THERMAL BOUNDARY LAYER EQUATION FOR TURBULENT RAYLEIGH–BÉNARD CONVECTION

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Abstract We report a new thermal boundary layer equation for turbulent Rayleigh–Bénard convection for Prandtl number $\mathcal{P}r > 1$ that takes into account the effect of turbulent fluctuations. These fluctuations are neglected in existing equations, which are based on steady-state and laminar assumptions. Using this new equation, we derive analytically the mean temperature profiles in two limits: $\mathcal{P}r \gtrsim 1$ and $\mathcal{P}r \gg 1$. These two theoretical predictions are in excellent agreement with the results of our direct numerical simulations for $\mathcal{P}r = 4.38$ and $\mathcal{P}r = 2547.9$, respectively.

RAYLEIGH–BÉNARD CONVECTION

Turbulent Rayleigh–Bénard convection (RBC) [1, 2, 3, 5, 12], consisting of a fluid confined between two horizontal plates, heated from below and cooled from above, is a paradigm system for studying turbulent thermal convection, which is ubiquitous in nature. The state of fluid motion in RBC is determined by the Rayleigh number $\mathcal{R}a = \alpha g \Delta H^3/(\kappa\nu)$ and Prandtl number $\mathcal{P}r = \nu/\kappa$. Here α denotes the isobaric thermal expansion coefficient, ν the kinematic viscosity and κ the thermal diffusivity of the fluid, g the acceleration due to gravity, Δ the temperature difference between the bottom and top plates, and H the distance between the plates. In turbulent RBC, there are viscous boundary layers (BLs) near all rigid walls and two thermal BLs, one above the bottom plate and one below the top plate. Both viscous and thermal BLs play a critical role in the turbulent heat transfer of the system and in particular the thicknesses of the thermal BL λ is inversely proportional to the heat transport. Therefore the understanding of the processes within the BLs and prediction of the mean temperature and velocity profiles are very important [4].

PREDICTION OF THE TEMPERATURE PROFILES AND DNS RESULTS

In [8] we derived a new thermal BL equation for turbulent RBC that takes into account the effect of the turbulent fluctuations, which are neglected in the existing BL equations based on steady-state and laminar assumptions [10, 9, 7]. Using this equation, we derive analytically the mean temperature profiles for $\mathcal{P}r \gtrsim 1$ and $\mathcal{P}r \gg 1$, compare our theoretical predictions with our DNS results for $\mathcal{P}r = 4.38$ and $\mathcal{P}r = 2547.9$, and find excellent agreement.
For laminar BLs, where fluctuations are absent, the temperature Prandtl–Blasius–Pohlhausen (PBP) BL equation can be written as $\theta_{\xi\xi} + \omega\Gamma^\omega \left(1+\omega^{-1}\right)\xi^{\omega-1}\theta_\xi = 0$ with $\omega = 2$ for $\mathcal{P}r \ll 1$ and $\omega = 3$ for $\mathcal{P}r \gg 1$ and thus the laminar

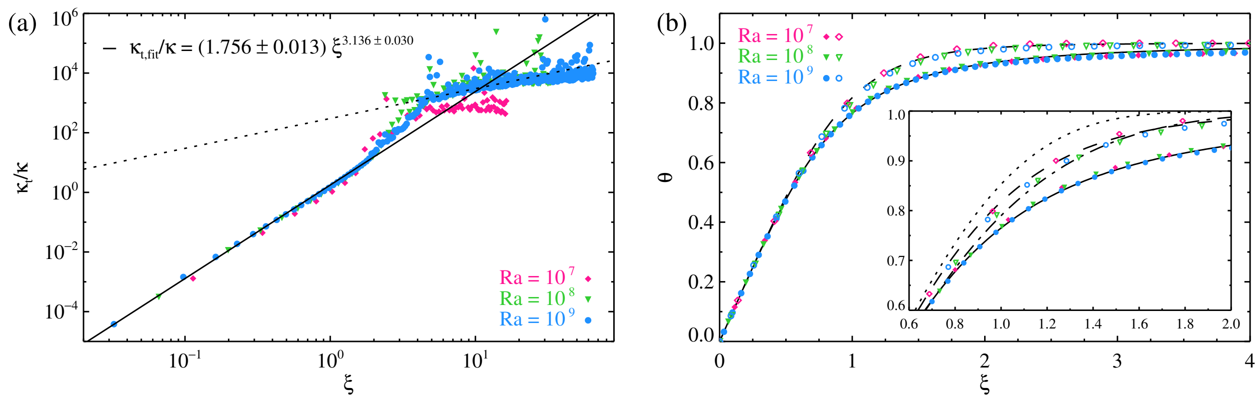

Figure 1. (a) Normalized eddy thermal diffusivity κ_t/κ, averaged in time and over horizontal cross-sections, as obtained in the DNS of RBC in a cylindrical container of the aspect ratio 1 for $\mathcal{P}r = 4.38$ and $\mathcal{R}a = 10^7$ (diamonds), 10^8 (triangles) and 10^9 (circles) together with a fit for $\mathcal{R}a = 10^9$ (line) show that close to the plate holds $\kappa_t/\kappa \propto \xi^3$. The dashed line shows the slope $\propto \xi$ that causes the logarithmical temperature profiles in the core part of the domain for sufficiently large $\mathcal{R}a$. (b) Mean temperature profiles obtained in the DNS of RBC for $\mathcal{P}r = 4.38$ (filled symbols) and $\mathcal{P}r = 2547.9$ (open symbols) for $\mathcal{R}a = 10^7$ (diamonds), 10^8 (triangles) and 10^9 (circles). Excellent agreement with the predictions (1) for $\mathcal{P}r \gtrsim 1$ (solid line) and (2) for $\mathcal{P}r \gg 1$ (dashed line) is demonstrated. An expanded view with the PBP prediction [11] for $\mathcal{P}r \gg 1$ (dotted line) and $\mathcal{P}r \ll 1$ (dot-dashed line) for comparison is shown in the inset.

temperature profiles for any $\mathcal{P}r$ are bounded by $\theta(\xi) = \int_0^\xi \exp\left[-\Gamma^\omega\left(1+\omega^{-1}\right)\chi^\omega\right]\,\mathrm{d}\chi$, with $2 \le \omega \le 3$ where Γ is the gamma function [11]. Here $\xi = z/\lambda$, z is the distance from the plate and λ the local thickness of the thermal BL.
To take into account the fluctuations, we need to know the eddy thermal diffusivity $\kappa_t = \kappa_t(x,z)$, defined as $\langle u_z' T'\rangle_t \equiv -\kappa_t \partial_z \langle T\rangle_t$, where $\langle\cdot\rangle_t$ denotes the average in time, u_z' and T' the fluctuations of the vertical velocity and temperature, respectively. A common approach for fully turbulent BLs is $\kappa_t \propto \xi$ [7] consequently leading to logarithmic temperature profiles. For moderate $\mathcal{R}a$, such log-profiles are also found but only in the turbulent bulk, which is at a relatively large distance from the plate. In the vicinity of the plate κ_t behaves rather as $\kappa_t \propto \xi^3$ (see Fig. 1a). In [8] it was derived that $\kappa_t/\kappa \approx a^3\xi^3$ holds for small ξ with some dimensionless constant a. For $\mathcal{P}r \ge$ the following thermal BL equation was also derived in [8]:

$$(1 + a^3\xi^3)\theta_{\xi\xi} + 3a^3 c\,\xi^2\theta_\xi = 0$$

with the solution

$$\theta(\xi) = \int_0^\xi (1 + a^3\eta^3)^{-c} d\eta.$$

Note that the constants a and c are related by the requirement $\theta(\infty) = 1$, which gives $a = B(c - 1/3, 1/3)/3$, where B is the beta function. Our DNS show that $0.52 < a^3 < 1.76$, with $a \sim 1.2$ for $\mathcal{P}r \gtrsim 1$ and $a \sim 0.8$ for $\mathcal{P}r \gg 1$. Thus we have $c \sim 1$ for $\mathcal{P}r \sim 1$ and $c \sim 2$ for $\mathcal{P}r \gg 1$. The analytical solutions of the BL equation for $c = 1$ and $c = 2$ read

$$\theta = \frac{\sqrt{3}}{4\pi}\log\frac{(1+a\xi)^3}{1+(a\xi)^3} + \frac{3}{2\pi}\arctan\left(\frac{4\pi}{9}\xi - \frac{1}{\sqrt{3}}\right) + \frac{1}{4}, \quad c = 1, \quad a = 2\pi/(3\sqrt{3}) \approx 1.2, \tag{1}$$

$$\theta = \frac{\sqrt{3}}{4\pi}\log\frac{(1+a\xi)^3}{1+(a\xi)^3} + \frac{3}{2\pi}\arctan\left(\frac{8\pi}{27}\xi - \frac{1}{\sqrt{3}}\right) + \frac{\xi}{3(1+(a\xi)^3)} + \frac{1}{4}, \quad c = 2, \quad a = 4\pi/(9\sqrt{3}) \approx 0.8. \tag{2}$$

Thus, all temperature profiles for $\mathcal{P}r > 1$ lie between (1) ($\mathcal{P}r \gtrsim 1$) and (2) ($\mathcal{P}r \gg 1$).
To check the predictions, eq. (1) and eq. (2), we conducted DNS of turbulent RBC for $\mathcal{P}r = 4.38$ and $\mathcal{P}r = 2547.9$ [6] for $\mathcal{R}a$ from 10^7 to 10^9. The DNS were conducted using the RBC-version [6] of the code [13]. The obtained mean temperature profiles collapse and depend only on $\mathcal{P}r$ and ξ (see Fig. 1b). The DNS profile for $\mathcal{P}r = 4.38$ is in perfect agreement with the predicted profile (1) for $\mathcal{P}r \gtrsim 1$ while the DNS profile for $\mathcal{P}r = 2547.9$ is in perfect agreement with the predicted profile (2). On the other hand, the PBP prediction [11] with $\omega = 3$ for $\mathcal{P}r \gg 1$ lies well above the DNS profile for $\mathcal{P}r = 2547.9$ and the DNS profile for $\mathcal{P}r = 4.38$ lies outside the bounds of the PBP predictions with $2 \le \omega \le 3$.
In summary, we have derived a new thermal BL equation for turbulent RBC for $\mathcal{P}r > 1$ using the idea of an eddy thermal diffusivity, which is shown to depend on the cubic power of the distance from the plate. We have solved the equation to obtain two analytical mean temperature profiles for $\mathcal{P}r \gtrsim 1$ and $\mathcal{P}r \gg 1$ respectively, and demonstrated that they are in excellent agreement with the DNS profiles.

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