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Device design and artificial material implementation

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**G U L I U X I N
J I N**

TOPOLOGY IN ENGINEERED QUANTUM SYSTEMS: DEVICE DESIGN AND ARTIFICIAL MATERIAL IMPLEMENTATION

TOPOLOGY IN ENGINEERED QUANTUM SYSTEMS: DEVICE DESIGN AND ARTIFICIAL MATERIAL IMPLEMENTATION

Dissertation

for the purpose of obtaining the degree of doctor
at Delft University of Technology
by the authority of the Rector Magnificus, prof. dr. ir. T.H.J.J. van der Hagen,
chair of the Board for Doctorates
to be defended publicly on
Tuesday, 13 January 2026 at 12:30

by

Guliuxin JIN

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Keywords: quantum physics, topological phases of matter, quantum entanglement, metamaterial, quantum random walk.

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Front & Back: An abstract representation of multiple one dimensional SSH chains, designed by Bin Wei.

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*In memory of my dear friend Shule Li (†2023),
with whom many paths were shared.*

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SUMMARY

Topological quantum matter offers robust, protected quantum states that could benefit quantum technologies. However, most work has focused on idealized systems rather than controllable platforms suitable for quantum information applications. This thesis investigates how topological features can be engineered in artificial quantum systems and what advantages they actually provide.

I study four questions through theoretical analysis and proposed experimental implementations: Can topological edge states stabilize quantum entanglement? How can topological phase transitions be controlled and measured? What are the practical limitations of topological protection in finite systems? How do topological quantum walks relate to quantum algorithms?

The results show that topological edge states do stabilize entangled Bell states against parameter fluctuations compared to trivial states. Real-time tuning of SSH arrays enables tracking of topological phase transitions and mid-gap state evolution. However, conventional topological invariants can mislead in finite systems, to which I propose a criterion bridging experimentally measurable quantities with real-space topology. Finally, discrete-time quantum walks on topological systems exhibit localization phenomena reminiscent of quantum search algorithms.

This work demonstrates that simple topological models can be implemented in realistic quantum hardware, revealing both the potential and limitations of topological protection in finite, noisy systems. The results inform future applications in quantum simulation and algorithm design while providing practical guidance for engineering topological features in quantum technologies.

1

INTRODUCTION

Since humanity first began to distinguish its thoughts, beliefs, and culture from the physical world governed by nature laws, we have been driven by an innate curiosity to understand the environment around us. This curiosity, rooted in our instincts, is as fundamental as the need to sleep, eat and stay alive.

Physics focuses to uncover the fundamental constitution of nature – matter, forces and their interactions. It's not just a passive learning process of observation and learning. The principle of physics is absorbed and internalized as tools, such that we can apply the knowledge in practical ways, or so to say that in modern words, we call it engineering.

Today, physics and engineering are often seen as separate disciplines. Engineers use the well established physics foundation to develop cutting-edge technologies, while physics, especially theoretical physics, often seems to be disconnected from the mundane world – sometimes even to physicist themselves. Physicist aim to pursue the pure beauty of natures, uncover truths about the natural world and build models to explain and predict phenomena. The epistemological orientations of physics deviates greatly from engineers' utilitarianism philosophy. Many view engineering as fundamentally dependent on physics, but this relationship is not one-sided.

From the perspective of this thesis, the relationship between physics and engineering resembles more of a symbiosis. Physics can gain a lot from engineering. Physics usually tries to answer questions about the fundamental nature of existence. While the nature being itself can be passively probed, the underlying method, theory and numerical tools, give great freedom to physicist to explore. This is where the engineering principles shine. The engineering mindset to design tools or achieve practical goals can give physics research a great incitement. Scientist can take initiative to design research method that extract the meaningful results. This interaction helps us better understand nature and find applications that benefit humanity. This thesis will demonstrate a few examples of this interplay and reflect on the physics research philosophy.

This dissertation begins by exploring the condensed matter physics, where matter behaves in ways that challenge our understanding of reality. It is a field where particles can show strange connections, where classical computation limit do not apply, and ma-

chines are designed to exploit the quantum mechanics. Through the lens of topological phases of matter, quantum experiments, and theoretical insights, this thesis will look at the frontier of quantum technology inspired by the beautiful concepts of topological physics and illustrate how these concepts inspire new paradigms in both science and engineering.

The works presented in this thesis follow two guiding principles: a global approach in condensed matter physics and bottom-up engineering in quantum physics. In condensed matter physics, the gigantic system sizes require a framework that describes all research objects collectively, rather than writing separate equations for individual components. Topological phases of matter, characterized by the global topology of their configurations, is an pearl of such examples. Topological phenomena are rooted in the global features, just like a knot in a buckled belt. There is no way to move the knot away unless you break the buckle [126]. In the quantum regime, bottom-up engineering serves as the guiding principle. Various technical challenges must be addressed, including device design [10, 38, 142], algorithm implementation [77, 169, 54] and quantum error correction [68, 175, 14]. The innovative aspect of this thesis lies in the combination of these two lines of research. First, we implement simple topological models in an engineered manner to stabilize quantum entanglement, a key resource for quantum computing. Second we implement simple topological physics in artificial material platforms and perform tuning tasks that triggers topological phase transitions. Third we bridge the gap between experimental measurement and mathematical structure to interpret the real world limitation of topology. Lastly, we implement a quantum random walk with topological properties and demonstrate its manifestation of a dynamical localization phenomenon, which can be viewed as a search algorithm.

1.1. AN OVERVIEW OF PHYSICS PRINCIPLES

s Physics studies matter, force and their interaction. The various subfields in physics are categorized by the size and energy scale of the system and the targets. In the field of condensed matter physics, physicists focus on low energy scale and small system sizes. As the name implies, it is usually at a temperature range where matters condensate. The typical temperature range studied in condensed matter physics spans a wide spectrum, from extremely low temperatures close to absolute zero (0 Kelvin or -273.15°C) to well above the room temperature. The scale of the number of particles typically considered in usually on the order of 10^{23} , corresponding to Avogadro's number. What is described as small number of particles, it still on the order of billions time trillions. Understanding the physics of these colossal systems requires more than simply writing exact equations for millions gliding one-dimensional balls hitting each other on a frictionless runway. Even if we can write those equations, we have no exact solutions. We therefore need a theory that describes collective phenomena that emerge at this scale.

1.1.1. TOPOLOGICAL PHASES OF MATTER

In topological phases, the many-body wavefunction cannot be described with purely localized orbitals, and the information is stored non-locally, spread over the entire system [78]. Multiple approaches have been proposed in order to describe the collective

behavior of many particles, where many body interactions between particles lead to emergent phenomena that can be described as quasi-particles [8, 115, 73]. "The key point is that it is not only that the magnitude of the microscopic objects prevent us from writing trillions of equations. As Robert Laughlin pointed out [117], nothing can be calculated exactly from the microscopic equations, and macroscopic systems are characterized by broken symmetry: the symmetry present in the microscopic equations is not present in the macroscopic system, due to phase transitions. The macroscopic systems are described in their own way with properties independent from some microscopic details [117]. One manifestation of such idea is the successful discovery of integer quantum hall effect [190, 119, 185] and fractional hall effect [187, 118] with strong magnetic fields that characterize the advent of topological phases of matter. These topological phases give rise to exotic phenomena include dissipationless transport and particles with fractional charge [137]. Laughlin stated that these robust physics phenomena in quantum Hall effects are protected from volatile details of microscopic physics such as quantum noise [117]. As the name suggests, topological phases of matter have particular physics features that are robust towards noise and disorder that respects the topology and symmetry of the physics systems. Later in the mid 2000s, researchers extended topological ideas beyond systems with external magnetic fields [79], with discoveries such as Quantum Spin Hall Effect [94, 140], 2D topological insulators [80, 95, 23, 112], 3D topological insulators [81, 71, 196, 85], Weyl and Dirac Semimetals [124, 9, 199], and Higher-Order Topological Phases [87, 162]. These discoveries have significantly impacted modern physics by revealing new phases of matter, enabling robust transport, and introducing exotic quasi-particles such as anyons in the fractional quantum Hall effect and Weyl fermions in semimetals – they challenge the fundamental ideas in particle physics [187, 118, 9, 124].

Despite its novelty and perceived strangeness, the exotic physics in topological phases of matter can be revealed in simple models with simple physics. We will exemplify the concept through the renowned SSH model as a minimal model to understand topological phases in the following sections [11, 178]. The field that emerged from these relatively simple considerations is now widespread and has multiple interesting applications that go beyond the native materials. Topological effects has been engineered in artificial material (meta-material) platforms like mechanical or phononic systems [167, 180, 179, 162], superconducting quantum circuits [174, 87, 92, 130, 200], ultracold atoms [121, 131, 96, 51, 91, 122], quantum dots [98, 146] and hybrid qubit-photon systems [99, 189]. These examples show the ideo of topological physics goes way beyond traditional electronic structure.

In this thesis, we take an engineering-minded approach and ask: How can we use the principles of topological physics to achieve something practically useful in quantum technologies?

1.1.2. GENERAL QUANTUM PHYSICS AND QUANTUM COMPUTING

If the energy scale of the system we are studying is restricted to be near absolute zero, the thermal fluctuations are greatly suppressed, making quantum fluctuation much more prominent. In this regime, quantum mechanical effects dominate, leading to phenomena such as quantum superposition, quantum entanglement, superconductivity, new

phases of matter, and quantum phase transitions [15, 48, 159]. Quantum physics studies the fundamental principles governing individual particles and their interactions at the smallest scales, where classical descriptions break down. Nature's behavior in quantum regime also sets the theoretical cornerstone of modern quantum technologies.

Building on this foundation, the idea of exploiting quantum principles for computation and information processing began to take shape in the 1980s, when the rudimentary quantum computer and quantum algorithms were proposed [65, 53]. What was once a theoretical projection at that time now has the foundation to grow towards technological realization. The technological progress over the past decades enabled us to pursue an idea of direct control of quantum degrees of freedom [65, 47, 171], and thus turning the fundamental understanding of quantum physics from the first half of the 20th century into concrete technological applications. Development of these applications gave rise to the fields of quantum computing [57, 58, 188, 109, 156, 29, 150], communication [198, 31, 153, 132], and simulation [172, 36, 17, 74, 100, 125].

In parallel to the advent of quantum computing, the field of classical computation faced the prediction of the empirical Moore's law, which states that the number of transistors on a chip doubles approximately every two years, leading to exponential growth in computational power. In an effort to fit the required number of transistors on chips, the size of each transistors is shrinking, in order to accommodate higher density. This path also leads to the inevitable situation where quantum effects like tunneling and Heisenberg uncertainty [149, 173] come into play as the electronic devices shrink in size. These effects not only hinder the computational reliability, but they also incur manufacturing challenges and economical concerns in miniaturizing our chips.

Although we know that Nature fundamentally operates according to the principles of quantum mechanics, there remain striking gaps in our understanding. For example, we still don't understand why the painkiller acetaminophen works, despite all the intellectual and resource dedication in the past decades. In the development of computers, instead of trying to circumvent the quantum effects, we could make computers that utilize the quantum effects. There are multiple benefits of doing quantum computations [57]. First and foremost, it is shown that the quantum systems cannot be efficiently and exactly simulated by a classical computer, due to exponential scaling of required computational resources [143, 37, 182]. Quantum computers are expected to be able to simulate quantum systems in a way beyond the reach of classical computers, especially in areas like quantum chemistry and strongly-coupled physics systems [143, 37, 182, 139]. This engineering need, combined with the theoretical developments in quantum information gave rise to the field of quantum computing as we know it today.

Quantum computing is a rapidly evolving field that has garnered intense attention from both academia and industry. A variety of physical platforms have demonstrated quantum computational capabilities, including superconducting qubits [25, 108, 113], trapped ions [195, 38, 147, 116], neutral atoms [60, 84] and photonic systems [201, 67, 18] nitrogen-vacancy centers in diamond [32], and semiconductor quantum dots [193, 197, 191]. Beyond traditional qubits, higher-dimensional computational units such as qudits have also been explored for enhanced encoding and computational power [154, 44].

Despite this progress, significant challenges remain. Many platforms face limita-

tions in scalability, short coherence times, control complexity, and insufficient gate fidelity, which restrict the practical circuit depth before decoherence dominates [202, 40, 107, 22]. As a result, fault-tolerant quantum computation hinges critically on the implementation of effective quantum error correction schemes. A major milestone was the recent demonstration by Google's Willow processor of quantum error correction below the fault-tolerance threshold, showing that increasing the number of physical qubits per logical qubit can exponentially suppress logical errors [2, 4, 110, 101]. Neutral atom platforms, such as QuEra's 48-logical-qubit system, have also advanced quantum error correction techniques [27], alongside cat-qubit implementations based on harmonic oscillators [133, 152].

Pushing the boundaries of quantum computing has become a joint effort between academic institutions and private companies, with close collaborations accelerating progress in both fundamental research and engineering scalability. These efforts are collectively driving quantum applications across cryptography [127], materials discovery [50, 19], quantum chemistry [12], drug design [39], and optimization problems [63]. As mentioned above, simulating complex quantum systems is a natural and straightforward task for quantum computers. They enable studies of many-body and topological systems [129], quantum phase estimation [76], and entanglement in topologically ordered states [161, 170, 166]. Both analogue [136, 90] and digital [64] simulations provide access to quantum dynamics beyond classical capabilities.

One of the major challenges in the development of quantum computing is preventing errors. This challenge is being actively addressed in the field of error correction. Due to the nature of quantum states and their susceptibility to environmental noise, maintaining coherence and accuracy is far more demanding than in classical systems. Error correction codes have been proposed to address this, including the toric code [101], surface code [68], Bacon-Shor code [14] and qLDPC code [145, 56, 155]. However, they require substantial qubit overhead, posing a serious scalability challenges.

Topological quantum computing offers an alternative approach by encoding information in non-local degrees of freedom that are inherently protected against local errors [101, 114, 1, 160]. Just like the analogy of robust topological properties, it is like a twist in a buckled belt [126]. Perturbations are like manipulating the belt. Anyone can easily conclude that no manipulation can erase the twist unless the belt is unbuckled. Majorana zero modes, for example, can emerge in superconducting systems, and they are intrinsically robust against local noise and environmental disturbances [101, 160, 120]. The stability means it is possible to make quantum computers with fewer physical qubit overhead. These Majorana modes can be realized through engineered systems like the Kitaev chain [102], and recent experiments have demonstrated progress in this area [59, 183]. The non-Abelian statistics of Majorana modes also opens the door to topologically protected quantum gate operations [186, 181].

Despite these advances, it is still commonly recognized that realizing and controlling Majorana-based systems is extremely hard [160, 1]. Even to see their existence takes endless human effort and their observation is much debated subject by the physics community. This motivates the exploration of simpler topological models that may still offer partial protection against noise. In this thesis, we investigate whether topological features of more accessible systems, such as the Su-Schrieffer-Heeger (SSH) model (intro-

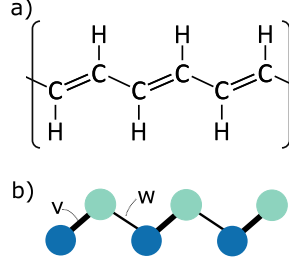


Figure 1.1: (a) A schematic plot of a section of polyacetylene polymer. Letter C represent the carbon atom and letter H represents the hydrogen atom. Single and double bonds are shown in single and double solid line. (b) Schematic plot of the SSH model in a finite system. The thick and thin lines represent the intra-cell coupling v and inter-cell coupling w respectively. The blue and green dots are type-A and B sublattice sites. The chain consists of four unit cells.

duced in Sec. 1.1.1), can be utilized to design quantum devices where certain operations benefit from topological robustness [89, 99, 69, 52, 176, 183].

1.2. THEORY

1.2.1. TOPOLOGY IN CONDENSED MATTER PHYSICS

SU-SCHRIEFER-HEEGER (SSH) MODEL

A classic starting point for understanding topological phases is the 1D Su-Schrieffer-Heeger (SSH) model. Originally proposed as a model for polyacetylene, it has become a textbook example of 1D topological insulators. Polyacetylene has a backbone of alternating single and double carbon bonds. The π -bonds allow electron delocalization, making it one of the earliest metallic polymers [83, 168]. This bond alternation leads to chiral (sublattice) symmetry, a key feature we will revisit later. The schematic plot is shown in Fig. 1.1 panel a). In Fig. 1.1b) we show the subtracted schematic plot of the corresponding SSH model. Each dot denotes a lattice site, and thick/thin lines represent intra-cell (v) and inter-cell (w) hopping. The alternating hoppings form a lattice of repeating unit cells with A/B sublattices (blue/green dots).

The tight banding Hamiltonian of the one dimensional SSH model [178] is given by

$$H = \sum_{n=1}^N (v_n c_{n,A}^\dagger c_{n,B} + w_n c_{n,B}^\dagger c_{n+1,A} + h.c.), \quad (1.1)$$

where $c_{n,\alpha}^\dagger$ ($c_{n,\alpha}$) is the creation (annihilation) of a particle on lattice site (n, α) with unit cell index $n \in [1, N]$ and sublattice index $\alpha \in A, B$. The alternating v_n and w_n define a dimerized chain of repeating A-B units. Translation invariance leads to the fact that $v_n = v$ and $w_n = w$. The intra-cell to inter-cell hopping ratio, v/w , controls the topological phase transition.

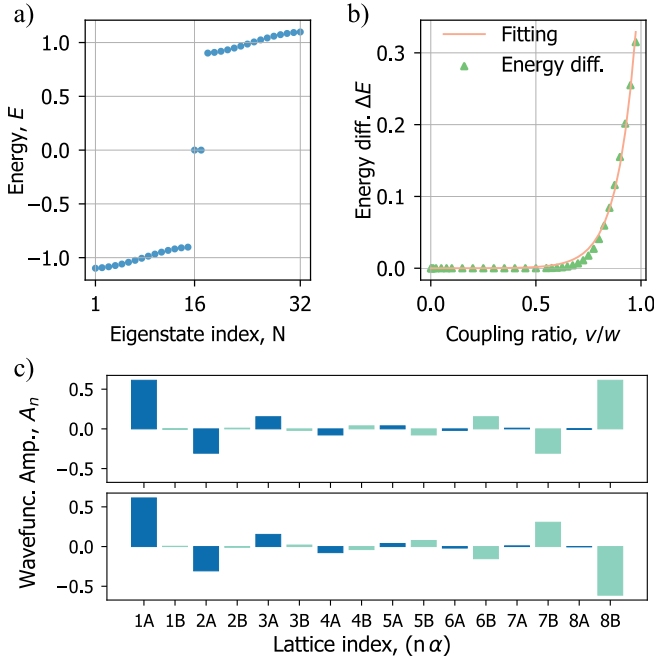


Figure 1.2: (a) Energy spectrum of the SSH model with $\nu = 1$, $w = 2$ for a chain of $N = 8$ lattice sites. (b) Energy splitting between the two mid-gap edge modes as a function of the hopping ratio ν/w , fitted with an exponential function. (c) Wavefunction amplitudes of symmetric and anti-symmetric edge modes. Upper panel: symmetric edge mode; lower panel: anti-symmetric edge mode.

TOPOLOGICAL EDGE MODES

In the topological phase, the SSH model supports edge-localized modes with energies in the band gap (Fig. 1.2c). The edge modes are exponentially localized at the edge of the systems. Their energy sits in the band gap. In infinitely large systems, each mode occupies one edge of the chain and their energies degenerate. In finite chains, the edge modes hybridize due to their spatial overlap, leading to an energy splitting that decays exponentially with system size as $\sim e^{-L/\xi}$, where ξ is the localization length. In the SSH model, the localization length ξ of the edge states is determined by the ratio of the intra-cell hopping ν and inter-cell hopping w , and is given by

$$\xi^{-1} = \left| \ln \left| \frac{\nu}{w} \right| \right|. \quad (1.2)$$

As $\nu/w \rightarrow 0$, $\xi \rightarrow 0$ which means perfect localization. As $\nu \rightarrow w$, the localization length diverges $\xi \rightarrow \infty$, signaling delocalization of the edge modes and proximity to the topological phase transition[11].

These edge modes are protected by chiral symmetry and cannot be removed without closing the gap or breaking the symmetry. The consequence of the symmetry is that they are robust against local perturbations that respect the underlying symmetry. The mid-

gap edge modes distinguishes the SSH model from a conventional insulator despite both being gapped in the bulk. It represents the family of topological insulators.

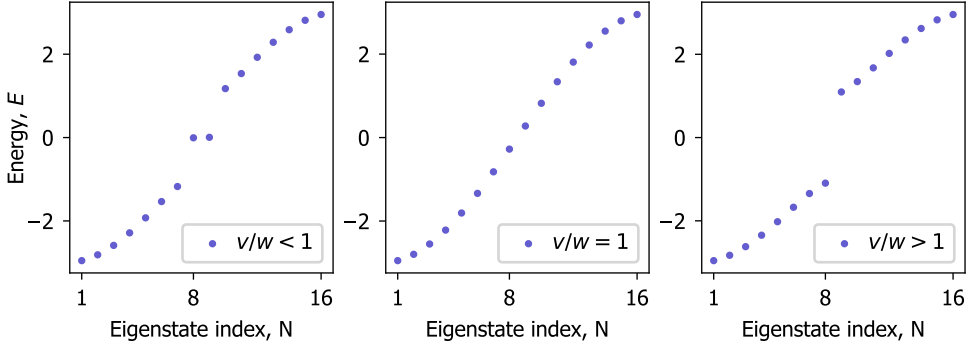


Figure 1.3: Energy spectrum of the SSH model showing the topological phase transition in an $N = 8$ lattice. Each panel displays the energy eigenvalues as a function of eigenstate index for different hopping ratios v/w : (left) $v/w = 0.5$ in topological phase, (middle) $v/w = 1$ is the critical point, and (right) $v/w = 2$ in trivial phase. The presence of mid-gap states in the topological phase and their disappearance in the trivial phase highlight the topological transition.

TOPOLOGICAL PHASE TRANSITION

Tuning v/w drives a phase transition. For $v < w$, the system is topologically non-trivial with edge states. At $v = w$, the gap closes, and for $v > w$, it becomes trivial (Fig. 1.3). For example, starting from $v = 0$ with any finite w value, it is expected to see two topological edge modes in the band gap. Increasing v relative to w causes the band gap to shrink and eventually closes at the critical point $v = w$, where the system becomes gapless (see Fig. 1.3 middle panel). Beyond this point ($v > w$), the system enters a topologically trivial phase in which the edge states vanish (see Fig. 1.3 right panel). In this regime, the bulk band structure no longer supports edge-localized zero modes under open boundary conditions, reflecting the change in topological property across the phase transition.

While the SSH model was originally proposed for spinless fermions, the model can readily be implemented as an array of superconducting resonators [99]. We will elaborate on this implementation in Chapter 2.

SYMMETRY IN TOPOLOGICAL PHASES

In this section, we will have a brief overview of symmetry in condensed matter physics. When we say that a Hamiltonian has a certain symmetry, it means that it has conservation laws and can be block diagonalized by this symmetry operator [11]. The symmetry operators can be either unitary or anti-unitary, and can commute or anti-commute with the Hamiltonian.

Unitary symmetries (e.g., spin rotation or translation) commute with the Hamiltonian and can simplify the analysis, such as reducing H to $H(k)$ in momentum space.

However, they usually do not protect topological features on their own. Exceptions include spatial symmetries like inversion or mirror symmetry, which constrain the global band structure and can protect nontrivial topology in crystals [71, 70]. These spatial symmetries connect different k -points, which gives it the ability to constrain the global structure of the Bloch bands. Such properties usually give nontrivial topology.

In quantum mechanics, there are three fundamental symmetries that enforce nontrivial constraints, which are either anti-unitary or involve conjugation. They are time reversal symmetry, particle hole symmetry and chiral symmetry. We give a short summary of their attributes in the following:

Time reversal symmetry (TRS) $\mathcal{T}$. As the name suggests, TRS flips the time. $\mathcal{T}$ is an anti-unitary operator which commutes with the Hamiltonian,

$$\mathcal{T} H(k) \mathcal{T}^{-1} = H(-k). \quad (1.3)$$

In reality its present can be broken by applying magnetic fields. Using spin as an example, it couples to the spin via the Zeeman term $\mathbf{B} \cdot \boldsymbol{\sigma}$ [66]. Zeeman term changes sign under time reversal, since time reversal flips spin direction but the external magnetic field remains unchanged. TRS also leads to Kramers degeneracy and protect topological insulators like the quantum spin Hall effect [94, 23].

Particle hole symmetry (PHS) $\mathcal{P}$ appears in BdG Hamiltonians for superconductors [28, 49]. It is an anti-unitary operator which anti-commutes with the Hamiltonian,

$$\mathcal{P} H(k) \mathcal{P}^{-1} = -H(-k). \quad (1.4)$$

This symmetry relates positive and negative energy states. It's crucial for topological superconductors [6].

Chiral symmetry (CS) Γ . It is a unitary operator which anti-commutes with the Hamiltonian. Chiral symmetry operator Γ acts on the Hamiltonian in the following way:

$$\Gamma H(k) \Gamma^{-1} = -H(k). \quad (1.5)$$

Note that Γ is not completely independent from $\mathcal{T}$ and $\mathcal{P}$. Chiral symmetry operator Γ can be expressed as the combination of TRS and PHS: $\Gamma = \mathcal{T} \cdot \mathcal{P}$, which means that when CS is present, both TRS and PHS need to be present or absent simultaneously [158].

Topological phases respond differently to symmetry. For example, the quantum Hall type insulator does not have any symmetry constraint, and it will break down when time-reversal symmetry is present [184, 79]. On the contrary, some topological phases are protected by symmetries. For instance, the quantum spin Hall insulator is protected by time-reversal symmetry [95, 23, 138, 71], Majorana zero mode in the Kitaev chain is protected by particle hole symmetry [102].

Topological phases of non-interacting fermionic systems are classified based on the presence or absence of the three fundamental symmetries: TRS, PHS, and CS. Each combination defines a symmetry class in the Tenfold Way classification with its topological invariant [6, 164, 103]. This framework provides a systematic way to understand which kinds of topological phases can exist in different dimensions under different symmetry constraints.

In order to demonstrate the effect of symmetry, we consider chiral symmetry in more detail. If a Hamiltonian satisfies chiral symmetry:

$$\Gamma H(k) \Gamma^{-1} = -H(k), \quad (1.6)$$

then the energy spectrum is symmetric around zero. That is, for every eigenstate with energy ϵ , there exists a partner with energy $-\epsilon$.

This has several implications. First, the presence of zero-energy modes is robust: chiral symmetry forbids any term in the Hamiltonian that can shift them away from zero. Second, in one dimension, systems with chiral symmetry belong to class AIII (or BDI depending on the symmetry constraint) in the Tenfold Way classification and are characterized by an integer-valued topological invariant — the winding number $\nu \in \mathbb{Z}$, which we will discuss in the following.

MOMENTUM-SPACE FORMULATION

In order to analyze bulk band structure and define topological invariants, we consider momentum space representation of the operators/Hamiltonian. Assuming translational invariance ($v_n = v$, $w_n = w$), we Fourier transform the creation (annihilation) operators $c_{n,\alpha}$ ($c_{n,\alpha}^\dagger$) as

$$c_{n,\alpha} = \frac{1}{\sqrt{N}} \sum_k e^{ikn} c_{k,\alpha}, \quad (1.7)$$

$$c_{n,\alpha}^\dagger = \frac{1}{\sqrt{N}} \sum_k e^{-ikn} c_{k,\alpha}^\dagger, \quad (1.8)$$

where $\alpha = A, B$ labels the sublattice indices, momentum variable $k \in [-\pi, \pi]$ runs over the first Brillouin zone and N labels the unit cell indices.

Substituting this into the real-space Hamiltonian Eq. (1.1), we obtain the Bloch Hamiltonian in momentum space given by

$$H = \sum_k \Psi_k^\dagger H(k) \Psi_k, \quad \Psi_k = \begin{pmatrix} c_{k,A} \\ c_{k,B} \end{pmatrix}, \quad (1.9)$$

and

$$H(k) = \begin{pmatrix} 0 & v + w e^{-ik} \\ v + w e^{ik} & 0 \end{pmatrix} \quad (1.10)$$

This 2×2 bulk momentum-space Hamiltonian $H(k)$ acts in the sublattice (A/B) basis. The general two band model can be expressed compactly using the Pauli matrices $\boldsymbol{\sigma}$:

$$H(k) = d_x(k) \sigma_x + d_y(k) \sigma_y + d_z(k) = \mathbf{d}(k) \cdot \boldsymbol{\sigma}, \quad (1.11)$$

where $\boldsymbol{\sigma}$ is the vector containing Pauli matrices and $\mathbf{d}(k)$ are the three dimensional coefficients which we will refer to as Bloch vector. In the case of the SSH model, $\mathbf{d}(k)$ is given by

$$d_x(k) = v + w \cos(k); \quad d_y(k) = w \sin(k); \quad d_z(k) = 0: \quad (1.12)$$

Bloch vector $\mathbf{d}(k)$ reflects the an internal structure of the eigen states of the SSH model. Specifically it is the change of the direction of this d-vector while the momentum k varies. Note that the absence of $d_z(k)$ component reveals the chiral symmetry of the model: the Hamiltonian anti-commutes with σ_z , i.e., $\{H(k), \sigma_z\} = 0$ [11]. This symmetry properties will be discussed further in the following sections.

Diagonalizing $H(k)$ gives the dispersion relation given by

$$E(k) = \pm |\mathbf{d}(k)| = \pm \sqrt{v^2 + w^2 + 2vw \cos(k)}. \quad (1.13)$$

The energy dispersion relation of the SSH model is plotted in Figure Fig. 1.4 (a-e) for five sets of different v/w ratio. A band gap exists for all k except when $v = w$, where the gap closes at $k = \pm\pi$. This gap closing signals a topological phase transition, as discussed in the earlier section.

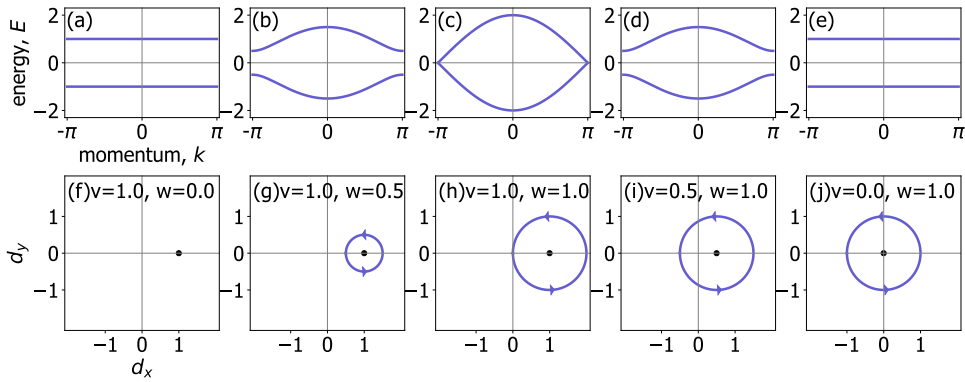


Figure 1.4: Energy bands and Bloch vectors of the SSH model for different hopping parameters. Top row (a-e): energy dispersion, as in Eq. (1.2.1), as a function of momentum k for different intra-cell v and inter-cell w hopping amplitudes. The value of each pair of v and w is denoted in the corresponding panel below. The gap closes when $v = w$ at $k = \pm\pi$, and reopens as the ratio v/w changes. Bottom row (f-j): corresponding $\mathbf{d}(k) = (d_x(k), d_y(k))$ (Eq. (1.2.1)) trajectories when momentum k runs over the Brillouin zone $k \in [0, 2\pi]$. The path of endpoints of the Bloch vector represents the momentum Hamiltonian $H(k)$. Loops that wind around the origin indicate a topological phase with nonzero winding number, and vice versa.

In the SSH model, the Bloch vector $\mathbf{d}(k)$ Eq. (1.2.1) traces a closed trajectory in the (d_x, d_y) plane as the momentum k varies across the Brillouin zone $[0, 2\pi]$. As illustrated in Fig. 1.4 (f-j), the topology of this trajectory depends on the relative values of the ratio between inter-cell and intra-cell hopping v/w . A useful quantity to characterize the system's phase is the winding number ν , which counts how many times the trajectory encircles the origin. Counter-clockwise encircling of the origin contributes $+1$ while clockwise encircling contributes -1 [11]. This topological invariant distinguishes different phases of the system. When $\nu > 0$, the trajectory does not enclose the origin (Fig. 1.4(f-g)), yielding a winding number $\nu = 0$ and indicating a trivial insulating phase.

Conversely, when $\nu < w$, the trajectory encloses the origin (Fig. 1.4(i-j)), corresponding to a winding number $\nu = 1$ and a nontrivial topological phase. At the critical point $\nu = w$, the trajectory intersects the origin (Fig. 1.4(h)), where the energy gap closes, marking a topological phase transition. The winding number thus provides a geometric and topological characterization of the SSH model's distinct phases, clearly visualized in Fig. 1.4. Note that this topological invariant is defined in momentum space. In Chap. 4 we will discuss the real space counterpart of this quantity and interpret the finite size effect of winding number.

TOPOLOGICAL INVARIANT: WINDING NUMBER

We now turn to the central concept that unifies these features discussed so far: topological invariants [80, 11]. These are quantized, global properties of the system's eigenstates that remain unchanged under smooth deformations of system parameters, as long as the bulk energy gap remains open. In the SSH model, the key topological invariant is the winding number, which classifies distinct topological phases in one dimension [11]. In contrast to conventional phases of matter, which are typically characterized by local order parameters and spontaneous symmetry breaking, topological phases are defined by such global invariants encoded in the band structure. These invariants serve as sharp labels distinguishing distinct topological phases that cannot be connected without undergoing a phase transition. A key physical consequence of this is the emergence of protected boundary states, as captured by the principle of bulk-boundary correspondence [80, 11].

In the context of the SSH model, the relevant topological invariant is the winding number ν . It characterizes how the Bloch Hamiltonian $H(k)$, when expressed in terms of the effective vector $\mathbf{d}(k)$, winds around the origin in the $d_x - d_y$ plane as the momentum k varies. Intuitively, this winding reflects the topological structure of the Bloch states: a nontrivial winding number indicates a topologically nontrivial phase, which supports edge states in a finite geometry. Mathematically the calculation of ν is given by the following integral:

$$\nu = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(\frac{\mathbf{d}(k)}{|\mathbf{d}(k)|} \times \frac{d}{dk} \left(\frac{\mathbf{d}(k)}{|\mathbf{d}(k)|} \right) \right) dk, \quad (1.14)$$

where $\frac{\mathbf{d}(k)}{|\mathbf{d}(k)|}$ is the normalized Bloch vector. This expression captures the total phase rotation of the vector $\mathbf{d}(k)$ around the origin as k evolves over the Brillouin zone. The cross product inside the integrand evaluates the rate of angular change, i.e., the differential arc traced by the unit vector on the unit circle.

Equivalently, if we define a complex function $q(k) = d_x(k) - i d_y(k)$, the winding number can also be written as the change in the phase of $q(k)$ over the Brillouin zone:

$$\nu = \frac{1}{2\pi} \int_{-\pi}^{\pi} \partial_k \arg[q(k)] dk. \quad (1.15)$$

This interpretation makes it clear: ν is the net number of times the phase of $q(k)$ winds by 2π as k goes from $-\pi$ to π .

The topological invariant defined in the bulk has direct physical implications at the boundaries that can be observed. In the SSH model, a nontrivial winding number $\nu =$

1 guarantees the presence of a pair of zero-energy edge states under open boundary conditions. More generally, the magnitude of a topological invariant often predicts the number of protected boundary modes: for instance, the number of zero modes in the SSH model equals $|\nu|$, and the number of chiral edge states in a Chern insulator equals the Chern number [158, 82, 135]. These boundary modes are robust as long as the bulk gap remains open and the protecting symmetry is preserved. This correspondence is a defining feature of topological phases of matter.

While the winding number is the central invariant for the SSH model and other one-dimensional chiral symmetric systems, higher-dimensional systems can exhibit other topological invariants, such as the Chern number in two-dimensional quantum Hall systems [184], or the $\mathbb{Z}_2$ invariants in time-reversal symmetric topological insulators [95]. Each invariant plays a role in classifying topological phases under the framework of the so-called tenfold way [158], depending on the symmetry class and spatial dimension.

EXPERIMENTAL ADVANCES AND GENERALIZATIONS

While the SSH model is a minimal toy model, its topological structure has inspired a wide range of experimental realizations across condensed matter and artificial material platforms. Variants of the SSH model have been implemented in photonic crystals [151, 148], mechanical metamaterials [180, 86, 141], ultracold atoms in optical lattices [13], and superconducting circuit chains [157, 174]. In these systems, the bulk-boundary correspondence manifests as robust boundary modes, directly observable through local spectroscopy, light transmission, or dynamical response. These engineered setups not only confirm the abstract bulk classification but also allow unprecedented tunability of parameters and symmetries, enabling exploration of topological phase transitions and non-Hermitian extensions. In later chapters, we revisit such generalizations, especially in the context of tunable quantum simulators.

1.2.2. QUANTUM TECHNOLOGY

Quantum systems exhibit exponential scaling in their computational state space, where N qubits can represent 2^N distinct computational states through quantum superposition [143, 104]. This exponential growth in accessible computational resources fundamentally distinguishes quantum computers from classical machines, where computational power scales only linearly with the number of bits. The ability to coherently manipulate exponentially large superposition states enables quantum computers to explore vast solution spaces, providing a unprecedented computational advantage [143]. This exponential resource scaling represents the core computational power of quantum computing, applicable across diverse problem domains beyond quantum system simulation.

Theoretical quantum algorithms exploit this exponential computational space to achieve provable advantages over classical approaches across various computational complexity classes [143]. Shor's polynomial time factoring algorithm [169] demonstrates exponential speedup for cryptographically relevant problems, while Grover's search algorithm [77] provides quadratic acceleration for unstructured search tasks. However, the practical utility of algorithms with only quadratic speed-ups, such as Grover's, remains under active investigation, particularly regarding whether such improvements translate

into real-world advantages when accounting for quantum overhead and error correction [177]. These complexity theoretic results establish the fundamental computational advantages that quantum mechanics can provide, laying the groundwork for understanding the basic building blocks of quantum information processing.

QUBIT

In quantum computation, the fundamental elementary computational unit is called the quantum bit, or the qubit, as the counterpart of the bit in classical computation. A classical bit can only assume discrete values of 0 or 1, representing a binary two-level information system. In principle, any physics system exhibiting two distinct states can function as a classical bit, from simple mechanical switches to the transistors that form the foundation of contemporary digital computers [143].

While a qubit also constitutes a two-level quantum system, it exploits quantum mechanical superposition to exist in a coherent linear combination of both basis states simultaneously. [143]. The general qubit state is

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle, \quad (1.16)$$

where α and β are complex probability amplitudes satisfying the normalization condition $|\alpha|^2 + |\beta|^2 = 1$. This superposition principle enables qubits to encode and process information in fundamentally different ways compared to classical bits.

Let's look at the immediate consequence of the different physics property between classical bits and qubit. Consider a system of three quantum particles, each capable of existing in spin up $|\uparrow\rangle$ or spin down $|\downarrow\rangle$ states. The complete three-particle system hosts in total $2^3 = 8$ computational basis states. Classical representation of this system requires explicit storage of all eight states, demanding computational resources scaling as 2^3 classical bits. On the contrary, three qubits can coherently represent a superposition over all eight basis states, such as the maximally entangled state

$$\psi = \frac{1}{\sqrt{8}}(|000\rangle + |001\rangle + |010\rangle + |100\rangle + |011\rangle + |101\rangle + |110\rangle + |111\rangle). \quad (1.17)$$

Classical simulation of N quantum particles requires 2^N bits, becoming intractable for modest system sizes [65]. Quantum computers naturally operate within this exponential Hilbert space, providing inherent advantages for such computations. This capability has profound implications for computational chemistry, condensed matter physics, and drug discovery, where conventional numerical approximation methods such as mean-field theory, density functional theory, and density matrix renormalization group approaches often fail to capture essential quantum correlations in strongly interacting many-body systems [128, 41, 12].

The physical constitution of qubits varies significantly across different quantum computing platforms, with implementations ranging from trapped ions and superconducting circuits to photonic systems and semiconductor quantum dots [17, 193, 60, 67]. These physical qubits, while capable of storing and manipulating quantum information, are inherently susceptible to decoherence and operational errors that accumulate during computation [150]. To address these limitations, the concept of logical qubits emerges as an essential abstraction layer, where multiple physical qubits are collectively

encoded using quantum error correction protocols to form a single, more robust computational unit [2, 27]. This hierarchical structure enables fault-tolerant quantum computation by distributing quantum information across many physical qubits, allowing the logical qubit to maintain coherence and computational fidelity even when individual physical qubits experience errors [75].

Quantum systems can exist in two fundamentally different types of states: pure states and mixed states. A pure state represents a quantum system with complete information, described by a single state vector $|\psi\rangle$ in the Hilbert space [143]. The system's evolution is deterministic and coherent, governed by unitary transformations. Pure states exhibit quantum interference effects and can demonstrate genuine quantum superposition, making them essential for quantum computational algorithms. In contrast, mixed states arise when the quantum system's preparation is uncertain or when the system has undergone decoherence through environmental interaction [163]. Mixed states cannot be described by a single state vector but require the density matrix formalism:

$$\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|, \quad (1.18)$$

where p_i represents the classical probability of finding the system in pure state $|\psi_i\rangle$, with $\sum_i p_i = 1$ [143]. The purity of a state can be quantified by $\text{Tr}(\rho^2)$, which equals unity for pure states and is less than unity for mixed states.

ENTANGLEMENT

Quantum entanglement emerges as a uniquely quantum phenomenon where multi-particle systems exhibit correlations that cannot be explained by any classical theory [165]. For a composite quantum system of two subsystems A and B, a pure state $|\psi\rangle_{AB}$ is separable if it can be written as a product state:

$$|\psi\rangle_{AB} = |\psi_A\rangle \otimes |\psi_B\rangle. \quad (1.19)$$

If no such decomposition exists, the state is entangled. Entangled states exhibit non-local correlations where measurement outcomes on spatially separated subsystems remain correlated in ways that violate classical intuition and local realism [20]. Bell states represent the paradigmatic examples of maximally entangled two-qubit pure states. The four Bell states form a complete orthonormal basis for two-qubit systems:

$$\begin{aligned} |\Phi^+\rangle &= \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle), & |\Phi^-\rangle &= \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle), \\ |\Psi^+\rangle &= \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle), & |\Psi^-\rangle &= \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle). \end{aligned} \quad (1.20)$$

These states cannot be decomposed into products of single-qubit states, demonstrating perfect quantum entanglement where measurement of one qubit instantaneously determines the state of the other, regardless of spatial separation [61]. Bell states serve as fundamental resources for quantum information protocols, including quantum teleportation, superdense coding, and quantum cryptography [31, 21].

QUANTUM GATES AND CIRCUITS

Quantum computation operates through the application of unitary transformations, known as quantum gates, which manipulate qubit states while preserving quantum coherence [143]. Single-qubit gates include the Pauli matrices (X , Y , Z), Hadamard gate (H), and rotation gates (for example T -gate represents the rotation around the Z -axis by an angle of $\pi/4$), while two-qubit gates such as the controlled-NOT (CNOT) enable entanglement generation between qubits [16].

The gates can be categorized into Clifford gate and non-Clifford gates. The above discussed gates are Clifford gate, which

These gates form universal gate sets, meaning any quantum computation can be decomposed into sequences of these fundamental operations [194]. Quantum circuits represent algorithms as directed acyclic graphs of gate operations applied to qubit registers, providing the computational framework for quantum algorithm implementation.

QUANTUM MEASUREMENT AND PARALLELISM

Quantum measurement projects quantum superposition states onto classical outcomes according to the Born rule, where measurement probabilities are given by the squared amplitudes of the corresponding basis states [30]. This fundamentally probabilistic process destroys quantum coherence, making the timing of measurement crucial for algorithm design [143]. However, extracting useful information requires careful arrangement of quantum interference, where different computational paths interfere constructively for correct answers while interfering destructively for incorrect ones, enabling quantum speedups over classical computation [77, 143, 169].

Modern Quantum Computers Contemporary quantum computers exist primarily in the Noisy Intermediate-Scale Quantum (NISQ) regime, characterized by limited qubit counts and significant error rates that preclude full error correction [150]. Major hardware platforms include superconducting transmon qubits operated at millikelvin (mK) temperatures, trapped atomic ions manipulated by laser fields, and photonic systems utilizing optical quantum states [10, 136, 201]. These systems require sophisticated classical control infrastructure for qubit initialization, gate operations, and measurement readout, with typical gate fidelity ranging from 99-99.9% and coherence times on the order of microseconds to milliseconds [113] while the range varies significantly by platform. Current quantum processors contain dozens to hundreds of physical qubits with limited connectivity, imposing constraints on circuit depth and algorithm implementation that significantly influence the design and feasibility of near-term quantum algorithms [24]. Currently, the quantum computing community continues to develop quantum error correction protocols and hardware implementations to address these fundamental challenges, with promising results demonstrating below-threshold quantum error correction where logical error rates decrease exponentially as the number of physical qubits increases [2, 152, 27].

QUANTUM ALGORITHMS AND APPLICATIONS

As quantum computing has matured, several quantum algorithms have demonstrated significant speedups over their classical counterparts. These algorithms can be broadly

organized into overlapping paradigms based on their underlying mechanisms. The first paradigm leverages quantum Fourier transforms and global quantum interference effects. Notable examples include Shor's algorithm for integer factorization [169], which achieves exponential speedup over the best-known classical algorithms, and the Deutsch–Jozsa algorithm, which provides exponential query complexity separation for specific promise problems [54]. Quantum phase estimation, a fundamental subroutine underlying many of these algorithms, enables efficient extraction of eigenvalue information from quantum systems [143].

A second paradigm encompasses amplitude amplification techniques, which systematically enhance the success probability of quantum subroutines. This framework includes Grover's algorithm for unstructured search [77] and quantum counting, which estimates the number of marked items in a database [33], both achieving quadratic speedups in query complexity. A third important class involves quantum Hamiltonian simulation, which enables efficient simulation of quantum many-body systems that would require exponential classical resources [123, 72]. Finally, quantum walk-based algorithms exploit quantum analogues of classical random walks to solve problems such as element distinctness and spatial search on graphs, often achieving optimal or near-optimal query complexities [7, 45].

Collectively, these algorithms highlight the diverse mechanisms by which quantum systems can outperform classical computation. Each approach is tailored to specific problem structures and computational models, ranging from gate-based quantum circuits to analog quantum simulators. In this thesis, we cover quantum walks and quantum search, so in the following paragraphs, we introduce Grover's algorithm in more detail to provide context for our subsequent analysis of localization-based search methods.

Grover's algorithm exemplifies the amplitude amplification framework, which systematically enhances the success probability of quantum subroutines through coherent interference. For a probabilistic quantum procedure with success probability p , amplitude amplification can boost this probability arbitrarily close to unity using $\mathcal{O}(1/\sqrt{p})$ iterations [33]. Applied to the unstructured database search problem, Grover's algorithm achieves a quadratic speedup over classical approaches.

Problem Formulation Given an unstructured database of $N = 2^n$ items, the objective is to identify a marked element w satisfying $f(w) = 1$, where $f : \{0, 1\}^n \rightarrow \{0, 1\}$ represents an oracle function distinguishing marked from unmarked items. Classical algorithms require $\mathcal{O}(N)$ oracle queries in the worst case, while Grover's algorithm achieves the same task using only $\mathcal{O}(\sqrt{N})$ oracle calls [77].

The algorithm operates in the N -dimensional computational basis $\{|x\rangle\}_{x=0}^{N-1}$ and begins by preparing the uniform superposition state:

$$|s\rangle = \frac{1}{\sqrt{N}} \sum_{x=0}^{N-1} |x\rangle.$$

The core iteration, known as the Grover operator G , implements a sequence of two reflections:

$$G = (2|s\rangle\langle s| - I)(I - 2|w\rangle\langle w|),$$

where the first reflection inverts the amplitude of the marked state $|w\rangle$, and the second performs an inversion about the average amplitude.

Geometrical Interpretation The geometric perspective provides crucial insight into Grover's algorithm operation. The algorithm evolution is confined to a two-dimensional subspace spanned by two orthogonal states:

- Unmarked superposition:

$$|s'\rangle = \frac{1}{\sqrt{N-1}} \sum_{x \neq w} |x\rangle \quad (1.21)$$

- Marked state: $|w\rangle$

The initial uniform superposition of all states, as shown in the left panel of Fig. 1.5, can be decomposed as:

$$|s\rangle = \sqrt{\frac{N-1}{N}} |s'\rangle + \frac{1}{\sqrt{N}} |w\rangle = \cos(\theta/2) |s'\rangle + \sin(\theta/2) |w\rangle, \quad (1.22)$$

where $\sin(\theta/2) = 1/\sqrt{N}$ and $\cos(\theta/2) = \sqrt{(N-1)/N}$.

Each Grover iteration performs a rotation by angle θ in this two-dimensional subspace. The state after k iterations becomes:

$$G^k |s\rangle = \cos\left(\frac{2k+1}{2}\theta\right) |s'\rangle + \sin\left(\frac{2k+1}{2}\theta\right) |w\rangle. \quad (1.23)$$

For large N , $\theta \approx 2/\sqrt{N}$, and the optimal number of iterations is approximately $\lfloor \frac{\pi}{4}\sqrt{N} \rfloor$, maximizing the amplitude on the marked state.

The Grover operator can be decomposed as $G = R_s R_w$, where:

$$R_w = I - 2|w\rangle\langle w|, \quad (1.24)$$

$$R_s = 2|s\rangle\langle s| - I. \quad (1.25)$$

R_w represents oracle reflection, which flips the amplitude of the marked state w.r.t the unmarked superposition $|s'\rangle$ (x-axis). In the next step, R_s represents diffusion operator, which reflects $R_w |s\rangle$ about the uniform superposition $|s\rangle$. Geometrically, R_w reflects the state vector about the axis perpendicular to $|w\rangle$, while R_s reflects about the axis $|s\rangle$. The composition of these reflections yields a rotation by θ toward the marked state. The geometrical relationship of all the vectors is shown in Fig. 1.5 (a). The accumulation of Grover steps is shown in Fig. 1.5 (b). k represents the number of applying Grover's operator G and Eq. (1.2.2) is represented by the vectors with different tinted color showing their iteration number k . Note that applying multiple iterations will lead to the saturation eventually the search probability will decrease as k increases. Therefore the behavior is periodic.

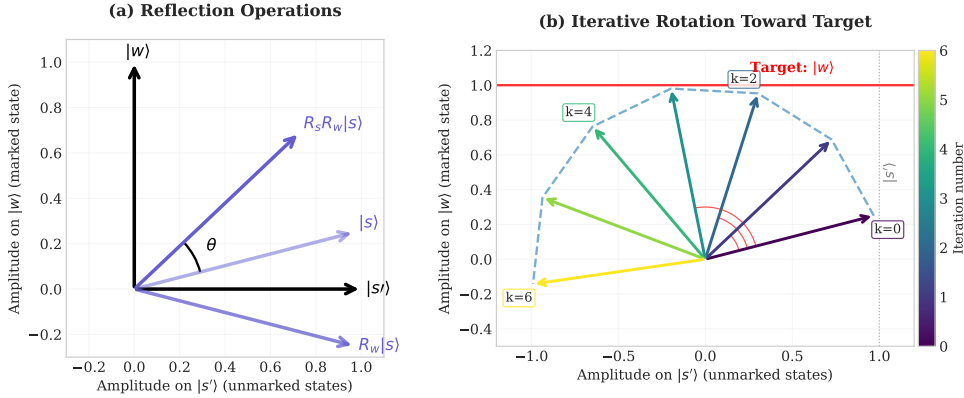


Figure 1.5: Figure: Geometric interpretation of Grover's algorithm in the two-dimensional subspace spanned by the unmarked superposition $|s'\rangle$ (x-axis) and marked state $|w\rangle$ (y-axis). (a) Step 1. Reflection operations: The Grover operator $G = R_s R_w$ consists of two sequential reflections. The oracle reflection R_w flips the initial state $|s\rangle$ through $|s'\rangle$, transforming $|s\rangle$ to $R_w|s\rangle$. Step 2: The diffusion operator R_s then reflects $R_w|s\rangle$ about the uniform superposition axis ($|s\rangle$), giving $G|s\rangle$. Two steps complete one iteration with a net rotation by angle $\theta = 2 \arcsin \frac{1}{\sqrt{N}}$. (b) Iterative rotation toward target: Successive applications of the Grover operator rotate the state vector by θ in each iteration, systematically increasing the overlap with the marked state $|w\rangle$. The trajectory (blue dashed curve) shows the evolution from the initial uniform superposition toward the target state, with optimal measurement occurring after approximately $\pi\sqrt{N}/4$ iterations. The color gradient indicates the iteration number, demonstrating the quadratic speedup over classical search. k represents the iteration of Grover's operator G .

Limitations and Robustness Considerations Grover's algorithm assumes ideal conditions: perfect oracle implementation, coherent quantum evolution, and precise knowledge of the number of marked items. In realistic scenarios, several factors degrade performance. Decoherence and gate errors accumulate over the $\mathcal{O}(\sqrt{N})$ iterations, potentially destroying the quantum advantage for sufficiently large databases. Additionally, when the number of marked items is unknown or multiple solutions exist, the algorithm requires modification to avoid overshooting the optimal rotation angle. These practical limitations have motivated investigation of alternative quantum search paradigms that maintain computational efficiency while exhibiting enhanced robustness. One promising direction exploits quantum localization phenomena in engineered many-body systems. Rather than implementing discrete unitary rotations, these approaches utilize ground-state preparation or adiabatic evolution in specifically designed Hamiltonians to concentrate amplitude on marked configurations [3]. Such methods can naturally accommodate analog quantum simulation platforms and may demonstrate superior resilience to environmental perturbations and imperfect control, representing a complementary approach to gate-based quantum search algorithms. The intricacy of Grover's algorithm and its relation to localization phenomena will be further discussed in later

chapters of this thesis.

QUANTUM SIMULATIONS IN cQED PARADIGMS

From now on we will discuss the physical realization to implement all the controlled quantum phenomena. Circuit quantum electrodynamics (cQED) represents a paradigm where artificial atoms—superconducting qubits—are coupled to quantized electromagnetic modes in superconducting microwave resonators [26]. This platform has emerged as one of the most promising architectures for quantum computation, quantum simulation, and quantum error correction due to its strong coupling strengths, long coherence times, and high degree of controllability. The fundamental Hamiltonian governing this interaction is the Jaynes-Cummings model [88], given by:

$$\hat{H} = \hat{H}_{atom} + \hat{H}_{field} + \hat{H}_{int} \quad (1.26)$$

$$= \hbar\omega_q\hat{\sigma}_z/2 + \hbar\omega_r\hat{a}^\dagger\hat{a} + \hbar g(\hat{\sigma}_+\hat{a} + \hat{\sigma}_-\hat{a}^\dagger), \quad (1.27)$$

with three terms consisting free field Hamiltonian, atomic Hamiltonian and the interaction Hamiltonian. In Eq. (1.2.2), ω_q is the qubit transition frequency, ω_r is the resonator frequency, g is the coupling strength, $\hat{\sigma}_\pm$ are the qubit raising/lowering operators, and $\hat{a}(\hat{a}^\dagger)$ are the bosonic annihilation (creation) operators.

Superconducting microwave resonators form the foundation of cQED systems, providing high-quality factor cavities that enable strong light-matter interactions [55, 25]. These resonators, typically implemented as coplanar waveguides or lumped-element circuits, can achieve quality factors $Q = \omega_r/\kappa$ exceeding 10^6 in well-designed systems, where κ is the resonator decay rate [25]. The electromagnetic field modes in a coplanar waveguide resonator of length L are characterized by:

$$\omega_n = \frac{n\pi c}{L\sqrt{\epsilon_{eff}}}, \quad (1.28)$$

where n is the mode number, c is the speed of light, and ϵ_{eff} is the effective permittivity. The zero-point fluctuations of the electric field create the vacuum Rabi frequency:

$$g = \frac{d_{ge}}{2} \sqrt{\frac{\hbar\omega_r}{2\epsilon_0 V_{eff}}}, \quad (1.29)$$

where d_{ge} is the electric dipole matrix element and V_{eff} is the effective mode volume.

The operation regime of cQED systems is fundamentally determined by the relationship between the coupling strength g and the detuning $\Delta = \omega_q - \omega_r$. When the system operates in the dispersive regime where $|\Delta| \gg g$, the dynamics change dramatically compared to the resonant case. This regime is of particular importance for quantum information processing as it enables quantum non-demolition measurements and forms the basis for most qubit readout schemes in superconducting quantum processors [26, 111].

To understand the dispersive regime, we begin with the Jaynes-Cummings Hamiltonian and apply a Schrieffer-Wolff transformation to eliminate the rapidly oscillating terms [34]. In the interaction picture with respect to the bare qubit and resonator Hamiltonians, the interaction term becomes:

$$\hat{H}_I = \hbar g(\hat{\sigma}_+\hat{a}e^{i\Delta t} + \hat{\sigma}_-\hat{a}^\dagger e^{-i\Delta t}) \quad (1.30)$$

For large detuning $|\Delta| \gg g$, these terms oscillate rapidly and their time-averaged effect can be treated perturbatively. Applying second-order perturbation theory, the effective Hamiltonian in the dispersive regime becomes [26]:

$$\hat{H}_{eff} = \hbar \tilde{\omega}_q \hat{\sigma}_z / 2 + \hbar \tilde{\omega}_r \hat{a}^\dagger \hat{a} + \hbar \chi \hat{\sigma}_z \hat{a}^\dagger \hat{a} \quad (1.31)$$

The key parameters are renormalized as follows:

- The effective qubit frequency: $\tilde{\omega}_q = \omega_q + \chi$.
- The effective resonator frequency: $\tilde{\omega}_r = \omega_r + \chi$.
- The dispersive coupling strength: $\chi = g^2 / \Delta$.

The dispersive interaction term $\hbar \chi \hat{\sigma}_z \hat{a}^\dagger \hat{a}$ represents a state-dependent frequency shift. When the qubit is in the ground state ($\hat{\sigma}_z = -1$), the resonator frequency becomes $\omega_r - \chi$, while in the excited state ($\hat{\sigma}_z = +1$), it becomes $\omega_r + \chi$. This 2χ frequency difference forms the basis for dispersive qubit readout.

The physical mechanism underlying dispersive coupling can be understood through the concept of the AC Stark shift. The qubit-resonator interaction creates virtual photon exchange processes that shift the energy levels of both systems. The magnitude of this shift depends on the photon number in the resonator, leading to the characteristic $\hat{n}$ -dependent term in the effective Hamiltonian.

For practical implementations, the dispersive shift enables quantum non-demolition (QND) measurements of the qubit state through frequency shifts of the resonator [111]. A microwave tone applied near the resonator frequency will exhibit different phase and amplitude responses depending on the qubit state, allowing for high-fidelity state discrimination. The measurement strength is characterized by the signal-to-noise (SNR) ratio [26]:

$$\text{SNR} = \frac{4\chi^2 \bar{n}}{\kappa}, \quad (1.32)$$

where $\bar{n}$ is the average photon number in the resonator during measurement. This relationship highlights the trade-off between measurement speed (larger $\bar{n}$) and measurement-induced dephasing.

The dispersive regime also introduces important corrections to qubit dynamics. The presence of thermal or coherent photons in the resonator leads to additional dephasing mechanisms through the dispersive coupling. The pure dephasing rate due to photon shot noise is given by:

$$\Gamma_\phi^{(shot)} = \chi^2 \bar{n} \frac{2\kappa}{(\Delta_d)^2 + \kappa^2}, \quad (1.33)$$

where Δ_d is the detuning between the drive frequency and the bare resonator frequency.

Building upon these foundational cQED principles, quantum error correction emerges as a critical requirement for fault-tolerant quantum computation. The surface code, a two-dimensional topological stabilizer code, has emerged as the leading candidate for superconducting quantum processors [68] due to its high threshold and compatibility with nearest-neighbor connectivity constraints typical of superconducting architectures.

A stabilizer code is defined by its stabilizer group $\mathcal{S} = \langle g_1, g_2, \dots, g_{n-k} \rangle$, where each generator g_i is a tensor product of Pauli operators. The logical subspace is the +1 eigenspace of all stabilizers:

$$|\psi_L\rangle \in \mathcal{H}_L \iff g_i |\psi_L\rangle = |\psi_L\rangle \quad \forall g_i \in \mathcal{S}. \quad (1.34)$$

The surface code encodes k logical qubits in $n = L^2$ physical qubits arranged on a 2D lattice with periodic boundary conditions. The code distance $d = L$ determines the error correction capability, allowing correction of up to $\lfloor (d-1)/2 \rfloor$ errors. The stabilizer generators consist of X-type stabilizers: $X_s = \prod_{i \in \partial s} X_i$ for each square plaquette s and Z-type stabilizers: $Z_v = \prod_{i \in \partial v} Z_i$ for each vertex v .

The syndrome measurement circuit involves ancilla qubits coupled to data qubits through controlled-NOT gates, enabling detection of both bit-flip and phase-flip errors. In superconducting implementations, these syndrome measurements are performed using the same dispersive readout techniques described above, creating an intimate connection between the fundamental cQED physics and error correction protocols.

The error threshold theorem states that quantum computation can be performed fault-tolerantly if the physical error rate p is below a threshold p_{th} . For the surface code under realistic noise models in superconducting systems:

$$\begin{aligned} p_{th} &\approx 1\% \text{ (depolarizing noise),} \\ p_{th} &\approx 0.1\% \text{ (biased noise).} \end{aligned} \quad (1.35)$$

The logical error rate scales as:

$$p_L \propto \left(\frac{p}{p_{th}} \right)^{(d+1)/2}. \quad (1.36)$$

The physical error sources in superconducting qubits that limit the approach to these thresholds include several decoherence mechanisms:

- Relaxation (T_1 process): Energy decay with time constant $T_1 = 1/\Gamma_1$
- Dephasing (T_2 process): Phase decoherence with $1/T_2 = 1/(2T_1) + 1/T_\phi$
- Gate errors: Coherent and incoherent errors during quantum operations.

The master equation describing the evolution under Markovian noise is:

$$\frac{d\rho}{dt} = -\frac{i}{\hbar} [\hat{H}, \rho] + \sum_{\alpha} \mathcal{D}[\hat{L}_{\alpha}] \rho, \quad (1.37)$$

where $\mathcal{D}[\hat{L}] \rho = \hat{L} \rho \hat{L}^\dagger - \frac{1}{2} \{ \hat{L}^\dagger \hat{L}, \rho \}$ is the Lindblad dissipator [35].

The same superconducting platform that enables quantum error correction also provides a powerful testbed for quantum simulations. Digital quantum simulation employs gate sequences to approximate the time evolution operator $\hat{U}(t) = e^{-i\hat{H}t/\hbar}$ using Trotterization [72]:

$$\hat{U}(t) = \lim_{n \rightarrow \infty} \left[\prod_j e^{-i\hat{H}_j \delta t / \hbar} \right]^n, \quad (1.38)$$

where $\hat{H} = \sum_j \hat{H}_j$ and $\delta t = t/n$. The first-order Trotter error scales as $O((\delta t)^2)$.

Many-body spin systems are natural targets for quantum simulation using superconducting circuits. The Heisenberg model with competing interactions:

$$\hat{H} = J_1 \sum_{\langle i,j \rangle} \vec{S}_i \cdot \vec{S}_j + J_2 \sum_{\langle\langle i,j \rangle\rangle} \vec{S}_i \cdot \vec{S}_j, \quad (1.39)$$

can be implemented using native two-qubit gates in superconducting processors. The ZZ interaction arises naturally from the always-on coupling between transmon qubits, while XY interactions can be engineered through parametric driving schemes.

Topological quantum states represent a particularly intriguing class of many-body systems that exhibit robust entanglement properties, making them attractive for both quantum simulation and error correction [192]. The connection between topological order and quantum error correction is profound: both rely on non-local correlations that protect quantum information from local perturbations. The topological entanglement entropy provides a measure of topological order:

$$S_{\text{topo}} = S_A + S_B + S_C - S_{AB} - S_{BC} - S_{AC} + S_{ABC}, \quad (1.40)$$

where S_X denotes the von Neumann entropy of the region X [192].

QUANTUM RANDOM WALKS

Quantum random walks (QRWs) are quantum analogs of classical random walks and are fundamental in areas such as quantum computing, simulation, and topological physics. There are two principal formulations: continuous-time and discrete-time quantum walks.

In a Continuous-Time Quantum Walk (CTQW), the state $|\psi(t)\rangle$ evolves according to the Schrödinger equation with a Hamiltonian H derived from the adjacency matrix A of a graph G :

$$|\psi(t)\rangle = e^{-iHt} |\psi(0)\rangle. \quad (1.41)$$

This model avoids the need for an auxiliary coin degree of freedom. Coherent transport arises purely from interference on the graph structure [62]. CTQWs have been applied to model transport in complex systems and biological networks [134].

Discrete-Time Quantum Walks (DTQWs), by contrast, include an internal coin space and alternate between coin and shift operations. The single-step unitary operator is

$$U = S \cdot (C \otimes \mathbb{I}), \quad (1.42)$$

where C is the coin operator and S is a conditional shift defined by

$$S = \sum_x (|0\rangle\langle 0| \otimes |x-1\rangle\langle x| + |1\rangle\langle 1| \otimes |x+1\rangle\langle x|). \quad (1.43)$$

After t steps, the walker's state is $|\psi(t)\rangle = U^t |\psi(0)\rangle$ [5]. Unlike classical walks (with variance $\sigma^2 \sim t$), DTQWs exhibit ballistic spreading ($\sigma^2 \sim t^2$) [97].

Dynamical localization emerges when disorder (e.g., spatially random coins) suppresses diffusion due to destructive interference. In such cases, the wavefunction remains confined to a finite region:

$$|\psi(t)\rangle \approx \sum_{x \in \text{local region}} \alpha_x(t) |x\rangle. \quad (1.44)$$

The localization length depends on the disorder strength [93, 144], akin to Anderson localization in static systems, and persists in periodically driven cases [43].

QRWs underpin a variety of quantum algorithms. CTQWs provide exponential speedups for graph traversal [46], while DTQWs offer quadratic improvements in search problems [7]. These speedups stem from coherent amplitude amplification and interference.

Topological features can also be embedded into DTQWs. In the split-step protocol, different coin rotations and conditional shifts generate effective topological band structures. A common unitary operator is:

$$U(\theta_1, \theta_2) = S_+ \cdot R(\theta_2) \cdot S_- \cdot R(\theta_1), \quad (1.45)$$

where $R(\theta)$ is a coin rotation and $S_{\pm}$ are direction-dependent shifts. These walks support edge states protected by topological invariants [105], and their robustness has been observed in photonic and cold atom experiments [106, 42].

1.3. STRUCTURE OF THE THESIS

This thesis is structured around four main chapters, each corresponding to an original research project that builds toward the goal of using engineered quantum systems to probe and harness topological and quantum information phenomena. The thesis explores how topological phases can be realized, controlled, and interpreted in artificial material platforms, with applications ranging from entanglement stabilization to quantum algorithms alike phenomena. Each chapter is linked to a peer-reviewed publication or preprint.

Chapter 2: Entanglement stabilization using topological edge states The main body of the thesis begins by investigating how simple topological edge states can serve as robust carriers of entanglement for quantum information processing. Focusing on an experimental proposal of circuit QED implementation of the SSH model, this chapter shows how localized edge states can be used to stabilize entangled states against local losses. The work demonstrates how topology can enhance quantum information protocols by protecting coherence in a driven-dissipative setting.
<https://journals.aps.org/prresearch/abstract/10.1103/PhysRevResearch.5.023088>

Chapter 3: Experimental realization of topological bands in superconducting circuits The next chapter focuses on the experimental realization of topological band structures in a tunable superconducting circuit. The platform allows for control over individual coupling strengths and on-site potentials, enabling a continuous transition between trivial and topological phases. Through local transmission measurements and dynamical response, the presence of topological edge modes is directly identified. This chapter focuses on how tunable superconducting resonator systems can be used to simulate static band topology and explores the topological phase transition.
<https://journals.aps.org/prresearch/abstract/10.1103/PhysRevResearch.6.043286>

Chapter 4: Local observables for real-space detection of topology The fourth chapter sets the theoretical foundation for interpretation of real space topology. Rather than relying solely on bulk invariants in the thermodynamic limit, the work introduces a physically motivated framework for diagnosing topology through experimentally accessible observables. Through analytical and numerical analyses, the behavior of conductivity and real-space topological invariants is examined, revealing practical insights for determining topology in finite systems. The results show how finite-size effects shape the topological signatures and propose operational criteria for identifying topological phases in intermediate-scale systems.

<https://arxiv.org/abs/2411.17822>

Dynamical localization in topological quantum walks The final chapter turns to time-dependent quantum dynamics by studying discrete-time quantum walks and discovered dynamical localization in a topological quantum walk as a search algorithm. A localized defect in the coin parameter creates robust, defect-bound states that localizes quantum walkers. The results provide insight into the mechanism of topological localization and draw a connection to quantum search algorithms, showing how topological control in time dependent systems can support computationally relevant dynamics.

<https://arxiv.org/abs/2406.18768>

1.4. LIST OF CONTRIBUTIONS

List of Contributions

This section outlines my contributions to the chapters in this dissertation and highlights the collaborative efforts behind each project.

In Chapter 2, the idea for the project came from Eliska Greplova, and I contributed during the initial discussions. I was responsible for running the numerical simulations and creating the plots, with feedback and guidance from Eliska. We analyzed the results together and collaborated closely on writing the manuscript.

Chapter 3 describes an experiment initially conceived by Lukas Splitthoff and Christian Kraglund Andersen, with support from Eliska Greplova and me. Lukas and Miguel Carrera Belo designed the experiment, collected and analyzed the data. Lukas also fabricated the device, and Yu Li provided the proximitized nanowires. The manuscript was written by Lukas, Miguel, and Christian, with input from the rest of us. Christian also supervised the project.

Chapter 4 is based on a project I initiated, with input from Dmytro Oriekhov, Lukas Splitthoff, and Eliska Greplova. I carried out the simulations and made the plots, again working closely with Dmytro, Lukas, and Eliska. Dmytro contributed the analytical calculations, and we collaborated on interpreting the results. Together with Eliska and Lukas, we co-wrote the manuscript. Eliska supervised the project. I also had the pleasure of working with bachelor student Naoual El Yazidi, and master students Sibren van der Meer and Stan Bergkamp, who contributed through helpful discussions and support.

In Chapter 5, the project was conceived by Eliska Greplova, with input from Dmytro Oriekhov and me. Dmytro and I worked together on the numerical simulations, while he also developed the analytical side of the quantum random walk. We discussed and

interpreted the results as a team—Dmytro, Eliska, and I—and co-wrote the manuscript. The project was supervised by Eliska.

During my PhD at TU delft, I published a paper containing the continuing work based on my masters thesis work at ETH Zurich (Physical Review A 105.2 (2022): 023302 <https://journals.aps.org/pr/abstract/10.1103/PhysRevA.105.023302>), which is not included in this thesis. This work focus on characterizing quantum simulation systems. As discussed in this introduction, quantum simulators can have unprecedented computational power that is intractable for classical computers. However create the challenge that the calibration of large-scale quantum devices is hard. In this work we developed a scalable algorithm based on neural networks for Hamiltonian tomography in out-of-equilibrium quantum systems. Our algorithm can reconstruct the underlying Hamiltonian parameters to a precision level beyond previous approaches.

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2

TOPOLOGICAL ENTANGLEMENT STABILIZATION IN SUPERCONDUCTING QUANTUM CIRCUITS

Topological properties of quantum systems are among the most intriguing emerging phenomena in condensed matter physics. A crucial property of topological systems is the symmetry-protected robustness towards local noise. Experiments have demonstrated topological phases of matter in various quantum systems. However, using the robustness of such modes to stabilize quantum correlations is still a highly sought-after milestone. In this work, we put forward a concept of using topological modes to stabilize fully entangled quantum states, and we demonstrate the stability of the entanglement with respect to parameter fluctuations. Specifically, we see that entanglement remains stable against parameter fluctuations in the topologically nontrivial regime, while entanglement in the trivial regime is highly susceptible to local noise. We supplement our scheme with an experimentally realistic and detailed proposal based on coupled superconducting resonators and qubits. Our proposal sets an approach for generating long-lived quantum modes with robustness towards disorder in the circuit parameters via a bottom-up experimental approach relying on easy-to-engineer building blocks.

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2.1. INTRODUCTION

Topological quantum states have been an object of great interest to physicists over the last few decades due to their curious properties [44, 2, 23]. Starting with the discovery of the quantized conductivity in the quantum Hall effect [42], the field has developed a series of exciting discoveries in modern topics like topological insulators [21, 4], Weyl semimetals [43, 38], and topological superconductors [33, 13, 34, 32]. At the same time, topological phenomena offer promising applications. For example, non-interacting topological models, represented by the band theory, give rise to the field of spintronics [46], while certain interacting models are predicted to lead to ground-breaking new applications such as topological quantum computing [1].

A great success of modern condensed matter theory is the discovery of that topological phases of matters are generally subject to topological invariants related to the global symmetries [3]. These invariants were originally connected to the phases of quantum systems. Still, recent studies have shown that a number of topological phenomena initially observed in non-interacting quantum systems are reproducible in purely classical systems [41, 40, 22, 30, 7, 8, 27, 37, 18, 26, 24]. Recently the straightforward engineering principles discovered through the field of classical metamaterials have been adopted to quantum systems, where such metamaterials were then reproduced [26, 24]. In this work, we put forward a new concept of topology-stabilized quantum entanglement, inspired by classical topological metamaterials, and thus introduce a bottom-up method to engineer topological metamaterials inherently manifesting quantum properties. Specifically, we propose a method to generate long-range entangled states of topological modes in a one-dimensional (1D) system and provide a detailed analysis of the noise robustness such modes possess.

This manuscript is organized as follows: In Sec. 1.2.1 we give a brief overview of the Su-Schrieffer-Heeger (SSH) model, a basic building bloc of our proposal. In Sec. 2.3 we propose a sequence to entangle two spatially separated topological edge modes. In Sec. 2.4 we statistically analyze the stability of such topological entanglement in the presence of parameter fluctuations. In Sec. 2.5 we discuss practical methods to address topological entanglement in disordered systems. Finally, in Sec. 2.6 we give a conclusion of our study and discuss the outlook.

2.2. SU-SCHRIEFFER-HEEGER MODEL

In Sec. 1.2.1, we introduced the SSH model, which is originally proposed for fermions. Nevertheless, the model can readily be implemented as an array of superconducting resonators [24].

For the sake of convenience, recall that the SSH Hamiltonian [39] is given by

$$H = \sum_{n=1}^N (v_n c_{n,A}^\dagger c_{n,B} + w_n c_{n,B}^\dagger c_{n+1,A} + h.c.), \quad (2.1)$$

where $c_{n,\alpha}^\dagger$ ($c_{n,\alpha}$) is the creation (annihilation) of a particle on lattice site (n, α) with unit cell index $n \in [1, N]$ and sublattice index $\alpha \in A, B$. The intra-cell to inter-cell hopping ratio, v/w , controls the topological phase transition. The model is in the topologically

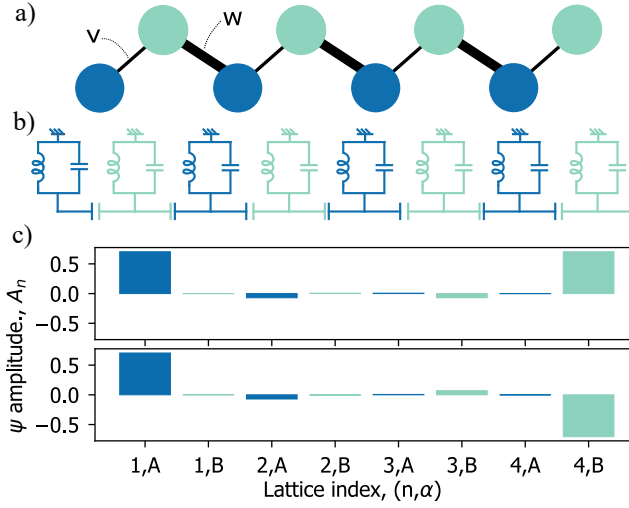


Figure 2.1: a) Schematic plot of the SSH model in a finite system. The blue and green dots are type- A and B sublattice sites. The thin and thick lines represent the intra-cell coupling v and inter-cell coupling w respectively. b) Experimental realization with capacitively coupled superconducting resonators. c) Wave-function ψ amplitude A_n of the two mid gap zero-energy edge modes with $v = 0.1$, $w = 1.0$. Upper panel: Plot of the symmetric edge mode; lower panel: Plot of the anti-symmetric edge mode.

non-trivial phase at $v/w < 1$ where two mid-gap states exist. Such states are exponentially localized at the edge lattices, as shown in Fig. 1.2(c). The two edge modes are robust towards arbitrary disorder as long as the disorder respects the chiral symmetry of the Hamiltonian. The energy gap closes at $v/w = 1$, which indicates the topological phase transition. When $v/w > 1$, the model is in the trivial phase where two mid-gap edge states merge into the bulk band, and the edge localization is no longer valid. In the context of superconducting circuits, resonator transmission and reflection measurements can readily confirm the appearance of the mid-gap modes in the topological phase of the SSH model. We provide a detailed discussion of the SSH model and its spectrum in Sec. 2.A.

2.3. ENTANGLING THE TOPOLOGICAL MODES

In the context of quantum entanglement, we propose a scheme to entangle two spatially separated topological SSH edge modes and investigate the robustness of such entanglement against disorders. This task can be accomplished by engineering a system consisting of a single qubit and two 1D SSH arrays, see Fig. 2.2(a). In Fig. 2.2(b) we show how this architecture can be engineered with capacitively coupled superconducting (SC) resonators arrays [6, 25, 45, 12]. The specific realization of the qubit is flexible and platform-dependent. It can be achieved, for example, by a transmon within a purely superconducting setting [25, 17, 36, 45] or by a quantum dot in the case of hybrid devices

[35, 12, 9, 10, 15, 28, 14].

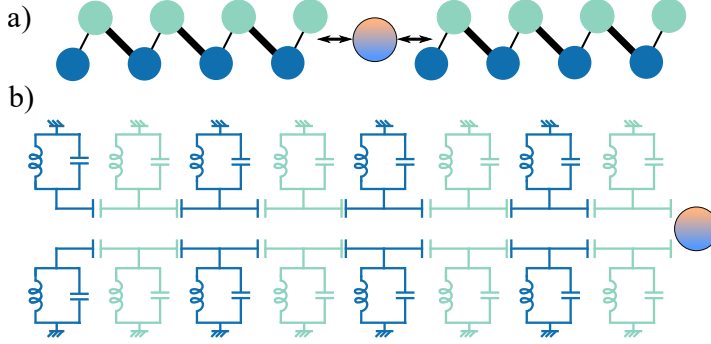


Figure 2.2: a) Schematic plot of two SSH arrays coupled to one qubit - each chain consisting of four unit cells of two sublattice sites. Blue and green colors represent the A and B types of sublattice. Orange-blue gradient circle represents the qubit. Arrows represent dispersive coupling between the qubit and the edge lattice sites. b). Each sublattice site represents one superconducting LC resonator. The varying capacitive coupling between resonators is indicated with the capacitor distance.

When both SSH chains are in the topologically non-trivial phase, i.e., inter-cell coupling w is larger than intra-cell coupling v , we can observe two zero-energy modes localize at the edges in each array. The system in Fig. 2.2(a) is described, in a frame rotating at the resonator and qubit frequencies, by the Hamiltonian

$$H = \sum_{\langle i,j \rangle} J_{1,ij} a_{1,i}^\dagger a_{1,j} + \sum_{\langle i,j \rangle} J_{2,ij} a_{2,i}^\dagger a_{2,j} + \xi_1 \sigma_z a_{1,N}^\dagger a_{1,N} + \xi_2 \sigma_z a_{2,1}^\dagger a_{2,1}, \quad (2.2)$$

where $a_{X,i}^\dagger$ ($a_{X,i}$) are creation (annihilation) operators in the SSH chains. $X = 1, 2$ represents the two SSH chains and i is the unit cell within a chain. In this setting, the modes excited by driving of the resonator array are bosons. The qubit is described by the operator $\sigma_z = |e\rangle\langle e| - |g\rangle\langle g|$. We assume that the qubit is coupled to the resonators in the dispersive regime with ξ the qubit state-dependent dispersive frequency-shift of the resonators, given by $\xi_X = g^2/\delta_X$, where g is the dipole-coupling between the resonator and the qubit. This assumption is valid when a large qubit-resonator detuning δ_X with respect to the coupling strength g is present [25, 6].

We start investigating the entanglement in the resonator-qubit system based on Hamiltonian Eq. (2.2). A tripartite quantum state is created by exciting this system to a state that couples the qubit to the topological edge modes in both resonator arrays. By three parties, we here refer to topological edge modes in first resonator array, the qubit, and the topological edge modes in the second resonator array. Below we show that the qubit projective measurement results in a maximally entangled state among the topological modes on SSH arrays, whose robustness we wish to analyze.

Numerical analysis of systems shown in Fig. 2.2 is done by the exact diagonalization, where the single particle basis of each SSH chain is deployed to numerically diagonalize the Hamiltonian Eq. (2.2). Note that the non-interacting nature of the bosonic excitations does not restrict each SSH chain to only hosting a single excitation, it merely means we are expressing our wave-function in the basis of single-particle excitations. The exact diagonalization gives 128 eigenstates of the Hamiltonian Eq. (2.2).

We proceed by performing projective measurement in the σ_z basis of the qubit, thereby projecting the qubit state either to the ground ($|g\rangle$) or excited ($|e\rangle$) state. The probability to measure the $|e\rangle$ and $|g\rangle$ states are equal. The details of the calculation are in App. 2.B. The qubit projection yields number of possible entangled states between the remaining two SSH arrays. When the desired eigenstate of the system is excited followed by a projective measurement yielding the state $|e\rangle$, the density matrix of the two SSH arrays shows the appearance of maximal entanglement, see Fig 2.3

2.4. ENTANGLEMENT STABILITY ANALYSIS

Negativity is a measure of bipartite quantum entanglement. It is derived from the positive partial transpose (PPT) criterion for separability of quantum states [47, 31, 16] and is an entanglement monotone. Therefore it reaches its maximum on the maximally entangled states. Zero negativity value indicates no distillable entanglement, while non-zero value indicates the existence of distillable entanglement. We use negativity to characterize the entanglement generated in the two SSH system above.

Consider a general system with density matrix ρ composed of two subsystems A and B . The partial transposition of ρ with respect to subsystem B is given by [47, 31, 16]

$$\rho^{T_B} := (I \otimes T)(\rho), \quad (2.3)$$

which is the identity map applied to A and the transposition map applied to B . Negativity can be calculated through the absolute sum of the negative eigenvalues of ρ^{T_B} [47, 31, 16], given by

$$\mathcal{N}(\rho) = \left| \sum_{\lambda_i < 0} \lambda_i \right| = \sum_i \frac{|\lambda_i| - \lambda_i}{2}, \quad (2.4)$$

where λ_i are all the eigenvalues of ρ^{T_B} . Note that $\mathcal{N} \in [0, 0.5]$ for bipartite entanglement. Maximal entanglement states, such as Bell states, reach the value $\mathcal{N} = 0.5$. The reconstruction of the density matrix ρ from experimental data is dependent on the specific implementation. The quantum state tomography in the context of superconducting qubits was shown in Refs. [5, 11].

By inspecting all the eigenstates of Hamiltonian Eq. (2.2), we find that the proposed system indeed hosts states that lead to entanglement between two SSH chains. For example, in Fig. 2.3 we show the density matrix ρ of a $\mathcal{N} = 0.5$ state hosted by the two SSH chains after the qubit has been measured and projected onto the state $|e\rangle$ (here both SSH chains are in the topological regime). From the structure of the density matrix it is visible that the amplitudes are located at the edges of the two arrays according to the definition of the occupational basis. We emphasize that in the two-SSH system, each chain acts as a

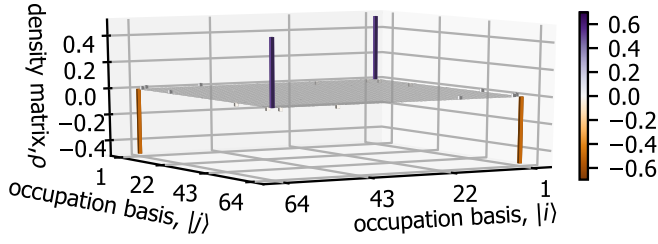


Figure 2.3: Density matrix ρ of a chosen entangled states after the projective qubit measurement. The chosen state is a eigenstate of Eq. (2.2) projected to qubit $|e\rangle$. The parameters are chosen to be: inter-cell coupling $w = 10$, intra-cell $v = 0.5$, dispersive coupling rate $\xi_1 = \xi_2 = 0.001$. The occupation basis $|1\rangle$ and $|64\rangle$ correspond to single particle excitation at the edges in both SSH1 and SSH2.

subsystem defined in Eq. (2.3). Therefore, the entanglement measured by the negativity is indeed between two subsystems, i.e., two SSH chains.

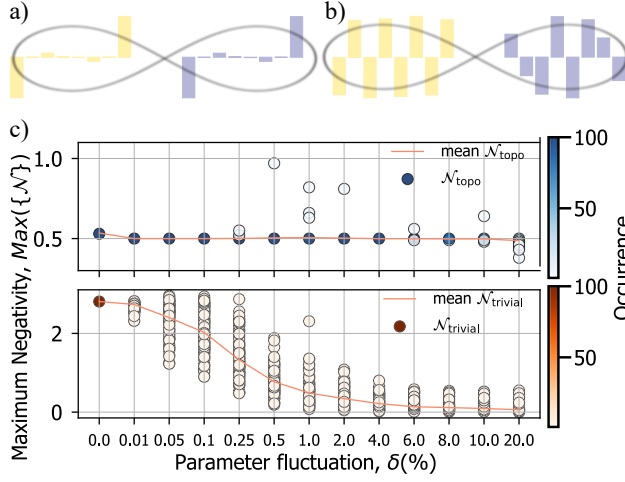


Figure 2.4: a) Schematic visualization of entanglement between two SSH chains in the topologically non-trivial regime. The state is chosen from a random Hamiltonian of parameter fluctuation $\delta = 0.1\%$ with $v = 0.1, w = 1$. b) Schematic visualization of entanglement in the trivial regime. The state is chosen from a random Hamiltonian of parameter fluctuation $\delta = 0.1\%$ with $v = 1, w = 0.1$. c) Maximal negativity $Max(\{\mathcal{N}\})$ as a function of parameter fluctuation strength δ in both topological and trivial regimes. At each parameter fluctuation strength δ , the maximal negativities for each random Hamiltonian are shown. 100 Hamiltonians are sampled at each δ . The color gradient indicates the number of occurrences per specific value of negativity. The upper panel shows the SSH chains in topological regime with $v = 0.1, w = 1$, while the lower panel is in trivial regime with $v = 1, w = 0.1$. The mean negativity over all random Hamiltonians is indicated with the orange line.

However, selecting the exact eigenstate that leads to maximal entanglement can be experimentally challenging because of the complicated spectrum under disorders. Let alone that noise can lead to decoherence of the entanglement itself. We will now analyze the selection feasibility when many nearly degenerate eigenstates are present, particularly the existence of entanglement states when disorder is presented.

In our entanglement analysis, the first step is to investigate the robustness of the entanglement against parameter fluctuation disorders δ . Now consider the case where the hopping parameters v, w have certain degrees of uncertainty which we numerically simulate by the random distributions. The Hamiltonian Eq. (2.2) is prepared such that both SSH arrays are in the topological phase with fluctuating v, w . Specifically, the Hamiltonian hopping amplitudes, given by value $v(w)$, can fluctuate uniformly within the interval $[v - \delta/2, v + \delta/2]$ ($[w - \delta/2, w + \delta/2]$). The exact diagonalization of a randomly sampled Hamiltonian yields its 128 eigenstates. For each eigenstate, we project out the qubit and evaluate the negativity $\mathcal{N}$ on the remaining two-SSH system.

Next, we identify the state with the highest negativity among all the eigenstates. In the topological regime, we find out that such highly entangled states occupy the edges

of each chain, the wave function of which is schematically shown in Fig. 2.4(a). In contrast, when a Hamiltonian is in the trivial regime, the maximally entangled state is distributed across the whole array, as shown in Fig. 2.4(b). Thus, we anticipate that the entanglement in the topological regime will remain stable against local parameter fluctuations since it shares the profile feature similar to the edge modes in the SSH model. On the other hand, the maximally entangled modes in the trivial regime is expected to be strongly susceptible to fluctuations. We quantitatively assess the anticipated robustness towards parameter fluctuations by calculating maximal negativity generated from 100 random Hamiltonians for each value of parameter fluctuation, δ . In each Hamiltonian, all 128 eigenstates are projected to $|e\rangle$ qubit state without loss of generality, and the corresponding 128 negativities, $\{\mathcal{N}\}$, are calculated. We then evaluate the maximum negativity value. While we are displaying the results of the $|e\rangle$ state projection, it is worth noting that projecting onto the $|g\rangle$ state yields similar results.

In the topological regime (upper panel of Fig. 2.4(c)) the maximal negativity $Max(\{\mathcal{N}\})$ distribution is centered around $\mathcal{N} = 0.5$. Additionally, this value remains robust with respect to increased parameter fluctuation δ . In the trivial regime (lower panel of Fig. 2.4(c)) we observe that the maximal negativity $Max(\{\mathcal{N}\})$ distribution changes rapidly with the parameter fluctuation and eventually decreases to zero. More importantly, the maximal negativity $Max(\{\mathcal{N}\})$ shows a large spread over the sampled Hamiltonians, for example, the $Max(\{\mathcal{N}\})$ distribution at $\delta = 0.25\%$ in the lower panel of Fig. 2.4(c). In the trivial regime at the fixed parameter fluctuation, the fact that the maximal negativity varies for different samples indicates that after performing the measurement, the states collapse to random states in the Hilbert space, and the structure of their correlations does not remain fixed. The resulting entanglement structure is discussed in App. 2.D. We have also observed that the maximal negativities $Max(\{\mathcal{N}\})$ in the trivial regime exceed the $\mathcal{N} = 0.5$ limit. This behavior indicates that such entanglement likely goes beyond the bipartite entanglement. We discuss this phenomenon in detail in App. 2.D.

In summary, we observe that the existence of a maximally entangled state is highly robust when the two SSH chains are in the topological regime, while the entangled states has random negativity value and are prone to disorder in the trivial regime. From now on, we consider the $\mathcal{N} = 0.5$ states in the topological regime as the target state.

2.5. TARGETING THE MAXIMUM ENTANGLEMENT

In order to properly address the target state, such as in Fig. 2.4(a), we first study the energy spectrum of the proposed system and the spectrum change as a function of hopping parameter fluctuations. We show two eigenenergy spectra of Eq. (2.2) with parameter fluctuation $\delta = 1.0\%$ and $\delta = 10.0\%$ in the topological regime ($v = 0.1, w = 1.0$) in Fig. 2.5. The eigenstates are sorted by their energies and we refer to the sorted eigenstates by their eigenstate index (x-axes in Fig. 2.5). From our previous study where we calculate the negativity $\mathcal{N}$ of each eigenstate after qubit projection, we find out that the target states with maximal entanglement $\mathcal{N} = 0.5$ exist in the near zero energy section (see triangles in Fig. 2.5). Our study also finds that the increased parameter fluctuation δ yields a broader splitting between each eigenstate in energy spectrum, see the right panels in Fig. 2.5(a) and (b). This means that a certain amount of parameter fluctuation lifts the degeneracy and makes it easier to address certain states in the frequency space.

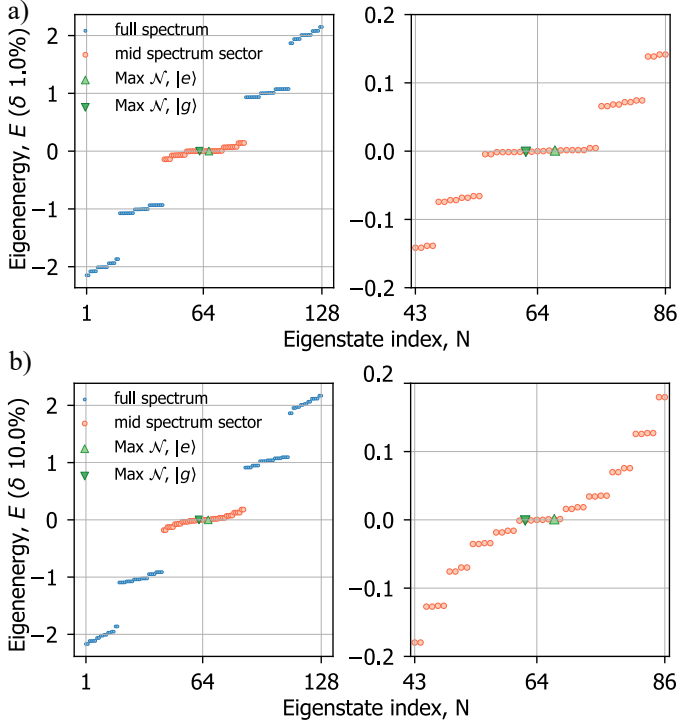


Figure 2.5: a) Eigenenergy spectrum with $\delta = 1\%$. b) Eigenenergy spectrum with $\delta = 10\%$. Both are in the topological regime ($\nu = 0.1, w = 1.0$). The two green upward and downward triangles represent the two maximally entangled states when the qubit is projected to $|e\rangle$ or $|g\rangle$, respectively. The middle near-zero-energy section in the spectrum (highlighted in orange) is enlarged in the right panel.

This fact indicates the experimental applicability of our scheme.

Our simulations show that the two states yielding maximum bipartite entanglement (green triangles) appear at the fixed eigenstate index of the non-disordered Hamiltonian. Thus we now investigate the eigenstate index stability to address the target maximum entanglement state. For instance, the non-disordered topological Hamiltonian of $\delta = 0.0\%$ shows maximal entanglement at eigenstate index 73 in the spectrum.

To keep the notation concise, we now refer to the eigenstate with index 73 as $|73\rangle$. Without loss of generality, we choose $|73\rangle$ to represent the $\mathcal{N} = 1/2$ topological entanglement state and analyze the stability of a specific eigenstate against parameter fluctuation. We trace this state starting from zero δ ($\delta = 0.0\%$) towards large δ and calculate three quantities: the negativity, the fidelity between $|73\rangle$ at disorder ($\delta > 0$) and $|73\rangle_{\delta=0}$, and the energy difference between $|73\rangle$ at disorder ($\delta > 0$) and $|73\rangle_{\delta=0}$, see Fig. 2.6. Generally, the fidelity between two quantum states, $|\psi_a\rangle$ and $|\psi_b\rangle$, is evaluated as

$$\mathcal{F}(a, b) = |\langle \psi_a | \psi_b \rangle|^2. \quad (2.5)$$

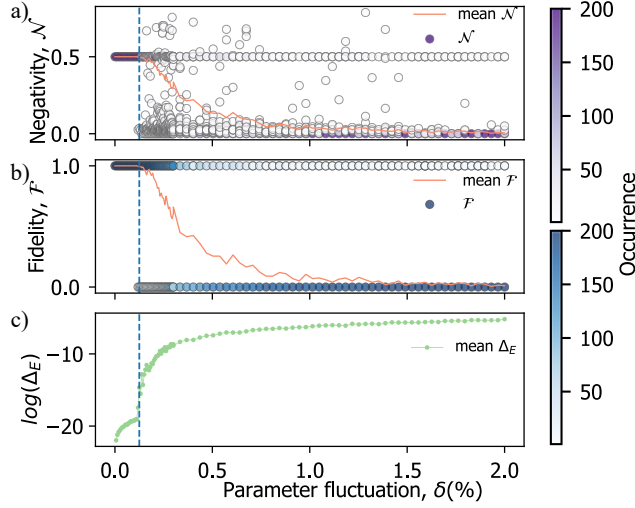


Figure 2.6: Tracking the state $|73\rangle$ across the parameter fluctuation δ . a) Negativity of state $|73\rangle$ projected to $|e\rangle$. b) Fidelity between the disordered $|73\rangle$ with respect to state $|73\rangle_{\delta=0}$. c) Logarithm energy difference between disordered $|73\rangle$ and state $|73\rangle_{\delta=0}$. Each dot represents a random Hamiltonian. At each parameter fluctuation δ , 200 random Hamiltonians are generated. The color gradient shows the occurrence of negativity/fidelity at each value, i.e., the overlapping Hamiltonians. The Hamiltonians are in the topological regime with $\nu = 0.1, w = 1$. The vertical dashed lines indicate that the system undergoes a cross-over at a parameter fluctuation around $\delta_t = 0.125\%$.

Since a eigenstate can be referred and therefore addressed by its eigenstate index, we statistically analyze the eigenstate index stability, we choose a parameter fluctuation δ grid evenly from $[0.0\%, 2.0\%]$. For each δ , we initialize 100 random Hamiltonians in the topological regime: $\nu = 0.1, w = 1.0$. As shown in Fig. 2.6, when parameter fluctuation δ grows beyond a certain threshold value (dashed vertical lines), the negativity distribution of $|73\rangle$ is no longer consistently at 0.5 and the mean value $\overline{\mathcal{N}}$ decreases. Similarly the fidelity $\mathcal{F}$ between $|73\rangle$ and $|73\rangle_{\delta=0}$ deviates from $\mathcal{F} = 1$ when passing the threshold δ . Moreover we also see a drastic change in the $\log(\Delta E)$ at the same threshold δ . Note that the exact value of dashed vertical lines ($\approx 0.125\%$) is only to show the instability of the target state eigenindex. We can conclude from these studies that although the existence of a state that leads to a negativity $\mathcal{N} = 0.5$ is validated by Fig. 2.4, such states cannot be excited by simply addressing the specific eigenstate index from the eigenspectra.

Despite the fact that the target topological entanglement state cannot be directly identified by the eigenindex, we observe that the state is generally located within a certain energy window. Moreover, when a random disordered topological Hamiltonian has an eigenstate that leads to $\mathcal{N} = 0.5$ states, there will only be one such state. When parameter fluctuation δ changes, the energy of this target state is stable in comparison to the scale of the energy window width. Thus it is possible to locate the target state within such energy windows. The details of energy window preparation can be found in

App. 2.E. In the case of $\nu = 0.1$, $w = 1.0$, the energy window has size $\Delta_E = 1.08 \times 10^{-6}$ and centers around the target energy $E_{0,|e\rangle} = 9.9 \times 10^{-4}$ or symmetrically $E_{0,|g\rangle} = -9.9 \times 10^{-4}$. The energy window parameters will change for different qubit projections as well as for different ν/w ratios. The complete parameters are given in Tab. 2.1.

Let us illustrate what these results would correspond to in the physical units in a typical superconducting experiment. Setting $w = 2\pi/50\text{MHz}$ will result in the difference between $E_{0,|e\rangle}$ and $E_{0,|g\rangle}$ of 100kHz and Δ of ca 50Hz. Assuming typical resonator frequency of 5GHz and Q-factor of 250000 we obtain a line-width of 20kHz. The mid-gap states are thus individually addressable and the fluctuation of the energy window also does not pose an issue for this addressability. Additionally, the line-width could be much smaller by increasing the Q-factor which is routinely brought up to several million in the current experiments.

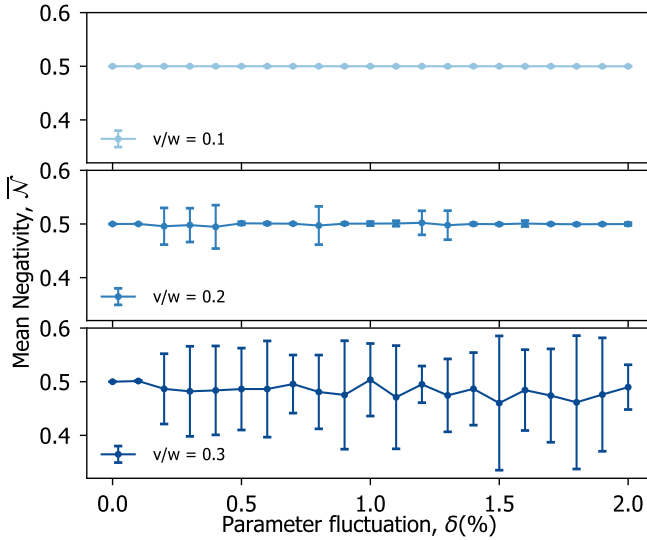


Figure 2.7: Mean negativity $\overline{\mathcal{N}}$ as a function of parameter fluctuation strength δ under different ν/w ratios. The mean negativity is calculated from all states with eigenenergy in the energy window. Each point represents the mean negativity value with the standard deviation indicated with the error bar. Each data point is calculated from 100 randomly sampled Hamiltonian under the specific δ .

When we address all the eigenstates within the prescribed energy window, Figure 2.7 shows the mean negativity $\overline{\mathcal{N}}$ and the corresponding standard deviation. If the energy window is empty, we will not enter the calculation of mean negativity. Each data point is calculated from 100 random Hamiltonians and averaged from both qubit projections. We find that the mean negativity $\overline{\mathcal{N}}$ is stable at small ν/w ratios as in the upper panel of Fig. 2.7. The negativity instability rises as the ν/w increases. The increased δ does not increase the instability. Thus, the non-trivial topology of the SSH model is critical in stabilizing the entanglement in the presence of disorders.

Besides the mean negativity $\overline{\mathcal{N}}$, we are also interested in the number of states within

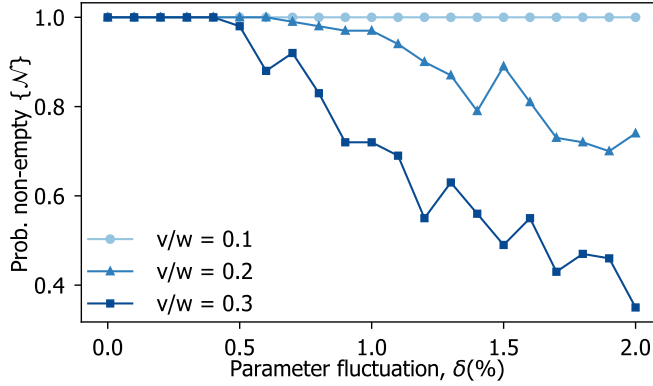


Figure 2.8: The probability of the v/w dependent energy window being non-empty in a Hamiltonian, at different parameter fluctuation δ . The v/w are chosen in consistence with Fig. 2.7. Each data point is calculated from 100 randomly sampled Hamiltonian under the specific δ .

the energy window since the negativity is ill-defined in Fig. 2.7 if the energy window contains zero state. Specifically, we study the probability of having an energy window being non-empty, i.e., the energy window addresses at least one state of arbitrary $\mathcal{N}$, at various δ and v/w . In Fig. 2.8, each data point is averaged over 100 random Hamiltonians. We find that when deep in the topological regime ($v/w = 0.1$), the prescribed energy window will always hit a specific state, the negativity of which is shown in Fig. 2.7. In contrast, we find that the number of states within the target window decreases when the system moves away from deep topological regime, as shown in the case of $v/w = 0.2$ and 0.3 . The probability of having a non-empty window also decreases with δ in the latter two cases. Overall, the topological entanglement can be systematically achieved by addressing the specific energy window for a random Hamiltonian deep in the topological regime ($v/w = 0.1$). In this setup, the parameter fluctuation will not hinder the desired topological entanglement.

2.6. CONCLUSION AND OUTLOOK

In this work, we proposed a protocol for realizing robust topological entanglement in superconducting circuits. Specifically, we theoretically construct an architecture containing SSH arrays and a single qubit. We showed that the Bell-like entanglement between topological edge modes could be achieved by projecting out the qubit. Such entanglement is robust against parameter fluctuations in the SSH resonator arrays. Additionally, we provided a detailed analysis of addressing the entanglement modes in the frequency space. We formulated a prescription on the rendered topologically stabilized entanglement as a function of parameter fluctuation. We conclude that there is always a unique way to prepare a maximally bipartite entanglement of topologically protected

edge modes by targeting a specific frequency window.

By proposing an experimentally accessible scheme for the proof of the principle of topology stabilized entanglement, we put forward the concept of classical metamaterial-inspired topological quantum devices engineering on firm quantum setups. This work can be used as a stepping stone for further topology stabilized quantum information processing and communication. The potential applications include more robust quantum communications links or robust on-chip entanglement.

The code needed to reproduce the results presented in this manuscript can be found in [19].

2.A. SSH MODEL SPECTRUM

The SSH model describes the non-interacting particle hopping on a one dimensional lattice composed of N unit cells of sublattice A and B . The lattice has staggered hopping amplitudes v and w . Depending on the ratio of the two hoppings v/w , the SSH model has a topological phase transition.

Given the SSH Hamiltonian Eq. (2.1), we analyze the spectrum of this model in topologically non-trivial case with hoppings $v = 0.5$, $w = 1.0$. In a lattice of 16 lattice sites with open boundary condition, the energy spectrum is given in panel (a) of Fig. 1.2. The two edge-states are shown in the middle of the gap. The two topological edge modes are degenerate in the periodic boundary condition, while the degeneracy is lifted in the finite systems with open boundary condition. We find that the energy difference between two mid-gap edge states increases when we move away from the deep topological non-trivial regime, i.e., small v/w values. Thus the degeneracy between two edge states is lifted when increasing v/w in finite systems. The energy difference grows exponentially with respect to the v/w ratio, as shown in the panel (b) of Fig. 1.2. This also indicates that the gap begins to close exponentially.

Although originally proposed to describe fermions on a one dimensional lattice with staggered hopping, the experimental implementations of the SSH model can go beyond the fermionic systems. The photonic analog is realized in systems where superconducting qubits are coupled to a metamaterial waveguide [24]. The phononic analog is demonstrated in mechanical metamaterials[7]. More complicated 2D models with topological phases are also implemented using metamaterials [29, 41].

2.B. EXACT DIAGONALIZATION AND NEGATIVITY

In order to numerically analyze the Hamiltonian Eq. (2.2) of the two SSH chains with a qubit, as shown in Fig. 2.2, we exact diagonalize the Hamiltonian in the single particle basis of each SSH chain with the coupled two-level qubit system. The full multi-particle Hilbert space is given by the tensor product space of two groups of SSH single particle basis and the two-level qubit system:

$$\{|b_{full}\rangle\} = \{|b_{ssh1}\rangle \otimes |b_{qubit}\rangle \otimes |b_{ssh2}\rangle\}. \quad (2.6)$$

The single-particle basis of each SSH chain (4 unit cells with 2 sublattice sites on each) is given by

$$\begin{aligned} \{|b_{sshX}\rangle\} = & \\ & \{|1, 0, 0, 0, 0, 0, 0, 0\rangle, |0, 1, 0, 0, 0, 0, 0, 0\rangle, \\ & |0, 0, 1, 0, 0, 0, 0, 0\rangle, |0, 0, 0, 1, 0, 0, 0, 0\rangle, \\ & |0, 0, 0, 0, 1, 0, 0, 0\rangle, |0, 0, 0, 0, 0, 1, 0, 0\rangle, \\ & |0, 0, 0, 0, 0, 0, 1, 0\rangle, |0, 0, 0, 0, 0, 0, 0, 1\rangle\}, \end{aligned} \quad (2.7)$$

where each digit represents a lattice site among the 8 sublattice sites. The qubit basis is given by

$$\{|b_{qubit}\rangle\} = \{|e\rangle, |g\rangle\}. \quad (2.8)$$

Therefore the Hilbert space size considered is $8 \times 8 \times 2 = 128$. Note that the single particle picture is deployed to describe the non-interacting particles in each SSH chain separately. In experimental implementation, we do not intend to restrict the entire circuit to have only single excitation.

The bipartite entanglement between two SSH chains is achieved via the qubit projection, after preparing the entire coupled system in an eigenstate. The qubit projection operators are given by

$$\begin{aligned} \hat{P}_{|e\rangle} &= \sum_{|b_{qubit}\rangle=|e\rangle} |b_{full}\rangle \langle b_{full}|, \\ \hat{P}_{|g\rangle} &= \sum_{|b_{qubit}\rangle=|g\rangle} |b_{full}\rangle \langle b_{full}|. \end{aligned} \quad (2.9)$$

Each eigenstate can be projected to one of the two qubit states. The projection of 128 eigenstates onto both qubit states respectively yields 256 two-SSH states with some states being identical to the others. The resulting Hilbert space of the two-SSH system has size 64. Here, the probability to measure the qubit state $|e\rangle$ is 50%, similarly for $|g\rangle$ the probability is also 50%.

After the qubit projection, the entanglement need to be measured by negativity. Here we elaborate the detail in the negativity calculation. Consider a general system composed of two subsystems A and B with a total density matrix ρ . The total Hilbert space is $\mathcal{H}_A \otimes \mathcal{H}_B$. The density matrix ρ is given by

$$\rho = \sum_{ijkl} p_{kl}^{ij} |i\rangle \langle j| \otimes |k\rangle \langle l|. \quad (2.10)$$

ρ^{T_B} is the partial transpose of density matrix ρ with respect to subsystem B , given by [47, 31, 16]

$$\begin{aligned} \rho^{T_B} &:= (I \otimes T)(\rho) = \sum_{ijkl} p_{kl}^{ij} |i\rangle \langle j| \otimes (|k\rangle \langle l|)^T \\ &= \sum_{ijkl} p_{kl}^{ij} |i\rangle \langle j| \otimes |l\rangle \langle k| = \sum_{ijkl} p_{lk}^{ij} |i\rangle \langle j| \otimes |k\rangle \langle l|, \end{aligned} \quad (2.11)$$

where $(I \otimes T)(\rho)$ is the identity map applied to the A party and the transposition map applied to the B party.

Negativity can be computed through the absolute sum of the negative eigenvalues of ρ^{T_B} [47, 31, 16], defined as the following,

$$\mathcal{N}(\rho) = \left| \sum_{\lambda_i < 0} \lambda_i \right| = \sum_i \frac{|\lambda_i| - \lambda_i}{2}, \quad (2.12)$$

where λ_i are all the eigenvalues of ρ^{T_B} . Negativity is a monotone and $\mathcal{N} \in [0, 0.5]$ for bipartite entanglement. Maximal entanglement states, such as Bell states, reach the value $\mathcal{N} = 0.5$. Note that the negativity is independent of which subsystem was partially transposed in Eq. 2.11 since $\rho^{T_A} = (\rho^{T_B})^T$.

2.C. ENTANGLEMENT AT DIFFERENT PARAMETER RATIO

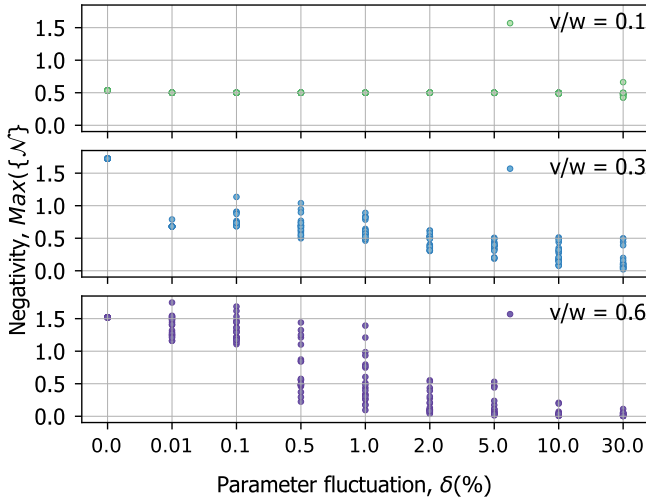


Figure 2.9: The maximal negativity $Max(\mathcal{N})$ vs parameter fluctuation δ under different v/w ratio. The three panels are in the topological regimes.

In the presence of parameter fluctuation, the topological phase transition might happen at a different v/w parameter value compared to fully ordered systems. Note the clear pattern difference between maximal negativity distributions in the topologically non-trivial and trivial regime as shown in Fig. 2.4. The broadening of the distribution and decreasing of the mean value characterize the trivial phase. From the three panels with an increasing v/w ratio, we find that although the systems are still in the topological regime, the negativity distribution patterns show a similarity towards the trivial phase. This cross-over behavior characterizes the change in the topological properties in the presence of disorders. The detail of this cross-over behavior is not the focus in this

work. We emphasize that in order to address the target entanglement between topological modes, it is advised to stay deep in the topological regime.

2.D. ENTANGLEMENT IN THE TOPOLOGICAL REGIME

Apart from the bipartite entanglement analyzed in the main text, there exist other types of entanglement in the topological regime where the negativity exceeds 0.5, e.g., in the upper panel of Fig. 2.4(c). These states are in the minority, which can be seen from the small amount of Hamiltonian population at these values. The fact of $\mathcal{N} > 0.5$ means that this is an indication that something goes beyond the bipartite entanglement. When $\mathcal{N} > 0.5$, it is only considered as an indicator of outlier entanglement, instead of the entanglement measure. We do not aim to fully understand such entanglement in this work. However, we can provide some insights into them.

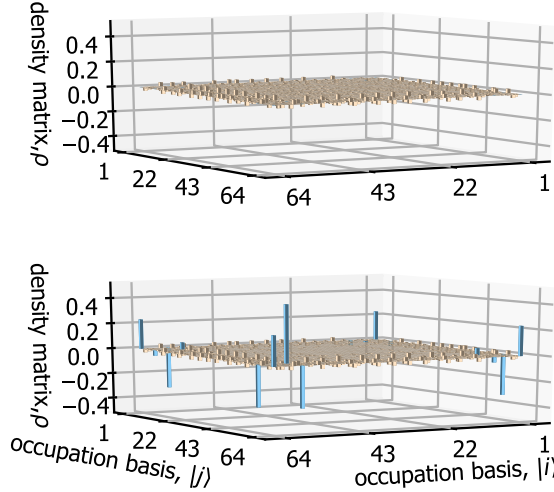


Figure 2.10: Reduced density matrix. In the upper panel, we show the case with eliminated the edge-edge component; in the lower panel, we show the case with eliminated the bulk-bulk components. The edge-edge components are highlighted in light blue color.

In such outlier entanglement states, the bulk of the SSH also contributes to the entanglement. Same as in Eq. (2.7), when the single excitation is at the edge sites, the basis is considered as the edge components, and vice versa. The basis of the two SSH systems can be written as the tensor product of the two:

$$|b\rangle = |b\rangle_{SSH1} \otimes |b\rangle_{SSH2}. \quad (2.13)$$

We can split the density matrix ρ elements into four types according to their basis components of SSH1 and SSH2: edge-edge, edge-bulk, bulk-edge, and bulk-bulk. We illustrate this component analysis by starting from a state in topological regime with $\mathcal{N} =$

0.707. The upper panel of Fig. 2.10 is the reduced density matrix with edge-edge components set to zero. It has negativity equals 0.671. Lower panel of Fig. 2.10 has bulk-bulk components set to zero with negativity = 0.658. The results show that the contributions from both the edge and bulk parts of the SSH are significant in the outlier entanglement states. These types of entanglement are complicated and beyond the scope of this work.

Table 2.1: Energy window width and center at different v/w ratios.

v/w	center, $ e\rangle$	center, $ g\rangle$	width
0.1	9.9×10^{-4}	-9.9×10^{-4}	1.08×10^{-6}
0.2	9.6×10^{-4}	-9.6×10^{-4}	4.50×10^{-6}
0.3	9.1×10^{-4}	-9.1×10^{-4}	2.87×10^{-6}

2.E. THE ADDRESSABLE ENERGY WINDOW

In order to target the $\mathcal{N} = 0.5$ topological entanglement properly in the frequency space, we prepare an energy window in the following steps:

- Step 1: Find out the target energies $E_{0,|qubit\rangle}$. This target energy varies for $|e\rangle$ and $|g\rangle$ qubit projection.
- Step 2: Find a trial energy window width Δ_E .
- Step 3: Prepare an energy window centered at $E_{0,|qubit\rangle}$ with width Δ_E for both qubit projection states.
- Step 4: Calculate $\mathcal{N}$ for states in this window at both qubit projections.
- Step 5: Note the target energy $E_{0,|qubit\rangle}$ shift with respect to v/w ratio.

In step 1, we locate the target eigenenergies, i.e., center of the energy windows. We sample 100 Hamiltonian at each δ value in the grid of parameter fluctuation. For each Hamiltonian, at each qubit projection, we filter out the state with $\mathcal{N} = 0.5$ (rounded to 3 digits) and their energies. We collect the specific one energy with minimal absolute value $E_{sample_i,abs,min}$. We average the energy $E_{sample_i,abs,min}$ over 100 Hamiltonian samples. Thus each δ grid has an energy value. We then choose the most representative energy according to the occurrence across the fluctuation grid. The most common energy is the target energy, denoted as $E_{0,|qubit\rangle}$. $E_{0,|qubit\rangle}$ represents the mean energy of states with $\mathcal{N} = 0.5$ ($\mathcal{N}$ rounded to 3 digits) at qubit state $|qubit\rangle$.

In step 2 and 3, we prepare an energy window centered at $E_{0,|qubit\rangle}$ corresponding to a practical experimental setup. The trial width is chosen to be the minimum energy difference between any two states lying on the middle spectrum sector in Fig. 2.5, i.e., the orange middle sector. The challenge here is such: if the energy window is too narrow, we might miss the target state. If the energy window is too broad, we will also excite unwanted states. The choice of energy window width is empirical and can be decided from numeric studies.

In step 4, we sample 100 Hamiltonian at each δ value in the δ grid. We calculate $\mathcal{N}$ for all states lie in the energy window. The statistics on $\mathbf{N}$ will characterize the probability of finding entangled topological states within this energy window as shown in Fig. 2.7 and Fig. 2.8.

The energy window we have used for the analysis in Fig. 2.7 and Fig. 2.8 is the Tab. 2.1. Note that the center of an energy window is relatively fixed for a specific v/w ratio; the width can be adjusted to half size or double the size without losing the high targeting probability.

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3

GATE-TUNABLE PHASE TRANSITION IN A BOSONIC SU-SCHRIEFFER-HEEGER CHAIN

Metamaterials engineered to host topological states of matter in controllable quantum systems hold promise for the advancement of quantum simulations and quantum computing technologies. In this context, the Su-Schrieffer-Heeger (SSH) model has gained prominence due to its simplicity and practical applications. Here, we present the implementation of a gate-tunable, five-unit-cell resonator-based SSH chain on a one-dimensional lattice of superconducting resonators. We achieve electrostatic control over the inductive intracell coupling using semiconductor nanowire junctions, which enables the spectroscopic observation of a transition from a trivial to a topological phase in the engineered metamaterial. In contrast to prior work, our approach offers precise and independent in situ tuning of the coupling parameters, which is critical to directly approximate ideal tight-binding Hamiltonians via the control over local variables. Our results supplement efforts towards gate-controlled superconducting electronics and large controllable resonator-based lattices to enable quantum simulations.

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3.1. INTRODUCTION

Metamaterials are engineered structures of simple constituents, which exhibit functionalities that go beyond those of the individual building blocks. For quantum systems, this transferable concept of metamaterials may be used in an efforts to engineer materials that host topological states of matter which remain robust against imperfections and are therefore suited for quantum computing and simulation [30, 60]. Recently, there has been great interest in topological metamaterials that resembles the Su-Schrieffer-Heeger (SSH) model [68, 18, 2, 4, 12]. The SSH model predicts a phase transition of a finite, 2N-site chain with alternating coupling strength between a band insulator and a topological insulator with localized edge states. The SSH model finds applications in the entanglement stabilization of quantum states [23], the long-range interaction of qubits [38, 29, 69, 80] and the study of non-hermitian light-matter interaction [16, 76, 22].

Due to the generality of the concept, topological metamaterials can be realized in various material platforms and efforts materialized in spin qubits [28], Rydberg atoms [34], adatoms [50] and integrated photonics [57]. The idea has also been taken to classical electrical circuits made out of surface mount components on printed circuit boards [36, 21, 81]. Here, we will focus on superconducting quantum circuits, which has emerged as one of the leading platforms for quantum simulation and computation [7]. Thanks to the mature fabrication and design techniques of superconducting circuits, the reliable large scale implementation of superconducting qubits and resonators is feasible which led to a number of experiments with engineered photonic baths [24, 35, 39, 55, 6, 8, 15, 45, 61, 13, 27], topological electrical circuits [33, 25], and qubit-to-topological waveguide coupling [77, 46, 12, 72, 29].

While all of the aforementioned efforts resemble static realizations of metamaterials, the engineering of topological metamaterials from superconducting circuits with tunable spectral band-gap and controllable interaction remains an open challenge due to the necessary integration of multiple tunable elements with low cross-talk. Those efforts would enable quantum simulation of novel states of matter [78, 26] or the development of new circuit components such as on-chip isolators [5] or long-range couplers. The required tunability could be provided via flux control [47] or current biased conductors [43, 70, 74, 48]. However, a challenge still persist since it has been shown that flux-crosstalk between SQUID loops significantly complicates the device tune-up [1, 11, 3]. It is also expected that long range supercurrents arising from current biased conductors may lead to crosstalk between otherwise independent sites. To alleviate the need for advanced control strategies and eliminate this type of crosstalk between mesoscopic circuit elements, one could instead leverage the local control over the microscopic properties of circuit elements. Such source of tunability became available with the local electrostatic control of the supercurrents in hybrid superconducting semiconducting structures [31, 9, 19, 65, 51] or other hybrid structures, which should enable the crosstalk-free integration of tunability. Based on this concept of gate-tunability, gate-tunable transmon qubits [31, 19], bus resonators [9] and parametric amplifiers [51, 66] have been realized in previous work.

In this work, we present a topological metamaterial with gate-tunable coupling within each unit cell. Specifically, we implement a five-unit-cell bosonic SSH chain comprised of two resonators per unit-cell forming a one-dimensional lattice of ten superconduct-

ing resonators. By taking the idea of electrostatic control from the single element level, as in a gatemon device, to scale, we achieve in-situ, electrostatic control over several inductive intra-cell coupling elements using semiconductor nanowire junctions, which enables the spectroscopic observation of a trivial-to-topological phase transition in the engineered topological metamaterial.

3.2. SSH CHAIN IMPLEMENTATION

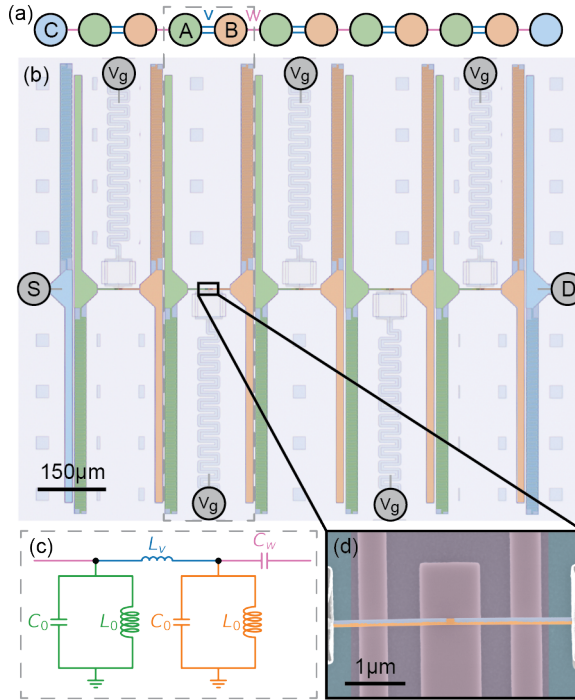


Figure 3.1: Superconducting resonator-based SSH chain implementation. (a) Illustration of the SSH tight-binding model with two-site unit cell A,B, intra-cell coupling ν and inter-cell coupling w . The finite chain is terminated to coupling sites C. The gray box highlights one unit cell. (b) False-colored microscope image of the five unit cell SSH chain composed of lumped element resonators with alternating variable inductive coupling and static capacitive coupling. The chain is measured from port S to port D in transmission. The inductive coupling, mediated by proximitized nanowire Josephson junctions, is controlled via five independence voltage gates. One unit cell is highlighted within the dashed box as in panel (a). (c) Equivalent circuit of a single unit cell in (b). The individual circuit elements are discussed in the main text. (d) False-colored micrograph images of a single gate controlled proximitized nanowire Josephson junction. Orange: InAs semiconductor, blue: Al thin film, purple: SiN gate dielectric on NbTiN gate electrodes, turquoise: Si substrate, grey: NbTiN contacts.

We implement the five unit-cell tight-binding SSH model, as shown in Fig. 3.1a, using a chain of 10 lumped-element, high-kinetic inductance, superconducting resonators with alternating variable inductive intra-cell coupling ν and static capacitive inter-cell coupling w , see Fig. 3.1b. One unit-cell, see Fig. 3.1(c), is comprised of two resonators with intra-cell coupling ν . The capacitive inter-cell coupling is set by the geometry of the neighboring resonators between unit cells. The inductive intra-cell coupling is realized by five gate-tunable Josephson junctions formed in five proximitized semiconducting nanowires, which intra-connect the two resonators of a unit cell.

The resonators are made from a high kinetic inductance film to reduce their footprints and avoid spurious modes. Within a unit cell, the resonators are spatially separated to minimize the residual capacitive intra-cell coupling and are arranged in a mirrored configuration to suppress the residual capacitive second nearest neighbor coupling and residual mutual inductances between neighbouring sites. The chain is terminated to either side by resonator-like coupling sites with an open inductor to maintain a coupling strength w on both ends to preserve the chiral symmetry. These coupling sites themselves connect directly to the input port S and the output port D, which enable transmission measurements.

The equivalent circuit model of a single unit cell is shown in Fig. 3.1(c) displaying the four relevant circuit parameters: the resonator capacitance C_0 , the resonator inductance L_0 , the capacitive inter-cell coupling C_w , and the inductive intra-cell coupling L_ν on the two sub-lattice sites A and B. The relation between ν and L_ν and between w and C_w is discussed in the Supplementary Material Sec. 3.A.2. In this approximation we neglect the residual capacitive intra-cell C_ν and the residual capacitive second nearest neighbor coupling C_{SNN} .

As mentioned, the inductive coupling are realized using nanowire Josephson junctions as shown in the zoomed in picture in Fig. 3.1(d). The gate voltages are applied via Chebyshev-filtered gate lines [see Fig. 3.1(b)] and control the Andreev bound states carrying the super-current between the superconducting leads of the nanowire junctions separately by changing the chemical potential which in turn set the respective Josephson energies E_J . Eventually, the coupling inductance is then given by the Josephson inductance $L_\nu(V_g) = \phi_0(2\pi I_c(V_g))^{-1} = \phi_0^2(E_J(V_g))^{-1}$, where I_c is the gate-tunable critical current, E_J the gate-tunable Josephson energy and $\phi_0 = h/2e$ the magnetic flux quantum. The inductance approaches infinite when the nanowire Josephson junction gated below the pinched-off voltage $V_g < V_p$, and it is finite on the order of a few nano-Henry otherwise when charge carriers are accumulated in the junction region. Physically, the junction is comprised of thin Aluminum film on two facets of the InAs nanowire which has been selectively etched to form a 110 nm long Josephson junction. The junction directly connects to resonators on either side via NbTiN contacts. The voltage gate is surrounded by two narrow ground lines to ensure equal ground potentials on the upper and lower half of the SSH chain.

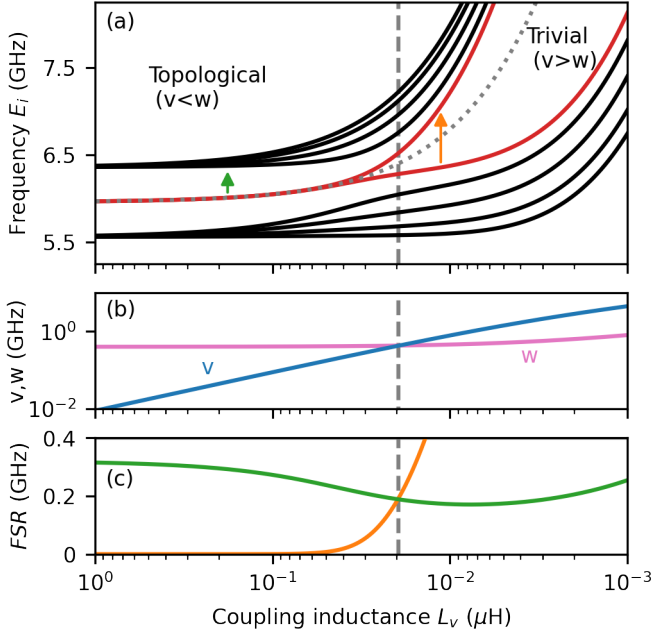


Figure 3.2: Simulated spectrum versus the coupling inductance L_v of a bosonic SSH chain with the circuit parameters used in the experiment. (a) Energy spectrum of the bulk modes (black) and the edge modes (red) versus the coupling inductance L_v . The dashed line indicates the phase transition point between the topological and the trivial phase where $v = w$. (b) Coupling coefficients v (blue), w (pink) versus coupling inductance L_v . (c) Free spectral range FSR between the edge states and the bulk (green, arrow in (a)) and between the two edge modes (orange, arrow in (a)).

Our circuit implementations shares the same tight-binding geometry as the SSH model with identical on-site potentials and alternating coupling strengths v and w , as highlighted with a corresponding color code in Fig. 3.1(a,b). To further affirm the correspondence between the SSH model and our implementation we derive the real space Hamiltonian from the circuit Lagrangian (see Supplementary Material Sec. 3.A.2). The diagonalization of the real space Hamiltonian for realistic circuit parameters yields the eigenvalues for uniformly varied coupling strength L_v , see Fig. 3.2.

As shown numerically in Fig. 3.2(a) for 10 circuit modes, in the case of large inductances, $L_v > 22 \text{ nH}$ corresponding to $v < w$, the spectrum exhibits two nearly degenerate mid-gap modes (red) centered around the eight bulk modes (black). Such mid-gap modes are the characteristic feature of a topological insulator state. Hence, we find the system to be in the topological insulator state if the nanowire junctions are pinched-off. As the coupling inductance decreases and the coupling ratio crosses over to $v > w$, see Fig. 3.2(b), the mid-gap modes split and approach the bulk modes, which eventually leads to a fully gaped spectrum corresponding to the trivial insulator state. Particular for this implementation is the frequency shift of the entire spectrum for smaller coupling

inductances as the change in L_ν also renormalizes the on-site potential, here the resonator frequency ω_r . Consequently, also w increases slightly following the increase in ω_r as seen in Fig. 3.2(b). Note that the spectrum remains symmetric around the instantaneous resonator frequency ω_r (see Supplementary Material Sec. 3.A.3), which is characteristic for a system obeying chiral symmetry [2]. To highlight the spectral evolution further, we present the free spectral range between one mid-gap mode and the nearest bulk mode as well as the separation between the two mid-gap modes in Fig. 3.2(c). The system undergoes a phase transition between the topological and the trivial state as the order of the coupling strength ν and w inverts. At the point where $\nu = w$, the free spectral ranges between the mid-gap modes and the nearest bulk modes are equal, see Fig. 3.2(c), but remains finite due to the finite system size. We refer to this intermediate regime with approximately equal FSR as a normal state.

Despite the presence of on-site and coupling strength disorder in this simulation, which represents the actual experimental circuit implementation, the spectrum is nearly indistinguishable from an ideal SSH chain, which shows the robustness of this implementation method. The only difference appears in the symmetry of the bulk modes (see Supplementary Material Sec. 3.A.2).

We experimentally resolve the energy spectrum of the 10 site SSH chain implementation via a transmission measurement at microwave frequencies through the chain from port S to port D for different gate voltages applied to nanowire junctions measured at the base temperature of a dilution refrigerator. In this measurement, eigenmodes of the SSH chain manifest as peaks in the transmission spectrum. Their linewidth is proportional to the respective wavefunction weight on the outer sites of the chain $|\psi_{0,2N}|^2$.

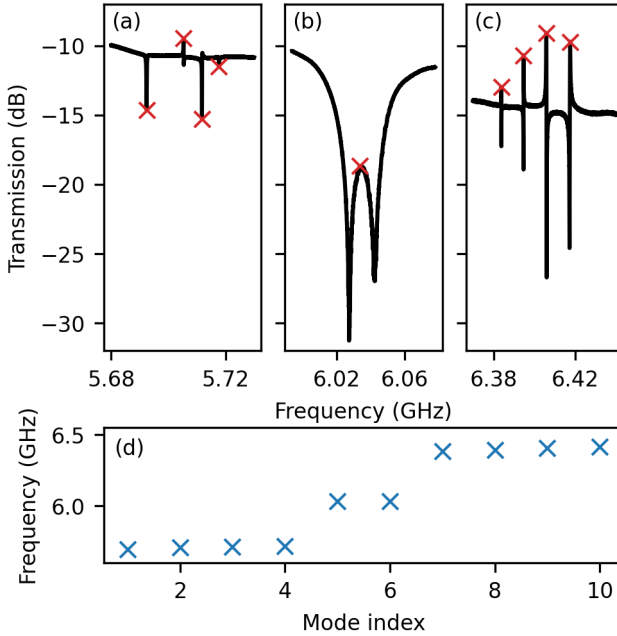


Figure 3.3: Parameter estimation of the SSH chain in the topological phase. (a-c) Transmission spectrum of the lower band, the gap and the upper band. The red crosses mark the 10 modes of the five unit cell SSH chain. (d) Eigenfrequency of the 10 modes in the topological phase sorted by mode index. The simulated eigenfrequencies based on the parameter extraction (orange) overlap with the measurement results (blue) with 1 kHz precision.

By pinching off the Josephson junctions, the intra-cell coupling ν is minimized and we measure the spectrum with a gate voltage of $V_g = -1$ V on all gates. As shown in Fig. 3.3, we observe a transmission spectrum with 9 peaks corresponding to the eight bulk modes and the two quasi-degenerate edge modes occupying the lower band, the gap, and the upper band of the spectrum. The red crosses indicates the frequency points of the individual modes. As discussed, the linewidth of a mode depends on its coupling strength to the measurement ports. Hence, modes with more wavefunction weight on lattice sites close to the edge of the chain couple more strongly to the measurement ports and appear as broader modes in the spectrum. Following this argument, we assign the narrow modes in ranges 5.68 GHz to 5.72 GHz and 6.38 GHz to 6.42 GHz as bulk modes. The wider, central peak around 6.04 GHz then corresponds to the quasi-degenerate mid-gap modes. The reduced transmission amplitude of the mid-gap modes compared to the maximal transmission results from the localization on either site of the chain. The asymmetric and distorted lineshapes of the SSH chain modes originate from Fano resonances of these relatively narrow modes with a broad spectral feature centered around 6 GHz formed by the sample mount.

The sorted eigenfrequencies versus mode index in Fig. 3.3(d) shows the gaped spec-

trum with quasi-degenerate zero energy modes, which is characteristic for the topological insulator state expected for this system. After the mode assignment, we can estimate the on-site energies and the coupling strength of each individual site along the SSH chain to obtain an estimate of the circuit parameters by fitting the full 10 dimensional Hamiltonian to the extracted eigenfrequencies (see Supplementary Material Sec. 3.A.2). The average values of the estimated circuit parameters C_0 , L_0 , C_w and L_v deviate from the design parameters by less than 10 %. Moreover, we find a parameter disorder of less than 1 % along the chain. The obtained parameters were used to generate the simulated data shown in Fig. 3.2.

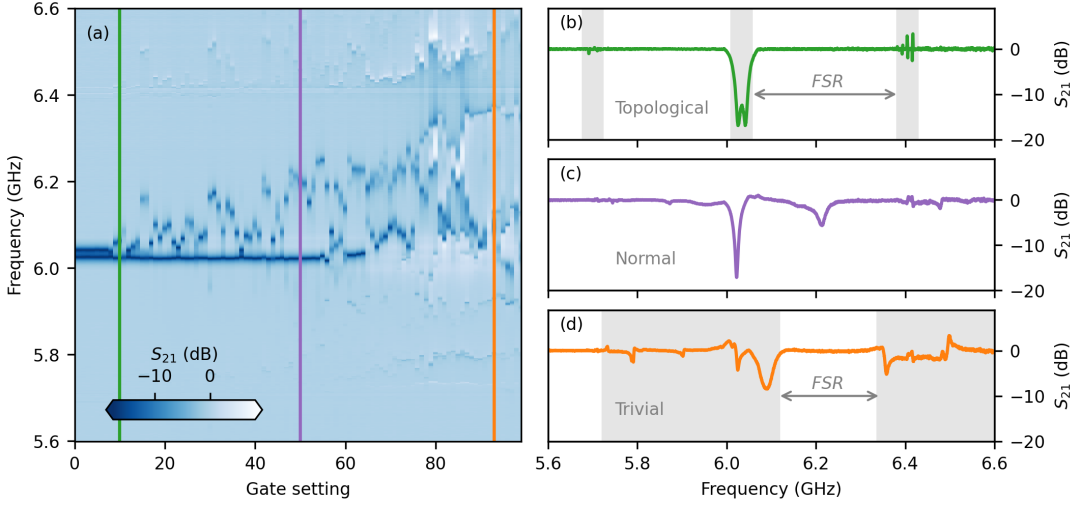


Figure 3.4: Gate-tunable phase transition. (a) SSH chain spectrum versus individual gate voltage V_{g_i} aggregated in a joint gate setting sweep (more information in the main text). The shades indicate the S_{21} transmission through the chain. (b-d) SSH chain spectrum for a given gate setting indicated as colored line in (a). The greyed out regions in (b) and (d) highlight the width of the bands. The arrows indicate the largest free spectral range in these traces.

Having established the circuit parameters in the topological regime, we now leverage the tunability per unit cell to observe the topological phase transition predicted for the SSH model. We first characterize the gate dependence of each individual nanowire junction. Specifically, we record the spectrum for a wide range of gate voltages per nanowire junction to identify the pinch-off voltage V_{p_i} that suppresses the supercurrent across each junction and the open voltage V_{o_i} that maximizes the supercurrent of the junction, see also Supplementary Material Sec. 3.C. Next, we measure the energy spectrum of the SSH chain along a synchronous evolution of the coupling inductances L_v as we linearly interpolate each junction between V_{o_i} and V_{p_i} . We assume that each nanowire junction opens at a similar rate between V_o and V_p such that the joint gate control mimics the synchronous tuning of L_v in all junctions. The resulting spectrum over 1 GHz with a

70 kHz resolution in the few photon regime is shown in Fig. 3.4(a) after background correction (see Supplementary Material Sec. 3.F). Three linecuts, colored relative to their respective gate setting in Fig. 3.4, exhibit three different regimes of the SSH chain and demonstrate that we can experimentally access each regime with in-situ tuning.

The spectral fan-out and overall increase of the eigenfrequencies of the SSH modes with increasing gate voltages, hence decreasing coupling inductances, qualitatively follows the simulation presented in Fig. 3.2(a). For the initial gate settings close to V_p shown in Fig. 3.4(b), we obtain a gaped spectrum with mid-gap modes and a free spectral range of $FSR = 350$ MHz, which we previously identified as topological insulator state. As the gate voltages increase towards V_o , the initially quasi-degenerate mid-gap modes with relatively wide linewidth and maximal prominence split and approach the lower and upper bundle of four modes. Eventually, close to V_o the lines in the spectrum form two bundles of five modes each, are narrow in linewidth and of low prominence, see Fig. 3.4(d). We identify this single gaped spectrum with a free spectral range of $FSR = 240$ MHz as trivial insulator state. Note that the two highest mode in Fig. 3.4(d) exceed the measurement window.

The phase transition of the SSH chain manifests in this experiment in the FSR and the linewidth of the modes: (i) Starting from the topological insulator state, the FSR between the mid-gap modes and the nearest bulk modes continuously closes while FSR between the initial mid-gap modes widens. Around the phase transition the system passes through an intermediate, normal regime corresponding to the trace in Fig. 3.4(c), where the FSR between the mid-gap modes and the FSR to the nearest bulk modes is approximately equal. This change in FSR is in qualitative agreement with the simulated trend shown in Fig. 3.2(c). (ii) The narrower linewidth together with the larger transmission amplitude of the initial mid-gap modes indicate that these modes transition from being localized at the edges to becoming more delocalized as the inductors open. Moreover, as it can be seen from the comparison of Fig. 3.4(b) and Fig. 3.4(d), the bulk mode resonances widen from the topological to the trivial state (see Supplementary Material Sec. 3.F).

Beyond these two indicators of the phase transition, we also numerically calculate a topological invariant for the system, see Supplementary Material Sec. 3.A.3. Specifically, we calculate the real space winding number (RSWN) [41, 63, 56, 54] and find a non-zero value in the topological regime ($\nu \approx 0$) while the RSWN approaches zero for more open junctions (increasing ν). The finite size of the system prohibits a sharp transition between the two regimes, but the RSWN calculations verify that the limiting cases of pinched off and fully open represent two topologically distinct phases.

While the nonlinear frequency shift of the system is not directly related to the phase transition and kept small by choosing a low probe power, it serves as additional tool to verify that the nanowire junctions are open. While the system in the topological regime remains nearly unchanged versus changes in the probe power, the system frequencies lower drastically with power in the trivial regime due to added Kerr nonlinearity arising from the presence of the open Josephson junctions.

A caveat, however, remains arising from the nature of nanowire Josephson junctions. As observed in earlier work on nanowire Josephson junctions [31, 42, 10, 53], the functional dependence of the inductance on the individual gate voltage is highly non-

monotonic and hysteretic over time, which prevents us from observing a smooth phase transition. The effect of disorder and disorder-induced phase transitions can further be studied, following ideas presented in Ref. [40, 49].

Conclusion. We have realized a macroscopic SSH chain out of a one-dimensional lattice of superconducting resonators, which inherits the tunability from the microscopic properties of nanowire Josephson junctions. We have characterized the system in the topological regime and mapped our bosonic implementation onto the original spin-less, single particle SSH Hamiltonian. Eventually, we leveraged the unit cell tunability and the engineered robustness against disorder to control the extended states along the SSH chain, which lead to the observation of the topological insulator phase transition in the macroscopic, gate-tunable resonator based SSH chain from the topological to the trivial insulator phase.

Our experiment takes the idea of gate-tunable superconducting circuits [73] from a single gate-tunable element to scale and demonstrates that the implementation of several nanowire Josephson junctions in circuit QED experiments is possible. This result encourages further research on gate-tunable qubits, such as $\cos(2\phi)$ gatemon qubits [32], Andreev spin qubits [53] or Kitaev chain qubits [14, 52]. Also the composition of two-dimensional lattices [37] of gate-tunable superconducting resonators seems feasible.

However, our findings also suggest that further material development of the hybrid superconductor-semiconductor stack is required to enable a smooth and reproducible gate control and exploit the full potential of the microscopic properties of nanowire junctions in superconducting circuits. In future realization of this tunable system we intend to probe the on-site wavefunction with qubits as local probes via their affected decoherence and qubit frequency arising from an AC Stark shift [75, 29, 23]. Going beyond the current implementation, one can study the nonlinear coupling [17, 44] between unit cells. Other work might focus on the long-range coupling of qubit to a chain or on implementing more complex systems like the Rice-Mele model [58] or the Kitaev model [79]. The microwave transmission spectroscopy used in this work may also find application in the characterization of topological insulators and other material systems.

ACKNOWLEDGEMENTS

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DATA AVAILABILITY

The raw data and the analysis script underlying all figures in this manuscript are available online [\[64\]](#).

SUPPLEMENTARY MATERIAL FOR “GATE-TUNABLE PHASE TRANSITION IN A BOSONIC SU-SCHRIEFFER-HEEGER CHAIN”

3.A. SSH CHAIN MODEL

Our implementation has been derived from the SSH model presented in Sec. 3.A.1. We compare the model with the ideal implementation in Sec. 3.A.2 and discuss its symmetries and the localisation of the wavefunctions in Sec. 3.A.3 and Sec. 3.A.3.

3

3.A.1. SSH HAMILTONIAN

The condensed matter system inspired tight-binding Hamiltonian of the SSH chain with on-site potential ϵ , intra-cell hopping ν and inter-cell hopping w yields a tri-diagonal real space Hamiltonian of dimension $2N$, where N is the number of unit cells in the system. The basis of this real space Hamiltonian is formed by the sites along the chain. The Hamiltonian reads

$$\begin{aligned}
 H_{CM} &= \sum_{n=1}^{2N} \epsilon |n\rangle \langle n| \\
 &+ \sum_n^N [\nu |2n-1\rangle \langle 2n| \\
 &+ w |2n\rangle \langle 2n+1| + h.c.] \\
 &= \begin{pmatrix} \epsilon & \nu & & & & \\ \nu & \epsilon & w & & & \\ & w & \epsilon & & & \\ & & & \ddots & & \\ & & & & \epsilon & w \\ & & & & w & \epsilon & \nu \\ & & & & & \nu & \epsilon \end{pmatrix} \quad (3.1)
 \end{aligned}$$

The spectral properties of this system and its eigenstates are displayed in Fig. 3.A.1. The key feature is the symmetrically gaped spectrum spanned by the bulk modes (black) with mid-gap modes in the topological insulator phase for $\nu < w$. The appearance of the symmetric spectrum is linked to the chiral or sublattice symmetry of the system, which can be tested by computing the anti-commutator

$$\{\Gamma, H\} = \Gamma H + H \Gamma = 0 \quad (3.2)$$

with

$$\Gamma = \mathbf{I}_5 \otimes \sigma_z \quad (3.3)$$

where the chiral symmetry operator Γ is given by the outer product of N dimensional identity matrix $\mathbf{I}_5$ and the Pauli-z matrix σ_z . For this tri-diagonal real space Hamiltonian described above the anti-commutator is indeed zero. The spectral gap never closes due to the finite size of the system, in contrast to the infinite limit in which the spectral gap is given by $|\nu - w|$. Instead, the transition point at $\nu = w$ is signaled by the equal free spectral range between the mid-gap modes and one of the mid-gap modes and the

first bulk mode. Beyond the transition, the system enters a trivial phase where $v > w$ eventually leading to a full dimerization. The corresponding eigenstates of the mid-gap modes are localized around the edges of the chain, as exemplified for a specific configuration in Fig. 3.A.1d, hence the name edge mode. An exponential fit to the wavefunction supported on either sublattice A or B yields the localization length ξ in Fig. 3.A.1b. As long as the edge modes are quasi-degenerate, the localization length extracted from the fit follows the theoretical limit $\log(w/v)^{-1}$, but the fit does not capture the divergence of the localization length well. For a finite system length, the inverse participation ratio appears to be a better measure for the localization of the wave function.

3.A.2. LUMPED ELEMENT, SUPERCONDUCTING CIRCUIT HAMILTONIAN

The design of the lumped element, superconducting circuit implementation of the SSH chain follows a simple guiding principle. We realize every site as lumped element resonator with frequency ω and the coupling terms as capacitors C_w or inductors L_v . We then derive the circuit Hamiltonian from a classical circuit Lagrangian formalism as part of a standard circuit quantization [71]. The resulting Hamiltonian H_{LC} indeed takes the form of the tight-binding Hamiltonian

$$\begin{aligned}
 H_{LC} &= \sum_{n=1}^{2N} \hbar\omega |n\rangle \langle n| \\
 &+ \hbar\omega \sum_n^N \left[\frac{L_T}{2L_v} |2n-1\rangle \langle 2n| \right. \\
 &+ \left. \frac{C_w}{2C_T} |2n\rangle \langle 2n+1| + h.c. \right] \\
 &= \hbar\omega \begin{pmatrix} 1 & \frac{L_T}{2L_v} & & & & \\ \frac{L_T}{2L_v} & 1 & \frac{C_w}{2C_T} & & & \\ & \frac{C_w}{2C_T} & 1 & & & \\ & & & \ddots & & \\ & & & & 1 & \frac{C_w}{2C_T} \\ & & & & \frac{C_w}{2C_T} & 1 & \frac{L_T}{2L_v} \\ & & & & & \frac{L_T}{2L_v} & 1 \end{pmatrix} \quad (3.4)
 \end{aligned}$$

Based on the correspondence between the two systems, we can compare the coefficients in the Hamiltonian and formalize the mapping between the on-site energy ϵ and the resonator frequency ω , between the coupling strength v and the Josephson inductance L_v , and between the coupling strength w and the capacitive coupling C_w

$$\begin{aligned}
 \epsilon &= \hbar\omega \\
 v &= \frac{\hbar\omega}{2} \frac{L_T}{L_v} \\
 w &= \frac{\hbar\omega}{2} \frac{C_w}{C_T}
 \end{aligned} \quad (3.5)$$

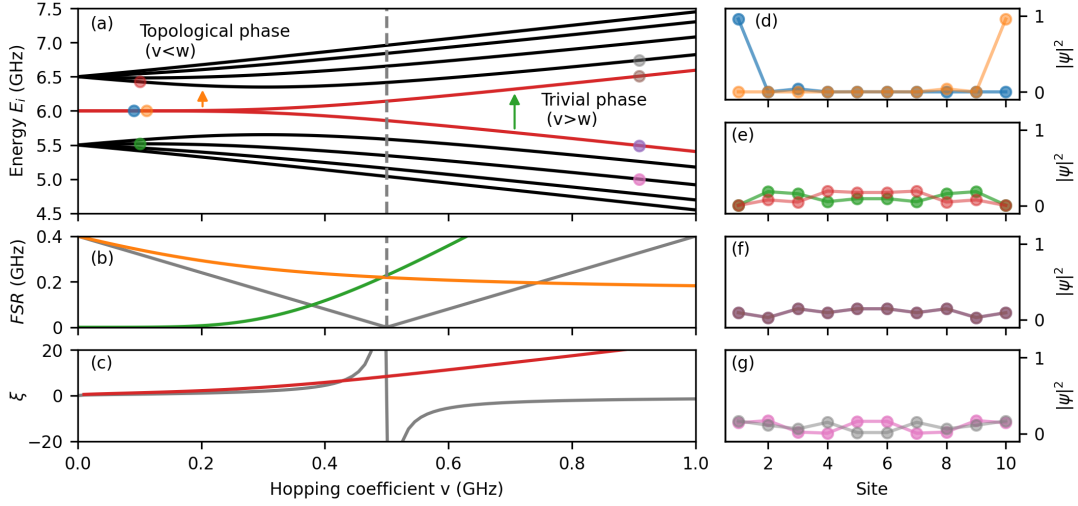


Figure 3.A.1: Simulated spectrum and states of a tight-binding SSH model with on-site potential $\epsilon = 6.5$ GHz and hopping coefficient $w = 0.5$ GHz. (a) Energy spectrum of the bulk modes (black) and the edge modes (red) versus the coupling strength v . (b) Free spectral range Δ between the edge states and the bulk (blue, arrow in (a)) and between the two edge modes (orange, arrow in (a)) versus hopping strength v . The dashed line indicates the phase transition point between the topological and the trivial phase where $v = w$. The spectral gap $|v - w|$ (grey) only closes in the infinite length system (c) Localization length ξ extracted as exponential decay fit to the wave functions of the edge modes. The grey line shows the theoretical limit $\log(w/v)^{-1}$. (d) Energy spectrum of the bulk modes (black) and the edge modes (red) versus the coupling strength v . (d-g) Real space wave function $|\psi|^2$ on the resonator lattice for the topological and the trivial phase of the bulk and the edge states. The colors correspond to the markers in (a).

It should be noted that $\omega = \omega(L_v)$ which leads to a small increase in w , but a significant shift in the on-site potential ϵ causing the upwards shift of the spectrum as L_v decreases. For further visual comparison we also show the spectral properties and the eigenstates of the ideal SSH chain implementation in Fig. 3.A.1.

3.A.3. TOPOLOGICAL PROPERTIES OF SSH IMPLEMENTATION

REAL SPACE WINDING NUMBER

Topological invariants characterize the global properties of topological insulators. In case of the SSH model, the winding number is used as the topological invariant, which takes a phase-dependent binary value: 0 in the trivial insulator phase and 1 in the topological insulator phase [59, 2]. In the finite-size SSH chain, the periodicity is absent, thus making momentum not a good quantum number and the winding number becomes an ill-defined invariant. Alternatively, the co-variant real space winding number (RSWN) serves as a good approximation for the momentum space winding number given the

large enough system size, and the RSWN remains quantized in the presence of disorder [41, 63, 56, 54].

We will discuss the RSWN extensively in later chapters. For now we only introduce the formula to calculate the RSWN, given by

$$\nu = -\frac{1}{L} \text{Tr} \{P_B Q P_A [X, P_A Q P_B]\}. \quad (3.6)$$

In our study, we compute the RSWN versus the coupling strength ν/w for various chain lengths N and for $w = 0.5$, see Fig. 3.A.2 (a). On either end of the coupling strength range ($\nu \in (0, 1)$), the system approaches the topological and trivial state with $\text{RSWN}=1$ and $\text{RSWN}=0$, respectively, regardless of the chain length. In-between $\nu \in (0, 1)$, the RSWN indicates a cross-over between the two different insulator states, but only in the thermodynamic limit $N \rightarrow \infty$ the RSWN correctly predicts the phase transition point as it approximates the k-space winding number. For small N , the RSWN does not capture the phase transition, which is spectroscopically still defined, due to finite size effects. Additionally, we investigate the robustness of the topological state by examining the RSWN under varying levels of disorder in the coupling parameter ν and w for three different intra-cell couplings $\nu = \{0.0, 0.2, 1\}$, see Fig. 3.A.2 (b-d). While the RSWN does not aim at benchmarking the phase transition for our finite size system, we see that in the limiting cases we have distinct topological phases which providing valuable insights into the topological phase transition that we observe experimentally.

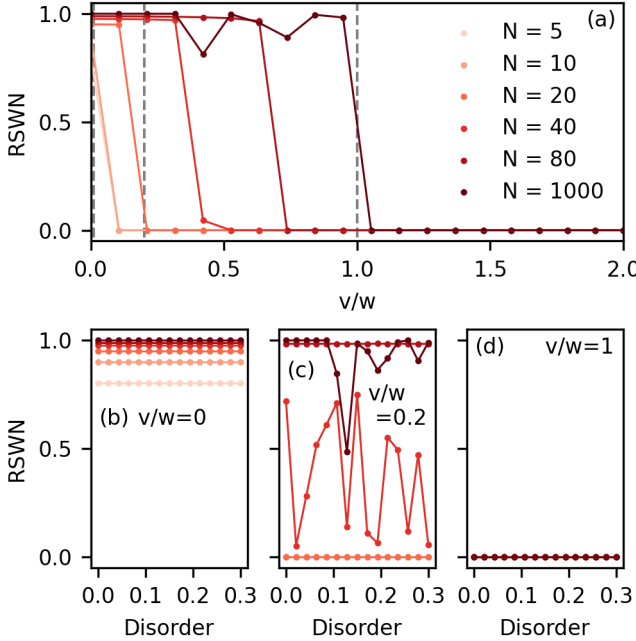


Figure 3.A.2: Real space winding number. (a) RSWN versus coupling strength ratio. The dashed vertical lines correspond to the ν/w ratio used in the disorder simulations shown in (b-d). (b-d) RSWN versus disorder strength for three different coupling strength.

SYMMETRIES

To further strengthen the claim that the chiral symmetry persists in the ideal SSH chain implementation, we present the spectrum of Fig. 3.A.1c normalized by the instantaneous resonator frequency $\omega(L_V)$ in Fig. 3.A.3(a). This normalized spectrum is symmetric around the on-site energy. Moreover, we can compute the anti-commutator and find that it vanishes $\{\Gamma, H_{LC}\} = 0$.

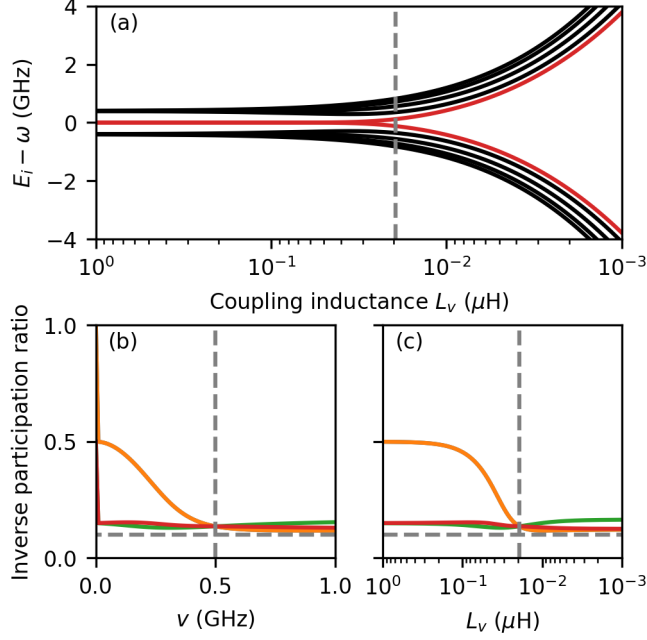


Figure 3.A.3: (a) Normalized simulated spectrum of the bosonic SSH chain with ideal the on-site and coupling energies as in Fig. 3.A.1. (b,c) Inverse participation ratio for different states for the tight-binding model (b) and the ideal SSH chain implementation (c) versus the coupling strength ν . The dashed line indicates the $1/2N$ limit. The colors correspond to the respective wave functions in Fig. 3.A.1.

LOCALIZATION

The localization of the wavefunctions of the edge states is a characteristic of the SSH chain. For finite chain lengths however, the localization length defined as exponential decay $\exp(-x/\xi)$ does not capture the delocalization of the edge modes as the system undergoes the phase transition, as seen in Fig. 3.A.1(b). Instead, for finite chain length we can define an inverse participation ratio (IPR) [62], which measures the support of a wave function on a specific site and is defined as

$$IPR = \frac{\sum_x |\psi(x)|^4}{(\sum_x |\psi(x)|^2)^2} \quad (3.7)$$

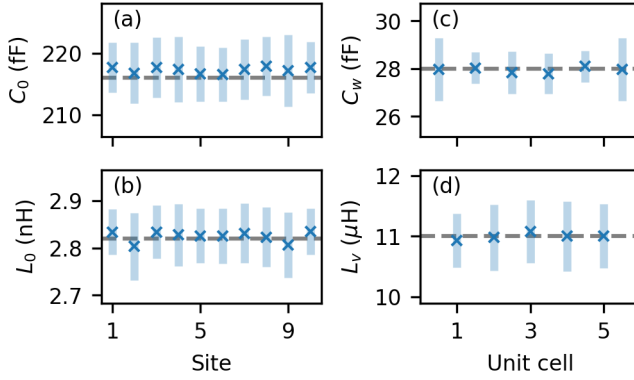


Figure 3.B.1: Extracted circuit parameters in the topological phase. (a,b) Resonator capacitance C_0 and inductance L_0 per site. (c,d) Coupling capacitance C_w and coupling inductance L_v per coupling site.

We observe that the IPR of the edge modes is finite and larger than for the bulk modes in the topological phase. As the system undergoes the phase transition, the IPR decreases and approaches the delocalization limit $1/2N = 1/10$ of the bulk modes at $\nu = w$, see Fig. 3.A.3b,c. The IPR of the localized modes is of value one only in the fully dimerized case $\nu/w = 0$, hence we observe a kink in the IPR in Fig. 3.A.3b close to $\nu = 0$.

3.B. CIRCUIT PARAMETER ESTIMATION

From the presence of the bulk and gap modes in the spectrum for a given L_v configuration, we can extract the circuit parameters C_0 , L_0 , C_w and L_v . First, we identify and list the eigenmodes of the system. Then, we fit the 10 dimensional Hamiltonian H_{LC} to the list of eigenmodes while allowing for free variation of all circuit parameters along the chain. For completeness, we also take the two C_w couplings to the coupling sites C into account, which expands the C_w list from four to six elements. Since the eigenfrequencies of the coupling sites are far detuned from the SSH spectrum, we do not consider them in the system Hamiltonian. The circuit parameters obtained from this optimization routine in the topological phase are shown in Fig. 3.B.1. We assume the design parameters as start parameters for the Nelder–Mead optimization. The variation of the obtained parameters is smaller than 1%. Finally, we input the obtained circuit parameters into the full Hamiltonian to compute the spectrum and the eigenstates to simulate the data in Fig. 3.2 and Fig. 3.3d. The agreement of this optimization with the measurement results is on the order of a few kHz, which is also the spectral resolution of the measurement. This parameter estimation yields the static parameters C_0 , L_0 , and C_w . The optimization routine can then be repeated for different L_v configurations to the trivial phase.

While any on-site disorder and second-nearest neighbor interaction due to residual capacitive coupling breaks the chiral symmetry, strictly speaking, the mid-gap modes remain localized and can therefore be understood as chiral for practical purposes [62].

3.C. SINGLE GATE DEPENDENCE

A transmission measurement through the five unit cell SSH chain allows for a site specific tune-up of every tunable coupling element. We run single gate scans for every nanowire Josephson junction over a wide gate voltage range, while we keep all other gates at the pinch-off voltage V_p , see Fig. 3.C.1(a-e). Hence, all scans begin deep in the topological regime. For the bulk gates NW_i with $i \in [2, 3, 4]$ we zoom on the low band in the topological state, while we focus on the gap for the edge gates NW_i with $i \in [1, 5]$, as we expect the biggest change in frequency in those spectral ranges.

We notice the non-monotonic single gate dependence of the modes in the respective test spectrum due to microscopic properties, most likely spurious quantum dots in the 100 nm long junction. In fact, we observe discontinuities due to jumps in the microscopic properties of the junctions. Repetitions of these scans reveal that the slopes change over time. Hence, even these single gate dependencies cannot be used to calibrate for a joint gate scan across the phase transition point with identical nanowire inductances along the SSH chain.

The measured single gate dependencies are in qualitative agreement with the simulated spectra on the right of Fig. 3.C.1. From the single gate dependencies we can extract the voltage V_o , which corresponds to the maximal supercurrent, hence minimal L_J or, expressed in Josephson energy, maximal E_J . This voltage point seem robust enough and constant over time and over several gate scans.

3.D. ADDITIONAL JOINT GATE DEPENDENCE

In Fig. 3.D.1, we repeat the measurement shown in Fig. 3.4 over the gate settings V_p and V_o defined as minimal and maximal value of the single gate dependencies shown in Fig. 3.C.1 at -30 dBm signal power. The splitting of the modes with increasing gate voltage, thus higher gate setting index, is in qualitative agreement with the equivalent scan from an earlier cooldown presented in Fig. 3.4, which uses similar gate settings $V_p = [400, 150, 430, 400, 1900]$ mV and $V_o = [1800, 1800, 1800, 1800, 4000]$ mV for the nanowire gates 1 to 5.

3.E. POWER-INDUCED PHASE TRANSITION

Fig. 3.E.1 shows the mid-gap spectrum versus signal power at the signal generator output for three gate voltage settings corresponding to the three gate settings [50, 100, 150] in Fig. 3.D.1, (a) close to the topological regime, (b) close to the normal regime and (c) deep in the trivial regime. The dark lines on the light background indicate the mid-gap modes. We observe that the two modes are quasi-degenerate at high signal powers and the spectrum is nearly indistinguishable from the topological regime regardless of the exact gate setting. As we reduce the signal power, the two modes split and approach the low power spectrum shown in Fig. 3.D.1. In the low power regime, the spectral evolution exhibits a gate dependence. For a gate set-point deep in the topological regime, the modes do not disperse with signal power (not shown here). We understand this power-induced phase transition as a result of the power-dependent change in the kinetic inductance of the

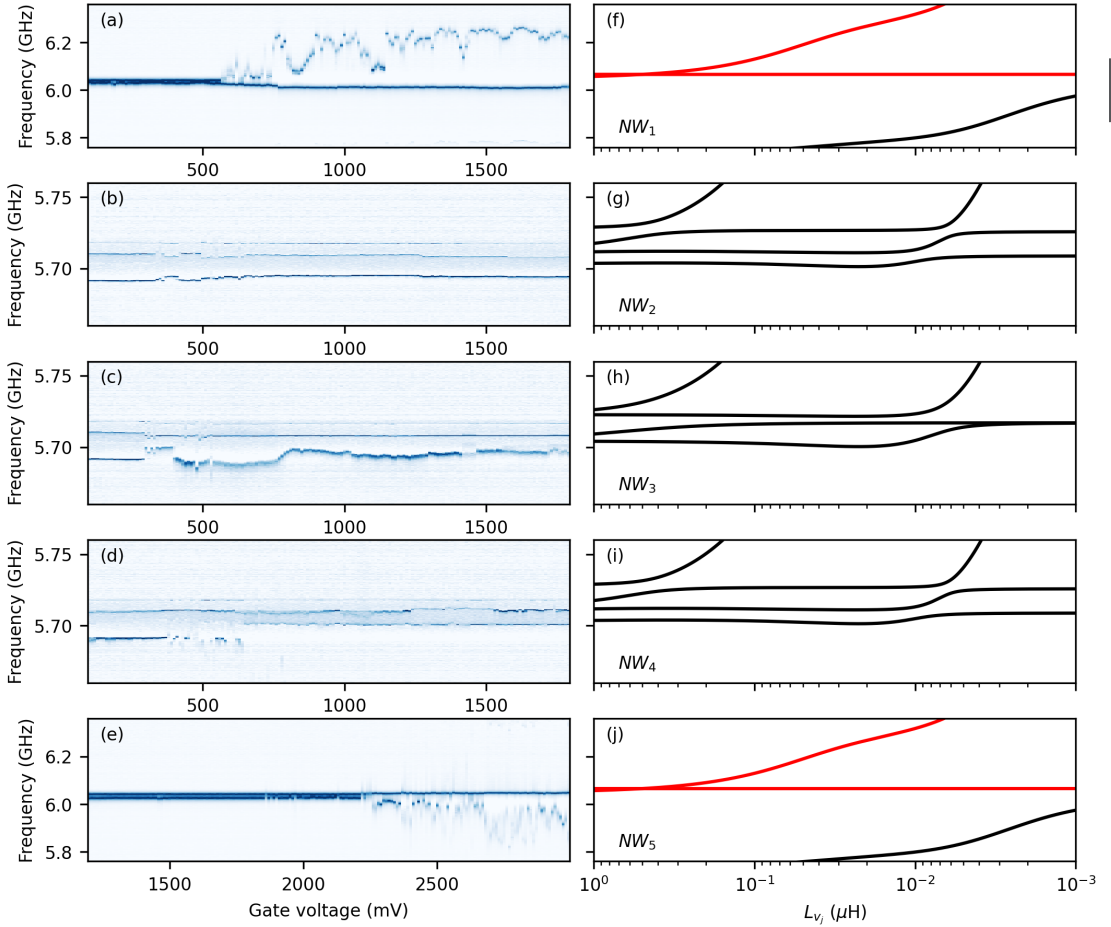


Figure 3.C.1: Single gate dependence. (a-e) Measured spectrum per gate while the other gates are set to V_p . Zoom-in on low band for bulk gates and zoom-in on gap for edge gates. The data is normalized on the background to enhance the visibility of the change. (f-j) Corresponding simulated spectrum.

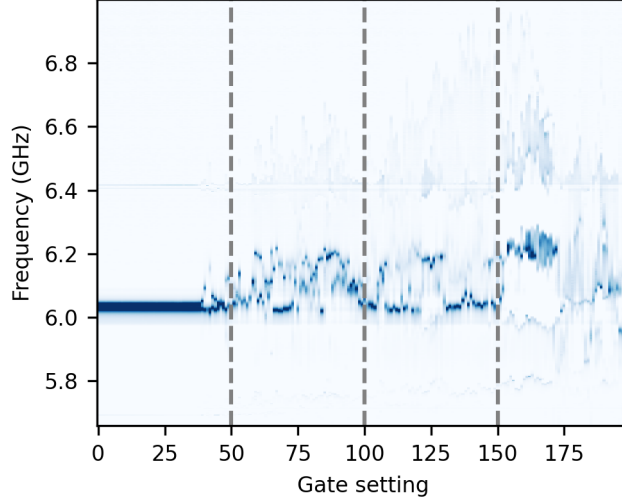


Figure 3.D.1: Additional joint gate dependence with V_p and V_o taken as minimal and maximal value of the single gate dependencies shown in Fig. 3.C.1. The dashed lines indicate the gate settings used for the power dependence shown in Fig. 3.E.1

proximitized nanowires. The inductance in these nanowires follows the equation

$$L_{NW} = L_0(V_g) \left[1 + \left(\frac{I_s}{I_*(V_g)} \right)^2 \right] \quad (3.8)$$

where $L_0(V_g)$ is the gate-dependent low-power inductance, I_s is the signal current through the chain, and $I_*(V_g)$ is the gate-dependent critical current. The signal power enters the equation via the signal current to which it is proportional $P_s \propto I_s^2$. Hence, an increasing signal power increases the inductance, which in turn transitions the SSH chain into the topological regime. The non-monotonic power dependence as well as the different transition points most likely arise from the different critical currents $I_*(V_g)$ per nanowire. Consequently, in order to recover the SSH spectrum versus gate voltage, the measurements must be acquired with a signal power below -30 dBm.

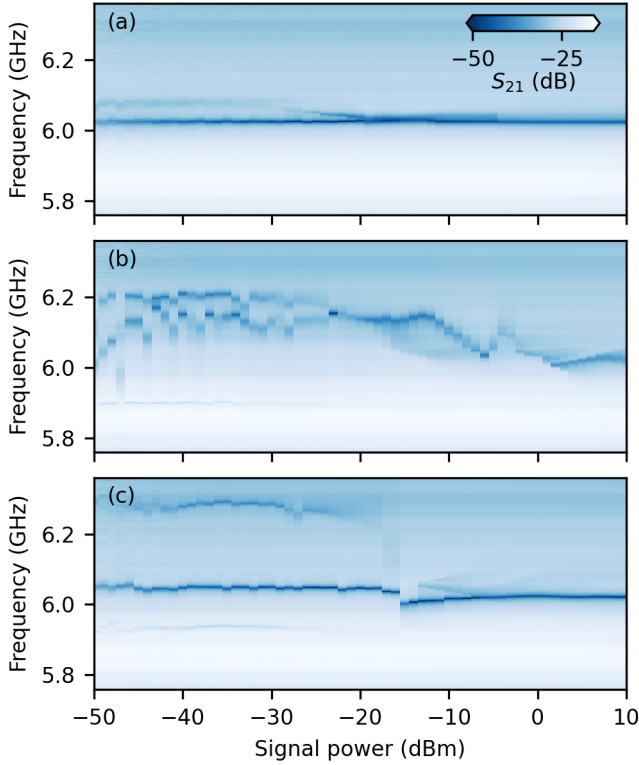


Figure 3.E.1: Power-induced phase transition. (a-c) Mid-gap spectrum versus signal power at the signal generator output for the three gate voltage settings [50, 100, 150] in Fig. 3.4.

3.F. SSH CHAIN ENVIRONMENT

We measure the transmission spectrum of the SSH chain in the topological state over the accessible frequency range (4 – 8 GHz), see Fig. 3.E.1. We observe a peaked background transmission with a finer modulation and with a maximal transmission around 6 GHz. We attribute the resonance-like peak at around 6 GHz to a spurious feature hosted by the PCB enclosure spanning between the two test ports S and D, which acts as a band pass filter in parallel to the SSH chain, which in turn affects the overall transmission. We also observe that the transmission spectrum recorded in several experimental runs and on different samples, but within the same enclosure differs from the expected transmission spectrum. We highlight the resemblance of the transmission spectrum with a broad band resonator with a fit to a Lorentzian (orange). The finer modulation of the transmission probably arises from impedance mismatches along the lines connecting to the measurement electronics. To simplify the data analysis in the presence of the broad spectral feature, we obtain a linearly interpolated background transmission spectrum to which we normalize the measurement data to better identify the transmission peaks and

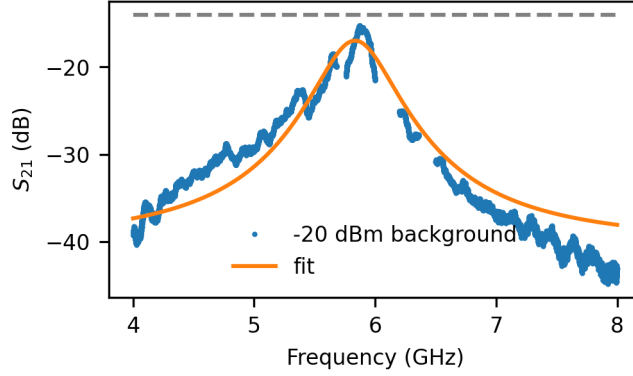


Figure 3.F1: Box mode. S_{21} transmission spectrum over the accessible frequency range 4 – 8GHz with SSH spectrum removed (blue). A Lorentzian fit highlights the presence of a box mode (orange). The dashed grey line indicates the expected transmission given the input attenuation and output amplification.

dips of the SSH chain spectrum.

While we cannot extract the exact effective inductance and capacitance leading to the spurious feature formed by the enclosure, we can approximate its effect in a lumped element simulation in which we assume a resonator in parallel to the SSH chain. The results of these lumped element simulations are presented in Fig. 3.E2(a) for the ideal case and in Fig. 3.E2(b) for the case in presence of a spurious resonance. In the ideal case, we expect two, or three well defined bands with in total 10 modes and otherwise suppressed background transmission. However, a spurious mode yields a higher overall transmission due its band pass feature and distorts the measured line shape of the modes due to its capacitive contribution on the rising edge and its inductive contribution on the falling edge. Consequently, the bulk modes appear as dips and peaks and the mid gap modes in the topological state appear as deep dip with a shallow peak in the center. The feature, which we modelled for simplicity as a single mode, does not affect the SSH chain, but it does complicate the interpretation of the overall transmission spectrum. To overcome the effect, a new sample enclosure should be carefully designed [20]. Going beyond the current experiment, one could also intentionally coupled a SSH chain to a resonator as described in Ref. [13].

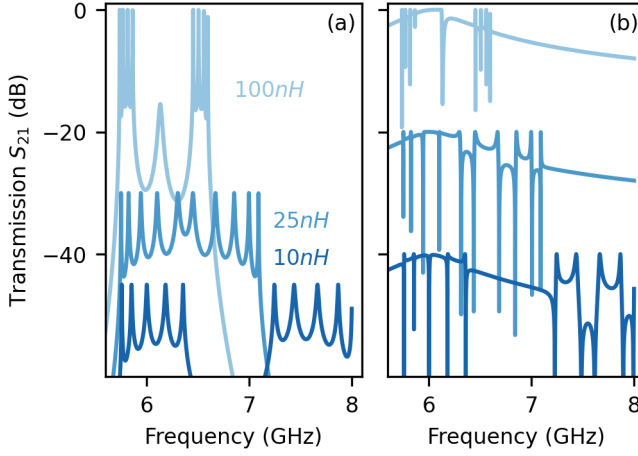


Figure 3.F.2: Lumped element simulation of the three regimes topological, normal, trivial for different tuning scenarios. (a) Expected, ideal SSH chain spectrum for a joint synchronous sweep of L_v . (b) Same device tuning as in (a), but accounting for the presence of a box mode modelled as transmission resonator in parallel to the SSH chain. Note that all spectra per panel are offset for better visibility.

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4

TOPOLOGICAL FINITE SIZE EFFECT IN ONE-DIMENSIONAL CHIRAL SYMMETRIC SYSTEMS

Topological phases of matter have been widely studied for their robustness against impurities and disorder. The broad applicability of topological materials relies on the reliable transition from idealized, mathematically perfect models to finite, real-world implementations. In this paper, we explore the effects of finite size and disorders on topological properties. We propose a new criterion for characterizing finite topological systems based on the bulk conductivity of topological edge modes. We analyze the behavior of bulk conductivity and real space topological invariants both analytically and numerically for the family of SSH models. We show that our approach offers practical insights for topology determination in contemporary intermediate scale experimental applications.

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4.1. INTRODUCTION

The discovery of topological phases of matter has revolutionized contemporary condensed matter and solid state physics [50, 22, 4, 54, 1, 25, 53, 43, 38, 14, 41, 37]. Significant attention has been given to the deep connection between physics and mathematics, particularly in the construction of topological invariants of different phases [2, 40, 5]. While topological invariants have been originally formulated in the momentum space [40, 5, 2, 16], there has been much interest in real space representations of these invariants, such that the topology can be readily identified and leveraged in finite-sized and disordered system setting [33, 45, 36, 30, 32, 55]. Recent experimental progress in realizing topological modes highlights the need for easily applicable criteria to determine whether the observed model retains the desired topological properties or if these are compromised by system size or noise [24, 46, 11, 26, 20, 10, 27, 31, 56, 3].

The robust topological protection of edge states makes topological materials essential for quantum information and quantum transport applications [18, 27, 13, 12, 47, 49]. It is crucial to analyze the model's topology under finite size and to keep track of edge states contributions. In one-dimensional chiral models, the edge states are localized at different edges in large systems, while the bulk remains insulating. However, in smaller systems, edge states can hybridize due to the significant overlap between them through the bulk, resulting in the formation of new symmetric and antisymmetric edge states. This phenomenon is referred to as the finite size effect [9, 34, 51, 17, 21, 29]. In large systems, the effect is suppressed by exponentially small overlap of states [2].

In order to assign the topological invariants to finite systems properly, the real space representation of winding number (RSWN) was introduced [33, 39, 36, 30, 55]. This formulation arises from substituting momentum derivatives in the Berry connection with a commutator involving the coordinate operator in real space. However, it is important to note that the existing RSWN expressions do not fully account for the potential overlap between edge states that may occur due to the finite size of the system. As a result, there are instances where this formula yields anomalous winding number values. One possible way to resolve this issue was suggested in Ref. [44] involving the truncation of edge sites, effectively mitigating the contributions from edge states.

In this work, we present a new approach to identify the real space topology. We investigate the contribution of finite size effect to real-space topological invariants and topological phase transition. Our analysis reveals that relying solely on the real space topological invariant is insufficient, as it may lead to misjudgments caused by finite size effects. In order to solve this issue, we develop criteria to characterize the real space topological phase transition through the bulk conductivity of the mid-gap edge states. Specifically we utilize real-space invariant in conjunction with the bulk conductivity of topological mid-gap edge states to characterize the topology. We demonstrate our criteria in two types of one dimensional chiral symmetric systems: the Su-Schrieffer-Heeger (SSH) model [48, 16, 2] and the extended SSH model [35]. Our criteria provide a robust framework for determining the topology relevant to real-world experimental applications, such as bosonic SSH model realized via Rydberg atoms [11], the in-situ gate-tunable SSH phase transition in superconducting resonators [46], synthetic Rydberg atom arrays [24], semiconductor quantum dots [26], superconducting metamaterials [20, 31, 56], hybrid qubit-photon systems [27, 52].

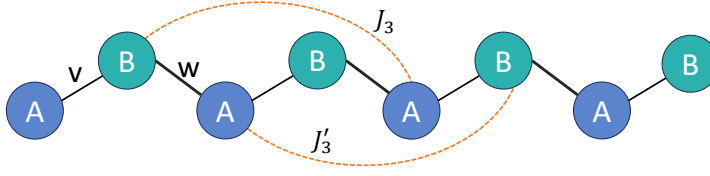


Figure 4.2.1: Lattice geometry of SSH model and extended SSH model. The SSH model contains only the nearest-neighbor hoppings, represented by v and w as solid lines. The extended SSH model contains the nearest-neighbor hopping and the third order hopping, represented by J_3 and J'_3 in dashed orange lines.

The paper is organized as follows: in Sec. 4.2 we first review one dimensional chiral models for which we will analyze size-dependent phase diagrams given by the RSWN formula. Furthermore, we establish the criteria of bulk conductivity as a measure of the insulating/conducting state of the topological edge modes. In Sec. 4.3 we present the background and definition of the RSWN. In Sec. 4.4 we numerically illustrate an efficacy of bulk conductivity for the determination of the topological phases. In Sec. 4.5 we provide analytic insights into how edge states influence the phase diagram from a continuum model perspective. Finally, in Sec. 4.6 we investigate the impact of disorder on the derived phase diagrams, examining how topological protection defined by RSWN is affected by various disorder. The conclusions are summarized in Sec. 4.7.

4

4.2. CHIRAL SYMMETRIC MODELS IN 1D

4.2.1. SU–SCHRIEFFER–HEEGER (SSH) MODEL

Inspired by the modeling of polyacetylene, a long chain of repeating -CH- molecular units with alternating single and double bonds between carbon atoms, the Su–Schrieffer–Heeger (SSH) model [48, 16, 2] has gained significant attention due to its profound connection between topology and physics.

The Hamiltonian of the SSH model is given by,

$$H = \sum_{n=1}^N (v c_{n,A}^\dagger c_{n,B} + w c_{n,B}^\dagger c_{n+1,A}) + h.c., \quad (4.1)$$

where $c_{n,\alpha}^\dagger$ ($c_{n,\alpha}$) denotes the creation (annihilation) of a particle at lattice site (n, α) , with the unit cell index $n \in [1, N]$ and the sublattice index $\alpha \in A, B$. The ratio of intra-cell to inter-cell hopping, v/w , controls the topological phase transition. The SSH model hosts two topologically distinct phases: winding number is $\nu = 1$ when $v < w$, which is called the topologically non-trivial (topological phase), while for $v > w$, $\nu = 0$ which is called the topologically trivial (trivial phase). The SSH model is classified as class AIII for spinful fermions or BDI for spinless bosons [23]. In the topologically non-trivial phase, two mid-gap edge states arise, which are exponentially localized at the edge lattices, as shown in Fig. 4.2.2.

The momentum-space Hamiltonian $H(k)$ of a two band model can be expressed in terms of Pauli matrices $\hat{\sigma}$ as $H(k) = d_0(k)\sigma_0 + \mathbf{d}(k)\hat{\sigma}$. The coefficients constitute the

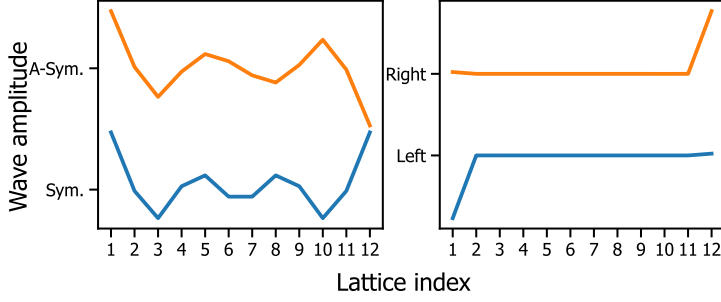


Figure 4.2.2: Example of hybridized SSH edge states (Left) with $v/w = 0.5, N = 6$ and localized edge states (right) with $v/w = 0.0005, N = 6$. The x-axis label of odd (even) number represents type A (B) sub-lattices. A-Sym.(Sym.) represents the anti-symmetric (symmetric) hybridization of two edges.

4

$\mathbf{d}$ -vector, often referred to as the Bloch vector. The SSH Hamiltonian corresponds to $d_x(k) = v + w\cos(k); d_y(k) = -w\sin(k); d_z(k) = d_0(k) = 0$.

4.2.2. EXTENDED SSH MODEL

The chiral symmetry permits more types of hopping, provided the hoppings connect different sublattices (for example A-B and B-A). These types of hoppings are called odd hoppings. The most straight forward generalization of the SSH model is the extended SSH model with the third order hoppings (next next neighbor hopping). The Hamiltonian of the extended SSH model is given by,

$$H = \sum_{n=1}^N (v c_{n,A}^\dagger c_{n,B} + w c_{n,B}^\dagger c_{n+1,A} + J_3 c_{n,B}^\dagger c_{n+2,A} + J'_3 c_{n,A}^\dagger c_{n+1,B}) + h.c. \quad (4.2)$$

In addition to the SSH model, Eq. (4.1), J_3 and J'_3 are the third order hoppings, represented by orange dashed line in Fig. 4.2.1. When both first and third-neighbor hoppings are considered, the system can be configured to preserve time-reversal, particle-hole and chiral symmetry, thus it belongs to the BDI class, just as the standard SSH model. The corresponding topological phases are characterized by winding number $|\nu| = 0, 1, 2$ in this model.

In models with long-range odd hoppings, the symmetries of the standard SSH model are maintained. However, the extended SSH models allow for larger values of the topological invariant. For a given hopping order α , the maximum possible winding number is $\nu_{\max} = (\alpha + 1)/2$. The winding number also indicates the maximum number of edge mode pairs that can exist in the system, reflecting the expanded topological behavior of models with higher-order hopping terms [35].

Switching to the momentum space, the $\mathbf{d}$ -vector of the extended SSH model with

third order hopping is given by [35],

$$\begin{aligned} d_x(k) &= v + (w + J'_3) \cos(k) + J_3 \cos(2k), \\ d_y(k) &= (w - J'_3) \sin(k) + J_3 \sin(2k), \quad d_z(k) = 0. \end{aligned} \quad (4.3)$$

We will calculate the k-space winding number later with this expression.

The extended SSH model can support up to two pairs of edge states, corresponding to the highest winding number $\nu = 2$. These edge states are characterized by their wave function amplitudes, which are localized at the left or right edges of the system and on either the type-*A* or type-*B* sublattices. The exact position of the maximum amplitude depends on the tuning parameters v, w, J_3 , and J'_3 . Consequently, the edge modes can be predominantly localized at the first, third, or fifth site of the edge of the chain [35].

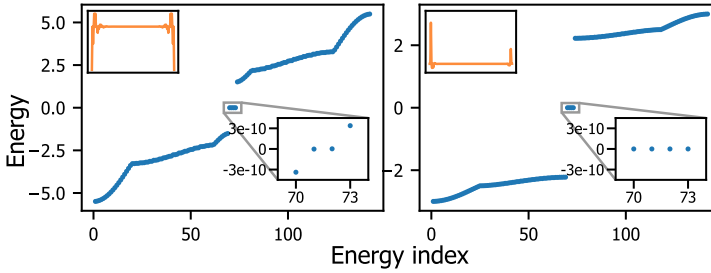


Figure 4.2.3: Energy spectrum of extended SSH models. The system size is $N = 71$. Left: $\nu = 0.5, w = 1, j_3 = 3, j_{3p} = 1, N = 71$. Right: $\nu = 0.25, w = 0.25, j_3 = 2.5, j_{3p} = 0.0, N = 71$. Both panels are in the $\nu = 2$ phase. The inset of each panel shows a zoom in of the zero energy region. Each insets shows four nearly degenerate mid-gap modes as a manifestation of $\nu = 2$ phase. The corresponding mid-gap state wavefunctions are plotted in Fig. 4.2.4 respectively.

4.2.3. BULK CONDUCTIVITY OF TOPOLOGICAL EDGE MODES

The successful application of topological materials relies on the bulk conductivity of topological edge modes in transport experiments [46]. We introduced the term "bulk conductivity" to describe the scenario when the two localized edge states have non-negligible overlaps in the bulk such that the hybridized mid-gap states become conducting. Note that usually bulk transport refers to transport caused by bulk states. Here, by conductivity we refer to transport induced by the hybridization in the bulk. However, in theory the topological edge states do not inherently guarantee conductivity, for example in Fig. 4.2.2, the right panel shows the localized edge states, with negligible overlap in the bulk - this type of edge modes does not have a conducting bulk and therefore the transport measurement will yield an insulator. In Fig. 4.2.3 we show the energy spectrum of two extended SSH Hamiltonians of different parameters. Both models are in the topologically non-trivial regime with winding number $\nu = 2$ and a set of four mid-gap edge states. The topological gap in the bulk insulating type of extended SSH (right panel) is larger compared to the bulk conducting type of extended SSH (left panel). The degeneracy is lifted in the

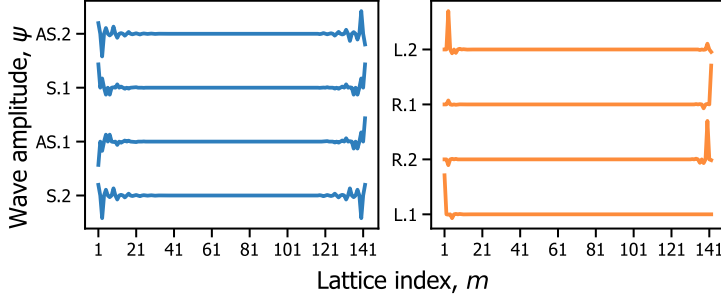


Figure 4.2.4: Examples of edge states in Extended SSH model. $N = 71$. Left panel: hybridized edge states derived from $\nu = 0.5, w = 1, j_3 = 3, j_{3p} = 1$. Right panel: localized edge states derived from $\nu = w = 0.25, j_3 = 2.5, j_{3p} = 0$. Lattice index m represents sublattices. In y-axis label, AS (S) represents anti-symmetric (symmetric) edge states, while L (R) represents the localized edge states on left (right) edges. Similarly, 1 (2) represents the location of the peak on first (second) edge lattice unit cells. E.g., AS.1 represents the first anti-symmetric mode (peak on the edge unit cell). S.2 represents the second symmetric edge state (peak on the second unit cell from the edge).

bulk conducting type (left panel). In Fig. 4.2.4 we show the exact wave function profile of the set of four edge states in Fig. 4.2.3 respectively. In the following we discuss the difference between edge localization/hybridization and bulk conductance/insulation.

The insulating edge states usually appear in larger systems or in systems deep in the topological phase. In our numerical analysis, we establish a criterion by putting a threshold for the averaged amplitude across the central two unit cells, defined as follows:

$$(|\Psi_{N/2,A}| + |\Psi_{N/2,B}| + |\Psi_{N/2+1,A}| + |\Psi_{N/2+1,b}|)/4 < \delta. \quad (4.4)$$

Edge states that meet (or fail to meet) this criterion are classified as bulk insulators (or conductors), respectively. The threshold value $\delta = 10^{-15}$ is set according to the numerical precision of the python package NumPy and its data type float64 [15], given by `np.finfo(float).resolution = 1e-15`. This threshold relates to the concept of machine precision, which defines the smallest distinguishable difference from 0 in floating-point arithmetic. This means that differences smaller than δ may be affected by rounding errors. It is useful to note this concept differs from the smallest representable number, `np.finfo(float).tiny = 2.2250738585072014e-308`, which is much lower. The manual threshold δ in Eq. 4.4 ensures our classification remains unaffected by numerical artifacts.

In the standard SSH model, we observe that edge states satisfying bulk conductivity condition, Eq. (4.4), are consistently localized on a single edge, while edge states that do not satisfy Eq. (4.4) exhibit hybridization across both edges. We show both types of SSH edge states in Fig. 4.2.2. Conversely, in the extended SSH model, we find that the localization and hybridization of pairs of edge states, e.g. Fig. 4.2.4, show no straightforward dependence on whether the criteria in Eq. (4.4) are met or violated. Therefore, we expect

that, rather than edge hybridization, the bulk conductivity condition Eq. (4.4) is closely related to the finite size topology. We will elaborate on this distinction between models and validate our assumption in the following sections.

4.3. COVARIANT REPRESENTATION OF WINDING NUMBER

The covariant real space representation of winding number was first proposed in Ref. [33]. The resulting formula has been corroborated in several subsequent studies [45, 39, 36]. In particular, this real space formulation can be useful for disordered systems that respect the underlying symmetry. The real space topological invariant defined in Eq. (4.16) is a global quantity, and importantly, it retains its quantization even in the presence of strong disorder [33, 45, 36].

While the calculation of the RSWN is often presented in a complex form, we offer a streamlined and pedagogical approach to make this calculation more accessible. For a generic chiral symmetric system, we begin by recalling the anti-commutation relation between the Hamiltonian H and the chiral operator, which satisfies $\Gamma \{ \Gamma, H \} = 0$. As a consequence, the Bloch Hamiltonian $H(k)$ can be written in an off-diagonal form when expressed in the basis of the chiral symmetry operator, where the operator assumes the form of σ_z in the case of SSH model [42].

$$H(k) = \begin{pmatrix} 0 & h(k) \\ h(k)^\dagger & 0 \end{pmatrix},$$

where $h(k) = d_x(k) - i d_y(k)$.

In parallel, the original formulation of the k-space winding number is given by [42]

$$\nu = \frac{1}{2\pi i} \int_{BZ} \text{Tr}\{[h(k)^{-1} \partial_k h(k)]\} \in \mathbb{Z}. \quad (4.5)$$

The expression in Eq. (4.5) has a direct covariant form in real space [33]. Starting from the real space SSH Hamiltonian H , the homotopically equivalent flatband version of H is given by

$$Q = \frac{H}{|H|} = \begin{pmatrix} 0 & Q_0 \\ Q_0^\dagger & 0 \end{pmatrix}, \quad (4.6)$$

where the off-diagonal matrix Q_0 will enter the real space winding number calculation. The covariant real-space form of the winding number can be obtained by applying the Bloch-Floquet transformation of Eq. (4.5) written in terms of flatband Hamiltonian [33]

$$\nu = -\text{Tr}_{\text{volume}}\{Q_0^{-1}[X, Q_0]\}, \quad (4.7)$$

where X is the position operator and $\text{Tr}_{\text{volume}}$ denotes the trace per volume. In the context of SSH model, position eigenstates $|x, c\rangle$ are labeled by the unit cell index $x = 1, \dots, N$ and the sublattice degree of freedom $c = A, B$. The position operator X therefore takes the form

$$X = \sum_{x \in \mathbb{Z}} \sum_{c=A,B} x |x, c\rangle \langle x, c|. \quad (4.8)$$

The primary task is obtaining the off-diagonal term Q_0 . Note that the chiral symmetry for lattice Hamiltonians is also referred to as sublattice symmetry. It can be defined

by grouping all internal states into two sets, the sublattices A and B . In order to obtain Q_0 , we begin by defining the spectral projectors of chiral symmetry as follows:

$$\begin{aligned} P_A &= \sum_{x \in \mathbb{Z}} \sum_{c=A} |x, c\rangle \langle x, c|, \\ P_B &= \sum_{x \in \mathbb{Z}} \sum_{c=B} |x, c\rangle \langle x, c| = \hat{I} - P_A. \end{aligned} \quad (4.9)$$

The chiral operator Γ is given by

$$\Gamma = P_A - P_B. \quad (4.10)$$

Chiral symmetry anti-commutes with the Hamiltonian, meaning the Hamiltonian has no matrix elements between states on the same type of sublattice. Γ acts on each unit cell separately and it satisfies the unitary condition $\Gamma^\dagger = \Gamma^{-1} = \Gamma$. An immediate consequence of chiral symmetry is that the eigenstates come in pairs $\{|n\rangle, \Gamma|n\rangle\}$, with energies $\{-E_n, E_n\}$. To complement this framework, we introduce another set of projectors that partition the energy spectrum into upper and lower halves, given by,

$$\begin{aligned} P_- &= \sum_{E_n \leq 0} |n\rangle \langle n|, \\ P_+ &= \sum_{E_n \geq 0} |n\rangle \langle n| = \Gamma P_- \Gamma. \end{aligned} \quad (4.11)$$

Recall the definition of the flatband Hamiltonian Q in Eq. (4.6) and note that in the eigenstates basis it can be expressed as

$$Q = \frac{H}{|H|} = \frac{\sum_n E_n |n\rangle \langle n|}{\sum_n E_n} = \sum_n |n\rangle \langle n|. \quad (4.12)$$

Therefore, the flatband Hamiltonian in Eq. (4.6) can be written as

$$Q = P_+ - P_-. \quad (4.13)$$

We assume the half-filling condition in the construction of Q . Note that Q is constructed from the spectral flattening of the occupied and unoccupied states, which relies on a well-defined Fermi level within a band gap. As a chiral-symmetric operator, Q satisfies the condition

$$Q = P_A Q P_B + P_B Q P_A, \quad (4.14)$$

the off-diagonal terms in Eq. (4.6) can be written in terms of the above defined operators as

$$Q_0 = P_A Q P_B, \quad (Q_0)^{-1} = P_B Q P_A. \quad (4.15)$$

Finally, one can express real space winding number Eq. (4.7) as

$$\nu = -\frac{1}{N} \text{Tr} \{P_B Q P_A [X, P_A Q P_B]\}. \quad (4.16)$$

This real space representation reduces to the conventional k -space form in the presence of translational invariance, and it remains valid and quantized even in the presence of disorders [33, 45]. In the subsequent discussion, we will refer to Eq. (4.16) as the

real-space winding number (RSWN). This formula is known to distinguish topological phases in chiral symmetric system like SSH model[48] and the Kitaev chain[28].

The RSWN (4.16) serves as a global real space topological invariant. There also exist well-defined local topological invariants. The topological winding marker[32]/ Chern marker[6, 8] are such invariants and they can locally distinguish topological phases in 1D/2D. The 1D winding marker has a very similar constitution to RSWN, given by [32]

$$w(x) = \frac{-1}{V_{uc}} \sum_{\alpha} \langle x, \alpha | Q_0^{-1} [X, Q_0] | x, \alpha \rangle, \quad (4.17)$$

where x are positions, α are sublattice indices and V_{uc} is the unit cell volume.

4.4. RSWN AND BULK CONDUCTIVITY IN FINITE SIZE SYSTEMS

4.4.1. RSWN IN SSH MODEL

In many cases, the SSH Hamiltonian itself already reveals most information we need. However, this is not the case when one thinks about the bulk conductivity of the edge states. One needs to diagonalize the Hamiltonian and check the mig-dap modes in order to obtain that bulk conductivity information. We find interesting relationship between the finite size topology and the bulk conductivity, which we elaborate in the following.

When calculating the RSWN ν from Eq. (4.16) for finite-size SSH chains, the results reveal a deviation from the expected topological phase transitions. Figure 4.4.1 shows the RSWN Eq. (4.16) obtained from SSH chain of size $N \in \{5, 1000\}$. Interestingly, the RSWN transition occurs at a smaller ν/w , rather than at $\nu/w = 1$. For $N = 5$, RSWN $\nu = 0$ in most cases where it is expected to be $\nu = 1$. However, for $N = 1000$, the transition happens at $\nu/w = 1$, as predicted by the winding number formulated in k-space. This example illustrates that in finite size systems, the RSWN, as the counterpart of k-space winding number in real space, becomes unreliable in the topologically non-trivial regime.

In order to investigate the anomalous values in RSWN, we analyze the bulk transport properties of two topological edge modes, defined in Eq. (4.4). We first observe that the bulk conductivity of the two edge modes from the same Hamiltonian remains identical, meaning they either conduct or insulate under the same parameter set. Additionally, we find that the transition in bulk transport aligns with the transition in RSWN, as shown in Fig. 4.4.2, with both transitions exhibiting a clear dependence on system size. This phenomenon will be further analyzed in the following sections. Notably, several methods have been proposed to address the anomalous RSWN values. For example, Song, F., et al. [44] considered taking the trace only over the bulk. Similarly, winding marker Eq. (4.17) incorporates even more localized information.

4.4.2. RSWN IN THE EXTENDED SSH MODEL

In the previous section, we demonstrated that the RSWN becomes unreliable in the topologically non-trivial regime for the SSH model in finite size systems. A natural question arises: how does this formula perform in systems with multiple topological phases, such as those with higher winding numbers? In order to answer this question, we apply the same RSWN formula Eq. (4.16) to the extended SSH model, which includes third

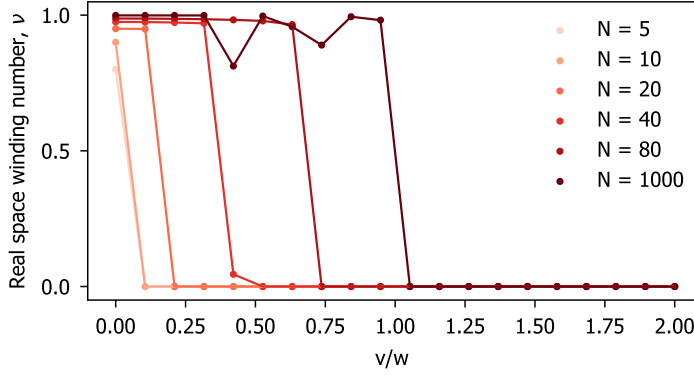


Figure 4.4.1: Real space winding number(RSWN) ν calculated for different system sizes $N \in \{5, 1000\}$. Each dot represents the RSWN ν of an SSH Hamiltonian rendered at corresponding ν/w value on the x-axis. The Hamiltonian parameters are set to be $w = 0.5$ and $\nu \in \{0, 1.0\}$.

order hopping, allowing the winding number to reach $|\nu| = 2$. The corresponding winding number is calculated and shown in Fig. 4.4.3. We set the ν, w parameters to be fixed at $\nu = 1/2, w = 1$ and J_3 and J_3' are varied in the range of $[0, 4]$. For small system size, such as $N = 48$, the RSWN prediction can be completely inaccurate. It still restores the k-space prediction as the large system sizes grows to $N = 512$. We notice that there is a $\nu = 0$ region even in the $N = 512$ case that is absent in k-space plot. This indicates that higher ordered topological phases are more susceptible to finite size effect.

Again, we examine the bulk transport properties of sets of mid gap topological edge states in the extended SSH mode, the result is shown in Fig. 4.4.3 and Fig. 4.4.4. The key message from these two figures is that, although the RSWN is affected by finite system size, it still reflects the bulk transport behavior. In Fig. 4.4.4, the purple region characterizes the conducting bulk, which perfectly overlaps with the RSWN $\nu = 0$ region in Fig. 4.4.3. Likewise, the yellow region in Fig. 4.4.4, characterizing the bulk insulating phase, fully overlaps with the non-zero RSWN region in Fig. 4.4.3.

In conclusion, our numerical study strongly indicates that the bulk conductance serves as an experimentally accessible indicator of the RSWN. In the following section, we will support our reasoning by an analytical analysis.

4.5. ANALYTICS OF RSWN PHASE DIAGRAMS

In this Section we present analytic explanation of phase diagrams obtained above. The calculations are based on perturbative estimation of overlap between edge states.

We start from SSH model, where all calculations could be performed exactly. At phase transition point $\nu = w$ the gap closing for translation-invariant model happens at the points $k = \pm\pi$ of the first Brillouin zone. Decomposing the tight-binding Hamiltonian

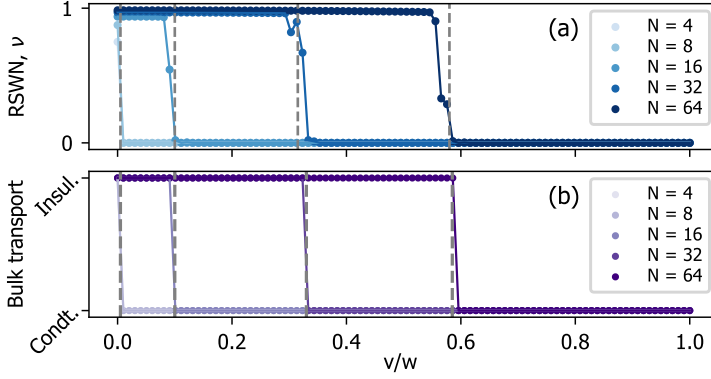


Figure 4.4.2: (a) Real space winding number(RSWN) ν calculated for small system sizes. Each dot represents the RSWN ν of an SSH Hamiltonian rendered at corresponding ν/w value on the x -axis with $\nu/w \in \{0, 1.0\}$. The topological transition is marked by a dashed vertical line in different system size N . (b) Bulk transport properties of the SSH topological edge states. y -axis shows the two states of the edge mode: Conduction (Condt.) and insulation (Insul.). The criterion of transition from conduction to insulation is the overlapping of wave function amplitude at the system bulk. Our definition in Eq. (4.4) is: the averaged probability amplitude of middle two unit cells $< 10^{-15}$. Python package ‘numpy’ and ‘numpy.linalg.eig’ are used.

up to linear order in q for $k = \pi + q$, we obtain:

$$H_{lin,\pi}(k) = \begin{pmatrix} 0 & v - (1 + iq)w \\ v - (1 - iq)w & 0 \end{pmatrix}. \quad (4.18)$$

This Hamiltonian describes effective low-energy model, which predicts linear dispersion of quasi-particles at phase transitions $\nu = w$. Away from phase transition, the dispersion $\pm\sqrt{(v-w)^2 + w^2 q^2}$ has a gap of size 2Δ with $\Delta = |\nu - w|$. To find the eigenstates in real space system, we transform to coordinate space representation by substituting $q \rightarrow -i\partial_x$. Then, the Schroedinger equation $H\Psi = E\Psi$ becomes a system of two coupled differential equations for $\Psi = (a(x), b(x))$:

$$\begin{cases} (v - w)b(x) + w\dot{b}(x) = Ea(x), \\ (v - w)a(x) - w\dot{a}(x) = Eb(x). \end{cases} \quad (4.19)$$

The general solution of this system of differential equations for $E \neq 0$ reads

$$\begin{aligned} a(x) &= C_1 \sin \left[\sqrt{-(v/w - 1)^2 + E^2/w^2} x \right] \\ &\quad + C_2 \cos \left[\sqrt{-(v/w - 1)^2 + E^2/w^2} x \right], \\ b(x) &= \frac{(v - w)a(x) - w\dot{a}(x)}{E}. \end{aligned} \quad (4.20)$$

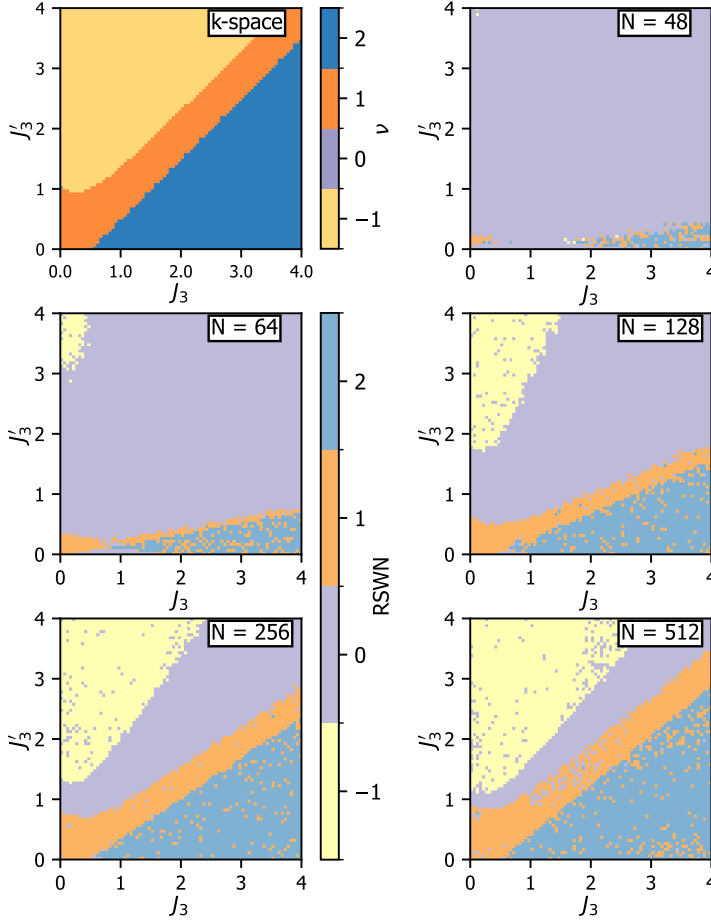


Figure 4.4.3: Winding number calculated from the Extended SSH model in finite size systems and the theoretical k-space limit. The parameters $\nu = 1/2, w = 1$ are set to be fixed values. J_3 and J'_3 are varied in the range of $[0, 4]$.

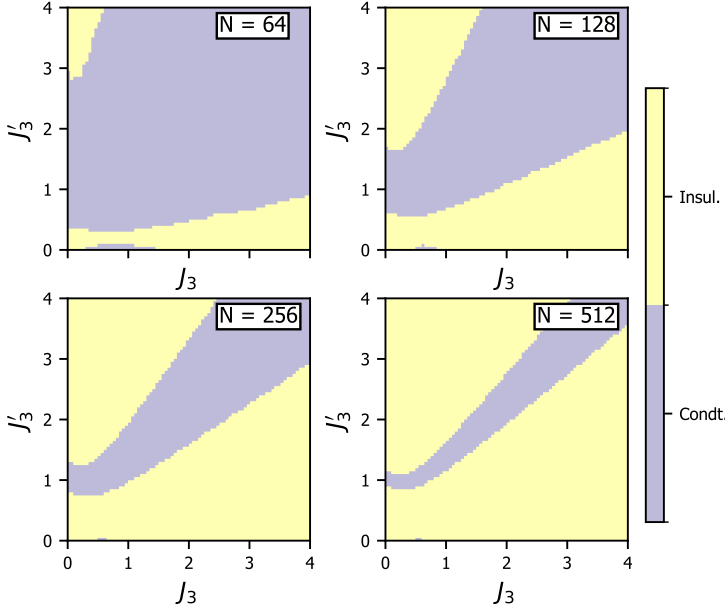


Figure 4.4.4: Mid-gap edge states bulk conductivity phase diagram in the extended SSH model of finite size systems. The parameters $\nu = 1/2, w = 1$ are set to be fixed values. J_3 and J_3' are varied in the range of $[0, 4]$. Label **Insul.** represents the edge state with no overlap in the bulk, while label **Condt.** represents the case where there exists non-negligible overlap between two edges in the topological edge states.

The boundary conditions in the finite system of length L (even number, $L = 2N$ with N being the number of unit cells) are $a(0) = 0$, and $b(L) = 0$. The exactly zero energy solutions of this system of equation do not satisfy these boundary conditions, since they have the form $a(x) = C_a e^{(v/w-1)x}$, $b(x) = C_b e^{-(v/w-1)x}$. Substituting general solution into boundary conditions, we get a spectral equation:

$$\tan \left[\sqrt{-\left(\frac{v}{w} - 1\right)^2 + \frac{E^2}{w^2}} L \right] = \frac{\sqrt{-(v-w)^2 + E^2}}{w-v}. \quad (4.21)$$

To find the solution for edge states within the gap range $|E| < \Delta$ in the topological phase with $w > v$, we perform a series expansion of both sides up to leading order in $|E|/\Delta$. Approximating the square root

$$\sqrt{-(v-w)^2 + E^2} \approx i|v-w|(1 - E^2/2\Delta^2), \quad (4.22)$$

we find the solution for the energy of edge states:

$$E_0^\pm \approx \pm \sqrt{\frac{2\Delta^2 \left(e^{-2L\frac{\Delta}{w}} + 1 \right)}{\cosh\left(2L\frac{\Delta}{w}\right) - 2L\frac{\Delta}{w} + 1}} \sim \pm \sqrt{2}\Delta e^{-L\frac{\Delta}{w}}. \quad (4.23)$$

The solutions that correspond to these energies have approximate form $\Psi_{\pm E_0}(x) = (a(x), b(x))$ with

$$a_{\pm E_0}(x) = C_1 \sinh[\sqrt{\Delta^2 - E_0^2} x/w], \quad (4.24)$$

$$b_{\pm E_0}(x) = \pm \frac{C_1}{|E_0|} \left(\sqrt{\Delta^2 - E_0^2} \cosh[\sqrt{\Delta^2 - E_0^2} x/w] + \Delta \sinh[\sqrt{\Delta^2 - E_0^2} x/w] \right). \quad (4.25)$$

These solutions are symmetric and antisymmetric and are localized near both edges. Thus, in the continuum theory there are no states localized only on single edge. To estimate the contribution of such states into RSWN formula, we substitute only these states into Eq.(4.16). Due to particle-hole symmetry, the states' vectors are $|\pm E_0\rangle = (a(x), \pm b(x))$ and the Q -operators read $Q = | + E_0\rangle\langle + E_0| - | - E_0\rangle\langle - E_0|$. Applying the projectors on sublattices, we obtain expression for trace

$$\begin{aligned} \text{Tr} P_B Q P_A [X, P_A Q P_B] &= \text{Tr}(|0, b\rangle\langle a, 0| - |0, -b\rangle\langle a, 0|) \times \\ &(\hat{X}(|a, 0\rangle\langle 0, b| - |a, 0\rangle\langle 0, -b|) - \\ &(|a, 0\rangle\langle 0, b| - |a, 0\rangle\langle 0, -b|)\hat{X}) \\ &= \text{tr}(X_A) - \text{tr}(X_B), \end{aligned} \quad (4.26)$$

where in the last line we evaluated all matrix elements between different bra and ket states, and introduced the notation for coordinate operator matrix elements:

$$X_A = \langle a, 0|\hat{X}|a, 0\rangle = \text{diag}(0, \dots, La^2(L)) \quad (4.27)$$

$$X_B = \langle 0, b|\hat{X}|0, b\rangle = \text{diag}(b^2(0), \dots, 0). \quad (4.28)$$

The trace of the diagonal matrix in the first line creates a contribution $\sim -L/L$ that cancels all other bulk state contributions in the winding number. This demonstrates the general principle of how pairs of edge states with symmetric/antisymmetric shape on both sides of system reduce RSWN value by 1. It is important to note that for localized edge states on one side, projection operation $P_A Q P_B$ directly gives zero, and thus RSWN value is not reduced. We note that since the values on the diagonal of matrices $X_{A,B}$ depend on exact distribution of $|a(x)|^2$ over coordinate space, the procedure of restoring back the value of winding number by removing few edge sites is not well defined and would depend on exact ratio of hopping parameters (as v/w in SSH model) [45]. The possibility of selecting in-gap eigenstates localized each at single edge is allowed by inversion symmetry only when both of them are exactly at zero energy. The inversion symmetry operator switches $x \rightarrow -x$ with respect to central coordinate of the system, and $a \rightarrow b$ sublattices. In the case of edge states being localized only on one side, such operation swaps edge states between each other, while other eigenstates are even or odd under inversion transformation[40, 2, 35]. When the two states have different energies, each of them should satisfy inversion symmetry individually. That way the new symmetric and antisymmetric edge states that reduce RSWN value by 1 appear.

Thus, several elements at the start or end of the diagonal cancel out all other contributions of winding markers, resulting in zero value of winding number. To explain why

the RSWN formula can be still used, the special attention should be paid to the meaning of topological protection in the real system.

The edge states are topologically protected if the crosstalk with the bulk states and between states is negligibly small comparing to the corresponding level broadening or coherence time. The level broadening could happen when the temperature changes or due to the coupling to the environment. In other words, if the energy $E_0 \ll \Gamma$, with Γ being a level broadening, both basis selections $\Psi_{\pm E_0}$ and $1/2(\Psi_{+E_0} \pm \Psi_{-E_0})$ are allowed. The second selection results in a value, predicted by RSWN formula, being close to k -space winding number up to $1/L$ -corrections. In numerical calculations with precision regulated by Eq. 4.4, the level broadening could be replaced by the accuracy of numerical solver. Together all these effects force the region of topologically trivial phase to spread into topologically nontrivial phase near the phase transition, effectively changing phase transition point in parameter space.

Having explained the general principle, we now extend the discussion to extended SSH model. Each phase transition line corresponds to gap closing at particular k -point, and these points are different for each transition. According to bulk-boundary correspondence, the change of winding number in translation-invariant model corresponds to the change of number of pairs of edge states in finite size systems. Near the phase transition lines, the energy of edge states that disappear is determined similarly to Eq. (4.23). Thus, each pair of edge states that disappear lowers RSWN by 1. This qualitative description covers all examples shown in Fig. 4.4.3. Note that the phases are still ordered by increasing winding number by the same steps as in k -space. That is because the gaps near single phase transition line could be ordered as $\Delta_0 < \Delta_1 \leq \Delta_2 \dots$ where $\Delta_0 = 0$ exactly at phase separation line. While one pair of edge states that corresponds to Δ_0 contributes to decreasing RSWN, the remaining ones are not. That is because their overlap in the bulk, governed mainly by exponential $e^{-\Delta L/w}$ as in Eq. (4.23), is much lower than the level broadening and does not lead to hybridization.

Before proceeding to analysis of the role of disorder, we note that the calculation of RSWN with the level broadening estimated for particular system might lead to the understanding of the quality of topological protection. In particular, while some pairs of edge states might have a crosstalk over bulk, the rest are still localized and would not lose the coherence. This could be applied to extended SSH model with phase diagrams presented in Figs. 4.4.3, 4.4.4, where one should expect topological protection only for very small range of parameters in small system $N = 48$, and different numbers of topologically protected states in longer systems.

4.6. DISORDER

4.6.1. DISORDER IN HOPPING PARAMETERS

Topological materials are well known for their robustness against disorder, a property rooted in the topological protection derived from symmetries in momentum space. Since disorder breaks translational invariance—a key condition for defining topological invariants in momentum space—this robustness cannot be directly demonstrated through standard momentum-space calculations. However, using the RSWN, we can capture this robustness in finite-sized disordered systems.

In the SSH family, there are two types of disorders: hopping disorders, which preserves the chiral symmetry; and the disorders in the chemical potentials, which break the chiral symmetry [2]. We begin by discussing hopping disorders. Previous studies have shown that the RSWN calculated for hopping disordered SSH model retains the discrete values expected by its k-space counterpart [46, 45, 36].

To further reinforce the confidence in topological physics, we investigate the behavior of RSWN on extended SSH under hopping disorders. Fig. 4.6.1 shows the phase diagrams of the extended SSH model with 5% hopping disorders applied to the hopping parameters ν, w, j_3, j_{3p} , i.e., the parameters fluctuate in the $\pm 2.5\%$ region of their desired values. At each value of a J_3 and J'_3 configuration, we simulated one randomly disordered Hamiltonian of this configuration. The RSWN remains stable even with this 5% uncertainty, as shown in Fig. 4.4.3. In numeric, we set the designated values of hopping parameters to be: $\nu = 1/2, w = 1$, while J_3 and J'_3 are varied in the range of $[0, 4]$. The obtained phase diagrams truthfully resembles the original phase diagram in Fig. 4.4.3.

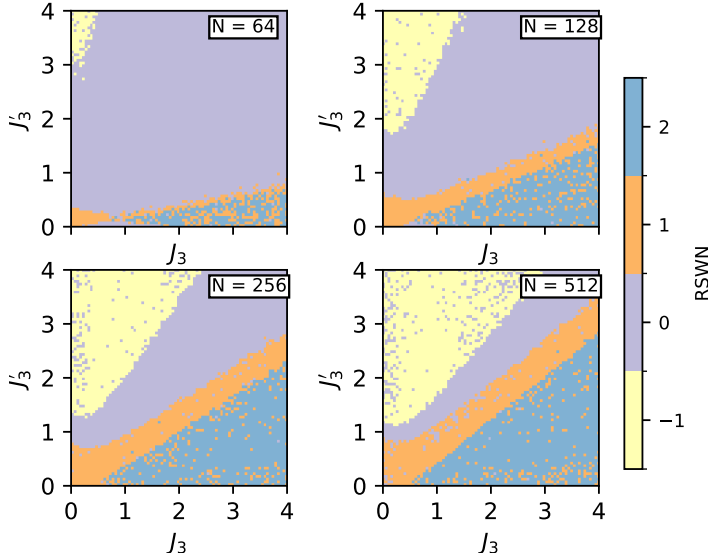


Figure 4.6.1: Real space winding number ν of extended SSH model with 5% uncertainty in the hopping parameters ν, w, j_3, j_{3p} . Similar to previous simulations, the parameters $\nu = 1/2, w = 1$ are set to be fixed values. J_3 and J'_3 are varied in the range of $[0, 4]$.

4.6.2. DISORDER IN CHEMICAL POTENTIAL μ_i

A fundamental question in implementing topological phases of matter in real-world experiments is: *What happens when perfect symmetry is slightly broken?* In the SSH model, the chiral symmetry requires equal local chemical potentials μ_i , but achieving this exact condition in reality—whether in engineered meta-materials or natural condensed matter systems—is virtually impossible due to inevitable fluctuations. This raises the question: *Does chiral symmetry ever truly exist in physical systems?* This is where the RSWN

formula can give insight.

We find that chemical potential disorder $\delta\mu$ can push the RSWN transition in finite size systems to the theoretical limit. For example, in Fig. 4.4.1, the transition of $N = 80$ occurs around $\sim \nu/w \sim 0.75$ in disorder free systems. However, when chemical potential disorder $\delta\mu$ is introduced, the transition of $N = 80$ shifts closer to $\nu/w \sim 1$ as shown in Fig. 4.6.2.

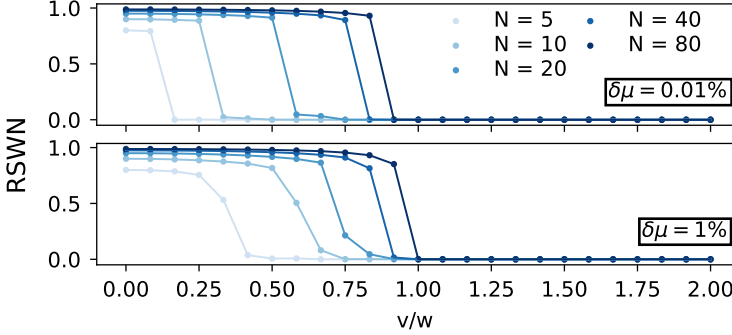


Figure 4.6.2: RSWN of SSH model vs ν/w when the chemical potentials μ_i are disturbed. Upper panel: the fluctuation in chemical potential is $\delta\mu = 0.01\%$. Lower panel: the fluctuation in chemical potential is $\delta\mu = 1\%$. The parameters are $\nu \in [0, 1]$, $w = 0.5$.

Similarly, in the presence of chemical potential disorder $\delta\mu$, the phase transition in finite sized extended SSH model shifts closer to ideal values, as shown in Fig. 4.6.3. Additionally, the anomalous $\nu = 0$ region (the plateau in Fig. 4.6.3) is narrowed. Thus, we conclude that the presence of chemical potential disorder $\delta\mu$ enhanced the observation possibility of topological phase transition in experimental implementations.

4.7. CONCLUSION AND DISCUSSION

In this work, we addressed the critical limitations in the existing real space winding number Eq. (4.16) based on one dimensional chiral symmetric models of different winding number thresholds. We demonstrated that while the RSWN deviates from expected values in small systems, it ultimately converges to theoretical predictions in larger systems. We developed bulk conductivity criteria to reliably characterize the topology in finite sized and disordered topological systems. We showed that the combination of RSWN and the bulk conductivity serves as a good measure of topological protection. Our proposal compensates for the inaccurate prediction of topological phase transition from the RSWN alone.

We supported our conclusions with both numerical and analytical studies of the effect of edge states on the phase transition. The real space invariant together with edge states transport properties provides a wholesome overview of the topological properties of the system. Furthermore, we investigated the impact of hopping disorder and chemical potential disorder on RSWN predicted topological phase transitions. We numerically

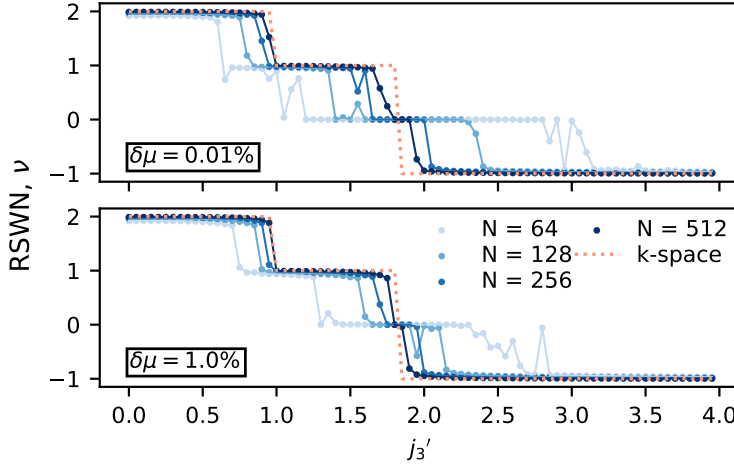


Figure 4.6.3: RSWN of Extended SSH model vs j_3' when the chemical potentials μ_i are disturbed. Upper panel: the fluctuation in chemical potential is $\delta\mu = 0.01\%$. Lower panel: $\delta\mu = 1\%$. The theoretical phase transition predicted by k-space winding number is plotted in the orange dotted line. The parameters used are $\nu = 0.5$, $w = 1$, $j_3 = 1.5$.

show that the RSWN is robust against symmetry respecting disorders. Additionally, we found the chemical potential disorder, which breaks the chiral symmetry, can shift the transition points closer back to the ideal values, thus supporting experimental implementations of topological materials.

In the near term, our results are immediately relevant to designing and optimizing topological materials for experimental setups that rely on robust edge states, such as quantum dots, superconducting resonators, and hybrid qubit-photon systems [24, 46, 20, 10, 56, 3, 27, 13, 12, 47, 49]. It is worthy to note that the RSWN is usually hard to be measured directly from experiments, but can be learned with machine learning techniques [7]. Our proposed criteria involving bulk conductivity will help experimentalists verify and refine the topology of their systems even in disordered configurations. The potential experimental verification of, for example, the lifetime of topological modes could reveal the real life topology discussed in this work. Our result also opens the door to controlled disorder engineering as a tool to fine-tune topological phase transitions. This approach could significantly impact fields like quantum computing and topological quantum states, where scalable and resilient topological properties are crucial.

The future works can develop in the following two direction. First one can study the similar effect in higher dimensional topological materials, for instance two dimensional topological insulators [49, 38], and for different types of topological invariants [8, 6]. Symmetries of different kinds should also be considered. In our case, the chiral symmetry is broken while the chemical potential is added, while in reality it does not completely destroy the topological phase transition. It is highly intriguing if the similar situation will happen when one moves to the topological crystalline insulator where

the crystalline symmetry can be easily broken when one introduced local perturbations. We do believe the our transport-integrated criteria is vita when characterizing the finite sized systems. We think it is interesting to investigate new benchmarking method of real life topology.

Overall, our results suggest that the bulk conductivity is an indispensable compensation to the anomalous RSWN in finite sized systems with disorders. With the support from local bulk conductivity information, RSWN shows a clear robustness towards perturbations and suggests potential benefits of previously harmful disorders that disrespects the symmetry. Our work contributes to create a better designing principle of topological materials and a deeper understanding of topological physics. Our approach also aligns with the practical requirements of quantum information and quantum transport applications, where topological protection of edge states is essential.

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5

DYNAMICAL LOCALIZATIONS IN 2D TOPOLOGICAL QUANTUM RANDOM WALKS

We study the dynamical localization of discrete time evolution of topological split-step quantum random walk (QRW) on a single-site defect starting from a uniform distribution. Using analytical and numerical calculations, we determine the high localization probability regions in the parameter space of the quantum walker. These regions contain two or more pairs of trapped states, forming near a lattice defect. By investigating the spectral properties of the discrete-time evolution operators, we show that these trapped states have large overlap with the initial uniformly distributed state, thus offering a simple interpretation of the localization effect. As this localization scheme could be interpreted as a variation of spatial quantum search algorithm, we compare the localization probability and time with other types of two-dimensional quantum walks that do not have topological phases and realize localization time scaling similar to Grover's algorithm. Finally we show that mechanism of localization we identified is robust against the influence of disorder.

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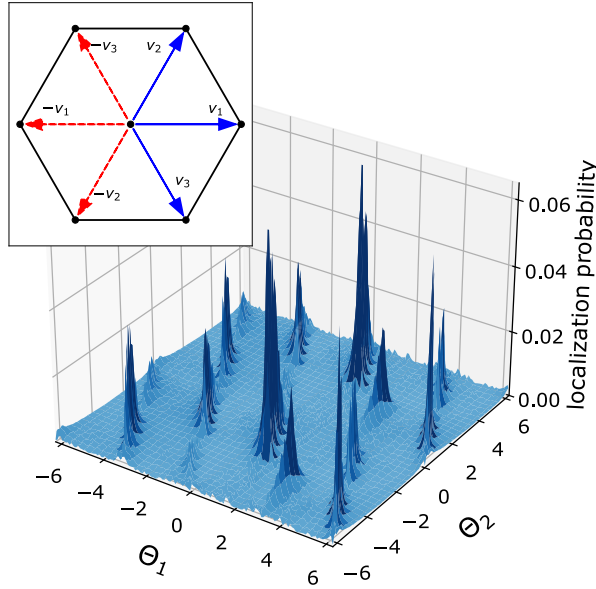


Figure 5.1.1: The localization probability of topological quantum walk as a function of spin rotation parameters $\Theta_{1,2}$ (see Eq. (5.2)) for a defect with $\Theta_1^{def} = 5\pi/8$, $\Theta_2^{def} = \pi/2$. High peaks are of two orders of magnitude larger than average localization probability and correspond to parameters for which quantum walker performs localization efficiently. Inset: geometry of split-step QRW on a triangular lattice with three shift vectors $\vec{v}_i$ - solid ones correspond to shift of spin up components, dashed $-\vec{v}_i$ for spin down.

5.1. INTRODUCTION

Quantum random walks were introduced in Ref. [1] as a natural quantum version of classical random walks. Due to interference effects and state superposition, the average travel distance of the walkers is much longer than that of their classical counterparts. The clear structure of possible physical systems that could realize quantum walks [1] has attracted much attention. In particular, it has been shown that quantum walks provide a possible way to implement Grover's algorithm [13] in the case of absence of high-quality multi-qubit quantum computer [26, 4, 2, 15, 24, 27]. It has been shown that the so-called Grover's oracle operator has a natural realization in quantum walks and is related to dynamical localization phenomena [26, 4, 2, 15, 24, 27]. In particular, discrete-time quantum random walks are defined by a set of shift and coin-flip operators applied to the entire system at each time step [2, 7, 1, 26]. The successful localization event is defined as a time of maximum localization of quantum walker on a defect, starting from a uniform distribution over the whole system. Going forward, we will refer to the time at which the first maximum in localization probability appears as the localization time.

In the most common mechanism of localization, several eigenstates of QRW evolution operator create a constructive superposition on a defect at a certain time [15, 2]. Little is known about the precise values of the optimal defect parameters that maximize

localization relative to the surrounding system. In addition, it has been shown that certain positioning of defects can significantly decrease the localization probability [25]. Here, we study two-dimensional split-step quantum random walks, analogous to Chern type topological insulators in crystalline solids [18]. For such quantum walks, it is expected that topology creates a new mechanism of localization, similar to mid-gap states in solids. Following this motivation, we explore how topology affects the localization mechanism. Topological split-step QRWs in 1D space were studied in Ref. [6, 7]. In this paper, we analyze the parameter spaces of the quantum walker and the defect. We find that there exist sets of parameters that deterministically demonstrate a high localization probability at a given defect (see a localization probability map in Fig. 5.1.1). In Sec. 5.2 we show that these regions in parameter space fall into two distinct classes based on their position inside the topological phase or on the phase boundary lines. Notably, we find the localization time for the parameters in both classes of regions saturates to a constant with growing system sizes. The saturation of localization time happens at different system sizes, while before this saturation, the localization time exhibits square-root growth behavior.

The localization after a constant time, regardless of system size, was previously identified only in electric Dirac QRW with non-local marking of defect in the form of Coulomb potential [29, 9]. The non-locality of defects creates an important difference from our scenario: our protocol does not require a large number of nodes to form a defect structure. In the algorithmic terminology, in our work, the constant localization time is achieved with local marking, in contrast to non-local extended marking of Coulomb potential.

In Sec. 5.3, we analyze the time evolution of QRW states across a range of parameters and study the overlap of the initial state with the eigenstates of the unitary operator corresponding to a single time step of QRW. Particularly, we demonstrate that the localization speedup with constant time asymptotic is related to the appearance of trapped states of a specific structure near the defect: two pairs of trapped states should simultaneously have a large matrix element with the coordinate operator of the defect, and a large overlap with the uniformly distributed initial state. In mathematical terms, the trapped states are defined as states that have compact support and zero density elsewhere. Their analogue in condensed matter systems are the strongly localized bound states on impurities. We note that a number of mathematically rigorous studies have been made for spectral properties of quantum random walks and trapped states [15, 16, 6, 12, 20, 11] together with their dynamics [10, 21]. However, the possible efficiency of the localization mechanism that leverages the particular structure of QRW was not previously addressed. Furthermore, the strong dependence of the trapped states structure on the defect and walker parameters also suggests a scenario of protecting quantum localization against random disorder, by an appropriate fine-tuning of the marking parameters for the defect element. The corresponding study is presented in Sec. 5.4 and the concluding remarks are given in Sec. 5.5.

5.2. THE QRW MODEL AND LOCALIZATION PHENOMENON

The quantum random walk analyzed in this work was first introduced in Ref.[18] as a way to simulate topological insulators. Its single time step

$$\Psi(t+1) = U(\Theta_1, \Theta_2)\Psi(t) \quad (5.1)$$

5

is defined via three spin rotations and subsequent translations:

$$U(\Theta_1, \Theta_2) = T_{\vec{v}_3} R(\Theta_1) T_{\vec{v}_2} R(\Theta_2) T_{\vec{v}_1} R(\Theta_1), \quad (5.2)$$

$$R(\Theta) = \begin{bmatrix} \cos(\Theta/2) & -\sin(\Theta/2) \\ \sin(\Theta/2) & \cos(\Theta/2) \end{bmatrix}.$$

The translation operations shift the spin-up components by $+\vec{v}_i$ and the spin-down by $-\vec{v}_i$. The vectors $\vec{v}_{1,2,3}$ can be defined on a triangular lattice as $\nu_1 = (1, 0)$, $\nu_2 = (1/2, \sqrt{3}/2)$ and $\nu_3 = (1/2, -\sqrt{3}/2)$, see inset in Fig.5.1.1.

The spectrum of quasi-energies of this QRW without defects has two particle-hole symmetric bands, due to the fact that matrix $U(\Theta_{1,2})$ is real. The corresponding eigenstates with energy $\pm E$ are related by complex conjugation $\Psi^{-E} = (\Psi^E)^*$. Depending on the parameters Θ_1 and Θ_2 , such QRW realizes distinct topological phases with Chern numbers of Floquet-type bands $C = \pm 1, 0$. In contrast to the same kind of topological insulators in crystalline solids [17, 14], we note that QRW spectrum is defined in terms of Floquet quasi-energies, and is thus limited to $(-\pi, \pi)$ interval. The bang gap closing happens either at zero energy or at the ends of this interval along the phase boundary lines indicated in Fig. 5.2.1(a). Note that at $\Theta_2 = 0$ line, the gap closes everywhere, as discussed in Appendix 5.A.

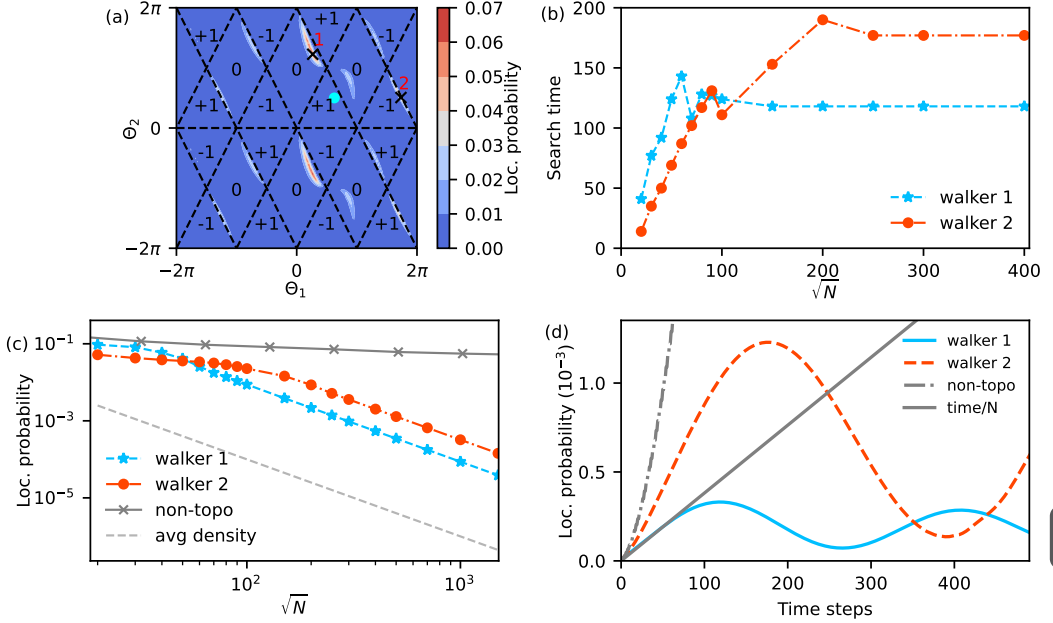


Figure 5.2.1: (a) A color map representation of a maximum localization probability on a defect site over the evolution time $T = 1000$ for a system of size $N = 40 \times 40$ starting from a fully uniform distribution. Black numbers indicate Chern numbers of the corresponding parameter regions and dashed lines correspond to phase separation lines. Cyan dot marks the defect parameters $(5\pi/8, \pi/2)$. (b) localization time measured as a position of first probability peak as a function of system size for walker parameters indicated by crosses '1' and '2' in panel (a). (c) Localization probability dependence on system size. The two regions of different dependence are present where probability follows 'Grover-type' scaling $P \sim 1/\log(N)$ as non-topological QRW from Ref.[25] until the time asymptotic changes to constant in (b) and then switches to system size scaling $P \sim 1/N$ but having few orders of magnitude larger value than average density in the system. (d) Comparison of time-dependent localization probability with non-topological QRW [25] and repeated classical localization for system size $\sqrt{N} = 512$. Walker 2 shows an advantage over classical. Note the scaling factor of $1e-3$ in the panel (d) for localization probabilities.

In our setup, the localization phenomena are expected to occur after a certain number of time steps at a single defect node on a finite 2D lattice of size $\sqrt{N} \times \sqrt{N}$ unit cells, starting from initial state $|i\rangle$. We apply the periodic boundary condition in order to avoid the appearance of edge states. The defect node is marked by parameters $\Theta_{1,2}^{def}$ that are different from surrounding $\Theta_{1,2}$. To estimate the possibility of a given quantum walker to localize efficiently with high probability, we first scan the entire parameter range $\Theta_{1,2}$ for a given defect values $\Theta_{1,2}^{def}$. The measure of localization probability that is relevant for quantum search applications is defined as a peak $\max_{t \in [0, T]} P_{def}(t)$ in the

time-dependent density of the system state at the defect node,

$$P_{def}(t) = |\langle d | \Psi(t) \rangle|^2, \quad (5.3)$$

where $|d\rangle = \sum_{\sigma=\uparrow,\downarrow} |x=def, \sigma\rangle$ is the defect coordinate operator with both components of spin σ . The initial state $|i\rangle$ of the system is defined as $|i\rangle = \frac{1}{\sqrt{2N}} \sum_{\sigma,x} |x, \sigma\rangle$ with summation over entire system.

The plot of the maximum value of density $P_{def}(t)$ on the defect over the interval of time $T = 1000$ for the test system size $N = 40 \times 40$ sites is shown in Fig. 5.1.1 for $\Theta_{1,2}^{def} = (5\pi/8, \pi/2)$. In Fig. 5.2.1(a), we show the color map representation of the same localization probability distribution on top of the phase diagram in parameter space. For other positions of defect $\Theta_{1,2}^{def}$ in parameter space, the colormaps of localization probability are shown in Supplemental material [5].

The islands of efficient localization in parameter space fill small areas placed in two distinct types of regions: inside topological phases and on phase separation lines. We compare the localization time and localization probability behavior with growing system size for the walker parameters marked by '1' and '2' in Fig. 5.2.1(a), which realize maximal probability values within their respective regions. In Fig. 5.2.1(b), the localization time is plotted against the square root of the system for the walkers '1' and '2'. With the growing system size, the behavior of localization time demonstrates a qualitative change - from a $T \sim \sqrt{N}$ dependence to $T \sim const.$ for both walkers. As noted previously, such a time dependence of the localization maximum was observed only for the Dirac QRW with a particular type of Coulomb-like disorder potential [29, 9]. In Fig. 5.2.1(c), the localization probability of walkers '1' and '2' is compared to the average density at non-defect nodes. We find that the constant localization time regime corresponds to $P_{def}(t) = O(1/N)$ with a large numerical prefactor. For the algorithmic comparison, it is important to note that this decrease of localization probability with system size N is faster than for non-topological QRW discussed in Ref. [25], where $P_{def}^{Grover} = O(1/\log N)$. The localization probability dependence in the form $O(1/N)$ arises due to the finite support of the trapped states. When the system size grows beyond this support, the localization time asymptotically saturates. We then observe a clear dependence of the localization probability on the system size, see Fig. 5.2.1(c).

For the fixed parameters of quantum walker '1', we visualized the distribution of localization probability in the space of defect parameters $\Theta_{1,2}^{def} \in [-2\pi, 2\pi]$. The corresponding discussion in Appendix 5.A shows that the localization probability depends only on Θ_1^{def} parameter, and remains constant along Θ_2^{def} . Thus, it is more informative to present a dependence on Θ_1^{def} as shown in Fig. 5.3.1(a), to compare with overlap criteria introduced below.

The asymptotic scaling of localization probability behaves as $O(const/N)$ in the large N limit, where $const. \in [100, 1000]$ depends on the parameters of the quantum walker and the defect. This asymptotic is different from the fine-tuned regime in some non-topological QRWs, where probability can scale as $O(1/\log N)$ (see Ref. [25]). At the same time, it outperforms the classical $1/N$ asymptotic by a constant factor. In Fig. 5.2.1(d), we show how localization probability changes with time and compare it to $time/N$, which

corresponds to classical localization phenomena. Additionally, we compare it to the $1/\log N$ scaling of non-topological localization. So while topological QRWs do not reach scaling in the Grover's search algorithm, they can still locally outperform classical counterparts. In Sec. 5.4, we show that the topological QRW behavior shown in Fig. 5.2.1 can be further tuned by the presence of disorder.

5.3. CONTRIBUTION OF TRAPPED STATES

To describe the origin of dramatically different localization behaviors for various parameters of the topological QRW - and to explain why the regions of efficient localization occupy only a small fraction of the parameter space - we analyze the spectrum and localization properties of eigenvectors of a unitary operator of the system with a single defect.

The probability evolution at a defect site at discrete times t can be rewritten using the unitary operator of QRW,

$$P_{def}(t) = |\langle d | \bar{U}^t(\Theta_1, \Theta_2) | i \rangle|^2. \quad (5.4)$$

The $\bar{U}$ notation is introduced to emphasize that the unitary operator includes the defect site. Substituting the eigenbasis decomposition of the $\bar{U}$ operator, $\bar{U}|n, \pm\rangle = e^{i\pm E_n}|n, \pm\rangle$, we have

$$P_{def}(t) = \left| \langle d | \sum_{n, \lambda=\pm} e^{i\lambda E_n t} |n, \lambda\rangle \langle n, \lambda | i \rangle \right|^2. \quad (5.5)$$

The saturation of localization time to constant scaling - after system size becomes sufficiently large - suggests that the properties of the states that contribute significantly to the Eq.(5.5) do not change with system size. This feature directly corresponds to the definition of trapped states previously introduced in Refs. [15, 16], in the context of QRWs that remain localized around its initial starting node over infinite evolution time. We investigate the contribution of different eigenstates of $\bar{U}$ to the evolution of probability, $P(t)$, in Fig. 5.3.1(b), by comparing the product of overlaps $|\langle d | n, \lambda \rangle \langle n, \lambda | i \rangle|^2$. The panels in Fig. 5.3.1(b) correspond to different parameters of the defect (listed above each panel) denoted by vertical dashed lines for walker '1' in Fig. 5.3.1(a).

The scale of x -axis in each panel of Fig. 5.3.1(b) shows that defect parameters with higher localization probability also correspond to a higher overlap product for several states - in this case, two pairs of particle-hole symmetric states. To further verify this observation, we compare the maximal value of overlap product

$$W_{\mathcal{M}} = \left| \sum_{n \in \mathcal{M}, \lambda=\pm} \langle d | n, \lambda \rangle \langle n, \lambda | i \rangle \right|^2, \quad (5.6)$$

of a single eigenstate of $\bar{U}$ with the normalized probability dependence $\max[P_{def}(t)]$ as a function of Θ_1^{def} . The comparison is made by taking either a single pair of states that maximize the overlap with a set of indices $\mathcal{M} = \{n_{max}\}$ - or two pairs of states, $\mathcal{M} = \{n_{max}, n_{second\ max}\}$. The results, presented in Fig. 5.3.1(a), indicate that while $\mathcal{M} = \{n_{max}\}$ tends to overestimates the region of good localization instances, $\mathcal{M} = \{n_{max}, n_{second\ max}\}$

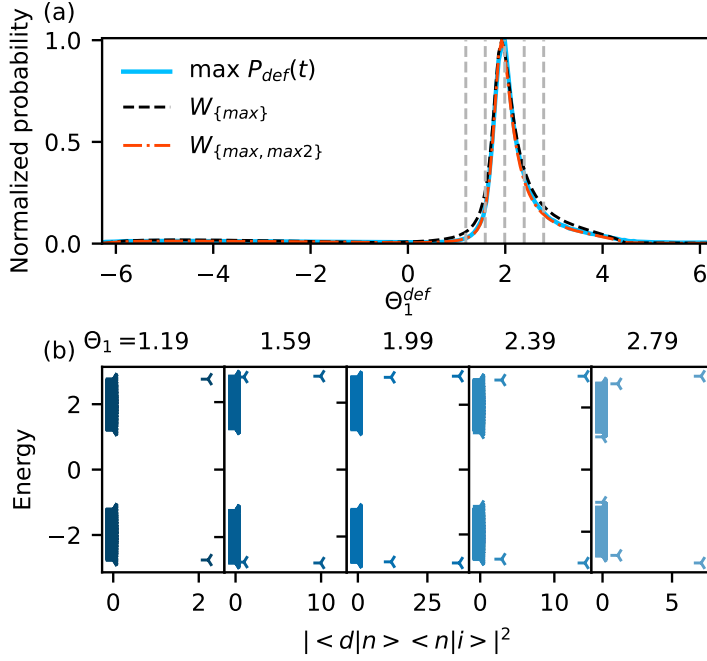


Figure 5.3.1: (a) Comparison of maximum value of probability on defect for a walker '1' normalized by numerically achieved maximum on the interval from Fig.5.2.1 as a function of defect parameter Θ_1^{def} with the maximal value of squared sum of overlaps defined in Eq.(5.6) with $\mathcal{M}$ containing one and two states respectively. All quantities are normalized to scale (0,1) for the purpose of comparison. The summation over two trapped states gives a very accurate approximation of localization probability peak. (b) The squared product of overlaps plotted as a function of the energy of each eigenstate of the operator $\tilde{U}$ for walker parameters '1' and defect parameter Θ_1 indicated on top of each panel. Trapped states differ from other states by significantly higher values of overlap.

very precisely describes the high localization probability peak. To further check this correspondence between the evolution described by two pairs of states with maximal overlap and the maximum probability distribution, we show the same comparison for a defect $\Theta_{1,2}^{def} = (5\pi/8, \pi/2)$ along phase separation line and the line with walker '1', in Appendix 5.B. The correspondence for a test system size with $\sqrt{N} = 40$ shows that efficient localization event for such split-step QRW are largely determined by the presence of two pairs of trapped states with a large overlap with both defect and initial state.

Finally, we perform compare the time evolution defined by two pairs of trapped states that maximize the overlap criteria with the full evolution of probability on the defect. For this purpose, we note that the $\bar{U}$ has the same particle-hole symmetry as $U(\Theta_{1,2})$ because it is a real matrix. In addition, by calculating overlap products, denoted as $w_{\pm,j}^\sigma = \langle d, \sigma | j \rangle \langle j | i \rangle$, for each spin component $\sigma = \uparrow, \downarrow$ and for each pair of trapped states j at the defect site separately, we find that for each trapped state there is a phase difference $w_{+,j}^\downarrow = i w_{+,j}^\uparrow$ and corresponding $w_{-,j}^\downarrow = -i w_{-,j}^\uparrow$ up to numerical precision. The former symmetry is present only at the defect site and is a characteristic feature of trapped states localized around the defect. Thus, the evolution of both spin components at the defect site, as generated by the trapped states, can be described as follows.

$$\Psi_{\uparrow,d}(t) = 2 \sum_{j=1,2} \left| w_{+,j}^\uparrow \right| \cos \left(E_j t + \arg \left(w_{+,j}^\uparrow \right) \right), \quad (5.7)$$

$$\Psi_{\downarrow,d}(t) = 2 \sum_{j=1,2} \left| w_{+,j}^\uparrow \right| \sin \left(E_j t + \arg \left(w_{+,j}^\uparrow \right) \right), \quad (5.8)$$

where we took into account the particle-hole symmetry relation $w_{-,j}^\sigma = (w_{+,j}^\sigma)^*$. It is important to emphasize that after calculating probability at a defect site, one finds that the two spin components exactly cancel oscillating parts of each other for a single value of j . That happens due to the different phases of oscillations between $\sin$ and $\cos$ in Eqs.(5.7) and (5.8) above, and results in:

$$\begin{aligned} |\Psi_{\sigma,d}^j(t)|^2 &= 2 \left| w_{+,j}^\uparrow \right|^2 \\ &\times \left(1 + \text{sign}(\sigma) \cos \left(2E_j t + 2 \arg(w_{+,j}^\uparrow) \right) \right), \end{aligned} \quad (5.9)$$

with $\text{sign}(\sigma) = \pm$. Thus, the contribution of each single pair of trapped states is constant and due to the initial condition of uniformly distributed state over the entire system it could not describe the high peak in localization probability.

However, the crucial contribution comes from the superposition between two pairs of trapped states and is manifested in the last oscillating term in the total density evolution at the defect:

$$\begin{aligned} |\Psi_{\uparrow,d}(t)|^2 + |\Psi_{\downarrow,d}(t)|^2 &= 4 \left(\left| w_{+,1}^\uparrow \right|^2 + \left| w_{+,2}^\uparrow \right|^2 \right) + 8 \\ &\times \left| w_{+,1}^\uparrow \right| \left| w_{+,2}^\uparrow \right| \cos \left((E_1 - E_2) t + \arg w_{+,1}^\uparrow - \arg w_{+,2}^\uparrow \right). \end{aligned} \quad (5.10)$$

The period of this oscillation is different from the fast oscillations of each pair of trapped states and is defined by the energy difference between the positive energy levels of two

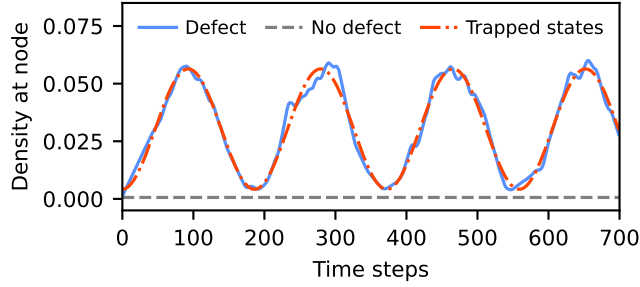


Figure 5.3.2: The comparison of density evolution at marked defect node (blue solid line) and another distanced node without defect (gray dashed line) with time for a walker parameters ‘1’ and defect marked by $(5\pi/8, \pi/2)$. The red dash-dotted line corresponds to the evolution described by only two pairs of trapped states that maximize the product of overlaps and given by Eq. (5.10). The evolution from trapped states approximates with very good precision the first few peaks in probability and the time of their occurrence.

pairs of trapped states ($E_1 - E_2$). Fig. 5.3.2 shows that expression in Eq. (5.10) captures very well the structure of oscillations of localization probability despite its simplicity.

In Fig. 5.3.2 the results are presented for a walker ‘1’. We show the equivalent result walker ‘2’ in Appendix 5.B. To further verify that the predicted oscillation period agrees with the actual probability oscillation at the defect node, we performed a numerical Fourier transform. The agreement with period defined by $(E_1 - E_2)$ has errors below 2% on the timescale of evolution $T = 5000$ for system with $\sqrt{N} = 40$. It was observed [28, 25] that in non-topological QRWs, that realize localization mechanism similar to Grover’s algorithm, and, the period of oscillation is defined by an energy difference within the same particle-hole symmetric pair of states at lowest energies. The difference between the two definitions of the period shows that the mechanism of localization in topological QRWs is different compared to non-topological QRWs studied in the literature.

The change of time asymptotic that is shown in Fig. 5.2.1(b) happens when the system size becomes larger than diameters of both trapped state pairs. In Appendix 5.C we show that the state from the pair with a smaller overlap in Fig. 5.2.1(c) has a larger radius of localization (95% of the probability density). This pair of states determines the chance in the localization time asymptotic as shown in Fig. 5.2.1b.

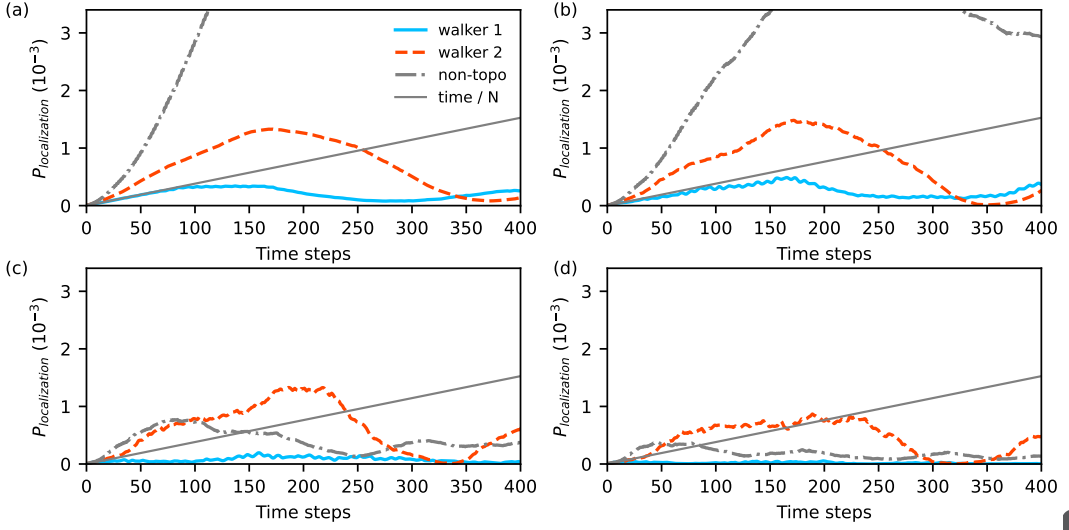


Figure 5.3.3: The localization probability calculated for two walker parameters analyzed in the main text (see Fig.2) and compared with the localization probability for QRW from Ref.[25] and with classical localization phenomena with linearly increasing probability. The system size is 512×512 . Disorder strength varies from small in panel (a) $\Theta_{dis} = 0.05$, to (b) $\Theta_{dis} = 0.15$, (c) $\Theta_{dis} = 0.35$, and large in panel (d) $\Theta_{dis} = 0.5$. The walker 2 shows the best performance at large disorder out of all compared search procedures.

5.4. DYNAMICAL LOCALIZATION WITH DISORDER

Now we move on to analyzing the stability of localization on a single-node defect with disorder. For the topological QRW, the disorder is added to rotation operators on the nodes in Eq.(5.2), via replacement $R(\Theta) \rightarrow R(\Theta + \delta\Theta)$ where $\delta\Theta \in [-\Theta_{dis}, \Theta_{dis}]$ is a uniformly distributed random value centered around zero in a finite interval. The defect node is not disordered. As an example of another mechanism, the localization of non-topological discrete-time QRW from Ref. [25] is studied for the same system sizes. The corresponding time step operator is defined as:

$$U = T_y C_y T_x C_x, \quad T_i |\pm\rangle |i\rangle = |\pm\rangle |i \pm 1\rangle. \quad (5.11)$$

In this case, the disorder is implemented as phases in the coin rotation operators:

$$C_x(\delta\Theta) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \exp(i\delta\Theta) \\ i \exp(-i\delta\Theta) & 1 \end{pmatrix}, \quad (5.12)$$

$$C_y(\delta\Theta) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \exp(i\delta\Theta) \\ -i \exp(-i\delta\Theta) & 1 \end{pmatrix}. \quad (5.13)$$

While different selections of disorder could be made, the one introduced above is motivated by the possible loss of phase coherence on the nodes of large systems (described by slightly different phases acquired by passing each node). The sizes of the test systems are taken to be the same - 512×512 nodes for a single configuration of disorder,

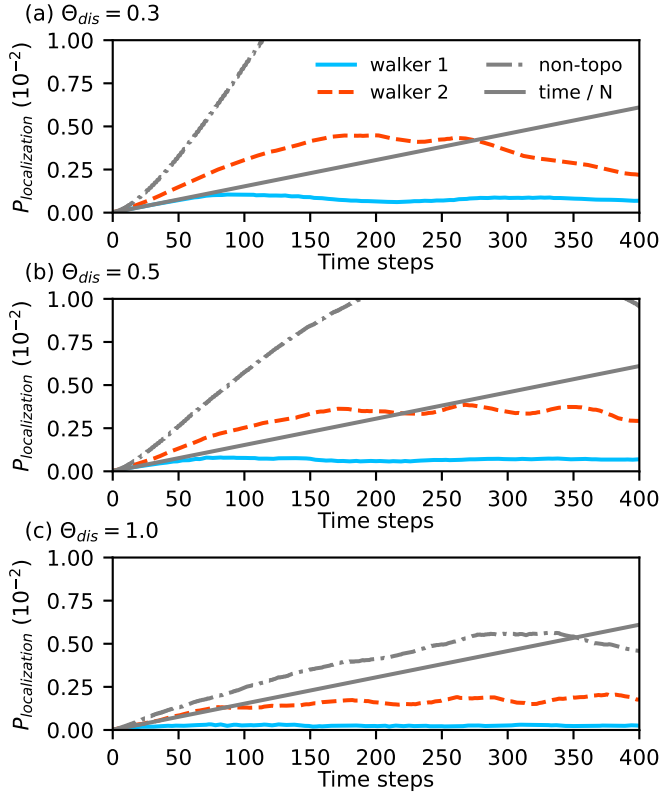


Figure 5.4.1: The dynamical localization phenomena on a single-node defect averaged over 20 configurations of disorder. The averaged localization in non-topological split-step QRW from Ref.[25] has a higher density of defect node rather than of a topological QRW due to a larger amount of states contributing to maximum localization. The size of the system is 256×256 nodes.

and 256×256 for averaging over 20 configurations. Both sizes are well above the saturation time threshold for example walker parameters '1' and '2' described in the previous sections.

The results of simulation for different sizes of interval distributions $\delta\Theta \in [-\Theta_{dis}, \Theta_{dis}]$ of angle disorder and for single configuration in coordinate space are presented in Fig. 5.3.3. The values Θ_{dis} are equal for topological and non-topological QRWs. The disorder reduces the height of localization probability peak for both quantum walks, and consequently the corresponding efficiency of localizations, as it is shown in panels (a)-(d) for Θ_{dis} varying from 0.05 to 0.5. However, for one of the two topological walkers, the localization probability always remains higher than the classical localization probability during the first period of oscillation. Notably, the probability of quantum localization (non-topological QRW) drops dramatically as the disorder increases.

While the example shown in Fig. 5.3.3 constitutes one specific disorder configuration and type, the trapped states demonstrate relatively stable behavior, even locally outperforming existing quantum search for large disorder. Still, we point out that more systematic studies should be made to define the optimal performance conditions of different quantum and classical random walk localizations. One should note, however, that our random walk localizations still outperforms classical counterparts in localization probability maximum at a fixed time. The counterparts always behaves as $P_{classical}(t) = t/N$ with t - number of checkups of individual nodes.

Finally, the three panels of Fig. 5.4.1 demonstrate a study of the disorder effect on two mechanisms of localization with the average over 20 spatial configurations. We show three different values of disorder interval limit $\Theta_{dis} \in \{0.3, 0.5, 1.0\}$. The small oscillations of the localization probability are not present in these plots, and the curves are smoother. The effect of reducing the maximum value of $P_{localization}$ is present, especially for large disorder with $\Theta_{dis} = 1$ in panel (c). The localization of non-topological QRW demonstrates a higher probability than both example walkers from topological QRW.

5.5. CONCLUSIONS

In conclusion, we demonstrated that a fast dynamical localization phenomenon takes place in the split-step discrete-time topological quantum random walks. It was shown that the localization mechanism is based on just two pairs of localized trapped states associated with the defect. Each pair of trapped states is particle-hole symmetry in the quasienergy spectrum. The time of highest localization is defined by the gap that separates two positive-energy trapped states. Due to a finite radius of localization, the position of these states in the spectrum is not influenced in large enough systems. We should note that the overlap product criteria served as an efficient tool to identify these trapped states, capture the evolution of the walker, and uncover the principle behind the distribution of high-localization peaks in the parameter space of the walker.

The described phenomena open an avenue towards the realization of the efficient dynamical localization based on topological QRWs as well as further studies on the explainability of localization phenomena in quantum random walks. The localization mechanism, as formulated in this work, can be immediately tested in photonic and synthetic lattice quantum walk experiments that realize topological QRW [19, 3, 22, 8].

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5.A. PROPERTIES OF SPLIT-STEP QUANTUM RANDOM WALK

In this section, we present a number of properties for split-step QRWs that realize topological phases in the case without defect nodes. Using Eq.(1) from the main text, and assuming the infinite system in both directions, we find the following equation for the spectrum of the translation-invariant unit cell:

$$\lambda^2 + 2\lambda \cos \frac{\Theta_2}{2} \left(\sin^2 \frac{\Theta_1}{2} \cos(2k_y) - \cos^2 \frac{\Theta_1}{2} \cos(k_x) \right) \quad (5.14)$$

$$+ \lambda \sin(\Theta_1) \sin \frac{\Theta_2}{2} (\cos(k_x - 2k_y) + 1) + 1 = 0. \quad (5.15)$$

Here, the momentum k_y was rescaled by a factor $\sqrt{3}/2$, as it enters the dynamical phases of shift operators only multiplied by y-components of vectors ν_2 and ν_3 . The solution of this equation, $\lambda_{\pm} = e^{iE}$, gives a particle-hole symmetric spectrum of quasi-energies. The quasi-energy is given by

$$E = \pm \arccot \frac{b}{\sqrt{4 - b^2}}, \quad (5.16)$$

$$b = 2 \cos \frac{\Theta_2}{2} \left(\sin^2 \frac{\Theta_1}{2} \cos(2k_y) - \cos^2 \frac{\Theta_1}{2} \cos(k_x) \right) \quad (5.17)$$

$$+ \sin \Theta_1 \sin \frac{\Theta_2}{2} (\cos(k_x - 2k_y) + 1) \quad (5.18)$$

The topological phases and corresponding Chern numbers were defined in Ref.[18]. The phase separation lines are characterized by the gap closing. For the technical purpose of identifying different types of localization speed up discussed in the main text, we plot a map of gap value: the minimal gap value between two bands at $E = 0$ or $E = \pm\pi$ (see Fig.5.A.1(a)) and the maximal gap value around the same energies. From such maps, we find that the full gap closing happens only at lines $\Theta_2 = 2\pi n$ with n being integer.

The colormap of localization dependence on defect parameters for the walker ‘1’ shown in Fig.2 of the main text is presented in Fig.5.A.2. The dependence on Θ_2^{def} is absent, which motivates the study of only different Θ_1^{def} values in the main text.

5.B. PRODUCT OF OVERLAPS WITH OPTIMAL REGIONS

In this Appendix, we analyze how precise the correspondence between maximization of $|\langle d|n\rangle\langle n|i\rangle|^2$ and $(\sum_{n=\max_1, \max_2} |\langle d|n\rangle\langle n|i\rangle|)^2$ is compared to the regions of optimal

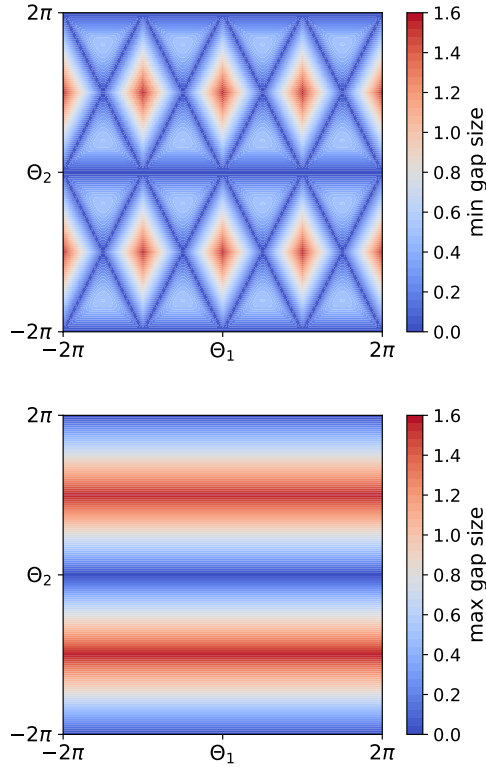


Figure 5.A.1: The regions of one gap closing (upper panel) and both gaps closing (lower panel). Note the agreement between the upper panel and phase separation lines position in Fig. 5.2.1a.

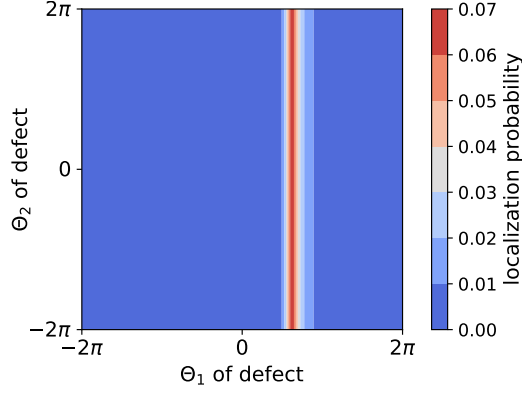


Figure 5.A.2: 2D color map of dependence on defect parameters for a walker '1'. The optimal localization event is performed on a line - which demonstrates that the dependence on Θ_2^{def} is absent.

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localization in parameter space for a fixed defect parameters. For this purpose, we compare the max localization probability curves plotted along two lines depicted as dashed cyan in the first panel of Fig.5.B.1. One of these lines is a phase separation line and the second one is parallel to phase separation and contains a walker '1' discussed in the main text. This second line crosses different topological phases. In the second and third panels of Fig.5.B.1 we plot the comparison of max probability found on time interval $T = 1000$ for the system size with $\sqrt{N} = 40$ with values of different overlaps. The second and third panels in Fig.5.B.1 clearly demonstrate that the combination of maximum and second maximum products of overlaps - the contribution of two pairs of trapped states - describes the probability peak most precisely. This leads to a conclusion that efficient localization is performed by two pairs of trapped states in this system in all cases studied.

Next, we analyze the structure of trapped states and their position in the spectrum for example walker parameters labeled as '2'. The analysis is similar to what is done in the main text and is presented in three panels of Fig.5.B.2: comparison of localization probability with overlap criteria in the first panel, the set of overlaps for each eigenstate of $\bar{U}$ in second panel and the comparison of probability evolution at the defect with the one defined by the two pairs of the trapped states in the third panel. However, in this case, we find that the two pairs of trapped states that mainly contribute to the efficient localization are placed near zero energy. Also, there is no gap in spectrum around this energy, but the in-gap trapped states placed around $E = \pm\pi$ do not contribute to the constructive interference relevant for successful localization.

5.C. STRUCTURE OF TRAPPED STATES IN COORDINATE SPACE

In this section we analyze how extended are the trapped states that mainly contribute to the localization events. From the localization time dependence on system size shown in Fig.2(b) in the main text, we conclude that the more extended trapped state has radius

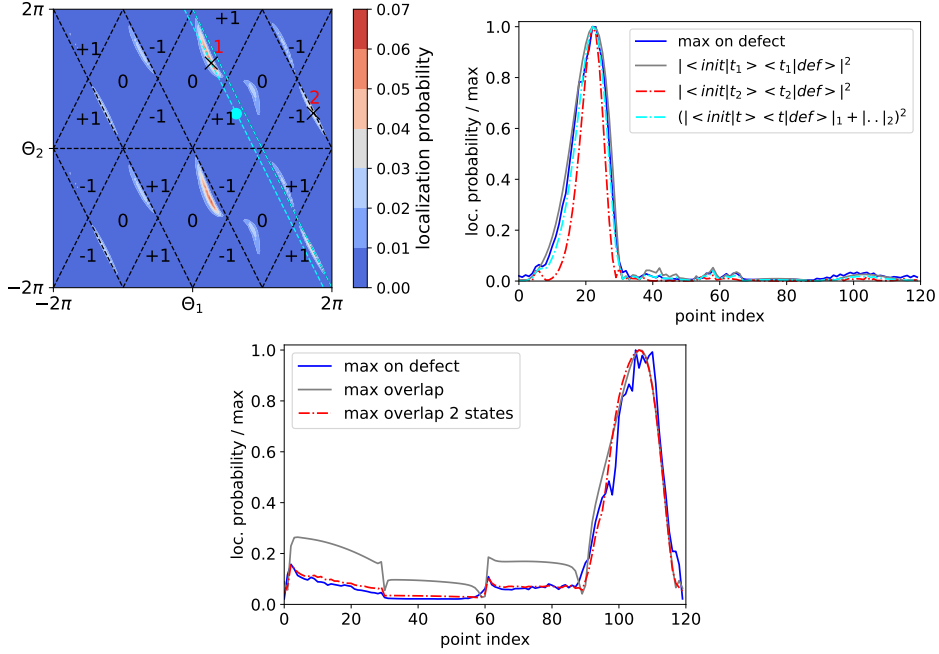


Figure 5.B.1: Comparison of maximized over time localization probability with a maximal product of overlaps described in the main text found overall eigenstates of evolution operator $\bar{U}$ for two lines in parameter space plotted as dashed cyan lines in the first panel. The second panel corresponds to a line going through walker '1' and the third panel corresponds to a phase separation line.

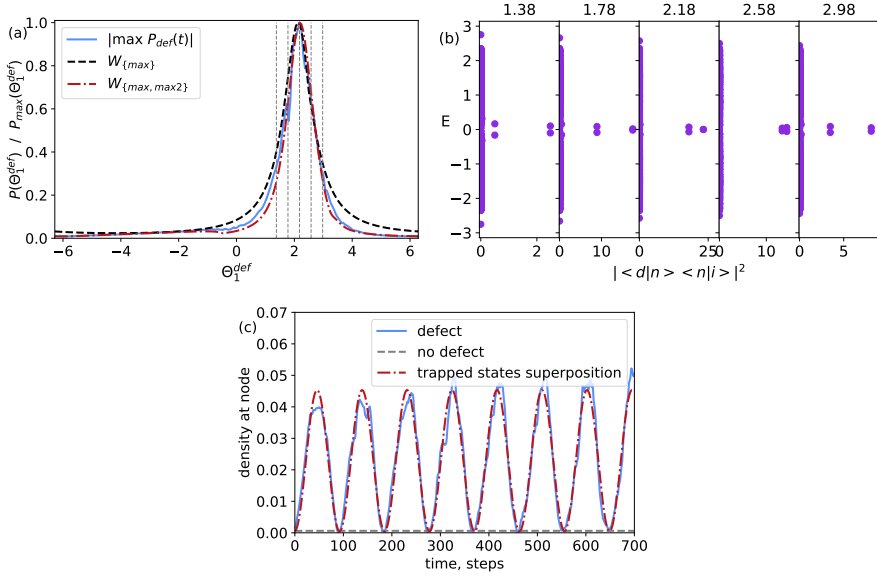


Figure 5.B.2: (a) The comparison of maximized over time localization probability with overlap criteria for walker ‘2’, with all quantities being normalized by their maximum value on the same grid of Θ_1^{def} parameters. (b) The identification of states that mainly contribute to constructive superposition on a marked defect node for walker ‘2’ parameters. (c) The comparison of density evolution at the defect node (blue solid line) with the superposition (red dashed line) of trapped states identified from panel (b).

$R \approx 50$ for walker '1' and $R \approx 100$ for walker '2' and defect parameter $\Theta_1^{def} = 5\pi/8$. The radius of the trapped state was estimated as twice smaller than the system size where the transition in time asymptotic happens. To further analyze the trapped state radius dependence on system size, we plot it in Fig.5.C.1. The set of tested system sizes covers the region where the trapped states are having overlap with itself due to periodic boundary condition. To reduce the numerical precision issue related to identifying the density of a quickly decaying state far from its center, we plot a dependence of radius which covers 80% and 95% of density. Two plots show results for a state taken from each pair of trapped states discussed in the main text in Fig.3(b). The states from the second pair have larger radii and it stronger depending on system size. Thus, the change in localization time asymptotic is mostly defined by this second pair of trapped states, which has a second maximal overlap value.

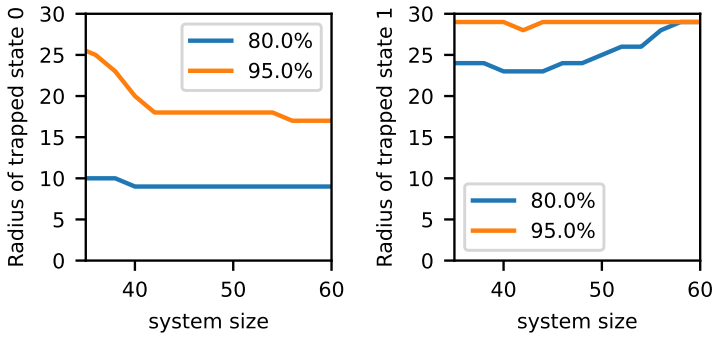


Figure 5.C.1: The radius of the trapped state as a function of system size. Left panel corresponds to a trapped state with largest overlap $|\langle d|n\rangle\langle n|i\rangle|$ in Fig.3(b) of the main text, and right panel corresponds to the state with the second largest overlap. Two curves correspond to the radius of the area in which 80% and 95% of state density are concentrated.

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6

CONCLUSION AND OUTLOOK

6.1. CONCLUSION

This thesis has explored the rich landscape of engineered topological quantum systems, with a particular focus on their controllability, interpretation and localization phenomena across various platforms and dimensions. The central research question guiding this work has been: *How can topological properties be engineered to stabilize quantum states and control quantum phenomena in realistic, finite-size systems?* Through four comprehensive studies spanning entanglement stabilization scheme, superconducting quantum circuits implementation, interpretation of finite size topology, and quantum random walks, this thesis has demonstrated that topological physics extends beyond theoretical idealization to practical quantum technologies.

The central research question of this thesis has been comprehensively addressed through both numerical analysis and experimental implementation. Topological properties can indeed be engineered to stabilize quantum states and control quantum phenomena in experimentally realizable system, but this engineering requires a nuanced understanding of how topology manifests in finite-size, imperfect systems. The key insight emerging from this work is that topological protection is not an all-or-nothing phenomenon but rather exists on a spectrum that depends on system parameters, and the specific physics properties one seeks to protect.

The stabilization of entangled states through topological modes demonstrates that quantum correlations can benefit from topological protection. Similarly, the gate-tunable phase transitions show that this topological property can be dynamically controlled, enabling a new method of quantum state manipulation. The finite-size effects studies reveal the practical and subtle boundaries of topologically trivial and non-trivial states, while the topological quantum walk investigations illustrate how quantum dynamics benefits from topology in complex ways.

6.2. OUTLOOK

The foundations established in this thesis open numerous exciting research directions that promise to further advances in both fundamental theory and device-level engineering of topological quantum systems.

6.2.1. NEAR-TERM RESEARCH OPPORTUNITIES

Scaling Topological Quantum State Stabilization: Building upon the demonstrated stabilization of quantum entanglement, a natural next step is to implement the experimental proposal outlined in Chap. 2, for example using superconducting circuits coupled via a transmon qubit. Additionally, the emergence of multipartite entanglement in topological systems presents an intriguing direction [18, 24]. By engineering such entangled topological states, one could explore new schemes for encoding quantum information. However, the scalability of these approaches remains an open question. The relationship between noise and system size must be carefully characterized before considering practical applications. In particular, we propose to quantify the noise scaling law in the form $\Gamma_{\text{deph}} \propto N^\alpha$, where Γ_{deph} is the dephasing rate and N the number of sites, using spectroscopy and process tomography.

Regarding the future research steps based on this project, one promising direction involves extending these ideas toward full quantum error correction protocols. While theoretical results indicate that topological codes can enable fault-tolerant quantum computation [15, 2, 20], but experimental realizations remain limited [23]. The superconducting platform developed in this thesis may serve as a suitable testbed for further investigations in this direction.

Exploit Quantum Walks for Quantum Information Purpose: The dynamical localization phenomena observed in two-dimensional topological quantum walks represent a compelling bridge between fundamental stochastic process theory and practical quantum information applications. The demonstration of robust localization through trapped states near lattice defects opens several promising research directions that extend naturally from the current work. The extension from two-dimensional to higher-dimensional quantum walk studies is particularly motivated by the fundamental dimensionality advantage in spatial search problems. As established by Aaronson and Ambainis [1, 4], quantum walk-based search algorithms achieve optimal $O(\sqrt{N})$ speedup for $d \geq 3$ dimensional lattices, while requiring $O(\sqrt{N} \log N)$ time in $d = 2$, and offers no speedup possible in $d = 1$. This dimensional scaling suggests that the topological localization mechanisms identified in our 2D studies could be significantly enhanced in three-dimensional systems, potentially offering both improved search efficiency and richer topological protection. Furthermore, continuous-time quantum walk approaches [7] provide an alternative framework where the interplay between topological properties and dimensionality could yield novel localization phenomena with distinct advantages over discrete-time implementations.

Beyond dimensional scaling, the multi-walker paradigm presents opportunities to explore multi-partite entanglement phenomena between quantum walkers, though the specific mechanisms by which topological protection influences entanglement dynamics remain to be established. The interaction between multiple quantum walkers in

topological environments may generate complex entanglement structures, but quantifying their computational utility and decoherence resilience requires careful analysis of the underlying Hamiltonian dynamics and environmental coupling. We propose modeling interacting coined-walk Hamiltonians with tunable defect potentials, and quantifying entanglement via logarithmic negativity or Rényi entropy, to identify when topological protection enhances coherence. Multiple defect encoding schemes could store distributed information, but the interference between localized states must be understood.

Furthermore, the localization properties demonstrated in single-defect scenarios suggest potential applications in multiple node marking schemes, where different topological defects could simultaneously encode distinct pieces of information. However, the scalability of such encoding schemes depends critically on understanding the interference effects between multiple localized states and their collective response to disorder. While the demonstrated robustness against disorder is promising, practical quantum search and database applications will require quantitative bounds on fidelity degradation and careful optimization of the trade-off between localization strength and search efficiency [7, 1]

Machine Learning-Enhanced Topological Characterization: The challenges in the understanding of finite-size topological systems revealed in this work suggests that machine learning approaches could be used to enhance our ability to characterize and predict topological properties. Recent work [19, 13, 21] show that physics-informed neural networks (PINNs) and graph neural networks (GNNs) trained on momentum-space data, with constraints enforcing gauge invariance and symmetries, could reliably extract global geometric information from noisy data. In the context of topological matter, this enables the extraction of topological features (like Berry curvature profiles or entanglement structure) even in noisy or partially observed data.

Meanwhile, the interpretability of machine learning models is becoming a growing concern in the physics community [8, 10]. Rather than applying machine learning methods as a generic classifier with minimal insight, there is a growing interest in aligning the algorithmic structure with the mathematical foundations of topological invariants. For example, in our QMAI group, we are developing models that are explicitly constructed on the mathematical components of topological quantities, such as winding numbers or Chern numbers, and we constrain the learning process to respect these known structures. This avoids the pitfalls of treating machine learning methods as a black box and enables us to more precisely assess what kind of topological information the model learns, and under what conditions it succeeds or fails. This physically constrained formulation will deepen our understanding of how topology emerges from data, and how robust such learning is to noise, symmetry breaking, or finite-size effects.

6.2.2. LONG-TERM RESEARCH DIRECTIONS

Topological Quantum Sensing: The robustness of topological states against local perturbations paradoxically makes them excellent candidates for sensing global changes in system parameters. Topological quantum sensors could achieve unprecedented sensitivity to magnetic fields, electric fields, and other physical quantities by exploiting the

sharp transitions between distinct topological phases, where small parameter changes can cause dramatic shifts in measurable quantities such as quantized conductance, mid-gap edge states, or topological invariants like Chern numbers [6, 12]. While the fundamental physics supports this approach, significant challenges remain in translating topological sensitivity into practical sensing devices, including the need for precise control over system parameters, maintaining coherence at operating temperatures, and developing robust readout mechanism that can distinguish topological signals from conventional noise sources. Nevertheless, the potential for quantum-enhanced sensitivity through topological protection represents a promising frontier where fundamental condensed matter physics could enable transformative sensing technologies.

Topological Quantum Internet: One of the ultimate vision emerging from this field is the development of a topological quantum internet where quantum information is transmitted and processed using topologically protected channels. Such a network would combine the long-range entanglement distribution capabilities of quantum networks [14, 22] with the local robustness of topological protection.

Key technical challenges include the development of topological quantum repeaters, for in stance using the base designed presented in Chap. 2, creating well-defined interfaces between topological and conventional quantum systems, and establishing protocols for topological quantum communication. We outline a three-layer architecture: (1) physical layer—protected transmission via edge channels; (2) link layer—topological quantum repeaters based on Chap. 2; (3) application layer—fault-tolerant distributed quantum computing.

Emergent Topological Phases in Quantum Many-Body Systems: Future research should explore how topological phases emerge dynamically in interacting quantum many-body systems far from equilibrium, including many-body localized regimes [17, 3] and pre-thermal topological states [9, 5], characterized via out-of-time-order correlators and entanglement growth rates.

6.2.3. SPECIFIC TECHNICAL RESEARCH DIRECTIONS

Based on the findings and methodologies developed in this thesis, several specific technical research directions warrant immediate attention:

Advanced Characterization Techniques: Develop new experimental probes for topological properties in finite-size systems. The bulk conductivity criterion introduced in this thesis could be extended to include dynamic measurements, spectroscopic techniques, and transport-based characterization methods. Particular attention should be paid to developing techniques that can distinguish between topological protection and other forms of robustness.

Disorder Engineering: Systematically investigate how controlled disorder can be used to enhance rather than degrade topological properties. Recent theoretical work suggests that certain types of disorder can drive trivial systems into topological phases [16,

11]. Both uncorrelated (Anderson-type) and correlated disorder should be taken into account. The controllable platforms developed in this thesis provide ideal systems for testing these predictions.

6.2.4. FINAL CONCLUSION

Overall, This thesis has demonstrated that topological quantum systems represent a convergence of fundamental physics and practical quantum technology. The robustness properties that make topological systems fascinating from a theoretical perspective also make them valuable for technological applications. However, realizing this potential requires careful attention to the details of how topology manifests in real, finite-size systems with imperfections and external control requirements.

The path forward involves both deepening our fundamental understanding of topological phenomena and developing new engineering capabilities that are necessary in order to transform these phenomena into concrete quantum technologies. The controllable platforms and characterization techniques developed in this work provide important tools for this endeavor, but significant challenges remain.

The ultimate success of topological quantum technologies will depend on our ability to bridge the gap between idealized theoretical models and practical implementations. This thesis has shown that this bridge can be built, but it requires careful attention to finite-size effects, disorder, controllability, and the intricate relationships between different types of quantum correlations.

As we look toward the future, topological quantum systems represent one of the most promising pathways toward robust, scalable quantum technologies. The foundations established in this work provide a solid base for the continued development of this exciting field, with implications that extend far beyond the specific systems and phenomena studied here. The journey from topological theory to topological quantum technology is well underway, and the work presented in this thesis represents important milestones along this path.

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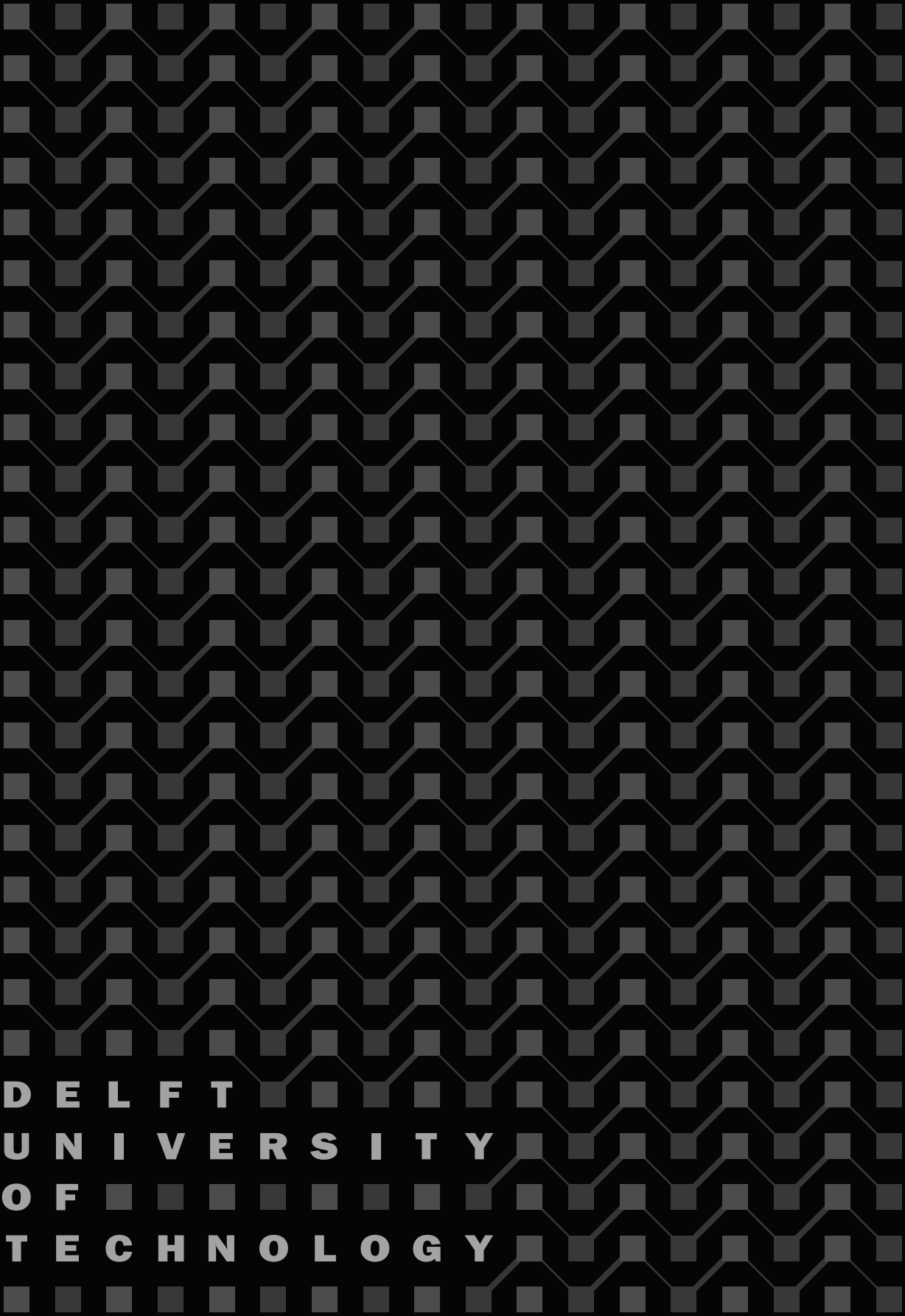
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LIST OF PUBLICATIONS

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