eliminates redundancy, and ensures grammatical accuracy. To analyze the dynamic behavior of various kinds of INNs, the researchers frequently use the variable substitution approach by which second-order systems are reduced to first-order systems as [12, 23–33]. The obtained criteria are also effective, but they are more complex and challenging to implement due to the increased number of variables. Nowadays, the authors frequently use the non-reduction approach to study the INNs, which keeps the originality of the system and makes the analysis approach easy and find tremendous results [34–36]. Keeping these things in mind for INNs, the current study has employed the without-reduction order technique for QVINNs.

In the year 1970s, Willems proposed dissipativity since it has been used in numerous areas, including chaos, robust control, state estimation, and stability theory. Certainly, dissipativity can be viewed as a generalization or extension of Lyapunov stability, as it encompasses not only the stability of equilibrium points but also a broader range of dynamical behaviors [37]. The most significant advantage of the dissipative theory is that it can be used to discuss complicated systems and offer a reliable framework for stability analysis. One must identify the attractive global set in the subject of dissipativity, an estimated range of periodic states. The equilibrium point and chaos may be calculated once the set has been found. Many researchers are interested in the dissipativity analysis of NNs, and many significant conclusions have been obtained [7, 12, 38–41]. Unfortunately, these works are mainly based on traditional NNs. To date, Still, up to now, there are very few results of GD on INNs [12, 42–44]. The authors of the current article directed their attention toward the INNs and considered the GD of INNs with time delays in the quaternion domain.

Due to the limited signal transmission rate and the finite switching rates of neuron amplifiers, time delays should be considered in NNs and electronic circuits, as they might cause performance degradation [45, 46]. Recent achievements in investigating the dynamical behavior of QVNNs and INNs with time delays can be found in the research articles [7, 12, 38–44, 47]. It is important to note that the above research primarily considers discrete-time delays as bounded delays. When time delays are finite, this suggests that a neuron's current dynamics are only partially related to its past. We understand that a neuron's current actions are linked to its prior states. As a result, one should consider time delays in NNs to be unbounded, which can more accurately reflect how neurons behave in human brains [48]. However, it should be noted that unbounded time-varying delays without a separation approach and GED with an unbounded rate of convergence factor have not yet been investigated in the context of QVINNs. On the other hand, considering the existence of numerous parallel pathways characterized by varying axon sizes and lengths, it becomes more reasonable to incorporate continuously distributed delays into neural network models [10, 49, 50]. This choice aligns with the intricate nature of biological neural networks, where information transmission occurs through diverse pathways with distinct temporal characteristics. The introduction of continuously distributed delays allows for a more accurate representation of the heterogeneous delays in these parallel pathways, enhancing the model's realism and capturing the complexities inherent in biological neural systems. Inspired by the above discussion, the main contribution of this article is to study the GD of QVINNs with unbounded discrete and distributed time delays as outlined below.

1. Contrary to different results on the GD of traditional RVNNs, CVNNs, and QVNNs [3, 40, 46, 51], the present article deals with the GD and the attractive global set for QVINNs and the concerned mathematical model is the more general complex model.
2. Unlike the common method that used order reduction and quaternion separation approach for QVNNs and QVINNs [8–10, 26, 28, 51, 52], this article is based on the non-separation non-reduction and methods and is reducing the computation complexity and making the analysis approach easy.
3. Unlike the bounded time-varying delays for QVNNs and QVINNs [2, 8–12, 26, 28, 38, 39, 51, 52], the present article is based on the unbounded discrete time-varying delays, which relate the to all of the previous states of neurons' rather than the bounded time delays which partially relates of the neurons dynamics history.
4. To guarantee the GD and GED of the respective models, which encompass certain recent findings as specific instances and possess broader applicability. Some valid requirements are derived using testable algebraic inequalities.
5. The numerical simulation validates the effectiveness of quaternion domain neural networks in successfully storing and retrieving true color images.

The remainder of this paper can be outlined as follows: In Section 2, we introduce the QVINNs system along with pertinent definitions and lemmas. The primary findings, encompassing GD and GD, are expounded upon in Section 3. Section 4 showcases numerous numerical simulations that corroborate the principal outcomes. Finally, Section 5 depicts a brief conclusion and outlines potential avenues for future research.

Notations: Throughout this paper, the following notation is employed: I signifies the set $\{1, 2, 3, \dots, n\}$, while $\mathbf{R}$, $\mathbf{C}$, and $\mathbf{Q}$ correspond to the real, complex, and quaternion skew field, respectively. Additionally, $\mathbf{R}^+$ denotes the set of positive real numbers, and $\mathbf{R}^n$ denotes n-dimensional real vectors, while $\mathbf{Q}^n$ represents n-dimensional quaternion-valued vectors. A quaternion number can be expressed as $q = q^R + q^I i + q^J + q^K \in Q$, where q^R, q^I, q^J, and q^K are real, i, j, and k are imaginary units, and the modulus $|q| = \sqrt{(q^R)^2 + (q^I)^2 + (q^J)^2 + (q^K)^2}$. For a quaternion-valued vector $q = (q_1, q_2, q_3, \dots, q_n) \in \mathbf{Q}^n$ with $q_l = q_l^R + q_l^I i + q_l^J + q_l^K$, the vector norm is defined as $||q|| = \sqrt{(q_l^R)^2 + (q_l^I)^2 + (q_l^J)^2 + (q_l^K)^2}$. The notation $\phi((-\infty, 0], \mathbf{Q^n})$ represents the set of all continuous functions mapping from $(-\infty, 0]$ to $\mathbf{Q^n}$.

2 | Model Description

In this article, consider the class of QVINNs with unbounded time-varying delays as follows:

$$\begin{aligned}\frac{d^2 r_\iota(t)}{dt^2} = &-\alpha_\iota \frac{dr_\iota(t)}{dt} - \beta_\iota r_\iota(t) + \sum_{\jmath=1}^{n} a_{\iota\jmath} f_\jmath(r_\jmath(t)) \\ &+ \sum_{\jmath=1}^{n} b_{\iota\jmath} g_\jmath(r_\jmath(t - \tau_\jmath(t))) \\ &+ \sum_{\jmath=1}^{n} d_{\iota\jmath} \int_{t-\sigma_\jmath(t)}^{t} h_\jmath(r_\jmath(s))ds + u_\iota(t), \quad \iota \in I\end{aligned} \tag{1}$$

where $r_\iota(t)$ is the state of the ιth neurons, $\frac{d^2 r_\iota(t)}{dt^2}$ is the inertial terms, $\alpha_\iota, \beta_\iota \in \mathbf{R}^+$, $a_{\iota\jmath}, b_{\iota\jmath}$, and $d_{\iota\jmath} \in \mathbf{Q}$ are weight connection matrices respectively, $f_\jmath(.)$, $g_\jmath(.)$, and $h_\jmath(.)$ are the activation functions, $\tau_\jmath(t), \sigma_\jmath$ denotes the delays from $\jmath$-th neurons at a time t, $u_\iota(t)$ is the external input.

In this article, the following are taken into account:

Lemma 1. ([12]). *For any $x, y \in Q$, the following properties holds:*
$(I)\ \overline{x+y} = \overline{x} + \overline{y}; (II) \overline{xy} = \overline{y}\overline{x}$.

Lemma 2. ([12]). *If $x, y \in Q$, and $\epsilon \in \mathbf{R}^+$, then $yx + \overline{y}\,\overline{x} \leq \epsilon \overline{x}x + \frac{1}{\epsilon} y\overline{y}$.*

Assumption 1. The discrete time-varying delays $\tau_\jmath(t), \jmath \in I$ are unbounded, differentiable and satisfy $\dot{\tau}_\jmath(t) \leq \tau < 1$, where τ is constant.

Assumption 2. For any $\jmath \in I, \forall s_1, s_2 \in \mathbf{Q}$, there exist non-negative constants $M_\jmath, L_\jmath$ and $K_\jmath$ the nonlinear activation functions are satisfied as

$$\begin{aligned}|f_\jmath(s_1) - f_\jmath(s_2)| &\leq M_\jmath |s_1 - s_2|, \\ |g_\jmath(s_1) - g_\jmath(s_2)| &\leq L_\jmath |s_1 - s_2|, \\ |h_\jmath(s_1) - h_\jmath(s_2)| &\leq K_\jmath |s_1 - s_2|.\end{aligned}$$

Assumption 3. For any $j \in I, \forall s_1, s_2 \in \mathbf{Q}$, there exist non-negative constants M_j, L_j, and, K_j such that the nonlinear activation functions are satisfied as

$$|f_j(s_1) - f_j(s_2)| \le M_j |s_1 - s_2|,$$
$$|g_j(s_1) - g_j(s_2)| \le L_j |s_1 - s_2| e^{-\overline{\sigma}\tau_j(t)},$$
$$|h_j(s_1) - h_j(s_2)| \le K_j |s_1 - s_2|,$$

where $\tau_j(t)$ is the time delay in the system (1) and $a > 0$ is sufficiently large number.

Definition 1. If there is a compact set $\Upsilon \subset \mathbf{Q}^n$, for $\forall\, \Psi(s)$, $\Psi^*(s) \in \mathbf{Q}^n - \Upsilon$ for $s \in (-\infty, t_0]$, $\exists\, T > 0$ such that $r(t, t_0, \Psi, \Psi^*) \subset \Upsilon$ for $t \ge T + t_0$, then QVINNs (1) is referred to as a GD system. The set Υ is the globally attractive set (GAS) of (1). A set Υ is called to be the positive invariant set if for $\forall\, \Psi(s), \Psi^*(s) \in \Upsilon$ for $s \in (-\infty, t_0]$ which implies that $r(t, t_0, \Psi, \Psi^*) \subset \Upsilon$ for $t \ge t_0$.

Definition 2. Let Υ be a GAS of the QVINNs model (1). Then the model (1) is said to be a globally exponentially dissipative system if there exists a compact set $\Upsilon \subset \mathbf{Q}^n$, for all $\Psi(s), \Psi^*(s) \in \mathbf{Q}^n - \Upsilon$, $s \in (-\infty, t_0]$, there exist constants $K(\Psi, \Psi^*) > 0$, $\lambda > 0$ such that $\inf_{u \in \mathbf{Q}^n - \Upsilon}\{\|r(t, t_0, \Psi, \Psi^*) - u^*\| | u^* \in \Upsilon\} \le K(\Psi, \Psi^*) e^{-\lambda(t - t_0)}$. Meanwhile, the set Υ is a globally exponential attractive set of (1).

Remark 1. Most of the results regarding GD and stability for QVINNs and QVNNs rely on the method of quaternion separation, as seen in [7–11, 51, 52]. As a result, the number of variables and dimensions increases, leading to higher computational complexity. Instead of using this technique, we propose to perform the analysis directly, which better fits the requirements of actual applications while preserving the originality of the addressed systems.

Remark 2. It's worth highlighting that Assumption 3 covers a range of time delay types as specific instances. For instance, it encompasses constant discrete delays as discussed in references like [26, 28], proportional delays as explored in [38], and bounded discrete-time delays, as examined in works such as [12, 23, 26–29, 34, 51, 52]. This characteristic renders our model more comprehensive and adaptable. Conversely, investigating GED commonly poses challenges when involving unbounded time delays. However, this article introduces an innovative assumption for the activation function $g_l(\cdot)$, which simplifies addressing this challenge.

Remark 3. According to the majority of the existing research, the most popular approach for analyzing the dynamic characteristics of INNs is transforming the INNs into first-order differential equations by appropriate selection of variable substitutions. However, this work directly analyzes the GD of the proposed QVINNs by utilizing a novel Lyapunov functional without converting the suggested system into a conventional one. The considered Lyapunov functional, unlike the usual Lyapunov functional, directly includes both the state variables and their derivatives. This approach is more effective for examining the asymptotic characteristics of different INNs as it minimizes unnecessary computational steps and simplifies the theoretical analysis.

3 | Main Results

This section will analyze GD results by employing Lyapunov theory along with various analytic techniques, which are alternatives to the conventional reduced order and separation technique.

Theorem 1. *Presume Assumptions (1) and (2), there exist $\delta_\iota > 0$ and the system (1), such that $X_\iota < 0$, $Y_\iota < 0$, $Z_\iota^2 \le X_\iota Y_\iota$, $i \in I$, where $Z_\iota = \delta_\iota - \alpha_\iota - \beta_\iota + 1$, $Y_\iota = 2 - 2\alpha_\iota + \epsilon_5 + \frac{n}{\epsilon_3} + \sum_{j=1}^n \left(\frac{1}{\epsilon_1} a_{\iota j} \overline{a}_{\iota j} + \frac{\sigma_j}{\epsilon_7} K_j^2 d_{\iota j} \overline{d}_{\iota j} \right)$ and $X_\iota = \epsilon_6 + \frac{n}{\epsilon_4} + \tau \gamma_\iota M_\iota^2 - 2\beta_\iota + n\epsilon_1 M_\iota^2 + n\epsilon_2 M_\iota^2 + \sum_{j=1}^n \left(\frac{1}{\epsilon\, 2} a_{\iota j} \overline{a}_{\iota j} + \frac{\sigma_j}{\epsilon_7} K_j^2 d_{\iota j} \overline{d}_{\iota j} + \frac{(\epsilon_3 + \epsilon_4)|b_{j\iota}|^2 L_\iota^2}{(1 - \tau)} \right)$.*

Then QVINN model (1) is a global dissipative system, and the set $\Upsilon_1 = \left\{ r(t) \in \mathbf{Q}^n | \overline{r}_\iota(t) r_\iota(t) \le \frac{\epsilon^*}{\kappa} \overline{u}_\iota(t) u_\iota(t), \quad \iota \in I \right\}$ is a global attractive set and a positively invariant set of QVINNs (1) where $\kappa = min_{1 \le \iota \le n} \left\{ \frac{Z_\iota^2}{Y_\iota} - X_\iota \right\}$ and $u(t) = \sum_{\iota=1}^n \overline{u}_\iota(t) u_\iota(t)$.

Proof. To establish the proof for this theorem, let us employ a Lyapunov functional in the following manner:

$$\begin{aligned} V(t) = & \sum_{\iota=1}^n \delta_\iota \overline{r}_\iota(t) r_\iota(t) + \sum_{\iota=1}^n \left(\overline{\dot{r}_\iota(t) + r_\iota(t)} \right) \left(\dot{r}_\iota(t) + r_\iota(t) \right) \\ & + \sum_{\iota=1}^n \sum_{j=1}^n \int_{t - \tau_j(t)}^t \frac{(\epsilon_3 + \epsilon_4)|b_{\iota j}|^2}{(1 - \tau)} \overline{g}_j(r_j(s)) g_j(r_j(s)) ds \\ & + \sum_{\iota=1}^n \sum_{j=1}^n 2\rho_{\iota j} \int_{-\sigma_j(t)}^0 \int_{t+s}^t \overline{r}_\iota(\vartheta) r_\iota(\vartheta) d s d\vartheta \\ & + \sum_{\iota=1}^n \int_{-\tau}^0 \int_{t+\theta}^t \gamma_\iota \overline{f}_\iota(r_\iota(s)) f_\iota(r_\iota(s)) d s d\theta \end{aligned} \tag{2}$$

Taking derivative along Equation (1), one can get

$$\begin{aligned} \frac{dV(t)}{dt} = & \sum_{\iota=1}^n \delta_\iota \overline{\dot{r}}_\iota(t) r_\iota(t) + \sum_{\iota=1}^n \delta_\iota \overline{r}_\iota(t) \dot{r}_\iota(t) + \sum_{\iota=1}^n (\overline{\ddot{r}}_\iota(t) + \overline{\dot{r}}_\iota(t))(\dot{r}_\iota(t) + r_\iota(t)) \\ & + \sum_{\iota=1}^n (\overline{\dot{r}}_\iota(t) + \overline{r}_\iota(t))(\ddot{r}_\iota(t) + \dot{r}_\iota(t)) + \sum_{\iota=1}^n \sum_{j=1}^n \frac{(\epsilon_3 + \epsilon_4)|b_{\iota j}|^2}{(1 - \tau)} \\ & (\overline{g}_j(r_j(t)) g_j(r_j(t)) - \overline{g}_j(r_j(t - \tau_j)) g_j\left(r_j(t - \tau_j(t))(1 - \dot{\tau}_j(t))\right) \\ & + \sum_{\iota=1}^n \sum_{j=1}^n 2\rho_{\iota j} \left(\sigma_j(t) \overline{r}_j(t) r_j(t) - (1 - \dot{\sigma}_j(t)) \int_{t - \sigma_j(t)}^t \overline{r}_j(s) r_j(s) ds \right) \\ & + \sum_{\iota=1}^n \tau \gamma_\iota \overline{f}_\iota(r_\iota(t)) f_\iota(r_\iota(t)) - \sum_{\iota=1}^n \gamma_\iota \int_{t-\tau}^t \overline{f}_\iota(r_\iota(s)) f_\iota(r_\iota(s)) ds, \end{aligned}$$

$$\begin{aligned} \frac{dV(t)}{dt} \le & \sum_{\iota=1}^n \delta_\iota \overline{\dot{r}}_\iota(t) r_\iota(t) + \sum_{\iota=1}^n \delta_\iota \overline{r}_\iota(t) \dot{r}_\iota(t) + \sum_{\iota=1}^n (\overline{\dot{r}}_\iota(t) \dot{r}_\iota(t) + \overline{\dot{r}}_\iota(t) r_\iota(t)) \\ & + \sum_{\iota=1}^n (\overline{r}_\iota(t) \dot{r}_\iota(t) + \overline{r}_\iota(t) \dot{r}_\iota(t)) + \sum_{\iota=1}^n \Bigg\{ - \alpha_\iota \overline{\dot{r}}_\iota(t) - \beta_\iota \overline{r}_\iota(t) \\ & + \sum_{j=1}^n \overline{f}_j(r_j(t)) \overline{a}_{\iota j} + \sum_{j=1}^n \overline{g}_j(r_j(t - \tau_j(t))) \overline{b}_{\iota j} \\ & + \sum_{j=1}^n \int_{t - \sigma_j(t)}^t \overline{h}_j(r_j(s)) \overline{d}_{\iota j} ds + \overline{u}_\iota(t) \Bigg\} (\dot{r}_\iota(t) + r_\iota(t)) \end{aligned}$$

$$
\begin{aligned}
&+\sum_{i=1}^{n}(\bar{\dot{r}}_i(t)+\bar{r}_i(t))\Bigg\{-\alpha_i\dot{r}_i(t)-\beta_i r_i(t)+\sum_{j=1}^{n}a_{ij}f_j(r_j(t))\\
&+\sum_{j=1}^{n}b_{ij}g_j(r_j(t-\tau_j(t)))\sum_{j=1}^{n}\int_{t-\sigma_j(t)}^{t}h_j(r_j(s))d_{ij}ds+u_i(t)\Bigg\}\\
&+\sum_{i=1}^{n}\tau\gamma_i\bar{f}_i(r_i(t))f_i(r_i(t))+\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{(\epsilon_3+\epsilon_4)|b_{ij}|^2}{(1-\tau)}\bar{g}_j(r_j(t))g_j(r_j(t))\\
&+\sum_{i=1}^{n}\sum_{j=1}^{n}2\rho_{ij}\left(\sigma_j(t)\bar{r}_j(t)r_j(t)-(1-\dot{\sigma}_j(t))\int_{t-\sigma_j(t)}^{t}\bar{r}_j(s)r_j(s)ds\right)\\
&-\sum_{i=1}^{n}\sum_{j=1}^{n}(\epsilon_3+\epsilon_4)|b_{ij}|^2\bar{g}_j(r_j(t-\tau_j(t)))\times g_j(r_j(t-\tau_j(t))),
\end{aligned}
$$

$$
\begin{aligned}
\frac{dV(t)}{dt}\leq&\sum_{i=1}^{n}\delta_i\bar{\dot{r}}_i(t)r_i(t)+\sum_{i=1}^{n}\delta_i\bar{r}_i(t)\dot{r}_i(t)+\sum_{i=1}^{n}(\bar{\dot{r}}_i(t)\dot{r}_i(t)+\bar{\dot{r}}_i(t)r_i(t))\\
&+\sum_{i=1}^{n}(\bar{\dot{r}}_i(t)\dot{r}_i(t)+\bar{r}_i(t)\dot{r}_i(t))+\sum_{i=1}^{n}\Big(-2\alpha_i\bar{\dot{r}}_i(t)\dot{r}_i(t)-\alpha_i\bar{\dot{r}}_i(t)r_i(t)\\
&-\beta_i\bar{r}_i(t)\dot{r}_i-2\beta_i\bar{r}_i(t)r_i(t)-\alpha_i\bar{r}_i(t)\dot{r}_i(t)-\beta_i\bar{\dot{r}}_i(t)r_i(t)\Big)\\
&+\sum_{i=1}^{n}\sum_{j=1}^{n}\Big(\bar{f}_j(r_j(t))\bar{a}_{ij}\dot{r}_i(t)+\bar{\dot{r}}_i(t)a_{ij}f_j(r_j(t))\Big)\\
&+\sum_{i=1}^{n}\sum_{j=1}^{n}\Big(\bar{f}_j(r_j(t))\bar{a}_{ij}r_i(t)+\bar{r}_i(t)a_{ij}f_j(r_j(t))\Big)\\
&+\sum_{i=1}^{n}\sum_{j=1}^{n}\Big(\bar{g}_j(r_j(t-\tau_j(t))\bar{b}_{ij}\dot{r}_i(t)+\bar{\dot{r}}_i(t)b_{ij}g_j\big(r_j(t-\tau_j(t))\big)\Big)\\
&+\sum_{i=1}^{n}\sum_{j=1}^{n}\Big(\bar{g}_j(r_j(t-\tau_j(t))\bar{b}_{ij}r_i(t)\ +\bar{r}_i(t)b_{ij}g_j\big(r_j(t-\tau_j(t))\big)\Big)\\
&+\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\bar{\dot{r}}_i(t)d_{ij}\int_{t-\sigma_j(t)}^{t}h_j(r_j(s)ds+\int_{t-\sigma_j(t)}^{t}\bar{h}_j\Big(r_j(s)ds\bar{d}_{ij}\dot{r}_i(t)ds\right)\\
&+\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\bar{r}_i(t)d_{ij}\int_{t-\sigma_j(t)}^{t}h_j(r_j(s)ds+\int_{t-\sigma_j(t)}^{t}\bar{h}_j\Big(r_j(s)ds\bar{d}_{ij}r_i(t)ds\right)\\
&+\sum_{i=1}^{n}\Big(\bar{\dot{r}}_i(t)u_i(t)+\bar{u}_i(t)\dot{r}_i(t)\Big)+\sum_{i=1}^{n}\big(\bar{r}_i(t)u_i(t)+\bar{u}_i(t)r_i(t)\big)\\
&+\sum_{i=1}^{n}\tau\bar{f}_i(r_i(t))f_i(r_i(t))+\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{(\epsilon_3+\epsilon_4)|b_{ij}|^2}{(1-\tau)}\bar{g}_j(r_j(t))g_j(r_j(t))\\
&+\sum_{i=1}^{n}\sum_{j=1}^{n}2\rho_{ij}\left(\sigma_j(t)\bar{r}_j(t)r_j(t)-(1-\dot{\sigma}_j(t))\int_{t-\sigma_j(t)}^{t}\bar{r}_j(s)r_j(s)ds\right)\\
&-\sum_{i=1}^{n}\sum_{j=1}^{n}(\epsilon_3+\epsilon_4)|b_{ij}|^2\bar{g}_j(r_j(t-\tau_j(t)))g_j(r_j(t-\tau_j(t))).
\end{aligned}
$$

On behalf of Lemma 2, one can get

$$
\begin{aligned}
\frac{dV(t)}{dt}\leq&\sum_{i=1}^{n}(\delta_i-\alpha_i-\beta_i+1)\bar{\dot{r}}_i(t)r_i(t)+\sum_{i=1}^{n}(\delta_i-\alpha_i-\beta_i+1)\bar{r}_i(t)\dot{r}_i(t)\\
&+\sum_{i=1}^{n}(2-2\alpha_i)\bar{\dot{r}}_i(t)\dot{r}_i(t)+\sum_{i=1}^{n}(-2\beta_i)\bar{r}_i(t)r_i(t)\\
&+\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\epsilon_1\bar{f}_j(r_j(t))f_j(r_j(t))+\frac{1}{\epsilon_1}\bar{\dot{r}}_i(t)a_{ij}\bar{a}_{ij}\dot{r}_i(t)\right)\\
&+\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\epsilon_2\bar{f}_j(r_j(t))\times f_j(r_j(t))+\frac{1}{\epsilon_2}\bar{r}_i(t)a_{ij}\bar{a}_{ij}r_i(t)\right)\\
&+\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\epsilon_3\bar{g}_j(r_j(t-\tau_j(t)))\bar{b}_{ij}b_{ij}g_j(r_j(t-\tau_j(t)))+\frac{1}{\epsilon_3}\bar{\dot{r}}_i(t)\dot{r}_i(t)\right)\\
&+\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\epsilon_4\bar{g}_j(r_j(t-\tau_j(t)))\bar{b}_{ij}b_{ij}g_j(r_j(t-\tau_j(t)))+\frac{1}{\epsilon_4}\bar{r}_i(t)r_i(t)\right)\\
&+\sum_{i=1}^{n}\left(\epsilon_5\bar{\dot{r}}_i(t)\dot{r}_i(t)+\frac{1}{\epsilon_5}\bar{u}_i(t)u_i(t)\right)+\sum_{i=1}^{n}\left(\epsilon_6\bar{r}_i(t)r_i(t)+\frac{1}{\epsilon_6}\bar{u}_i(t)u_i(t)\right)\\
&+\sum_{i=1}^{n}\tau\gamma_i\bar{f}_i(r_i(t))f_i(r_i(t))+\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{(\epsilon_3+\epsilon_4)|b_{ij}|^2}{(1-\tau)}\bar{g}_j(r_j(t))g_j(r_j(t))\\
&-\sum_{i=1}^{n}\sum_{j=1}^{n}(\epsilon_3+\epsilon_4)|b_{ij}|^2\bar{g}_j(r_j(t-\tau_j(t)))g_j(r_j(t-\tau_j(t)))\\
&+\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\int_{t-\sigma_j(t)}^{t}\epsilon_7\bar{h}_j(r_j(s))h_j(r_j(s))ds\ +\ \frac{\sigma_j(t)}{\epsilon_7}\bar{\dot{r}}_i(t)d_{ij}\bar{d}_{ij}\dot{r}_i(t)\right)\\
&+\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\int_{t-\sigma_j(t)}^{t}\epsilon_8\bar{h}_j(r_j(s))h_j(r_j(s))ds+\frac{\sigma_j(t)}{\epsilon_8}\bar{r}_i(t)d_{ij}\bar{d}_{ij}r_i(t)\right)\\
&+\sum_{i=1}^{n}\sum_{j=1}^{n}2\rho_{ij}\left(\sigma_j(t)\bar{r}_j(t)r_j(t)-(1-\dot{\sigma}_j(t))\int_{t-\sigma_j(t)}^{t}\bar{r}_j(s)r_j(s)ds\right).
\end{aligned}
$$

Following Assumptions 1 and 2 and rearranging the terms, one can get

$$
\begin{aligned}
\frac{dV(t)}{dt}\leq&\sum_{i=1}^{n}(\delta_i-\alpha_i-\beta_i+1)\bar{\dot{r}}_i(t)r_i(t)+\sum_{i=1}^{n}(\delta_i-\alpha_i-\beta_i+1)\bar{r}_i(t)\dot{r}_i(t)\\
&+\sum_{i=1}^{n}(2-2\alpha_i)\bar{\dot{r}}_i(t)\dot{r}_i(t)+\sum_{i=1}^{n}(-2\beta_i)\bar{r}_i(t)r_i(t)\\
&+\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\epsilon_1M_j^2\bar{r}_j(t)r_j(t)+\frac{1}{\epsilon_1}\bar{\dot{r}}_i(t)a_{ij}\bar{a}_{ij}\dot{r}_i(t)\right)\\
&+\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\epsilon_2M_j^2\bar{r}_j(t)\Big)r_j(t)+\frac{1}{\epsilon_2}\bar{r}_i(t)a_{ij}\bar{a}_{ij}r_i(t)\right)\\
&+\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\epsilon_3L_j^2\bar{r}_j(t-\tau_j(t))\bar{b}_{ij}b_{ij}r_j(t-\tau_j(t))+\frac{1}{\epsilon_3}\bar{\dot{r}}_i(t)\dot{r}_i(t)\right)\\
&+\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\epsilon_4L_j^2\bar{r}_j(t-\tau_j(t))\bar{b}_{ij}b_{ij}r_j(t-\tau_j(t))+\frac{1}{\epsilon_4}\bar{r}_i(t)r_i(t)\right)\\
&+\sum_{i=1}^{n}\left(\epsilon_5\bar{\dot{r}}_i(t)\dot{r}_i(t)+\frac{1}{\epsilon_5}\bar{u}_i(t)u_i(t)\right)\\
&+\sum_{i=1}^{n}\left(\epsilon_6\bar{r}_i(t)r_i(t)+\frac{1}{\epsilon_6}\bar{u}_i(t)u_i(t)\right)+\sum_{i=1}^{n}\tau\gamma_iM_i^2\bar{r}_i(t))r_i(t)\\
&+\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{(\epsilon_3+\epsilon_4)|b_{ij}|^2}{(1-\tau)}L_j^2\bar{r}_j(t))r_j(t))\\
&+\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\int_{t-\sigma_j(t)}^{t}\epsilon_7K_j^2\bar{r}_j(s)r_j(s)ds+\frac{\sigma_j}{\epsilon_7}\bar{\dot{r}}_i(t)d_{ij}\bar{d}_{ij}\dot{r}_i(t)\right)\\
&+\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\int_{t-\sigma_j(t)}^{t}\epsilon_8K_j^2\bar{r}_j(s)\times r_j(s)ds+\frac{\sigma_j}{\epsilon_8}\bar{r}_i(t)d_{ij}\bar{d}_{ij}r_i(t)\right)\\
&-\sum_{i=1}^{n}\sum_{j=1}^{n}(\epsilon_3+\epsilon_4)|b_{ij}|^2L_j^2\bar{r}_j(t-\tau_j(t)))r_j(t-\tau_j(t)))\\
&+\sum_{i=1}^{n}\sum_{j=1}^{n}2\rho_{ij}\left(\sigma_j\bar{r}_j(t)r_j(t)-0.5\int_{t-\sigma_j(t)}^{t}\bar{r}_j(s)r_j(s)ds\right),
\end{aligned}
$$

Select $\rho_{\iota j} = (\epsilon_7 + \epsilon_8)K_j^2$, and $\epsilon^* = \left(\frac{1}{\epsilon_5} + \frac{1}{\epsilon_6}\right)$, we obtain

$$\frac{dV(t)}{dt} \le \sum_{\iota=1}^{n}(\delta_\iota - \alpha_\iota - \beta_\iota + 1)\bar{\dot{r}}_\iota(t)r_\iota(t) + \sum_{\iota=1}^{n}(\delta_\iota - \alpha_\iota - \beta_\iota + 1)\bar{r}_\iota(t)\dot{r}_\iota(t) + \sum_{\iota=1}^{n}\Bigg(2 - 2\alpha_\iota + \frac{n}{\epsilon_3} + \epsilon_5 + \sum_{j=1}^{n}\left(\frac{1}{\epsilon_1}a_{\iota j}\bar{a}_{\iota j} + \frac{\sigma_j}{\epsilon_7}K_j^2 d_{\iota j}\bar{d}_{\iota j}\right)\Bigg)\bar{\dot{r}}_\iota(t)\dot{r}_\iota(t) + \sum_{\iota=1}^{n}\Bigg(\epsilon_6 + \frac{n}{\epsilon_4} + \tau\gamma_\iota M_\iota^2 - 2\beta_\iota + n\epsilon_1 M_\iota^2 + n\epsilon_2 M_\iota^2 + \sum_{j=1}^{n}\left(\frac{1}{\epsilon_2}a_{\iota j}\bar{a}_{\iota j} + \frac{\sigma_j}{\epsilon_8}d_{\iota j}\bar{d}_{\iota j} + \frac{(\epsilon_3+\epsilon_4)|b_{j\iota}|^2 L_\iota^2}{(1-\tau)}\right)\Bigg) \bar{r}_\iota(t)r_\iota(t) + \sum_{\iota=1}^{n}\epsilon^*\bar{u}_\iota(t)u_\iota(t).$$

$$\le \sum_{\iota=1}^{n} Z_\iota\left(\bar{\dot{r}}_\iota(t)r_\iota(t) + \bar{r}_\iota(t)\dot{r}_\iota(t)\right) + \sum_{\iota=1}^{n} Y_\iota \bar{\dot{r}}_\iota(t)\dot{r}_\iota(t) + \sum_{\iota=1}^{n} X_\iota \bar{r}_\iota(t)r_\iota(t) + \sum_{\iota=1}^{n}\epsilon^*\bar{u}_\iota(t)u_\iota(t)$$

$$\text{or, } \frac{dV(t)}{dt} \le \sum_{\iota=1}^{n} Y_\iota \overline{\left(\dot{r}_\iota(t) + \frac{Z_\iota}{Y_\iota}r_\iota(t)\right)}\left(\dot{r}_\iota(t) + \frac{Z_\iota}{Y_\iota}r_\iota(t)\right) + \sum_{\iota=1}^{n}\left(X_\iota - \frac{Z_\iota^2}{Y_\iota}\right)\bar{r}_\iota(t)r_\iota(t) + \sum_{\iota=1}^{n}\epsilon^*\bar{u}_\iota(t)u_\iota(t) \tag{3}$$

$$\text{or, } \frac{dV(t)}{dt} \le \sum_{\iota=1}^{n}\left\{\left(X_\iota - \frac{Z_\iota^2}{Y_\iota}\right)\bar{r}_\iota(t)r_\iota(t) + \epsilon^*\bar{u}_\iota(t)u_\iota(t)\right\}$$

$$\text{or, } \frac{dV(t)}{dt} \le -\kappa\|r(t)\|^2 + \epsilon^*\|u(t)\|^2 < 0 \tag{4}$$

for $\|r(t)\|^2 > \frac{\epsilon^*\|u(t)\|^2}{\kappa}$.

That is, $r(t) \notin \Upsilon_1$. This implies that if $\Psi(s), \Psi^*(s) \in \Upsilon_1$, then $r(t) \subset \Upsilon_1$ for $t > t_0$, which means that the set Υ is a positive invariant set of (1). When $\Psi(s), \Psi^*(s) \notin \Upsilon_1$, there is a $T > 0$ such that $r(t) \subset \Upsilon_1$, for $t > T + t_0$. This means Υ_1 is a global attractive set, and then the QVINN system is a GD system, Υ_1 is a global attractive set, and it is a positively invariant set of (1). □

Corollary 1. *If $d_{\iota j} = 0$ in system (1) and presume Assumptions 1 and 2, there exist $\delta_\iota > 0$ such that $X_\iota < 0$, $Y_\iota < 0$, $Z_\iota^2 \le X_\iota Y_\iota$, $\iota \in I$, where $Z_\iota = \delta_\iota - \alpha_\iota - \beta_\iota + 1$, $Y_\iota = 2 - 2\alpha_\iota + \epsilon_5 + \frac{n}{\epsilon_3} + \sum_{j=1}^{n}\frac{1}{\epsilon_1}a_{\iota j}\bar{a}_{\iota j}$ and $X_\iota = \epsilon_6 + \frac{n}{\epsilon_4} + \tau\gamma_\iota M_\iota^2 - 2\beta_\iota + n\epsilon_1 M_\iota^2 + n\epsilon_2 M_\iota^2 + \sum_{j=1}^{n}\left(\frac{1}{\epsilon_2}a_{\iota j}\bar{a}_{\iota j} + \frac{(\epsilon_3+\epsilon_4)|b_{j\iota}|^2 L_\iota^2}{(1-\tau)}\right)$.*

Then, the QVINN model (1) is a global dissipative system, and the set $\Upsilon_1 = \left\{r(t) \in \mathbf{Q}^n | \bar{r}_\iota(t)r_\iota(t) \le \frac{\epsilon^*}{\kappa}\bar{u}_\iota(t)u_\iota(t), \quad \iota \in I\right\}$ is a global attractive set and a positively invariant set of QVINNs (1)

Corollary 2. *Presume Assumptions 1 and 2, and if the time delay is bounded, that is, $\tau_j(t) \le \tau$ and $\dot{\tau}_j(t) < 1$, where $\tau > 0$. Then there exist $\delta_\iota > 0$ and the systems (1), such that $X_\iota < 0$, $Y_\iota < 0$, $Z_\iota^2 \le 4X_\iota Y_\iota$ $\iota \in I$, where $Z_\iota = \delta_\iota - \alpha_\iota - \beta_\iota + 1$, $Y_\iota = 2 - 2\alpha_\iota + \epsilon_5 + \frac{n}{\epsilon_3} + \sum_{j=1}^{n}\left(\frac{1}{\epsilon_1}a_{\iota j}\bar{a}_{\iota j} + \frac{\sigma_j}{\epsilon_7}K_j^2 d_{\iota j}\bar{d}_{\iota j}\right)$ and $X_\iota = \epsilon_6 + \frac{n}{\epsilon_4} + \tau\gamma_\iota M_\iota^2 - 2\beta_\iota + n\epsilon_1 M_\iota^2 + n\epsilon_2 M_\iota^2 + \sum_{j=1}^{n}\left(\frac{1}{\epsilon_2}a_{\iota j}\bar{a}_{\iota j} + \frac{\sigma_j}{\epsilon_7}K_j^2 d_{\iota j}\bar{d}_{\iota j} + \frac{(\epsilon_3+\epsilon_4)|b_{j\iota}|^2 L_\iota^2}{(1-\tau)}\right)$.*

The system (1) is global dissipative under the same conditions of the Theorem 1.

Corollary 3. *Presume Assumptions 1 and 2, such that, if $\delta_\iota = \alpha_\iota + \beta_\iota - 1$ and $Y_\iota < 0, Z_\iota < 0$, then the system (1) is global dissipative and the set $\Upsilon_2 = \left\{r(t) \in \mathbf{Q}^n | \bar{r}_\iota(t)r_\iota(t) \le \left(\frac{\pi\epsilon^*}{\min_{1\le\iota\le n}|X_\iota|}\right)\bar{u}_\iota(t)u_\iota(t), \ \iota \in I\right\}$ is the global attractive set.*

Theorem 2. *Followed from Assumptions 1 and 3, the considered system is GED for $\lambda < \bar{\sigma}$ such that $X_\iota < 0$, $Z_\iota^2 \le 4X_\iota Y_\iota$ $\iota \in I$, where $Z_\iota = \lambda + \delta_\iota + 1 - \alpha_\iota - \beta_\iota$, $Y_\iota = \lambda + 2 - 2\alpha_\iota + \frac{n}{\epsilon_3} + \epsilon_5 + \sum_{j=1}^{n}\left(\frac{1}{\epsilon_1}a_{\iota j}\bar{a}_{\iota j} + \frac{\sigma_j}{\epsilon_7}d_{\iota j}\bar{d}_{\iota j}\right)$, $X_\iota = \lambda(\delta_\iota + 1) - 2\beta_\iota + n\epsilon_1 M_\iota^2 + n\epsilon_2 M_\iota^2 + \frac{n}{\epsilon_4} + \epsilon_6 + \sum_{j=1}^{n}\left(\frac{1}{\epsilon_2}a_{\iota j}\bar{a}_{\iota j} + \frac{2\epsilon L_\iota^2|b_{j\iota}|^2}{(1-\tau)} + \frac{\sigma_j}{\epsilon_8}d_{\iota j}\bar{d}_{\iota j}\right)$, where σ is the sufficient large positive number. Then QVINN (1) is a global dissipative system, and the set $\Upsilon_1^* = \left\{r(t) \in \mathbf{Q}^n | \bar{r}_\iota(t)r_\iota(t) \le \frac{\epsilon^*}{\kappa_1}\bar{u}_\iota(t)u_\iota(t), \quad \iota \in I\right\}$ is a global attractive set and a positively invariant set of QVINN model (1), where $\kappa_1 = min_{1\le\iota\le n}\left\{\frac{Z_\iota^2}{Y_\iota} - X_\iota\right\}$.*

Proof. We can consider the Lyapunov functional as follows:

$$V(t) = \sum_{\iota=1}^{n}\delta_\iota\bar{r}_\iota(t)r_\iota(t)e^{\lambda t} + \sum_{\iota=1}^{n}\left(\overline{\dot{r}_\iota(t) + r_\iota(t)}\right)\left(\dot{r}_\iota(t) + r_\iota(t)\right)e^{\lambda t} + \sum_{\iota=1}^{n}\sum_{j=1}^{n}\frac{2\epsilon L_j^2|b_{\iota j}|^2}{(1-\tau)}\int_{t-\tau_j(t)}^{t}\bar{r}_j(s)r_j(s)e^{\lambda s} + \sum_{\iota=1}^{n}\sum_{j=1}^{n}2\rho_{\iota j}\int_{-\sigma_j(t)}^{0}\int_{t+s}^{t}\bar{r}_\iota(\vartheta)r_\iota(\vartheta)e^{\lambda\vartheta}\,ds\,d\vartheta \tag{5}$$

Taking derivative along Equation (1), we have

$$\frac{dV(t)}{dt} = \sum_{\iota=1}^{n}\left(\lambda\delta_\iota\bar{r}_\iota(t)r_\iota(t) + \delta_\iota\bar{\dot{r}}_\iota(t)r_\iota(t) + \delta_\iota\bar{r}_\iota(t)\dot{r}_\iota(t)\right)e^{\lambda t} + \sum_{\iota=1}^{n}\lambda\left(\left(\bar{\dot{r}}_\iota(t) + \bar{r}_\iota(t)\right)\right)\left(\dot{r}_\iota(t) + r_\iota(t)\right)e^{\lambda t} + \sum_{\iota=1}^{n}(\bar{\ddot{r}}_\iota(t) + \bar{\dot{r}}_\iota(t))(\dot{r}_\iota(t) + r_\iota(t))e^{\lambda t} + \sum_{\iota=1}^{n}\left(\bar{\dot{r}}_\iota(t) + \bar{r}_\iota(t)\right)\left(\ddot{r}_\iota(t) + \dot{r}_\iota(t)\right)e^{\lambda t} + \sum_{\iota=1}^{n}\sum_{j=1}^{n}\frac{2\epsilon L_j^2|b_{\iota j}|^2}{(1-\tau)}\Big(\bar{r}_j(t)r_j(t)e^{\lambda t} - \bar{r}_j(t-\tau_j(t)) r_j(t-\tau_j(t))(1-\dot{\tau}_j(t))e^{\lambda(t-\tau_j(t))}\Big) + \sum_{\iota=1}^{n}\sum_{j=1}^{n}2\rho_{\iota j}\left(\sigma_j(t)\bar{r}_j(t)r_j(t)e^{\lambda t} - (1-\dot{\sigma}_j(t))\int_{t-\sigma_j(t)}^{t}\bar{r}_j(s)r_j(s)e^{\lambda s}ds\right) \tag{6}$$

$$
= \sum_{\imath=1}^{n}\Big(\lambda(\delta_\imath+1)\overline{r}_\imath(t)r_\imath(t)+(\lambda+\delta_\imath+1)\dot{\overline{r}}_\imath(t)r_\imath(t)
$$

$$
+(\lambda+\delta_\imath+1)\overline{r}_\imath(t)\dot{r}_\imath(t)+(\lambda+2)\dot{\overline{r}}_\imath(t)\dot{r}_\imath(t)\Big)e^{\lambda t}
$$

$$
+\sum_{\imath=1}^{n}\Bigg\{-\alpha_\imath\dot{\overline{r}}_\imath(t)-\beta_\imath\overline{r}_\imath(t)+\sum_{\jmath=1}^{n}\overline{f}_\jmath(r_\jmath(t))\overline{a}_{\imath\jmath}
$$

$$
+\sum_{\jmath=1}^{n}\overline{g}_\jmath(r_\jmath(t-\tau_\jmath(t)))\overline{b}_{\imath\jmath}+\sum_{\jmath=1}^{n}\int_{t-\sigma_\jmath(t)}^{t}\overline{h}_\jmath(r_\jmath(s))\overline{d}_{\imath\jmath}ds+\overline{u}_\imath(t)\Bigg\}
$$

$$
\times(\dot{r}_\imath(t)+r_\imath(t))e^{\lambda t}+\sum_{\imath=1}^{n}(\dot{\overline{r}}_\imath(t)+\overline{r}_\imath(t))
$$

$$
\Bigg\{-\alpha_\imath\dot{r}_\imath(t)-\beta_\imath r_\imath(t)+\sum_{\jmath=1}^{n}a_{\imath\jmath}f_\jmath(r_\jmath(t))+\sum_{\jmath=1}^{n}b_{\imath\jmath}g_\jmath(r_\jmath(t-\tau_\jmath(t)))
$$

$$
+\sum_{\jmath=1}^{n}\int_{t-\sigma_\jmath(t)}^{t}d_{\imath\jmath}h_\jmath(r_\jmath(s))ds+u_\imath(t)\Bigg\}e^{\lambda t}
$$

$$
+\sum_{\imath=1}^{n}\sum_{\jmath=1}^{n}\frac{2L_\jmath^2\epsilon|b_{\imath\jmath}|^2}{(1-\tau)}\Big(\overline{r}_\jmath(t)r_\jmath(t)e^{\lambda t}-\overline{r}_\jmath(t-\tau_\jmath(t))r_\jmath(t-\tau_\jmath(t)
$$

$$
\times(1-\dot{\tau}_\jmath(t)))e^{\lambda(t-\tau_\jmath(t))}\Big)+\sum_{\imath=1}^{n}\sum_{\jmath=1}^{n}2\rho_{\imath\jmath}
$$

$$
\left(\sigma_\jmath(t)\overline{r}_\jmath(t)r_\jmath(t)e^{\lambda t}-(1-\dot{\sigma}_\jmath(t))\int_{t-\sigma_\jmath(t)}^{t}\overline{r}_\jmath(s)r_\jmath(s)e^{\lambda s}ds\right),
$$

$$
=\sum_{\imath=1}^{n}\Big(\big(\lambda(\delta_\imath+1)-2\beta_\imath\big)\overline{r}_\imath(t)r_\imath(t)+(\lambda+\delta_\imath+1-\alpha_\imath-\beta_\imath)\dot{\overline{r}}_\imath(t)r_\imath(t)
$$

$$
+(\lambda+\delta_\imath+1-\alpha_\imath-\beta_\imath)\overline{r}_\imath(t)\dot{r}_\imath(t)+(\lambda+\delta_\imath+1-2\alpha_\imath)\dot{\overline{r}}_\imath(t)\dot{r}_\imath(t)\Big)e^{\lambda t}
$$

$$
+\sum_{\imath=1}^{n}\sum_{\jmath=1}^{n}\Big(\overline{f}_\jmath(r_\jmath(t))\overline{a}_{\imath\jmath}\dot{r}_\imath(t)+\dot{\overline{r}}_\imath(t)a_{\imath\jmath}f_\jmath(r_\jmath(t))\Big)e^{\lambda t}
$$

$$
+\sum_{\imath=1}^{n}\sum_{\jmath=1}^{n}\Big(\overline{f}_\jmath(r_\jmath(t))\times\overline{a}_{\imath\jmath}r_\imath(t)+\overline{r}_\imath(t)a_{\imath\jmath}f_\jmath(r_\jmath(t))\Big)e^{\lambda t}
$$

$$
+\sum_{\imath=1}^{n}\sum_{\jmath=1}^{n}\Big(\overline{g}_\jmath(r_\jmath(t-\tau_\jmath(t)))\overline{b}_{\imath\jmath}\dot{r}_\imath(t)+\dot{\overline{r}}_\imath(t)b_{\imath\jmath}\overline{g}_\jmath(r_\jmath(t-\tau_\jmath(t)))\Big)e^{\lambda t}
$$

$$
+\sum_{\imath=1}^{n}\sum_{\jmath=1}^{n}\Big(\overline{g}_\jmath(r_\jmath(t-\tau_\jmath(t)))\overline{b}_{\imath\jmath}r_\imath(t)+\overline{r}_\imath(t)b_{\imath\jmath}\overline{g}_\jmath(r_\jmath(t-\tau_\jmath(t)))\Big)e^{\lambda t}
$$

$$
+\sum_{\imath=1}^{n}\Big(\overline{u}_\imath(t)\dot{r}_\imath(t)+\dot{\overline{r}}_\imath(t)u_\imath(t)\Big)e^{\lambda t}+\sum_{\imath=1}^{n}\Big(\overline{u}_\imath(t)r_\imath(t)+\overline{r}_\imath(t)u_\imath(t)\Big)e^{\lambda t}
$$

$$
+\sum_{\imath=1}^{n}\sum_{\jmath=1}^{n}\frac{2\epsilon L_\jmath^2|b_{\imath\jmath}|^2}{(1-\tau)}\Big(\overline{r}_\jmath(t)r_\jmath(t)e^{\lambda t}-\overline{r}_\jmath(t-\tau_\jmath(t))r_\jmath(t-\tau_\jmath(t))
$$

$$
\times(1-\dot{\tau}_\jmath(t))e^{\lambda(t-\tau_\jmath(t))}\Big)+\sum_{\imath=1}^{n}\sum_{\jmath=1}^{n}\Bigg(\dot{\overline{r}}_\imath(t)d_{\imath\jmath}\int_{t-\sigma_\jmath(t)}^{t}h_\jmath(r_\jmath(s)ds
$$

$$
+\int_{t-\sigma_\jmath(t)}^{t}\overline{h}_\jmath\Big(r_\jmath(s)ds\overline{d}_{\imath\jmath}r_\imath(t)ds\Big)e^{\lambda t}+\sum_{\imath=1}^{n}\sum_{\jmath=1}^{n}\Bigg(\overline{r}_\imath(t)d_{\imath\jmath}\int_{t-\sigma_\jmath(t)}^{t}h_\jmath(r_\jmath(s)ds
$$

$$
+\int_{t-\sigma_\jmath(t)}^{t}\overline{h}_\jmath\Big(r_\jmath(s)ds\overline{d}_{\imath\jmath}\dot{r}_\imath(t)ds\Big)e^{\lambda t}+\sum_{\imath=1}^{n}\sum_{\jmath=1}^{n}2\rho_{\imath\jmath}\big(\sigma_\jmath(t)\overline{r}_\jmath(t)r_\jmath(t)e^{\lambda t}
$$

$$
-(1-\dot{\sigma}_\jmath(t))\int_{t-\sigma_\jmath(t)}^{t}\overline{r}_\jmath(s)r_\jmath(s)e^{\lambda s}ds\Bigg).
$$

Now by using Lemma 2 and Assumptions 1 and 3, one can have

$$
\frac{dV}{dt}\leq\sum_{\imath=1}^{n}\Big(\big(\lambda(\delta_\imath+1)-2\beta_\imath\big)\overline{r}_\imath(t)r_\imath(t)+(\lambda+1+\delta_\imath-\alpha_\imath-\beta_\imath)\dot{\overline{r}}_\imath(t)r_\imath(t)
$$

$$
+(\lambda+1+\delta_\imath-\alpha_\imath-\beta_\imath)\overline{r}_\imath(t)\dot{r}_\imath(t)+(\lambda+2-2\alpha_\imath)\dot{\overline{r}}_\imath(t)\dot{r}_\imath(t)\Big)e^{\lambda t}
$$

$$
+\sum_{\imath=1}^{n}\sum_{\jmath=1}^{n}\Big(\epsilon_1M_\jmath^2\overline{r}_\jmath(t)r_\jmath(t)++\frac{1}{\epsilon_1}\dot{\overline{r}}_\imath(t)a_{\imath\jmath}\overline{a}_{\imath\jmath}\dot{r}_\imath(t)\Big)e^{\lambda t}
$$

$$
+\sum_{\imath=1}^{n}\sum_{\jmath=1}^{n}\Big(\epsilon_2M_\jmath^2\overline{r_\jmath(t)}r_\jmath(t)+\frac{1}{\epsilon_2}\overline{r}_\imath(t)a_{\imath\jmath}\overline{a}_{\imath\jmath}r_\imath(t)\Big)e^{\lambda t}
$$

$$
+\sum_{\imath=1}^{n}\sum_{\jmath=1}^{n}\Big(\epsilon_3L_\jmath^2\overline{r}_\jmath(t-\tau_\jmath(t))\overline{b}_{\imath\jmath}b_{\imath\jmath}r_\jmath(t-\tau_\jmath(t))e^{-2\overline{\sigma}\tau_\jmath(t)}+\frac{1}{\epsilon_3}\dot{\overline{r}}_\imath(t)\dot{r}_\imath(t)\Big)
$$

$$
+\sum_{\imath=1}^{n}\sum_{\jmath=1}^{n}\Big(\epsilon_4L_\jmath^2\overline{r}_\jmath(t-\tau_\jmath(t))\overline{b}_{\imath\jmath}b_{\imath\jmath}r_\jmath(t-\tau_\jmath(t))e^{-2\overline{\sigma}\tau_\jmath(t)}+\frac{1}{\epsilon_4}\overline{r}_\imath(t)r_\imath(t)\Big)e^{\lambda t}
$$

$$
+\sum_{\imath=1}^{n}\Big(\epsilon_5\dot{\overline{r}}_\imath(t)\dot{r}_\imath(t)+\frac{1}{\epsilon_5}\overline{u}_\imath(t)u_\imath(t)\Big)e^{\lambda t}
$$

$$
+\sum_{\imath=1}^{n}\Big(\epsilon_6\overline{r}_\imath(t)r_\imath(t)+\frac{1}{\epsilon_6}\overline{u}_\imath(t)u_\imath(t)\Big)e^{\lambda t}+\sum_{\imath=1}^{n}\sum_{\jmath=1}^{n}\frac{2\epsilon L_\jmath^2|b_{\imath\jmath}|^2}{(1-\tau)}\overline{r}_\jmath(t)r_\jmath(t)e^{\lambda t}
$$

$$
+\sum_{\imath=1}^{n}\sum_{\jmath=1}^{n}\Bigg(\int_{t-\sigma_\jmath(t)}^{t}\epsilon_7K_\jmath^2\overline{r}_\jmath(s)r_\jmath(s)ds+\frac{\sigma_\jmath}{\epsilon_7}\dot{\overline{r}}_\imath(t)d_{\imath\jmath}\overline{d}_{\imath\jmath}\dot{r}_\imath(t)\Bigg)e^{\lambda t}
$$

$$
+\sum_{\imath=1}^{n}\sum_{\jmath=1}^{n}\Bigg(\int_{t-\sigma_\jmath(t)}^{t}\epsilon_8K_\jmath^2\overline{r}_\jmath(s)r_\jmath(s)ds+\frac{\sigma_\jmath}{\epsilon_8}\overline{r}_\imath(t)d_{\imath\jmath}\overline{d}_{\imath\jmath}r_\imath(t)\Bigg)e^{\lambda t}
$$

$$
-\sum_{\imath=1}^{n}\sum_{\jmath=1}^{n}2\epsilon L_\jmath^2|b_{\imath\jmath}|^2\overline{r}_\jmath(t-\tau_\jmath(t))r_\jmath(t-\tau_\jmath(t))e^{\lambda(t-\tau_\jmath(t))}
$$

$$
+\sum_{\imath=1}^{n}\sum_{\jmath=1}^{n}2\rho_{\imath\jmath}\Bigg(\sigma_\jmath(t)\overline{r}_\jmath(t)r_\jmath(t)e^{\lambda t}-0.5\int_{t-\sigma_\jmath(t)}^{t}\overline{r}_\jmath(s)r_\jmath(s)e^{\lambda s}ds\Bigg).
$$

Since $\frac{e^{\lambda t-2\overline{\sigma}\tau_\jmath(t)}}{e^{\lambda(t-\tau_\jmath(t))}}=e^{\tau_\jmath(t)(\lambda-2\overline{\sigma})}<1$, choose $\rho_{\imath\jmath}=(\epsilon_7+\epsilon_8)K_\jmath^2$, and $\epsilon=$ Max $\left(\epsilon_3,\epsilon_4\right)$

$$
\begin{aligned}
\frac{dV(t)}{dt}\leq&\sum_{\imath=1}^{n}\Big(\big(\lambda+\delta_\imath+1-\alpha_\imath-\beta_\imath\big)\Big(\dot{\overline{r}}_\imath(t)r_\imath(t)+\overline{r}_\imath(t)\dot{r}_\imath(t)\Big)\Big)e^{\lambda t}\\
&+\sum_{\imath=1}^{n}\Bigg(\lambda(\delta_\imath+1)-2\beta_\imath+n\epsilon_1M_\imath^2+n\epsilon_2M_\imath^2\\
&+\frac{n}{\epsilon_4}+\epsilon_6+\sum_{\jmath=1}^{n}\Bigg(\frac{1}{\epsilon_2}a_{\imath\jmath}\overline{a}_{\imath\jmath}+\frac{2\epsilon L_\imath^2|b_{\imath\jmath}|^2}{(1-\tau)}+\frac{\sigma_\jmath}{\epsilon_8}d_{\imath\jmath}\overline{d}_{\imath\jmath}\Bigg)\Bigg)\\
&e^{\lambda t}\overline{r}_\imath(t)r_\imath(t)+\sum_{\imath=1}^{n}\Bigg(\lambda+2-2\alpha_\imath+\frac{n}{\epsilon_3}+\epsilon_5\\
&+\sum_{\jmath=1}^{n}\Bigg(\frac{1}{\epsilon_1}a_{\imath\jmath}\overline{a}_{\imath\jmath}+\frac{\sigma_\jmath}{\epsilon_7}d_{\imath\jmath}\overline{d}_{\imath\jmath}\Bigg)\Bigg)e^{\lambda t}\dot{\overline{r}}_\imath(t)\dot{r}_\imath(t)\\
&+\sum_{\imath=1}^{n}\Bigg(\frac{1}{\epsilon_5}+\frac{1}{\epsilon_6}\Bigg)e^{\lambda t}\overline{u}_\imath(t)u_\imath(t)
\end{aligned}\tag{7}
$$

$$
\begin{aligned}
&=e^{\lambda t}\sum_{\imath=1}^{n}\Bigg\{Z_\imath\Big(\dot{\overline{r}}_\imath(t)r_\imath(t)+\overline{r}_\imath(t)\dot{r}_\imath(t)\Big)+Y_\imath\dot{\overline{r}}_\imath(t)\dot{r}_\imath(t)\\
&\quad+X_\imath\overline{r}_\imath(t)r_\imath(t)+\epsilon^*\overline{u}_\imath(t)u_\imath(t)\Bigg\}\\
&=e^{\lambda t}\sum_{\imath=1}^{n}\Bigg\{Y_\imath\overline{\Big(\dot{r}_\imath(t)+\frac{Z_\imath}{X_\imath}r_\imath(t)\Big)}\Big(\dot{r}_\imath(t)+\frac{Z_\imath}{X_\imath}r_\imath(t)\Big)\\
&\quad+\Bigg(X_\imath-\frac{Z_\imath^2}{Y_\imath}\Bigg)\overline{r}_\imath(t)r_\imath(t)+\epsilon^*\overline{u}_\imath(t)u_\imath(t)\Bigg\}
\end{aligned}\tag{8}
$$

$$\text{or, } \frac{dV(t)}{dt} \leq e^{\lambda t} \sum_{\iota=1}^{n} \left\{ \left(X_\iota - \frac{Z_\iota^2}{Y_\iota} \right) \overline{r}_\iota(t) r_\iota(t) + \epsilon^* \overline{u}_\iota(t) u_\iota(t) \right\} \quad (9)$$

$$\text{or, } \frac{dV(t)}{dt} \leq e^{\lambda t} \left(-\kappa \|r(t)\|^2 + \epsilon^* \|u(t)\|^2 \right) < 0 \quad (10)$$

For $\|r(t)\|^2 > \frac{\epsilon^* \|u(t)\|^2}{\kappa}$, that is, $r(t) \notin \Upsilon_2$. From Equation (10), one has $V(t) \leq V(t_0)$, then from Equation (5), we have $\sum_{\iota=1}^{n} \delta_\iota \overline{r}_\iota(t) r_\iota(t) \leq e^{-\lambda t} V(t) \leq e^{-\lambda t} V(t_0)$.

Let $\delta = \min_{1 \leq \iota \leq n}$, $M(\Psi, \Psi^*) = \frac{1}{\delta} \sup_{s \in (-\infty, t_0]} V(s)$, then

$$\inf_{r^* \in \Upsilon_2} \{ \|r(t) - r*\| \} \leq \|r(t) - 0\| \leq M(\Psi, \Psi^*) e^{-\lambda(t - t_0)}, \quad t \geq t_0 \quad (11)$$

Based on the definition provided in Definition 2, it can be deduced that the system described by Equation (1) qualifies as a global exponential dissipative system, and the set Υ_1^* is a global exponentially attractive set. □

Corollary 4. *Followed from Assumptions 1 and 3, the considered system is GED without distributive delay term for* $\lambda < \sigma$ *such that* $X_\iota < 0$, $Z_\iota^2 \leq 4X_\iota Y_\iota$ $\iota \in I$, *where* $Z_\iota = \lambda + \delta_\iota + 1 - \alpha_\iota - \beta_\iota$, $Y_\iota = \lambda + 2 - 2\alpha_\iota + \frac{n}{\epsilon_3} + \epsilon_5 + \sum_{\jmath=1}^{n} \frac{1}{\epsilon_1} a_{\iota\jmath} \overline{a}_{\iota\jmath}$, $X_\iota = \lambda(\delta_\iota + 1) - 2\beta_\iota + n \epsilon_1 M_\iota^2 + n\epsilon_2 M_\iota^2 + \frac{n}{\epsilon_4} + \epsilon_6 + \sum_{\jmath=1}^{n} \left(\frac{1}{\epsilon_2} a_{\iota\jmath} \overline{a}_{\iota\jmath} + \frac{2\epsilon L_\iota^2 |b_{\iota\jmath}|^2}{(1-\tau)} \right)$, *where* σ *is the sufficient large positive number. Then QVINN (1) is a global dissipative system, and the set* $\Upsilon_1^* = \left\{ r(t) \in \mathbf{Q}^n | \overline{r}_\iota(t) r_\iota(t) \leq \frac{\epsilon^*}{\kappa_1} \overline{u}_\iota(t) u_\iota(t), \quad \iota \in I \right\}$ *is a global attractive set and a positively invariant set of QVINN model (1) without distributive delay.*

Corollary 5. *Followed from Assumptions 1 and 3, the considered system (1) is GED, for* $\lambda > 0$ *such that* $X_\iota < 0$, $Z_\iota^2 \leq 4X_\iota Y_\iota$, $\iota \in I$, *where* $Z_\iota = \lambda + \delta_\iota + 1 - \alpha_\iota - \beta_\iota$, $Y_\iota = \lambda + 2 - 2\alpha_\iota + \frac{n}{\epsilon_3} + \epsilon_5 + \sum_{\jmath=1}^{n} \frac{1}{\epsilon_1} a_{\iota\jmath} \overline{a}_{\iota\jmath}$ $X_\iota = \lambda(\delta_\iota + 1) - 2\beta_\iota + n\epsilon_1 M_\iota^2 + n\epsilon_2 M_\iota^2 + \frac{n}{\epsilon_4} + \epsilon_6 + \sum_{\jmath=1}^{n} \left(\frac{1}{\epsilon_2} a_{\iota\jmath} \overline{a}_{\iota\jmath} + \frac{2\epsilon L_\iota^2 |b_{\iota\jmath}|^2}{(1-\tau)} \right)$, *and* $\Upsilon_1^* = \left\{ r(t) \in \mathbf{Q}^n | \overline{r}_\iota(t) r_\iota(t) \leq \frac{\pi\epsilon^*}{\kappa_1} \overline{u}_\iota(t) u_\iota(t) \quad \iota \in I \right\}$ *is a global attractive set, which is a positively invariant set of QVINN model (1).*

Corollary 6. *Followed from Assumptions 1 and 2, if* $\delta_\iota = (\lambda + 1 - \alpha_\iota - \beta_\iota)$ and $X_\iota < 0, Y_\iota < 0$, *then the system (1) is global dissipative system and the set* $\Upsilon_2^* = \left\{ r(t) \in \mathbf{Q}^n | \overline{r}_\iota(t) r_\iota(t) \leq \left(\frac{\pi\epsilon^*}{\min_{1 \leq \iota \leq n} |X_\iota|} \right) \overline{u}_\iota(t) u_\iota(t), \quad \iota \in I \right\}$ *is the global attractive set.*

Remark 4. In many studies, the focus has been on exponential stability and synchronization [24, 34, 44, 53], where researchers have commonly imposed a bound on the rate of convergence denoted by λ. Specifically, they consider $\lambda \in (0, 1)$, indicating that a lower convergence rate corresponds to a slower stabilization speed. However, the authors of the present article deviate from this convention by eliminating the restriction on the convergence rate. Instead, they have derived criteria that allow researchers to choose any large value for λ without jeopardizing the validity of the results. This departure offers flexibility in selecting higher values for λ and provides potential benefits in terms of stabilization speed. By removing the constraint on λ, the authors have expanded the range of possibilities and opened up new avenues for research in the field of stability and synchronization. This approach allows researchers to explore scenarios with faster convergence rates, potentially leading to more efficient and practical applications in various domains.

Remark 5. Our results yield interesting and significant implications. Specifically, when the external inputs are set to zero ($u_\iota(t) = 0$), and the parameters of the system satisfy the conditions outlined in Theorem 1, it can be concluded that the equilibrium point $r = 0$ of the QVNN model (1) is globally asymptotically stable. This finding implies that under these circumstances, the system will converge to the equilibrium point for any initial condition. Furthermore, if the system parameters fulfill the conditions specified in Theorem 2, the equilibrium point $r = 0$ exhibits global exponential stability. This means that not only does the system converge to the equilibrium point, but it does so at an exponential rate, ensuring faster and more robust stabilization. Hence, this article proposes a systematic method for examining the Lyapunov stability of QVNNs. By identifying the conditions for global asymptotic and exponential stability, the authors offer a valuable approach to analyzing and understanding the stability properties of QVNN models. These results have practical implications and contribute to the advancement of stability analysis in the field of NNs.

Remark 6. This article introduces a novel perspective by considering the dissipativity of QVINNs, representing a broader concept of stability beyond the traditional Lyapunov stability. Unlike Lyapunov stability, dissipativity focuses on the dynamics of the entire system rather than solely on equilibrium points. It considers the behavior of the system's orbits, which may not necessarily converge to equilibrium points in some instances. Furthermore, dissipativity is applicable even in situations where certain networks do not possess equilibrium points at all. This characteristic significantly broadens the scope of applicability for dissipativity compared to traditional stability analysis. By incorporating dissipativity analysis, the article expands the understanding and evaluation of stability in QVINNs, considering the system dynamics beyond equilibrium points. This approach offers valuable insights and a more comprehensive understanding of the behavior and properties of QVINNs.

Remark 7. In addition to dissipativity analysis, this article presents a detailed estimation of the global attractive set for QVINNs. This estimation is highly valuable as it simplifies the study of QVNN dynamics to a significant extent. It enables researchers to investigate the dynamics within the identified global attractive set. The concept of the global attractive set, as explained by [54], encompasses all equilibria, periodic solutions, and chaos attractors that exist within the QVNN system. By confining the analysis to this attractive set, researchers can comprehensively understand the system's dynamics without needing to explore regions outside this set. Consequently, the article provides an effective and practical approach to examining the dynamics of QVNNs. Researchers can concentrate their efforts on understanding and analyzing the behaviors occurring within the GAS.

Remark 8. The proof derivation process in Theorems 1 and 2 is relatively straightforward due to the utilization of the direct method. This approach involves the application of two inequalities and standard operations with real numbers. By employing this direct method, the proof's complexity is significantly reduced compared to the separation technique. One notable advantage of the direct method is that it avoids decomposing the quaternion system into four real-valued systems. This decomposition process is typically required when utilizing the separation technique, which increases the computational complexity of the analysis. By circumventing this decomposition step, the direct method offers a more streamlined and efficient approach to proving the desired results. The simplicity and computational efficiency of the direct method make it an attractive choice for analyzing the stability and dissipative properties of QVINNs. By minimizing complexity and reducing computational burden, this method allows for more efficient analysis and facilitates the practical application of the derived results.

Remark 9. Up to this point, numerous outcomes have been obtained regarding the GD of QVNNs. These outcomes have been established for scenarios involving either constant delay or bounded delays, as referenced in [7, 12, 40, 51, 52, 55]. In contrast to these existing studies, the present article focuses on the GD of QVNNs with inertial terms. Specifically, it considers QVNNs with unbounded time delays and employs a non-separation, non-reduction order approach that is more realistic and practical.

Remark 10. Based on the conditions outlined in Theorems 1 and 2 and Corollaries 1–6, it can be concluded that the zero solution of the model described by Equation (1), with $u_1(t) = u_2(t) = 0$, will demonstrate both global attractiveness and global exponential attractiveness, respectively.

4 | Numerical Examples

Example 1. The QVINNs with time delays in a two-dimen sional setting are to be thought of as

$$\frac{d^2 r_\iota(t)}{dt^2} = -\alpha_\iota \frac{dr_\iota(t)}{dt} - \beta_\iota r_\iota(t) + \sum_{j=1}^{n} a_{\iota j} f_j(s_j(t)) + \sum_{j=1}^{n} b_{\iota j} g_j(s_j(t-\tau_j(t))) + \sum_{j=1}^{n} d_{\iota j} \int_{t-\sigma_j(t)}^{t} h_j(r_j(s))ds + u_\iota(t), \quad \iota \in I \tag{12}$$

where the parametric values are $\alpha_1 = 10, \alpha_2 = 11, \beta_1 = 15, \beta_2 = 16, \delta_1 = 12,\ \delta_2 = 15, \gamma_1 = \gamma_2 = 0.5,\ \epsilon_1 = \epsilon_2 = \epsilon_3 = \epsilon_4 = \epsilon_5 = \epsilon_6 = \epsilon_7 = \epsilon_8 == 1, f_1 = f_2 = \frac{1}{2}\tanh(r(t)),\ g_1 = g_2 = \frac{1}{2}\tanh(r(t)), h_1 = h_2 = tanh(r(t)), \tau_1(t) = \tau_2(t) = \frac{t}{2},\quad u_1(t) = 0.1sin(t) + 0.6sin(t)i - 1.2sin(t)j +\ 0.2cos(t)k,\ u_2(t) = 1.7sin(t) - 0.5sin(t)i\ + 0.7sin(t)j - 0.4cos(t)k$

Now, consider the weight connection matrices as

$$[a]_{2\times2} = \begin{pmatrix} 0.5 + 1.2i - 0.4j + 0.6k & 0.6 + 0.9i + 0.6j + k \\ 1.1 + 0.9i - 0.5j + 0.8k & 0.7 - 1.2i - 0.9j + 0.5k \end{pmatrix},$$

$$[b]_{2\times2} = \begin{pmatrix} 0.8 + i - 0.9j - 0.5k & 0.9 - 1.1i - j + 0.1k \\ 0.7 - 0.8i + 1.1j - 1.2k & 1 - 0.5i - 1.4j - k \end{pmatrix},$$

$$[d]_{2\times2} = \begin{pmatrix} -0.25 - 0.2i + 0.3j + 0.12k & 0.1 - 0.3i - 0.5j - 0.1k \\ 0.15 + 0.25i - 0.2j - 0.3k & -0.1 + 0.2i - 0.4j - 0.5k \end{pmatrix}.$$

Case I. When distributive delays are considered, the model becomes more general and complex. In the context of neural networks, distributive delays refer to delays that are not confined to a specific connection but spread out or distributed across the network. This more comprehensive model accounts for a broader range of temporal interactions within the system, making it more applicable to real-world scenarios where delays are not instantaneous. In summary, while the model with distributive delays is more general and realistic, it is also more complex, requiring advanced mathematical techniques for analysis and interpretation. Then, by simple computations, one can get $L_1 = L_2 = M_1 = M_2 = \frac{1}{2}, K_1 = K_2 = 1, \tau = \frac{1}{2}, \epsilon^* = 2,\ \kappa = 1.0936$ and $Z_1 = -12 < 0, Y_1 = -10.118, X_1 = -15.32571$. It is obvious that $Z_1^2 = 144 < Y_1 X_1 = 155.065$ also

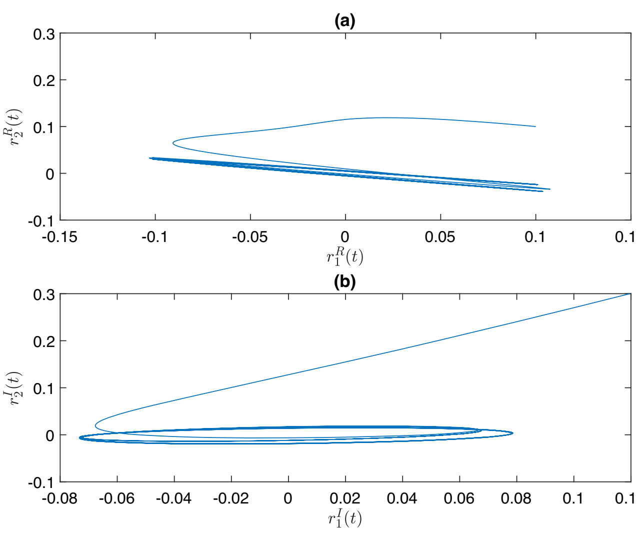

FIGURE 1 | Phase plot of state variables (a) r_1^R, r_2^R and (b) r_1^I, r_2^I of the system (12). [Colour figure can be viewed at wileyonlinelibrary.com]

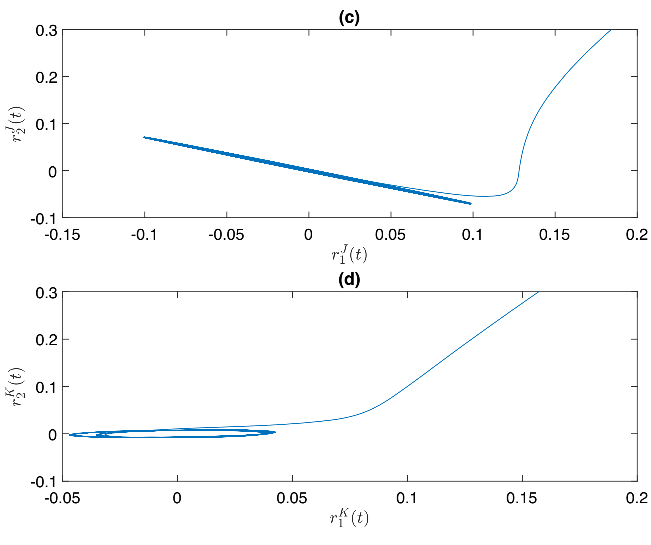

FIGURE 2 | Phase plot of state variables (c) r_1^J, r_2^J and (d) r_1^J, r_2^J of the system (12). [Colour figure can be viewed at wileyonlinelibrary.com]

$Z_2 = -11 < 0, Y_2 = -10.9312, X_2 = -13.8787$, and therefore, $Z_2^2 = 121 < Y_2X_2 = 151.7120$. Hence, Theorem 1 is satisfied, our considered system (12) is globally dissipative, and the set $\Upsilon_1 = \{r_1(t), r_2(t) \in \mathbf{Q}^2 | \|r_1(t)\| \leq 1.819, \ \|r_2(t)\| \leq 2.6326\}$ is a GAS and a positive invariant set of (12). For the initial conditions $\{\Psi_1(s), \Psi_2(s), \Psi_1^*(s), \Psi_2^*(s)\} = \{0.1, 0.1, -0.8, 0.3\}$,

Case II. When contemplating the scenario where $d_y = 0$, it implies that there are no distributive delays in the considered model. This condition simplifies the model by eliminating the contribution of distributive delays between the nodes. Then, by simple computations, one can get $L_1 = L_2 = M_1 = M_2 = \frac{1}{2}, \tau = \frac{1}{2}, \epsilon^* = 2, \ \kappa = 1.432$ and $Z_1 = -12 < 0, Y_1 = -10.26, X_1 = -15.4675$. It is obvious that $Z_1^2 = 144 < Y_1X_1 = 158.6965$ also $Z_2 = -11 < 0, Y_2 = -11.1, X_2 = -14.0475$, and therefore, $Z_2^2 = 121 < Y_2X_2 = 155.927$. Hence, Corollary 1 is satisfied, our considered system (12) is globally dissipative, and the set $\Upsilon_1 = \{r_1(t), r_2(t) \in \mathbf{Q}^2 | \|r_1(t)\| \leq 1.5899, \ \|r_2(t)\| \leq 2.2516\}$ is a GAS and a positive invariant set of (12). For the initial conditions $\{\Psi_1(s), \Psi_2(s), \Psi_1^*(s), \Psi_2^*(s)\} = \{0.1, 0.1, -0.8, 0.3\}$.

Figures 1 and 2 show the phase trajectories of the system (12), and Figures 3 and 4 depict the time changes of the state variables $r_1(t), r_2(t)$. Figures 5 and 6 show the global attractiveness of the zero solution for the Equation (12), which also validates the accuracy of Remark 10.

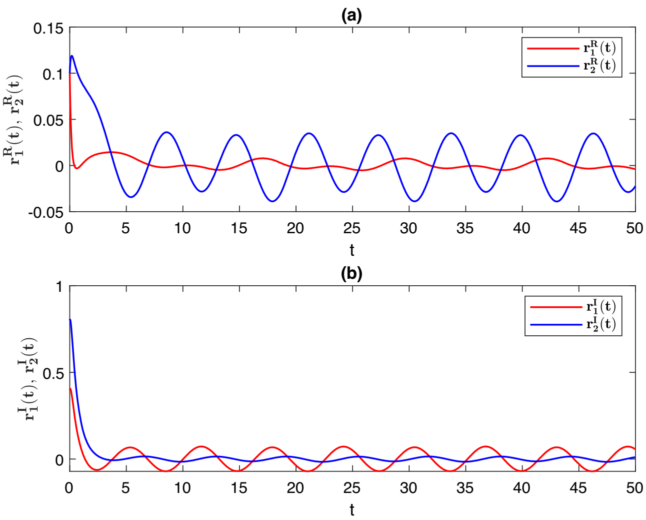

FIGURE 3 | Time response of state variables (a) r_1^R, r_2^R and (b) r_1^I, r_2^I of the system (12). [Colour figure can be viewed at wileyonlinelibrary.com]

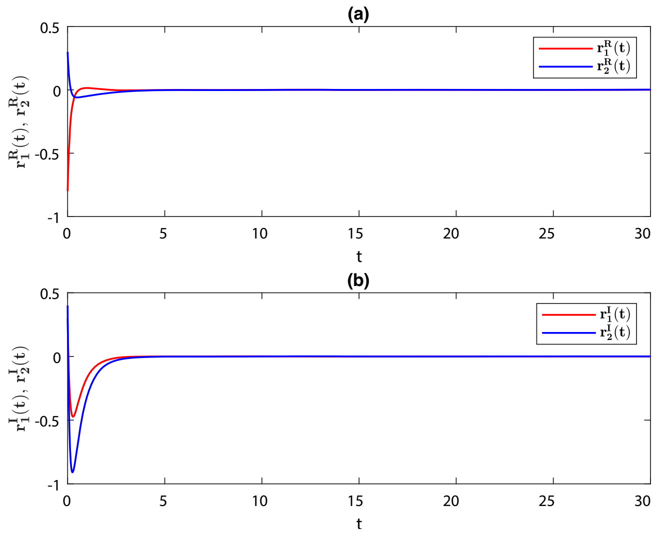

FIGURE 5 | Time response of state variables (a) r_1^R, r_2^R and (b) r_1^I, r_2^I of the system (12) with J_1^R, J_2^R, J_1^I and $J_2^I = 0$. [Colour figure can be viewed at wileyonlinelibrary.com]

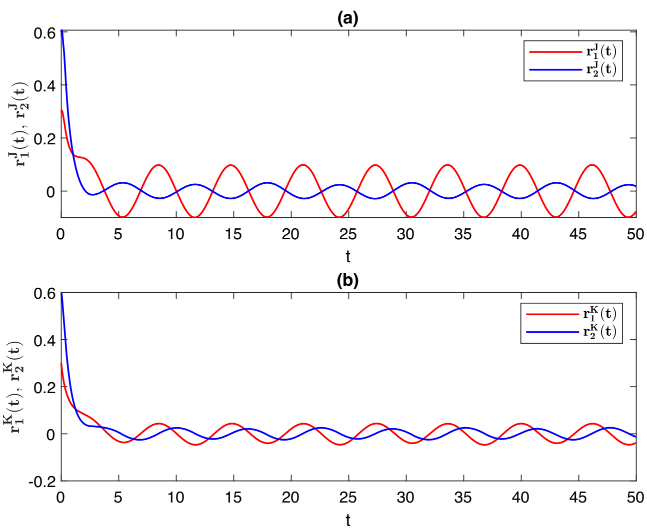

FIGURE 4 | Time response of state variables (a) r_1^J, r_2^J and (b) r_1^K, r_2^K of the system (12). [Colour figure can be viewed at wileyonlinelibrary.com]

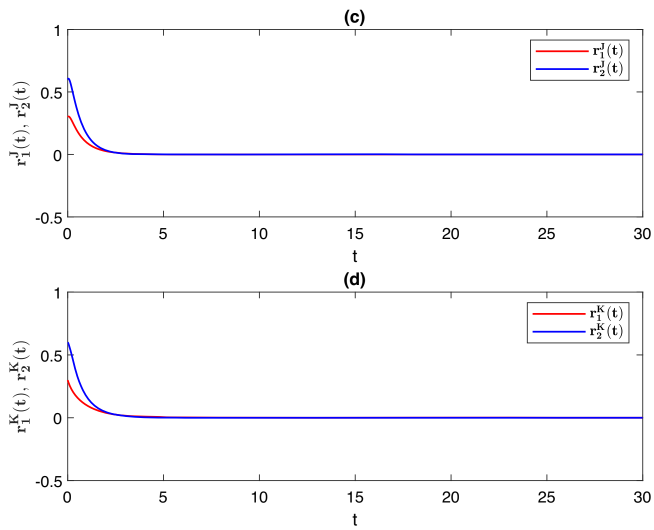

FIGURE 6 | Time response of state variables (c) r_1^J, r_2^J and (d) r_1^K, r_2^K of the system (12) with J_1^J, J_2^J, J_1^K and $J_2^K = 0$. [Colour figure can be viewed at wileyonlinelibrary.com]

Example 2. The two-dimensional QVINNs with time delays are to be thought of as

$$\frac{d^2 r_\iota}{dt^2} = -\alpha_\iota \frac{dr_\iota}{dt} - \beta_\iota r_\iota(t) + \sum_{j=1}^{n} a_{\iota j} f_j(s_j(t)) + \sum_{j=1}^{n} b_{\iota j} g_j(s_j(t - \tau_j(t))) + \sum_{j=1}^{n} d_{\iota j} \int_{t-\sigma_j(t)}^{t} h_j(r_j(s))ds + u_\iota(t), \quad \iota \in I \tag{13}$$

where the parametric values are $\lambda = 1, \alpha_1 = 12, \alpha_2 = 14, \beta_1 = 17, \beta_2 = 19, \delta_1 = 13, \delta_2 = 15, \epsilon_1 = \epsilon_2 = \epsilon_3 = \epsilon_4 = \epsilon_5 = \epsilon_6 = \epsilon_7 = \epsilon_8 == 1, f_1 = f_2 = \frac{1}{2}\tanh(r_j(t)), g_1 = g_2 = \frac{1}{2}\tanh(r_j(t))e^{-2\tau_j(t)}, \tau_1(t) = \tau_2(t) = 0.5\ln(1+t), \sigma_j(t) = 0.25\sin^2(t), \overline{\sigma} = 2, u_1(t) = -1.2sin(t) + 0.9cos(t)i + 0.9cos(t)j - 0.5sin(t)k, u_2(t) = 0.8sin(t) - 0.7sin(t)i + 0.3sin(t)j - 0.9cos(t)k.$

Now, consider the weight connection matrices as

$$[a]_{2\times 2} = \begin{pmatrix} 0.51 - 0.35i - 0.6j - 0.1k & 0.2 + 0.5i - 0.7j - 0.2k \\ 0.25 - 0.1i - 0.4j + 0.6k & 0.3 - 0.25i - 0.5j - 0.6k \end{pmatrix},$$

$$[b]_{2\times 2} = \begin{pmatrix} 0.5 - 0.2i + 0.3j - 0.7k & -0.1 - 0.4i + 0.6j - 0.9k \\ 0.3 - 0.35i - 0.5j + 0.6k & -0.7 - 0.3i + 0.5j + 0.6k \end{pmatrix}.$$

$$[d]_{2\times 2} = \begin{pmatrix} -0.1 - 0.3i - 0.5j + 0.2k & -0.7 + 0.3i + 0.4j - 0.6k \\ -0.2 + 0.5i + 0.1j - 0.2k & -0.1 + 0.4i - 0.8j - 0.2k \end{pmatrix}.$$

Case I. When distributive delays are considered, the model becomes more general. Then, by simple computations, one can get $M_1 = M_2 = L_1 = L_2 = \frac{1}{2}, K_1 = K_2 = 1, \tau = \frac{1}{2}, \sigma = 0.25, \dot{\sigma} = 0.5, \epsilon^* = 2, \kappa_1 = \min\{0.4894, 1.4519\} = 0.4894, Z_1 = -14 < 0, Y_1 = -16.32975 < 0, X_1 = -12.11975,$ then it is obvious

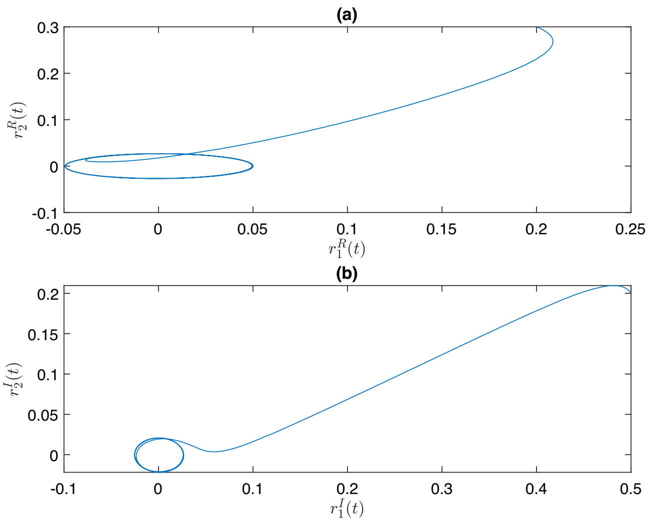

FIGURE 7 | Phase plot of state variables (a) r_1^R, r_2^R and (b) r_1^I, r_2^I of the system (13). [Colour figure can be viewed at wileyonlinelibrary.com]

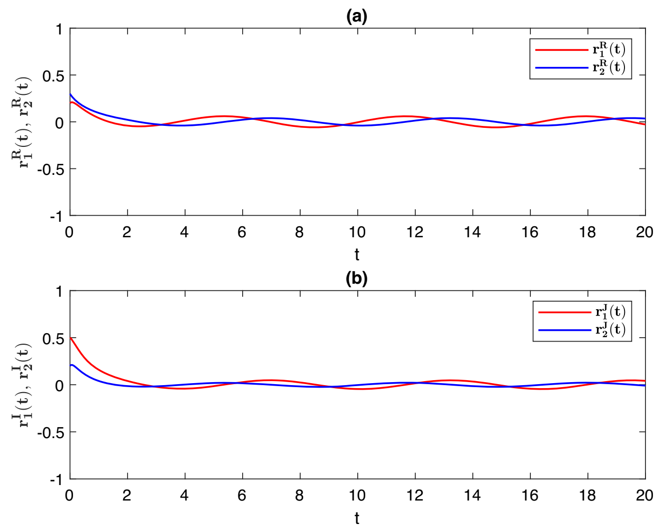

FIGURE 9 | Time response of state variables (a) r_1^R, r_2^R and (b) r_1^I, r_2^I of the system (13). [Colour figure can be viewed at wileyonlinelibrary.com]

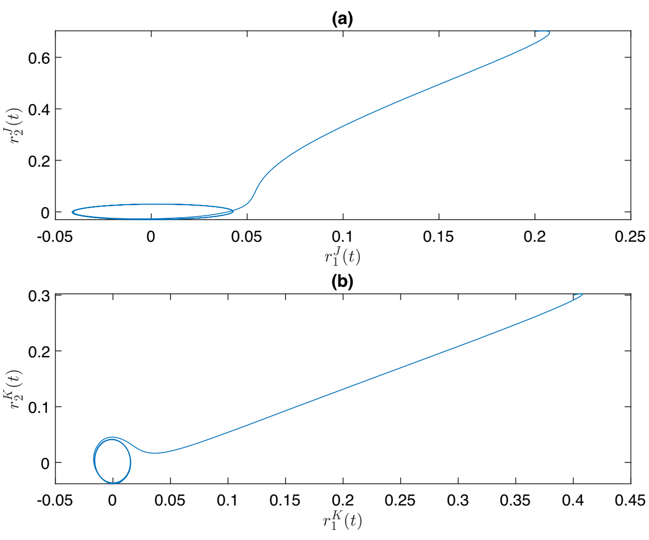

FIGURE 8 | Phase plot of state variables (a) r_1^J, r_2^J and (b) r_1^K, r_2^K of the system (13). [Colour figure can be viewed at wileyonlinelibrary.com]

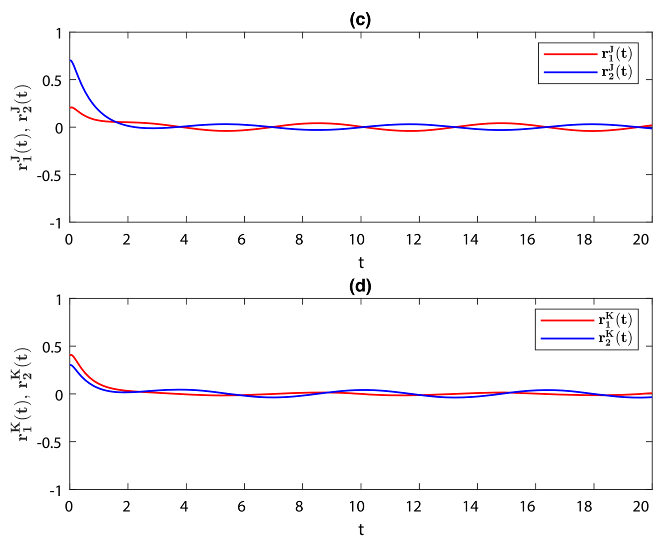

FIGURE 10 | Time response of state variables (a) r_1^J, r_2^J and (b) r_1^K, r_2^K of the system (13). [Colour figure can be viewed at wileyonlinelibrary.com]

that $Z_1^2 = 196 < Y_1X_1 = 197.9124$. Similarly, $Z_2 = -16 < 0, Y_2 = -20.3475 < 0, X_2 = -13.525 < 0$, which shows that $Z_2^2 = 256 < Y_2X_2 = 275.1999$. Hence, conditions of Theorem 2 are satisfied, and our considered system (13) is globally exponential dissipative. The set $\Upsilon_1^* = \{r_1(t), r_2(t) \in \mathbf{Q}^2 | \|r_1(t)\| \leq 2.9084, \ \|r_2(t)\| \leq 2.232\}$ is a GAS and a positive invariant set of (13).

Case II. When contemplating the scenario where $d_{ij} = 0$, it implies that there are no distributive delays in the considered model, this condition simplifies the model by eliminating the contribution of distributive delays between the nodes. Then, by simple computations, one can get $M_1 = M_2 = L_1 = L_2 = \frac{1}{2}$, $\tau = \frac{1}{2}$, $\epsilon^* = 2$, $\kappa_1 = \min\{1.4324, 3.1465\} = 1.4324$, $Z_1 = -14 < 0, Y_1 = -16.7025 < 0, X_1 = -12.492$, then it is obvious that $Z_1^2 = 196 < Y_1X_1 = 208.656$. Similarly, $Z_2 = -16 < 0, Y_2 = -20.645 < 0, X_2 = -13.822 < 0$, which shows that $Z_2^2 = 256 < Y_2X_2 = 285.365$. Hence conditions of Corollary 4 are satisfied, and our considered system (13) is globally exponential dissipative. The set $\Upsilon_1^* = \{r_1(t), r_2(t) \in \mathbf{Q}^2 | \|r_1(t)\| \leq 1.5360, \ \|r_2(t)\| \leq 1.3471\}$ is a GAS and a positive invariant set of (13). For the initial conditions $\{\Psi_1(s), \Psi_2(s)\Psi_1^*(s), \Psi_2^*(s)\} = \{0.2, 0.3, 0.3, -0.5\}$.

Figures 7 and 8 are the phase trajectory of the system (13), and Figures 9 and 10 are the time changes of the state variables $r_1(t), r_2(t)$. Figures 11 and 12 depict the global exponential attractiveness of the zero solution for Equation (13), which also validates the correctness of Remark 10.

Example 3. To illustrate the application of traditional QVNNs, let us focus on the example of a 12×12 pixel image pattern denoted as "T," and its corresponding color image representation is shown in Figure 13. In this context, a set of QVNNs in the format of (14) are configured, comprising 144 neurons. These QVNNs possess 144-dimensional equilibrium points, which facilitate the storage of colored "T" patterns. Therefore, we shall now consider the traditional QVNNs to proceed.

$$\frac{dr_\iota}{dt} = -\beta_\iota r_\iota(t) + \sum_{j=1}^{n} a_{\iota j} f_j(r_j(t)) + \sum_{j=1}^{n} b_{\iota j} g_j(r_j(t - \tau_j(t))) + u_\iota, \quad \iota \in I \tag{14}$$

in this context, where $\beta_j > 0$ and a_{ij} and b_{ij} represent weight connection matrices, while u_ι denotes the external input of (14). The chosen parameter values for the QVNNs defined by Equation (14) are taken to be as

$$\beta_j = 1 \tag{15}$$

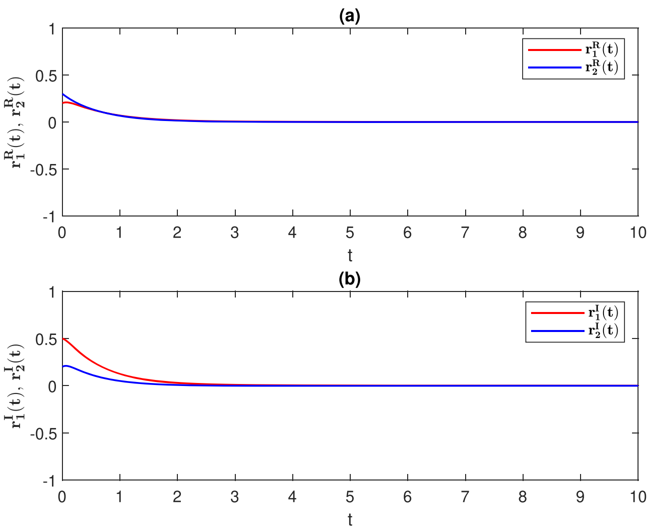

FIGURE 11 | Time response of state variables (a) r_1^R, r_2^R and (b) r_1^I, r_2^I of the system (13) with J_1^R, J_2^R, J_1^I and $J_2^I = 0$. [Colour figure can be viewed at wileyonlinelibrary.com]

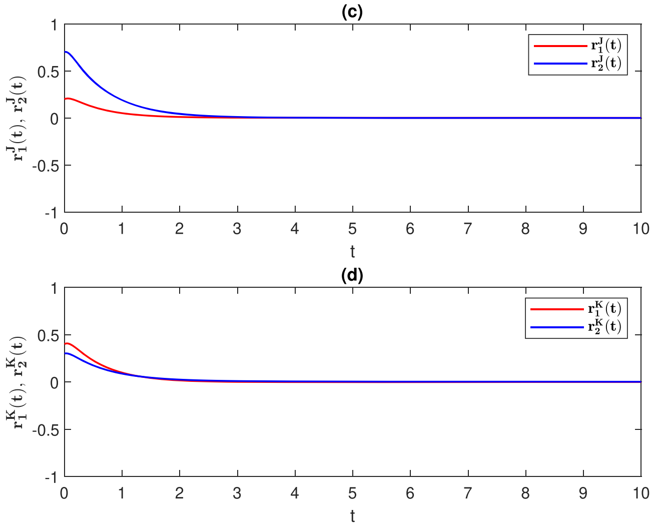

FIGURE 12 | Time response of state variables (c) r_1^J, r_2^J and d r_1^K, r_2^K of the system (13) with J_1^J, J_2^J, J_1^K and $J_2^K = 0$. [Colour figure can be viewed at wileyonlinelibrary.com]

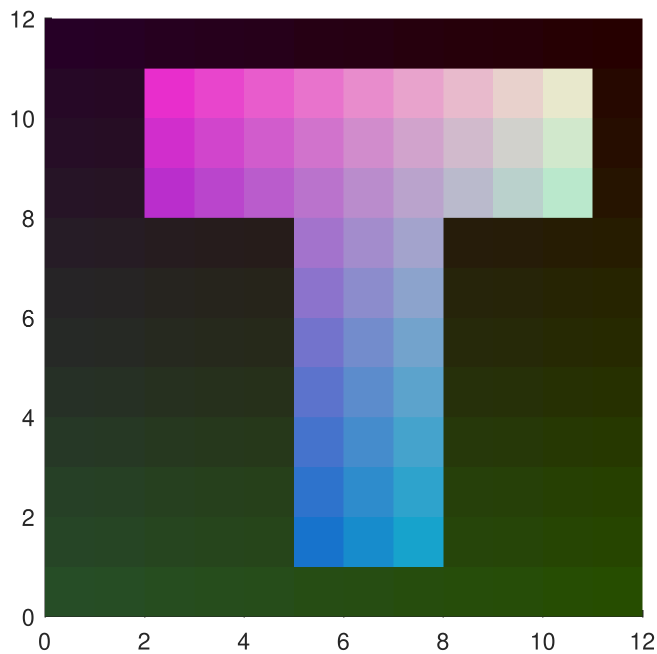

FIGURE 13 | Original color image of pattern "T." [Colour figure can be viewed at wileyonlinelibrary.com]

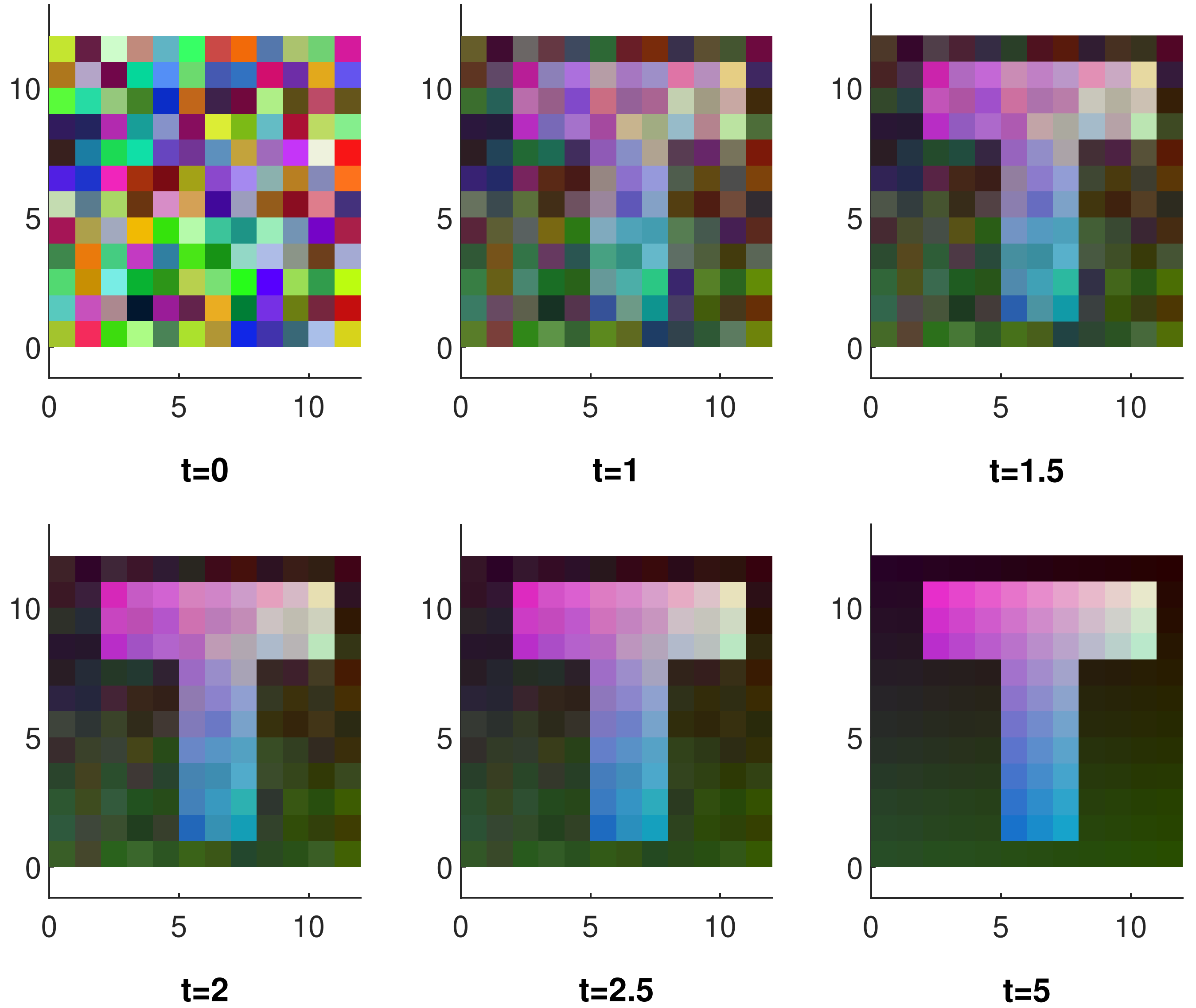

FIGURE 14 | Simulation of retrieving image "T" with random initial values of time t. [Colour figure can be viewed at wileyonlinelibrary.com]

$$a_{ij} = \begin{cases} 4.0 + 4.0 \times 10^{-1} i - 3.0 \times 10^{-1} j + 5.0 \times 10^{-1} k, & i = j \\ 4.0 \times 10^{-1} - 5.0 \times 10^{-1} i + 5.0 \times 10^{-1} j - 3.0 \times 10^{-1} k, & i \neq j, \end{cases} \tag{16}$$

$$b_{ij} = \begin{cases} -2.0 \times 10^{-1} + 2.0 \times 10^{-1} i - 5.0 \times 10^{-1} j + 4.0 \times 10^{-1} k, & i < j \\ 2 + 3.0 \times 10^{-1} i + 2.0 \times 10^{-1} j - 3.0 \times 10^{-1} k, & i = j \\ -1.0 \times 10^{-1} + 2.0 \times 10^{-1} i + 3.0 \times 10^{-1} j - 5.0 \times 10^{-1} k, & i > j, \end{cases} \tag{17}$$

$$f_j^x(r) = \tanh(r) \tag{18}$$

where $i, j = 1, 2, \dots, 144$ and $x = 0, 1, 2, 3$. In order to recall the image pattern "T," The equilibrium point of the designed QVNNs should be $r = (r_1, r_2, \dots, r_{144})^T \in \mathbf{Q}^{144}$ where $r_1 = 0 + 15.0 \times 10^{-2} i + 3.0 \times 10^{-1} j + 15.0 \times 10^{-2} k$, $r_2 = 0 + 15.0 \times 10^{-2} i + 3.0 \times 10^{-1} j + 14.0 \times 10^{-2} k$, $\dots$, $r_144 = 0 + 15.0 \times 10^{-2} i + 0j + 0k$,which correspond to the color $(15.0 \times 10^{-2}, 3.0 \times 10^{-1}, 0.150)$, $(15.0 \times 10^{-2}, 3.0 \times 10^{-1}, 14.0 \times 10^{-2})$, $\dots$, $(15.0 \times 10^{-2}, 0, 0)$ of the pixels in color image pattern "T." Figure 14 shows a simulation with random initial values. Based on the equilibrium points r, we can easily calculate the external input parameter u that is calculated as $U = (u_1, u_2, \dots, u_{144}) \in \mathbf{Q}^{144}$, where $u_1 = -628.0 \times 10^{-1} + 2275.0 \times 10^{-2} i - 905.0 \times 10^{-1} j - 725.0 \times 10^{-2} k, u_2 = -63 + 2095.0 \times 10^{-2} i - 905.0 \times 10^{-1} j - 566.0 \times 10^{-2} k, \dots, u_{144} = -914.0 \times 10^{-1} - 2346.0 \times 10^{-1} i - 908.0 \times 10^{-1} j + 2214.0 \times 10^{-1} k$. Owing to spatial constraints, we list only three elements of r and u. One simulation results, conducted with random initial data, are presented in Figure 14. The parameters extracted from Equations (15–18) indicate that the examined system (14) possesses the capability to recover the aforementioned "T" pattern consistently.

Remark 11. Example 3 demonstrates that the proposed system (14), defined by Equations (15–18) for its parametric values, necessitates the use of 144.00 neurons to store a 12×12 pixel image pattern. In contrast, according to the findings in the article [56], a comparable image stored in a CVNN would require 432 neurons, a significantly larger quantity. This distinction highlights the greater storage capacity of QVNNs over CVNNs. This article deals specifically with QVNNs employing the "tanh" activation functions. Remarkably, the image "T" reconstruction can be approximated to occur within a time span of $t = 2.5$.

5 | Conclusion and Future Work

In this article, the authors have comprehensively examined the GD and GED of QVINNs with unbounded time-varying delays. The Lyapunov function and inequality approaches have been employed to investigate these properties. The authors have focused on the without-separation approach of QVINNs and the non-reduction order method for inertial terms, providing a novel perspective for analyzing these complex systems. The first contribution of this article, as presented in Theorem 1, pertains to the GD of QVINNs. By utilizing inequality techniques, the authors have established a global attractive set for the system (1). Additionally, two corollaries were derived to refine and extend the findings of the theorem. Moving on to Theorem 2, the GED and global exponential attractive set of QVINNs have been investigated. This theorem highlighted the unbounded nature of the convergence rate, further enhancing the understanding of the system's stability properties. Two corollaries were proposed to provide additional insights based on these results. Furthermore, the article has presented several algebraic criteria for assessing the GD of QVINNs with unbounded time-varying delays. This consideration of unbounded time delays is particularly relevant as it aligns with real-world scenarios more accurately than bounded delays. To illustrate the practical implications of the proposed results, the authors have provided three consecutive examples. The first two examples satisfied the conditions outlined in the first two theorems and corollaries, demonstrating the applicability of the derived criteria. The third example demonstrates the potential of QVNNs in efficiently storing large amounts of data using fewer neurons. Overall, this article contributes to the field of QVINNs by offering novel insights into GD and exponential dissipativity, providing algebraic criteria and demonstrating their practical applications through illustrative examples.

In future work, the authors have planned to investigate the pre-assigned fixed-time stability of a specific class of QVINNs with unbounded and non-differentiable time delays. This research direction addresses an important aspect of stability analysis by considering fixed-time stability, which imposes a pre-determined time constraint on achieving stability rather than relying on convergence over an arbitrary duration. Including unbounded and non-differentiable time delays the complexity of the analysis may be enhanced to reflect the, reflecting realistic scenarios in various practical applications. By considering these types of delays, the authors aim to provide a more comprehensive understanding of the dynamics and stability properties of QVINNs. Examining fixed-time stability in QVINNs with unbounded and non-differentiable time delays, a significant extension of the existing research can be made. Furthermore, our work on INNs can be extended to the hypercomplex domain, as discussed in [57], and we will explore true color image applications in the hypercomplex domain by utilizing the direct results.

Author Contributions

Sunny Singh: conceptualization, investigation, methodology, formal analysis, software, validation, writing – review and editing, writing – original draft. **Subir Das:** formal analysis, supervision, validation, investigation. **Jinde Cao:** conceptualization, visualization, supervision, formal analysis, software.

Acknowledgments

The authors express their sincere gratitude to the institute for providing the research opportunity. Furthermore, the authors extend their heartfelt thanks to the reviewers and editors of the article for their valuable suggestions. The authors express their gratitude to the Indian Institute of Technology (BHU) for supporting this research with the necessary funding and facilities.

Conflicts of Interest

The authors declare no conflicts of interest.

Data Availability Statement

The authors have nothing to report.

References

1. W. R. Hamilton, *Lectures on Quaternions: Containing a Systematic Statement of a New Mathematical Method; of Which the Principles Were Communicated in 1843 to the Royal Irish Academy; and Which Has Since Formed the Subject of Successive Courses of Lectures, Delivered in 1848 and Subsequent Years in the Halls of Trinity Collegee, Dublin: With Numerous Illustrative Diagrams, and With Some Geometrical and Physical Applications* (Hodges and Smith, 1853).

2. B. C. Ujang, C. C. Took, and D. P. Mandic, "Quaternion-Valued Nonlinear Adaptive Filtering," *IEEE Transactions on Neural Networks* 22, no. 8 (2011): 1193–1206.

3. C. Zou, K. I. Kou, and Y. Wang, "Quaternion Collaborative and Sparse Representation With Application to Color Face Recognition," *IEEE Transactions on Image Processing* 25, no. 7 (2016): 3287–3302.

4. T. Isokawa, T. Kusakabe, N. Matsui, and F. Peper, "Quaternion Neural Network and Its Application," in *Knowledge-Based Intelligent Information and Engineering Systems: 7th International Conference, KES 2003, Oxford, UK, September 2003. Proceedings, Part II 7* (Springer, 2003), 318–324.

5. S. L. Adler, *Quaternionic Quantum Mechanics and Quantum Fields*, vol. 88 (Oxford University Press on Demand, 1995).

6. T. Isokawa, N. Matsui, and H. Nishimura, "Quaternionic Neural Networks: Fundamental Properties and Applications," in *Complex-Valued Neural Networks: Utilizing High-Dimensional Parameters* (IGI global, 2009), 411–439.

7. X. Liu and Z. Li, "Global μ-Stability of Quaternion-Valued Neural Networks With Unbounded and Asynchronous Time-Varying Delays," *IEEE Access* 7 (2019): 9128–9141.

8. S. S. Chouhan, S. Das, S. Singh, and H. Shen, "Multiple μ-Stability Analysis of Time-Varying Delayed Quaternion-Valued Neural Networks," *Mathematical Methods in the Applied Sciences* 46, no. 9 (2023): 9853–9875.

9. X. Xiaohui, X. Quan, J. Yang, H. Xue, and X. Yanhai, "Further Research on Exponential Stability for Quaternion-Valued Neural Networks With Mixed Delays," *Neurocomputing* 400 (2020): 186–205.

10. S. Singh, U. Kumar, F. Subir Das, F. Alsaadi, and J. Cao, "Synchronization of Quaternion Valued Neural Networks With Mixed Time Delays Using Lyapunov Function Method," *Neural Processing Letters* 54, no. 2 (2022): 785–801.

11. Y. Zhang, L. Yang, K. I. Kou, and Y. Liu, "Fixed-Time Synchronization for Quaternion-Valued Memristor-Based Neural Networks With Mixed Delays," *arXiv preprint arXiv:2301.01275*, (2023).

12. T. Zhengwen, J. Cao, A. Alsaedi, and T. Hayat, "Global Dissipativity Analysis for Delayed Quaternion-Valued Neural Networks," *Neural Networks* 89 (2017): 97–104.

13. J. Zhang, X. Ma, Y. Li, Q. Gan, and C. Wang, "Synchronization in Fixed/Preassigned-Time of Delayed Fully Quaternion-Valued Memristive Neural Networks via Non-Separation Method," *Communications in Nonlinear Science and Numerical Simulation* 113 (2022): 106581.

14. M. Pang, Z. Zhang, X. Wang, Z. Wang, and C. Lin, "Fixed/Preassigned-Time Synchronization of High-Dimension-Valued Fuzzy Neural Networks With Time-Varying Delays via Nonseparation Approach," *Knowledge-Based Systems* 255 (2022): 109774.

15. H.-L. Li, H. Cheng, L. Zhang, H. Jiang, and J. Cao, "Non-Separation Method-Based Robust Finite-Time Synchronization of Uncertain Fractional-Order Quaternion-Valued Neural Networks," *Applied Mathematics and Computation* 409 (2021): 126377.

16. K. L. Babcock and R. M. Westervelt, "Stability and Dynamics of Simple Electronic Neural Networks With Added Inertia," *Physica D: Nonlinear Phenomena* 23, no. 1-3 (1986): 464–469.

17. K. L. Babcock and R. M. Westervelt, "Dynamics of Simple Electronic Neural Networks," *Physica D: Nonlinear Phenomena* 28, no. 3 (1987): 305–316.

18. C. Koch, "Cable Theory in Neurons With Active, Linearized Membranes," *Biological Cybernetics* 50, no. 1 (1984): 15–33.

19. A. Mauro, F. Conti, F. Dodge, and R. Schor, "Subthreshold Behavior and Phenomenological Impedance of the Squid Giant Axon," *Journal of General Physiology* 55, no. 4 (1970): 497–523.

20. D. E. Angelaki and M. J. Correia, "Models of Membrane Resonance in Pigeon Semicircular Canal Type II Hair Cells," *Biological Cybernetics* 65, no. 1 (1991): 1–10.

21. J. Qi, C. Li, and T. Huang, "Stability of Inertial bam Neural Network With Time-Varying Delay via Impulsive Control," *Neurocomputing* 161 (2015): 162–167.

22. L. Wang, Z. Zeng, M.-F. Ge, and H. Junhao, "Global Stabilization Analysis of Inertial Memristive Recurrent Neural Networks With Discrete and Distributed Delays," *Neural Networks* 105 (2018): 65–74.

23. X. Chen, D. Lin, and W. Lan, "Global Dissipativity of Delayed Discrete-Time Inertial Neural Networks," *Neurocomputing* 390 (2020): 131–138.

24. J. Shi and Z. Zeng, "Global Exponential Stabilization and Lag Synchronization Control of Inertial Neural Networks With Time Delays," *Neural Networks* 126 (2020): 11–20.

25. D. Lin, X. Chen, Y. Guoping, Z. Li, and Y. Xia, "Global Exponential Synchronization via Nonlinear Feedback Control for Delayed Inertial Memristor-Based Quaternion-Valued Neural Networks With Impulses," *Applied Mathematics and Computation* 401 (2021): 126093.

26. H. Wei, W. Baowei, and T. Zhengwen, "Exponential Synchronization and State Estimation of Inertial Quaternion-Valued Cohen-Grossberg Neural Networks: Lexicographical Order Method," *International Journal of Robust and Nonlinear Control* 30, no. 6 (2020): 2171–2185.

27. Z. Zhang, S. Wang, X. Wang, Z. Wang, and C. Lin, "Event-Triggered Synchronization for Delayed Quaternion-Valued Inertial Fuzzy Neural Networks via Non-Reduced Order Approach," *IEEE Transactions on Fuzzy Systems* 31, no. 9 (2023): 3000–3014.

28. R. Li and J. Cao, "Exponential Stabilization of Inertial Quaternion-Valued Cohen-Grossberg Neural Networks: Lexicographical Order Method," *International Journal of Robust and Nonlinear Control* 30, no. 13 (2020): 5205–5220.

29. R. Zhao, B. Wang, and J. Jian, "Global μ-Stabilization of Quaternion-Valued Inertial Bam Neural Networks With Time-Varying Delays via Time-Delayed Impulsive Control," *Mathematics and Computers in Simulation* 202 (2022): 223–245.

30. Y. Yaning, Z. Zhang, M. Zhong, and Z. Wang, "Pinning Synchronization and Adaptive Synchronization of Complex-Valued Inertial Neural Networks With Time-Varying Delays in Fixed-Time Interval," *Journal of the Franklin Institute* 359, no. 2 (2022): 1434–1456.

31. T. Fang, R. Tingting, F. Dongmei, S. Lei, and J. Wang, "Extended Dissipative Filtering for Markov Jump Bam Inertial Neural Networks Under Weighted Try-Once-Discard Protocol," *Journal of the Franklin Institute* 358, no. 7 (2021): 4103–4117.

32. W. Zhang, J. Qi, and X. He, "Input-to-State Stability of Impulsive Inertial Memristive Neural Networks With Time-Varying Delayed," *Journal of the Franklin Institute* 355, no. 17 (2018): 8971–8988.

33. A. Pratap, R. Raja, J. Alzabut, J. Cao, G. Rajchakit, and C. Huang, "Mittag-Leffler Stability and Adaptive Impulsive Synchronization of Fractional Order Neural Networks in Quaternion Field," *Mathematical Methods in the Applied Sciences* 43, no. 10 (2020): 6223–6253.

34. S. Singh, U. Kumar, S. Das, and J. Cao, "Global Exponential Stability of Inertial Cohen–Grossberg Neural Networks With Time-Varying Delays via Feedback and Adaptive Control Schemes: Non-Reduction Order Approach," *Neural Processing Letters* 55, no. 4 (2023): 4347–4363.

35. Y. Zhang and L. Zhou, "Stabilization and Lag Synchronization of Proportional Delayed Impulsive Complex-Valued Inertial Neural Networks," *Neurocomputing* 507 (2022): 428–440.

36. X. Li, X. Li, and H. Cheng, "Some New Results on Stability and Synchronization for Delayed Inertial Neural Networks Based on Non-Reduced Order Method," *Neural Networks* 96 (2017): 91–100.

37. X. Liao and J. Wang, "Global Dissipativity of Continuous-Time Recurrent Neural Networks With Time Delay," *Physical Review E* 68, no. 1 (2003): 16118.

38. N. Li and J. Cao, "Global Dissipativity Analysis of Quaternion-Valued Memristor-Based Neural Networks With Proportional Delay," *Neurocomputing* 321 (2018): 103–113.

39. L. Duan, L. Huang, and Z. Guo, "Global Robust Dissipativity of Interval Recurrent Neural Networks With Time-Varying Delay and Discontinuous Activations," *Chaos: An Interdisciplinary Journal of Nonlinear Science* 26, no. 7 (2016): 073101.

40. R. Li, X. Gao, J. Cao, and K. Zhang, "Dissipativity and Exponential State Estimation for Quaternion-Valued Memristive Neural Networks," *Neurocomputing* 363 (2019): 236–245.

41. C. Aouiti, R. Sakthivel, and F. Touati, "Global Dissipativity of Fuzzy Cellular Neural Networks With Inertial Term and Proportional Delays," *International Journal of Systems Science* 51, no. 8 (2020): 1392–1405.

42. T. Zhengwen, J. Cao, and T. Hayat, "Matrix Measure Based Dissipativity Analysis for Inertial Delayed Uncertain Neural Networks," *Neural Networks* 75 (2016): 47–55.

43. W. Kai and J. Jian, "Non-reduced Order Strategies for Global Dissipativity of Memristive Neutral-Type Inertial Neural Networks With Mixed Time-Varying Delays," *Neurocomputing* 436 (2021): 174–183.

44. G. Zhang, Z. Zeng, and H. Junhao, "New Results on Global Exponential Dissipativity Analysis of Memristive Inertial Neural Networks With Distributed Time-Varying Delays," *Neural Networks* 97 (2018): 183–191.

45. Y. Sheng, Y. Shen, and M. Zhu, "Delay-Dependent Global Exponential Stability for Delayed Recurrent Neural Networks," *IEEE Transactions on Neural Networks and Learning Systems* 28, no. 12 (2016): 2974–2984.

46. X. Lv and X. Li, "Delay-Dependent Dissipativity of Neural Networks With Mixed Non-Differentiable Interval Delays," *Neurocomputing* 267 (2017): 85–94.

47. S. Singh, S. Das, S. S. Chouhan, and J. Cao, "Anti-Synchronization of Inertial Neural Networks With Quaternion-Valued and Unbounded Delays: Non-Reduction and Non-Separation Approach," *Knowledge-Based Systems* 278 (2023): 110903.

48. Y. Sheng, H. Zhang, and Z. Zeng, "Synchronization of Reaction–Diffusion Neural Networks With Dirichlet Boundary Conditions and Infinite Delays," *IEEE Transactions on Cybernetics* 47, no. 10 (2017): 3005–3017.

49. A. Kumar, S. Das, S. Singh, et al., "Quasi-Projective Synchronization of Inertial Complex-Valued Recurrent Neural Networks With Mixed Time-Varying Delay and Mismatched Parameters," *Chaos, Solitons & Fractals* 166 (2023): 112948.

50. L. Zhang, R. Wei, J. Cao, and X. Ding, "Synchronization Control of Quaternion-Valued Inertial Memristor-Based Neural Networks via Adaptive Method," *Mathematical Methods in the Applied Sciences* 47, no. 2 (2024): 581–599.

51. M. Syed Ali, G. Narayanan, S. Nahavandi, J.-L. Wang, and J. Cao, "Global Dissipativity Analysis and Stability Analysis for Fractional-Order Quaternion-Valued Neural Networks With Time Delays," *IEEE Transactions on Systems, Man, and Cybernetics: Systems* 52, no. 7 (2021): 4046–4056.

52. C. Aouiti and F. Touati, "Global Dissipativity of Quaternion-Valued Fuzzy Cellular Fractional-Order Neural Networks With Time Delays," *Neural Processing Letters* 55, no. 1 (2023): 481–503.

53. G. Zhang, Z. Zeng, and D. Ning, "Novel Results on Synchronization for a Class of Switched Inertial Neural Networks With Distributed Delays," *Information Sciences* 511 (2020): 114–126.

54. X. Liao, Q. Luo, and Z. Zeng, "Positive Invariant and Global Exponential Attractive Sets of Neural Networks With Time-Varying Delays," *Neurocomputing* 71, no. 4-6 (2008): 513–518.

55. R. Li and J. Cao, "Passivity and Dissipativity of Fractional-Order Quaternion-Valued Fuzzy Memristive Neural Networks: Nonlinear Scalarization Approach," *IEEE Transactions on Cybernetics* 52, no. 5 (2020): 2821–2832.

56. Q. Song and X. Chen, "Multistability Analysis of Quaternion-Valued Neural Networks With Time Delays," *IEEE Transactions on Neural Networks and Learning Systems* 29, no. 11 (2018): 5430–5440.

57. S. Baluni, V. K. Yadav, and S. Das, "Lagrange Stability Criteria for Hypercomplex Neural Networks With Time Varying Delays," *Communications in Nonlinear Science and Numerical Simulation* 131 (2024): 107765.