System Identification in Dynamic Networks

A.G. Dankers

SYSTEM IDENTIFICATION IN DYNAMIC NETWORKS

PROEFSCHRIFT

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The last four years have not been without ups and downs, but I look back at the experience with pleasure. I have gained a much deeper understanding about system identification, and I see many opportunities for further developments, and applications of the theory to tackle problems that engineers and scientists are facing. I start the next step in my life with a lot of motivation and optimism.

Delft, August 2014
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Chapter 1

INTRODUCTION

Due to advancing technology, systems in engineering are becoming increasingly complex and interconnected. Despite the ubiquity of systems that can be modelled as interconnected systems, the field of system identification is still mainly focused on open-loop and closed-loop systems, limiting the application of system identification tools beyond a classical control systems framework. This thesis aims to extend the theory of closed-loop identification to the identification in dynamic networks. The main question that guides the research is: under what conditions is it possible to consistently identify the dynamics of a particular module of interest that is embedded in the network? As the reader progresses through the thesis the conditions/assumptions become less restrictive. The types of conditions and assumptions that are considered are: the presence of process noise, presence of measurement noise, which variables are measured/measureable, and the intersample behavior. In this first chapter dynamic networks are introduced from a broad point of view. The notion of causality is briefly discussed since it is at the foundation of modeling a dynamic network. The chapter concludes with a detailed presentation of the goals of this thesis.

1.1 INTRODUCTION

SYSTEMS IN ENGINEERING are becoming more complex and interconnected. Consider for example, power systems (Kundur, 1994), telecommunication systems (Proakis & Salehi, 2001), distributed control systems (Ren & Beard, 2008), windfarms (Johnson & Thomas, 2009; Soleimanzadeh & Wisniewski, 2011), etc.. Many of these systems form part of the foundation of our modern society. Consequently, their seamless operation is paramount. However, the increasing complexity and size of the systems poses real engineering challenges. Consider for example, maintaining stability of the electrical power grid and increasing data throughput of telecommunication networks. These systems cannot be operated, designed, and maintained without the help of models.

Also due to advancing technology, it is becoming possible/easier to collect an increasing array of different types of data. Sensors are becoming cheaper, more sensitive, and able to measure many different types of variables such as temperature,

pressure, flow rates, etc. For example, due to dropping costs, phase measurement units (PMUs) are being deployed in the power grid to synchronously sample and measure voltage and current at many locations in the electrical grid. Similarly, in reservoir engineering, new sensors have been developed that can be permanently placed at the bottom of a wellbore so that it is possible to continuously measure pressure, temperature, flow rate, and fluid density directly at the interface between the reservoir and the wellbore. The point is that in many practical situations measurements of many interrelated variables (i.e. variables that form part of a dynamic network) are available.

The main topic of this thesis is to use the data collected from a dynamic network to construct a model of the system.

It is worth spending some time on the notion of a model. A model describes the interaction between the variables of the system. In engineering, common examples of variables that form part of a model are *flow variables* such as current, fluid flow rate, and velocity, *potential variables* such as voltage, pressure, and force, and other types of variables such as temperatures, altitudes, etc. A model is a tool that is an abstraction of reality, constructed with a specific purpose in mind.

There are many different purposes for models in engineering such as system design, prediction, simulation, measurement and diagnosis. (Ljung, 1999).

System Design. A model of a system can be used to design the system such that it achieves a behavior that is (close to) the desired behavior. For instance, if a mechanical system appears to be vulnerable to mechanical vibration, the mass of the system could be increased. Another method to achieve a desired behavior is the use of controllers. In order to design a controller, a model of the system to be controlled is required. Using this technique an active vibration controller could be designed to reduce mechanical vibrations in a system.

Prediction. A model could be used to make predictions about the future behavior of a system given a particular set of inputs and operating conditions. For instance, electricity usage, the effect of various inputs and operating conditions on the efficiency of a power plant, etc. Such models are used so that an operator can make informed decisions as to how to best manage a system.

Simulation. A model can be used to simulate the behavior of a system. Such a model can be used to train operators of complex systems (flight simulators, power plant simulators, etc.).

Measurement. A model can be used to estimate the value of an unmeasurable variable (such as estimating the acceleration of a system by measuring its velocity and position).

Diagnosis. A model can be used to detect faults in a system (deviations from normal operation). In an electrical system examples of faults are transmission lines breaking. Often the type and the location of the fault must be detected.

Models are, of course, not used exclusively in engineering, but have wide usage in all domains of science. In science models are constructed based on theories, and the models are compared to what is observed in nature. Or, vice versa, a model is constructed that fits a set of observations and then the model is interpreted in order to explain what mechanisms cause the observations. The latter methodology is common in fields like economics, where fundamental laws governing the system

are difficult to define. The typical purpose of models in science is to explain how particular variables interact, in order to gain insight into how the world works.

> "Inferring models from observations and studying their properties is really what science is about. The models ('hypotheses', 'laws of nature', 'paradigms', etc.) may be of more or less formal character, but they have the basic feature that they attempt to link observations together into some pattern" (Ljung, 1999)

When modeling large, complex systems, that describe the inter-relationships between many different variables, it is invaluable to impose a structure on the system. The variables are partitioned into subgroups, such that each subgroup only directly interacts with a small subset of other subgroups. The result is a network. When the variables are also temporally related, the result is a *dynamic network*.

1.1.1 Dynamic Networks

In this thesis we consider dynamic networks of the following form. A network consists of a collection of *internal variables*. Each internal variable is dynamically related to other variables in the network. Each internal variable is assumed to be *measureable* in the sense that it can be measured using an appropriate measurement device. Examples of internal variables are pressure, flow rate, voltage, current, position, force, etc. There may also be *unmeasured disturbances* present that affect the value of an internal variable. Examples of unmeasured disturbances are thermal noise in an electrical circuit, and wind affecting a mechanical structure. A third type of variable that may be present in a dynamic network is an *external variable*. This is a variable that can be directly manipulated by an operator of the system. For example a valve may be opened or closed by an operator or a voltage in a circuit may be controlled using a waveform generator.

These concepts will be formally defined in Chapter 2. For now, consider the following examples of dynamic networks taken from different engineering domains: control, power and reservoir engineering.

The first example is a distributed control system. Consider the case of a team of mobile robots as shown in Figure 1.1. Each vehicle can communicate with other vehicles within a limited range.

Suppose that the objective of the team is to move in a certain formation along a pre-specified path. One control scheme is that each vehicle communicates with a centralized controller. This controller co-ordinates the manoeuvres for each vehicle. The centralized control scheme requires that each vehicle is constantly in contact with the central controller, which could be an unrealistic or burdonesome requirement. Secondly, this scheme does not scale well with additional vehicles.

An alternative approach is a *distributed control scheme*, where each vehicle only communicates with its neighbors (Ren & Beard, 2008). The vehicles communicate amongst themselves to achieve a consensus as to what the next manoeuvres should be. Extra vehicles can easily enter and leave the team, and only a limited communication range is required. Further reading on distributed control of multi-agent systems can be found in Ren & Beard (2008).

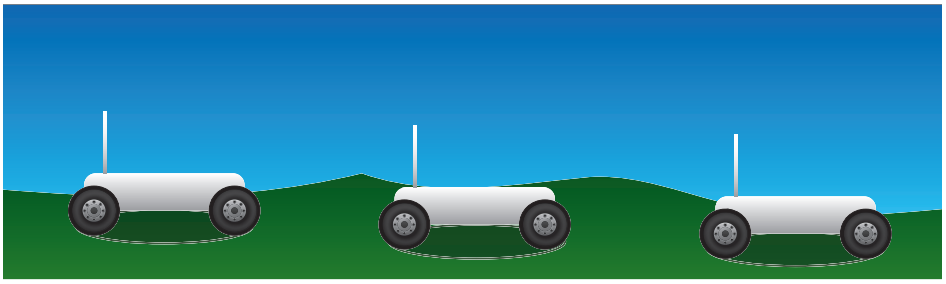

Figure 1.1: Diagram of a team of mobile robots.

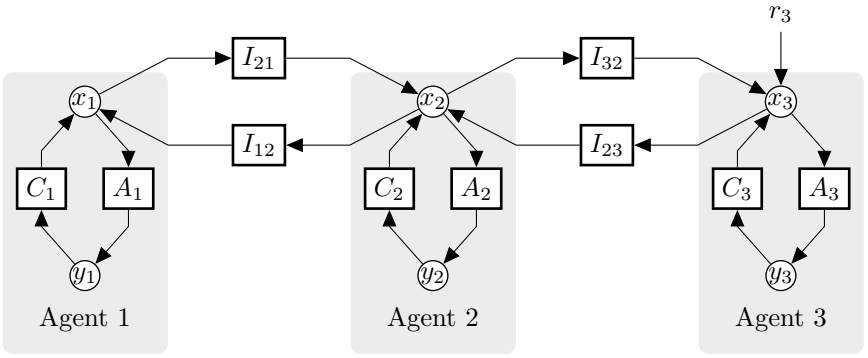

Figure 1.2: Model of the multi-agent system shown in Figure 1.1 using transfer functions. The variable x_i denotes the state of agent i (could be position and velocity for instance); y_i denotes the output of agent i; and r_3 is an external reference variable. The transfer function A_i denotes the internal dynamics of agent i, C_i denotes the controller dynamics of agent i, and I_{ij} denotes the dynamics of the interconnection between agents.

In Figure 1.2 a model of a distributed control system is shown. Each block in the figure represents a transfer function and circles denote internal variables. The main point is that a distributed control system can be modelled as a dynamic network.

A second example of a dynamic network is taken from reservoir engineering, shown in Figure 1.3. In the figure, three wells are shown that are drilled into a reservoir. The pressure of the fluid in the reservoir is very high, driving the fluid up through the wellbores. The rate of the flow in the wellbores is controlled by a valve at the wellhead. The variables that are used to model the system are the pressure and flow rate at the well heads and the pressure and flow rates at the bottom holes of the wellbores. The properties of the fluid in the reservoir creates a dynamic interconnection between the flow rates in the three wellbores. For instance, suppose that the flow rate in one of the wellbores is allowed to be very large, this would cause the flow rate to drop in the surrounding wellbores. Reservoir engineers would like

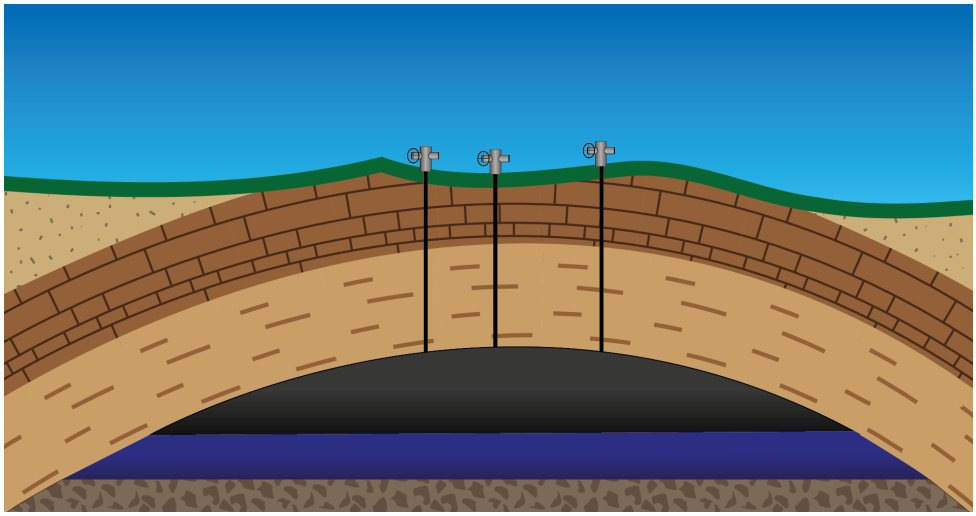

Figure 1.3: Diagram of three wells. The brown layers denote layers of rock, the black layer denotes a permeable layer where oil has gathered, and the blue layer denotes a water layer. The rock formation shown is an *anticline* formation. Reservoirs often are found in this type of formation (Dake, 1983).

to infer characteristics of the reservoir such as depth of the reservoir, permeability of the rock in the reservoir, shape of the reservoir, presence of faults in the geology of the reservoir, etc., based on the observed pressure and flow rate data.

A model of the three wells is shown in Figure 1.4. Again, the main point is that this situation can be described by a dynamic network. This particular example will be further worked out in Chapter 8.

The third example of a dynamic network is taken from power engineering. A simple power system consists of generators, transmission lines, and loads as shown in Figure 1.5. The transmission lines provide an interconnection between the generators and the loads. The variables that can be included in a model of a power system are voltage, current at different locations in the transmission line, and the torque and radial velocity of the shaft of the generator.

Again, this situation can be modelled as a dynamic network as shown in Figure 1.6.

The main point of this section is that we have illustrated that systems in engineering can be modelled as systems of interconnected transfer functions, i.e. dynamic networks.

In Willems (2008) a nomenclature is proposed to talk about dynamic networks. This nomenclature is adopted in this thesis. Thus, each transfer function is referred to as a *module*. Each module is *embedded* in a dynamic network via an *interconnection structure*.

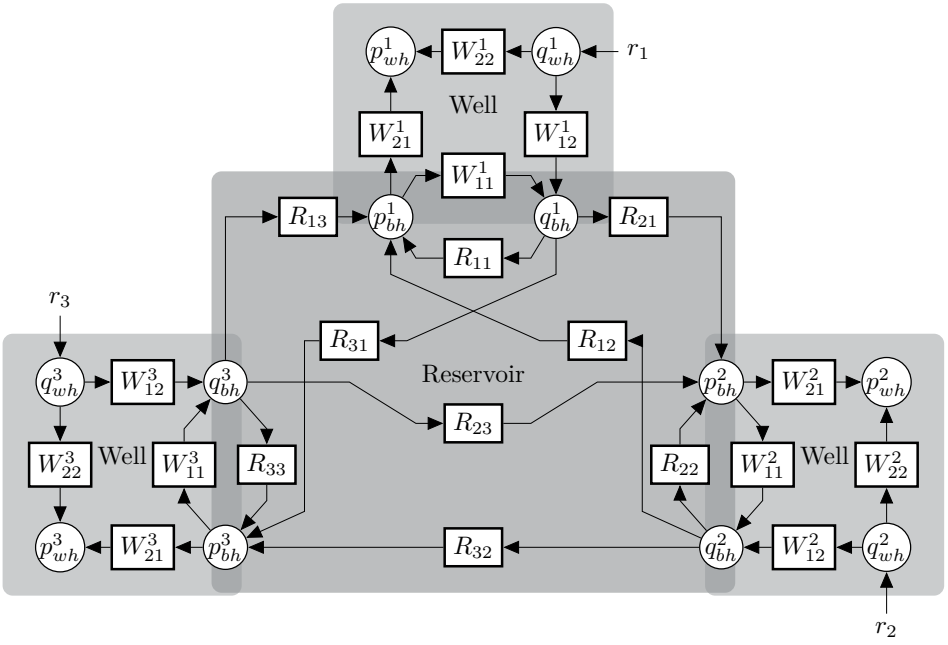

Figure 1.4: Model of the three wells and reservoir shown in Figure 1.3 using transfer functions. The variables p^i_{bh} and p^i_{wh} denote bottomhole and wellhead pressures of well i respectively; q^i_{bh} and q^i_{wh} denote bottomhole and wellhead flow rates respectively; r_i denotes the flowrate that can be externally controlled using a valve. The transfer functions W_{ji} denote the dynamics of wellbore k, and R_{ji} denote the dynamics of the reservoir.

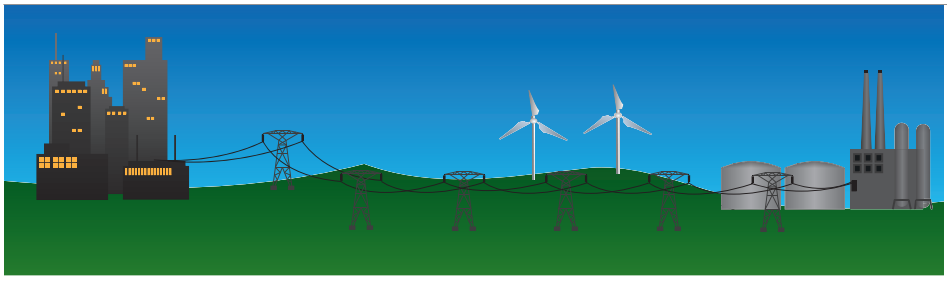

Figure 1.5: Diagram of a power system. Power is generated by power plants and wind turbines. The load (city) is connected to the power source by transmission lines.

1.1.2 System Modeling

There are many ways to determine the dynamics of each module embedded in the network. One could attempt to construct a model based on the physics of the

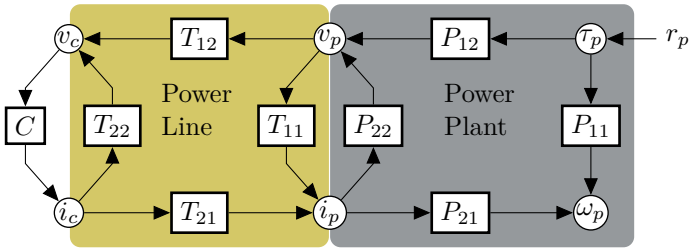

Figure 1.6: Model of the power system of Figure 1.5 using transfer functions. The variables v_p and v_c denote the voltage on the transmission line at the plant and city respectively; i_p and i_c denote the current on the transmission line at the plant and city respectively, τ_p and ω_p denote the torque and radial velocity of the turbine shaft of the power plant. The transfer functions T_{ji} denote the dynamics of the transmission line; P_{ji} denote the dynamics of the power plant; C denotes the dynamics of the load (i.e. the city).

system (sometimes referred to as *first principles modeling*). In this case the equations relating all the variables of the system are derived based on principles in physics (such as the laws of conservation of energy and mass, Newton's first law, etc). This approach may not be feasible for complex systems, or for systems where the physics are not known (to a sufficiently accurate degree). Using this procedure good insight into the system is developed in the sense that it is clear which parameters (such as temperature, permeability, etc.) affect the dynamics of the system.

1.1.3 System Identification

A different approach is to use measurements of the variables of the system to infer a model of the system. This is called *system identification*.

There are three main components in the system identification procedure: (1) the data set, (2) the set of candidate models and (3) the selection of the "best´´ model from the set of candidate models (Ljung, 1999; Söderström & Stoica, 1989a; Pintelon & Schoukens, 2012b). Each component is briefly described below.

The Data Set. In this step the user must decide which variables to measure, and how to measure the variables. It may be possible to excite the system using a signal which is designed by the user.

The Model Set. In this step the user must choose a model structure. The model structure is a parameterized mapping from the inputs and past outputs to the output at the current time. The user has many choices to make in this step. For instance one must choose linear/nonlinear, discrete/continuous, gray box/black box, parametric/non-parametric, what type of noise model to include, how many poles and zeros to include in the model, etc..

Identification Criterion. In this step the user must choose a rule which can discriminate between the candidate models and select the best model. A typical choice is to asses the candidate model quality using the sum of squared prediction

errors.

Once all the choices in the three steps have been made numerical algorithms can be applied to the data to obtain a model.

A large portion of the identification literature deals with the analysis of the obtained models. The two most important properties of an estimated model are its *bias* and its *variance*. It is crucial to know under what conditions a proposed identification procedure will (or will not) lead to an unbiased model. Similarly, it is just as important to understand how to user choices affect (reduce) the variance of an estimated model.

An important factor (if not dealt with properly) that can contribute to the bias of an estimated model is the presence of feedback in the data generating system. Consequently several *closed-loop identification* methods have been proposed in the literature (Forssell & Ljung, 1999; Van den Hof, 1998; Ljung, 1999).

The majority of the identification literature deals with systems that are operating in either open-loop or closed-loop. The main topic of this thesis is to move beyond identifying open and closed loop systems to identifying modules that are embedded in dynamic networks.

1.1.4 System Identification and Dynamic Networks

System identification as presented in the previous section seems to offer huge potential in a world where data collection is becoming easier and cheaper for a wide variety of systems, and the demand (need) for accurate models of these systems is growing. One of the main aspects that limits the applicability of system identification tools to a broader range of systems is the lack of tools that are dedicated to dynamic networks. In the current system identification literature data generating systems are typically considered to be either open or closed loop. There are considerable advantages to taking into consideration the structure of the system under investigation, as will be shown in this thesis. For example, the ease with which prior knowledge (such as known controllers) can be incorporated into the model[1], the increased flexibility in which variables need to be measured[2], and the (some what surprising) fact that sensor noise is easily dealt with in a network setting (unlike in the classical open-loop errors-in-variables problem)[3]. In addition, a network setting offers interesting opportunities for reducing the variance of the estimates that are not available in a classical open or closed-loop setting Wahlberg et al. (2009); Everitt et al. (2013); Gunes et al. (2014).

When given a data set generated by a dynamic network, one of the first questions that the user is confronted with is how to determine which variables cause which. Does a change in voltage cause a change in current, or vice versa? If there are feedback loops present in the system, what does this mean in terms of causality? Which variables should be classified as “inputs” and which variables should be classified as “outputs”. Does there exist a causal structure in the data, or can a causal structure be imposed by the user? For this reason, although this is not the main topic of

[1] This is discussed in Chapter 4.
[2] This is discussed in Chapter 5.
[3] This is discussed in Chapter 6.

this thesis, we include a short section on causality in this introductory chapter. The presence of a causal structure in the data is at the very foundation of identification in dynamic networks, and so it is worth spending some time on investigating what role causality takes in both system modeling and system identification.

The rest of this introductory chapter is structured as follows. After a section briefly discussing causality in Section 1.2 the current state of the art of dynamic network identification is summarized in Section 1.3. The literature on identification and dynamic networks can be split into two categories: those that assume that the interconnection structure is known (referred to as *identification in dynamic networks*), and those that assume it is unknown (referred to as *topology detection*). In scientific applications it is perhaps more common that the interconnection structure is unknown, and the goal is to discover the interconnection structure in order to gain a deeper understanding of the system under investigation (for example in systems biology, ecology, economics, etc.). In engineering applications, it is more common that the interconnection structure is known since it has been explicitly designed (for example power systems, telecommunication systems, etc.). Both categories of the literature are summarized.

This chapter ends with Section 1.4 where the problem statement of this thesis is presented in detail. The motivating question is stated and the main contributions contained in this thesis are presented.

1.2 CAUSALITY

In this section we briefly discus the notion of causality. Causality is not the main topic of this thesis, however it implicitly plays a central role in a dynamic network model. The direction of the arrows in each of the Figures 1.2, 1.4 and 1.6 denotes a causal relationship.

The topic of causality is a very interesting one, and quickly leads down a very philosophical path. It is not without controversy. There are many different definitions (Granger proposes 7 different definitions in just one article (Granger, 1980)). The debate ranges from "there is no such thing as causality" to "causality is the most basic foundation of the world we live in". In his book *Causality*, Pearl (2000) says:

> "Though it is basic to human thought, Causality is a notion shrouded in mystery, controversy, and caution, because scientists and philosophers have had difficulties defining when one event truly causes another. We all understand that the rooster's crow does not cause the sun to rise, but even this simple fact cannot easily be translated into a mathematical equation."

Willems (2007), on the other hand argues that cause and effect should not play a role in system modeling, even going so far as to quote Bertrand Russell:

> "The law of causality, I believe, like much that passes muster among philosophers, is a relic of a bygone age, surviving, like the monarchy, only because it is erroneously supposed to do no harm" (Russell, 1912)

In the literature, causality has been referred to as: “due to”; “temporally prior”; “temporally interrelated” (Granger, 1980); “feedback free” (Caines & Chan, 1975); and “non-anticipatory” (Polderman & Willems, 1991). Identification in dynamic networks is fundamentally based on the notion that we are able to determine which variables cause which. So what should we make of this controversy? How should we define causality?

In this section we explore what some of the leading thinkers have thought about causality and we attempt to consolidate their seemingly contradictory beliefs into one line of reasoning. This will give insight into foundation of identification in dynamic networks.

Willems’ argument is that from a system modeling perspective there is no reason to label some variables as inputs, and others as outputs. In fact, he proves that any variable in a linear time invariant system can be considered as both an input or an output (Polderman & Willems, 1991). For instance, consider an electrical circuit. The variables of the system are the current and voltage at different locations in the circuit. One could choose current as the “input” to a particular component, which implicitly means that voltage is the “output” of the component. However, one could just as easily chose voltage as the “input”, which means that current is the “output”. Since the choice of input/output partition is a free choice, the idea of causality does not make sense in this setting.

Now consider a system identification perspective. Consider a set of measurements from a system with no additional knowledge about the data generating system. No additional knowledge, does not mean no knowledge! Given that fact that a set of measurements has been collected from the system already says something about the system: (1) the system forms part of/is connected to the real world, i.e. boundary conditions have been applied to the relevant equations, (2) something is driving the system, i.e. something is causing the variables to be non-zero and (3) a transfer of energy has occurred.

The main point is that items (1) - (3) imply that a particular input/output partition of the variables has generated the data. Once an input/output partition has been chosen, and the proper boundary conditions have been applied, it makes sense to talk about causality. Thus, a causal structure is present in a data set.

This idea is illustrated by an example using a transmission line.

1.2.1 A Transmission Line

This example is largely taken from personal communication with Jan Willems and partially from Willems (2010) where Willems discusses the behavior of transmission lines. More of his ideas on the behavioral framework in interconnected systems is found in Willems (1997, 2007, 2008). The main point of this subsection is to illustrate that the concept of causality may not be very meaningful at a system equations level, however at a measurement level it is.

Consider a transmission line as shown in Figure 1.7. Intuitively, we are used to thinking about transmission lines as simple input/output devices (modelled as transfer functions). For instance, suppose a voltage signal is applied to the left hand side of the line, then the voltage signal will appear, slightly delayed, on the other

end of the transmission line. In fact, for simplicity, let us assume that the transfer function of the transmission line is simply a delay, i.e. $G(q) = q^{-1}$. Denote the input voltage as $u = P_1 - P_2$ and denote the output voltage (i.e. the voltage at the other end of the transmission line) as $y = P_3 - P_4$. Then $y = Gu$, or as expected $y(t) = u(t-1)$ (i.e. the output y is simply a delayed version of the input). Intuitively this is how a transmission line works.

By this reasoning, and by the usual rules for manipulating transfer functions, it is then also possible to say that $u = G^{-1}y$. In this case, since G is simply a delay G^{-1} is a non-causal transfer function. This is not as expected! Intuitively, it should be that if a signal y is applied to the right hand side, the signal appearing on the left should be simply a delayed version of y, i.e. $u = Gy$, not $u = G^{-1}y$ as suggested by the transfer function reasoning. Where is the error in this line of reasoning?

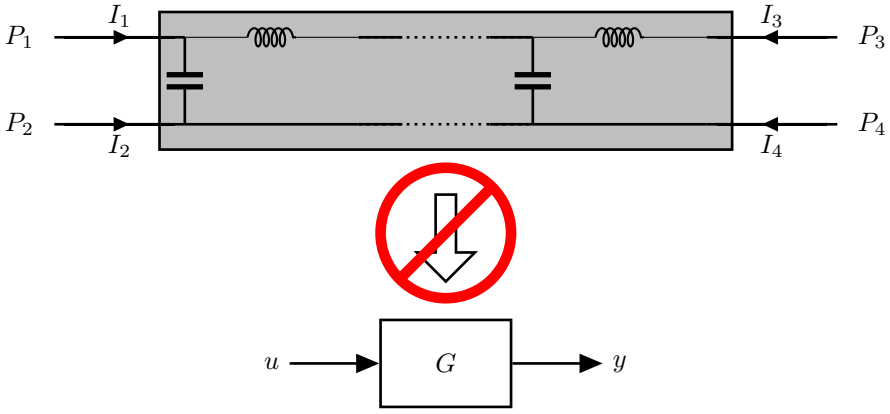

Figure 1.7: A lossless transmission line. I's denote currents, and P's denote voltage potentials and a transfer function representation of the transmission line.

In order to find the mistake in the reasoning, consider how the transfer function representation of a system is derived from the governing differential equations. The equations governing a lossless transmission line are (O'Neil, 2007):

$$\frac{\partial}{\partial x}V(t,x) = \frac{\partial}{\partial t}I(t,x)$$
$$\frac{\partial}{\partial x}I(t,x) = \frac{\partial}{\partial t}V(t,x)$$

where unit inductance and capacitance are chosen for simplicity, $V(t,x)$ is the voltage at time t and position x measured with respect to the return wire, and $I(t,x)$ is the current in the transmission line are time t and position x. It can be shown that (O'Neil, 2007)

$$\begin{aligned} V(t,x) &= f_+(t-x) + f_-(t+x) \\ I(t,x) &= -f_+(t-x) + f_-(t+x), \end{aligned} \tag{1.1}$$

where f_- and f_+ are any functions. The function f_+ represents a wave traveling in the forward direction, whereas f_- represents a wave traveling in the reverse direction. The voltage at any point in the line is a sum of both the forward and the backwards waves. Let the voltage at the start of the line (i.e. at $x = 0$) be denoted $w_1(t) = V(t, 0)$, and the voltage at the other end of the transmission line of length 1 be denoted $w_2(t) = V(t, 1)$. Then,

$$\begin{aligned} w_1(t) &= f_+(t) + f_-(t) \\ w_2(t) &= f_+(t-1) + f_-(t+1) \end{aligned}$$

which shows that there is a noncausal relationship between w_1 and w_2.

Where does the intuition that a transmission line acts as a causal transfer function come from? A voltage signal is injected into the line at one end, and appears delayed at the other end. This however, requires a proper termination (a boundary condition).

Suppose that the transmission line is terminated at $x = 1$ with a unit resistor. By Ohm's Law, this imposes the following constraint:

$$V(t, 1) = -I(t, 1).$$

By (1.1) this is equivalent to imposing:

$$f_+(t-1) + f_-(t+1) = f_+(t-1) - f_-(t+1),$$

resulting in $f_-(t + 1) = 0$. Here, the transmission line has been terminated by its characteristic impedance, with the result that there is no reflected wave.

What is the moral of the story? The equations governing a transmission line are non-causal. However, if a voltage input is imposed at $x = 0$, and the voltage across a terminating resistor is measured, the result is a causal relationship $y(t) = f(u(t-1))$. Conclusion: it is not the transmission line that acts like an input/output device, but the transmission line together with a termination. This architecture breaks the perceived symmetry of a transmission line as shown in Figure 1.8.

To be able to collect a data set from a transmission line, it follows that there must be a source connected to the line, and there must be a termination on the line. Thus, given a data set, if one were to identify the transfer function from $u \to y$, one would obtain an estimate of G. Alternatively, if one were to identify a transfer function from $y \to u$, one would obtain an estimate of G^{-1}, as originally expected from the transfer function reasoning.

The main point is that, given a data set, the conditions required to talk about causality in a meaningful way are implicitly satisfied.

In summary, what is the main insight offered by this example?

- In order to generate data, a generating mechanism needs to be in place. In the case of a transmission line that means terminating one end of the line and attaching a signal generator to the other end.

- The generating mechanism destroys any symmetry that the differential equations may have.

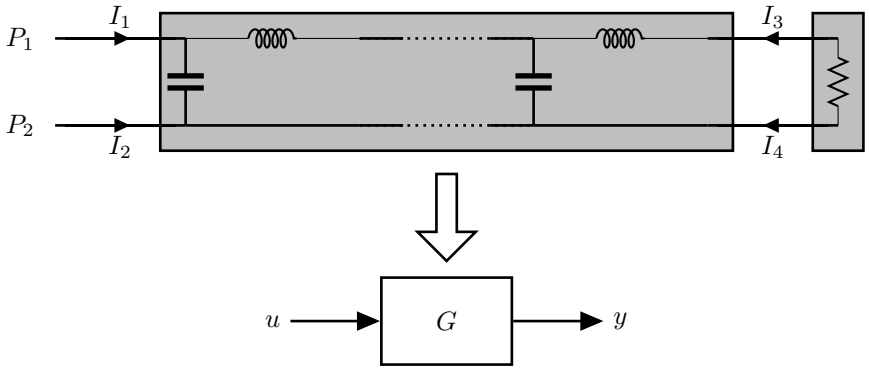

Figure 1.8: A lossless transmission line with a termination. I's denote currents, and P's denote voltage potentials and a transfer function representation of the transmission line.

- The notion of a transfer function can only be used once a generating mechanism has been defined.

The main conclusion from a system identification point of view: although the notion of causality may not be present in the differential equations used to model a system, a causal structure is present in the data collected from the system.

Given a data set of measured variables, just because it is possible to relate two variables through a (proper) transfer function does not necessarily mean that the input is a cause of the output. This, along with a proposed definition for causality is briefly discussed in the next subsection.

1.2.2 Granger Causality and Causal Inference

Two large bodies of literature that attempt to define operational definitions of causality are *Granger Causality* and *Causal Inference*. An *operational definition* of causality means that the definition can be used for detecting causality from a data set. Detecting which variables cause others from a data set is not trivial. The difficulty falls under the "rubric of spurious correlations, namely correlations that do not imply causation" (Pearl, 2000). Consider for instance an example:

> "A rooster crow stands in constant conjunction to the sunrise, yet it does not cause the sun to rise. Regularity of succession is not sufficient; what would be sufficient? What patterns of experience would justify calling a connection 'causal'? ... If causal information has an empirical meaning beyond regularity of succession, then that information should show up in the laws of physics. But it does not! ... This may come as a surprise to some of you but the word 'cause' is not in the vocabulary of probability theory; we cannot express in the language of probabilities the sentence, 'mud does not cause rain' - all we can say is that the two are mutually

correlated, or dependent - meaning if we find one, we can expect the other" (Pearl, 2000).

The idea of Causal Inference is built on Bayesian Networks. In Pearl (1988) and Pearl (2000) a calculus of causality is presented. However, since the reasoning is fundamentally based on Bayesian Networks, the theory has troubles dealing with loops or feedback, and both books focus exclusively on networks without loops. In recent literature there is a push to move towards network containing loops, see Mooij et al. (2011) for instance. Also from the system identification side, recently there are some results using a Bayesian framework (Aravkin et al., 2011, 2012; Chuiso & Pillonetto, 2012). Thus, in the future, it may be possible to make some very nice connections between Bayesian system identification and causal inference. However, this is not the approach taken in this thesis.

Most of the literature on topology detection is based on the idea of Granger Causality. This approach is more in-line with classical system identification concepts.

Granger was concerned with formulating a testable definition of causality. He determined three axioms of causality that he thought were incontestable truths. Then he attempted to turn these truths into testable conditions. Before presenting the axioms, consider the following notation. Let Ω_n denote all knowledge in the universe available at time n, and denote by $\Omega_n - Y_n$ this information except the values taken by a variable Y up to time n (Granger, 1980). Then:

Axiom A. The past and the present may cause the future, but the future cannot cause the past.

Axiom B. Ω_n contains no redundant information.

Axiom C. All causal relationships remain constant in direction throughout time.

His starting point for defining causality is the following general definition.

Definition 1.1 (General Definition (Granger, 1980)) *Y_n is said to* cause *X_{n+1} if $Prob(X_{n+1} \in A|\Omega_n) \neq Prob(X_{n+1} \in A|\Omega_n - Y_n)$ for some A.*

In other words, for Y_n to cause X_{n+1}, the variable Y_n needs to have some unique information (not contained anywhere else in the universe) about what value X_{n+1} will take in the immediate future (Granger, 1980). In his view, a cause is a"difference to the normal course which accounts for the difference in the outcome" (Granger, 1980).

Definition 1.1 cannot be used with actual data (all knowledge in the universe up to time n is not available!). Thus, Granger imposed several constraints to render the definition operational. Let J_n be an information set available at time n consisting of the vector series Z_n. Suppose that Z_n does not include any components of Y_n. Let $F(X_{n+1}|J_n)$ denote the conditional distribution function of X_{n+1} given J_n. Consider the following two operational definitions:

Definition 1.2 ((Granger, 1980)) *Y_n does* not cause *X_{n+1} with respect to $J_n + Y_n$ if*

$$F(X_{n+1}|J_n) = F(X_{n+1}|J_n + Y_n),$$

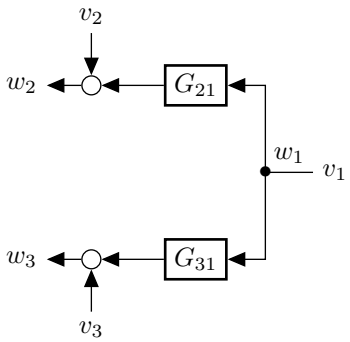

Figure 1.9: Diagram of the dynamic network of Example 1.4.

so that the extra information in $J_n + Y_n$ has not affected the conditional distribution. □

Definition 1.3 ((Granger, 1980)) *If*

$$F(X_{n+1}|J_n) \neq F(X_{n+1}|J_n + Y_n),$$

then Y_n is said to be the prima facie cause *of X_{n+1} with respect to the information set $J_n + Y_n$.* □

The main difference between the general Definition 1.1 and the operational Definitions 1.2 and 1.3 is that not all information in the universe is assumed to be known in the operational definitions. There are two main consequences of this change: (1) it becomes possible to formulate a practical test of causality, and (2) true causality is no longer detectable since there may be missing variables which make it seem like there is a causal relationship between two variables, when in fact there is not.

Granger devoted a lot of attention to the situation where missing variables could cause a mis-interpretation of a causal link between two variables. The problem is illustrated by an example.

Example 1.4 *Consider the simple dynamic network shown in Figure 1.9. The equations for the network are:*

$$\begin{aligned} w_1(t) &= v_1(t) \\ w_2(t) &= G_{21}(q)w_1(t) + v_2(t) \\ w_3(t) &= G_{31}(q)w_1(t) + v_3(t) \end{aligned}$$

where v_1, v_2 and v_3 are stationary stochastic process with rational power spectral densities. Suppose that $G_{21}(q) = q^{-1}$ and $G_{31}(q) = q^{-2}$. Then the network equations become

$$\begin{aligned} w_1(t) &= v_1(t) \\ w_2(t) &= v_1(t-1) + v_2(t) \\ w_3(t) &= v_1(t-2) + v_3(t). \end{aligned} \tag{1.2}$$

From the figure, it is clear that w_1 is causing both w_2 and w_3 with respect to the information set $J_n(w_1, w_2, w_3)$.

However, consider the information set $J_n(w_2, w_3)$. From (1.2), the variable w_3 is a prima facie cause of w_2 with respect to the information set $J_n(w_2, w_3)$, since $w_3(t)$ contains unique information about what the value of $w_2(t+1)$ will be.

Thus, we must conclude that w_3 does not cause w_2 with respect to information set $J_n(w_1, w_2, w_3)$, but that w_3 does cause w_2 using information set $J_n(w_2, w_3)$. Apparently missing variables can cause spurious links between known variables. □

In the next section it is shown how these operational definitions of causality form the foundation of the dynamic network identification theory.

1.3 CURRENT STATE OF THE ART OF DYNAMIC NETWORK IDENTIFICATION

In this section we present a review of the current literature related to dynamic network identification. The literature is split into two categories based on the whether the interconnection structure of the network is known or not. The situation when the interconnection structure is assumed to be unknown is referred to as *topology detection*. The situation where the interconnection structure of the system under investigation is known is referred to as *identification in dynamic networks*. It seems that the topology detection problem has attracted more attention judging by the number of papers written on the topic. However, the results of the papers for identification in dynamic networks are more readily implemented in practice (many of the papers directly address specific applications to test their methods).

1.3.1 Topology Detection

In this section a brief review of the topology detection literature is presented. As mentioned in Section 1.2, the topology detection literature can be split into two categories, those based on Granger Causality and those based on Causal Inference. Only the literature based on Granger Causality will be presented in this section because it is very closely connected to the prediction-error identification methods that will form the basis of the methods in this thesis. The main disadvantage with the methods based on Causal Inference is that they have trouble dealing with the presence of loops in the data generating system. However, this is a very active area of research, and in the near future it is likely that topology detection methods based on Causal Inference will be developed that can deal with the presence of loops in the data generating system (see Mooij et al. (2011) for instance).

First the algorithm devised in Granger (1969, 1980) is presented since it forms the foundation of the topology detection methods presented in this section. After presenting Granger's algorithm, it is shown how various modern tools are used to slightly adapt/improve the algorithm.

Granger's Method

Consider a data set consisting of measured internal variables w_1, ..., w_L. Suppose that a predictor has been constructed in order to predict/estimate the current value of the internal variable $w_j(t)$ using current and past values of a set of internal variables $w_{k_1}, \ldots, w_{k_n}$ and past values of w_j. Let $\{k_1, \ldots, k_n\} = \mathcal{D}_j$ and refer to the set of internal variables $w_{k_1}, \ldots, w_{k_n}$ as *predictor inputs*. Let $M(\theta, \mathcal{D}_j)$ denote the predictor model with predictor inputs w_k, $k \in \mathcal{D}_j$, and parameter vector θ. For the present discussion, it is not important how the predictor model is obtained, however we assume that it satisfies an optimality criterion (for example, θ_0 is such that the predictor $M(\theta_0)$ has the smallest sum of squared prediction errors out of the set of candidate predictor models).

Algorithm 1.5 *Testing for Granger Causality (Granger, 1969, 1980)*

1. *Construct a predictor model to predict w_j using all other internal variables that are available, i.e. construct $M(\theta_0, \mathcal{D}_j)$, where $\mathcal{D}_j = \{1, \ldots, L\} \setminus \{j\}$.*

2. *Examine if there is evidence of possible causal relationships. If there is a (non-zero) dynamic relationship between w_j and a predictor input w_i in $M(\theta_0, \mathcal{D}_j)$ then w_i is a candidate cause for w_j. Denote the set of indices of the candidate causes as $\mathcal{C}_j$.*

3. *Identify a new predictor model using only $w_{\ell_1}, \ldots, w_{\ell_{n-1}}$, $\{\ell_1, \ldots, \ell_{n-1}\} = \mathcal{D}_j \setminus \{i\}$, where i is an element in $\mathcal{C}_j$ (i.e. w_i is a candidate cause) as the set of predictor inputs.*

4. *The forecasting ability (prediction error) of both models $M(\theta_0, \mathcal{D}_j)$ and $M(\theta_0, \mathcal{D}_j \setminus \{i\})$ are then compared. If removing the internal variable w_i from the model $M(\theta_0, \mathcal{D}_j)$ results in significantly worse forecasting ability, then evidence of causation is found.*

5. *Repeat Steps 3-4 for every candidate cause (i.e. repeat for every element in $\mathcal{C}_j$).*

Topology Detection Based on Granger's Method

In this section a brief overview of the current topology detection literature based on Granger's Method is presented. A new feature of some of these algorithms is that they attempt to combine steps 2-5 of Algorithm 1.5 into step 1 by using some form of regularization (such as ℓ_1 regularization).

Early contributions to this problem date back to Caines & Chan (1975); Caines (1976); Anderson & Gevers (1982); Gevers & Anderson (1981, 1982) who address the question whether an open-loop or closed-loop structure is present between two measured signals w_1 and w_2. They consider a stochastic system with two observed variables as shown in Figure 1.10. They conclude that indeed the two different data generating systems shown in Figure 1.10 can be distinguished based on observations of the variables w_1 and w_2. The main assumptions that they make are: (a) the data generating system is either the open-loop system shown in Figure 1.10a, or the one shown in Figure 1.10b, and (b) the stochastic variables v_1 and v_2 are uncorrelated

for all lags. The method developed in these papers has later been referred to as the *Joint IO Method* of closed-loop identification. Steps 3-4 of Algorithm 1.5 are not dealt with in these papers.

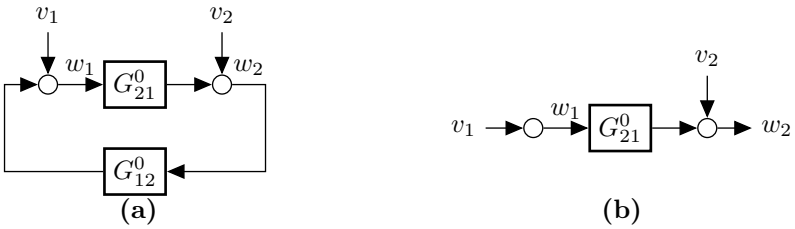

Figure 1.10: Open and Closed-loop data generating systems considered as candidate data generating systems considered in Caines & Chan (1975); Caines (1976); Anderson & Gevers (1982); Gevers & Anderson (1981, 1982). The v's denote stochastic processes with rational power spectral density.

Seneviratne & Solo (2012a,b) exactly implement Granger's Algorithm using a Laguerre Basis Function Model.

Friedman et al. (2010); Julius et al. (2009) propose using an FIR model structure, in conjunction with a Least Absolute Shrinkage and Selection Operator (LASSO, (Tibshirani, 1994)) approach. The idea is that the method will automatically detect which transfer functions (causal links) are not present and force them to exactly zero. Thus, steps 2-5 in Algorithm 1.5 are automatically taken care of.

Sanandaji et al. (2011, 2012) also uses an FIR model structure, however, they combine it with a compressed sensing approach. The advantage of this approach is that it can be used for large numbers of variables and relatively small data sets, i.e. even in the case where that are more variables than the length of the data records.

Innocenti & Materassi (2009); Materassi & Innocenti (2010); Materassi et al. (2011); Materassi & Salapaka (2012) use a non-parametric approach, which they show is equivalent to calculating the optimal Wiener filter. They show that this approach can correctly detect the interconnection structure of a dynamic network as long as every transfer function in the network is strictly proper (strictly causal). They do not deal with steps 2-5 of Algorithm 1.5. citetTan11 use a similar non-parametric approach.

Torres (2014); Torres et al. (2014a) present a subspace implementation. However, they do not allow for loops in the data generating system.

Gonçalves & Warnick (2008) consider the case where the network is driven only by known (deterministic) inputs. They derive conditions on the interconnection structure of the network and the location and number of the external inputs to ensure a unique representation of the network. The results are equally applicable to a stochastic network. The results of Gonçalves & Warnick (2008) are applied to the structure detection problem in Yuan et al. (2009, 2010, 2011). In Yuan et al. (2011) they explicitly look into how to apply steps 2-5 of Granger's Algorithm 1.5 (in modern language, one would call this an ℓ_0 regularization). In Hayden et al. (2014a,b) they present the Joint IO method extended to include results for non-minimum phase

data generating systems. In Yeung et al. (2010, 2011); Adebayo et al. (2012) they have attempted to further extended the idea of identifiability in networks. They define several levels of interconnection (such as a interconnection between states, interconnection between variables, interconnection between larger subsystems, etc), and state which level of interconnection is identifiable (i.e. detectable).

In Bottegal & Picci (2013, 2014) results are presented where each subsystem is assumed to be largely the same. The dynamics that are shared by each subsystem are called the *flock dynamics*. In addition to the flock dynamics each subsystem has a component that is unique to that subsystem.

In Dankers et al. (2012b) we propose that instead of using the Joint IO method as suggested by Granger (1969); Caines & Chan (1975); Caines (1976); Anderson & Gevers (1982); Gevers & Anderson (1981, 1982), other Prediction Error Methods can be used as well such as the Direct closed loop method (these methods will be properly introduced lated in this thesis). One of the advantages is that the Direct Method is easier to implement in practice than the Joint IO method.

Marques et al. (2013) propose additional tests that can be used in Step 2 of the algorithm. Similar to Dankers et al. (2012b) they also use the Direct Method. They propose that in addition to looking at whether a transfer function is zero or not, cross-correlation tests can be used to detect if feedback is present. It provides an extra clue/tool to detect possible causal relationships.

Most of the papers cited above assume that the system under investigation is a stochastic network in the sense that the data is generated by a set of unknown stochastic processes. All internal variables in the network are measured, and each internal variable is driven by an independent noise source. Most papers assume that all transfer functions G_{jk}^0 in the data generating system are strictly proper. Under these conditions it is shown that structure detection is possible.

1.3.2 Identification in Dynamic Networks

There seem to be two main lines of research that deal with the identification in dynamic networks: (1) very large systems are considered and the objective is to identify all the transfer functions in the network, (2) the objective is to identify only part of the dynamic network (the size of the entire network is not so important in this line of research).

The first line of research is more developed. The types of networks that are often considered in this line of research are *spatially distributed systems*. Such systems result from the discretization of partial differential equations for instance. For example, a model of a beam, or thermal conductivity along a metal plate. Another application is in wind farms, where the turbines are placed in a grid, and are interconnected by the wind passing through the farm (Annoni et al., 2014). The number of subsystems considered in this line of research is very large, and can range from more than 100 in wind farm applications to more than 100000 when the system under investigation is the result of the discretization of a partial differential equation over a spatial domain. Because of the large number of modules that make up the network, and because the objective is to identify all the modules, the emphasis is on the numerical implementation of the methods. Often simplifying assumptions are

made such as assuming that each module is only connected to its direct neighbors, or assuming that all modules have the same dynamics. The assumptions are made in order to develop numerically efficient methods.

In the second line of research the objective is to identify one module embedded in a dynamic network. The emphasis in this line of research is in (1) determining if it possible to obtain consistent estimates of the module of interest for the setup under consideration, and (2) studying if it is possible to reduce the variance of the estimate of the module of interest for the setup under consideration. There are only a few results available in this line of research for very simple dynamic network such as cascaded systems, and two closed-loops with an interaction. Although there are only a few results available in the literature, there seems to be many areas where such results/tools could be applied. Consider the three examples of dynamic networks given in Section 1.1.1

Consider the team of mobile robots. Each robot has its own controller. Just as it may be beneficial to identify a plant operating in a closed loop (to ensure that the model is targeted to the frequency range in which the plant is operating for instance), it may be beneficial to identify a model for the robot while it is operating in a distributed control setting (i.e. receiving inputs from neighboring robots, and from its own on board controller). The objective then is to identify the transfer function A_3 in Figure 1.2.

In the case of a power system, a possible objective is to identify the transfer function of a power plant. Most models of power plants are based on first principles modeling and then linearized about an operating point. Many decisions in the operation of the power grid are based on this model. However, over time (due to aging of the power plant) the first principles model may no longer be an accurate description of the power plant. Thus an alternative approach is to estimate a model of the (linearized) power plant from data collected from the power system. Since it is not desirable to disconnect the power plant from the grid in order to do an identification experiment, the identification must be done in a networked setting. In this case the objective is to obtain an estimate of the transfer functions P_{12} and P_{22} in the system shown in Figure 1.6.

Instead of identifying the dynamics of one power plant as in the previous example, it may also be attractive to identify the dynamics of an aggregate of power plants and generators. For instance the mix of power generation (wind, solar and conventional) may change over time, and consequently the dynamics of the aggregated power plants will change over time. The operators need to know what the dynamics of the aggregated power generating unit are inorder to make informed decisions to operate the power grid. First principles modeling does not seem to be a viable option in this case.

Some results of identification in a power system can be found in Virag et al. (2013); Dosiek & Pierre (2013); Wu et al. (2012); Pierre et al. (2012).

Lastly, consider the reservoir engineering example. In this case the objective of the reservoir engineer is to determine some characteristics of the reservoir using pressure and flow rate data collected from the wellbore. In particular, reservoir engineers are interested in estimating the thickness of the reservoir at the location of the wellbore; the permeability of the rock at the location of the wellbore; and

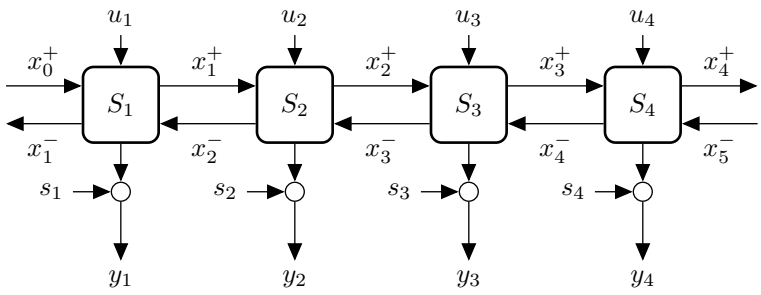

Figure 1.11: A spatially interconnected system. The x's represent the states of each of the subsystems, u's denote external inputs, s's denote sensor noise which is modelled as a stochastic process with rational power spectral density, and the y's denote the measured outputs of the subsystems.

the distance from the wellbore to the boundary of the reservoir. All three of these features can be obtained from the transfer function from the pressure to the flow rate at the bottomhole of the wellbore. In other words, the objective could be to estimate R_{11} in Figure 1.3.

In the following subsections, the literature for both lines of research is reviewed.

Identification of all modules in large networks

A diagram of a spatially distributed system is shown in Figure 1.11. In such a system, it is the state that forms the interconnection. Often the modules are interconnected according to a well defined pattern. For example, each module is only interconnected to its direct neighbors. Each module is assumed to have a (exactly) known external input u_k and an output y_k. The state interconnection in a positive direction is labeled x_k^+ and the state interconnection in a negative direction is labeled x_k^-. The inputs are assumed to be exactly known, and the outputs are measured with noise (called *sensor noise*). A key feature of sensor noise is that it does not propagate through the network. The result is that open-loop identification techniques can be applied to obtain consistent estimates. Since the number of modules is very large, and the objective is to identify all of them, the algorithms must be computationally efficient.

Fraanje & Verhaegen (2005); Massioni & Verhaegen (2008, 2009) assume that each subsystem is identical. In Ali et al. (2009, 2011c,b,a) the reasoning is extended to more complex noise models, and also to the case where each subsystem is modelled as a linear parameter varying system.

Sarwar et al. (2010); Van Wingerden & Torres (2012); Torres et al. (2014b) propose methods to identify all the subsystems in the dynamic network that scale linearly with each added subsystem. In these papers, they do not make the assumption that each subsystem is identical.

In Haber et al. (2011); Haber & Verhaegen (2012, 2013) a variant of a spatially distributed system is considered where each subsystem is assumed to interact only

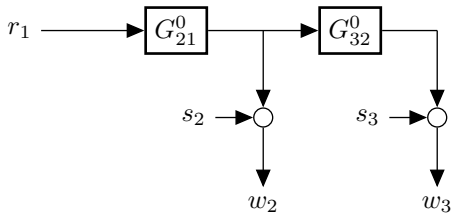

Figure 1.12: Cascaded system that is considered in Wahlberg et al. (2009); Everitt et al. (2013, 2014), where r_1 denotes an external variable, s_2 and s_3 denote sensor noise.

with other subsystems within a limited distance. This assumption is used to implement numerically efficient algorithms. The subsystems are not constrained to be identical in this case.

Identification of a module embedded in a network

There are only a few results available in this line of research. Moreover, none of the results are very general. They only develop consistency or variance results for very basic dynamic networks.

In Gevers et al. (2006); Wahlberg et al. (2009); Everitt et al. (2013, 2014) some interesting questions are posed. They consider a cascaded system as shown in Figure 1.12. There is one external excitation variable r_1 which is exactly known to the user. Each of the internal variables w_2 and w_3 are measured with noise. A consistent estimate of G_{21}^0 can be obtained using any open loop identification method. The question is: how can the measurement w_3 help in order to reduce the variance of the estimate of G_{21}^0 (even though w_3 is not directly connected to G_{21}^0)? The main conclusion of the work is that usually the extra measurement w_3 can be used to reduce the variance of the estimate of G_{21}^0. In Gunes et al. (2014) it is shown that the reasoning can be extended to general interconnection structures.

Lastly, there are some results which consider networks that are slightly more complex than a single closed-loop, such as two closed-loops with an interaction (Gudi & Rawlings, 2006) and a multi-variable closed-loop (Leskens & Van den Hof, 2007). In these papers it is shown how the closed-loop methods can be adapted to use in the network configuration under investigation in order to obtain consistent estimates of the plant.

1.4 PROBLEM STATEMENT

As can be seen from the literature review in Section 1.3 the field of dynamic network identification has been very active with many papers being published in the last 5-6 years. Currently the papers form a disconnected patchwork of results instead of a strong theory of dynamic network identification. For instance, each paper makes assumptions suited to the method presented; consistency is not always rigorously

treated; there is almost no mention of the variance of the obtained estimates in many of the papers. However, almost all the results can be easily "translated" into extensions of (closed-loop) system identification methods. In addition to providing a common foundation to all the current results, viewing the results through the lens of system identification also offers new insights and opportunities for further development of the field.

Not only does system identification offer a vast array of tools and a foundation for the dynamic network identification literature, but the integration of networks and system identification presents an opportunity for system identification to move (beyond control) into many new application domains. This opportunity should not be missed by the system identification community.

There are many possible research questions when considering system identification and dynamic networks. The aim of this thesis is to start the development of a rigorous theory of dynamic network identification based on the foundations of system identification.

> The main motivating question for the research presented in this thesis is: under what conditions is it possible to consistently identify a particular module that is embedded in a dynamic network?

The question is very broad. Clearly, the motivating question encompasses the identification in dynamic networks problem. However, it also encompasses the topology detection problem: by determining that a particular transfer function is zero, the topology of the network is implicitly detected (as long as the estimated transfer function is consistent).

What does the motivating question not encompass? Variance analysis of the estimates obtained is not discussed. This is not to say this is not important. Since the number of parameters that need to be estimated is large, issues associated with the variance should certainly be addressed. In fact, as shown by Wahlberg et al. (2009); Everitt et al. (2013) there are unique opportunities when to moving to a network setting to reduce the variance of the obtained estimates by using extra measurements collected from the system. Secondly, the numerical implementation of the methods is not discussed in this thesis.

In the next subsection the research question is further analyzed, and the scope of the research in this thesis is further sharpened.

1.4.1 Analysis of Research Question

One of the main distinctions that has been made thus far is the case when the interconnection structure is known vs. the case when it is unknown. From the consistency analysis point of view the two cases are not that different. If a method results in a consistent estimate a transfer function, then it does not matter if the estimated transfer function is non-zero, very small, or zero. Of course, from a practical implementation point of view, there could be many additional challenges that need to be tackled in the topology detection case. Thus, the approach that is taken in this thesis is to first assume that the interconnection structure is known, and investigate

under what conditions a module embedded in a dynamic network can be consistently estimated. Then the consistency results can be transferred straightforwardly to the case where the interconnection structure is unknown.

A second fundamental choice that we make in this thesis is to choose closed-loop identification methods as a starting point for analysing dynamic network identification. This is not such a bad starting point, dynamic networks may contain many loops, and in fact it can be shown that any dynamic network can be expressed as a single multi-variable feedback loop.[4] However, there are certain assumptions rooted in the closed-loop identification literature that become unrealistic when applied to dynamic networks. The two main assumptions that become untenable are the assumption that the predictor inputs are measured without sensor noise and that the predictor inputs are zero-order-hold signals. The closed-loop system that is typically considered in the identification literature is shown in Figure 1.13.

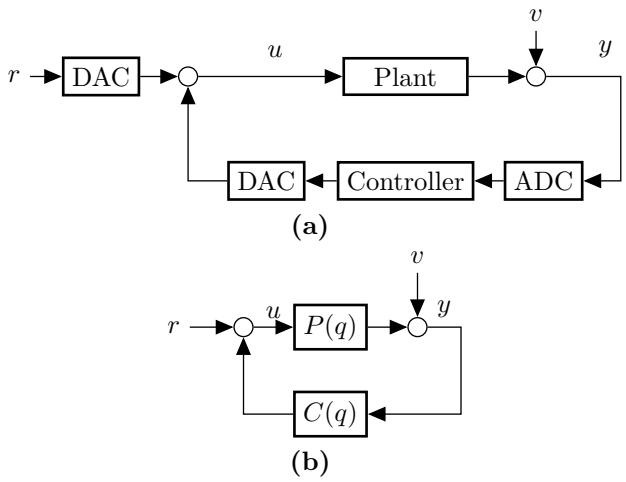

Figure 1.13: A closed-loop system as considered in the closed-loop identification literature.

In this closed-loop system the feedback path consists of (1) an *analog to digital converter* (ADC) which samples the output y, (2) a digital controller, and (3) a *digital to analog converter* (DAC) which converts the digital output of the controller back to an analog signal typically by using a *zero order hold*. The first feature of this setup is that it is reasonable to suppose that the input to the plant, u, is known noise free. It is the sum of two signals that are exactly known to the experimenter, i.e. the reference r and the output of the controller. Secondly, the input to the plant is a piece-wise constant signal (as long as both DACs are equipped with zero-order-hold circuits).

Consider now the case of a dynamic network. Consider for example a power system where voltage and current are measured at various locations in the network. Suppose that the dynamics of a transmission line are to be identified. The "input"

[4]This statement is formally proved in Proposition 4.2 in Chapter 4.

is the voltage at one end of the line, and the "output" is the voltage at the other end of the line. Firstly, neither the input nor the output are known, noise free. Both variables are measured using sensors, and so only noisy versions of the variables are known. Secondly, neither signal is generated by a DAC, and so neither signal is constant during the sampling interval. Thus, for this example, both assumptions that are typically made in closed-loop identification are not realistic. The same reasoning holds for many types of dynamic networks.

Since we have chosen to use the closed-loop methods as a starting point for a theory of dynamic network identification, both of these issues must be dealt with in an appropriate manner in order to develop methods that are applicable in practice.

A third fundamental choice that we make in this thesis is to base our reasoning on the Prediction-Error methodology. The reason for this choice is that we are mainly interested in studying the consistency of the estimated transfer functions, and the Prediction-Error framework is equipped with very useful tools when analyzing consistency.

In light of these three fundamental choices, the following types of conditions are considered in this thesis when analyzing consistency. The aim is to address conditions that render the methods more applicable to practical situations. As the thesis progresses the assumptions become more realistic.

1. Presence and correlation of process noise. In the prediction error methods it is common to consider process noise on the output. Process noise can be thought of as thermal noise, noise due to electromagnetic radiation, etc., i.e. it is noise that affects the value of a variable. In a network, process noise can enter anywhere, not just at the variable considered as "output". Moreover, as in the case of electromagnetic radiation affecting an electric circuit, it seems likely that many of the process noise terms will be correlated. How does this affect the estimates obtained by various identification methods? This question is addressed in Chapter 4.

2. Presence of variables that can be controlled/changed by the user. In certain situations it may be possible to excite the network by manipulating a particular variable. For example, in reservoir engineering it may be possible to change/manipulate particular valve settings on some of the wellbores for instance. How can these external variables be used to our advantage? This question is addressed in Chapter 4.

3. Measurability or availability of the internal variables in the network. Is it possible to measure all variables in the network, or are several variables unmeasurable? Or perhaps several variables are expensive/difficult/unsafe to measure, thus the user would prefer to avoid using those particular variables. Which variables need to be measured in order to guarantee that it is possible to obtain consistent estimates of a particular module embedded in the network? This question is addressed in Chapter 5.

4. Presence and correlation of sensor noise. Another source of noise is due to measurement errors. This type of noise does not propagate through the network, however it has fundamental consequences for the consistency of several

methods. How can this issue be addressed in a network setting? This question is addressed in Chapter 6.

5. As discused, the zero order hold assumption on the intersample behavior of the variables in a dynamic network is typically not a realistic assumption. It implies that almost the entire network consists of digital components. Many dynamic networks consist almost entirely of continuous-time dynamics. What is the consequence of modeling a continuous-time data generating system as a discrete-time dynamic network? Under what conditions can a continuous-time module embedded in a (continuous-time) dynamic network be identified? Some preliminary results addressing these questions are presented in Chapter 7.

1.4.2 Isn't This Just MIMO Identification?

One of the main distinctions of the approach we take in this thesis and standard MIMO identification is that we are not attempting to identify all the transfer functions in the dynamic network, only a small subset. The result of our approach is that the conditions under which we can identify one module embedded in the dynamic network are considerably less restrictive than those required in a MIMO approach. For instance, we do not need to measure all internal variables; external variables do not need to be present at each node; and correlation between (some) noise terms is allowed. The result of our approach versus a MIMO approach is analogous to a distributed/local approach vs. a centralized/global approach. To identify a particular module embedded in a dynamic network, we only require local measurements. The main feature of our approach that enables the relaxation of the conditions under which consistent estimates are possible is the introduction of the interconnection structure of the dynamic network into the identification problem. The main points are briefly illustrated in the following text.

Consider the data generating system[5]:

$$\begin{aligned}\begin{bmatrix} w_1(t) \\ w_2(t) \\ \vdots \\ w_L(t) \end{bmatrix} &= \begin{bmatrix} 0 & G_{12}^0(q) & \cdots & G_{1L}^0(q) \\ G_{21}^0(q) & 0 & \ddots & G_{2L}^0(q) \\ \vdots & \ddots & \ddots & \vdots \\ G_{L1}^0(q) & G_{L2}^0(q) & \cdots & 0 \end{bmatrix} \begin{bmatrix} w_1(t) \\ w_2(t) \\ \vdots \\ w_L(t) \end{bmatrix} + \begin{bmatrix} r_1(t) \\ r_2(t) \\ \vdots \\ r_L(t) \end{bmatrix} + \begin{bmatrix} v_1(t) \\ v_2(t) \\ \vdots \\ v_L(t) \end{bmatrix} \\ &= G^0(q)w(t) + r(t) + v(t), \end{aligned} \tag{1.3}$$

where each r_k is a known external variable (that can be manipulated by the user), v_k is process noise (modelled as a stochastic process with rational power spectral density) and each w_k is a measured internal variable. The transfer functions G_{jk}^0 are non-zero if and only if there is a direct connection from w_k to w_j. Suppose that $I - G^0$ is invertible so that w can be expressed as:

$$w(t) = (I - G^0(q))^{-1}(r(t) + v(t)) \tag{1.4}$$

[5]Dynamic network models will be presented more formally and in more detail in Chapter 2

Option 1. One could take all the r's as inputs, and all the w's as outputs and obtain an estimate of $(I - G^0)^{-1}$ using tools from MIMO identification. This is simply an open loop MIMO identification problem. Even if G^0 has a structure of zero entries, $(I - G^0)^{-1}$ does not have any/many zero entries due to feedback loops. The matrix G^0 can be recovered from the estimate of $(I - G^0)^{-1}$ by algebraic manipulations. There are several issues with this proposed procedure:

1. There must be a (sufficiently exciting) signal present at every node in the network. In otherwords, for each measured variable w_k, there must be an r_k present.
2. Each r_k is only allowed to directly affect one node.
3. All L internal variables, $w_1, \ldots, w_L$ in the network must be available (measured).
4. As this is a indirect method, it leads to high order estimates, i.e. inaccuracies can be introduced by the algebraic manipulations required to recover G^0 from the estimated transfer function $(I - G^0)^{-1}$.
5. All L^2 transfer functions of the network must be identified in order to obtain an estimate of the one transfer function of interest.

Consequently, this approach may not be a feasible or recommended approach.
Option 2. A second MIMO approach could be to (attempt to) directly identify the matrix G^0 in (1.3) by taking the vector w as the output, and also as the input (and parameterizing the model with zeros on the diagonal). In this way, the problem of requiring an external variable to be present at each node is avoided, and also since the matrix G^0 is directly identified, no algebraic manipulations are required which result in high order estimates. The main problem with this approach is that the input w is correlated to the noise v. Thus, this is not an open-loop identification problem, and the estimate of G^0 obtained using this approach will not be consistent.

The solution to this problem is to move to closed-loop identification methods. In this thesis we show that indeed closed-loop tools can be used to address the identification problem sketched in Option 2. However, we also show that by only attempting to identify one module embedded in a dynamic network conditions under which consistent estimates are possible can be significantly relaxed compared to a MIMO approach.

Perhaps most importantly, the approach that we take in this thesis results in a local identification method in the sense that only local measurements are required to identify a module embedded in the dynamic network. As an additional advantage, a local method is such that it can be easily parallelized so that multiple modules can be estimated in parallel.

1.5 STRUCTURE OF THESIS

The remainder of the thesis is structured as follows. Chapters 2 and 3 contain background information on dynamic network models and system identification. The

subsequent chapters present methods of identification in dynamic networks with increasingly less restrictive conditions. Thus the methods of the last chapter are the most general, i.e. can be applied in the largest variety of situations. The structure is graphically shown in Fig. 1.14.

Chapter 2 - Dynamic Network Models. In this chapter a discrete-time dynamic network model is presented. In particular several assumptions are made regarding what are deemed to be "realistic" models.

Chapter 3 - System Identification. In this chapter several closed loop system identification methods are presented. Since dynamic networks may contain loops, it seems natural that the closed-loop identification methods can be modified such that they can be applied in networks. In particular the Direct, Two-Stage and Joint IO and Basic Closed Loop Instrumental Variable Prediction-Error Methods are presented. Lastly the Errors-in-Variables framework is presented, with some closed-loop identification results.

Chapter 4 - From Identification in Closed-Loops to Identification in Networks. In this chapter the first results of identification in dynamic networks are presented. The presented methods are straightforward extensions of the Direct, Two-Stage and Joint IO Methods. The material in this chapter is based on the following conference and journal papers:

> Dankers, A.G.,Van den Hof, P.M.J., Heuberger, P.S.C., and Bombois, X., (2012). Dynamic network identification using the direct prediction error method. In *Proceedings of the 51st IEEE Conference on Decision and Control* (pp. 901-906).
>
> Van den Hof, P.M.J., Dankers, A.G., Heuberger, P.S.C., and Bombois, X. (2012). Identification in dynamic network with known interconnection topology. In *Proceedings of the 51st IEEE Conference on Decision and Control* (pp. 895-900).
>
> Van den Hof, P.M.J., Dankers, A.G., Heuberger, P.S.C., and Bombois, X. (2012). Identification of dynamic models in complex networks with prediction error methods - basic methods for consistent module estimates. *Automatica, 49*, 2994-3006.

Chapter 5 - Predictor Input Selection. In this chapter conditions regarding which variables need to be measured in order to guarantee that a particular module can be consistently identified are presented. The conditions can be used to check whether it is possible to consistently estimate a particular module embedded in the network while avoiding the need to measure particular variables. The conditions can also be used to design a sensor placement scheme to ensure that it is possible to consistently estimate a particular module. The material in this chapter is based on the following conference and journal papers:

> Dankers, A.G., Van den Hof, P.M.J., Bombois, X., Heuberger, P.S.C. (2013). Predictor input selection for two stage identification in dynamic networks. In *Proceedings of the European Control Conference 2013.* (pp. 1422-1427). Zürich, Switzerland.

Dankers, A.G., Van den Hof, P.M.J., Heuberger, P.S.C. (2013). Predictor input selection for direct identification in dynamic networks. In *Proceedings of the 52nd IEEE Conference on Decision and Control*. Florence, Italy.

Dankers, A.G., Van den Hof, P.M.J., Bombois, X., Heuberger, P.S.C. (2014). Identification of dynamic models in complex networks with prediction error methods - predictor input selection. *IEEE Transactions on Automatic Control*. Under Review.

Chapter 6 - Dealing with Sensor Noise. In this chapter two methods are presented that can deal with sensor noise on all the internal variables. The methods can be framed as hybrids between the Direct closed-loop identification method and the closed loop instrumental variable method. The material in this chapter is based on the following conference and journal papers:

Dankers, A.G., Van den Hof, P.M.J., Bombois, X., Heuberger, P.S.C. (2014). Errors-in-variables Identification in Dynamic Networks Using an Instrumental Variable Approach. In *Proceedings of the 19th IFAC World Congress*. Cape Town, South Africa. Accepted.

Dankers, A.G., Van den Hof, P.M.J., Bombois, X., Heuberger, P.S.C. (2014). Errors-in-variables Identification in Dynamic Networks - Consistency Results for an Instrumental Variable Approach . *Automatica*. Under Review.

Chapter 7 - Continuous-Time Identification in Dynamic Networks. In this chapter the assumptions on the intersample behavior of the measured variables is addressed. A continuous-time identification method is proposed. The material in this chapter is based on the following conference paper:

Dankers, A.G., Van den Hof, P.M.J., Bombois, X. (2014). An Instrumental Variable Method for Continuous-Time Identification in Dynamic Networks. In *Proceedings of the 53th Conference on Decision and Control*. Los Angeles, USA. Under Review.

Chapter 8 - Case Study - Well Test Analysis. In this chapter a detailed case study is presented to illustrate some of the methods presented in this thesis. The example is taken from reservoir engineering. The material in this chapter is based on the conference paper:

Mansoori, M., Dankers, A.G., Van den Hof, P.M.J. (2014). Errors-in-Variables identification in bilaterally coupled systems with application to oil well testing. In *Proceedings of the 19th IFAC World Congress*. Cape Town, South Africa. Accepted.

Chapter 9 - Conclusions. In this chapter conclusions are formulated and some suggestions for future work are made.

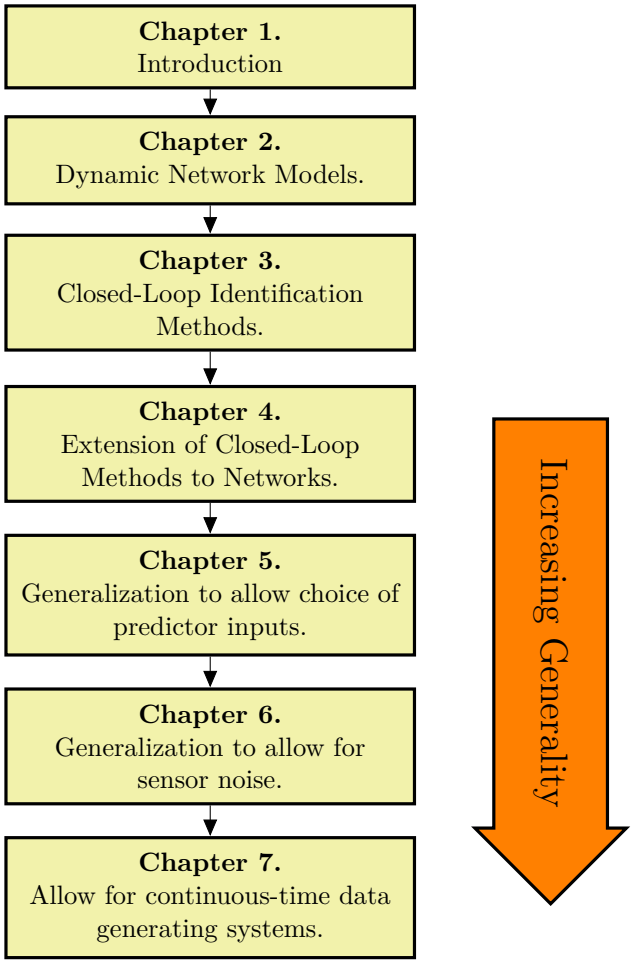

Figure 1.14: Schematic diagram of the structure of the chapters

Chapter 2

DYNAMIC NETWORK MODELS

In this chapter a dynamic network model is presented. The dynamic network model is a straightforward extension of the classical closed-loop model and as such, it consists of internal, external and noise variables. The internal variables define the interconnection structure of the network. In this chapter we demonstrate that a wide variety of physical phenomena can be modeled as a dynamic network, suggesting that it is a useful modeling tool. The property of well-posedness of a dynamic network model is also discussed. Lastly, it is shown how a graph of a dynamic network model can be constructed. This is useful since it enables the use of graph theoretical tools.

2.1 INTRODUCTION

AS ILLUSTRATED IN Chapter 1 a set of variables can be modelled as a dynamic network. In this chapter a dynamic network model is formally defined. A dynamic network model is a tool that can help to gain insight into the operation of a system. For instance, by modeling a set of variables as a dynamic network, it becomes clear if there is a feedback mechanism present. This insight could then be used to control the system.

In this thesis, we will only consider Linear Time Invariant (LTI) models of dynamic networks.

A dynamic network model may consist of three classes of variables: internal, external, and noise variables.

Internal Variables. These are the system variables of the model. It is these variables that define the interconnection structure of the dynamic network. Examples of internal variables for various types of systems are: voltage, current, pressure, flow rate, temperature, etc. The internal variable are assumed to be *measurable* (at least in principle) in the sense that a measurement device can be used to record the value of the variable.

External Variables. These are variables that can be directly manipulated by an experimenter. For instance a voltage in an electrical circuit can be manipulated using a waveform generator, or a flow rate can be manipulated using a value setting.

Process Noise/Disturbance Variables. These variables are used to model unknown external variables. For example, wind affecting an airplane, or electromagnetic radiation inducing current in the wires of an electrical circuit.

The dynamic network model that is presented in this chapter is a straightforward extension of the classical closed-loop model. In fact, it is possible to express any dynamic network as a multi-variable closed-loop system.[1] However doing so results in a loss of information about the interconnection structure. An important feature of any model is a characterization as to what types of systems can be modelled using the proposed model. In Section 2.2 it is illustrated that a wide variety of physical phenomena can be modelled using dynamic network model.

In Section 2.4 the notion of a well-posed dynamic network model is presented. There are certain features that a physical system has (or does not have), for instance, variables cannot take on a value that is infinite, and a system cannot react to an external excitation before the excitation is applied. These properties are grouped together in the concept of well-posedness. Since a physical system is well posed, a model should be as well.

The chapter concludes with Section 2.5 where it is shown how a graph of a dynamic network model can be constructed. In Graph Theory many tools are available to analyze interconnections between nodes. Consequently, it will be very useful to create a graph of a dynamic network model in order to be able to use the available tools from Graph Theory.

2.2 A DYNAMIC NETWORK MODEL

First some basic notation will be introduced. In this thesis, t will be used to denote time (except for the chapter on continuous-time identification (Chapter 7), t is always a discrete-time variable). Let q denote a shift operator such that $q^{-1}u(t) = u(t-1)$.

Suppose that a linear time-invariant (LTI) discrete-time causal system has an impulse response $g_{21}(t)$, $t = 0, 1, \ldots$, and a input w_1 is applied to the system. Using the q-operator notation the output can be expressed as

$$\begin{aligned} w_2(t) &= \sum_{k=0}^{\infty} g_{21}(k) w_1(t-k) \\ &= \sum_{k=0}^{\infty} g_{21}(k) q^{-k} w_1(t) \\ &= G_{21}(q) w_1(t). \end{aligned}$$

Informally, $G_{21}(q)$ is the z-transform of the impulse response g_{21} with z replaced by q.

Now a dynamic network model is presented. The network structure that we consider in this thesis is built up of L elements or nodes, related to L *internal variables*, w_j, $j = 1, \cdots L$. Each internal variable w_j in this network can be expressed

[1] This statement is proved in Proposition 4.2 in Chapter 4.

as:

$$w_j(t) = \sum_{k \in \mathcal{N}_j} G_{jk}(q) w_k(t) + r_j(t) + v_j(t), \tag{2.1}$$

where

- $\mathcal{N}_j$ is the set of indices of internal variables w_k, $k \neq j$, for which $G^0_{jk} \neq 0$, i.e. the set of indices of internal variables with direct causal connections to w_j.
- v_j is an unmeasured disturbance, referred to as the *process noise* (this variable will be characterized in more detail later on).
- r_j an *external variable* that can be manipulated by the experimenter.

The variables r_j and v_j may or may not be present at each node.

We choose not to allow a G_{jj} term in (2.1) (i.e. $j \notin \mathcal{N}_j$). This is equivalent to choosing not to allow self loops. This choice does not impose any restrictions on the generality of a dynamic network model. It can be viewed as a normalization.

Remark 2.1 *Throughout this thesis all the variables w_k, r_k, and v_k, $k = 1, \ldots, L$ are treated as scalar variables. However, there is no reason that they cannot be vectors of variables.*

Graphically, (2.1) can be interpreted as a single building block of a dynamic network. This is illustrated in Fig. 2.1. A dynamic network is then constructed by interconnecting several blocks.

The following sets will be useful throughout the remainder of this thesis.

- $\mathcal{V}$ denotes the set of indeces of process noise variables that are present in the network model.
- $\mathcal{R}$ denotes the set of indeces of external variables that are present in the network model.

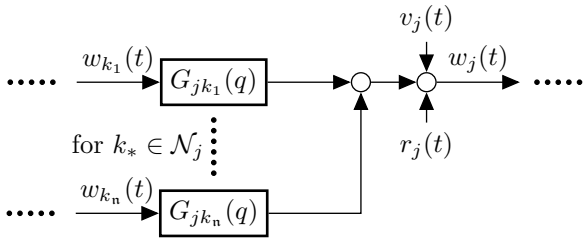

Figure 2.1: Graphical Representation of a single measured variable.

All the internal variables can be written in a single matrix equation as:

$$\begin{bmatrix} w_1(t) \\ w_2(t) \\ \vdots \\ w_L(t) \end{bmatrix} = \begin{bmatrix} 0 & G_{12}(q) & \cdots & G_{1L}(q) \\ G_{21}(q) & 0 & \ddots & G_{2L}(q) \\ \vdots & \ddots & \ddots & \vdots \\ G_{L1}(q) & G_{L2}(q) & \cdots & 0 \end{bmatrix} \begin{bmatrix} w_1(t) \\ w_2(t) \\ \vdots \\ w_L(t) \end{bmatrix} + \begin{bmatrix} r_1(t) \\ r_2(t) \\ \vdots \\ r_L(t) \end{bmatrix} + \begin{bmatrix} v_1(t) \\ v_2(t) \\ \vdots \\ v_L(t) \end{bmatrix}$$
$$= G(q)w(t) + r(t) + v(t), \tag{2.2}$$

with obvious definitions for G, w, r and v, and where the (j,k)th entry of G is non-zero if and only if $k \in \mathcal{N}_j$; the kth entry of r is non-zero if and only if $k \in \mathcal{R}$; and the kth entry of v is non-zero if and only if $k \in \mathcal{V}$.

The interconnection structure of the dynamic network is encoded by the matrix G. This is illustrated in the following example.

Example 2.2 *Consider a network defined by:*

$$\begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \\ w_5 \\ w_6 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & G_{14} & 0 & 0 \\ G_{21} & 0 & G_{23} & 0 & 0 & 0 \\ 0 & G_{32} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & G_{46} \\ 0 & G_{52} & 0 & G_{54} & 0 & G_{56} \\ 0 & 0 & G_{63} & 0 & G_{65} & 0 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \\ w_5 \\ w_6 \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{bmatrix}$$

Three equivalent diagrams of the network are shown in Fig. 2.2. The diagram shown in Figure 2.2a is a traditional block diagram as is commonly used in the control systems field. Here a circle represents a summation and a rectangle represents a transfer function.

In many fields such as artificial intelligence, machine learning (Materassi et al., 2011) and systems biology (Yuan et al., 2010) node-and-link style diagrams are more common to represent dynamic networks. In this style of diagram each internal variable is represented as a node, and arrows between nodes represent causal dynamic relationships between two internal variables. This style is illustrated in Fig. 2.2b.

The node-and-link style of diagram (shown in Figure 2.2b) emphasizes the internal variables (measurement-centric), whereas the style commonly used in control (shown in Figure 2.2a) emphasizes the transfer functions (module-centric).

In this thesis a style is used that is a hybrid between the two styles discussed. We wish to emphasize the interconnection structure (which is more clearly represented using the node-and-link style) and we also wish to emphasize the dynamics of the system (which is more clearly represented using the rectangles of the control systems style). Thus, we move the labels of the internal variables inside each summation, which denotes that the outgoing edges of that summation are the denoted internal variables. This style is shown in Figure 2.2c. □

The process noise variable v_j represents unknown or unmodeled variables that affect w_j. Since the process noise is inherently something that is unknown, it is modelled as a stochastic process. It is common to model v_k as filtered white noise:

$$v_k(t) = H_k(q)e_k(t)$$

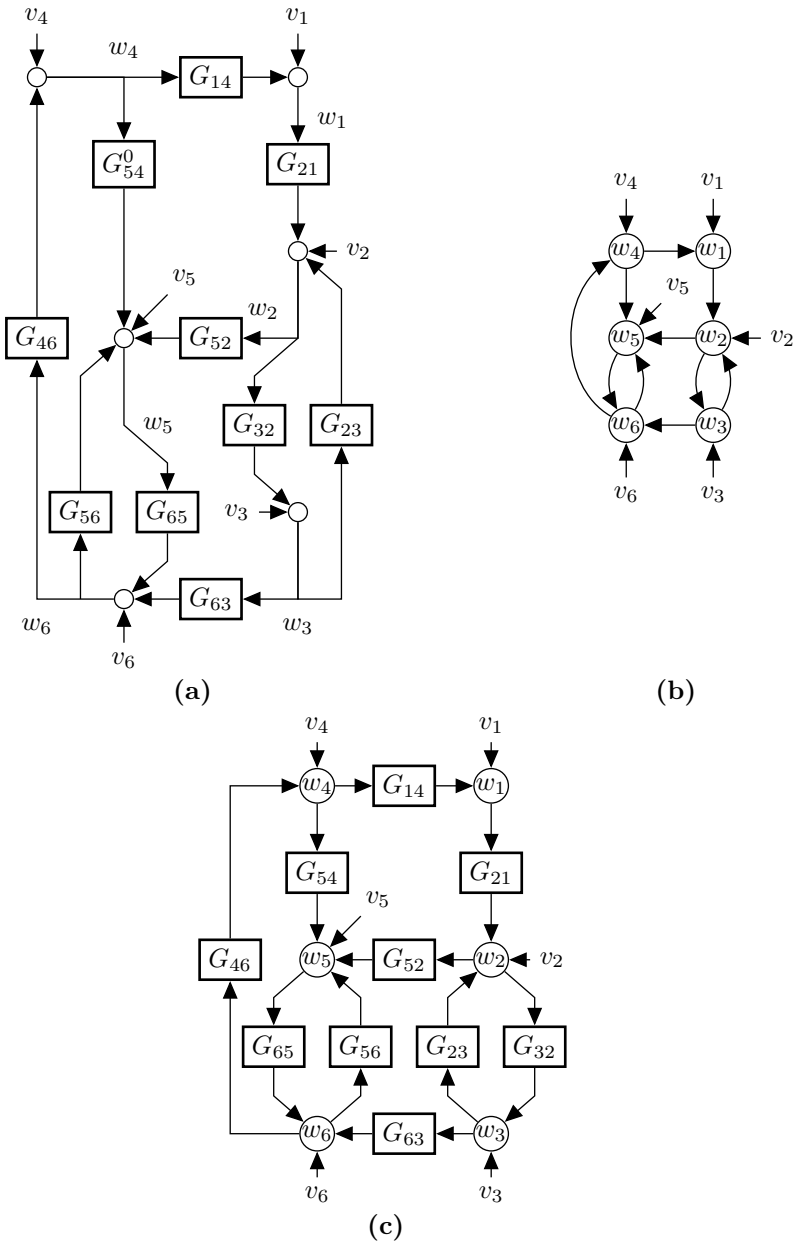

Figure 2.2: Three equivalent visual representations of the network considered in Example 2.2. In (a), each rectangle represents a transfer function, and each circle represents a summation. In (b) a node-and-link representation of the same network is shown where a transfer function is denoted by a link, and an internal variable is denoted by a circle. In (c) the internal variables are placed inside the circles which denotes that the edges leaving the circle are equal to the internal variable denoted inside the circle. Rectangles represent transfer functions.

where e is white noise in accordance with the standard Prediction-Error identification framework (Ljung, 1999). The variance of e is denoted $\sigma^2_{e_k}$, and H_k is a monic, stable, minimum phase filter.

Remark 2.3 *What is the physical basis for modeling the noise as a sequence of Gaussian processes?*

In engineering many types of noise can be described as Brownian Motion *(Åström, 1970). This type of stochastic process was first studied in the motion of small particles. Thermal noise, for instance is a classical example of a Brownian Motion Process.*

Of course, not all noise-generating mechanisms are Brownian. However, approximations to many other noise generating mechanisms can be made sufficiently small by considering filtered versions of Brownian Motion Processes (Pintelon & Schoukens, 2012b).

It can be shown that by sampling a Brownian Motion Process at periodic intervals, a sequence of Gaussian processes results (Åström, 1970).

Examples of process noise are: thermal noise in an electrical circuit, electromagnetic radiation inducing current in wires of an electrical network, vibrations in mechanical structures, and turbulent wind flow hitting a mechanical structure.

Some of the process noise terms may be correlated. Consider an electrical network for instance. If the process noise is (predominantly) caused by external electromagnetic radiation, most likely all variables related to the current in the wires are somehow affected by this radiation. The result is that all the v_k terms are correlated in this case.

2.3 MODELING OF PHYSICAL SYSTEMS USING DYNAMIC NETWORK MODELS

In the previous section a dynamic network model was presented. What types of systems can be modeled using a dynamic network model? The framework that is chosen in this thesis is a straight-forward extension of the closed-loop models that are common in closed-loop identification, i.e. the closed loop system shown in Figure 2.3 is a special case of a dynamic network model. Is this extension justified? In this section we show that many systems in engineering can be modeled using the dynamic network model proposed in this thesis. A systematic method is presented to model a physical system using interconnected transfer functions.

Bond graph modeling is a common first principles modeling technique for large interconnected systems that was developed in the early 1970's (Shearer et al., 1971; Takahashi et al., 1970). A similar modeling technique is presented in Willems (1997, 2007, 2008, 2010) which he calls *Tearing, Zooming and Linking.* A third, closely related, first principles modeling technique for large interconnected systems is *Port-Hamiltonian Modeling* (Duindam et al., 2009). The method of Willems is slightly more general than Bond graph modeling. All three methods are modularized methods of modeling. That is, a system is supposed to be made up of many different subsystems, or modules, that are interconnected (to form a network).

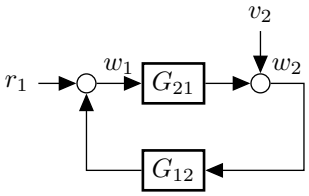

Figure 2.3: Typical closed-loop model in system identification

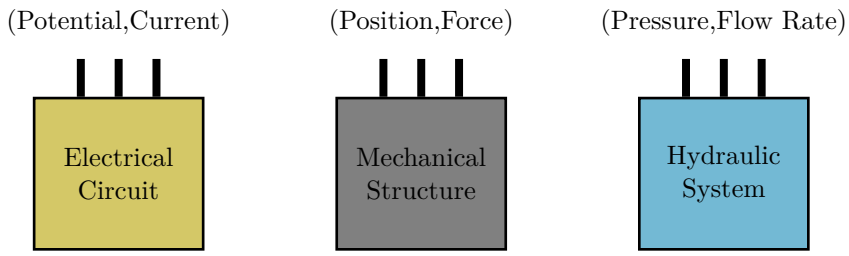

Figure 2.4: Examples of systems with terminals (Willems, 2007). The terminals are shown in thick lines, and the variables living on those terminals are indicated in brackets.

A *module* can be viewed as a black-box with *terminals* (see Figure 2.4). The *terminals* are where the variables "live", and it is how the modules interact with their environment (Willems, 2007). Each terminal is of a certain *type*, characterized by the variables that live there. For instance current and voltage are associated with electrical terminals, flow rate and pressures are associated with hydraulic terminals, and forces and displacements are associated with mechanical terminals. Typically a terminal has a "flow" variable and a "potential" variable. A terminal of a certain type can be connected to another terminal of the same type, thus forming interconnections between modules. Forming an interconnection imposes constraints on the terminal variables. Typically, the potential of each of the terminals must be the same after interconnection, and, the flow into the interconnection must equal the flow out of the interconnection. Consider the following illustrative example.

Example 2.4 *Consider a hydraulic system as shown in Figure 2.5a. The system consists of two tanks of water connected by a pipe. One of the tanks is connected to a constant flow pump which injects water. The system can be viewed as composed of five modules that are interconnected, a module for the pump, each of the tanks, the pipe, and a plug as shown in Figure 2.5b. Each module has terminals of hydraulic type. Thus the variables that live on the terminals are pressure, denoted* p *and flow rate, denoted* f*.*

Differential equations can be derived to describe the behavior of the variables of each of the modules.

Lastly, the modules can be interconnected in series in order to form the complete

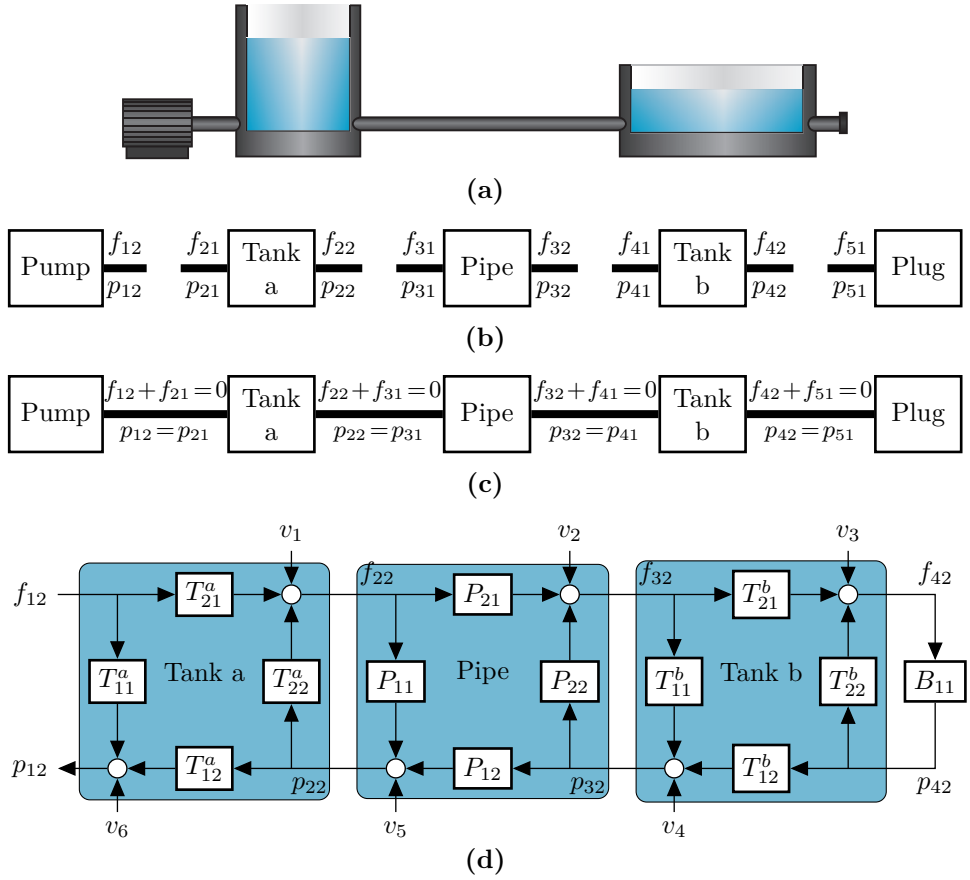

Figure 2.5: Simple hydraulic system.

system as shown in Figure 2.5c. The result is a (continuous time) model of the hydraulic system.

For the system to exhibit a behavior (other than a transient due to the initial conditions) a source and a boundary condition must be included in the model. In the example of Figure 2.5 a constant flow pump can be attached to the left-hand terminal of the left tank, and a plug can be attached to the right-hand terminal of the tank on the right.

By connecting a source to the model, a direction of causality is defined. This is exactly in line with the discussion in Section 1.2 in Chapter 1 where, once a data generating mechanism is present, causality becomes a well-defined notion.

The important feature is that once a direction of causality exists, transfer functions can be used to model each module (Takahashi et al., 1970). By connecting a source to a terminal, one of the variables on that terminal is transformed to an "input". For instance, by connecting a constant flow pump to the nozzle of a tank, the flow rate of that terminal becomes an input to the system. Since the second

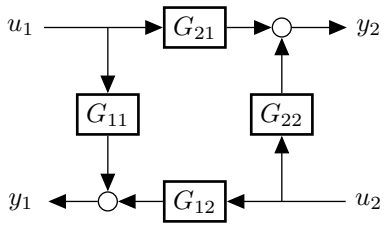

Figure 2.6: Graphical illustration of a bilaterally coupled system.

variable on the terminal is completely determined by the value of the first variable, it becomes an "output". As a consequence each module can be represented as a *bilaterally coupled system* (Mansoori et al., 2014). Once a direction of causality is known, (for a linear system) four transfer functions relate the inputs and outputs of a two-terminal (or two port) subsystem:

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} G_{11}(q) & G_{12}(q) \\ G_{21}(q) & G_{22}(q) \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} \tag{2.3}$$

where u_1 and y_1 are the (input and output) variables that live on terminal 1, and u_2 and y_2 are the (input and output) variables that live on terminal 2. If the subsystem is not linear, then (2.3) represents a linear approximation of the module. Graphically, (2.3) can be expressed as shown in Figure 2.6.

Remark 2.5 *In (2.3) we have assumed a discrete-time representation of the two-port system. However, in physical modeling, one usually deals with differential equations and so the two-port system should be modelled in the Laplace domain. The discussion of the relationship between the continuous-time transfer functions and the discrete-time transfer functions is delayed until Chapter 7. There it is shown that under certain conditions each discrete-time transfer function G_{jk} in (2.3) is simply the discrete-time version of its continuous-time counterpart.*

Consequently, each module of the overall model can be expressed as a bilaterally coupled system, resulting in a dynamic network model as proposed in the previous section. Process noise variables can be added to each sum in order to model/compensate for unknown disturbances that may affect the output.

Example 2.6 *Consider again the system shown in Figure 2.5a. Once the constant flow pump is attached to the left hand terminal of the tank on the left, a direction of causality is defined for the system.*

Consequently, the four transfer function model of (2.3) can be substituted into each module resulting in the dynamic network model shown in Figure 2.5d.

In this section a systematic modeling procedure has been introduced in order to illustrate that many systems in engineering can be modelled as a dynamic network model. In the next section an important property of a dynamic network model is discussed.

2.4 WELL-POSEDNESS OF A DYNAMIC NETWORK MODEL

To characterize the suitability of the equations (2.2) in describing a physical system, the property of well-posedness is used. The topic of well-posedness was first studied in the context of continuous-time closed-loop systems in the late 1960's by Willems (1971); Zames (1964), and for interconnected dynamical systems in the late 1970's and early 1980's by Vitacco & Michel (1977); Vidyasagar (1980); Araki & Saeki (1983). It has remained an important topic in the field of control ever since (see (Zhou et al., 1996; D'Andrea & Dullerud, 2003) for instance). The following quote encapsulates the essence of well-posedness:

> "Well-posedness is essentially a modeling problem. It expresses that a mathematical model is, at least in principle, adequate as a description of a physical system. ... Well-posedness thus imposes a regularity condition on feasible mathematical models for physical systems. ... In other words, since exact mathematical models would always be well posed, one thus requires this property to be preserved in the modeling" (Willems, 1971)

The following pair of definitions are intended to be in the spirit of Willems (1971) and capture the physical reasons why well-posedness is a critical issue. In the following text we will gradually build up the definition of wellposedness from a simple closed loop to a dynamic network model.

Consider the simple closed loop as shown in Figure 2.7. The property of well-posedness for this system is defined as follows. The conditions are stated in a non-formal manner inorder to convey the main message of well-posedness. For a technical definition, see Willems (1971).

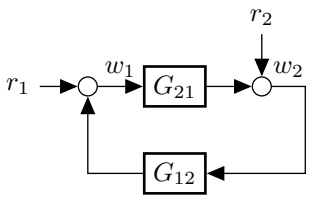

Figure 2.7: Closed-loop system considered by Willems (1971) when discussing well-posedness.

Definition 2.7 *The closed-loop model shown in Figure 2.7 is* well-posed *if the following conditions are satisfied.*

(a) The internal variables w_1 and w_2 are completely (uniquely) determined by r_1 and r_2.

(b) The internal variables w_1 and w_2 depend causally on the external variables r_1 and r_2.

(c) The internal variables w_1 and w_2 depend on r_1 and r_2 in a continuous manner.

(d) Small[2] *changes in the model should not result in a model that does not satisfy Conditions (a) - (c).*

The equations for the system shown in Figure 2.7 are

$$\begin{bmatrix} w_1(t) \\ w_2(t) \end{bmatrix} = \begin{bmatrix} 0 & G_{12}(q) \\ G_{21}(q) & 0 \end{bmatrix} \begin{bmatrix} w_1(t) \\ w_2(t) \end{bmatrix} + \begin{bmatrix} r_1(t) \\ r_2(t) \end{bmatrix}$$
$$w = Gw + r. \tag{2.4}$$

Condition (a) ensures that the operator $(I - G)$ is one-to-one and onto, i.e. it is *invertible*. Conditions (b) and (c) ensure that the operator $(I - G)^{-1}$ is causal and continuous respectively.

Consider the following illustration of Condition (d). Introducing a tiny delay into the loop of the closed-loop model should not result in a model that does not satisfy Conditions (a)-(c) of Definition 2.7. Consider the following examples of illposed models.

Example 2.8 *Consider the closed-loop model shown in Figure 2.7. From (2.4) the transfer function from the variables, r_1, and r_2 to the internal variables w_1 and w_2 is:*

$$\begin{bmatrix} w_1(t) \\ w_2(t) \end{bmatrix} = \frac{1}{1 - G_{12}(q)G_{21}(q)} \begin{bmatrix} 1 & G_{12}(q) \\ G_{21}(q) & 1 \end{bmatrix} \begin{bmatrix} r_1(t) \\ r_2(t) \end{bmatrix}. \tag{2.5}$$

Suppose that $G_{12} = G_{21}^{-1}$, where G_{12} is a proper transfer function, and G_{12}^{-1} is also a proper transfer function. In this case, the term $1 - G_{12}G_{21} = 0$, thus from (2.5), the operator $(I - G)$ is not invertible. Consequently, by the first condition of Definition 2.7 this model is not well posed. It seems reasonable to exclude this system as a realistic model. Since the matrix $I - G$ is not full rank, the internal variables are not uniquely determined by the variables r_1 and r_2. As a model of a physical system, this does not make sense.

Example 2.9 *Consider again the closed loop system shown in Figure 2.7. Suppose G_{12} and G_{21} are both monic transfer functions. In this case, the term $1 - G_{12}(q)G_{21}(q)$ in the denominator of (2.5) is a strictly proper transfer function. Consequently, $1/(1 - G_{12}(q)G_{21}(q))$ is not proper. The result is that the value of w_1 at time t depends on the value of r_1 and r_2 at the future time $t + 1$. In this case the second condition of Definition 2.7 is violated. Again, it seems reasonable to exclude such models as accurate descriptions of physical systems.*

Next, consider a slight generalization of Definition 2.7 to a loop that is embedded in a dynamic network.

Definition 2.10 *Consider a dynamic network as defined in (2.2). Consider a loop embedded in the network. Suppose the loop passes through the internal variables $w_{\ell_1}, \ldots, w_{\ell_n}$. Let $\mathcal{L} = \{w_{\ell_1}, \ldots, w_{\ell_n}\}$. If a variable in the network has a direct*

[2] We use the informal term "small" here to avoid technicalities that distract from the main message. See Willems (1971) for the exact technical statement of the condition.

connection to any $w_\ell \in \mathcal{L}$ denote this variable as a 'variable that affects the loop $\mathcal{L}$'. Let $\mathcal{A}_\mathcal{L}$ denote the set of variables that affect the loop. Note that $\mathcal{A}_\mathcal{L}$ consists of internal variables, external variables and process noise variables. The loop defined by $\mathcal{L}$ is well posed *if the following conditions are satisfied:*

(a) The internal variables of the loop (i.e. all $w_\ell \in \mathcal{L}$) are completely (uniquely) determined by the variables that affect the loop, i.e. all variables in $\mathcal{A}_\mathcal{L}$.

(b) The internal variables of the loop depend causally on the variables that affect the loop.

(c) The internal variables of the loop depend on the variables that affect the loop in a continuous manner.

(d) Small changes in the model should not result in a loop $\mathcal{L}$ that does not satisfy Conditions (a) - (c).

Similar to the case of the definition of well-posedness for a simple closed-loop, the first condition of Definition 2.10 ensures that given all knowledge of everything that is external to the loop, it is possible to uniquely determine the value of the loop variables. The second condition ensures that only past and present knowledge of everything that is external to the loop is needed to determine that value of the internal variables in the loop. Finally, consider the definition of well-posedness for a dynamic network model.

Definition 2.11 *Consider a dynamic network as defined in (2.2). The dynamic network model is* well-posed *if every loop in the network is well-posed.*

By Definition 2.11 it follows that if a dynamic network model as defined in (2.2) is well posed, then the matrix $I - G$ is invertible and moreover, the inverted matrix $(I - G)^{-1}$ only consists of causal transfer functions. Also note that dynamic network models with algebraic loops (i.e. loops such that the loop transfer function, $G_{\ell_1\ell_2} \cdots G_{\ell_n\ell_1}$, has a direct feed-through term) can still be well posed. Thus, well posedness does not place a restriction on the presence of algebraic loops in the dynamic network model.

Remark 2.12 *In some literature (Zhou et al., 1996; D'Andrea & Dullerud, 2003) the definition of well-posedness is stated in terms of a condition on a matrix. Here a justification for the matrix condition is sought, grounded in physical phenomena. Later in this chapter, in Proposition 2.14 it is shown that Definition 2.11 implies the condition on the matrix that (Zhou et al., 1996; D'Andrea & Dullerud, 2003) use as the definition for well-posedness.*

Example 2.13 *Consider the dynamic network model shown in Figure 2.8. The equations for the model are:*

$$\begin{bmatrix} w_1(t) \\ w_2(t) \\ w_3(t) \\ w_4(t) \end{bmatrix} = \begin{bmatrix} 0 & G_{12} & 0 & 0 \\ G_{21} & 0 & 0 & G_{24} \\ G_{31} & 0 & 0 & G_{34} \\ 0 & 0 & G_{43} & 0 \end{bmatrix} \begin{bmatrix} w_1(t) \\ w_2(t) \\ w_3(t) \\ w_4(t) \end{bmatrix} + \begin{bmatrix} v_1(t) \\ v_2(t) \\ v_3(t) \\ v_4(t) \end{bmatrix}. \tag{2.6}$$

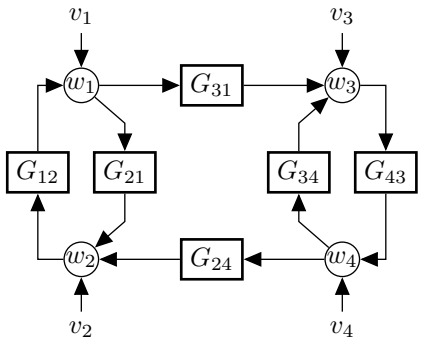

Figure 2.8: Dynamic Network Model considered in Example 2.13.

Suppose that $G_{21} = G_{12}^{-1}$.

There are three loops in this model:

$$w_1 \to w_2 \to w_1$$
$$w_1 \to w_3 \to w_4 \to w_2 \to w_1$$
$$w_3 \to w_4 \to w_3$$

The map from $[v_1\ v_2\ v_3\ v_4]^T$ *to* $[w_1\ w_2\ w_3\ w_4]^T$ *exists and is causal, since the matrix* G *is full rank. Thus the outer loop is well-posed.*

Consider however, the loop through w_1 *and* w_2. *The variables that affect this loop are* $\mathcal{A}_c = \{v_1, v_2, w_4\}$. *The map from these variables to the loop variables is*

$$\begin{bmatrix} w_1(t) \\ w_2(t) \end{bmatrix} = \begin{bmatrix} 1 & -G_{12} \\ -G_{21} & 1 \end{bmatrix}^{-1} \left(\begin{bmatrix} 0 \\ G_{24} \end{bmatrix} w_4 + \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \right).$$

Since $G_{21} = G_{12}^{-1}$ *the matrix inversion is not possible, thus this map is not well defined, and the loop is not well-posed.*

Consequently, by Definition 2.11 the dynamic network (2.6) is not well-posed.

It is useful to have a way of checking whether a dynamic network model is well-posed. The following proposition characterizes the property of well-posedness of a dynamic network model in terms of a condition on the matrix G in (2.2).

Proposition 2.14 *A dynamic network model as defined in (2.2) is well-posed if all principal minors of* $\lim_{z\to\infty}(I - G(z))$ *are non-zero.*

For a proof see Appendix 2.7.1. In fact the condition for wellposedness of Proposition 2.14 is essentially all but necessary. It is only Condition (d) of Definition 2.10 that is not necessarily satisfied if all principal minors of $\lim_{z\ to\infty}(I - G(z)$ are non-zero (see Willems (1971) for details).

2.5 THE GRAPH OF A DYNAMIC NETWORK MODEL

Graph Theory provides many tools for analyzing interconnections between nodes. Throughout this thesis tools from graph theory will be used to analyse properties of dynamic network models. Thus, in this section rules are presented in order to construct a graph of a dynamic network model.

A *directed graph* $\mathbb{G}$ is a collection of nodes (denoted $V(\mathbb{G})$) that are connected together by directed edges (denoted $E(\mathbb{G})$). A directed graph of a dynamic network model can be constructed as follows:

1. Let all w_k, $k = \{1, \ldots, L\}$ be nodes.
2. Let all v_k, $k \in \mathcal{V}$ and r_k, $k \in \mathcal{R}$ be nodes.
3. For all $i, j \in \{1, \ldots, L\}$ if $G_{ji} \neq 0$, then add a directed edge from node w_i to node w_j.
4. For all v_k, $k \in \mathcal{V}$ add a directed edge from v_k to w_k.
5. For all r_k, $k \in \mathcal{R}$ add a directed edge from r_k to w_k.

Example 2.15 *Consider the system of Example 2.2. For convenience the equations of the dynamic network model are repeated here:*

$$\begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \\ w_5 \\ w_6 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & G_{14} & 0 & 0 \\ G_{21} & 0 & G_{23} & 0 & 0 & 0 \\ 0 & G_{32} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & G_{46} \\ 0 & G_{52} & 0 & G_{54} & 0 & G_{56} \\ 0 & 0 & G_{63} & 0 & G_{65} & 0 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \\ w_5 \\ w_6 \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{bmatrix}$$

A graph of the dynamic network model constructed using the rules presented above is shown in Figure 2.9.

There exists a *path* from $w_i \to w_j$ if there exist integers $n_1, \ldots, n_k$ such that $G^0_{jn_1} G^0_{n_1 n_2} \cdots G^0_{n_k i}$ is non-zero. Likewise there exists a path from $r_i \to w_j$ (or $v_i \to w_j$) if there exist integers $n_1, \ldots, n_k$ such that $G^0_{jn_1} G^0_{n_1 n_2} \cdots G^0_{n_k i}$ is non-zero, and $i \in \mathcal{R}$ (or $i \in \mathcal{V}$). A *loop* is a path that starts and ends at the same node.

A graph can be fully represented by its *adjacency matrix*. This matrix, denoted A is a square matrix of size $(L + \text{card}(\mathcal{V}) + \text{card}(\mathcal{R})) \times (L + \text{card}(\mathcal{V}) + \text{card}(\mathcal{R}))$ is defined as

$$A(j, i) = 0 \text{ if there is no path from node } i \text{ to node } j$$
$$A(j, i) = 1 \text{ if there is no path from node } i \text{ to node } j.$$

Because of the interconnection structure that we consider in this thesis (see (2.2)) it follows that $A(i, i) = 0$, for all nodes in the graph.

The following lemma from graph theory will be very useful.

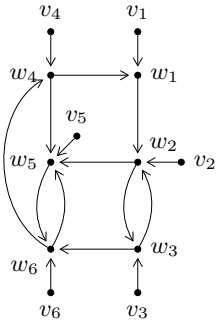

Figure 2.9: Graph of the dynamic network shown in Fig. 2.2

Proposition 2.16 (Diestel (1997)) *Consider a directed graph with adjacency matrix A. Then for $k \geq 1$, $[A^k]_{ji}$ indicates the number of distinct paths of length k from node i to node j.* □

In addition to the adjacency matrix A defined above, we will also consider a related delay-adjacency matrix A_d of which the elements have three possible values: 0 (no link), 1 (a link with no delay), and d (a link with a delay). Through the use of the following rules for addition and multiplication:

$$\begin{array}{llll} 0+0=0 & 0+d=d & 0\cdot 0=0 & 0\cdot d=0 \\ 0+1=1 & 1+d=1 & 0\cdot 1=0 & 1\cdot d=d \\ 1+1=1 & d+d=d & 1\cdot 1=1 & d\cdot d=d \end{array}$$

summation and multiplication of matrices A_d can be defined, and one can evaluate $[A_d^k]_{ji}$. The following proposition is helpful in characterizing the presence of delays in particular paths in a dynamic network model.

Proposition 2.17 *Consider a directed graph with delay-adjacency matrix A_d and the rules of multiplication and addition with d. Then for $k \geq 1$,*

- $[A_d^k]_{ji} = 1$ *indicates that there is a path of length k from i to j without a delay,*
- $[A_d^k]_{ji} = d$ *indicates that all paths of length k from i to j have a delay,*
- $[A_d^k]_{ji} = 0$ *indicates that there is no path of length k from i to j.*

The following proposition presents a property of the adjacency matrix of an acyclic graph (i.e. a graph with no loops).

Proposition 2.18 (Deo (1974)) *A directed graph is acyclic if and only if its vertices can be ordered such that the adjacency matrix is an upper (or lower) triangular matrix.*

More concepts from graph theory will be used throughout the thesis, but they will be presented where they are applicable.

2.6 SUMMARY

In this chapter a dynamic network model has been presented. It has been shown that this type of model can be used to model many different physical phenomena. The property of well-posedness has been investigated. Based on the discussions in this chapter the following general assumption will be made for all dynamic network models from this point on in the thesis.

Assumption 2.19 *General Conditions.*

(a) The dynamic network model is well-posed.

(b) The dynamic network is stable in the sense that $(I-G)^{-1}$ *consists only of stable transfer functions (transfer functions with poles inside the unit circle).*

(c) All r_m, $m \in \mathcal{R}$ *are uncorrelated to all* v_k, $k \in \mathcal{V}$. □

In the following chapters it is shown how this model can be used in a system identification framework. Before presenting how to identify transfer functions in a dynamic network model, some of the basics of system identification are presented in the next chapter.

2.7 APPENDIX

2.7.1 Proof of Proposition 2.14

Before presenting the proof of Proposition 2.14 consider the following useful result from Control Theory (Scherer, 2001). A transfer function matrix is called *proper* if all its elements are proper transfer functions.

Proposition 2.20 *let* $G(z)$ *be a proper, invertible transfer function matrix. The transfer function matrix* $G(z)^{-1}$ *is proper if and only if* $\lim_{z\to\infty} G(z)$ *is non-singular.*

Proof: It is easiest to proceed using a state-space representation of $G(z)$. Since, $G(z)$ is proper, it can be expressed as $G(z) = C(Iz - A)^{-1}B + D$. It follows that

$$\lim_{z\to\infty} G(z) = D.$$

By assumption, G is invertible, thus, it follows that D is non-singular. Since, D is invertible, $G(z)^{-1}$ can be expressed as

$$G(z)^{-1} = -D^{-1}C(Iz - A + BD^{-1}C)^{-1}BD^{-1} + D^{-1}, \tag{2.7}$$

which is also a proper transfer function matrix.

Next, a proof of Proposition 2.14 is presented.

Proof: It must be shown that every loop in the dynamic network model is well-posed. Consider a loop embedded in the dynamic network consisting of the internal variables $w_{d_1}, \ldots, w_{d_n}$, $\{d_1, \ldots, d_n\} = \mathcal{D}$. Let $\mathcal{Z}$ denote the indices of the internal variables not in the loop, i.e. $\mathcal{Z} = \{1, \ldots, L\} \setminus \mathcal{D}$. Let $w_{\mathcal{D}}$ denote the vector

$[w_{d_1} \ \cdots \ w_{d_n}]^T$. Let $w_{\mathcal{Z}}$ be similarly defined. Using this notation the equations of the dynamic network model (2.2) can be expressed as:

$$\begin{bmatrix} w_{\mathcal{D}} \\ w_{\mathcal{Z}} \end{bmatrix} = \begin{bmatrix} G_{\mathcal{D}\mathcal{D}} & G_{\mathcal{D}\mathcal{Z}} \\ G_{\mathcal{Z}\mathcal{D}} & G_{\mathcal{Z}\mathcal{Z}} \end{bmatrix} \begin{bmatrix} w_{\mathcal{D}} \\ w_{\mathcal{Z}} \end{bmatrix} + \begin{bmatrix} r_{\mathcal{D}} \\ r_{\mathcal{Z}} \end{bmatrix} \begin{bmatrix} v_{\mathcal{D}} \\ v_{\mathcal{Z}} \end{bmatrix}$$

with obvious definitions for $G_{\mathcal{D}\mathcal{D}}$, $G_{\mathcal{D}\mathcal{Z}}$, etc., and $r_{\mathcal{D}}$, $r_{\mathcal{Z}}$, etc. Consequently, the internal variables of the loop can be expressed as:

$$w_{\mathcal{D}} = G_{\mathcal{D}\mathcal{D}} w_{\mathcal{D}} + G_{\mathcal{D}\mathcal{Z}} w_{\mathcal{Z}} + r_{\mathcal{D}} + v_{\mathcal{D}}. \tag{2.8}$$

Since all the principal minors of $\lim_{z\to\infty}(I - G(z))$ are assumed to be non-zero, it follows that $\lim_{z\to\infty}(I - G_{\mathcal{D}\mathcal{D}}(z))$ is non-singular. Thus, by Proposition 2.20 $(I - G_{\mathcal{D}\mathcal{D}}(z))$ is causally invertible (i.e. it is invertible and the inverse consists only of proper transfer functions). Thus, (2.8) can be expressed as:

$$w_{\mathcal{D}} = (I - G_{\mathcal{D}\mathcal{D}})^{-1}\big(G_{\mathcal{D}\mathcal{Z}} w_{\mathcal{Z}} + r_{\mathcal{D}} + v_{\mathcal{D}}\big), \tag{2.9}$$

where all transfer functions are proper.

The first important point is that since $(I - G_{\mathcal{D}\mathcal{D}})$ is invertible, the internal variables $w_{\mathcal{D}}$ can be uniquely determined by the variables that affect the loop, i.e. any $w_{\mathcal{Z}}$, $r_{\mathcal{D}}$ and $v_{\mathcal{D}}$ with direct connections to and internal variables in $w_{\mathcal{D}}$. Thus Condition (a) of Definition 2.10 is satisfied.

Secondly, since $(I - G_{\mathcal{D}\mathcal{D}})$ is causally invertible, Condition (b) of Definition 2.10 is satisfied.

Thirdly, since all transfer functions in (2.9) are assumed to be rational by the definition of a dynamic network model, Condition (c) of Definition 2.10 is satisfied.

That Condition (d) of Definition 2.10 also holds is not proved here. However, the reasoning of Willems (1971); Vidyasagar (1980) can be extended to the case of (2.9). The technical details are beyond the scope of this thesis.

Thus all the conditions for well-posedness of a loop embedded in a dynamic network are satisfied. Since, all principal minors of $\lim_{z\to\infty}(I - G(z))$ are assumed to be non-zero, the reasoning applies to all loops embedded in the dynamic network, and thus the dynamic network model is well posed. □

Chapter 3

SYSTEM IDENTIFICATION

In this chapter the prediction-error framework for system identification is presented (Ljung, 1999). After a brief overview of system identification, a data generating system is defined. The idea of a data generating system is useful for the analysis of system identification methods. Then the prediction-error framework is presented with an emphasis on the concept of consistency, which will be the main focus throughout the thesis. As mentioned in the previous chapters, the approach that we take in this thesis is to extend closed-loop identification methods to the case of identification in dynamic networks. In this section several closed-loop identification methods are presented. Namely the Direct, Two-Stage, Joint IO and Instrumental Variable Prediction-Error methods. Each of these methods can be used under different circumstances. The chapter ends with an overview of the open- and closed-loop identification problem in the presence of sensor noise. (i.e. a closed-loop errors-in-variables framework).

3.1 INTRODUCTION

SYSTEM IDENTIFICATION refers to the procedure of obtaining a model from a given data set. There exist many different methods in the system identification literature. The most suitable method depends on considerations such as:

- is the data set generated in open-loop, closed-loop, or dynamic network?
- how is the data collected? (is sensor noise a problem?)
- are there external signals present that are exactly known?
- is it possible to actively interact with the system? In other words, can a signal of choice be used to excite the system? Or is the user a passive observer collecting data from the system?
- are multiple experiments possible?
- what is the purpose of the obtained model? For instance, simulation, prediction, controller design, physical parameter estimation?

- what type of model is desired, i.e. continuous time, discrete time, rational transfer function, non-parametric frequency domain models, etc.?

In this chapter various methods of system identification are presented that will be useful in view of the dynamic network identification problem. In this thesis only the prediction-error methods of system identification are studied. The main reason being that there are well established tools to analyze the properties of the prediction-error methods. The focus of this thesis is to analyze under which conditions it is possible to obtain consistent estimates of a transfer function. Albeit important, the focus is not on variance analysis of the obtained estimates, or the practical implementation of the methods.

In the system identification field it is well known that special attention is required by the user when identifying a plant that is operating in a closed loop (i.e. not every open-loop method will automatically work when it is used in a closed loop setting). Since in networks, there is the possibility of loops, it seems natural that the same problems that are encountered in identification in closed-loop systems are also present in identification in networks. Consequently, the starting point for this thesis is to extend various closed-loop methods to the network setting. A typical closed loop system is shown in Figure 3.1.

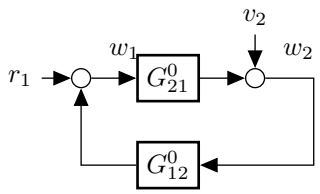

Figure 3.1: A typical closed loop system considered in the identification literature.

The main problem when identifying a plant operating in a closed-loop is that the input to the plant (w_1 in Figure 3.1) is correlated to the noise on the output of the plant (v_2 in Figure 3.1) due to the feedback path. A variety of methods have been developed to obtain consistent estimates even when a plant is operating in a closed-loop including

- Direct Methods, where the idea is to directly identify transfer functions between internal variables (Ljung, 1999; Van den Hof et al., 1992).
- Indirect methods such as the Two-Stage and Projection Method where first the internal variables are projected onto external excitation variables, and then the transfer function between the projected signals is estimated (Van den Hof & Schrama, 1993; Forssell & Ljung, 2000).
- Joint Input-Output methods, where an estimate of $(I - G^0)^{-1}$ is obtained, and then using algebraic manipulations the matrix G^0 is recovered (Granger, 1969; Caines & Chan, 1975; Caines, 1976; Ng et al., 1977; Granger, 1980; Gevers & Anderson, 1981; Anderson & Gevers, 1982; Gevers & Anderson, 1982).

- Closed-loop Instrumental Variable Methods(Söderström & Stoica, 1983; Söderström et al., 1988; Söderström & Stoica, 1989b; Gilson & Van den Hof, 2005; Gilson et al., 2009, 2011).
- Controller Design Relevant Methods (Zhu & Stoorvogel, 1992; de Callafon et al., 1993; Van den Hof et al., 1995; Van den Hof & de Callafon, 1996; Van Donkelaar & Van den Hof, 2000).
- Closed-Loop State Space Methods (Verhaegen, 1993; Ljung & McKelvey, 1996; Chuiso & Picci, 2003; Chiuso & Picci, 2005; Chiuso, 2007; Van der Veen et al., 2010, 2013).
- Closed-Loop Frequency Domain Methods (Schoukens et al., 1997; Pintelon & Schoukens, 2006, 2012b)
- Closed-loop Errors-in-Variables Methods where there is assumed to be sensor noise present on the input to the module (Söderström et al., 2013).
- Closed-Loop Continuous-time identification (Gilson & Garnier, 2003; Gilson et al., 2008).

In this thesis each of the methods in the list above will be addressed except the control-relevant and frequency-domain methods. State space methods will also not be dealt with explicitly.

Remark 3.1 *Although the state-space methods are not directly dealt with, there are strong connections between state space methods and the prediction-error methods. The sub-space method presented in Verhaegen (1993) can be seen as a Joint-IO method. In addition, in Ljung & McKelvey (1996); Chuiso & Picci (2003); Chiuso & Picci (2005) it is shown that sub-space methods are equivalent to using a high-order ARX model-structure. In Chiuso (2007); Van der Veen et al. (2010) this idea is incorporated to develop so called* predictor based subspace identification *methods for closed-loop systems. Thus, because the state-space methods are closely related to the prediction-error methods, indirectly, statements about the consistency of the state-space methods can be made, however, this is not explicitly done in this thesis.*

In order to analyze the consistency of a method, it is useful to make assumptions about a data generating system. In particular we will assume that the data generating system has the form of a dynamic network model as presented in Chapter 2. This should not be seen as a *true system*, but rather a useful tool to enable the analysis of an identification method. This is what Ljung (1999) refers to as the fiction of a true system:

> "The real-life actual system is an object of a different kind than our mathematical models. In a sense there is an impenetrable screen between our world of mathematical descriptions and the real world. We can look through this window and compare certain aspects of the physical system with its mathematical description, but we can never establish any exact connection between them. The question of nature's susceptibility

> to mathematical description has some deep philosophical aspects, and in practical terms we have to take a more pragmatic view of models" (Ljung, 1999)

Nevertheless, the fiction of a true system is useful (and is heavily employed in this thesis) in order to analyze the properties of an identification method.

The remainder of this chapter is organized as follows. First, in Section 3.2 the data generating system is defined. The measurement setup is considered to be part of the data generating system. In Section 3.3 the Prediction-Error framework is presented. In Section 3.4, closed-loop identification is presented, and the Direct, Two-Stage, Joint IO and Instrumental Variable closed-loop identification methods are presented. Finally, in Section 3.5 the closed-loop Errors-in-Variables framework is presented, which is a framework that incorporates sensor noise.

3.2 THE DATA GENERATING SYSTEM

As mentioned in the introduction, the data generating system is assumed to be a dynamic network model as presented in Chapter 2. Thus, the internal variables are assumed to satisfy:

$$\begin{aligned}\begin{bmatrix} w_1(t) \\ w_2(t) \\ \vdots \\ w_L(t) \end{bmatrix} &= \begin{bmatrix} 0 & G^0_{12}(q) & \cdots & G^0_{1L}(q) \\ G^0_{21}(q) & 0 & \ddots & G^0_{2L}(q) \\ \vdots & \ddots & \ddots & \vdots \\ G^0_{L1}(q) & G^0_{L2}(q) & \cdots & 0 \end{bmatrix} \begin{bmatrix} w_1(t) \\ w_2(t) \\ \vdots \\ w_L(t) \end{bmatrix} + \begin{bmatrix} r_1(t) \\ r_2(t) \\ \vdots \\ r_L(t) \end{bmatrix} + \begin{bmatrix} v_1(t) \\ v_2(t) \\ \vdots \\ v_L(t) \end{bmatrix} \\ &= G^0(q)w(t) + r(t) + v(t), \end{aligned} \tag{3.1}$$

with obvious definitions for G^0, w, r and v. In this thesis, the transfer functions of the data generating system will be denoted with a superscript 0. A single internal variable can be expressed as:

$$w_j(t) = \sum_{k \in \mathcal{N}_j} G^0_{jk}(q)w_k(t) + r_j(t) + v_j(t) \tag{3.2}$$

As with the dynamic network models presented in Chapter 2 the data generating system (3.1) is assumed to satisfy Assumption 2.19 (see Section 2.6 of Chapter 2).

In practice, the data is collected using measurement devices (sensors or probes for instance). Let $\tilde{w}_j$ denote the measurement of w_j. It is typically not possible to exactly record the true value of a variable; a measurement intrinsically has some error. This error will be referred to as the *measurement error*, or *sensor noise*. The measurement $\tilde{w}_j$ of w_j is modeled as

$$\tilde{w}_j(t) = w_j(t) + s_j(t)$$

where s_j represents the measurement error and is modeled as a stochastic process with rational power spectral density (it is not necessarily assumed to be white). The

measurements of the variables of the data generating system (3.1) are then

$$\begin{bmatrix} \tilde{w}_1(t) \\ \vdots \\ \tilde{w}_L(t) \end{bmatrix} = \begin{bmatrix} w_1(t) \\ \vdots \\ w_L(t) \end{bmatrix} + \begin{bmatrix} s_1(t) \\ \vdots \\ s_L(t) \end{bmatrix}$$
$$\tilde{w}(t) = w(t) + s(t)$$

with obvious definitions for $\tilde{w}$, w and s. Suppose that the measurements are obtained using physically different sensors, then it seems reasonable to assume that the errors each sensor is making in recording the variable are uncorrelated to each other. On the other hand, if there is an external phenomenon affecting all sensors, such as a mechanical vibration, then it is possible that the measurement errors are all affected by the mechanical vibration, resulting in a correlation between measurement errors.

Remark 3.2 *There is a very interesting difference between the process noise, v and the sensor noise s. Sensor noise does not affect the internal variables, it is just an error made in recording the value of the internal variable. Consequently, a measurement of the internal variable, w_k is not affected by sensor noise on w_j, $j \neq k$. On the other hand, process noise does affect the value of an internal variable. Consequently, an internal variable w_k is affected by the process noise v_j, if there is a path from w_j to w_k.*

In the following section, we shift our attention from the data generating system, to the identification methods.

3.3 PREDICTION-ERROR IDENTIFICATION

In this section, the prediction-error framework is briefly presented with a focus on evaluating the consistency of a model. For a more in depth treatment of the material, see Ljung (1999) for instance. Essentially, the philosophy of the prediction-error framework is that candidate models are evaluated based on their predictive capability. Before presenting the prediction-error framework, a framework for dealing with mixed deterministic and stochastic variables is briefly presented.

3.3.1 Quasi-Stationary Variables

Many of the variables in a data generating system as defined in Section 3.2 have a deterministic component and a stochastic component, thus they are said to be *quasi-stationary* variables (Ljung, 1999). The following tools are useful when dealing with quasi-stationary variables. The generalized expected value operator is (Ljung, 1999)

$$\bar{\mathbb{E}}[\,\cdot\,] = \lim_{t\to\infty} \frac{1}{N} \sum_{t=0}^{N-1} \mathbb{E}[\,\cdot\,]$$

where $\mathbb{E}[\,\cdot\,]$ is the (standard) expected value operator. Note that this is a combination of the ensemble mean and time-average of a variable.

Using this operator, the auto- and cross-covariance functions are defined as:

$$R_x(\tau) = \bar{\mathbb{E}}[x(t)x(t-\tau)] \text{ and } R_{xy}(\tau) = \bar{\mathbb{E}}[x(t)y(t-\tau)].$$

The power spectral density and cross power spectral densities are consequently defined as

$$\Phi_x(\omega) = \mathcal{F}[R_x(\tau)] \text{ and } \Phi_{xy}(\omega) = \mathcal{F}[R_{xy}(\tau)]$$

where $\mathcal{F}[\,\cdot\,]$ denotes the Discrete-Time Fourier Transform,

$$X(\omega) = \mathcal{F}[x(t)] = \sum_{t=-\infty}^{\infty} x(t)e^{-j\omega t}.$$

3.3.2 Prediction-Error Framework

There are three main pillars in any system identification framework. The data record, the model set, and the identification criterion. The data record must contain enough "excitation" so that the identification criterion can discriminate between the candidate models (the data set must be *informative enough with respect to the model structure*). The model structure places constraints on the set of candidate models. The identification criterion selects one model out of the possible sets of candidate models. Each of the three pillars are briefly presented in the context of the Prediction-Error framework.

First, consider the model set. In this thesis we will use the one-step-ahead predictor model. The point of a predictor model is to predict the value of a particular variable using the past values of that variable and other variables available in the data set. Let $w_j(t)$ denote the (internal) variable to be predicted[1]. Let $w_k(t)$ $k \in \mathcal{D}_j$ denote the internal variables that will be used to predict the value of $w_j(t)$. Suppose that the relationship between $w_j(t)$ and w_k, $k \in \mathcal{D}_j$ is:

$$w_j(t) = \sum_{k\in\mathcal{D}_j} G^0_{jk}(q)w_k(t) + v_j(t),$$

where $v_j(t) = H^0_j(q)e_j(t)$, and e_j is a white noise process. Next, a predictor is constructed in order to predict the current value of w_j using only the past values of $w_j(t)$, i.e. $\{w_j(t-1), w_j(t-2), \ldots\}$ and the current and past values of w_k, $k \in \mathcal{D}_j$, i.e. $\{w_k(t), w_k(t-1), \ldots\}$. In Ljung (1999) it is shown that such a *one-step-ahead predictor* of $w_j(t)$ has the form:

$$\hat{w}_j(t|t-1) = \sum_{k\in\mathcal{D}_j} H_j^{0^{-1}}(q)G^0_{jk}(q)w_k(t) + \big(1 - H^{0^{-1}}(q)\big)w_j(t).$$

The transfer functions $G^0_{jk}(q)$, $k \in \mathcal{D}_j$ and $H^0_j(q)$ are not known of course, and so they are modelled using parameterized transfer functions $G_{jk}(q,\theta)$, $k \in \mathcal{D}_j$ and

[1] Here we use the dynamic network notation in order to denote inputs and outputs. This is done to avoid having to re-introduce a new notation later in the thesis.

$H_j(q,\theta)$, where θ is a parameter vector. The one-step-ahead prediction model is then:

$$\hat{w}_j(t|t-1,\theta) = \sum_{k\in\mathcal{D}_j} H_j^{-1}(q,\theta)G_{jk}(q,\theta)w_k(t) + \big(1 - H_j^{-1}(q,\theta)\big)w_j(t). \tag{3.3}$$

The prediction-error for a model is:

$$\begin{aligned}\varepsilon_j(t,\theta) &= w_j(t) - \hat{w}_j(t|t-1,\theta)\\ &= H_j^{-1}(q,\theta)\Big(w_j(t) - \sum_{k\in\mathcal{D}_j} G_{jk}(q,\theta)w_k(t)\Big).\end{aligned} \tag{3.4}$$

In this thesis, the transfer functions $G_{jk}(q,\theta)$, $k \in \mathcal{D}_j$ and $H_j(q,\theta)$ are parameterized as rational functions:

$$G_{jk}(q,\theta) = \frac{q^{-n_k}(b_0^{jk} + b_1^{jk}q^{-1} + \cdots + b_{n_b}^{jk}q^{-n_b})}{1 + f_1^{jk}q^{-1} + \cdots + f_{n_f}^{jk}q^{-n_f}}, \text{ and}$$

$$H_j(q,\theta) = \frac{1 + c_1^{j}q^{-1} + \cdots + c_{n_c}^{j}q^{-n_c}}{1 + d_1^{j}q^{-1} + \cdots + d_{n_d}^{j}q^{-n_d}}.$$

where the parameter vector

$$\begin{aligned}\theta_{jk} &= [b_0^{jk} \ \cdots \ b_{n_b}^{jk} \ f_1^{jk} \ \cdots \ f_{n_f}^{jk}]^T,\\ \theta_j &= [c_1^{j} \ \cdots \ c_{n_c}^{j} \ d_1^{j} \ \cdots \ d_{n_d}^{j}]^T,\\ \theta &= [\theta_{jk_1} \ \cdots \ \theta_{jk_n} \ \theta_j]^T, \quad \{k_1,\ldots,k_n\} = \mathcal{D}_j.\end{aligned}$$

This parameterization is completely characterized by n_b, n_f, n_k, n_c and n_d. Let $\mathcal{M}$ denote the set of all candidate models. Let M_i denote a model in $\mathcal{M}$.

Now consider the data record. The second pillar of identification is that the data must satisfy certain conditions to ensure that "different" models result in "different" predicted outputs. A data set that meets this requirement is said to be *informative enough with respect to the model structure*. This statement is formalized in the following two definitions. First it is defined what "different models" are, and secondly informative data is defined.

Definition 3.3 *Let $G_{jk}^{M_i}$ and $H_j^{M_i}$ denote the transfer functions G_{jk} and H_j respectively in (3.3) of the model M_i. Two models $M_1, M_2 \in \mathcal{M}$ are equal if*

$$G_{jk}^{M_1}(e^{j\omega}) = G_{jk}^{M_2}(e^{j\omega}), \forall k \in \mathcal{D}_j, \text{ and } H_j^{M_1}(e^{j\omega}) = H_j^{M_2}(e^{j\omega})$$

for almost all $\omega \in [0,\pi]$. □

In otherwords two models are equal if the frequency response functions of the transfer functions defining the two models are the same. Let Z^N denote the data set where N is the data length.

Definition 3.4 *Let $\hat{w}_j^{M_i}$ denote the output predicted by a model M_i. A data set Z^∞ is* informative w.r.t. a model structure *if for any M_1, $M_2 \in \mathcal{M}$,*

$$\bar{\mathbb{E}}[(\hat{w}_j^{M_1}(t|Z^\infty) - \hat{w}_j^{M_2}(t|Z^\infty))] = 0 \implies M_1 = M_2$$

where equality of M_1 and M_2 is defined according to Definition 3.3.

In otherwords, when the data set is informative with respect to the model structure, if two models predict the same output, then these two models must be the same. Informative data ensures that it is possible to discriminate between all the possible candidate models.

The last pillar of identification is the criterion that is used to select one model from the set of all candidate models. The *selection criterion*, or *identification criterion* selects the model with the best predictive capability. It is common to choose the model with the smallest sum of squared prediction errors. In this way a scalar number is attached to each model in the model set:

$$V_N(\theta) = \frac{1}{N}\sum_{t=0}^{N-1} \varepsilon_j(t,\theta)^2. \tag{3.5}$$

The optimal model is then the model with

$$\hat{\theta}_N = \arg\min_\theta V_N(\theta).$$

The function $V_N(\theta)$ is the *identification criterion*. Sometimes V_N is referred to as the *objective function*. Note that the sum of squared prediction errors is only one possible identification criterion.

Under standard (weak) assumptions the estimated transfer function converges in the number of data N, to satisfy (Ljung (1999)):

$$G_{jk}(q,\hat{\theta}_N) \to G_{jk}(q,\theta^*) \text{ w.p. 1 as } N \to \infty$$

where

$$\theta^* = \arg\min_\theta \bar{V}(\theta), \text{ where } \bar{V}(\theta) = \bar{\mathbb{E}}[\varepsilon_j(\theta)^2]$$

If $G_{jk}(q,\theta^*) = G_{jk}^0(q)$ the estimated module transfers are said to be *consistent*. Consistent estimates are possible under several different conditions dependent on the experimental circumstances, and the chosen model parameterization. As mentioned, consistency plays a central role in the remainder of this thesis.

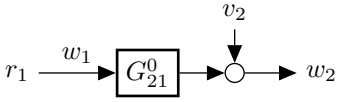

Figure 3.2: A open-loop data generating system.

The following open-loop result will help to illustrate the difference between open- and closed-loop identification. The open-loop system under consideration is shown in Figure 3.2, where

$$w_2(t) = G_{21}^0(q)w_1(t) + v_2(t).$$

Note that for this system the input is exactly known (i.e. there is no sensor noise on w_1. The case with sensor noise on w_1 is called an *Error-in-Variables* system, and is dealt with later in Section 3.5.

Proposition 3.5 *Consider the open-loop data generating system shown in Figure 3.2. Consider a predictor model of the form (3.3). Suppose that the data is informative with respect to the model structure. Consider an estimate $G(q, \hat{\theta}_N)$ obtained by minimizing (3.5). The estimate $G(q, \hat{\theta}_N)$ is consistent if the following conditions hold:*

(a) The noise v_2 is uncorrelated to the input w_1.

(b) The parameterization is chosen flexible enough such that there exists a θ^0 such that $G(q, \theta^0) = G^0$.

(c) The noise model $H_2(\theta)$ is independently parameterized of $G_{21}(\theta)$.

The main condition that we wish to highlight is that as long as the noise is uncorrelated to the input, then consistent estimates of G^0 are possible (as long as the other conditions are also satisfied). Secondly, note that the noise model $H(q, \theta)$ does not need to be parameterized flexible enough to be able to represent the dynamics of H^0.

In the following section some features of the closed-loop identification problem are presented.

3.4 CLOSED-LOOP IDENTIFICATION

The closed-loop identification setup is shown in Figure 3.3. In this figure the symmetry between the predictor model and the data generating system becomes apparent. The closed-loop data generating system is defined as:

$$\begin{aligned} w_1(t) &= G_{12}^0(q)w_2(t) + r_1(t) \\ w_2(t) &= G_{21}^0(q)w_1(t) + v_2(t). \end{aligned}$$

Why is closed-loop identification challenging? What is the critical difference between open- and closed-loop identification? In open-loop identification the process noise is assumed to be uncorrelated to the input of the system (see Condition (a) of Proposition 3.5). This fundamental assumption does not hold in a closed-loop setting since there is a path from the output back to the input. From the expression of the "input" w_1:

$$w_1(t) = \frac{G_{12}^0(q)}{1 - G_{12}^0(q)G_{21}^0(q)}v_2(t) + \frac{1}{1 - G_{12}^0(q)G_{21}^0(q)}r_1(t)$$

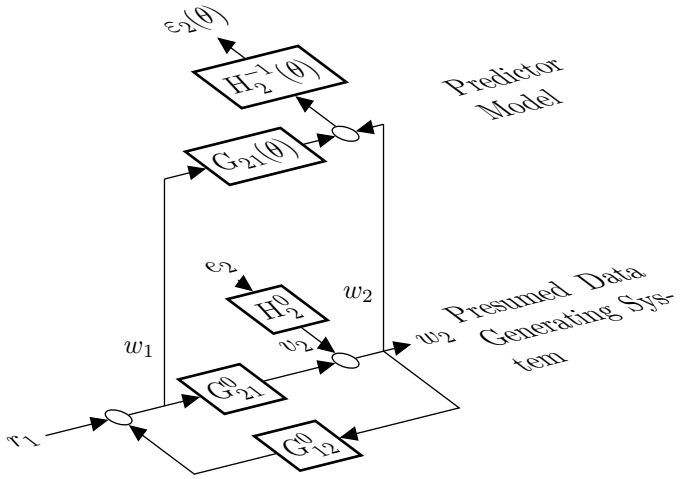

Figure 3.3: Closed-loop identification setup.

it is clear that it is correlated to the process noise on the output v_2. Why is this a problem? It means that it is tough to distinguish which component of the output is due to the noise, and which component is due to the input.

Over the last few decades several technique have been devised to deal with this problem (see the survey papers Van den Hof (1998); Forssell & Ljung (1999) for for an overview). The techniques can all be interpreted as somehow "decorrelating" the input from the process noise in order to reduce the problem back to the open loop situation (where it is easily possible to obtain a consistent estimate). Four of the techniques are presented in the following sections: the Direct, Two-Stage, Joint Input/Output, and Instrumental Variable Methods.

3.4.1 Direct Method of Closed-Loop Identification

In this method, one simply chooses w_1 as the input, w_2 as the output, constructs the predictor (3.3) and obtains an estimate of G_{21}^0 by minimizing the objective function (3.5).

> "Actually, the presence of feedback is simply discarded; the data is treated as if there was no feedback at all. This is of course an attractive approach if it works, since one does not have to bother about the presence of feedback at all" (Van den Hof, 2006).

Thus the predictor model that is used is:

$$\hat{w}_2(t|t-1,\theta) = H_2^{-1}(q,\theta)G_{21}(q,\theta)w_1(t) + \big(1 - H_2^{-1}(q,\theta)\big)w_2(t). \tag{3.6}$$

This approach results in consistent estimates under certain conditions. These conditions are briefly presented in this section.

The following proposition is a closed-loop counterpart to Proposition 3.5. The main point is that instead of requiring that the process noise is uncorrelated to the input (as was the case in Condition (a) of Proposition 3.5), we now require that there exists a noise model $H_2(\theta^*)$ such that the filtered version of the process noise is uncorrelated to the input. In other words, at the optimal parameter vector, the (filtered) noise is uncorrelated to the input.

Proposition 3.6 *Consider a closed-loop data generating system that satisfies Assumption 2.19. Consider the predictor model (3.6). Suppose that the data is informative w.r.t. the model structure. The estimate $G_{21}(q, \hat{\theta}_N)$ obtained by minimizing (3.5) is consistent if the following conditions hold:*

(a) There exists an $H_2(\theta^*)$ such that $H_2^{-1}(q, \theta^*)v_2(t)$ is white.

(b) $\bar{\mathbb{E}}[H_2^{-1}(q, \theta^*)v_2(t) \cdot \Delta G_{21}(q, \theta)w_1] = 0$ for all θ, where $\Delta G_{21}(q, \theta) = G_{21}^0(q) - G_{21}(q, \theta)$.

(c) There exists a θ^0 such that $G_{21}(q, \theta^0) = G_{21}^0(q)$. □

Thus, from Condition (b) of Proposition 3.6 it follows that if the (filtered) noise term $H_2^{-1}(q, \theta^*)$ is uncorrelated to the (filtered) input term $\Delta G_{21}(q, \theta)w_1$ then consistent estimates of $G_{21}^0(q)$ are possible. This is a statement that is analogous to the open-loop statement of the previous section. A typical way to ensure that Condition (b) holds is by assuming that there is a delay present in the loop (i.e. there is no *algebraic loop* in closed-loop system). Then $\Delta G_{21}(q, \theta)w_1(t)$ is a function of only delayed versions of $e_2(t)$, which is uncorrelated to $H_2(q, \theta^*)v_2(t) = e_2(t)$. This reasoning is emphasized in the following Corollary to Proposition 3.6

Corollary 3.7 *Consider a closed-loop data generating system that satisfies Assumption 2.19. Consider the predictor model (3.6). Suppose that the data is informative w.r.t. the model structure. Suppose that the data is informative w.r.t. the model structure. The estimate $G_{21}(q, \hat{\theta}_N)$ obtained by minimizing (3.5) is consistent if the following conditions hold:*

(a) There is a delay present in the loop (no algebraic loop).

(b) If G_{21}^0 has a delay, then $G_{21}(q, \theta)$ is parameterized with a delay.

(c) There exists a θ such that $G_{21}(q, \theta) = G_{21}^0(q)$ and $H_j(q, \theta) = H_j^0(q)$. □

For the Direct Method, several variables can contribute to the informativity of the data: the external variable r_1, the noise v_2, the order of the feedback path (the controller), and a controller that switches between several setting during the data collection experiment. Detailed conditions to ensure the data is informative can be found in Gevers et al. (2009a,b); Bazanella et al. (2010).

3.4.2 Two-stage method of closed-loop identification

In the Two-Stage Method a different approach is used to "decorrelate" the input and the process noise. The idea is to project the input orthogonal to the noise in a first stage, and then in a second stage obtain an estimate of G_{21}^0 using the projected input (which now by construction is uncorrelated to the process noise). The method was first proposed in Van den Hof & Schrama (1993) and a modification of the method is proposed in Forssell & Ljung (1999).

In the second stage, the same predictor structure (3.6), is used to address the closed loop identification problem. However, instead of using predictor input w_1, it uses only the part of w_1 that is correlated to the external signal r_1.

First it is shown how to project the input w_1 orthogonal to the noise v_2. Note that r_1 and w_1 are quasi-stationary signals (Ljung (1999)) such that the cross-correlation function

$$R_{w_1 r_1}(\tau) := \bar{\mathbb{E}}[w_1(t) r_1(t-\tau)]$$

is zero for $\tau < 0$ and non-zero for $\tau \geq 0$. Then there exists a proper transfer function $F_{w_1 r_1}^0$ such that

$$w_1(t) = F_{w_1 r_1}^0(q) r_1(t) + z(t)$$

with z uncorrelated to r_1. This provides a decomposition

$$w_1(t) = w_1^{(r_1)}(t) + w_1^{(\perp r_1)}(t)$$

with $w_1^{(\perp r_1)}(t) = z(t)$. Note that $w_1^{(r_1)}$ is the projection of signal w_1 onto the space of (causally) time-shifted versions of r_1, however, we will simply refer to $w_1^{(r_1)}$ as the *projection of* w_1 *onto* r_1.

If r_1 and w_1 are available from measurements then $F_{w_1 r_1}^0(q)$ can be consistently estimated from data, provided that the signal r_1 is persistently exciting of a sufficiently high order. This consistent estimation can be done without the necessity to model the noise dynamics of z, because it is open-loop identification problem (r_1 and z are uncorrelated thus by Proposition 3.5 consistent estimates are possible without exact noise models). Subsequently the projection

$$\hat{w}_1^{(r_1)}(t) := \hat{F}_{w_1 r_1}(q) r_1(t) \tag{3.7}$$

can be calculated, with $\hat{F}_{w_1 r_1}(q)$ the estimated transfer. This estimate then can serve as an accurate estimate of $w_1^{(r_1)}(t)$.

In the second stage of the algorithm $\hat{w}_1^{(r_1)}$ is used as an input in the predictor model (3.3):

$$\hat{w}_2(t|t-1,\theta) = H^{-1}(q,\theta) G_{21}(q,\theta) \hat{w}_1^{(r_1)}(t) + \big(1 - H^{-1}(q,\theta)\big) w_2(t). \tag{3.8}$$

Consequently, estimates of G_{21}^0 are obtained by minimizing the sum of squared prediction errors (3.5).

Typical conditions for consistency of the estimate of the module transfer G_{21} are

- In the first step a consistent estimate should be obtained of $F_{w_1 r_1}$. This is typically achieved by high order modeling.
- If in the second stage $G_{21}(q,\theta)$ and $H_2(q,\theta)$ are parameterized independently, then only G_{21}^0 needs to be an element in the parameterized model set.
- the feedback loop is sufficiently excited by external variable r_1. For the Two-Stage Method the noise does not contribute to the excitation of the data, since it is projected away in the first stage. This is in contrast to the Direct Method where the noise adds to the informativity of the data (the external variable r_1 is not even required to be present in the Direct Method as long as the loop is sufficiently excited by the noise v_2).
- There are no conditions on (the absence of) algebraic loops in the feedback system (in contrast to the Direct Method).

These statements are formalized in the following proposition.

Proposition 3.8 *Consider a closed-loop data generating system that satisfies Assumption 2.19. Consider the predictor model (3.8). Suppose that the data is informative w.r.t. the model structure. The estimate $G_{21}(q,\hat{\theta}_N)$ obtained by minimizing (3.5) is consistent if the following conditions hold:*

(a) The estimate $\hat{F}_{w_1 r_1}$ in (3.7) is consistent.

(b) If $G_{21}(q,\theta)$ and $H_2(q,\theta)$ are parameterized independently, then there exists a θ such that $G_{21}(q,\theta) = G_{21}^0(q)$. Otherwise, there must exist a θ such that $G_{21}(q,\theta) = G_{21}^0(q)$ and $H_2(q,\theta) = H_2^0(q)$. □

For a proof see Van den Hof & Schrama (1993); Forssell & Ljung (1999).

3.4.3 Joint-IO method of closed-loop identification

The Joint Input/Output method is an indirect identification method. First an estimate of the transfer function $W^0 = (I - G^0)^{-1}$ is obtained, then using algebraic manipulations, an estimate of G_{21}^0 is obtained.

The Joint input-output method was first proposed by Granger (1969); Caines & Chan (1975); Caines (1976); Ng et al. (1977); Granger (1980); Gevers & Anderson (1981); Anderson & Gevers (1982); Gevers & Anderson (1982). Typically in these papers the data generating system is assumed to be entirely stochastic in the sense that there are no external variables present. In this case the system is assumed to be entirely driven by stochastic processes. Thus the data generating system can be expressed as:

$$\begin{aligned}\begin{bmatrix} w_1(t) \\ w_2(t) \end{bmatrix} &= \begin{bmatrix} 0 & G_{12}^0(q) \\ G_{21}^0(q) & 0 \end{bmatrix} \begin{bmatrix} w_1(t) \\ w_2(t) \end{bmatrix} + \begin{bmatrix} v_1(t) \\ v_2(t) \end{bmatrix} \\ &= \begin{bmatrix} \frac{H_1^0(q)}{1-G_{12}^0(q)G_{21}^0(q)} & \frac{G_{12}^0(q)H_2^0(q)}{1-G_{12}^0(q)G_{21}^0(q)} \\ \frac{G_{21}^0(q)H_1^0(q)}{1-G_{12}^0(q)G_{21}^0(q)} & \frac{H_2^0(q)}{1-G_{12}^0(q)G_{21}^0(q)} \end{bmatrix} \begin{bmatrix} e_1(t) \\ e_2(t) \end{bmatrix}. \end{aligned} \tag{3.9}$$

where the inverse exists provided that the system is well-posed. Let

$$W^0 = \begin{bmatrix} W_{11}^0 & W_{12}^0 \\ W_{21}^0 & W_{22}^0 \end{bmatrix} = \begin{bmatrix} \frac{H_1^0(q)}{1-G_{12}^0(q)G_{21}^0(q)} & \frac{G_{12}^0(q)H_2^0(q)}{1-G_{12}^0(q)G_{21}^0(q)} \\ \frac{G_{21}^0(q)H_1^0(q)}{1-G_{12}^0(q)G_{21}^0(q)} & \frac{H_2^0(q)}{1-G_{12}^0(q)G_{21}^0(q)} \end{bmatrix} \tag{3.10}$$

Once an estimate of W^0 is obtained, an estimate of G_{12}^0 is obtained as follows:

$$G_{12}^0 = \frac{W_{12}^0}{W_{22}^0}, \tag{3.11}$$

where it is clear that the equality holds from inspection of (3.10).

The prediction-error method can be used inorder to obtain an estimate of W^0. In (3.9) the process $w(t) = [w_1(t)\ w_2(t)]^T$ is expressed as an multivariable Autoregressive Moving Average (ARMA) process (i.e. a multivariable stochastic process with no external inputs). As such, only a noise model can be identified. The one-step-ahead predictor is then:

$$\hat{w}(t|t-1,\theta) = (I - H(q,\theta)^{-1})w(t) \tag{3.12}$$

where $\hat{w}(t|t-1,\theta)$ and $w(t)$ are vectors, and the model $H(q,\theta)$ is a matrix. Typically $H(q,\theta)$ is parameterized such that $\lim_{z\to\infty} H(z,\theta)$ is either identity (Gevers & Anderson, 1982; Van Overschee et al., 1997; Mari & Stoica, 2000) or can be permuted to a triangular matrix (Anderson & Gevers, 1982; Van den Hof, 2006). Since the prediction error in this case is a vector, the objective function is changed to:

$$V_N(\theta) = \sum_{t=0}^{N-1} \varepsilon^T(t,\theta)\varepsilon(t,\theta). \tag{3.13}$$

Remark 3.9 *It is possible to include a weighting matrix in the objective function:*

$$V_N(\theta) = \sum_{t=0}^{N-1} \varepsilon^T(t,\theta)\Xi\varepsilon(t,\theta),$$

where Ξ is the weighting matrix. This matrix can be used to tune the method. For instance the user may choose to put more emphasis on measurements with low noise levels. In this thesis however we will simply set $\Xi = I$, since this results in simpler expressions, and because the choice of Ξ does not affect the consistency analysis.

For a closed-loop system, the conditions under which using the Joint-IO Method leads to consistent estimates of G_{21}^0 are very similar to those of the Direct Method, i.e. sufficient excitation of the closed-loop, and an absence of algebraic loops. The only difference is that in the case of the Joint IO Method, the noise term v_1 should be present.

Proposition 3.10 *Consider the closed-loop data generating system (3.9) that satisfies Assumption 2.19. Consider the predictor model (3.12). Suppose that the data is informative w.r.t. the model structure. Suppose that the data is informative w.r.t. the model structure. The estimate $G_{21}(q,\hat{\theta}_N)$ obtained using the Joint IO Method is consistent if the following conditions hold:*

(a) There is a delay present in the loop (no algebraic loop).

(b) There exists a θ such that $H(q,\theta) = W^0(q)$.

(c) The parameterization of $H(q,\theta)$ is such that the matrix $\lim_{z\to\infty} H(z,\theta)$ can be permuted to a triangular matrix with ones on the diagonal. □

For a proof see Anderson & Gevers (1982). Since $\lim_{z\to\infty} H(z,\theta)$ is equal to the impulse response of $H(q,\theta)$ at time $t = 0$, Condition (c) ensures that the model does not have any algebraic loops.

A disadvantage of the Joint IO method is that a high-order model is obtained. The estimate of G_{21}^0 is obtained by dividing two estimated transfer functions, as shown in (3.11). In practice, not all (or none) of the common poles and zeros of W_{12}^0 and W_{22}^0 will cancel, with the result that $G_{21}(q,\theta)$ has many extra poles and zeros (i.e. is of very high order).

Another difficulty with the Joint IO method as it has been presented here is the practical implementation.[2] Using the Joint IO method in practice requires parameterizing the matrix $H(q,\theta)$ such that Conditions (b) and (c) hold, and subsequently finding the global minimum of the objective function (3.13). This is a non-convex optimization problem with many parameters and many local minima.

Fortunately, in the early 1990's a sub-space like method (i.e. based on singular value decompositions and solving Riccati Equations) was proposed which could efficiently obtain a spectral factorization of a given power-spectral density. The point is that the matrix W^0 is exactly the spectral factor of the power-spectral density of the data. Consequently, the Joint IO method has enjoyed a rebirth in the literature, this time in a form that is "easily" implementable (Verhaegen, 1993; Lindquist & Picci, 1996; Van Overschee et al., 1997; Mari & Stoica, 2000; Hinnen et al., 2005). Note however, that although the spectral factorization approach of obtaining an estimate of W^0 is easier to implement in practice, the analysis of the consistency of the Joint IO method is easier using tools from the Prediction-Error framework.

In the following text it is shown that $H(q,\theta)$ is in fact an estimate of the spectral factor of Φ_w, the power spectral density of the vector w. Consider the objective function (3.13):

$$\begin{aligned}
\bar{V}(\theta) &= \bar{\mathbb{E}}[\varepsilon^T(\theta)\varepsilon(t,\theta)] \\
&= \operatorname{trace}\{\bar{\mathbb{E}}[\varepsilon(t,\theta)\varepsilon(t,\theta)^T]\} \\
&= \operatorname{trace}\{\bar{\mathbb{E}}[H(q,\theta)^{-1}w(t)w(t)^T H(q,\theta)^{-1}]\} \\
&= \operatorname{trace}\Big\{\int_{-\pi}^{\pi} H(e^{j\omega},\theta)^{-1}\Phi_w(\omega)H(e^{-j\omega},\theta)^{-1}\mathrm{d}\omega\Big\}.
\end{aligned}$$

Methods to obtain a spectral factor from a given power spectral density matrix have been developing since the late 1960's (Anderson, 1967, 1973, 1974, 1975; Boyd et al., 1994; Van Overschee et al., 1997; Hinnen et al., 2005). The key point is that a given power-spectral density matrix has a unique spectral factor that is monic, stable,

[2] Note that the MATLAB System Identification Toolbox cannot be used to minimize (3.13) since the toolbox only allows for diagonal noise models.

and minimum phase (Youla, 1961). This is exactly equivalent to parameterizing $\lim_{z\to\infty} H(z,\theta) = I$. The advantage of the spectral factorization approach is that there exist convex methods to obtain the spectral factors. Secondly, the estimate $H(q,\theta)$ does not need to be parameterized a priori.

3.4.4 Closed-Loop Instrumental Variable Methods

The last closed-loop Prediction-Error method that is presented is the Basic Closed-Loop Instrumental Variable method (Söderström & Stoica, 1983; Söderström et al., 1988; Söderström & Stoica, 1989b; Gilson & Van den Hof, 2005; Gilson et al., 2009, 2011). The IV methods are closely related to the Two-Stage Method since they both involve a "projection" onto the external variable r_1. A big advantage of the IV method is that it results in a predictor that is linear in the parameters. Consequently, the parameters can be obtained by a linear regression. This is unlike the previous three methods.

In order to present the basic closed-loop IV method, it is useful to explicitly define the parameter vector as $\theta = [a_1^{21} \ \cdots \ a_{n_a}^{21} \ b_0^{21} \ \cdots \ b_{n_b}^{21}]^T$. In addition, the following regressor will be very useful:

$$\phi_{21}^T(t) = \begin{bmatrix} -w_2(t-1) & \cdots & -w_2(t-n_a) & w_1(t) & \cdots & w_1(t-n_b) \end{bmatrix}$$

The output w_2 can now be expressed as

$$\begin{aligned} w_2(t) &= B_{21}^0(q)w_1(t) + \big(1 - F_{21}^0(q)\big)w_2(t) + F_{21}^0(q)v_2(t) \\ &= \phi_{21}^T(t)\theta_{21}^0 + \breve{v}_2(t) \end{aligned} \tag{3.14}$$

where B_{21}^0 and F_{21}^0 denote the numerator and denominator of G_{21}^0, θ_{21}^0 denotes the true parameters, and

$$\breve{v}_2(t) = F_{21}^0(q)v_2(t). \tag{3.15}$$

The IV estimate of θ^0 is the solution to (Gilson & Van den Hof, 2005)

$$\frac{1}{N}\sum_{t=0}^{N-1} z(t)\big(w_2(t) - \phi_{21}^T(t)\hat{\theta}_{IV}\big) = 0,$$

where $z(t)$ is a vector of so called *instruments*. If $\sum_{t=0}^{N-1} z(t)\phi_{21}^T(t)$ is nonsingular, then

$$\hat{\theta}_{IV} = \Big(\frac{1}{N}\sum_{t=0}^{N-1} z(t)\phi_{21}^T(t)\Big)^{-1}\Big(\sum_{t=0}^{N-1} z(t)w_2(t)\Big). \tag{3.16}$$

An expression of $\hat{\theta}_{IV}$ in terms of θ_0 can be obtained by substituting (3.14) into (3.16):

$$\hat{\theta}_{IV} = \theta^0 + \Big(\frac{1}{N}\sum_{t=0}^{N-1} z(t)\phi_{21}^T(t)\Big)^{-1}\Big(\sum_{t=0}^{N-1} z(t)\breve{v}_2(t)\Big). \tag{3.17}$$

From (3.16) and (3.17) the conditions for consistent estimation of θ^0 are exposed.

Proposition 3.11 *Consider a closed-loop data generating system that satisfies Assumption 2.19. The estimate $\hat{\theta}_{IV}$ obtained as in (3.16) is consistent if the following conditions hold:*

(a) $\bar{\mathbb{E}}[z(t)\phi_{21}^T(t)]$ *is nonsingular.*

(b) $\bar{\mathbb{E}}[z(t)\breve{v}_2(t)] = 0.$ □

Similar to the Two-Stage Method there is no condition on the absence of algebraic loops. Condition (a) essentially ensures that the data is informative with respect to the model structure.

The choice of the instrumental variable z is critical with respect to the consistency of the estimates. Typically in the closed-loop IV methods, the variable r_1 is chosen as the instrumental variable:

$$z(t) = \begin{bmatrix} r_1(t) & \cdots & r_1(t - n_a - n_b) \end{bmatrix}^T.$$

By Assumption 2.19 and (3.15), Condition (b) is met. Consequently, consistent estimates of G_{21}^0 are possible using this instrument. Note that, by (3.16), in order to calculate the estimate $\hat{\theta}_{IV}$ only a linear regression needs to be evaluated. This is a big advantage of the Instrumental Variable Methods.

3.5 ERRORS-IN-VARIABLES FRAMEWORK

A main assumption in the previous methods was that either the input w_1 or a reference signal r_1 is measured noise free. In this section that assumption is relaxed, leading to the so called Errors-in-Variables (EIV) framework. The majority of the EIV literature is for the open loop case (see the survey papers Söderström (2007, 2011)), however recently the closed-loop EIV problem has been considered in Pintelon & Schoukens (2012b); Söderström et al. (2013). In this section, first the open loop EIV problem is presented, and then the closed-loop results.

3.5.1 Open-Loop EIV

The open-loop EIV data generating system is illustrated in Fig. 3.4. The main problem is that the input to G_{21}^0 is not exactly known. Only a noisy version of w_1 is known. The equations for the data generating system are:

$$w_2(t) = G_{21}^0(q)w_1(t) \tag{3.18}$$

where the available measurements satisfy:

$$\tilde{w}_2(t) = w_2(t) + s_2(t)$$
$$\tilde{w}_1(t) = w_1(t) + s_1(t).$$

There are several different ways to obtain an estimate of G_{21}^0 using the data $\tilde{w}_1$ and $\tilde{w}_2$ (Söderström, 2007, 2012). The methods generally differ based on what assumptions can be made about s_1 and s_2. A common assumption is that s_1 is

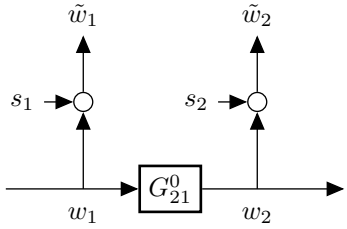

Figure 3.4: An open loop data generating system with sensor noise on both the input and the output.

white noise. However, a more general situation is to suppose that both s_1 and s_2 are stochastic processes with rational power spectral density (i.e neither s_1 or s_2 are necessarily white noise).

Several methods to obtain estimates of G_{21}^0 are based on analyzing the power spectral density of $[\tilde{w}_2 \ \tilde{w}_1]^T$. These methods will be briefly presented here. For a more detailed analysis, see Agüero & Goodwin (2008).

The power spectral density of $z = [\tilde{w}_2 \ \tilde{w}_1]$ is

$$\Phi_z = \begin{bmatrix} \Phi_{s_2} + G_{21}^0 \Phi_{w_1} G_{21}^{0^*} & G_{21}^0 \Phi_{w_1} \\ \Phi_{w_1} G_{21}^{0^*} & \Phi_{s_1} + \Phi_{w_1} \end{bmatrix}. \tag{3.19}$$

Let the spectral density Φ_{w_1} be expressed as

$$\Phi_{w_1}(z) = L_0(z) Q L_0(z^{-1})$$

where $L_0(z)$ is the stable, minimum phase, monic spectral factor of $\Phi_{w_1}(z)$, and Q is a positive constant. By Youla (1961) such a spectral factorization is unique.

The following (weak) assumption will be made about the data generating system (3.18).

Assumption 3.12 *Consider an open-loop data generating system as shown in Fig. 3.4. The following conditions are assumed to hold (Agüero & Goodwin, 2008):*

1. *$G_{21}(z)$ has no pole that is also a pole of $G_{21}(z^{-1})$.*
2. *$G_{21}(z)$ has no zero that is also a zero of $G_{21}(z^{-1})$.*
3. *$G_{21}(z)$ does not have a non-minimum phase zero that is a pole of Φ_{w_1}. No poles of $G_{21}(z)$ are zeros of Φ_{w_1}.*

Under this assumption, the poles and zeros of $G_{21}^0(z)$ can be obtained from $\Phi_{w_2 w_1}(z)$ as follows. Let $G_{21}^0(z)$ be factored as $b_0 \bar{G}_{21}^0(z)$, where $\bar{G}_{21}^0$ is a monic transfer function. Thus, $\Phi_{w_2 w_1}(z)$ can be expressed as:

$$\begin{aligned} \Phi_{w_2 w_1}(z) &= G_{21}^0(z) \Phi_{w_1}(z) \\ &= b_0 Q \bar{G}_{21}^0(z) L_0(z) L_0(z^{-1}). \end{aligned} \tag{3.20}$$

Let $\alpha_0 = b_0 Q$. From (3.20) the poles and zeros that appear in pairs that are symmetrically placed with respect to the unit circle (i.e. $z - a$ and $1 - az$ form a pair) belong to Φ_{w_1} and the remaining poles and zeros belong to $G_{21}^0(z)$. In Agüero & Goodwin (2008) it is shown that under Assumption 3.12 the poles and zeros of G_{21}^0 can be uniquely determined using this procedure (i.e. $\bar{G}_{21}^0$ can be uniquely determined).

Since $G_{21}^0 = b_0 \bar{G}_{21}^0$, there only remains one more parameter to estimate in order to obtain an estimate of $G_{21}^0(z)$ (namely, b_0). With no further prior knowledge about the system, b_0 cannot be uniquely determined. However, bounds on b_0 can be calculated (Agüero & Goodwin, 2008).

Proposition 3.13 *Consider the open-loop data generating system shown in Figure 3.2. Suppose that Assumption 3.12 holds. Suppose $\bar{G}_{21}^0(z)$, α, and $L_0(z)$ have been obtained from $\Phi_{w_2 w_1}(z)$. Then the following bound on b_0 holds:*

$$\min_{\omega} \frac{\alpha |L_0(e^{j\omega})|^2}{\Phi_{w_1}(e^{j\omega})} \leq b_0 \leq \max_{\omega} \frac{\alpha \Phi_{w_2}(e^{j\omega}) |L_0(e^{j\omega})|^2}{|\Phi_{w_2 w_1}(e^{j\omega})|^2} \tag{3.21}$$

The proof is in Agüero & Goodwin (2006); Agüero & Goodwin (2008). With additional assumptions on the data generating system, it is possible to obtain a unique estimate of G_{21}^0. Typical assumptions include (Söderström, 2007):

- If one of the sensor noise terms s_1 and/or s_2 are not Gaussian distributed, then the higher order statistics of the noise can be used to obtain more information about the system (Deistler, 1986).
- If more detailed assumptions are made about the poles and zeros of L_0, H_{s_1} and H_{s_2} it may be possible to obtain a unique estimate. See Theorem 14 in Agüero & Goodwin (2008).
- If the system can be excited by a periodic signal, or if repeated (identical) experiments are possible, then a unique estimate of G_{21}^0 can be obtained (Schoukens et al., 1997; Söderström & Hong, 2005).

Next, consider the closed-loop EIV problem.

3.5.2 Closed-Loop EIV

The closed-loop set-up that is considered in Söderström et al. (2013) is shown in Figure 3.5.

The difference between the data generating system shown in Figure 3.5 and the one studied in the section on the closed-loop Prediction-Error methods (i.e. Section 3.4) is the presence of sensor noise s_2 and s_3. The conclusion of Söderström et al. (2013) is that in this case G_{21}^0 is identifiable using a data set consisting of r_1, w_2 and w_3. This is not surprising since r_1 is still assumed to be known, noise free. Consequently, the Two-Stage and the Closed-Loop IV methods can still be used without any alterations to obtain consistent estimates of G_{21}^0.

A more challenging data generating system is when r_1 is not present, and w_1 is measured with noise. In this case the Two-Stage method will not result in consistent

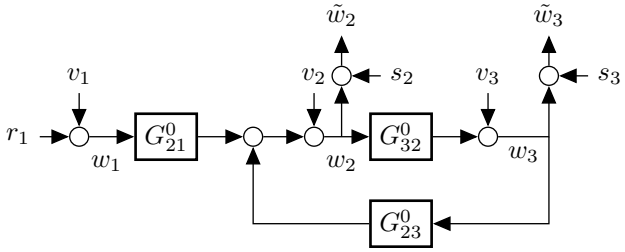

Figure 3.5: Closed-Loop data generating system with sensor noise.

estimates of G_{21}^0. However, it can be shown that the Closed-Loop IV method still results in consistent estimates. This result will be further discussed in Chapter 6.

3.6 SUMMARY

In this chapter the system identification framework that is used for the remainder of this thesis has been presented. First a data generating system was defined as a tool to analyse the properties of various identification methods. Then the prediction-error framework for system identification was presented. Subsequently, the Direct, Two Stage, Joint IO and Closed-Loop IV methods were presented. Each of these methods will be generalized in the remainder of this thesis in order to be able to estimate a transfer function that is embedded in a dynamic network.

Finally the Errors-in-Variables framework is presented which incorporates sensor noise into the framework. In Chapter 6 this framework will be extended to the case of dynamic network models.

Chapter 4

FROM IDENTIFICATION IN CLOSED-LOOPS TO IDENTIFICATION IN NETWORKS

The problem of identifying dynamical models on the basis of measurement data is usually considered in a classical open-loop or closed-loop setting. In this chapter this problem is generalized to dynamical systems that operate in a complex interconnection structure and the objective is to consistently identify the dynamics of a particular module in the network. For a known interconnection structure it is shown that classical prediction error methods for closed-loop identification can be generalized to provide consistent model estimates, under specified experimental circumstances. Graph theoretical tools are presented to verify the topological conditions under which the methods lead to consistent module estimates.[1]

4.1 INTRODUCTION

IN THIS CHAPTER we make a first attempt at identification in dynamic networks. Each of the closed-loop identification methods presented in Chapter 3 are extended to the case of identification in dynamic networks. This can be considered as a natural extension of the situation of open-loop data, closed-loop data in a single loop, towards data that is obtained from systems operating in a predefined network structure. Since dynamic networks typically contain (feedback) loops, it is shown that methods for closed-loop identification are an appropriate basis for developing more generalized tools to deal with complex networks.

The question that is addressed in this chapter is: given the interconnection structure of the dynamic network under what conditions can a particular module transfer function G_{ji}^0 be estimated consistently?

Several considerations are taken into account. Firstly, in this chapter we assume

[1] This chapter is based on the papers Van den Hof et al. (2013, 2012); Dankers et al. (2012a).

that every internal variable is in principle measurable/known. Moreover, we assume that the internal variables are measurable without sensor noise. In other words, noise free measurements of every internal variable w_k in the network are available. Although these assumptions are restrictive, they serve as a good starting point. In subsequent chapters these assumptions will be significantly relaxed.

Secondly, in this chapter we consider the possibility that particular modules in the network may be known a priori. For instance, there may be controllers embedded in the network. If so, how can this extra knowledge about the network be incorporated into the algorithms and be used as an advantage to obtain better estimates?

Throughout the chapter we will focus on conditions on the interconnection structure as well as on the presence of noise sources and excitation/probing signals for consistently identifying a particular module in the network.

Just as in the closed-loop case, the generalized Direct and Joint IO Methods rely on exact noise models (system in the model set). On the other hand, the generalized Two-Stage and IV Methods rely on the presence of external excitation variables.

The chapter proceeds as follows. In Section 4.2 some of the key equations from Chapter 3 are repeated for the convenience of the reader and some additional notation is introduced. In Section 4.3 some network properties are presented and in Sections 4.4, 4.5 and 4.6 network versions of the Direct, Two-Stage and Joint IO Methods respectively are presented and analyzed.

4.2 PRELIMINARIES AND NOTATION

The data generating system assumed to be of the form

$$\begin{bmatrix} w_1(t) \\ w_2(t) \\ \vdots \\ w_L(t) \end{bmatrix} = \begin{bmatrix} 0 & G_{12}^0(q) & \cdots & G_{1L}^0(q) \\ G_{21}^0(q) & 0 & \ddots & G_{2L}^0(q) \\ \vdots & \ddots & \ddots & \vdots \\ G_{L1}^0(q) & G_{L2}^0(q) & \cdots & 0 \end{bmatrix} \begin{bmatrix} w_1(t) \\ w_2(t) \\ \vdots \\ w_L(t) \end{bmatrix} + \begin{bmatrix} r_1(t) \\ r_2(t) \\ \vdots \\ r_L(t) \end{bmatrix} + \begin{bmatrix} v_1(t) \\ v_2(t) \\ \vdots \\ v_L(t) \end{bmatrix}$$
$$= G^0(q)w(t) + r(t) + v(t). \tag{4.1}$$

The objective considered in this chapter is to estimate a particular transfer function embedded in the network. This transfer function is denoted G_{ji}^0.

The generalized predictor model for the internal variable w_j is:

$$\hat{w}_j(t|t-1,\theta) = H_j^{-1}(q,\theta)\Big(\sum_{x_k \in \mathcal{X}} G_{jk}(q,\theta)x_k(t) + r_j(t)\Big) + \big(1 - H_j^{-1}(q,\theta)\big)w_j(t) \tag{4.2}$$

where the set of input variables $\mathcal{X}$ is left unspecified for the moment. Typically, $\mathcal{X}$ consists of internal and external variables. If $\mathcal{X}$ contains only one element, then (4.2) is referred to as a single-input, single-output (SISO) predictor. If $\mathcal{X}$ contains more than one element, then (4.2) is referred to as a multiple-input, single-output (MISO) predictor.

The identification criterion considered in this chapter is:

$$V_N(\theta) = \frac{1}{N}\sum_{t=0}^{N-1} \varepsilon_j^2(t,\theta) \tag{4.3}$$

where ε_j is the prediction error: $\varepsilon_j(t,\theta) = w_j(t) - \hat{w}_j(t|t-1,\theta)$.

The following sets are used throughout the chapter to denote parts of a dynamic network:

- $\mathcal{N}_j$ denote the set of indices of internal variables with direct connections to w_j, i.e. $k \in \mathcal{N}_j$ if and only if $G_{jk} \neq 0$.
- $\mathcal{K}_j$ denotes the set of indices of internal variables w_k, $k \in \mathcal{N}_j$ for which the module transfer functions G^0_{jk} of the data generating system are known. We assume that $i \notin \mathcal{K}_j$ since if G^0_{ji} is known, there is no need to obtain an estimate of G^0_{ji}.
- $\mathcal{U}_j^i$ denotes the set of indices of internal variables w_k, $k \in \mathcal{N}_j$, $k \neq i$ for which the module transfer functions G^0_{jk} of the data generating system are unknown.
- $\mathcal{R}$ denotes the set of indices of all external variables present in the network.
- $\mathcal{R}_j$ denotes the set of indices of all external variables present in the network that have a path (not necessarily a direct path) to the internal variable w_j.
- $\mathcal{V}$ denotes the set of indices of all process noise variables present in the network.
- $\mathcal{V}_j$ denotes the set of indices of all process noise variables present in the network that have a path (not necessarily a direct path) to the internal variable w_j.

Note that $\mathcal{N}_j = i \cup \mathcal{K}_j \cup \mathcal{U}_j^i$.

4.3 PROPERTIES OF DYNAMIC NETWORKS

Two properties of dynamic networks are presented in this section. The first property establishes a relationship between the transfer functions of the matrix $(I - G^0)^{-1}$ and the interconnection structure of a dynamic network model. This helps give an interpretation of the transfer functions in $(I - G^0)^{-1}$. This property will be used extensively throughout the remainder of this thesis.

The second property that is presented here establishes a relationship between a closed-loop model and a dynamic network model.

The first property is formulated in the following lemma. As mentioned, a relationship is established between the transfer functions of the matrix $(I - G^0)^{-1}$ and the interconnection structure of a network.

Lemma 4.1 *Consider a dynamic network model that satisfies Assumption 2.19, with transfer matrix G^0 (4.1). Let $\mathcal{G}^0_{ji}$ be the (j,i)th entry of $(I - G^0)^{-1}$. If every path from w_i to w_j has a delay, then $\mathcal{G}^0_{ji}$ has a delay. If there is no path from w_i to w_j then $\mathcal{G}^0_{ji} = 0$*

This lemma can be proved using Mason's Rules (Mason, 1953, 1956). For completeness, an alternative proof is included in Appendix 4.8.1.

One of the properties of a dynamic network is the possible occurrence of algebraic loops, i.e. loops for which the transfer function has a direct feed-through term. For analyzing the properties of direct identification algorithms it is attractive to be able to characterize these loops. Lemma 4.1 enables this analysis.

In addition it is often useful to determine if there is a path from an external input to an internal variable. The existence of such a path can be determined from the matrix $(I - G^0)^{-1}$ using Lemma 4.1.

The next proposition shows that the dynamic network can be rewritten in a classical feedback structure by denoting one particular w_j as an "output variable". Any node signal can serve this purpose. This equivalent structure will facilitate the understanding and analysis of the several identification results.

Proposition 4.2 *Consider a dynamic network that satisfies Assumption 2.19, and select one particular internal variable w_j to be referred to as "output". Classify the remaining variables $w_i, i \in \mathcal{D}$ with $\mathcal{D} := \{1, \ldots, L\} \setminus \{j\}$ as "inputs", denoted as $w_{\mathcal{D}} = [w_{k_1} \ \cdots w_{k_n}]^T$, $\{k_1, \ldots, k_n\} = \mathcal{D}$. The vectors $r_{\mathcal{D}}$ and $v_{\mathcal{D}}$ and defined analogously.*

Let $G^0_{j\mathcal{D}}$ denote the row vector $[G^0_{jk_1} \ \cdots \ G^0_{jk_n}]$, $\{k_1, \ldots, k_n\} = \mathcal{D}$, let $G^0_{\mathcal{D}j}$ denote the column vector $[G^0_{k_1 j} \ \cdots \ G^0_{k_n j}]^T$, $\{k_1, \ldots, k_n\} = \mathcal{D}$, and let $G^0_{\mathcal{D}\mathcal{D}}$ be a matrix constructed analogously.

The internal variables $\{w_k\}_{k=1,\cdots L}$ are equivalently described by the feedback connection structure as indicated in Figure 4.1, with w_j interpreted as output, and $w_{\mathcal{D}}$ as input, determined by

$$\begin{aligned} w_j &= G^0_{j\mathcal{D}}(q) w_{\mathcal{D}} + r_j + v_j \\ w_{\mathcal{D}} &= \breve{G}^0_{\mathcal{D}j}(q) w_j + \breve{G}^v(q) v_{\mathcal{D}} + \breve{G}^r(q) r_{\mathcal{D}}, \end{aligned}$$

where

$$\begin{aligned} \breve{G}^v = \breve{G}^r &= (I - G^0_{\mathcal{D}\mathcal{D}})^{-1} \\ \breve{G}^0_{\mathcal{D}j} &= (I - G^0_{\mathcal{D}\mathcal{D}})^{-1} G^0_{\mathcal{D}j} \end{aligned}$$

where transfer matrices $\breve{G}^v, \breve{G}^r \in \mathbb{R}^{(L-1)\times(L-1)}(z)$, and the transfer vector $\breve{G}^0_{\mathcal{D}j} \in \mathbb{R}^{(L-1)\times 1}(z)$.

Proof: Using the introduced notation the dynamic network model equations (4.1) can be written as

$$\begin{bmatrix} w_j \\ w_{\mathcal{D}} \end{bmatrix} = \begin{bmatrix} 0 & G^0_{j\mathcal{D}} \\ G^0_{\mathcal{D}j} & G^0_{\mathcal{D}\mathcal{D}} \end{bmatrix} \begin{bmatrix} w_j \\ w_{\mathcal{D}} \end{bmatrix} + \begin{bmatrix} r_j + v_j \\ r_{\mathcal{D}} + v_{\mathcal{D}} \end{bmatrix}$$

leading to the equations:

$$\begin{aligned} w_j &= G^0_{j\mathcal{D}} w_{\mathcal{D}} + r_j + v_j \\ w_{\mathcal{D}} &= (I - G^0_{\mathcal{D}\mathcal{D}})^{-1} G^0_{\mathcal{D}j} w_j + (I - G^0_{\mathcal{D}\mathcal{D}})^{-1}(r_j + v_j) \end{aligned}$$

provided that the inverse $(I - G^0_{\mathcal{D}\mathcal{D}})^{-1}$ exists and is proper. This is guaranteed by condition a of Assumption 2.19.

One of the important observations from the presented feedback structure is that the disturbance/excitation terms that directly affect the "output" w_j, do not appear as disturbance/excitation signals directly acting on the "input signals" $w_{\mathcal{D}}$; they only contribute to these inputs through the "feedback" operation $\breve{G}^0_{\mathcal{D}j}$.

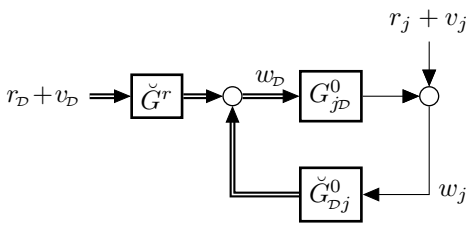

Figure 4.1: Closed Loop representation of w_j. Double lines denote multivariable paths.

4.4 IDENTIFICATION IN DYNAMIC NETWORKS USING THE DIRECT METHOD

The direct method for closed-loop identification can rather simply be generalized to the situation of dynamic networks. To this end we consider the one-step ahead predictor that was formulated in (4.2). The principal choice that has to be made is the set of input signals $x_k \in \mathcal{X}$ that has to be taken into account in the predictor. If the module transfer function G^0_{ji} needs to be identified it is tempting to choose only w_i as input for the predictor. However in most cases this will lead to biased results of the estimates due to the fact that other (neglected) variables with direct connections to w_j also affect the output. Therefore the safest situation is to choose in the predictor all variables that have a direct link to the output w_j, i.e. $\mathcal{X} = \mathcal{N}_j$ in (4.2), leading to the predictor:

$$\hat{w}_j(t|t-1,\theta) = H_j^{-1}(q,\theta)\Big(\sum_{k\in\mathcal{N}_j} G_{jk}(q,\theta)w_k(t) + r_j(t)\Big) + \big(1 - H_j^{-1}(q,\theta)\big)w_j(t). \quad (4.4)$$

For this predictor the following result is obtained.

Proposition 4.3 *Consider a dynamic network that satisfies Assumption 2.19, and consider a direct prediction error identification according to (4.3) with predictor (4.4). Then the module transfer functions G^0_{jk}, $k \in \mathcal{N}_j$ as well as H^0_j are estimated consistently under the following conditions:*

(a) The noise v_j is uncorrelated to all noise signals v_k, $k \in \mathcal{V}_j \backslash \{j\}$.

(b) For both the data generating system and the parametrized model, every loop through w_j has a delay.

(c) The spectral density of $[w_j \; w_{n_1} \; \cdots w_{n_n}]^T$, $n_* \in \mathcal{N}_j$, *denoted as* $\Phi_{j,\mathcal{N}_j}(\omega)$ *is positive definite for* $\omega \in [-\pi, \pi]$.

(d) The system is in the model set, i.e. there exists a θ^0 *such that* $G_{jk}(z, \theta^0) = G^0_{jk}(z)$ *for all* $k \in \mathcal{N}_j$, *and* $H_j(z, \theta^0) = H^0_j(z)$.

The proof can be found in Appendix 4.8.2.

Note that in the considered situation all transfers G^0_{jk}, $k \in \mathcal{N}_j$ and H^0_j need to be estimated simultaneously in order for the result to hold, and that the dynamics of noise source v_j needs to be modeled correctly through a noise model H_j. Note also that both the noise signal v_j and the probing signal r_j provide excitation to the loop that is going to be identified. The excitation condition (c) is a rather generic condition for informative data (Ljung, 1999). A further specification for particular finite dimensional model structures can be made along the results for classical feedback loops as developed in Gevers et al. (2009b).

Whereas in classical closed loop identification with the direct method there is a condition on absence of algebraic loops in the full feedback system (Van den Hof et al., 1992), this is further specified here in condition (b) by limiting that condition to only apply to the output signal that is considered for identification.

Remark 4.4 *In the Proposition above the predictor that is used employs all possible inputs that directly connect to the output signal* w_j. *If some of these transfers are known already, e.g. they could be controllers with known dynamics, then the result above can simply be generalized. Then the predictor (4.4) can be generalized as*

$$\hat{w}_j(t, \theta) = H_j^{-1}(q, \theta)\Big(\sum_{k \in \mathcal{N}_j \setminus \mathcal{K}_j} G_{jk}(q, \theta) w_k(t) + \sum_{k \in \mathcal{K}_j} G^0_{jk}(q) w_k(t) + r_j(t)\Big) + (1 - H_j^{-1}(q, \theta)) w_j(t), \tag{4.5}$$

leading to consistent estimates of the transfers G^0_{jk}, $k \in \mathcal{N}_j \setminus \mathcal{K}_j$, *while in the formulation of the conditions of Proposition 4.3, the set* $\mathcal{N}_j$ *is replaced by the set* $\mathcal{N}_j \setminus \mathcal{K}_j$. □

Next, an algorithm for checking Condition (b) will be presented. The other conditions are straightforward to check and do not need an algorithm. Recall that the matrix A_d is the adjacency matrix with d's in the entries with strictly proper module transfer functions, and 1's in the entries with proper module transfer functions (see Section 2.5 of Chapter 2).

Algorithm 4.5 *Check if all loops through* w_j *have a delay*

1. *Evaluate* A_d^ℓ *for* $\ell = 1, \ldots, L$ *using the multiplication and addition rules defined in Section 2.5 of Chapter 2.*

2. *If for any considered power* ℓ *entry* (j, j) *equals* 1, *Condition (b) is not met.*

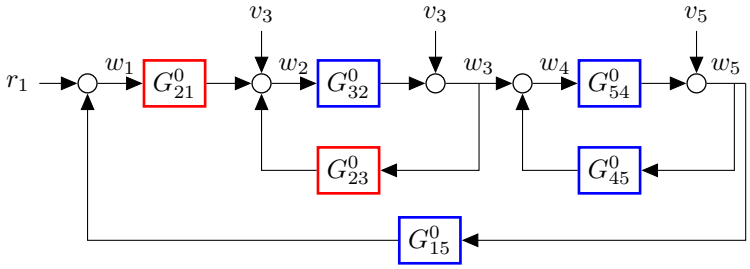

Figure 4.2: Dynamic network with 5 internal variables, of which 2 (red-colored) transfer functions G_{21}^0 and G_{23}^0 can be consistently identified with the direct method using a MISO predictor. The blue-colored transfer functions can be identified with SISO predictors.

Example 4.6 *If we apply the result of the direct method to the network example of Figure 4.2, it appears that the direct method can be applied to each of the internal variables* $w_1, \cdots w_5$. *Note that in this data generating system* $G_{43}^0 = 1$. *The transfer functions* G_{15}^0, G_{32}^0 *can be identified provided that the loop* $(G_{54}^0 G_{32}^0 G_{21}^0 G_{15}^0)$ *has a delay and an appropriate noise model is used. The transfer functions* G_{54}^0 *and* G_{45}^0 *can be identified provided that the loop* $(G_{54}^0 G_{45}^0)$ *has a delay and an appropriate noise model is used. The transfers* G_{21}^0 *and* G_{23}^0 *can only be estimated simultaneously by considering both* w_1 *and* w_3 *as inputs and* w_2 *as output. Under the condition that a delay is present in the loops* $(G_{32}^0 G_{23}^0)$ *and* $(G_{54}^0 G_{32}^0 G_{21}^0 G_{15}^0)$ *and by the use of an appropriate model set that includes accurate noise modeling, the transfers* G_{21}^0 *and* G_{23}^0 *can be estimated consistently.*

Remark 4.7 *It is not always necessary to include all* w_k, $k \in \mathcal{N}_j$ *as input in the predictor. For instance consider the case shown in Fig. 4.3a. Suppose that the objective is to obtain consistent estimates of* G_{21}^0. *According to Proposition 4.3 both* w_1 *and* w_3 *must be included as inputs in the predictor. However, from the figure, it can be seen that* w_3 *only acts as a (uncorrelated) disturbance on* w_2, *and does not need to be modeled for consistent estimation of* G_{21}^0. *This idea is illustrated in Fig. 4.3b where* $\tilde{v}_2 = v_2 + w_3$. *This idea is formalized and extended in the next chapter.*

Note that for using signal w_j as an output, it is not strictly necessary that a noise source v_j is present. This special case is considered in the next Corollary.

Corollary 4.8 *Consider the situation of Proposition 4.3. If the noise source* v_j *is* not *present. Then the module transfer functions* $G_{ji}^0, i \in \mathcal{N}_j$ *can be estimated consistently, under the conditions of Proposition 4.3, where the excitation condition (c) is replaced by:*

(c') *The spectrum of* $[w_{n_1} \ \cdots \ w_{n_n}]^T$, $n_* \in \mathcal{N}_j$, $\Phi_{\mathcal{N}_j}(\omega)$ *is positive definite for* $\omega \in [-\pi, \pi]$,

the delay condition (b) is removed, and the noise model is fixed to 1, *thereby focusing condition (d) on the module transfer property only.*

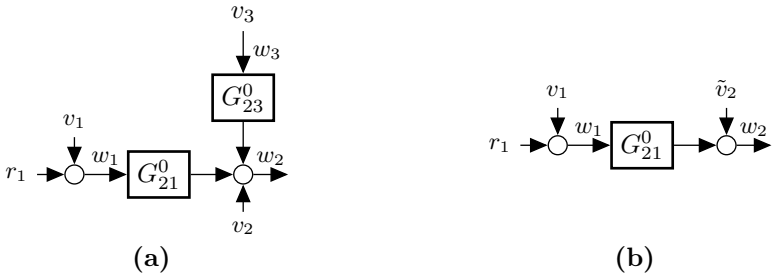

Figure 4.3: Example of a system where not all w_k, $k \in \mathcal{N}_j$ need to be included as inputs in the predictor (w_3 can just be considered as an uncorrelated disturbance).

Proof The same line of reasoning as in the proof of Proposition 4.3 can be followed starting with (4.24) and plugging in $v_j = 0$, $H(\theta) = 1$, and $\sigma_{e_j}^2 = 0$. □

4.5 IDENTIFICATION IN DYNAMIC NETWORKS USING THE 2-STAGE METHOD

The Two-Stage Method for closed-loop identification as described in Section 3.4.2 of Chapter 3 follows a different approach than the Direct Method. It explicitly utilizes the presence of (known) external variables, and has the potential to consistently identify module transfers without the necessity to consistently identify noise models also. Consider the scheme depicted in Figure 4.4. The key is that the internal variable w_j and w_i are excited by an external variable r_m that is present somewhere in the network. The following strategy is pursued in an attempt to consistently identify the module transfer G_{ji}^0. Note that here we attempt to use only w_i as an input in the predictor model.

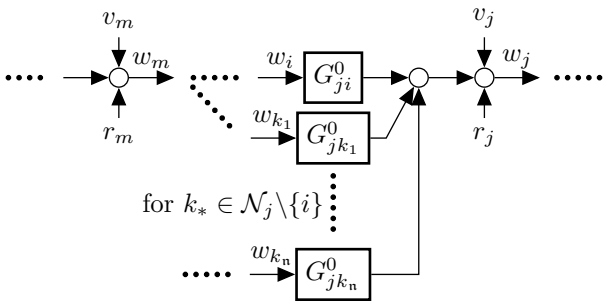

Figure 4.4: Graphical representation of a data generating system where the internal variable w_i is excited through an external variable r_m.

Algorithm 4.9 (Two-stage SISO model)

1. *Select a set of external variables* $\{r_m\}$, *with* $m \in \mathcal{T}_j \subseteq \mathcal{R}_i$, *each of them correlated with* w_i.

2. *On the basis of the variables* $\{r_m\}$, $m \in \mathcal{T}_j$ *and* w_i, *determine* $w_i^{(\mathcal{T}_j)}$:

$$w_i^{(\mathcal{T}_j)} := \sum_{m \in \mathcal{T}_j} w_i^{(r_m)} = \sum_{m \in \mathcal{T}_j} F_{km}^0 r_m.$$

   An estimate of $w_i^{(\mathcal{T}_j)}$ *can be obtained by estimating* $\{F_{im}^0\}$, $m \in \mathcal{T}_j$ *using parameterized models* $F_{im}(q,\gamma)$, *with* γ *a parameter vector, resulting in estimated models* $F_{im}(q,\hat{\gamma}_N)$. *These models are used to generate the simulated signal:*

$$\hat{w}_i^{(\mathcal{T}_j)}(t,\hat{\gamma}_N) = \sum_{m \in \mathcal{T}_j} F_{im}(q,\hat{\gamma}_N) r_m(t)$$

3. *Construct* $\tilde{w}_j(t) = w_j(t) - \sum_{k \in \mathcal{K}_j} G_{jk}^0(q) w_k(t) - r_j(t)$, *i.e. correct* w_j *with all known terms.*

4. *Identify the transfer function* G_{ji}^0 *on the basis of a predictor model with prediction error*

$$\varepsilon_j(t,\theta) = H_j(q,\eta)^{-1}\big(\tilde{w}_j(t) - G_{ji}(q,\theta) w_i^{(\mathcal{T}_j)}(t)\big)$$

   using $\tilde{w}_j$ *and* $w_i^{(\mathcal{T}_j)}$, *an identification criterion (4.3), and where* H_j *is a fixed noise model or parametrized independently of* θ.

For this algorithm the following result can be obtained.

Proposition 4.10 *Consider a dynamic network that satisfies Assumption 2.19. Then the module transfer function* G_{ji}^0 *can be consistently estimated with Algorithm 4.9 if the following conditions are satisfied:*

(a) The set $\mathcal{T}_j$ *is non-empty.*

(b) The external variables r_m $m \in \mathcal{T}_j$ *are uncorrelated to all noise variables* v_k, $k \in \{j, \mathcal{U}_j^i\}$;

(c) The projection of w_i *onto the set of external variables,* $w_i^{(\mathcal{T}_j)}$ *is persistently exciting of a sufficiently high order*[2];

(d) All internal variables w_k, $k \in \mathcal{U}_j^i, k \neq i$, *are uncorrelated to all* r_m, $m \in \mathcal{T}_j$.

(e) The module transfer function G_{ji}^0 *is in the model set, i.e. there exists a parameter* θ^0 *such that* $G_{ji}(q,\theta^0) = G_{ji}^0(q)$. □

[2]Within the classical prediction error framework (Ljung, 1999), $w_i^{(\mathcal{T}_j)}(t)$ will need to be persistently exciting of an order at least equal to the number of unknown parameters that is estimated in $G_{ji}(q,\theta)$.

Proof: Note that w_j can be expressed as

$$\begin{aligned} w_j(t) &= G^0_{ji}(q)w_i(t) + \sum_{k\in\mathcal{K}_j} G^0_{jk}(q)w_k(t) + \sum_{k\in\mathcal{U}^i_j} G^0_{jk}(q)w_k(t) + r_j(t) + v_j(t) \\ &= G^0_{ji}(q)w_i(t) + p_j(t) + x_j(t) + v_j(t) \end{aligned}$$

where p_j reflects the contributions of all components $G^0_{jk}(q)w_k$ that are known because of the fact that the dynamics G^0_{jk} are known, as well as $r_j(t)$; and $x_j(t)$ similarly reflects the contributions of all components $G^0_{jk}(q)w_k$ that are unknown, because the dynamics G^0_{jk} are unknown.

Subsequently

$$w_j(t) - p_j(t) = G^0_{ji}(q)w_i(t) + x_j(t) + v_j(t)$$

where the left hand side is a known variable.

Condition (b) together with the fact that by construction all r_m, $m \in \mathcal{T}_j$ are correlated to w_i, guarantee that w_i can be decomposed as $w_i = w_i^{(\mathcal{T}_j)} + w_i^{(\perp\mathcal{T}_j)}$. Then,

$$w_j - p_j = G^0_{ji}(q)\big(w_i^{(\mathcal{T}_j)} + w_i^{(\perp\mathcal{T}_j)}\big) + x_j + v_j. \tag{4.6}$$

Conditions (b) and (d) guarantee that x_j is uncorrelated to all r_m, $m \in \mathcal{T}_j$. And by condition (b) the noise v_j is uncorrelated to all r_m, $m \in \mathcal{T}_j$, while $w_i^{(\perp\mathcal{T}_j)}$ is uncorrelated to all r_m, $m \in \mathcal{T}_j$ by construction.

As a result a prediction error identification on the basis of input $w_i^{(\mathcal{T}_j)}$ and output $w_j - p_j$ will provide a consistent estimate of G^0_{ji}, provided that the input $w_i^{(\mathcal{T}_j)}(t)$ is persistently exciting of a degree at least equal to the number of parameters in $G_{ji}(q,\theta)$, see the classical conditions on consistency of prediction error estimates in Ljung (1999). □

Note that as an alternative for the Two-Stage method, also the IV method could have been used, using r_m as instrument, w_i as input and $w_j - p_i$ as output, leading to the same consistency result (Gilson & Van den Hof, 2005).

The next question is how to check whether the conditions of Proposition 4.10 are satisfied. Both the appropriate construction of the set $\mathcal{T}_j$ and Condition (d) can be checked mainly on the basis of the adjacency matrix A of the network.

Algorithm 4.11
Check for candidate external variables that are correlated to w_i.

1. *Evaluate element (i,m) of A^ℓ for $\ell = 1, \cdots L$.*
2. *If for any considered power ℓ this element is non-zero, then the external variable r_m qualifies as a candidate excitation source that excites the input w_i.*[3]

[3] In the case that $\sum_{\ell=1}^{N}[A^\ell]_{im} > 1$, there is a hypothetical option that different path connections cancel each other. Since the actual correlation between r_m and w_i always needs to be checked, this situation will not be dealt with separately.

Check whether all w_k, $k \in \mathcal{U}_j^i$ are uncorrelated to all r_m, $m \in \mathcal{T}_j$.
Check whether there is no path from m to k:

1. *Evaluate A^ℓ for $\ell = 1, \ldots, L$.*
2. *For all $k \in \mathcal{U}_j^i$, $k \neq i$, check whether the entries (k, m) of A^ℓ are zero for all powers ℓ.*

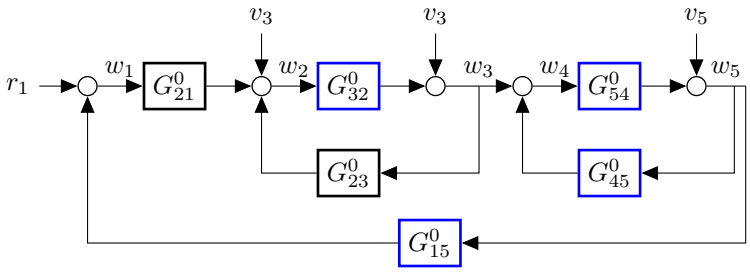

Figure 4.5: Dynamic network with 5 internal variables, of which 4 (blue-colored) transfer functions can be consistently identified using the Two-Stage Method presented of Algorithm 4.9.

Example 4.12 *Consider the dynamic network from Example 4.6, depicted in Figure 4.5. When applying the conditions of Proposition 4.10 it appears that the blue-colored transfers, G_{32}^0, G_{54}^0, G_{15}^0 and G_{45}^0 can be consistently identified with the two-stage approach presented in this section (i.e. using only w_i as a predictor input). These four transfers satisfy the conditions that their inputs are correlated to r_1, while their outputs do not include non-modeled terms that are correlated with r_1.*

Note that the transfers G_{21}^0 and G_{23}^0 do not satisfy the conditions of the Proposition because there are unknown contributions to w_2 that are correlated to r_1.

Actually the conditions that are formulated for Proposition 4.10 are very restrictive and it may be very well possible that even in case of networks that have several external variables present, there is no choice of $\mathcal{T}_j$ possible that satisfies the conditions. Additionally, by limiting attention to SISO predictors, i.e. by only considering w_i as input in the predictor, the effect of all other inputs that affect w_j will be modeled as noise and therefore will increase the variance of the parameter estimate. An alternative reasoning, that matches the situation of the direct method, is then to extend the predictor to a MISO format, as reflected in the following algorithm.

Algorithm 4.13 (Two-stage MISO model)

1. *Select a set of external variables $\{r_m\}$, with $m \in \mathcal{T}_j \subseteq \mathcal{R}_i$, each of them correlated with w_i.*
2. *Determine the set of internal variables w_k, $k \in \{\mathcal{U}_j^i, i\}$ that is correlated to any of the external variables $\{r_m\}$, with $m \in \mathcal{T}_j$. Denote this set as $\{w_k\}$, $k \in \mathcal{U}_{is}$.*

3. *Determine* $w_k^{(\mathcal{T}_j)}$, *for all* $k \in \mathcal{U}_{is}$ *(see step 2 of Algorithm 4.9 for details on how to obtain an estimate of* $w_k^{(\mathcal{T}_j)}$*).*

4. *Construct* $\tilde{w}_j(t) = w_j(t) - \sum_{k\in\mathcal{K}_j} G_{jk}^0(q)w_k(t) - r_j(t)$, *i.e. correct* w_j *with all known terms;*

5. *Identify the transfers* G_{jk}^0, $k \in \mathcal{U}_{is}$ *on the basis of a predictor model with prediction error*

$$\varepsilon_j(t,\theta) = H_j(q,\eta)^{-1}\big(\tilde{w}_j(t) - \sum_{k\in\mathcal{U}_{js}} G_{jk}(q,\theta)w_k^{(\mathcal{T}_j)}(t)\big)$$

using $\tilde{w}_j$ *and* $w_k^{(\mathcal{T}_j)}$, *an identification criterion (4.3), and where* H_j *is a fixed noise model or parametrized independently of* θ*.*

For this algorithm the following result can be obtained:

Proposition 4.14 *Consider a dynamic network that satisfies Assumption 2.19. Then the module transfer function* G_{ji}^0 *can be consistently estimated with algorithm 4.13 if the following conditions are satisfied:*

(a) The set $\mathcal{T}_j$ *is non-empty.*

(b) The external variables r_m, $m \in \mathcal{T}_j$ *are uncorrelated to noise variables* v_k, $k \in \{j, \mathcal{U}_j^i\}$.

(c) The power spectral density of $[w_{n_1}^{(\mathcal{T}_j)} \ \cdots \ w_{n_{\mathrm{n}}}^{(\mathcal{T}_j)}]^T$, $n_* \in \mathcal{U}_{is}$ *is positive definite for* $\omega \in [-\pi, \pi]$.

(d) The module transfers G_{jk}^0 *are in the model set, i.e. there exists a parameter* θ^0 *such that* $G_{jk}(q,\theta^0) = G_{jk}^0(q)$ *for all* $k \in \mathcal{U}_{is}$. □

Under the considered conditions, all model transfer functions G_{jk}^0, $k \in \mathcal{U}_{is}$ *are estimated consistently.*

Proof: The proof follows the same line as reasoning as the proof of Proposition 4.10 with appropriate change of notation. □

Example 4.15 *Returning now to the situation of Example 4.12, it can be observed that with Algorithm 4.13, the remaining module transfers* G_{21}^0 *and* G_{23}^0 *can be identified by using a MISO predictor with inputs* w_1 *and* w_3 *and output* w_2*. The external excitation variable* r_1 *excites both inputs. It only has to be checked whether this excitation is sufficiently informative. Adding a second excitation variable could be helpful in this respect.*

Moving from a SISO to a MISO predictor further increases the complexity of the identification procedure, in terms of number of models and parameters to be estimated. However it also can substantially reduce the variance of the estimates by improving the effective signal-to-noise ratio in the output. The choice for which

inputs to use in the predictor, and which external variables to project upon, leaves more freedom here to choose from.

Although in this chapter, we are dealing with noise-free measurements of the internal variables, it has to be noted that the two-stage method can simply be generalized to deal with the situation of having measurement noise on the internal variables also. This is caused by the property that measurement noise will disappear when the internal variables are projected onto the external variables.

4.5.1 Two-Stage Method With Reconstructible Noise Variables

Whereas in the two-stage method external variables serve as a basis for removing noise influences from inputs by way of projection, a similar mechanism can be realized under particular circumstances by noise variables. Consider the situation that somewhere in the network there is a noise variable v_m present, that can be reconstructed on the basis of internal variables and known transfers, and that provides excitation for the internal variable w_i that is an input to the transfer function G^0_{ji}. Then a reasoning that is completely similar to the two-stage method of the previous section can be applied by treating this reconstructible noise variable as an external variable.

The situation is depicted in Figure 4.6, where noise variable v_m is reconstructible if all transfers G^0_{mk}, $k \in \mathcal{N}_m$ are known. Then x_m can be calculated and v_m can be reconstructed according to $v_m = w_m - x_m$. From this moment onwards v_m can act as as an external variable that can be used in both the SISO and MIMO predictor of the two-stage method.

An algorithm for checking whether a noise variable is reconstructible is easily generated. For every index $m \in \mathcal{V}$: check if $\mathcal{K}_m = \mathcal{N}_m$. If so, v_m qualifies as a reconstructible noise variable. Algorithms for checking whether v_m satisfies the appropriate correlation properties with respect to the inputs w_i and w_k, $k \in \mathcal{N}_j$ are equivalent to the ones provided in the previous section.

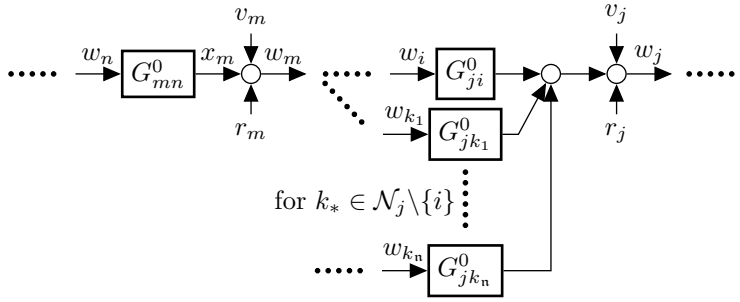

Figure 4.6: Single building block in a network structure, where the internal variable w_i is excited through a reconstructed noise variable v_m.

Example 4.16 *If we consider the network example of Figure 4.5, it appears that both v_3 and v_5 qualify as a reconstructible noise variables, provided that the transfers*

G_{32}^0 and G_{54}^0 are known a priori. However in the considered situation none of the remaining transfer functions satisfies the other condition of Proposition 4.10 that the outputs should not be disturbed by unknown terms that are correlated to the (reconstructible) noise source. However if we remove the outer loop connection G_{15}^0, as depicted in Figure 4.7, then G_{23}^0 can be identified consistently through reconstructible noise signal v_3 if G_{32}^0 is known. In Figure 4.7 this transfer is indicated in red. Similarly, using a two-input predictor the Two-Stage method can now be applied to the internal variable w_2 with inputs w_1, w_3 and external variables r and v_3.

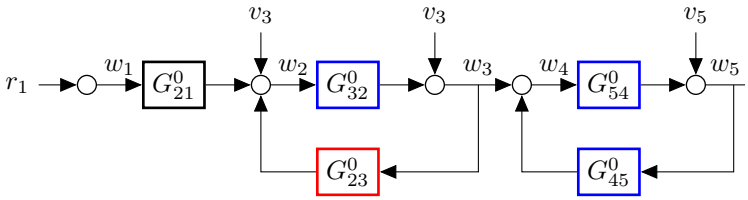

Figure 4.7: Dynamic network with 5 internal variables, of which 1 (red-colored) transfer function can be consistently identified with the two-stage method presented in this section 4.5.1 based on reconstructed noise signals.

The special phenomenon with reconstructible noise variables, is the appealing mechanism that a noise signal with variance-increasing effects on the model estimates, by the use of a prior knowledge of particular module transfers, can be turned into an external excitation variable that *reduces* the variance of the estimates.

4.6 IDENTIFICATION IN DYNAMIC NETWORKS USING THE JOINT IO METHOD

Also the joint IO method can be generalized to the situation of dynamic networks. As with the other methods presented before, we focus on a particular internal variable w_j, for which we intend to identify the module transfer G_{ji}^0. When isolating the internal variables w_i and w_j, and modeling the vector process $(w_j^T, w_i^T)^T$ as the output of a stationary stochastic process, it is very unlikely that the resulting process will allow to determine consistent estimates of G_{ji}^0, if the two internal variables are part of a complex network topology. Like in the direct method, we have to extend the number of internal variables that we take into account.

Consider the following partition of measured variables: $w = \{w_j, w_{\mathcal{N}_j}, w_{\mathcal{Z}_j}\}$ where $\mathcal{N}_j$ has the usual meaning, and $\mathcal{Z}_j$ is a set of indices of all remaining variables, i.e. $\mathcal{Z}_j = \{1, \ldots, L\} \setminus \{\{j\} \cup \mathcal{N}_j\}$. Let $w_{\mathcal{N}}$ denote the vector $[w_{k_1} \ \cdots \ w_{k_n}]^T$, where $\{k_1, \ldots, k_n\} = \mathcal{N}_j$. Let $v_{\mathcal{N}}$ denote the vector $[v_{k_1} \ \cdots \ v_{k_n}]^T$, where $\{k_1, \ldots, k_n\} = \mathcal{N}_j$, and where the ℓth entry is zero if v_ℓ is not present in the network (i.e. $\ell \notin \mathcal{V}$). The vectors $w_{\mathcal{Z}}$, $v_{\mathcal{Z}}$ are defined analogously. The ordering of the elements of $w_{\mathcal{N}}$ and $v_{\mathcal{N}}$ is not important, as long as it is the same for both these vectors (the same holds for

$w_{\mathcal{Z}}$ and $v_{\mathcal{Z}}$). The transfer function matrix between $w_{\mathcal{N}}$ and w_j is denoted $G^0_{j\mathcal{N}}$. The other transfer function matrices are defined analogously.

Since in the Joint IO method no explicit use is made of external variables, we assume that no r-variables are present, and that all external excitation originates from noise variables.

Using these partitions the data generating system (4.1) can be written as

$$\begin{bmatrix} w_j \\ w_{\mathcal{N}} \\ w_{\mathcal{Z}} \end{bmatrix} = \begin{bmatrix} 0 & G^0_{j\mathcal{N}} & 0 \\ G^0_{\mathcal{N}j} & G^0_{\mathcal{N}\mathcal{N}} & G^0_{\mathcal{N}\mathcal{Z}} \\ G^0_{\mathcal{Z}j} & G^0_{\mathcal{Z}\mathcal{N}} & G^0_{\mathcal{Z}\mathcal{Z}} \end{bmatrix} \begin{bmatrix} w_j \\ w_{\mathcal{N}} \\ w_{\mathcal{Z}} \end{bmatrix} + \begin{bmatrix} v_j \\ v_{\mathcal{N}} \\ v_{\mathcal{Z}} \end{bmatrix}. \tag{4.7}$$

First we are going to formalize the properties of the vector process $[w_j \ w_{\mathcal{N}}]^T$ in the next Lemma.

Lemma 4.17 *Consider a dynamic network (4.7) that satisfies Assumption 2.19. Suppose that v_j is present and is uncorrelated to all other noise variables v_k, $k \in \mathcal{V}_j$. The spectral density $\Phi_w(\omega)$ of the vector process $[w_j \ w_{\mathcal{N}}]^T$ is:*

$$\begin{aligned} \Phi_w &= \begin{bmatrix} \breve{G}^0_{jj} H^0_j & G^0_{j\mathcal{N}} \breve{G}^0_{\mathcal{N}\mathcal{N}} \tilde{H}^0_{\mathcal{N}} \\ \breve{G}^0_{\mathcal{N}\mathcal{N}} \tilde{G}^0_{\mathcal{N}j} H^0_j & \breve{G}^0_{\mathcal{N}\mathcal{N}} \tilde{H}^0_{\mathcal{N}} \end{bmatrix} \begin{bmatrix} \sigma^2_{e_j} & \\ & Q_s \end{bmatrix} \begin{bmatrix} \breve{G}^0_{jj} H^0_j & G^0_{j\mathcal{N}} \breve{G}^0_{\mathcal{N}\mathcal{N}} \tilde{H}^0_{\mathcal{N}} \\ \breve{G}^0_{\mathcal{N}\mathcal{N}} \tilde{G}^0_{\mathcal{N}j} H^0_j & \breve{G}^0_{\mathcal{N}\mathcal{N}} \tilde{H}^0_{\mathcal{N}} \end{bmatrix}^* \\ &= \begin{bmatrix} W^0_{jj} & W^0_{j\mathcal{N}} \\ W^0_{\mathcal{N}j} & W^0_{\mathcal{N}\mathcal{N}} \end{bmatrix} \begin{bmatrix} \sigma^2_{e_j} & \\ & Q_s \end{bmatrix} \begin{bmatrix} W^0_{jj} & W^0_{j\mathcal{N}} \\ W^0_{\mathcal{N}j} & W^0_{\mathcal{N}\mathcal{N}} \end{bmatrix}^* \\ &= W^0 Q W^{0*} \end{aligned} \tag{4.8}$$

where $\sigma^2_{e_j}$ is the variance of e_j, Q_s is a positive definite matrix, and

$$\begin{aligned} \breve{G}^0_{jj} &= (1 - G^0_{j\mathcal{N}} (I - \tilde{G}^0_{\mathcal{N}\mathcal{N}})^{-1} \tilde{G}^0_{\mathcal{N}j})^{-1}, \\ \breve{G}^0_{\mathcal{N}\mathcal{N}} &= (I - \tilde{G}^0_{\mathcal{N}\mathcal{N}} - \tilde{G}^0_{\mathcal{N}j} G^0_{j\mathcal{N}})^{-1}, \\ \tilde{G}^0_{\mathcal{N}\mathcal{N}} &= G^0_{\mathcal{N}\mathcal{N}} + G^0_{\mathcal{N}\mathcal{D}} (I - G^0_{\mathcal{D}\mathcal{D}})^{-1} G^0_{\mathcal{D}\mathcal{N}}, \\ \tilde{G}^0_{\mathcal{N}j} &= G^0_{\mathcal{N}j} + G^0_{\mathcal{N}\mathcal{D}} (I - G^0_{\mathcal{D}\mathcal{D}})^{-1} G^0_{\mathcal{D}j}, \end{aligned}$$

and $\tilde{H}^0_{\mathcal{N}}$ is a monic, stable, minimum phase spectral factor of the stochastic process $v_{\mathcal{N}} + G^0_{\mathcal{N}\mathcal{D}} (I - G^0_{\mathcal{D}\mathcal{D}})^{-1} v_{\mathcal{D}}$.

There are a few important things to note about this lemma. The matrix W^0 is a spectral factor of Φ_w. However, W^0 may not be monic. This is due to the fact that some transfers G^0_{ji} may not have delays. In particular the following two statements can be verified using Lemmas 4.17 and 4.1:

- If every loop through $k \in \{\mathcal{N}, j\}$ in the data generating system has a delay, then the diagonal entries of both W^0 and ${W^0}^{-1}$ are monic transfer functions.
- If every path from w_i to w_j has a delay (is zero), then W^0_{ji} has a delay (is zero).

If the matrix W^0 as defined in Lemma 4.17 is available (or an estimate thereof) then it is possible to obtain expressions for (estimates of) $G^0_{j\mathcal{N}}$ and H^0_j:

$$G^0_{j\mathcal{N}} = W^0_{j\mathcal{N}} {W^0_{\mathcal{N}\mathcal{N}}}^{-1} \tag{4.9}$$

$$H^0_j = W^0_{jj} - W^0_{j\mathcal{N}} {W^0_{\mathcal{N}\mathcal{N}}}^{-1} W^0_{\mathcal{N}j}. \tag{4.10}$$

The main effort when using the Joint IO method goes into obtaining an estimate of W^0. There are two ways to proceed. The first option is to use the prediction error methods. In this case the measured signals $(w_j, w_{\mathcal{N}})$ are modeled as outputs of an ARMA process. An estimate of W^0 is then obtained by estimating a noise model. The second option is to first obtain an estimate of the power spectral density of $[w_j \ w_{\mathcal{N}}^T]^T$, denoted Φ_w, and then obtain an estimate of W^0 by calculating the spectral factor of Φ_w.

In the following sections the two options are compared.

4.6.1 Joint IO Method - Prediction Error Approach

The algorithm is summarized as:

Algorithm 4.18 (Joint IO Method - Prediction Error Approach)

1. *Choose parameterization of* $W(\theta)$.

2. *Minimize the sum of squared prediction errors to obtain* $\hat{\theta}$, *i.e., minimize*

$$\hat{\theta}_N = \arg\min_{\theta} V_N(\theta), \ \textit{where} \ V_N(\theta) = \sum_{t=0}^{N-1} \varepsilon^T(t)\varepsilon(t).$$

3. *Calculate* $G_{j\mathcal{N}}(\hat{\theta}_N) = W_{j\mathcal{N}}(\hat{\theta}_N) W^{-1}_{\mathcal{N}\mathcal{N}}(\hat{\theta}_N)$.

A key element of the Prediction Error approach is choosing the parameterization. The parameterization must be chosen such that (a) the data generating system is in the model set (there exists a parameter such that $W(\theta_0) = W^0$), and (b) the parameter space is sufficiently restricted in order to guarantee uniqueness of θ_0. In this case, since we are only dealing with ARMA model structures, only the noise model needs to be parameterized.

It is common in the prediction error method to parameterize H as a monic, stable, and minimum phase transfer matrix. This guarantees uniqueness of H^0. However, using this choice of parameterization will not lead to consistent estimates in the situation that we are considering. This is because we are using H to model not only the noise component of the signal, but also the module dynamics. From Lemma 4.17 it can be seen that if certain transfers G^0_{ji} have delays, then W^0 will not be monic (i.e. off-diagonal elements of $\lim_{z\to\infty} W^0(z)$ will be non-zero).

In order to ensure that the data generating system is in the model set, every transfer without a delay in the data generating system must be parameterized without a delay in the model. Note that the converse statement does not need to hold.

A transfer with a delay need not be parameterized with a delay in order for the data generating system to be in the model set.

The following two lemmas will provide insight as to how to parameterize the model such that both the data generating system is in the model set, and uniqueness can be guaranteed.

Lemma 4.19 *Consider a dynamic network that satisfies Assumption 2.19. If every path from w_{k_2} to w_{k_1}, $k_1 \neq k_2$, $k_1, k_2 \in \{j, \mathcal{N}_j\}$ has a delay (is zero) then $W^0_{k_1 k_2}$ has a delay (is zero). If every loop through w_k $k \in \{j, \mathcal{N}_j\}$ has a delay (is zero) then W^0_{kk} is a monic transfer function (is one).*

Lemma 4.20 *Consider a dynamic network that satisfies Assumption 2.19. Suppose that every loop involving w_k, $k \in \{j, \mathcal{N}_j\}$ has a delay. Then there exists a permutation matrix*

$$P = \begin{bmatrix} 1 & 0 \\ 0 & P_{\mathcal{N}\mathcal{N}} \end{bmatrix}$$

such that

$$\begin{bmatrix} 1 & 0 \\ 0 & P_{\mathcal{N}\mathcal{N}} \end{bmatrix} \begin{bmatrix} W_{jj}(\infty) & W_{j\mathcal{N}}(\infty) \\ W_{\mathcal{N}j}(\infty) & W_{\mathcal{N}\mathcal{N}}(\infty) \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & P_{\mathcal{N}\mathcal{N}} \end{bmatrix}^T = \begin{bmatrix} 1 & [d_{j\mathcal{N}}\ 0] \\ \begin{bmatrix} 0 \\ d_{\mathcal{N}j} \end{bmatrix} & L^0_{\mathcal{N}\mathcal{N}} \end{bmatrix} \tag{4.11}$$

where length$(d_{j\mathcal{N}})$+length$(d_{\mathcal{N}j}) = n$, where $n = card(\mathcal{N}_j)$ and $L^0_{\mathcal{N}\mathcal{N}}$ is lower triangular with ones on the diagonal.

The proof can be found in Appendix 4.8.4. Here we have used the notation $W_{jj}(\infty)$ as a short-hand to denote $\lim_{z\to\infty} W_{jj}(z)$.

In light of Lemma 4.20, consider the following parameterization scheme. Choose

$$W(\theta) = \begin{bmatrix} W_{jj}(\theta) & W_{j\mathcal{N}}(\theta) \\ W_{\mathcal{N}j}(\theta) & W_{\mathcal{N}\mathcal{N}}(\theta) \end{bmatrix} \tag{4.12}$$

such that $\mathcal{S} \in \mathcal{M}$ and

$$W(\infty, \theta) = \begin{bmatrix} 1 & [d_{j\mathcal{N}}(\theta)\ 0] \\ \begin{bmatrix} 0 \\ d_{\mathcal{N}j}(\theta) \end{bmatrix} & L(\theta) \end{bmatrix} \tag{4.13}$$

where length$(d_{j\mathcal{N}}(\theta))$ is equal to the number of non-zero elements in $G^0_{j\mathcal{N}}(\infty)$, and length$(d_{\mathcal{N}j})$ is equal to $n -$ length$(d_{j\mathcal{N}}(\theta))$, and $L(\theta)$ is a fully parameterized lower triangular matrix with ones on the diagonal.

As mentioned, one role that the parameterization plays is to ensure uniqueness of the estimates. An important feature of the parameterization (4.12) is that both $W(\infty, \theta)$ and $W^{-1}(\infty, \theta)$ have ones on the diagonal. This feature ends up ensuring the uniqueness of the estimates.

Finally, consider the following Proposition which shows that an ARMA model parameterized as in (4.12) and (4.13) results in consistent estimates of $G^0_{j\mathcal{N}}$ using Algorithm 4.18.

Proposition 4.21 *Consider a data generating system (4.1) that satisfies Assumtion 2.19. Consistent estimates of $G^0_{j\mathcal{N}}$ can be obtained using Algorithm 4.18 if the following conditions hold:*

(a) v_j is present, and is uncorrelated to all v_k, $k \in \mathcal{V}_j$.

(b) The process $v_{\mathcal{N}} + \tilde{G}_{\mathcal{N}\mathcal{D}} v_{\mathcal{D}}$ is full rank.

(c) For all $k \in \{j, \mathcal{N}\}$, every loop through w_k in the data generating system has a delay.

(d) The model $W(\theta)$ is parameterized according to (4.12) and (4.13).

The proof can be found in Appendix 4.8.5.

4.6.2 Joint IO Method - Spectral Factorization Approach

Although, by Proposition 4.21, Algorithm 4.18 leads to a consistent estimate of $G^0_{j\mathcal{N}}$, it may be difficult to implement in practice. The key points are that choosing the parameterization (4.12) such that Condition (d holds could be difficult. Secondly, since there are a large number of parameters involved, finding the global minimum of $V_N(\theta)$ could also be difficult.[4]

Fortunately, as stated in Section 3.4.3 of Chapter 3 there exists an alternative approach to finding an estimate of W^0. Namely the spectral factorization approach. By exactly the same reasoning as in Section 3.4.3 it follows that W^0 is a spectral factor of the power spectral density of $[w_j \; w_{\mathcal{N}}]^T$.

Before proceeding to the result, consider the following nomenclature and notation. Let W be a matrix of transfer functions. Let $\lim_{z\to\infty} W(z)$ be denoted $W(\infty)$. The matrix W is *monic* if $W(\infty) = I$ (i.e. has ones on the diagonal and zeros elsewhere). The matrix W is *stable* if all transfer functions in W are stable. The matrix W is *minimum phase* W^{-1} exists and is stable.

Consider a data generating system where all transfer functions have a delay. Then by Lemmas 4.19 and 4.17, W^0 is monic, stable and minimum phase. By the Spectral Factorization Theorem[5] of Youla (1961) every power spectral density matrix has a unique monic, stable and minimum phase spectral factor. Consequently, in this case W^0 is the unique monic, stable, minimum phase spectral factor. Moreover, there exist several methods to obtain this spectral factor from data (Van Overschee et al., 1997; Mari & Stoica, 2000; Hinnen et al., 2005). Thus, in this case the following algorithm can be used to obtain an estimate of W^0.

Algorithm 4.22 *Assume that all transfer functions in the data generating system have a delay. Assume the power spectral density of $[w_j \; w_{\mathcal{N}}^T]^T$, or an estimate thereof is available. Denote it Φ_w.*

[4] The MATLAB system identification toolbox cannot be used to find models with this parameterization since the toolbox only allows for diagonal noise models, thus dedicated code must be written.

[5] See Appendix 4.8.6 for the exact statement of the Spectral Factorization Theorem.

1. *Obtain an estimate of the unique monic, stable, and minimum phase spectral factor of Φ_w. This is an estimate of W^0.*

2. *Use (4.9) and the estimate of W^0 to obtain an estimate of $G^0_{j\mathcal{N}}$.*

However, it is too restrictive to only consider data generating systems for which all transfer functions have a delay. As in the previous section, consider a data generating system such that all loops through all w_k, $k \in \{j\} \cup \mathcal{N}_j$ have a delay. In this case W^0 is not monic - there may be paths from some w_{k_1} to w_{k_2} that don't have a delay, which, by Lemma 4.19, results in $W^0(\infty)$ having a non-zero off-diagonal element. Therefore the following Algorithm is proposed to estimate $G^0_{j\mathcal{N}}$.

Algorithm 4.23 (Joint IO Method - Spectral Factorization Approach) *Assume all loops through w_k, $k \in \mathcal{N} \cup \{j\}$ have a delay. Assume the power spectral density of $[w_j \; w_{\mathcal{N}}^T]^T$, or an estimate thereof is available. Denote it Φ_w.*

1. *Obtain an estimate of the unique monic, stable, and minimum phase spectral factor of the stochastic process Φ_w. Denote this estimate as M, and denote the estimated covariance matrix as T (i.e. $\Phi_w = M^0 T^0 M^{0*}$).*

2. *Use M and T to obtain an estimate of W^0.*

3. *Use (4.9) and the estimate of W^0 to obtain an estimate of $G^0_{j\mathcal{N}}$.*

In the remaining text, we focus on Step 2 of Algorithm 4.23. The proceedure that is presented is algorithmic in nature, and thus can be easily implemented in practice (using MATLAB for instance).

First, we establish a connection between W^0, the non-monic spectral factor that we need in order to obtain an estimate of $G^0_{j\mathcal{N}}$ and M^0, the unique monic, stable, minimum phase spectral factor of Φ_w. Let D^0 denote the direct feed-through term of W^0:

$$D^0 = \lim_{z \to \infty} W^0(z).$$

Then

$$\begin{aligned} \Phi_w &= W^0 Q^0 W^{0*} \\ &= W^0 D^{0^{-1}} D^0 Q^0 D^{0^T} D^{0^{-T}} W^{0*} \\ &= M^0 T^0 M^{0*} \end{aligned} \tag{4.14}$$

where $M^0 = W^0 D^{0^{-1}}$ is monic, stable and minimum phase, and $T^0 = D^0 Q^0 D^{0^T}$ is a positive definite matrix.

From (4.14) it follows that once an estimate of D^0 is available, W^0 can be reconstructed from M^0 (i.e. $W^0 = M^0 D^0$).

However, it is not necessary to *exactly* estimate D^0 in order to obtain an estimate of $G^0_{j\mathcal{N}}$. The matrix D^0 need only be known up to a similarity transform X where

$$X = \begin{bmatrix} 1 & 0 \\ 0 & X_{\mathcal{N}\mathcal{N}} \end{bmatrix} \tag{4.15}$$

where $X_{\mathcal{N}\mathcal{N}}$ is a full rank $n \times n$ matrix of real numbers, and n is the cardinality of $\mathcal{N}_j$. To see why this is so, suppose that $\tilde{D}^0 = D^0 X$ is known. Thus an estimate of W^0 is known up to a similarity transform of the form (4.15) (since we have an estimate of M^0 is available from Step 1 of Algorithm 4.23):

$$\begin{aligned}\tilde{W}^0 &= M^0 \tilde{D}^0 \\ &= M^0 D^0 X \\ &= W^0 X \\ &= \begin{bmatrix} \breve{G}^0_{jj} H^0_j & G^0_{j\mathcal{N}} \breve{G}^0_{\mathcal{N}\mathcal{N}} \tilde{H}^0_{\mathcal{N}} X_{\mathcal{N}\mathcal{N}} \\ \breve{G}^0_{\mathcal{N}\mathcal{N}} \tilde{G}^0_{\mathcal{N}j} H^0_j & \breve{G}^0_{\mathcal{N}\mathcal{N}} \tilde{H}^0_{\mathcal{N}} X_{\mathcal{N}\mathcal{N}} \end{bmatrix}\end{aligned} \tag{4.16}$$

Using $\tilde{W}^0$ to obtain an estimate of $G^0_{j\mathcal{N}}$ according to (4.9) results in:

$$\begin{aligned}\tilde{W}^0_{j\mathcal{N}} \tilde{W}^{0^{-1}}_{\mathcal{N}\mathcal{N}} &= G^0_{j\mathcal{N}} \breve{G}^0_{\mathcal{N}\mathcal{N}} \tilde{H}^0_{\mathcal{N}} X_{\mathcal{N}\mathcal{N}} \left(\breve{G}^0_{\mathcal{N}\mathcal{N}} \tilde{H}^0_{\mathcal{N}} X_{\mathcal{N}\mathcal{N}}\right)^{-1} \\ &= G^0_{j\mathcal{N}}\end{aligned} \tag{4.17}$$

as desired.

In light of this similarity transform, we focus on the following problem: *given an estimate of T^0, obtain an estimate of $\tilde{D}^0$* (recall that T^0 is available from Step 1 in Algorithm 4.23). Once $\tilde{D}^0$ is known, an estimate of $G^0_{j\mathcal{N}}$ can be obtained as shown above.

To this end, recall the connection between T^0 and D^0. From (4.14) we have $T^0 = D^0 Q^0 D^{0^T}$ where T^0 is known, but Q^0 and D^0 are unknown. Although Q^0 is unknown, we do know from Lemma 4.17 that Q^0 is block diagonal (i.e. the off-diagonal terms of the first row and column are all zero). Now consider the connection between T^0 and $\tilde{D}^0$ (i.e. the connection between T^0 and D^0 up to a similarity transform X of the form (4.15)):

$$\begin{aligned}T^0 &= D^0 X X^{-1} Q^0 X^{-T} X^T D^{0^T} \\ &= \tilde{D}^0 \tilde{Q}^0 \tilde{D}^{0^T}\end{aligned} \tag{4.18}$$

where X is any matrix of the form (4.15) and $\tilde{Q}^0 = X^{-1} Q^0 X^{-T}$. Note that since Q^0 is block diagonal, $\tilde{Q}^0$ is also block diagonal.

From Lemma 4.20 we see that D^0 (recall that D^0 is defined as $W^0(\infty)$) can be permuted so that it has the form (4.11) (as long as every loop through each w_k, $k \in \mathcal{N}_j \cup \{j\}$ has a delay). Thus, without loss of generality we can consider $\tilde{D}^0$ to have the form:

$$\tilde{D}^0 = \begin{bmatrix} 1 & \tilde{d}_{j\mathcal{N}} & 0 \\ 0 & I & 0 \\ \tilde{d}_{\mathcal{N}j} & * & I \end{bmatrix}. \tag{4.19}$$

Now we present an algorithm to obtain $\tilde{D}^0$ from the known matrix T^0. The algorithm is based on (4.18) and the following facts: it is known that the off diagonal entries of the first row and column of $\tilde{Q}^0$ are all zero; and it is known that $\tilde{D}^0$ has the form

(4.19). The algorithm proceeds by performing row and column operations on T^0 to place zeros in the correct places according to the known locations of zeros in $\tilde{D}^0$ and $\tilde{Q}^0$. The result of these operations is an estimate of $\tilde{D}^0$.

Let T^0 be denoted/partitioned as:

$$T^0 = \begin{bmatrix} T^0_{11} & T^{0^T}_{21} & T^{0^T}_{31} \\ T^0_{21} & T^0_{22} & T^{0^T}_{32} \\ T^0_{31} & T^0_{32} & T^0_{33} \end{bmatrix}$$

where the partition is such that T^0_{11} is a scalar, $T^{0^T}_{12}$ has the same dimensions as $\tilde{d}_{j\mathcal{N}}$ in (4.19) and T^0_{31} has the same dimensions as $\tilde{d}_{\mathcal{N}j}$ in (4.19).

Algorithm 4.24 (Obtain $\tilde{D}^0$ from T^0)

1. *Place a vector of zeros in the* $(2,1)$ *and* $(1,2)$ *partitions of* T^0*:*

$$T^0 = \begin{bmatrix} 1 & T^{0^T}_{21} T^{0^{-1}}_{22} & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} \tilde{T}^0_{11} & 0 & \tilde{T}^{0^T}_{31} \\ 0 & T^0_{22} & T^{0^T}_{32} \\ \tilde{T}^0_{31} & T^0_{32} & T^0_{33} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ T^{0^{-1}}_{22} T^0_{21} & I & 0 \\ 0 & 0 & I \end{bmatrix} \tag{4.20}$$

where

$$\tilde{T}^0_{11} = T^0_{11} - T^{0^T}_{21} T^{0^{-1}}_{22} T^0_{21},$$
$$\tilde{T}^0_{31} = T^0_{21} - T^{0^T}_{21} T^{0^{-1}}_{22} T^{0^T}_{32}.$$

2. *Set the remaining off-diagonal non-zero entries in the top row and first column of the middle matrix in (4.20) to zero:*

$$T^0 = \begin{bmatrix} 1 & T^{0^T}_{21} T^{0^{-1}}_{22} & \\ & I & \\ & & I \end{bmatrix} \begin{bmatrix} 1 & & \\ & I & \\ \tilde{T}^0_{31} \tilde{T}^{0^{-1}}_{11} & & I \end{bmatrix} \begin{bmatrix} \tilde{T}^0_{11} & 0 & 0 \\ 0 & T^0_{22} & T^{0^T}_{32} \\ 0 & T^0_{32} & \tilde{T}^0_{33} \end{bmatrix} \begin{bmatrix} 1 & \tilde{T}^{0^{-1}}_{11} \tilde{T}^{0^T}_{31} & \\ & I & \\ & & I \end{bmatrix} \begin{bmatrix} 1 & & \\ T^{0^{-1}}_{22} T^0_{21} & I & \\ & & I \end{bmatrix}$$

where

$$\tilde{T}_{33} = T^0_{33} - \tilde{T}^{0^T}_{31} \tilde{T}^{0^{-1}}_{11} \tilde{T}^0_{31}.$$

3. *Multiply the first two matrices and the last two matrices resulting in the following equation:*

$$T^0 = \begin{bmatrix} 1 & T^{0^T}_{21} T^{0^{-1}}_{22} & 0 \\ 0 & I & 0 \\ \tilde{T}^0_{31} \tilde{T}^{0^{-1}}_{11} & 0 & I \end{bmatrix} \begin{bmatrix} \tilde{T}^0_{11} & 0 & 0 \\ 0 & T^0_{22} & T^{0^T}_{32} \\ 0 & T^0_{32} & \tilde{T}^0_{33} \end{bmatrix} \begin{bmatrix} 1 & 0 & \tilde{T}^{0^{-1}}_{11} \tilde{T}^{0^T}_{31} \\ T^{0^{-1}}_{22} T^0_{21} & I & 0 \\ 0 & 0 & I \end{bmatrix}, \tag{4.21}$$

where the first matrix has zero entries in the same locations as $\tilde{D}^0$ *in (4.19) and the middle matrix has zero entries in the same locations as* $\tilde{Q}^0$ *in (4.18). Thus, by (4.18), the first matrix in (4.21) is* $\tilde{D}^0$ *and the middle matrix is* $\tilde{Q}^0$ *(up to a similarity transforms of the form (4.15)).*

Using this estimate of $\tilde{D}^0$, it is possible to construct an estimate of $\tilde{W}^0$ in (4.16). Then from the estimate of $\tilde{W}^0$ an estimate of $G^0_{j\mathcal{N}}$ can be obtained as shown in (4.16).

In this section we have shown that it is possible to use the Joint IO method for identification in dynamic networks. The conditions under which it is possible to consistently identify G^0_{ji} are very similar to the Direct Method in the sense that we require $\mathcal{S} \in \mathcal{M}$ and that there is a condition on the presence of algebraic loops. However, the condition on the absence of algebraic loops is slightly stricter for the Joint IO method since for this method it is required that all loops through all w_k, $k \in \mathcal{N} \cup \{j\}$ have a delay, whereas for the Direct Method this condition only had to hold for loops through w_j. In addition a weakness of the Prediction-Error formulation of the Joint IO method is the practical implementation. However, using tools from spectral factorization theory, it is possible to develop a practical implementation of the method.

4.7 SUMMARY

Several methods for closed-loop identification have been generalized to become applicable to systems that operate in a general network configuration. In the current setting we have focused on networks in which noise free measurements of all internal variables are available, and where our intention is to model one particular module. Complex networks can be handled and effective use can be made of external variables. These external variables limit the necessity to perform exhaustive consistent modeling of all noise sources in the network. The several prediction error methods presented (direct method, two-stage method based on either external variables or on reconstructible noise signals, and joint-IO method) are shown to be able to estimate particular subparts of the network. It opens questions as to where and how many external and internal variables are required to identify particular parts of the network.

4.8 APPENDIX

4.8.1 Proof of Lemma 4.1

Let $G^0(\infty)$ denote $\lim_{z\to\infty} G^0(z)$, and let $G^0(\infty)$ represent a directed graph, denoted by $\mathbb{G}$. If every path from $b \to a$ has a delay then there is no path from b to a in the graph defined by $G^0(\infty)$. We can now separate the nodes of $\mathbb{G}$ into two groups, one called $\mathcal{A}$, containing node a and all nodes that have a path to a, and a second group of nodes called $\mathcal{B}$, containing b and all remaining nodes that have no path to a. By reordering the nodes in the graph $\mathbb{G}$, the matrix $G^0(\infty)$ related to this reordered representation can be written as

$$G^0(\infty) = \begin{bmatrix} G_{\mathcal{AA}} & 0 \\ G_{\mathcal{BA}} & G_{\mathcal{BB}} \end{bmatrix}$$

where $G_{\mathcal{AA}}$ and $G_{\mathcal{BB}}$ both have zeros on the diagonals.
With the inversion rule for block matrices it follows that:

$$\big(I - G(\infty)\big)^{-1} = \begin{bmatrix} I - G_{\mathcal{AA}} & 0 \\ -G_{\mathcal{BA}} & I - G_{\mathcal{BB}} \end{bmatrix}^{-1} = \begin{bmatrix} * & 0 \\ * & * \end{bmatrix}$$

which shows that the (a, b) entry in $(I - G^0(\infty))^{-1}$ is zero. Since $(I - G^0)^{-1}$ is proper, this implies that the (b, a) entry in $(I - G^0)^{-1}$ has a delay.
The reasoning to show that if there is no path from b to a then $\mathcal{G}^0_{ab} = 0$ is completely analogous except that instead of working with $G^0(\infty)$, it is necessary to work with G^0. □

4.8.2 Proof of Proposition 4.3

Proof:

1. Show that the lower bound of the objective function $\bar{V}_j(\theta) := \bar{\mathbb{E}}\varepsilon_j^2(t, \theta)$ is $\sigma^2_{e_j}$, the variance of e_j.
2. Show that $\bar{V}_j(\theta) = \sigma^2_{e_j}$ implies that $\theta = \theta_0$ (i.e the global minimum is attainable and unique).

Step 1. Throughout the proof, it will be useful to expand the measured variable w_i in terms of all noise sources and external inputs that affect w_i. From (4.1) and using the notation from Lemma 4.1 we have:

$$w_i = \sum_{k=1}^{L} \mathcal{G}^0_{ik}(v_k + r_k) = \sum_{k \in \mathcal{V}_i} \mathcal{G}^0_{ik} v_k + \sum_{k \in \mathcal{R}_i} \mathcal{G}^0_{ik} r_k \tag{4.22}$$

where the second equality holds by Lemma 4.1 and the definitions of $\mathcal{V}_i$ and $\mathcal{R}_i$.

Now, (4.22) will be used to express the objective function in terms of only noise sources and external inputs. With the predictor (4.4) it follows that

$$\begin{aligned}
\bar{V}_j(\theta) &= \bar{\mathbb{E}}\Big[\Big(H_j^{-1}(\theta)\Big(v_j + \sum_{i \in \mathcal{N}_j} \big(G^0_{ji} - G_{ji}(\theta)\big) w_i\Big)\Big)^2\Big] \\
&= \bar{\mathbb{E}}\Big[\Big(H_j^{-1}(\theta)\Big(v_j + \sum_{i \in \mathcal{N}_j} \Delta G_{ji}(\theta)\Big(\sum_{k \in \mathcal{V}_i} \mathcal{G}^0_{ik} v_k + \sum_{k \in \mathcal{R}_i} \mathcal{G}^0_{ik} r_k\Big)\Big)\Big)^2\Big] \\
&= \bar{\mathbb{E}}\Big[\Big(\Delta H_j(\theta) v_j + H_j^{-1}(\theta) \sum_{i \in \mathcal{N}_j} \sum_{k \in \mathcal{V}_i} \Delta G_{ji}(\theta) \mathcal{G}^0_{ik} v_k \\
&\qquad + H_j^{-1}(\theta) \sum_{i \in \mathcal{N}_j} \sum_{k \in \mathcal{R}_i} \Delta G_{ji}(\theta) \mathcal{G}^0_{ik} r_k + e_j\Big)^2\Big]
\end{aligned} \tag{4.23}$$

where $\Delta G_{ji}(\theta) = G^0_{ji} - G_{ji}(\theta)$, and $\Delta H_j(\theta) = H_j^{-1}(\theta) - H_j^{0^{-1}}$. Next Condition (b) will be used to simplify this expression.

By (b) if G^0_{ji} has a delay, then $G_{ji}(\theta)$ will be parameterized with a delay (i.e. $\Delta G_{ji}(\theta)$ has a delay if G^0_{ji} has a delay). Moreover, by Lemma 4.1 the term $G^0_{ji}\mathcal{G}_{ij}$

has a delay if all paths from j to j have a delay. By Condition (b), every path from j to j has a delay, therefore, $\Delta G_{ji}(\theta)\mathcal{G}_{ij}^0$ has a delay for all i.

Consequently every term in (4.23) is uncorrelated to e_j:

- since $H_j(\theta)$ and H_j^0 are both monic, $\Delta H_j(\theta)v_j$ is a function of $v_j(t-k)$, $k>1$;
- as described above, $\Delta G_{ji}(\theta)\mathcal{G}_{ij}^0 v_j$ is also a function of $v_j(t-k)$ $k>1$;
- by Condition (a) any term involving v_k, $k\in\mathcal{V}_j$, $k\neq j$ is uncorrelated to e_j;
- by Condition c of Assumption 2.19, e_j is uncorrelated to r_k for all k.

Using this reasoning to simplify (4.23) results in:

$$\begin{aligned}\bar{V}_j(\theta) &= \bar{\mathbb{E}}\Big[\Big(\Delta H_j(\theta)v_j + H_j^{-1}(\theta)\sum_{i\in\mathcal{N}_j}\sum_{k\in\mathcal{V}_i}\Delta G_{ji}(\theta)\mathcal{G}_{ik}^0 v_k \\ &\qquad + H_j^{-1}(\theta)\sum_{i\in\mathcal{N}_j}\sum_{k\in\mathcal{R}_i}\Delta G_{ji}(\theta)\mathcal{G}_{ik}^0 r_k\Big)^2\Big] + \sigma_{e_j}^2 \\ &= \bar{\mathbb{E}}\Big[\Big(\Delta H_j(\theta)v_j + H_j^{-1}(\theta)\sum_{i\in\mathcal{N}_j}\Delta G_{ji}(\theta)w_i\Big)^2\Big] + \sigma_{e_j}^2 \end{aligned} \tag{4.24}$$

where $\sigma_{e_j}^2$ is the variance of e_j. From (4.24), it is clear that $\bar{V}_j(\theta)\geq\sigma_{e_j}^2$. This concludes the first step.

Step 2. Next it must be shown that the global minimum of $\bar{V}_j(\theta)$ is attainable and unique. This will be done by showing

$$\bar{V}_j(\theta)=\sigma_{e_j}^2 \;\Rightarrow\; \theta=\theta_0.$$

Using (4.24), $\bar{V}_j(\theta)=\sigma_{e_j}^2$ can be written as

$$\bar{\mathbb{E}}\Big[\sum_{i\in\mathcal{N}_j}\frac{\Delta G_{ji}(\theta)}{H_j(\theta)}w_i + \Delta H_j(\theta)v_j\Big)^2\Big] + \sigma_{e_j}^2 = \sigma_{e_j}^2$$

or equivalently

$$\bar{\mathbb{E}}\left[\left(\left[\Delta H_j(\theta)\;\; \frac{\Delta G_{jn_1}(\theta)}{H_j(\theta)}\;\; \cdots\;\; \frac{\Delta G_{jn_{\mathrm{n}}}(\theta)}{H_j(\theta)}\right]\begin{bmatrix} v_j \\ w_{n_1}\\ \vdots \\ w_{n_{\mathrm{n}}}\end{bmatrix}\right)^2\right]=0$$

$$\bar{\mathbb{E}}\left[\left(\Delta x(\theta)\begin{bmatrix}1 & -G_{jn_1}^0 & \cdots & -G_{jn_{\mathrm{n}}}^0\\ & 1 & & \\ & & \ddots & \\ & & & 1\end{bmatrix}\begin{bmatrix} w_j \\ w_{n_1}\\ \vdots \\ w_{n_{\mathrm{n}}}\end{bmatrix}\right)^2\right]=0$$

$$\bar{\mathbb{E}}\Big[\Big(\Delta x(\theta)^T J w_{\{j,\mathcal{N}_j\}}\Big)^2\Big]=0 \tag{4.25}$$

where

$$\Delta x(\theta)^T = \Big[\Delta H_j(\theta)\ \frac{\Delta G_{jn_1}(\theta)}{H_j(\theta)}\ \cdots\ \frac{\Delta G_{jn_{\mathrm{n}}}(\theta)}{H_j(\theta)}\Big],$$
$$w^T_{\{i,\mathcal{N}_j\}} = [w_j\ w_{n_1}\ \cdots\ w_{n_{\mathrm{n}}}],\ n_k \in \mathcal{N}_j.$$

Using Parseval's Theorem results in:

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \Delta x(e^{j\omega},\theta)^T J \Phi_{\{j,\mathcal{N}_j\}}(\omega) J^* \Delta x(e^{-j\omega},\theta)\mathrm{d}\omega = 0$$

for $\omega \in [-\pi,\pi)$, where J^* denotes the conjugate transpose of J. By Condition (c), $\Phi_{\{j,\mathcal{N}_j\}}(\omega)$ is positive definite. Moreover, $J(e^{j\omega})$ is full rank for all ω. Thus the only way the equation can hold is if each entry of $[\Delta H_j\ \Delta G_{jn_1}\ \cdots\ \Delta G_{jn_{\mathrm{n}_1}}]$ is equal to zero for all ω. Therefore, by Condition (d) and if the parameterization of $G_{ji}(\theta)$ is such that the only way that $G^0_{ji} - G_{ji}(\theta)$ is equal to zero is when $G_{ji}(\theta) = G^0_{ji}$, the global minimum of $\bar{V}_j(\theta)$ is unique. □

Remark 4.25 *There exists an alternative reasoning to prove the proposition, by utilizing the equivalent feedback structure as presented in Proposition 4.2, combined with the reasoning in Van den Hof et al. (1992) concerning absence of algebraic loops. However the proof presented above naturally includes the excitation conditions also.*

4.8.3 Proof of Lemma 4.17

Proof: The variables w_z can be eliminated from the equations:

$$\begin{aligned}\begin{bmatrix} w_j \\ w_{\mathcal{N}} \end{bmatrix} &= \begin{bmatrix} 0 & G^0_{j\mathcal{N}} \\ G^0_{\mathcal{N}j} & G^0_{\mathcal{N}\mathcal{N}} \end{bmatrix}\begin{bmatrix} w_j \\ w_{\mathcal{N}} \end{bmatrix} + \begin{bmatrix} 0 \\ G^0_{\mathcal{N}z} \end{bmatrix}(I - G^0_{zz})^{-1}\begin{bmatrix} G^0_{zj} & G^0_{z\mathcal{N}} \end{bmatrix}\begin{bmatrix} w_j \\ w_{\mathcal{N}} \end{bmatrix} \\ &\quad + \begin{bmatrix} 0 \\ G^0_{\mathcal{N}z} \end{bmatrix}(I - G^0_{zz})^{-1} v_z + \begin{bmatrix} v_j \\ v_{\mathcal{N}} \end{bmatrix} \\ &= \begin{bmatrix} 0 & G^0_{j\mathcal{N}} \\ \tilde{G}^0_{\mathcal{N}j} & \tilde{G}^0_{\mathcal{N}\mathcal{N}} \end{bmatrix}\begin{bmatrix} w_j \\ w_{\mathcal{N}} \end{bmatrix} + \begin{bmatrix} I & 0 & 0 \\ 0 & I & \tilde{G}^0_{\mathcal{N}z} \end{bmatrix}\begin{bmatrix} v_j \\ v_{\mathcal{N}} \\ v_z \end{bmatrix},\end{aligned}$$

where the several matrices $\tilde{G}$ are implicitly defined through the equations. The transfer from $w_{\mathcal{N}}$ to w_j is still $G^0_{j\mathcal{N}}$, whereas the transfer from w_j to $w_{\mathcal{N}}$ has become a composite function of various tranfers (denoted $\tilde{G}^0_{\mathcal{N}j}$). Subsequently the map from v to $[w_j\ w_{\mathcal{N}}]^T$ is

$$\begin{bmatrix} w_j \\ w_{\mathcal{N}} \end{bmatrix} = \begin{bmatrix} 1 & -G^0_{j\mathcal{N}} \\ -\tilde{G}^0_{\mathcal{N}j} & I - \tilde{G}^0_{\mathcal{N}\mathcal{N}} \end{bmatrix}^{-1}\begin{bmatrix} I & 0 & 0 \\ 0 & I & \tilde{G}^0_{\mathcal{N}z} \end{bmatrix}\begin{bmatrix} v_j \\ v_{\mathcal{N}} \\ v_z \end{bmatrix}.$$

Consider the stochastic process $v_{\mathcal{N}} + \tilde{G}^0_{\mathcal{N}z} v_z$ which appears as part of $w_{\mathcal{N}}$. Denote the power spectral density of this process as $\tilde{\Phi}_{\mathcal{N}}(\omega)$, and let $\tilde{H}^0_{\mathcal{N}}$ be its monic, stable and

minimum-phase spectral factor. Substituting this into the expression of the data generating system results in

$$\begin{bmatrix} w_j \\ w_{\mathcal{N}} \end{bmatrix} = \begin{bmatrix} \breve{G}^0_{jj} & G^0_{j\mathcal{N}}\breve{G}^0_{\mathcal{N}\mathcal{N}} \\ \breve{G}^0_{\mathcal{N}\mathcal{N}}G^0_{\mathcal{N}j} & \breve{G}^0_{\mathcal{N}\mathcal{N}} \end{bmatrix} \begin{bmatrix} H^0_j & \\ & \tilde{H}^0_{\mathcal{N}} \end{bmatrix} \begin{bmatrix} e_j \\ \tilde{e}_{\mathcal{N}} \end{bmatrix}$$

where e_j and $\tilde{e}_{\mathcal{N}}$ are uncorrelated since v_j and $v_{\mathcal{N}}$ and v_z are uncorrelated. □

4.8.4 Proof of Lemma 4.20

Proof: Since we are assuming that every loop through w_k, $k \in \{j, \mathcal{N}_j\}$ has a delay, it follows by Lemma 4.19 that the diagonal entries of W^0 are monic transfers. This proves the top left entry of (4.11).

Consider a graph of $W^0_{\mathcal{N}\mathcal{N}}(\infty) - I$. By Lemma 4.19, since every loop involving w_k, $k \in \{j, \mathcal{N}_j\}$ has a delay this is an acyclic graph. By Theorem 2.18 it follows that there exists a permutation matrix $P_{\mathcal{N}\mathcal{N}}$ such that $P_{\mathcal{N}\mathcal{N}}W^0_{\mathcal{N}\mathcal{N}}(\infty)P^T_{\mathcal{N}\mathcal{N}}$ is lower triangular. Consequently, the lower right entry of (4.11) holds. Note that $P_{\mathcal{N}\mathcal{N}}$ may not be unique. Let $\mathcal{P}_{\mathcal{N}\mathcal{N}}$ denote the set of permutation matrices such that the lower right entry of (4.11) holds.

Finally it must be shown that for at least one $P_{\mathcal{N}\mathcal{N}} \in \mathcal{P}_{\mathcal{N}\mathcal{N}}$, the off diagonal entries of (4.11) also hold. The reasoning will be split into two steps. First it will be shown that there exists a $P_{\mathcal{N}\mathcal{N}} \in \mathcal{P}_{\mathcal{N}\mathcal{N}}$ such that $P_{\mathcal{N}\mathcal{N}}W^0_{\mathcal{N}j}W^0_{j\mathcal{N}}P^T_{\mathcal{N}\mathcal{N}}$ is strictly lower triangular. Secondly, it will be shown that for this matrix to be strictly lower triangular, the off diagonal entries of (4.11) must hold.

Consider the graph of $W^0_{\mathcal{N}\mathcal{N}}(\infty) + W^0_{\mathcal{N}j}(\infty)W^0_{j\mathcal{N}}(\infty) - I$. This graph is equal to the original graph of $W^0_{\mathcal{N}\mathcal{N}}(\infty) - I$ with some new egdes added. The set of permutation matrices that triangularizes $W^0_{\mathcal{N}\mathcal{N}}(\infty) + W^0_{\mathcal{N}j}(\infty)W^0_{j\mathcal{N}}(\infty) - I$ will be a subset of $\mathcal{P}_{\mathcal{N}\mathcal{N}}$ since edges have only been added to $W^0_{\mathcal{N}\mathcal{N}}(\infty) - I$ and none have been removed. This implies that if $W^0_{\mathcal{N}\mathcal{N}}(\infty) + W^0_{\mathcal{N}j}(\infty)W^0_{j\mathcal{N}}(\infty) - I$ is not triangularizable by any $P_{\mathcal{N}\mathcal{N}} \in \mathcal{P}_{\mathcal{N}\mathcal{N}}$ then there does not exist a permutation matrix such that it is triangularizable.

Denote, $P'_{\mathcal{N}\mathcal{N}} \in \mathcal{P}'_{\mathcal{N}\mathcal{N}} \subseteq \mathcal{P}_{\mathcal{N}\mathcal{N}}$ as the set of permutation matrices that triangularize $W^0_{\mathcal{N}\mathcal{N}}(\infty) + W^0_{\mathcal{N}j}(\infty)W^0_{j\mathcal{N}}(\infty) - I$

By the condition that all loops passing through w_k, $k \in \{j, \mathcal{N}_j\}$ have a delay, the graph $W^0_{\mathcal{N}\mathcal{N}}(\infty) + W^0_{\mathcal{N}j}(\infty)W^0_{j\mathcal{N}}(\infty) - I$ is acyclic. By Proposition 2.18 this implies that there exists a permutation matrix $P'_{\mathcal{N}\mathcal{N}}$ such that

$$P'_{\mathcal{N}\mathcal{N}}\big(W^0_{\mathcal{N}\mathcal{N}}(\infty) + W^0_{\mathcal{N}j}(\infty)W^0_{j\mathcal{N}}(\infty) - I\big)P'^T_{\mathcal{N}\mathcal{N}}$$

is lower triangular. Consequently, $\mathcal{P}'_{\mathcal{N}\mathcal{N}}$ is not empty. Since $\mathcal{P}'_{\mathcal{N}\mathcal{N}} \subseteq \mathcal{P}_{\mathcal{N}\mathcal{N}}$ it follows that there exists a permutation matrix such that $W^0_{\mathcal{N}\mathcal{N}}(\infty)$ and $W^0_{\mathcal{N}j}(\infty)W^0_{j\mathcal{N}}(\infty)$ are both lower triangular.

From Lemma 4.19 it follows that the diagonal entries of $P'_{\mathcal{N}\mathcal{N}}W^0_{\mathcal{N}j}(\infty)W^0_{j\mathcal{N}}(\infty)P'^T_{\mathcal{N}\mathcal{N}}$ are zero and therefore this matrix is strictly lower triangular.

Next it will be shown that the fact that $P_{\mathcal{N}\mathcal{N}}W^0_{\mathcal{N}j}W^0_{j\mathcal{N}}P^T_{\mathcal{N}\mathcal{N}}$ is strictly lower triangular implies that the off diagonal entries of (4.11) hold. Consider two vectors,

$x^T = [x_1^T \; x_2^T]$ and $y^T = [y_1^T \; y_2^T]$. Then,

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \begin{bmatrix} y_1^T & y_2^T \end{bmatrix} = \begin{bmatrix} x_1 y_1^T & x_1 y_2^T \\ x_2 y_1^T & x_2 y_2^T \end{bmatrix}.$$

The only way this this matrix can be strictly lower triangular is if both x_1 and y_2 are zero. Let $x = P_{\mathcal{N}\mathcal{N}} W^0_{\mathcal{N}j}(\infty)$ and $y = W^0_{j\mathcal{N}}(\infty) P^T_{\mathcal{N}\mathcal{N}}$, then by this reasoning, it follows that the off diagonal entries of (4.11) hold.

4.8.5 Proof of Proposition 4.21

Proof: Refer to Lemma 4.17, and let $Q = L_Q D L_Q^T$ be the LDL decompostion of Q. By Lemmas 4.19 and 4.20, and Condition (c) it is possible to assume, without loss of generality, that $W^0(\infty)$ is of the form:

$$W^0_{jj}(\infty) = 1 \tag{4.26}$$

$$W^0_{j\mathcal{N}}(\infty) = [d_{j\mathcal{N}} \; 0] \tag{4.27}$$

$$W^0_{\mathcal{N}j}(\infty) = \begin{bmatrix} 0 \\ d_{\mathcal{N}j} \end{bmatrix} \tag{4.28}$$

$$W^0_{\mathcal{N}\mathcal{N}}(\infty) = L^0_{\mathcal{N}\mathcal{N}} \tag{4.29}$$

In other words, the variables w_k, $k \in \mathcal{N}$ have been arranged such that (4.26) - (4.29) hold. The proof will proceed in the usual fashion:

1. Calculate a lower bound on $\bar{V}(\theta) \geq \text{trace}\{D\}$
2. Show that achieving this lower bound implies that $W(\theta^*) = W^0 L_Q$.

Using this expression of $W(\theta^*)$, consistent estimates of $G^0_{j\mathcal{N}}$ can be obtained.
Step 1. The expression for $\bar{V}(\theta)$ is

$$\begin{aligned}
\bar{V}(\theta) &= \bar{\mathbb{E}}[\varepsilon^T(t,\theta)\varepsilon(t,\theta)] \\
&= \bar{\mathbb{E}}\big[w^T W(\theta)^{-T} W(\theta)^{-1} w\big] \\
&= \text{tr}\big\{\bar{\mathbb{E}}[W(\theta)^{-1} w w^T W(\theta)^{-T}]\big\} \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} \text{tr}\big\{W(\theta)^{-1} W^0 Q (W(\theta)^{-1} W^0)^*\big\} \mathrm{d}\omega \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} \text{tr}\big\{W(\theta)^{-1} W^0 L_Q D L_Q^T (W(\theta)^{-1} W^0)^*\big\} \mathrm{d}\omega
\end{aligned} \tag{4.30}$$

where tr stands for trace and * denotes complex conjugate. By Condition (a) and Lemma 4.17 it follows that L_Q has the form

$$L_Q = \begin{bmatrix} 1 & 0 \\ 0 & L^{\mathcal{O}}_{\mathcal{N}\mathcal{N}} \end{bmatrix}$$

where $L^{\mathcal{O}}_{\mathcal{N}\mathcal{N}}$ is lower triangular with ones on the diagonal.

The expression (4.30) can be expanded as:

$$\bar{V}(\theta)=\frac{1}{2\pi}\int_{-\pi}^{\pi}\mathrm{tr}\big\{(W(\theta)^{-1}W^0L_Q-I)D(W(\theta)^{-1}W^0L_Q-I)^*+(W(\theta)^{-1}W^0L_Q-I)D \\ +D(W(\theta)^{-1}W^0L_Q-I)^*+D\big\}\mathrm{d}\omega \tag{4.31}$$

Two important properties of (4.31) is that the first term is always greater than zero for any θ, and secondly that the last term is not a function of θ. In the following text it will be shown that the second and third terms of (4.31) are zero. Consequently, $\bar{V}(\theta)\geq tr\{D\}$.

Consider the second term of (4.31):

$$(W(\theta)^{-1}W^0L_Q-I)D=\Bigg(\begin{bmatrix}W_{jj}(\theta)-W_{j\mathcal{N}}(\theta)W_{\mathcal{N}\mathcal{N}}^{-1}(\theta)W_{\mathcal{N}j}(\theta) & 0\\ 0 & W_{\mathcal{N}\mathcal{N}}(\theta)-W_{\mathcal{N}j}(\theta)W_{jj}^{-1}(\theta)W_{j\mathcal{N}}(\theta)\end{bmatrix}^{-1} \\ \cdot\begin{bmatrix}W_{jj}^0-W_{j\mathcal{N}}(\theta)W_{\mathcal{N}\mathcal{N}}^{-1}(\theta)W_{\mathcal{N}j}^0 & W_{j\mathcal{N}}^0-W_{j\mathcal{N}}(\theta)W_{\mathcal{N}\mathcal{N}}^{-1}(\theta)W_{\mathcal{N}\mathcal{N}}^0\\ W_{\mathcal{N}j}^0-W_{\mathcal{N}j}(\theta)W_{jj}^{-1}(\theta)W_{jj}^0 & W_{\mathcal{N}\mathcal{N}}^0-W_{\mathcal{N}j}(\theta)W_{jj}^{-1}(\theta)W_{j\mathcal{N}}^0\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & L_{\mathcal{N}\mathcal{N}}^{\mathcal{Q}}\end{bmatrix}-I\Bigg)D$$

By Lemmas 4.19 and 4.20 and Condition (d),

$$\big(W_{jj}(\theta)-W_{j\mathcal{N}}(\theta)W_{\mathcal{N}\mathcal{N}}^{-1}(\theta)W_{\mathcal{N}j}(\theta)\big)^{-1}\cdot\big(W_{jj}^0-W_{j\mathcal{N}}(\theta)W_{\mathcal{N}\mathcal{N}}^{-1}(\theta)W_{\mathcal{N}j}^0\big)-1$$

is a strictly proper transfer function, which means that the first diagonal element of $(W(\theta)^{-1}W^0L_Q-I)D$ is a strictly proper transfer function.

Secondly, Lemmas 4.19 and 4.20 and Condition (d),

$$\big(W_{\mathcal{N}\mathcal{N}}(\theta)-W_{\mathcal{N}j}(\theta)W_{jj}^{-1}(\theta)W_{j\mathcal{N}}(\theta)\big)^{-1}\cdot\big(W_{\mathcal{N}\mathcal{N}}^0-W_{\mathcal{N}j}(\theta)W_{jj}^{-1}(\theta)W_{j\mathcal{N}}^0\big)L_{\mathcal{N}\mathcal{N}}^{\mathcal{Q}}.$$

is a product of three lower triangular matrices with ones on the diagonals. In particular the first term is lower triangular due to Condtion (d), and the fact that the inverse of a lower triangular matrix with ones on the diagonal is a lower triangular matrix with ones on the diagonal. The second term is lower triangular with ones on the diagonal by Lemmas 4.19 and 4.20. The third term is lower triangular with ones on the diagonal by construction. The statement follows, since the product of triangular matrices with ones on the diagonal, is a lower triangular matrix with ones on the diagonal. Consequently, the diagonal terms of $(W(\theta)^{-1}W^0L_Q-I)D$ are all strictly proper transfer functions.

Finally, the integral in (4.30) is zero for strictly proper transfer fucntions. Consequently (4.31) can be simplified,

$$\bar{V}(\theta)=\frac{1}{2\pi}\int_{-\pi}^{\pi}\mathrm{tr}\big\{(W(\theta)^{-1}W^0L_Q-I)D(W(\theta)^{-1}W^0L_Q-I)^*+D\big\}\mathrm{d}\omega\geq\mathrm{tr}\{D\} \tag{4.32}$$

Step 2. From (4.32) $\bar{V}(\theta)=\mathrm{tr}\{D\}$ implies that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\mathrm{tr}\big\{(W(\theta)^{-1}W^0L_Q-I)D(W(\theta)^{-1}W^0L_Q-I)^*\big\}\mathrm{d}\omega=0 \tag{4.33}$$

Since D is positive definite $\forall\omega \in [-\pi, \pi)$ this implies that $(W(\theta)^{-1}W^0L_Q - I) = 0$ $\forall\omega \in [-\pi, \pi)$. Consquently,

$$\begin{aligned} W(\theta) &= W^0 L_Q \quad \text{for all } \omega \in [-\pi, \pi) \\ \begin{bmatrix} W_{jj}(\theta) & W_{j\mathcal{N}}(\theta) \\ W_{\mathcal{N}j}(\theta) & W_{\mathcal{N}\mathcal{N}}(\theta) \end{bmatrix} &= \begin{bmatrix} W^0_{jj} & W^0_{j\mathcal{N}} L^{\mathcal{Q}}_{\mathcal{N}\mathcal{N}} \\ W^0_{\mathcal{N}j} & W^0_{\mathcal{N}\mathcal{N}} L^{\mathcal{Q}}_{\mathcal{N}\mathcal{N}} \end{bmatrix} \end{aligned} \tag{4.34}$$

By Condition (d) and Lemma 4.20 the parameterization is such that there exists a solution to this equation. In particular the parameterization is such that the equality

$$W_{j\mathcal{N}}(\theta) = W^0_{j\mathcal{N}} L^{\mathcal{Q}}_{\mathcal{N}\mathcal{N}}$$

can hold. This completes the proof.

4.8.6 Spectral Factorization Theorem

Theorem 4.26 (Spectral Factorization Theorem Youla (1961)) *Let $\Phi(z)$ be a $n \times n$ real rational full rank spectral density matrix.*

(a) There exists a unique factorization of the form $\Phi(z) = \bar{W}(z)\bar{Q}\bar{W}^*(z)$, in which $\bar{W}(z)$ is $n \times n$, real, rational, stable, minimum phase and such that $\bar{W}(\infty) = I$, with $\bar{Q}$ positive definite.

(b) Any other factorization of the form $\Phi(z) = W(z)QW^*(z)$ in which $W(z)$ is real rational, and Q is positive semi-definite, is such that $W(z) = \bar{W}(z)V(z)$, where $V(z)$ is a real rational scaled paraunitary matrix, i.e. $V(z)QV^*(z) = \bar{Q}$. Moreover $V(z)$ is stable if and only if $W(z)$ is stable.

(c) Any other factorization of the form $\Phi(z) = W(z)QW^*(z)$ in which $W(\infty)$ is finite and nonsingular, $W(z)$ is $n \times n$ real rational, stable, and minimum phase, and Q is positive definite is such that $W(z) = \bar{W}(z)T$, where T is a real nonsingular constant matrix, with $TQT^T = \bar{Q}$.

Chapter 5

PREDICTOR INPUT SELECTION

In Chapter 4 several methods have been proposed to obtain consistent estimates of a module embedded in a dynamic network. However, in that chapter, the variables that are included in the predictor model are not considered as a user choice. In this chapter it is shown that there is considerable freedom as to which variables can be included as inputs to the predictor, while still obtaining consistent estimates of the particular module of interest. This freedom is encoded into sufficient conditions on the set of predictor inputs such that if the conditions are satisfied, consistent estimates of the the module of interest are possible. The conditions can be used to find the smallest number of predictor inputs, or they can be used to determine if it is possible to obtain consistent estimates without using certain variables for instance. Thus, the conditions can be used to design sensor placement schemes. In this chapter the Direct and Two Stage Prediction-Error methods are studied. Algorithms are presented for checking the conditions using tools from graph theory.[1]

5.1 INTRODUCTION

IN CHAPTER 4 it has been assumed that (noise free) measurements of all internal variables in the network are available. However, although it may be possible to take measurements at many different locations in the network, it may be expensive or inconvenient to do so. Thus, it may be attractive to use the minimum number of required variables or avoid using variables that are difficult (unsafe or practically unfeasible) to measure in order to obtain estimates of the module of interest.

The variables that are measured are available to use as *predictor inputs*, i.e. the predictor inputs are the variables that will be used to predict the value of a particular internal variable. Specifically, the question addressed in this chapter is: given a dynamic network with known interconnection structure, for which selection of predictor inputs can we guarantee that a particular module of interest can be estimated consistently? Conditions are presented for the set of predictor inputs

[1] This chapter is based on the papers Dankers et al. (2014d, 2013a,b).

that ensure this property. The conditions are derived for the Direct and Two-Stage Prediction-Error Methods.

If only a subset of variables in a network are measured, then measurement data can only reveal the dynamical links between the measured variables. In this chapter the related identifiability problem will be addressed by formalizing the notion of an *immersed network*. This is the network that (exactly) describes the dynamic relationships between the (subset) of measured/available variables. Under particular conditions on the set of available variables the relevant module of interest in the immersed network and the original network are the same.

The conditions point to a particular notion of *identifiability* of modules embedded in the network based on the availability of measurements. The conditions presented in this chapter can be used to answer the following questions. Given a set of variables from a network, which module transfer functions are identifiable? Or, conversely, which variables should be measured in order to ensure that a particular module in the network is identifiable?

In Section 5.2 a generalization of the Direct and Two-Stage methods is presented, along with some notation that will be used throughout the remainder of the chapter. In Section 5.3 the notion of the immersed network is presented and the relationship between the dynamics of the immersed network and the original network is investigated, and in Sections 5.4 and 5.5 the conditions that the predictor inputs must satisfy are derived for each method. In Section 5.6 an algorithm based on graph theory is presented to check the required conditions.

5.2 PRELIMINARIES AND NOTATION

The data generating system is written as

$$\begin{aligned}\begin{bmatrix} w_1(t) \\ w_2(t) \\ \vdots \\ w_L(t) \end{bmatrix} &= \begin{bmatrix} 0 & G_{12}^0(q) & \cdots & G_{1L}^0(q) \\ G_{21}^0(q) & 0 & \ddots & G_{2L}^0(q) \\ \vdots & \ddots & \ddots & \vdots \\ G_{L1}^0(q) & G_{L2}^0(q) & \cdots & 0 \end{bmatrix} \begin{bmatrix} w_1(t) \\ w_2(t) \\ \vdots \\ w_L(t) \end{bmatrix} + \begin{bmatrix} r_1(t) \\ r_2(t) \\ \vdots \\ r_L(t) \end{bmatrix} + \begin{bmatrix} v_1(t) \\ v_2(t) \\ \vdots \\ v_L(t) \end{bmatrix} \\ &= G^0(q)w(t) + r(t) + v(t). \end{aligned} \tag{5.1}$$

The objective considered in this chapter is to estimate a particular transfer function embedded in the network. This transfer function is denoted G_{ji}^0.

In this chapter the set of internal variables chosen as predictor inputs plays an important role. For this reason, it is convenient to partition (5.1) accordingly. Let $\mathcal{D}_j$ denote the set of indices of the internal variables that are chosen as predictor inputs used for the prediction of w_j, i.e. the internal variable w_k is a predictor input if and only if $k \in \mathcal{D}_j$. Let $\mathcal{Z}_j$ denote the set of indices not in $\{j\} \cup \mathcal{D}_j$, i.e. $\mathcal{Z}_j = \{1, \ldots, L\} \setminus \{\{j\} \cup \mathcal{D}_j\}$. Let $w_{\mathcal{D}}$ denote the vector $[w_{k_1} \ \cdots \ w_{k_n}]^T$, where $\{k_1, \ldots, k_n\} = \mathcal{D}_j$. Let $r_{\mathcal{D}}$ denote the vector $[r_{k_1} \ \cdots \ r_{k_n}]^T$, where $\{k_1, \ldots, k_n\} = \mathcal{D}_j$, and where the ℓth entry is zero if r_ℓ is not present in the network (i.e. $\ell \notin \mathcal{R}$). The vectors $w_{\mathcal{Z}}$, $v_{\mathcal{D}}$, $v_{\mathcal{Z}}$ and $r_{\mathcal{Z}}$ are defined analogously. The ordering of the elements of $w_{\mathcal{D}}$, $v_{\mathcal{D}}$, and $r_{\mathcal{D}}$ is not important, as long as it is the same for all these vectors (the

same holds for $w_{\mathcal{Z}}$, $v_{\mathcal{Z}}$, and $r_{\mathcal{Z}}$). The transfer function matrix between $w_{\mathcal{D}}$ and w_j is denoted $G^0_{j\mathcal{D}}$. The other transfer function matrices are defined analogously. Using this notation, the network equations (5.1) are rewritten as:

$$\begin{bmatrix} w_j \\ w_{\mathcal{D}} \\ w_{\mathcal{Z}} \end{bmatrix} = \begin{bmatrix} 0 & G^0_{j\mathcal{D}} & G^0_{j\mathcal{Z}} \\ G^0_{\mathcal{D}j} & G^0_{\mathcal{D}\mathcal{D}} & G^0_{\mathcal{D}\mathcal{Z}} \\ G^0_{\mathcal{Z}j} & G^0_{\mathcal{Z}\mathcal{D}} & G^0_{\mathcal{Z}\mathcal{Z}} \end{bmatrix} \begin{bmatrix} w_j \\ w_{\mathcal{D}} \\ w_{\mathcal{Z}} \end{bmatrix} + \begin{bmatrix} v_j \\ v_{\mathcal{D}} \\ v_{\mathcal{Z}} \end{bmatrix} + \begin{bmatrix} r_j \\ r_{\mathcal{D}} \\ r_{\mathcal{Z}} \end{bmatrix}, \tag{5.2}$$

where $G^0_{\mathcal{D}\mathcal{D}}$ and $G^0_{\mathcal{Z}\mathcal{Z}}$ have zeros on the diagonal. Note that the partition depends on j and $\mathcal{D}_j$.

5.2.1 Prediction Error Identification

In this section we briefly present the predictor model. The presentation is slightly more general than in the previous chapter. Let w_j denote the variable which is to be predicted, i.e. it is the output of the module of interest. The *predictor inputs* are those (known) variables that will be used to predict w_j. The sets $\mathcal{D}_j$ and $\mathcal{P}_j$ are used to denote the sets of indices of the internal and external variables respectively that are chosen as predictor inputs - w_k is a predictor input iff $k \in \mathcal{D}_j$, and r_k is a predictor input iff $k \in \mathcal{P}_j$. The one-step-ahead predictor for w_j is then (Ljung, 1999):

$$\hat{w}_j(t|t-1,\theta) = H_j^{-1}(q,\theta)\Big(\sum_{k\in\mathcal{D}_j} G_{jk}(q,\theta)w_k(t) + \sum_{k\in\mathcal{P}_j} F_{jk}(q,\theta)r_k(t)\Big) + \Big(1 - H_j^{-1}(q,\theta)\Big)w_j(t) \tag{5.3}$$

where $H_j(q,\theta)$ is the (monic) noise model, $G_{jk}(q,\theta)$ models the dynamics between w_k to w_j, $k \in \mathcal{D}_j$, and $F_{jk}(q,\theta)$ models the dynamics between r_k to w_j, $k \in \mathcal{P}_j$. Although a parameterization including $F_{jk}(q,\theta)$ may seem to add unnecessary complexity to the predictor, the importance will become apparent later in the chapter. The prediction error is:

$$\varepsilon_j(t,\theta) = H_j(q,\theta)^{-1}\Big(w_j(t) - \sum_{k\in\mathcal{D}_j} G_{jk}(q,\theta)w_k(t) - \sum_{k\in\mathcal{P}_j} F_{jk}(q,\theta)r_k(t)\Big). \tag{5.4}$$

The parameterized transfer functions $G_{jk}(\theta)$, $k \in \mathcal{D}_j$, $F_{jk}(\theta)$, $k \in \mathcal{P}_j$, and $H_j(\theta)$ are estimated by minimizing the sum of squared (prediction) errors:

$$V_j(\theta) = \frac{1}{N}\sum_{t=0}^{N-1} \varepsilon_j^2(t,\theta), \tag{5.5}$$

where N is the length of the data set.

In the following two sections we present slightly generalized versions of the Direct and Two-Stage Methods presented in Chapter 4.

5.2.2 The Direct Method

The Direct Method in this chapter is defined by the following algorithm. It is a generalization of the Direct Method as presented in Section 4.4 in Chapter 4 because here the choice of predictor inputs is up the the user.

Algorithm 5.1 *Direct Method. Objective: identify* G_{ji}^0.

1. *Select* w_j *as the variable to be predicted (the output).*
2. *Choose the internal and external variables to include as inputs to the predictor (5.3) (i.e. choose the sets* $\mathcal{D}_j$ *and* $\mathcal{P}_j$*).*
3. *Construct the predictor (5.3).*
4. *Obtain estimates* $G_{jk}(q,\hat{\theta}_N)$*, for all* $k \in \mathcal{D}_j$*,* $F_{jk}(q,\hat{\theta}_N)$*, for all* $k \in \mathcal{P}_j$ *and* $H_j(q,\hat{\theta}_N)$ *by minimizing the sum of squared prediction errors (5.5).*

In Chapter 4 Step 2 of the algorithm is replaced by a fixed choice, namely, $\mathcal{D}_j = \mathcal{N}_j$, and $\mathcal{P}_j = \emptyset$.

5.2.3 Two Stage Method

In the Two Stage Method, the predictor inputs are not internal variables, but *projections* of internal variables. As presented in Section 3.4.2 of Chapter 3 the term $w_k^{(r_m)}$ is the projection of w_k onto causally time shifted versions of r_m (referred to as simply the projection of w_k onto r_m). Recall also Algorithm 4.13 of Chapter 4 where the predictor inputs were projected onto a set of external variables $\{r_m\}$, $m \in \mathcal{T}_j$. The following algorithm is a generalization of the one presented in Chapter 4 (Algorithm 4.13) as the choice of predictor inputs is not fixed anymore.

Algorithm 5.2 *Two Stage Method. Objective: identify* G_{ji}^0.

1. *Select* w_j *as the variable to be predicted (the output).*
2. *Choose the external variables to project onto (choose* $\mathcal{T}_j$*).*
3. *Choose the internal and external variables to include as predictor inputs (i.e. choose* $\mathcal{D}_j$ *and* $\mathcal{P}_j$*).*
4. *Obtain estimates* $\hat{w}_k^{(\mathcal{T}_j)}$ *of* $w_k^{(\mathcal{T}_j)}$ *for each* $k \in \mathcal{D}_j$*.*
5. *Construct the predictor*

$$\hat{w}_j(t|t-1,\theta) = \sum_{k\in\mathcal{D}_j} G_{jk}(\theta)\hat{w}_k^{(\mathcal{T}_j)} + \sum_{k\in\mathcal{P}_j} F_{jk}(\theta)r_k. \tag{5.6}$$

6. *Obtain estimates* $G_{jk}(q,\hat{\theta}_N)$ *for all* $k \in \mathcal{D}_j$ *and* $F_{jk}(q,\hat{\theta}_N)$ *for all* $k \in \mathcal{P}_j$ *by minimizing the sum of squared prediction errors (5.5).*

For simplicity in (5.6) we do not include a noise model. However, a noise model can be included with no change in the results, as long as it is parameterized independently of the modules $G_{jk}(q,\theta)$, $k \in \mathcal{D}_j$.

5.3 CONSISTENT IDENTIFICATION ON THE BASIS OF A SUBSET OF INTERNAL VARIABLES

When only a subset of all internal variables in a network is available from measurements, a relevant question becomes: what are the dynamical relationships between the nodes in this subset of measured variables? In Section 5.3.1 it is shown that when only a selected subset of internal variables is considered, the dynamic relationships between these variables can be described by an immersed network. Several properties of the immersed network are investigated. Next, in Section 5.3.2 it is shown under which conditions the dynamics that appear between two internal variables remain invariant when reducing the original network to the immersed one. In Section 5.3.3 the results of identification in networks are characterized. It is shown that it is the dynamics of the modules in the immersed network that are being identified, and conditions for consistency of general identification results are formulated. The results presented in this section are independent of the particular identification method.

5.3.1 The Immersed Network

In this subsection, we show that there exists a unique dynamic network consisting only of a given subset of internal variables, that can be constructed by applying an algorithm from graph theory for constructing an immersed graph. Given the selected variables w_k, $k \in \{j\} \cup \mathcal{D}_j$, the remaining variables w_n, $n \in \mathcal{Z}_j$ are sequentially removed from the network.

The following proposition shows that there is a unique characterization of the dynamics between the selected variables.

Proposition 5.3 *Consider a dynamic network as defined in (5.1) that satisfies Assumption 2.19. Consider the set of internal variables $\{w_k\}$, $k \in \mathcal{D}_j \cup \{j\}$. There exists a network:*

$$\begin{bmatrix} w_j(t) \\ w_{\mathcal{D}}(t) \end{bmatrix} = \breve{G}^0(q,\mathcal{D}_j) \begin{bmatrix} w_j(t) \\ w_{\mathcal{D}(t)} \end{bmatrix} + \breve{F}^0(q,\mathcal{D}_j) \begin{bmatrix} r_j(t)+v_j(t) \\ r_{\mathcal{D}}(t)+v_{\mathcal{D}}(t) \\ r_{\mathcal{Z}}(t)+v_{\mathcal{Z}}(t) \end{bmatrix}, \tag{5.7}$$

where $\breve{G}^0$ and $\breve{F}^0$ are unique transfer matrices of the form (using a notation analogous to that of (5.2)):

$$\breve{G}^0 = \begin{bmatrix} 0 & \breve{G}^0_{j\mathcal{D}} \\ \breve{G}^0_{\mathcal{D}j} & \breve{G}^0_{\mathcal{D}\mathcal{D}} \end{bmatrix} \quad \text{and} \quad \breve{F}^0 = \begin{bmatrix} \breve{F}^0_{jj} & 0 & \breve{F}^0_{j\mathcal{Z}} \\ 0 & \breve{F}^0_{\mathcal{D}\mathcal{D}} & \breve{F}^0_{\mathcal{D}\mathcal{Z}} \end{bmatrix}, \tag{5.8}$$

where $\breve{G}^0_{\mathcal{D}\mathcal{D}}$ has zeros on the diagonal, $\breve{F}^0_{\mathcal{D}\mathcal{D}}$ is diagonal, and if there is an index ℓ such that both v_ℓ and r_ℓ are not present, then the corresponding column of $\breve{F}^0$ is set to all zeros. □

See Appendix 5.9.1 for the proof. Proposition 5.3 is based on a result of Gonçalves & Warnick (2008) where it is proven that a unique interconnection matrix $\breve{G}^0$ exists

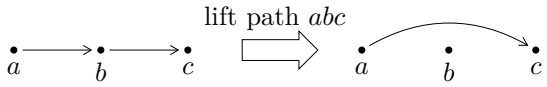

Figure 5.1: Simple illustration of lifting a path. In the graph on the left, the path through node b is lifted, resulting in the graph on the right.

if and only if enough entries of $\breve{G}^0$ and $\breve{F}^0$ are known (set to zero in our case). Enforcing $\breve{G}^0$ to have zeros on the diagonal results in a network that does not have any "self-loops", i.e. no paths that enter and leave the same node. This matches the assumptions imposed on the data generating system (5.1). Enforcing the leading square matrix of $\breve{F}^0$ to be diagonal results in a network where each r_k, $k \in \mathcal{D}_j \cup \{j\}$ only has a path to the corresponding internal variable w_k (again, this matches the interconnection structure of (5.1)). The external variables corresponding to internal variables w_n, $n \in \mathcal{Z}_j$ that were removed from the original network can have direct paths to one or more of the internal variables in the network defined by (5.7). This is encoded in (5.7) by allowing $\breve{F}^0_{j\mathcal{Z}}$ and $\breve{F}^0_{\mathcal{D}\mathcal{Z}}$ to be matrices with no pre-defined zero entries.

Denote the noise in (5.7) as:

$$\begin{bmatrix} \breve{v}_j \\ \breve{v}_{\mathcal{D}} \end{bmatrix} = \begin{bmatrix} \breve{F}^0_{jj} & 0 \\ 0 & \breve{F}^0_{\mathcal{D}\mathcal{D}} \end{bmatrix} \begin{bmatrix} v_j \\ v_{\mathcal{D}} \end{bmatrix} + \begin{bmatrix} \breve{F}^0_{j\mathcal{Z}} \\ \breve{F}^0_{\mathcal{D}\mathcal{Z}} \end{bmatrix} v_{\mathcal{Z}}. \tag{5.9}$$

Then by the Spectral Factorization Theorem (Youla, 1961), there exists a unique, monic, stable, minimum phase spectral factor $\breve{H}^0$:

$$\begin{bmatrix} \breve{v}_j \\ \breve{v}_{\mathcal{D}} \end{bmatrix} = \begin{bmatrix} \breve{H}^0_{jj} & \breve{H}^0_{j\mathcal{D}} \\ \breve{H}^0_{\mathcal{D}j} & \breve{H}^0_{\mathcal{D}\mathcal{D}} \end{bmatrix} \begin{bmatrix} \breve{e}_j \\ \breve{e}_{\mathcal{D}} \end{bmatrix}. \tag{5.10}$$

where $[\breve{e}_j \ \breve{e}^T_{\mathcal{D}}]^T$ is a white noise process.

In the following text it is shown that a network of the form (5.7) can be constructed using ideas from graph theory.

In graph theory, one way to remove nodes from a graph is by constructing an *immersed* graph. A graph $\mathbb{G}'$ is an immersion of $\mathbb{G}$ if $\mathbb{G}'$ can be constructed from $\mathbb{G}$ by lifting pairs of adjacent edges and then deleting isolated nodes (Langston & Plaut, 1998). *Lifting* an edge is defined as follows. Given three adjacent nodes a, b, c, connected by edges ab and bc, the lifting of path abc is defined as removing edges ab and bc and replacing them with the edge ac. This is shown graphically in Figure 5.1. If lifting an edge results in a self-loop, the self-loop is removed from the graph.

Constructing an immersed graph is illustrated in the following example.

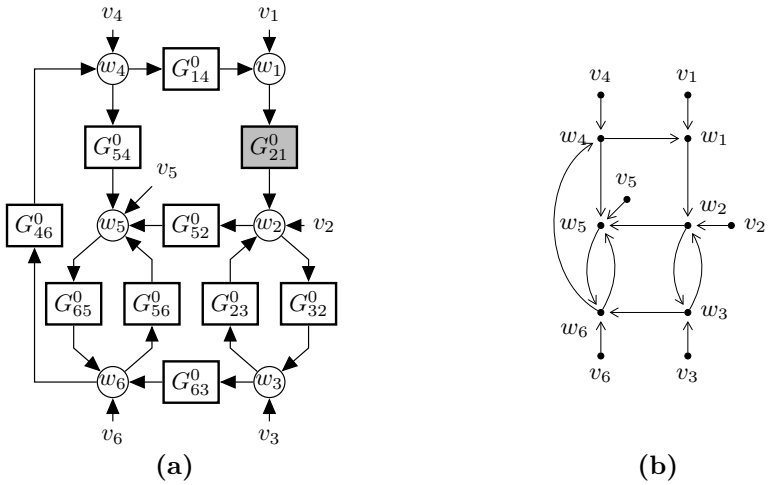

Figure 5.2: A diagram (a) and graph (b) of the network for Examples 5.4 and 5.7.

Example 5.4 *Consider a network defined by:*

$$\begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \\ w_5 \\ w_6 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & G^0_{14} & 0 & 0 \\ G^0_{21} & 0 & G^0_{23} & 0 & 0 & 0 \\ 0 & G^0_{32} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & G^0_{46} \\ 0 & G^0_{52} & 0 & G^0_{54} & 0 & G^0_{56} \\ 0 & 0 & G^0_{63} & 0 & G^0_{65} & 0 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \\ w_5 \\ w_6 \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{bmatrix}$$

shown in Fig. 5.2a. Its graph is shown in Fig. 5.2b. □

Consider the graph shown on the top left hand side of Figure 5.3 (it is the same graph shown in Figure 5.2). Suppose that the objective is to construct an immersed graph with $\{w_1, w_2, w_5, w_6\}$ as internal variables. In other words, w_3 and w_6 must be removed from the graph by lifting edges and removing isolated nodes.

First, w_3 will be removed from the graph. All paths through w_3 must be lifted. There are four distinct paths through w_3:

$$\begin{aligned} w_2 &\to w_3 \to w_6, \\ w_2 &\to w_3 \to w_2, \\ v_3 &\to w_3 \to w_6, \\ v_3 &\to w_3 \to w_2. \end{aligned}$$

Lifting each of these four paths results in the second graph of Figure 5.3. Lifting the path $w_2 \to w_3 \to w_2$ results in a self-loop around w_2, which by definition of lifting is removed from the graph.

Next, w_6 is removed from the second graph of Figure 5.3. The paths through w_6

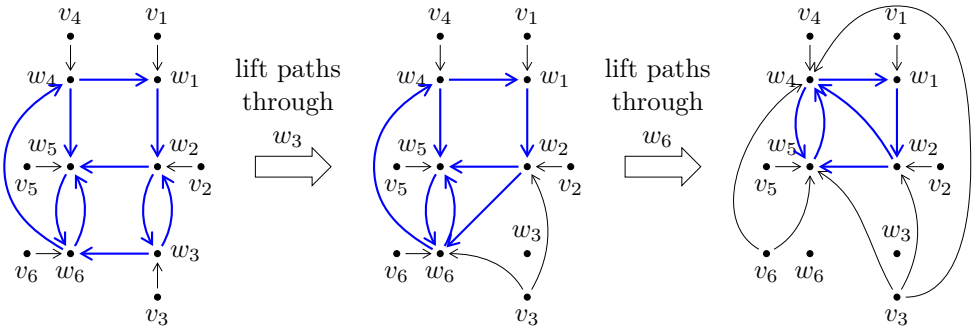

Figure 5.3: Example of constructing an immersion graph. The last step of removing the nodes w_3 and w_6 in the immersion graph is not shown. Edges between w's have been emphasized in thick blue lines since these connections define the interconnection structure of the corresponding dynamic network.

are:

$$\begin{aligned} w_2 &\to w_6 \to w_5, & w_2 &\to w_6 \to w_4,\\ w_5 &\to w_6 \to w_5, & w_5 &\to w_6 \to w_4,\\ v_6 &\to w_6 \to w_5, & v_6 &\to w_6 \to w_4,\\ v_3 &\to w_6 \to w_5, & v_3 &\to w_6 \to w_4. \end{aligned}$$

Lifting each of these paths results in the third graph of Figure 5.3.

Both w_3 and w_6 are isolated nodes in the final graph of Figure 5.3 and can thus be removed from the graph. This step is not shown graphically.

An interesting feature of the immersed graph in this example is that it has edges from nodes $w_2 \to w_4$ and $w_5 \to w_4$ whereas the original graph did not have these edges.

In this way an *immersed network* can be constructed by an algorithm that manipulates the dynamics of the network iteratively. To keep track of the changes in the transfer functions iteratively, let $G_{mn}^{(i)}$ and $F_{mn}^{(i)}$ denote the transfer functions of the direct connections w_n to w_m and from r_n and v_n to w_m, respectively, at iteration i of the algorithm.

Algorithm 5.5 *Constructing an immersed network.*

1. ***Initialize**. Start with the original network:*
   - $G_{mn}^{(0)} = G_{mn}^{0}$ *for all* $m, n \in \{1, \ldots, L\}$, *and*
   - $F_{kk}^{(0)} = 1$, *for all* $k \in \mathcal{R} \cup \mathcal{V}$, $F_{mn}^{(0)} = 0$ *otherwise.*
2. *Remove each w_k, $k \in \mathcal{Z}_j$ from the network, one at a time. Let $d = card(\mathcal{Z}_j)$. Let $\mathcal{Z}_j = \{k_1, \ldots, k_d\}$.*
   ***for** $i = 1 : d$*

(a) *Let $\mathcal{I}_{k_i}$ denote the set of internal variables with edges to w_{k_i}. Let $\mathcal{O}_{k_i}$ denote the set of nodes with edges from w_{k_i}. Lift all paths $w_n \to w_{k_i} \to w_m$, $n \in \mathcal{I}_{k_i}$, $m \in \mathcal{O}_{k_i}$. The transfer function of each new edge from $w_n \to w_m$ is $G^{(i)}_{mn} = G^{(i-1)}_{mk_i} G^{(i-1)}_{k_i n}$.*

(b) *Let $\mathcal{I}^r_{k_i}$ denote the set of external or disturbance variables with edges to w_{k_i}. Lift all paths $r_n \to w_{k_i} \to w_m$, $n \in \mathcal{I}^r_{k_i}$, $m \in \mathcal{O}_{k_i}$. The transfer function for each new edge from $r_n \to w_m$ is $F^{(i)}_{nm} = F^{(i-1)}_{nk_i} G^{(i-1)}_{k_i n}$.*

(c) *If there are multiple edges between two nodes, merge the edges into one edge. The transfer function of the merged edge is equal to the sum of the transfer functions of the edges that are merged.*

(d) *remove the node w_{k_i} from the network.*

end

3. *Remove all self-loops from the network. If node w_m has a self loop, then divide all the edges entering w_m by $(1 - G^{(d)}_{mm}(q))$ (i.e. one minus the loop transfer function).* □

Let $\breve{G}^{i^0}$ and $\breve{F}^{i^0}$ denote the final transfer matrices of the immersed network.

Remark 5.6 *Algorithm 5.5 has a close connection to Mason's Rules (Mason, 1953, 1956). However, Mason was mainly concerned with the calculation of the transfer function from the* sources *(external and noise variables) to a* sink *(internal variable). This is equivalent to obtaining the immersed network with $\mathcal{D}_j = \emptyset$, i.e. all internal variables except one are removed. Importantly, Algorithm 5.5 is an iterative algorithm which allows for easy implementation (even for large networks), whereas Mason's rules are not iterative and complicated even for small networks.*

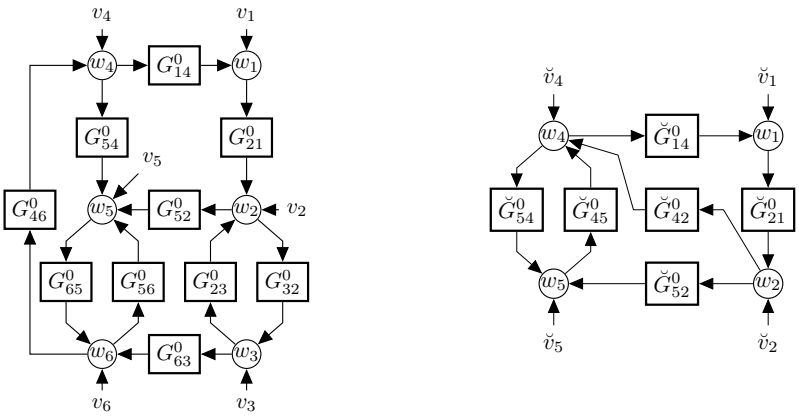

Figure 5.4: **(a)** Original dynamic network considered in Example 5.7. **(b)** Immersed network with w_3 and w_6 removed.

Example 5.7 *Consider the dynamic network shown in Figure 5.4a. The graph of this network is shown in the first graph of Fig. 5.3. Suppose w_3 and w_6 are to be removed from the network (i.e. $\mathcal{Z}_j = \{3, 6\}$). By Algorithm 5.5 the network shown in Figure 5.4b results. The transfer functions of the immersed network are:*

$$\breve{G}^{i^0}(q,\mathcal{D}_j)=\begin{bmatrix} 0 & 0 & G^0_{14} & 0 \\ \frac{G^0_{21}}{1-G^0_{23}G^0_{32}} & 0 & 0 & 0 \\ 0 & G^0_{32}G^0_{46}G^0_{63} & 0 & G^0_{46}G^0_{65} \\ 0 & \frac{G^0_{52}+G^0_{56}G^0_{63}G^0_{32}}{1-G^0_{56}G^0_{65}} & \frac{G^0_{54}}{1-G^0_{56}G^0_{65}} & 0 \end{bmatrix}$$

$$\breve{F}^{i^0}(q,\mathcal{D}_j)=\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{1-G^0_{23}G^0_{32}} & 0 & 0 & \frac{G^0_{23}}{1-G^0_{23}G^0_{32}} & 0 \\ 0 & 0 & 1 & 0 & G^0_{46}G^0_{63} & G^0_{46} \\ 0 & 0 & 0 & \frac{1}{1-G^0_{56}G^0_{65}} & \frac{G^0_{56}G^0_{63}}{1-G^0_{56}G^0_{65}} & \frac{G^0_{56}}{1-G^0_{56}G^0_{65}} \end{bmatrix}.$$

Note that the immersed network (shown in Figure 5.4b) is represented by the last graph shown in Figure 5.3. □

Interestingly, the matrix $\breve{F}^{i^0}$ in Example 5.7 has the same structure as that of $\breve{F}^0$ in Proposition 5.3. This alludes to a connection between the network characterized in (5.7) and immersed networks as defined by Algorithm 5.5.

Proposition 5.8 *The matrices $\breve{G}^0$ and $\breve{F}^0$ of the network characterized by (5.7) and the matrices $\breve{G}^{i^0}$ and $\breve{F}^{i^0}$ defined by Algorithm 5.5 are the same.* □

The proof is in Appendix 5.9.2. Since, by Proposition 5.8 the matrices in (5.7) are the same as those of the immersed network, the superscript i will be dropped from this point on in the matrices defined by Algorithm 5.5. An important consequence of Proposition 5.8 is that (by Proposition 5.3) the immersed network is unique.

Instead of calculating the matrices of the immersed network iteratively, it is also possible to derive analytic expressions for the matrices $\breve{G}^0$ and $\breve{F}^0$.

Proposition 5.9 *Consider a dynamic network as defined in (5.1) that satisfies Assumption 2.19. For a given set $\{j\} \cup \mathcal{D}_j$ the transfer function matrices $\breve{G}^0$ and $\breve{F}^0$ of the immersed network are:*[2]

$$\begin{bmatrix} 0 & \breve{G}^0_{j\mathcal{D}} \\ \breve{G}^0_{\mathcal{D}j} & \breve{G}^0_{\mathcal{D}\mathcal{D}} \end{bmatrix}=\begin{bmatrix} 1-\tilde{G}_{jj} & \\ & I-\mathrm{diag}(\tilde{G}^0_{\mathcal{D}\mathcal{D}}) \end{bmatrix}^{-1}\begin{bmatrix} 0 & \tilde{G}^0_{j\mathcal{D}} \\ \tilde{G}^0_{\mathcal{D}j} & \tilde{G}^0_{\mathcal{D}\mathcal{D}}-\mathrm{diag}(\tilde{G}^0_{\mathcal{D}\mathcal{D}}) \end{bmatrix}$$

$$\begin{bmatrix} \breve{F}^0_{jj} & 0 & \breve{F}^0_{j\mathcal{Z}} \\ 0 & \breve{F}^0_{\mathcal{D}\mathcal{D}} & \breve{F}^0_{\mathcal{D}\mathcal{Z}} \end{bmatrix}=\begin{bmatrix} 1-\tilde{G}_{jj} & \\ & I-\mathrm{diag}(\tilde{G}^0_{\mathcal{D}\mathcal{D}}) \end{bmatrix}^{-1}\begin{bmatrix} 1 & 0 & \tilde{F}^0_{j\mathcal{Z}} \\ 0 & I & \tilde{F}^0_{\mathcal{D}\mathcal{Z}} \end{bmatrix}$$

where

$$\begin{bmatrix} \tilde{G}_{jj} & \tilde{G}_{j\mathcal{D}} \\ \tilde{G}_{\mathcal{D}j} & \tilde{G}_{\mathcal{D}\mathcal{D}} \end{bmatrix} = \begin{bmatrix} 0 & G^0_{j\mathcal{D}} \\ G^0_{\mathcal{D}j} & G^0_{\mathcal{D}\mathcal{D}} \end{bmatrix}+\begin{bmatrix} G^0_{j\mathcal{Z}} \\ G^0_{\mathcal{D}\mathcal{Z}} \end{bmatrix}(I-G^0_{\mathcal{Z}\mathcal{Z}})^{-1}\begin{bmatrix} G^0_{\mathcal{Z}j} & G^0_{\mathcal{Z}\mathcal{D}} \end{bmatrix},$$

$$\begin{bmatrix} \tilde{F}_{j\mathcal{Z}} \\ \tilde{F}_{\mathcal{D}\mathcal{Z}} \end{bmatrix} = \begin{bmatrix} G^0_{j\mathcal{Z}} \\ G^0_{\mathcal{D}\mathcal{Z}} \end{bmatrix}(I-G^0_{\mathcal{Z}\mathcal{Z}})^{-1}.$$ □

[2] The arguments q or $\mathcal{D}_j$ (or both) of $\breve{G}^0_{jk}(q,\mathcal{D}_j)$ and $\breve{F}^0_{jk}(q,\mathcal{D}_j)$ are sometimes dropped for notational clarity.

The proof is in Appendix 5.9.3. The transfer functions $\breve{G}_{mn}$ correspond to $G^{(d)}_{mn}$ in Step 3 of Algorithm 5.5.

The immersed network inherits some useful properties from the original network.

Lemma 5.10 *Consider a dynamic network as defined in (5.1) that satisfies Assumption 2.19 and a given a set* $\{j\} \cup \mathcal{D}_j$.

1. *Consider the paths from* w_n *to* w_m, $n, m \in \mathcal{D}_j$ *that pass only through nodes* w_ℓ, $\ell \in \mathcal{Z}_j$ *in the original network. If all these paths and* $G^0_{mn}(q)$ *have a delay (are zero), then* $\breve{G}^0_{mn}(q, \mathcal{D}_j)$ *has a delay (is zero).*

2. *Consider the paths from* r_n *to* w_m *(or* v_n *to* w_m*),* $n \in \mathcal{Z}_j$, $m \in \mathcal{D}_j$. *If all these paths pass through at least one node* w_ℓ, $\ell \in \mathcal{D}_j$ *then* $\breve{F}^0_{mn}(q, \mathcal{D}_j) = 0$.

For a proof see Appendix 5.9.4.

Example 5.11 *Consider the network of Example 5.7 shown in Figure 5.4. Again, consider an immersed network that contains the internal variables* $\{w_1, w_2, w_4, w_5\}$, *i.e.* w_3 *and* w_6 *are removed from the original network. Using the set notation, suppose* $\{j\} = \{2\}$, *then* $\mathcal{D}_2 = \{1, 4, 5\}$ *and* $\mathcal{Z}_2 = \{3, 6\}$.

Suppose we want to know if there will be a direct path from w_2 *to* w_4 *in the immersed network. By Part 1 of Lemma 5.10 we need to check whether there are any paths from* w_2 *to* w_4 *that pass only through nodes* w_ℓ, $\ell in \mathcal{Z}_2$ *(i.e. the paths only pass through* w_3 *and* w_6*). From Figure 5.4a we see that there is such a path:*

$$w_2 \to w_3 \to w_6 \to w_4.$$

Consequently, by Lemma 5.10 there will be a direct path from $w_2 \to w_4$ *in the immersed network. By inspecting Figure 5.4b we see that indeed this is the case.*

5.3.2 Conditions to Ensure $\breve{G}^0_{ji}(q, \mathcal{D}_j) = G^0_{ji}(q)$

A central theme in the previous section was that the transfer function $\breve{G}^0_{ji}(\mathcal{D}_j)$ in the immersed network may not be the same as the transfer function G^0_{ji} in the original network. In other words, by selecting a subset of internal variables to be taken into account, the dynamics between two internal variables might change. In this section conditions are presented under which the module of interest, G^0_{ji}, remains unchanged in the immersed network, i.e. $\breve{G}^0_{ji}(q, \mathcal{D}_j) = G^0_{ji}(q)$.

The following two examples illustrate two different phenomena related to the interconnection structure that can cause the dynamics $\breve{G}^0_{ji}(q, \mathcal{D}_j)$ to be different from $G^0_{ji}(q)$.

Example 5.12 *Consider the dynamic network*

$$\begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \\ w_5 \end{bmatrix} = \begin{bmatrix} 0 & G^0_{12} & 0 & 0 & G^0_{15} \\ G^0_{21} & 0 & G^0_{23} & 0 & G^0_{25} \\ G^0_{31} & 0 & 0 & 0 & 0 \\ G^0_{41} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & G^0_{54} & 0 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \\ w_5 \end{bmatrix} + \begin{bmatrix} v_1 + r_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 + r_5 \end{bmatrix}.$$

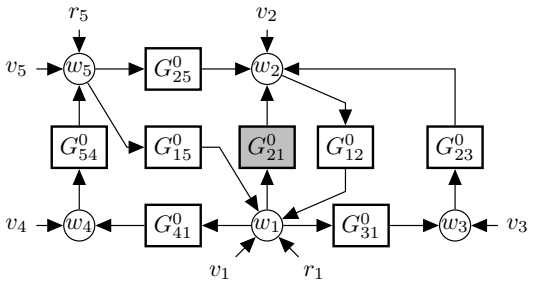

Figure 5.5: Network analyzed in Examples 5.12 and 5.29.

shown in Fig. 5.5. The objective of this example, is to choose $\mathcal{D}_2$ *such that in the immersed network* $\breve{G}^0_{21}(\mathcal{D}_2) = G^0_{21}$ *(denoted in gray). A key feature of the interconnection structure in this example is that there are multiple paths from* w_1 *to* w_2*:* $w_1 \to w_2$*,* $w_1 \to w_3 \to w_2$*,* $w_1 \to w_4 \to w_5 \to w_2$*, etc..*

Start by choosing $\mathcal{D}_2 = \{1\}$*, then by Proposition 5.9,*

$$\breve{G}^0_{21}(q,\{1\}) = G^0_{21}(q) + G^0_{23}(q)G^0_{31}(q) + G^0_{25}(q)G^0_{54}(q)G^0_{41}(q).$$

Two of the terms comprising this transfer function correspond to the two paths from w_1 *to* w_2 *that pass only through* w_k*,* $k \in \mathcal{Z}_2$ *(*$\mathcal{Z}_2 = \{3,4,5\}$*). From Algorithm 5.5 this is not surprising since the paths* $G^0_{23}G^0_{31}$ *and* $G^0_{25}G^0_{54}G^0_{41}$ *must be lifted to remove the nodes* w_3*,* w_4 *and* w_5 *from the original network. Clearly, for this choice of* $\mathcal{D}_2$*,* $\breve{G}^0_{21}(\mathcal{D}_2) \neq G^0_{21}$*.*

Now choose $\mathcal{D}_2 = \{1,5\}$*. By Proposition 5.9*

$$\breve{G}^0_{21}(q,\{1,5\}) = G^0_{21}(q) + G^0_{23}(q)G^0_{31}(q).$$

Again, one of the terms comprising $\breve{G}^0_{21}(q,\{1,5\})$ *corresponds to the (only) path from* w_1 *to* w_2 *that passes only through* w_k*,* $k \in \mathcal{Z}_2$ *(*$\mathcal{Z}_2 = \{3,4\}$*).*

Finally, choose $\mathcal{D}_2 = \{1,3,5\}$*. By Proposition 5.9* $\breve{G}^0_{21}(q,\{1,3,5\}) = G^0_{21}(q)$ *as desired.* □

In general, one internal variable w_k from every independent path w_i to w_j must be included in $\mathcal{D}_j$ to ensure that $\breve{G}^0_{ji}(q,\mathcal{D}_j) = G^0_{ji}(q)$. This is proved later in Proposition 5.14.

However, before presenting the proposition, there is a second phenomenon related to the interconnection structure of the network that can cause the dynamics $\breve{G}^0_{ji}(q,\mathcal{D}_j)$ to be different from $G^0_{ji}(q)$, as illustrated in the next example.

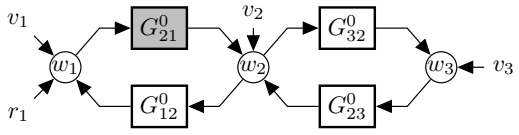

Figure 5.6: Network that is analyzed in Example 5.13.

Example 5.13 *Consider the network shown in Fig. 5.6. The objective of this example, is to choose $\mathcal{D}_2$ such that in the immersed network $\breve{G}^0_{21}(\mathcal{D}_2) = G^0_{21}$ (denoted in gray).*

Note that in this network there is only one independent path from w_1 to w_2. Choose $\mathcal{D}_2 = \{1\}$. By Proposition 5.9

$$\breve{G}^0_{21}(q, \{1\}) = \frac{G^0_{21}(q)}{1 - G^0_{23}(q)G^0_{32}(q)}$$

which is not equal to $G^0_{21}(q)$ as desired. The reason the factor $\frac{1}{1-G^0_{23}G^0_{32}}$ appears is because when lifting the path $G_{23}G_{32}$ a self-loop from w_2 to w_2 results. Thus, in step 3 of Algorithm 5.5 the transfer functions of the edges coming into w_2 are divided by the loop transfer function.

For the choice $\mathcal{D}_2 = \{1,3\}$, $\breve{G}^0_{21}(\{1,3\}) = G^0_{21}$ as desired. □

In general, if $\mathcal{D}_j$ is chosen such that no self-loops from w_j to w_j result due to the lifting of the paths when constructing the immersed network, the denominator in Step 3 of Algorithm 5.5 is reduced to 1. From these two examples we see that:

- Every parallel path from w_i to w_j should run through an input in the predictor model, and
- Every loop on the output w_j should run through an input in the predictor model.

This is formalized in the following proposition.

Proposition 5.14 *Consider a dynamic network as defined in (5.1) that satisfies Assumption 2.19. The transfer function $\breve{G}^0_{ji}(q, \mathcal{D}_j)$ in the immersed network is equal to $G^0_{ji}(q)$ if $\mathcal{D}_j$ satisfies the following conditions:*

(a) $i \in \mathcal{D}_j$, $j \notin \mathcal{D}_j$,

(b) every path w_i to w_j, excluding the path G^0_{ji}, goes through a node w_k, $k \in \mathcal{D}_j$,

(c) every loop w_j to w_j goes through a node w_k, $k \in \mathcal{D}_j$. □

The proof is in Appendix 5.9.5. The formulated conditions are used to make appropriate selections for the node variables that are to be measured and to be used as predictor inputs. In the following section it is shown that it is possible to identify the dynamics of the immersed network.

5.3.3 Estimated Dynamics in Predictor Model

In this section it is shown that the estimated dynamics between the predictor inputs and the module output w_j, are equal to $\breve{G}^0_{jk}(\mathcal{D}_j)$. The result confirms that the estimated dynamics are a consequence of the interconnection structure and the chosen predictor inputs. In addition conditions are presented that ensure that the estimates of $\breve{G}^0_{jk}(\mathcal{D}_j)$ are consistent. The results in this section are not specific to a particular identification method.

To concisely present the result, it is convenient to have a notation for a predictor which is a generalization of both the Direct and Two Stage Methods. Consider the predictor

$$\hat{w}_j(t|t-1,\theta) = H_j^{-1}(q,\theta)\Big(\sum_{k\in\mathcal{D}_j} G_{jk}(q,\theta) w_k^{(\mathcal{X})}(t) + \sum_{k\in\mathcal{P}_j} F_{jk}(q,\theta) r_k(t)\Big) + \Big(1 - H_j^{-1}(q,\theta)\Big) w_j(t) \quad (5.11)$$

where $\mathcal{X}$ denotes a (sub)set of the variables r_k, v_k, $k \in \{1,\dots,L\}$ and $H_j(q,\theta)$ is monic. Note that both predictors (5.3) and (5.6) are special cases of the predictor (5.11). For the Direct Method, choose $\mathcal{X} = \{r_{k_1},\dots,r_{k_n}, v_{\ell_1},\dots,v_{\ell_n}\}$, where $\{k_1,\dots,k_n\} = \mathcal{R}$, and $\{\ell_1,\dots,\ell_n\} = \mathcal{V}$. Then $w_k^{(\mathcal{X})} = w_k$. For the Two Stage Method, choose $\mathcal{X} = \{r_{k_1},\dots,r_{k_n}\}$, where $\{k_1,\dots,k_n\} = \mathcal{T}_j$.

A key concept in the analysis of this section is the *optimal output error residual*, which will be discussed next. From (5.7), w_j can be expressed in terms of w_k, $k \in \mathcal{D}_j$ as

$$w_j = \sum_{k\in\mathcal{D}_j} \breve{G}_{jk}^0 w_k + \sum_{k\in\mathcal{Z}_j\cap\mathcal{R}_j} \breve{F}_{jk}^0 r_k + \sum_{k\in\mathcal{Z}_j\cap\mathcal{V}_j} \breve{F}_{jk}^0 v_k + v_j + r_j. \quad (5.12)$$

Note that by Lemma 5.10 some $\breve{F}_{jk}^0(q,\mathcal{D}_j)$ may be zero depending on the interconnection structure. Let w_k be expressed interms of a component dependent on the variables in $\mathcal{X}$, and a component dependent on the remaining variables, denoted $w_k = w_k^{(\mathcal{X})} + w_k^{(\perp\mathcal{X})}$. In addition, split the sum involving the r_k-dependent terms according to whether r_k is in $\mathcal{P}_j$ or not. Then, from (5.12):

$$w_j = \sum_{k\in\mathcal{D}_j} \breve{G}_{jk}^0 w_k^{(\mathcal{X})} + \sum_{k\in\mathcal{D}_j} \breve{G}_{jk}^0 w_k^{(\perp\mathcal{X})} + \sum_{k\in\mathcal{P}_j} \breve{F}_{jk}^0 r_k + \sum_{k\in((\mathcal{Z}_j\cup\{j\})\cap\mathcal{R}_j)\setminus\mathcal{P}_j} \breve{F}_{jk}^0 r_k + \sum_{k\in\mathcal{Z}_j\cap\mathcal{V}_j} \breve{F}_{jk}^0 v_k + v_j. \quad (5.13)$$

When choosing an Output Error predictor (i.e. $H_j(q,\theta) = 1$), with predictor inputs $w_k^{(\mathcal{X})}$, $k \in \mathcal{D}_j$ and r_k, $k \in \mathcal{P}_j$, the part of (5.13) that is not modeled can be lumped together into one term. This term is the *optimal output error residual* of w_j, and is denoted p_j:

$$p_j(\mathcal{D}_j) := \sum_{k\in\mathcal{D}_j} \breve{G}_{jk}^0 w_k^{(\perp\mathcal{X})} + \sum_{k\in((\mathcal{Z}_j\cup\{j\})\cap\mathcal{R}_j)\setminus\mathcal{P}_j} \breve{F}_{jk}^0 r_k + \breve{v}_j, \quad (5.14)$$

where $\breve{v}_j$ is given by $\sum_{k\in\mathcal{Z}_j\cap\mathcal{V}_j} \breve{F}_{jk}^0 v_k + v_j$ in accordance with (5.9). Consequently, w_j equals:

$$w_j = \sum_{k\in\mathcal{D}_j} \breve{G}_{jk}^0 w_k^{(\mathcal{X})} + \sum_{k\in\mathcal{P}_j} \breve{F}_{jk}^0 r_k + p_j. \quad (5.15)$$

In a system identification setting, the optimal output error residual of w_j acts as the effective "noise" affecting w_j (this is clear from (5.15)). It also corresponds to the unmodeled component of w_j.

The following theorem is the main result of this section. It characterizes conditions that correlation between the optimal output error residual of w_j and the predictor inputs must satisfy so that it is possible to obtain consistent estimates of the dynamics between the predictor inputs. Such conditions are common in the identification literature. In open-loop identification for instance it is well known that if the innovation is uncorrelated to the input consistent estimates are possible (Ljung, 1999). Similarly, it is known (Ljung, 1999) that for the Direct Method in closed-loop, if the output noise is whitened and the whitened noise is uncorrelated to the plant input then consistent estimates of the plant are possible. The result that follows is an analogue to that reasoning adapted to identification in networks.

Proposition 5.15 *Consider a dynamic network as defined in (5.1) that satisfies Assumption 2.19. Consider model structures with independently parameterized noise and module models. For given sets $\mathcal{D}_j$, $\mathcal{P}_j$, and $\mathcal{X}$ construct the predictor (5.11). Suppose the power spectral density of*

$$[w_j \; w_{k_1}^{(\mathcal{X})} \; \dots \; w_{k_n}^{(\mathcal{X})} \; r_{\ell_1} \; \dots \; r_{\ell_m}]^T$$

where $\{k_1, \dots, k_n\} = \mathcal{D}_j$, $\{\ell_1, \dots, \ell_m\} = \mathcal{P}_j$ is positive definite for a sufficiently large number of frequencies $\omega_k \in (-\pi, \pi]$. Consider the conditions:

(a) $\bar{\mathbb{E}}[H_j^{-1}(q,\eta)p_j(t,\mathcal{D}_j)\cdot\Delta G_{jk}(q,\theta,\mathcal{D}_j)w_k^{(\mathcal{X})}(t)]=0,\ \forall k\in\mathcal{D}_j,$

(b) $\bar{\mathbb{E}}[H_j^{-1}(q,\eta)p_j(t,\mathcal{D}_j)\cdot\Delta F_{jk}(q,\theta,\mathcal{D}_j)r_k(t)] = 0,\ \forall k\in\mathcal{P}_j,$

where $\Delta G_{jk}(\theta,\mathcal{D}_j)=\breve{G}_{jk}^0(\mathcal{D}_j)-G_{jk}(\theta)$, and $\Delta F_{jk}(\theta,\mathcal{D}_j)=\breve{F}_{jk}^0(\mathcal{D}_j)-F_{jk}(\theta)$. Then $G_{jk}(q,\theta^*) = \breve{G}_{jk}^0(q,\mathcal{D}_j)$, where $\breve{G}_{jk}^0(q,\mathcal{D}_j)$ is defined in Proposition 5.9, if for all $\theta \in \Theta$:

1. *Conditions (a) and (b) hold for all η, or*
2. *The equations of Conditions (a) and (b) hold for η^* only, where*

$$\eta^* = \arg\min \bar{\mathbb{E}}[\big(H_j^{-1}(q,\eta)s_j(t,\mathcal{D}_j)\big)^2],$$

and $H_j^{-1}(q,\eta^*)p_j(t,\mathcal{D}_j)$ is white noise. □

The proof can be found in Appendix 5.9.6. The theorem can be interpreted as follows. In Case 1, consistent estimates are possible if the predictor inputs are uncorrelated to the optimal output error residual of w_j. This is analogous to the open loop situation. In Case 2, consistent estimates are possible if the whitened version of the optimal output error residual of w_j is uncorrelated to the predictor inputs. This is analogous to the closed-loop Direct Method reasoning.

The condition on the power spectral density of $[w_j \; w_{k_1}^{(\mathcal{X})} \; \dots \; w_{k_n}^{(\mathcal{X})} \; r_{\ell_1} \; \dots \; r_{\ell_m}]^T$ is basically a condition on the informativity of the data (Söderström & Stoica, 1989a) (i.e. the data must be persistently exciting of sufficiently high order).

The main point of Proposition 5.15 is twofold:

1. The estimated transfer functions $G_{jk}(q, \theta^*)$ are consequences of the choice of $\mathcal{D}_j$. In particular, they are estimates of the transfer functions $\breve{G}^0_{jk}(q, \mathcal{D}_j)$ specified by the immersed network.

2. To present general conditions under which consistent estimates are possible.

Proposition 5.15 points to a notion of identifiability. For a given set $\mathcal{D}_j$, a particular module G^0_{ji} is identifiable if $\breve{G}^0_{ji} = G^0_{ji}$. Thus, if the conditions of Proposition 5.14 are satisfied for a given set $\mathcal{D}_j$, then G^0_{ji} is identifiable.

Conditions (a) and (b) are enforced by different mechanisms in the Direct and Two Stage methods, leading to different conditions to ensure consistency of the estimates. This is discussed in the next two sections.

5.4 PREDICTOR INPUT SELECTION - DIRECT METHOD

In this section it is shown how to satisfy the conditions of Proposition 5.15 using the Direct Method.

When using the Direct Method for identification in dynamic networks, there are three main mechanisms that ensure consistent estimates of G^0_{ji} (Dankers et al., 2013b; Van den Hof et al., 2013) (the same mechanisms are present in the closed-loop Direct Method (Ljung, 1999; Forssell & Ljung, 1999; Van den Hof, 1998)):

1. the noise v_j affecting the output w_j is uncorrelated to all other noise terms v_n, $n \in \mathcal{V}_j$,

2. every loop that passes through w_j in the data generating system contains at least one delay, and

3. there exists a θ such that $H_j^{-1}(\theta)v_j = \breve{e}_j$ is white noise.

In Proposition 2 of Van den Hof et al. (2013) it is shown that for the choice $\mathcal{D}_j = \mathcal{N}_j$ and $\mathcal{P}_j = \emptyset$, these conditions plus a condition on the informativity of the data are sufficient in order to obtain consistent estimates of a module G^0_{ji} embedded in the network. In the setup considered in this chapter an additional mechanism plays a role, namely the choice of predictor inputs.

The following proposition presents conditions on the immersed network that ensure that Case 2 of Proposition 5.15 holds. The conditions reflect the three mechanisms presented above.

Proposition 5.16 *Consider a dynamic network as defined in (5.1) that satisfies Assumption 2.19. Consider the immersed network constructed by removing w_n, $n \in \mathcal{Z}_j$ from the original network. The situation of Case 2 of Proposition 5.15 holds for the immersed network if:*

(a) $\breve{v}_j$ is uncorrelated to all $\breve{v}_k$, $k \in \mathcal{D}_j$.

(b) There is a delay in every loop w_j to w_j (in the immersed network).

(c) If $\breve{G}^0_{jk}$ has a delay, then $G_{jk}(\theta)$ is parameterized with a delay.

(d) p_j is not a function of any r_n, $n \in \mathcal{R}$.

(e) There exists a η such that $H_j^{-1}(q,\eta)p_j(t)$ is white noise.

The proof can be found in Appendix 5.9.7.

In the following subsections, the conditions of Proposition 5.16 are interpreted in terms of what they mean in the original network. In Subsection 5.4.1 it is shown what conditions can be imposed in the original network in order to ensure that $\breve{v}_j$ is uncorrelated to $\breve{v}_k$, $k \in \mathcal{D}_j$ (i.e Condition (a) of Proposition 5.16 holds).In Subsection 5.4.2 it is shown under which conditions p_j is not a function of external variables (i.e. Condition (d) of Proposition 5.16 holds). In Subsection 5.4.3 a version of Proposition 5.16 is presented where all the conditions are stated only in terms of the original network.

5.4.1 Correlation of Noise

In this section conditions are presented that ensure that $\breve{v}_j$ is uncorrelated to $\breve{v}_k$, $k \in \mathcal{D}_j$. The conditions are presented using only variables in the original network.

Recall from (5.9) that $\breve{v}_k$ is a filtered sum of v_n, $n \in \mathcal{Z}_j \cup \{k\}$,

$$\breve{v}_k(t) = \sum_{n \in \mathcal{Z}_j} \breve{F}^0_{jn}(q, \mathcal{D}_j) v_n + \breve{F}^0_{jj}(q, \mathcal{D}_j) v_j(t). \tag{5.16}$$

Consider 2 variables $\breve{v}_{k_1}$ and $\breve{v}_{k_2}$. Suppose that there is a path from another variable v_n, $n \in \mathcal{Z}_j$ to both w_{k_1} and w_{k_2}. By Lemma 5.10 both $\breve{F}^0_{k_1 n}$ and $\breve{F}^0_{k_2 n}$ are non-zero in this situation. Consequently, as can be see from (5.16) both $\breve{v}_{k_1}$ and $\breve{v}_{k_2}$ are functions of v_n, with the result that $\breve{v}_{k_1}$ and $\breve{v}_{k_2}$ are correlated. Thus, due to the presence of v_n and the interconnection structure of the network, $\breve{v}_{k_1}$ and $\breve{v}_{k_2}$ are correlated. In this case v_n is a *confounding variable*. In statistics, and in particular in statistical inference, a confounding variable is a variable that is not known (or measured) and causally affects both the output variable and the input variable (Pearl, 2009). The induced correlation between input and output is however not caused by a direct causal relation between the input and output. In the framework of this chapter consider the following definition.

Definition 5.17 *Consider a particular output variable w_j and a set $\mathcal{D}_j$ of predictor inputs. In this modeling setup, a variable v_ℓ is a* confounding *variable if the following conditions hold:*

(a) There is a path from v_ℓ to w_j that passes only through w_m, $m \in \mathcal{Z}_j$.

(b) There is a path from v_ℓ to one or more w_k, $k \in \mathcal{D}_j$ that passes only through w_m, $m \in \mathcal{Z}_j$. □

The following is an example of a confounding variable.

Example 5.18 *Consider the network shown in Fig. 5.7. Suppose that the objective is to obtain a consistent estimate of G^0_{21} (denoted in gray) using the Direct Method.*

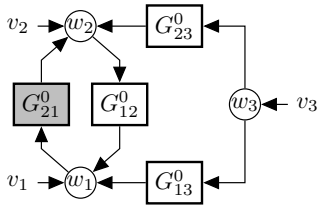

Figure 5.7: Network that is analyzed in Example 5.18.

Let $j = 2$, and choose $\mathcal{D}_2 = \{1\}$. By Definition 5.17, v_3 is a confounding variable. The expressions for $\breve{v}_1$ and $\breve{v}_2$ for this network are:

$$\breve{v}_1 = v_1 + G_{13}^0 v_3 \quad \text{and} \quad \breve{v}_2 = v_2 + G_{23}^0 v_3.$$

Clearly, the confounding variable v_3 induces a correlation between $\breve{v}_1$ and $\breve{v}_2$. □

The presence of confounding variables is not the only way that $\breve{v}_{k_1}$ and $\breve{v}_{k_2}$ could become correlated. Suppose that $\breve{v}_{k_1}$ is a function of v_n, and $\breve{v}_{k_2}$ is a function of v_m. If v_n and v_m are correlated, then $\breve{v}_{k_1}$ and $\breve{v}_{k_2}$ are correlated.

The following proposition presents conditions that ensure $\breve{v}_j$ is uncorrelated to all $\breve{v}_k$, $k \in \mathcal{D}_j$.

Proposition 5.19 *Consider a dynamic network as defined in (5.1) that satisfies Assumption 2.19. Consider the immersed network constructed from the internal variables, $\{w_k\}$, $k \in \mathcal{D}_j$. The disturbance term $\breve{v}_j$ (as defined in (5.9)) is uncorrelated to all $\breve{v}_k$, $k \in \mathcal{D}_j$ if the following conditions hold:*

(a) v_j is uncorrelated to all v_k, $k \in \mathcal{D}_j$ and to all variables v_n, $n \in \mathcal{Z}_j$ that have paths to any w_k, $k \in \mathcal{D}_j$ that pass only through nodes w_ℓ, $\ell \in \mathcal{Z}_j$.

(b) All v_k, $k \in \mathcal{D}_j$ are uncorrelated to all v_n, $n \in \mathcal{Z}_j$ that have a path to w_j that passes only through nodes in $\mathcal{Z}_j$.

(c) All v_n, $n \in \mathcal{Z}_j$ are uncorrelated to each other

(d) No variable v_k, $k \in \mathcal{Z}_j$ is a confounding variable.

The proof can be found in Appendix 5.9.8.

Remark 5.20 *Suppose that all v_k, $k \in \mathcal{V}$ are uncorrelated. Then Conditions (a) - (c) hold for any $\mathcal{D}_j$. However, whether Condition (d) holds depends on the interconnection structure and the choice of $\mathcal{D}_j$.* □

5.4.2 Adding External Excitation

External variables are not strictly necessary to ensure that the data is informative when using the direct method as long as the noise that is driving the system is sufficiently exciting. However, external excitation can be beneficial in order to reduce

the variance of the estimates, or provide extra excitation in a frequency range of interest.

Whenever there is an external variable r_k acting as a "disturbance" on the output variable w_j (i.e. p_j contains an element which is due to the external variable r_k), it makes sense to model that component. This happens whenever there is a path r_k to w_j that passes only through w_k, $k \in \mathcal{Z}_j$. Thus, in this case, choose the set $\mathcal{P}_j = \{k\}$ so that r_k is included as a predictor input (i.e. the dynamics from r_k to w_j are modeled). The advantage of this scheme is that the power of the optimal output error residual is reduced by eliminating known variables from p_j, suggesting that a better estimate will result (the signal to noise ratio is increased). Secondly, the result is that p_j is only a function of v's and so Condition (d) of Proposition 5.16 holds.

5.4.3 Main Result - Direct Method

Conditions are presented so that the Direct Method will result in consistent estimates of $\breve{G}^0_{ji}(\mathcal{D}_j)$. In Proposition 5.16 the conditions were stated in terms of the immersed network. In the following proposition the conditions are stated in terms of the original network.

Proposition 5.21 *Consider a dynamic network as defined in (5.1) that satisfies Assumption 2.19. Let $\{w_k\}$, $k \in \mathcal{D}_j$ and $\{r_k\}$, $k \in \mathcal{P}_j$ be the set of internal and external variables respectively that are included as inputs to the predictor (5.3). The set $\mathcal{P}_j$ is constructed to satisfy the condition that $k \in \mathcal{P}_j$ if and only if there exists a path from r_k to w_j, that passes only through nodes in $\mathcal{Z}_j$. Consistent estimates of $\breve{G}^0_{ji}$ are obtained using the Direct Method formulated in Algorithm 5.1 if the following conditions are satisfied:*

(a) There is a delay in every loop w_j to w_j.

(b) v satisfies the conditions of Proposition 5.19.

(c) The power spectral density of $[w_j \ w_{k_1} \ \cdots \ w_{k_{n}} r_{\ell_1} \ \cdots \ r_{\ell_m}]^T$, $k_* \in \mathcal{D}_j$, $\ell_* \in \mathcal{P}_j$ is positive definite for a sufficiently large number of frequencies $\omega_k \in (-\pi, \pi]$.

(d) The parameterization is chosen flexible enough, i.e. there exist parameters θ and η such that $G_{jk}(q,\theta) = \breve{G}^0_{jk}(q,\mathcal{D}_j)$, $\forall k \in \mathcal{D}_j$, $F_{jk}(q,\theta) = \breve{F}^0_{jk}(q,\mathcal{D}_j)$, $\forall k \in \mathcal{P}_j$, and $H_j(q,\eta) = \breve{H}^0_j(q,\mathcal{D}_j)$.

(e) If $\breve{G}^0_{jk}$ has a delay, then $G_{jk}(\theta)$ is parameterized with a delay. □

Proof: The proof follows almost directly from Proposition 5.15 and Propositions 5.16 and 5.19. It remains to be shown that $p_j = \breve{v}_j$ (i.e. Condition (d) of Proposition 5.16 holds).

By Lemma 5.10 $\breve{F}^0_{jk}$, $k \in \mathcal{D}_j$ is zero unless there is a path from r_k to w_j which passes only through w_n, $n \in \mathcal{Z}_j$. From (5.14) and by the way $\mathcal{P}_j$ is constructed it follows that there are no r terms present in p_j. Consequently, $p_j = \breve{v}_j$.

Remark 5.22 *In Proposition 5.21 conditions have been presented which, if satisfied, ensure that consistent estimates of $\breve{G}^0_{jk}(q,\mathcal{D}_j)$, $k \in \mathcal{D}_j$ as defined by the immersed network are obtained. If the set $\mathcal{D}_j$ is chosen such that $\breve{G}^0_{ji}(q,\mathcal{D}_j) = G^0_{ji}(q)$ (i.e. the $\mathcal{D}_j$ is chosen such that the conditions of Proposition 5.14 are satisfied) then Proposition 5.21 shows under which conditions G^0_{ji} can be consistently identified.* □

The reason that Condition (a) and exact noise modeling are required is due to the presence of a (feedback) path from w_j to at least one w_k, $k \in \mathcal{D}_j$. If there is no such feedback, then the conditions of Proposition 5.21 simplify considerably. Similarly, since, it is the variable v_j that is causing the problems when there is such a feedback path, if it is not present, the conditions can be simplified.

Corollary 5.23 *Consider the situation of Proposition 5.21. If there is no path from w_j to any w_k, $k \in \mathcal{D}_j$, or if v_j is not present in the network, then Conditions (a) and (e) can be omitted, and Condition (d) can be changed to:*

(d') The parameterization is chosen flexible enough, i.e. there exists a parameter θ such that $G_{jk}(q,\theta) = \breve{G}^0_{jk}(q,\mathcal{D}_j)$, $\forall k \in \mathcal{D}_j$, $F_{jk}(q,\theta) = \breve{F}^0_{jk}(q,\mathcal{D}_j)$, $\forall k \in \mathcal{P}_j$. □

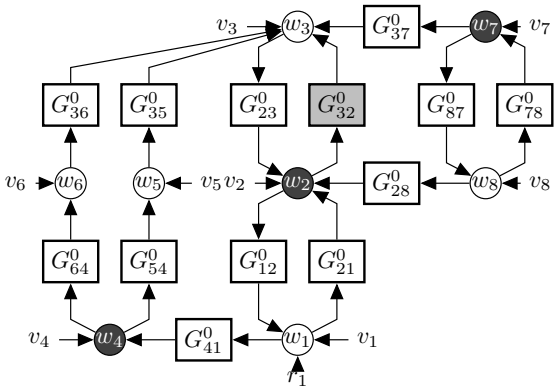

Figure 5.8: Network that is analyzed in Examples 5.24 and 5.30.

Example 5.24 *Consider the dynamic network shown in Fig. 5.8. Suppose the objective is to obtain consistent estimates of G^0_{32} (denoted in gray) using the Direct Method.*

First, we show how to choose the set $\mathcal{D}_3$ such that $\breve{G}^0_{32}(q,\mathcal{D}_j)$ in the immersed network is equal to $G^0_{32}(q)$ (i.e. $\mathcal{D}_j$ is chosen such that it satisfies the conditions of Proposition 5.14). Besides G^0_{32} there are several paths from w_2 to w_3:

$$w_2 \to w_1 \to w_4 \to w_5 \to w_3,$$
$$w_2 \to w_1 \to w_4 \to w_6 \to w_3$$

for instance. All paths from w_2 to w_3 (not including G^0_{32}) pass through either the nodes w_1 and w_2, the nodes w_4 and w_2. Thus, Condition (b) of Proposition 5.14 is satisfied for $\mathcal{D}_3 = \{1,2\}$ and $\mathcal{D}_3 = \{2,4\}$.

Since all loops from w_3 pass through w_2, Condition (c) of Proposition 5.14 is also satisfied for both these choices of $\mathcal{D}_3$.

For both of these choices, v_7 and v_8 are confounding variables (Condition (b) of Proposition 5.21 is not satisfied). However, if w_7 is included as a predictor input, then there are no more confounding variables.

By this reasoning two possible choices for $\mathcal{D}_3$ that lead to consistent estimates of G_{32}^0 are $\{2,4,7\}$ (denoted in dark gray), and $\{2,1,7\}$. In either case, $\mathcal{P}_3$ should be chosen as $\emptyset$.

Another possible choice for $\mathcal{D}_3 = \{2,5,6,7\} = \mathcal{N}_3$. It is interesting that the previous sets $\mathcal{D}_3$ are strictly smaller than $\mathcal{N}_3$, and are not even subsets of $\mathcal{N}_3$. □

The choice $\mathcal{D}_j = \mathcal{N}_j$, $\mathcal{P}_j = \emptyset$ always satisfies the Conditions of Proposition 5.14 and confounding variables are never present. This is the choice that is made in Van den Hof et al. (2013).

In the following section an analogue to Proposition 5.21 is presented for the Two-Stage Method.

5.5 PREDICTOR INPUT SELECTION - TWO STAGE METHOD

A guiding principle to ensure consistent estimates that has been presented in Proposition 5.15 is that the optimal output error residual of w_j should be uncorrelated to the predictor inputs. For the Two Stage Method this condition is enforced by projecting the predictor inputs onto the external variables. Consequently, the predictor inputs are only functions of r_m, $m \in \mathcal{T}_j$. As long as the unmodeled component of w_j is not a function of r_m, $m \in \mathcal{T}_j$ then Conditions (a) and (b) of Proposition 5.15 are satisfied.

Proposition 5.25 *Consider a dynamic network as defined in (5.1) that satisfies Assumption 2.19. Let $\{r_m\}$, $m \in \mathcal{T}_j$ be the external input(s) onto which will be projected. Let $\{w_k^{(\mathcal{T}_j)}\}$, $k \in \mathcal{D}_j$ and $\{r_k\}$, $k \in \mathcal{P}_j$ be the sets of (projections of) internal and external variables respectively that are included as inputs to the predictor (5.6). The set $\mathcal{P}_j$ is constructed to satisfy the condition that $k \in \mathcal{P}_j$ if and only if there exists a path from r_k to w_j, $k \in \mathcal{T}_j$, that passes only through nodes in $\mathcal{Z}_j$. Consistent estimates of G_{ji}^0 are obtained using the Two Stage Method (Algorithm 5.2) if the following conditions hold:*

(a) Every r_k, $k \in \mathcal{T}_j$ is uncorrelated to all r_m, $m \notin \mathcal{T}_j$, except those r_m for which there is no path to w_j.

(b) The power spectral density of $[w_{k_1}^{(\mathcal{T}_j)} \cdots w_{k_n}^{(\mathcal{T}_j)}\ r_{m_1} \cdots r_{m_n}]^T$, $k_* \in \mathcal{D}_j$, $m_* \in \mathcal{P}_j$, is positive definite for a sufficient number of frequencies $\omega_k \in (-\pi, \pi]$

(c) The parameterization is chosen flexible enough, i.e. there exists a parameter θ such that $G_{jk}(q,\theta) = \breve{G}_{jk}^0(q,\mathcal{D}_j)$, $\forall k \in \mathcal{D}_j$, $F_{jk}(q,\theta) = \breve{F}_{jk}^0(q,\mathcal{D}_j)$, $\forall k \in \mathcal{P}_j$. □

For a proof, see Appendix 5.9.9.

Note that in order for Condition (b) to hold, there must be a path from at least one

r_m, $m \in \mathcal{T}_j$ to w_i. If not, then $w_i^{(\mathcal{T}_j)} = 0$ and the power spectral density of Condition (b) will not be positive definite.

Remark 5.26 *The condition on the order of excitation of the data (Condition (b)) can be satisfied if there is one external variable present for each predictor input. This is however just a sufficient condition. For more information on how the network dynamics add excitation to the data so that fewer external variables are required see Gevers et al. (2009a) for instance.* □

Remark 5.27 *In the discussion thus far, we have not allowed the choice of w_j as a predictor input (by Condition (a) in Proposition 5.14, j is not allowed to be in $\mathcal{D}_j$). It can be shown that w_j can be used as a predictor input to consistently identify G_{ji}^0 using the Two-Stage method if r_j is present (and Conditions (a) - (c) of Proposition 5.25 are satisfied). Moreover, it can also be shown that if r_j is not present, then it is not possible to choose w_j as a predictor input to consistently identify G_{ji}^0 using the Two-Stage Method. These statements are proved in Appendix 5.9.10. The advantage of choosing w_j as a predictor input is that Condition (c) is automatically satisfied without the need to include any other variables.* □

Remark 5.28 *The Conditions presented in Proposition 5.25 do not change if there is measurement noise present on the measurements of w_k, $k \in \mathcal{D}_j$. The Two Stage method still results in consistent estimates of $\breve{G}_{ji}^0$ in the presence of measurement noise, as long as the r's are exactly known. This observation is further explored and generalized in Dankers et al. (2014c).* □

It is interesting to compare the conditions of the Direct and Two Stage Methods. For the Two Stage Method there are no restrictions on algebraic loops. Moreover, there are no conditions regarding the correlation of the noise terms, or the presence of confounding variables. However, to use the Two Stage Method at least one external variable r_m must be present that affects w_i (this is not the case for the Direct Method). Moreover, the excitation conditions of the Two Stage Method are much stricter than those of the Direct Method.

From the perspective of reducing the variance of an estimate, it is desirable to project onto as many external variables as possible, since this increases the power of the predictor inputs relative to the optimal output error residual (not projecting onto a particular external variable means that the power of the predictor inputs is less, and that particular external variable becomes part of the unmodeled component of the output, increasing the power of the optimal output error residual).

Example 5.29 *Recall the network of Example 5.12 shown in Fig. 5.5. Suppose that the objective is to obtain an estimate of G_{21}^0 (denoted in gray) using the Two Stage Method. Choose an output error model structure ($H_2(q, \theta) = 1$). Choose $\mathcal{D}_2 = \{1, 3, 4\}$. For this choice of $\mathcal{D}_2$ all conditions of Proposition 5.14 are satisfied, and therefore $\tilde{G}_{21}^0 = G_{21}^0$. To ensure that the estimate of $\tilde{G}_{21}^0$ is consistent, $\mathcal{P}_2$ must also be chosen properly.*

Choose to project the predictor inputs onto r_1 and r_5 ($\mathcal{T}_2 = \{1, 5\}$). Thus, by Proposition 5.25 $\mathcal{P}_2$ is set to $\{5\}$, since there is a path from r_5 to w_2 that passes only through w_n $n \in \mathcal{Z}_2 = \{5\}$.

Now consider projecting only onto r_1. In this case, by Proposition 5.25, $\mathcal{P}_2$ is set to $\emptyset$.

Finally, consider the choice $\mathcal{D}_2 = \{1, 2, 5\}$. Futhermore, choose to project onto both r_1 and r_5. In this case, by Proposition 5.25, $\mathcal{P}_2$ is set to $\emptyset$. In this case, due to the different choice of $\mathcal{D}_2$, $\mathcal{P}_2$ can be chosen as $\emptyset$ even though $\mathcal{T}_2 = \{1, 5\}$ just like in the first case considered in this example. □

Example 5.30 *Consider the same network as in Example 5.24, shown in Fig. 5.8. Suppose the objective is to obtain consistent estimates of G_{32}^0 (marked in gray) using the Two Stage Method. Choose r_1 as the external variable to project onto ($\mathcal{T}_3 = \{1\}$). By the same reasoning as in Example 5.24, choosing $\mathcal{D}_3 = \{1, 2\}$ or $\{2, 4\}$ satisfies the conditions of Proposition 5.14. However, in this case (unlike for the Direct Method) both these choices of $\mathcal{D}_3$ satisfy all the remaining conditions of Proposition 5.25 (since confounding variables are not an issue for the Two Stage Method).*

Finally, $\mathcal{P}_3$ must be chosen as stated in Proposition 5.25. There are two independent paths from r_1 to w_3,

$$r_1 \to w_4 \to w_6 \to w_3 \quad \text{and} \quad r_1 \to w_2 \to w_3$$

both of which pass through a variable w_n, $n \in \mathcal{D}_3$, so $\mathcal{P}_3$ should be chosen as $\emptyset$. □

Example 5.31 *In this example the Direct and Two Stage Methods will be compared using a simulation example. Data has been generated in Matlab for the network of Examples 5.24 and 5.30. Each transfer function G_{jk}^0, $j = 1, \ldots, 8$, $k \in \mathcal{N}_j$ is of second order with one delay. The transfer functions are chosen so that $(I - G_0)^{-1}$ is stable.*

The noise terms v_k, $k = 1, \ldots, 8$ and the external variable r_1 are constructed as

```
>>r = 3*randn(N,1);
>>e = randn(N,8);
>>v = lsim(H0,e,0:N-1);
```

where `H0` *is a diagonal matrix with first order monic transfer functions on the diagonal, and N is the number of data points used. For this simulation example $N = 5000$. As in Examples 5.24 and 5.30 suppose the objective is to identify G_{32}^0 (denoted in gray in the figure).*

Consider first the Direct Method. In Example 5.24 it was shown that choosing $\mathcal{D}_3 = \{2, 4, 7\}$ will result in consistent estimates of G_{32}^0. As discussed in Section 5.4, a key mechanism which ensures consistent estimates are possible using the Direct Method is that the term $\tilde{v}_3(t, \mathcal{D}_3)$ is correctly modeled. Choosing

$$H_3(q, \theta) = 1 + c_1 q^{-1} + \cdots + c_{50} q^{-50}$$

ensures that the noise model can model any H_3^0 that has an impulse response that is basically 0 for any lag greater than 50. Of course the noise model will not be exact,

but using this parameterization a very good approximation can be obtained for a wide range of possible noise models.

Secondly, the transfer functions $G_{3k}(q,\theta)$, $k \in \mathcal{D}_3$ must be parameterized. Since only the estimate $G_{32}(q,\theta)$ is of interest, choose the following parameterization:

$$G_{32}(q,\theta) = \frac{b_1^{32}q^{-1} + b_2^{32}q^{-2}}{1 + a_1^{32}q^{-1} + a_2^{32}q^{-2}}, \tag{5.17a}$$

$$G_{34}(q,\theta) = b_1^{34}q^{-1} + \cdots b_{50}^{34}q^{-50}, \tag{5.17b}$$

$$G_{37}(q,\theta) = b_1^{37}q^{-1} + \cdots b_{50}^{37}q^{-50}. \tag{5.17c}$$

The resulting estimates of 10 different identification experiments are plotted in Fig. 5.9. Note that the estimate of G_{32}^0 is quite good.

Next the Two Stage Method is used. In this case, as shown in Example 5.30, $\mathcal{D}_3 = \{2,4\}$ is sufficient to obtain consistent estimates of G_{32}^0. It is interesting to note that the parameterization of $G_{34}(q,\theta)$ in (5.17) cannot be used for the Two Stage Method since the Condition (b) is stricter than in the Direct Method. The condition places an upper bound on the number of parameters that can be estimated. Therefore, the following parameterization is used:

$$G_{32}(q,\theta) = \frac{b_1^{32}q^{-1} + b_2^{32}q^{-2}}{1 + a_1^{32}q^{-1} + a_2^{32}q^{-2}},$$

$$G_{34}(q,\theta) = \frac{b_2^{34}q^{-2} + \cdots + b_9^{34}q^{-9}}{1 + a_1^{34}q^{-1} + \cdots + a_8^{34}q^{-8}}.$$

The resulting estimates of 10 different identification experiments are plotted in Fig. 5.10. It is clear that the estimates of the Two-Stage method have a higher variance. This is as expected because only the projections of the internal variables onto the external variable are used as predictor inputs, thus the signal to noise ratio is much larger than in the Direct Method.

Example 5.32 *Consider now an illustration of the notion that the identified object is a function of the predictor inputs. Consider the same situation as in Example 5.31. Suppose that now we choose $\mathcal{D}_3 = \{2,5,7\}$. From Proposition 5.14 this choice of predictor inputs will not lead to a consistent estimate of G_{32}^0. Further suppose that v_4 and v_1 and r_1 are not present (otherwise the noise terms would be confounding variables, and r_1 would need to be chosen as a predictor input). Suppose that each transfer function (i.e. $G_{23}(\theta)$, $G_{25}(\theta)$, $G_{27}(\theta)$, and $H_3(\theta)$) is modeled using 50 parameters, just as in the previous simulation. The resulting estimates of 10 different identification experiments are plotted in Fig. 5.11. Note that all the identified transfer functions have changed from those shown in Fig. 5.9, simply because one predictor input was chosen differently. Now $G_{23}(\theta)$ is no longer a consistent estimate of G_{32}^0, but rather it is a consistent estimate of $\breve{G}_{32}^0$ which is equal to:*

$$\breve{G}_{32}^0(q,\mathcal{D}_3) = G_{32}^0(q) + G_{36}^0(q)G_{64}^0(q)G_{41}^0(q)G_{12}^0(q).$$

From Figure 5.11 it is clear that in fact consistent estimates of $\breve{G}_{32}^0$ are obtained. □

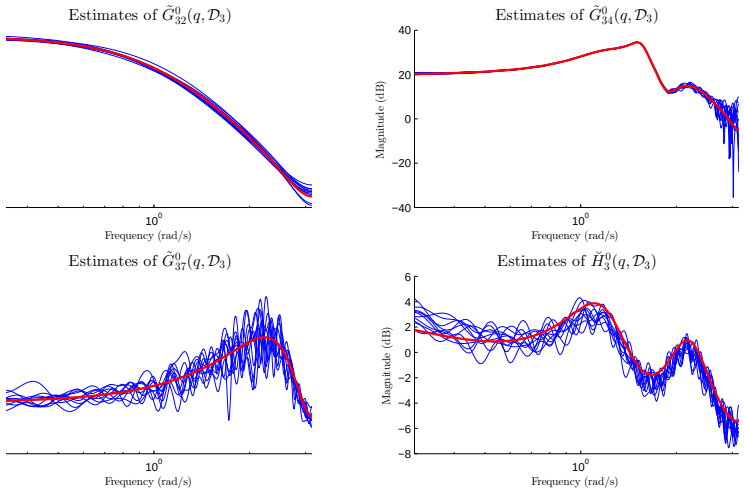

Figure 5.9: Results of Identification Experiments using the Direct Method as described in Example 5.31. The red lines denote the transfer functions of the immersed network for $\mathcal{D} = \{2, 4, 7\}$, the blue lines denote the estimates.

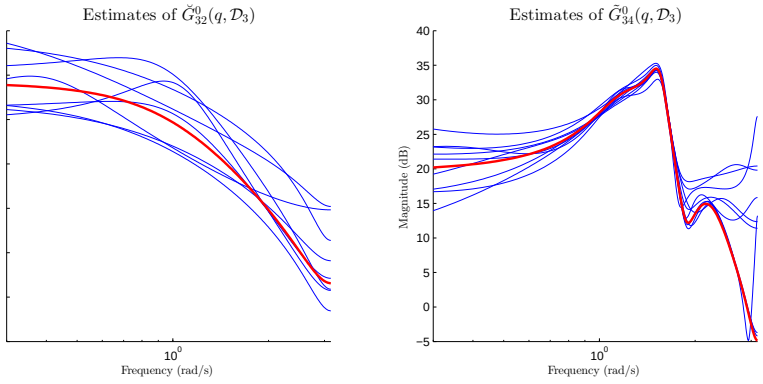

Figure 5.10: Results of Identification Experiments using the Two Stage Method as described in Example 5.31. The red lines denote the transfer functions of the immersed network for $\mathcal{D} = \{2, 4\}$, the blue lines denote the estimates.

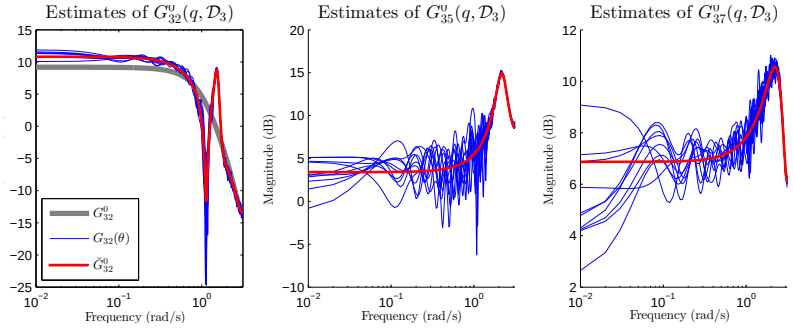

Figure 5.11: Results of Identification Experiments using the Direct Method as described in Example 5.31. The red lines denote the transfer functions of the immersed network for $\mathcal{D} = \{2, 5, 7\}$, the blue lines denote the estimates, and the gray line denotes the transfer function G_{32}^0 of the original network.

5.6 ALGORITHMIC ASPECTS

In this section an algorithm is presented that provides a practical way to check the conditions that the set $\mathcal{D}_j$ must satisfy in order to ensure that $\breve{G}_{ji}^0(q, \mathcal{D}_j)$ of the immersed network is equal to $G_{ji}^0(q)$ of the original network. In other words, an algorithm is presented to check whether, for a given set $\mathcal{D}_j$, the conditions of Proposition 5.14 are satisfied. The algorithm uses tools from graph theory, therefore, before presenting the result, consider the following definitions.

Definition 5.33 (A-B path (Diestel, 1997)) *Given a directed graph $\mathbb{G}$ and sets of nodes A and B. Denote the nodes in the graph by x_i. A path $\mathbb{P} = x_0 x_1 \cdots x_k$, where the x_i are all distinct, is an* A-B path *if $V(\mathbb{P}) \cap A = \{x_0\}$, and $V(\mathbb{P}) \cap B = \{x_k\}$.*

Definition 5.34 (A-B Separating Set (Diestel, 1997)) *Given a directed graph $\mathbb{G}$, and sets of nodes $A, B \subset V(\mathbb{G})$, a set $X \subseteq V(\mathbb{G})$ is an* A-B separating set *if the removal of the nodes in X results in a graph with no A-B paths.*

The following notation will be useful in order to reformulate the conditions of Proposition 5.14 using the notion of separating sets. Let the node w_j be split into two nodes, w_j^+ to which all incoming edges (of w_j) are connected and w_j^- to which all outgoing edges (of w_j) are connected. The new node w_j^+ is connected to w_j^- with the edge $G_{j^+j^-} = 1$. Let w_i^+ and w_i^- be defined analogously.

Proposition 5.35 *The conditions of Proposition 5.14 can be reformulated as: the set $\mathcal{D}_j$ is a $\{w_i^+, w_j^-\}$-$\{w_j^+\}$ separating set.*

Proof: The conditions of Proposition 5.14 can be rewritten as follows. The set $\mathcal{D}_j$ satisfies the following conditions:

1. $\mathcal{D}_j \setminus \{i\}$ is a $\{w_i\}$-$\{w_j\}$ separating set for the network with path G_{ji}^0 removed,

2. $\mathcal{D}_j$ is a $\{w_j^-\}$-$\{w_j^+\}$ separating set.

These two conditions can be formulated as the single condition of the proposition.

Note that w_i must always chosen to be in $\mathcal{D}_j$ to ensure that $\mathcal{D}_j$ is a $\{w_i^+, w_j^-\}$-$\{w_j^+\}$ separating set (i.e. Condition (a) of Proposition 5.14 is automatically satisfied). This is because there is always a path $w_i^+ \to w_i^- \to w_2^+$. Consequently, w_i^- must be chosen in the set $\mathcal{D}_j$.

The advantage of reformulating the conditions in terms of separating sets is that there exist tools from graph theory to check if a given set is a separating set or to find (the smallest possible) separating sets (Diestel, 1997; Kanevsky, 1993).

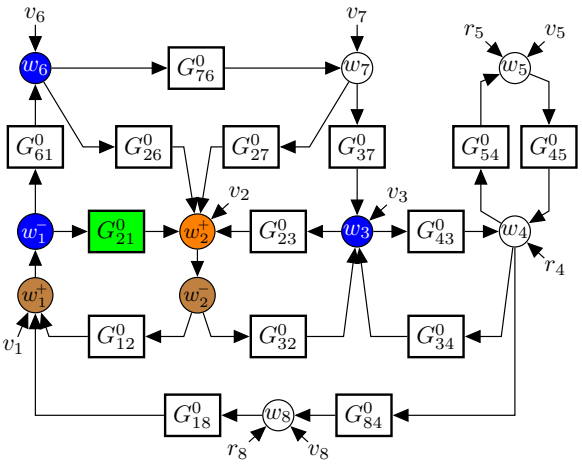

Figure 5.12: Example of an interconnected network used in Example 5.36.

Example 5.36 *Consider the network shown in Fig. 5.12. Suppose that the objective is to obtain consistent estimates of G^0_{21} (denoted in green). Both w_1 and w_2 have been split into two nodes as described above.*

By Proposition 5.35 the conditions of Proposition 5.14 are satisfied for the given network if $\mathcal{D}_2$ is a $\{w_1^+, w_2^-\}$-$\{w_2^+\}$ separating set. The outgoing set $\{w_1^+, w_2^-\}$ is denoted in brown, and the incoming set $\{w_2^+\}$ is denoted in orange in the figure.

There are many possible choices of $\mathcal{D}_2$, but the smallest choice, $\{w_1^-, w_6, w_3\}$, is denoted in black. It is easy to verify that all paths from the brown set to the orange set pass through a node in the black set. □

5.7 DISCUSSION

In this chapter we have not addressed algorithmic aspects of the identification methods themselves. The presented approach is a local approach in the sense that only a (small) subset of internal variables are required to identify a particular module embedded in the network. Therefore, even for large networks, the numerical complexity of obtaining an estimate of a particular module can be limited by proper choice of predictor inputs. If multiple modules need to be identified each module can be estimated in parallel, reducing the execution time of estimation routines. If

the number of predictor inputs is large it may be attractive to rely on linear regression schemes such as ARX, FIR (Ljung, 1999) and orthogonal basis function expansions (Heuberger et al., 2005), as well as IV-type and subspace algorithms (Ljung, 1999).

While we have restricted this chapter to dealing with questions of consistency, variance properties of estimates will be highly relevant to consider as a function of measured node signals as predictor inputs, as well as of external excitation signals present. This includes considering experiment design problems to increase the informativity of the measured data for the particular identification objective. For an analysis of related aspects in open-loop see e.g. Gevers et al. (2006).

The approach presented in this chapter can be extended to the situation where all measured node variables are measured with sensor noise. The resulting errors-in-variables problem can be handled in a dynamic network setting, as will be shown in the next chapter

5.8 SUMMARY

In this chapter, identification in dynamics networks has been investigated. In a dynamic network, unlike in open or closed loop systems, there are many options as to which variables to include as predictor inputs. It has been shown that when identifying in networks, the obtained estimates are consequences of the (chosen) set of predictor inputs. In particular, the obtained estimates are estimates of the dynamics defined by the immersed network. Conditions on the predictor inputs have been presented such that it is possible to obtain consistent estimates of a module embedded in a dynamic network using either the Direct or Two Stage methods of identification. These conditions are useful since they enable the user to design a least expensive sensor placement scheme or check if it is possible to avoid using particular variables in the identification experiment for instance. Moreover, it is shown that efficient algorithms exist, using separating sets, that can be used to find a set of predictor inputs that satisfies the conditions.

5.9 APPENDIX

5.9.1 Proof of Proposition 5.3

The following Lemma is used in proving Proposition 5.3.

Lemma 5.37 *Let $\mathcal{G}$ be a $n \times m$ matrix of transfer functions, with $n \leq m$. Suppose all principal minors of $\mathcal{G}$ are non-zero. The matrix $\mathcal{G}$ can be uniquely factored as $(I - G)^{-1}F$, where G and F have the structure defined in (5.8).*

Proof: The proof will proceed by using matrix operations to factor $\mathcal{G}$ into the form $(I-G)^{-1}F$, where G and F have the required form. Every matrix used in the proof is full rank, and consequently the operations are unique.

Partition $\mathcal{G}$ as $[\mathcal{G}_s \ \mathcal{G}_r]$ where $\mathcal{G}_s$ is square. Start by factoring out $\mathcal{G}_s$. Since, by assumption the principal minors of $\mathcal{G}$ are non-zero, this matrix is full rank.

$$\mathcal{G} = \mathcal{G}_s \begin{bmatrix} I & \mathcal{G}_s^{-1}\mathcal{G}_r \end{bmatrix}. \tag{5.18}$$

Let $\mathcal{W}_s^{-1} = M$. Then (5.18) can be expressed as:

$$\mathcal{G} = M^{-1} \begin{bmatrix} I & \mathcal{G}_s^{-1}\mathcal{G}_r \end{bmatrix}$$

Let D be a matrix of the diagonal entries of M. Since each diagonal entry is a principal minor, it follows that D is full rank. Removing the diagonal entries of M (so that it matches the structure of G) results in:

$$\begin{aligned} \mathcal{G} &= \big(D - (D - M)\big)^{-1} \begin{bmatrix} I & \mathcal{G}_s^{-1}\mathcal{G}_r \end{bmatrix} \\ &= \big(I - D^{-1}(D - M)\big)^{-1} D^{-1} \begin{bmatrix} I & \mathcal{G}_s^{-1}\mathcal{G}_r \end{bmatrix} \\ &= \big(I - D^{-1}(D - M)\big)^{-1} \begin{bmatrix} D^{-1} & D^{-1}\mathcal{G}_s^{-1}\mathcal{G}_r \end{bmatrix}. \end{aligned} \tag{5.19}$$

The main point is that $D^{-1}(D - M)$ has the required structure of G, and

$$\begin{bmatrix} D^{-1} & D^{-1}\mathcal{G}_s^{-1}\mathcal{G}_r \end{bmatrix}$$

has the required structure of F. This concludes the proof.
Now follows the proof of Proposition 5.3.

Proof: Any network can be expressed as

$$\begin{bmatrix} w_j(t) \\ w_{\mathcal{D}}(t) \end{bmatrix} = \mathcal{G}^0(q) \begin{bmatrix} r_j(t) + v_j(t) \\ r_{\mathcal{D}}(t) + v_{\mathcal{D}}(t) \\ r_{\mathcal{Z}}(t) + v_{\mathcal{Z}}(t) \end{bmatrix}.$$

Because the network is well posed, the principal minors of $\mathcal{G}$ are all non-zero. Thus, by Lemma 5.37, $\mathcal{G}$ can be uniquely factored into $\breve{G}^0$ and $\breve{F}^0$ with the structure (5.8).

If there is an index ℓ such that both v_ℓ and r_ℓ are not present, then setting the corresponding column of $\breve{F}^0$ to zero has no effect on in the validity of (5.7) with respect to the signals.

5.9.2 Proof of Proposition 5.8

Proof: The proof proceeds by showing that Algorithm 5.5 results in matrices $\breve{G}^0$ and $\breve{F}^0$ of the form in Proposition 5.3.

In Step 2c of Algorithm 5.5 no path starting from v_k (or r_k), $k \in \mathcal{D}_j$ is ever lifted. Moreover, in the framework considered in this chapter, in the original network, v_k, $k \in \mathcal{V}$ (or r_k, $k \in \mathcal{R}$) only has a path to w_k. It follows that in the immersed network, v_k (or r_k), $k \in \mathcal{D}_j$ only has a path to w_k. Thus, all the off-diagonal entries of the leading square matrix of $\breve{F}^{i^0}$ are zero, which shows that the form of $\breve{F}^{i^0}$ is the same as that of $\breve{F}^0$.

In Step 3 of the algorithm all self-loops are removed. Thus the diagonal entries of $\breve{G}^{i^0}$ are set to zero. This shows that $\breve{G}^{i^0}$ and $\breve{G}^0$ have the same form.

By the uniqueness result of Proposition 5.3 it follows that $\breve{F}^{i^0} = \breve{F}^0$ and $\breve{G}^0 = \breve{G}^{i^0}$

5.9.3 Proof of Proposition 5.9

Proof: The proof proceeds by starting with the original network (5.1) and removing the internal variables w_k, $k \in \mathcal{Z}_j$ from the equations. The proofs proceeds at a signal level. At the end of the proof, matrices $\breve{G}^0$ and $\breve{F}^0$ are obtained of the form required by Proposition 5.3. Consequently, uniqueness of the matrices is ensured.

Given a network of the form (5.2), the variables $w_{\mathcal{Z}}$ must be removed from the equation. This is done by expressing $w_{\mathcal{Z}}$ in terms of w_k, $k \in \{j\} \cup \mathcal{D}_j$, v_k, $k \in \mathcal{Z}_j$, and r_k, $k \in \mathcal{Z}_j$:

$$\begin{aligned} w_{\mathcal{Z}} &= G^0_{\mathcal{Z}j} w_j + G^0_{\mathcal{Z}\mathcal{D}} w_{\mathcal{D}} + G_{\mathcal{Z}\mathcal{Z}} w_{\mathcal{Z}} + v_{\mathcal{Z}} + r_{\mathcal{Z}} \\ &= (I - G_{\mathcal{Z}\mathcal{Z}})^{-1}(G_{\mathcal{Z}j} w_j + G_{\mathcal{Z}\mathcal{D}} w_{\mathcal{D}} + v_{\mathcal{Z}} + r_{\mathcal{Z}}). \end{aligned} \tag{5.20}$$

where the inverse exists by Assumption 2.19. In order to eliminate $w_{\mathcal{Z}}$ from the expression of $[w_j \ w_{\mathcal{D}}]$, first express $[w_j \ w_{\mathcal{D}}]$ in terms of $w_{\mathcal{Z}}$, and then substitute in (5.20):

$$\begin{aligned} \begin{bmatrix} w_j \\ w_{\mathcal{D}} \end{bmatrix} &= \begin{bmatrix} 0 & G_{j\mathcal{D}} \\ G_{\mathcal{D}j} & G_{\mathcal{D}\mathcal{D}} \end{bmatrix} \begin{bmatrix} w_j \\ w_{\mathcal{D}} \end{bmatrix} + \begin{bmatrix} G_{j\mathcal{Z}} \\ G_{\mathcal{D}\mathcal{Z}} \end{bmatrix} w_{\mathcal{Z}} + \begin{bmatrix} v_j \\ v_{\mathcal{D}} \end{bmatrix} + \begin{bmatrix} r_j \\ r_{\mathcal{D}} \end{bmatrix} \\ &= \begin{bmatrix} 0 & G_{j\mathcal{D}} \\ G_{\mathcal{D}j} & G_{\mathcal{D}\mathcal{D}} \end{bmatrix} \begin{bmatrix} w_j \\ w_{\mathcal{D}} \end{bmatrix} + \begin{bmatrix} G_{j\mathcal{Z}} \\ G_{\mathcal{D}\mathcal{Z}} \end{bmatrix} (I - G_{\mathcal{Z}\mathcal{Z}})^{-1} \begin{bmatrix} G_{\mathcal{Z}j} & G_{\mathcal{Z}\mathcal{D}} \end{bmatrix} \begin{bmatrix} w_j \\ w_{\mathcal{D}} \end{bmatrix} \\ &\quad + \begin{bmatrix} G_{j\mathcal{Z}} \\ G_{\mathcal{D}\mathcal{Z}} \end{bmatrix} (I - G_{\mathcal{Z}\mathcal{Z}})^{-1}(r_{\mathcal{Z}} + v_{\mathcal{Z}}) + \begin{bmatrix} v_j \\ v_{\mathcal{D}} \end{bmatrix} + \begin{bmatrix} r_j \\ r_{\mathcal{D}} \end{bmatrix}. \end{aligned} \tag{5.21}$$

Collect all the v's and r's into a single vector:

$$\begin{aligned} \begin{bmatrix} w_j \\ w_{\mathcal{D}} \end{bmatrix} &= \left(\begin{bmatrix} 0 & G_{j\mathcal{D}} \\ G_{\mathcal{D}j} & G_{\mathcal{D}\mathcal{D}} \end{bmatrix} + \begin{bmatrix} G_{j\mathcal{Z}} \\ G_{\mathcal{D}\mathcal{Z}} \end{bmatrix} (I - G_{\mathcal{Z}\mathcal{Z}})^{-1} \begin{bmatrix} G_{\mathcal{Z}j} & G_{\mathcal{Z}\mathcal{D}} \end{bmatrix} \right) \begin{bmatrix} w_j \\ w_{\mathcal{D}} \end{bmatrix} \\ &\quad + \begin{bmatrix} 1 & 0 & G_{j\mathcal{Z}}(I - G_{\mathcal{Z}\mathcal{Z}})^{-1} \\ 0 & I & G_{\mathcal{D}\mathcal{Z}}(I - G_{\mathcal{Z}\mathcal{Z}})^{-1} \end{bmatrix} \begin{bmatrix} r_j + v_j \\ r_{\mathcal{D}} + v_{\mathcal{D}} \\ r_{\mathcal{Z}} + v_{\mathcal{Z}} \end{bmatrix}. \end{aligned}$$

From the statement of the Proposition, the matrix preceding $[w_j \ w_{\mathcal{D}}]^T$ is $\tilde{G}^0$, and the matrix preceding the r and v terms is $\tilde{F}^0$. To put the matrices $\tilde{G}^0$ and $\tilde{F}^0$ into the form required by Proposition 5.3, the diagonals of $\tilde{G}^0$ must be removed. Let D denote the diagonal entries of $\tilde{G}^0$:

$$\begin{aligned} \begin{bmatrix} w_j \\ w_{\mathcal{D}} \end{bmatrix} &= \tilde{G}^0 \begin{bmatrix} w_j \\ w_{\mathcal{D}} \end{bmatrix} + \tilde{F}^0 \begin{bmatrix} r_j + v_j \\ r_{\mathcal{D}} + v_{\mathcal{D}} \\ r_{\mathcal{Z}} + v_{\mathcal{Z}} \end{bmatrix} \\ &= (I - D)^{-1}(\tilde{G}^0 - D) \begin{bmatrix} w_j \\ w_{\mathcal{D}} \end{bmatrix} + (I - D)^{-1} \tilde{F}^0 \begin{bmatrix} r_j + v_j \\ r_{\mathcal{D}} + v_{\mathcal{D}} \\ r_{\mathcal{Z}} + v_{\mathcal{Z}} \end{bmatrix}. \end{aligned} \tag{5.22}$$

Both matrices in (5.22 have the same form as $\breve{G}^0$, and $\breve{F}^0$ in (5.7). Thus, by Proposition 5.3, they are equal to $\breve{G}^0$ and $\breve{F}^0$.

5.9.4 Proof of Lemma 5.10

Proof: Consider part 1. From Proposition 5.9, the transfer function of the (m,n)th entry of $\breve{G}^0$ (where $m \neq n$) is

$$\breve{G}^0_{mn} = \frac{1}{1-\tilde{G}^0_{mm}}\Big(G^0_{mn} + \sum_{\ell_1 \in \mathcal{Z}_j}\sum_{\ell_2 \in \mathcal{Z}_j} G^0_{m\ell_1}\mathcal{G}^z_{\ell_1\ell_2}G^0_{\ell_2 n}\Big) \tag{5.23}$$

where $\mathcal{G}^z_{\ell_1\ell_2}$ denotes the (ℓ_2,ℓ_1) entry of $(I - G^0_{zz})^{-1}$. By Lemma 4.1 if every path from ℓ_2 to ℓ_1 passing only through nodes w_k, $k \in \mathcal{Z}_j$ has a delay then $\mathcal{G}^z_{\ell_1\ell_2}$ has a delay. Thus, if every path from w_{k_1} to w_{k_2} that passes only through nodes w_k, $k \in \mathcal{Z}_j$ has a delay, either $G^0_{m\ell_1}$, $\mathcal{G}^z_{\ell_1\ell_2}$, or $G^0_{\ell_2 n}$ has a delay (for every ℓ_1 and ℓ_2). By (5.23) the statement of the lemma follows.

To show that $\breve{G}^0_{mn} = 0$ when there is no path from w_m to w_n that passes through only nodes w_k, $k \in \mathcal{Z}_j$ follows the same reasoning, as does part 2 of the Lemma.

5.9.5 Proof of Proposition 5.14

Proof: From Algorithm 5.5 there are two ways that the transfer function $\breve{G}^0_{ji}$ can change to be different from G^0_{ji}: in Steps 2c and 3. Using the same notation as that in Algorithm 5.5, the proof will proceed by showing that Conditions (b) and (c) ensure that no change to $\breve{G}^{(k)}_{ji}$ occurs for all $k = 1:d$ in Steps 2c and 3 respectively.

Start by investigating Step 2c. A change to $G^{(k)}_{ji}$ occurs if a path has been lifted in Step 2a and resulted in an edge from w_i to w_j. By Condition (b) every path from w_i to w_j passes through a node w_n, $n \in \mathcal{D}_j$. Consequently, it will never occur at any iteration k that a node w_n is being removed that has an incoming edge from w_i and an outgoing edge to w_j. Thus, there will never be parallel edges generated from w_i to w_j that must be merged in Step 2c.

Similarly, by Condition (c) every path from w_j to w_j passes through a node w_n, $n \in \mathcal{D}_j$. Consequently, it will never occur at any iteration k of the algorithm that a node w_n is being removed that has an incoming edge from w_j and an outgoing edge to w_j. Thus there is never a self loop from w_j to w_j generated. Which means that the division in Step 3 will simply be a division by 1.

5.9.6 Proof of Proposition 5.15

The following Lemma will be used to prove Proposition 5.15.

Lemma 5.38 *Consider a vector of rational functions*

$$\Delta X(q,\theta) = [\Delta X_1(q,\theta_1) \ \cdots \ \Delta X_d(q,\theta_d)]^T$$

where $\Delta X_k(q,\theta_k) = L_k(q,\theta_k)(X^0_k(q) - X_k(q,\theta_k))$, *where* L_k *is a monic transfer function,* X^0_k *is a transfer function and* $X_k(\theta_k)$ *is a transfer function parameterized as:*

$$X_k(\theta_k) = \frac{b^k_0 + b^k_1 q^{-1} + \cdots + b^k_{n_b} q^{-n_b}}{1 + a^k_1 q^{-1} + \cdots + a^k_{n_a} q^{-n_a}},$$

where $\theta_k = [b_0^k \ \cdots \ b_{n_b}^k \ a_1^k \ \cdots \ a_{n_a}^k]^T$. *Suppose the parameterization is chosen such that for each* $\Delta X_k(\theta_k)$, *there exists a parameter vector* θ^* *such that* $\Delta X(\theta^*) = 0$. *Consider a* $(d \times d)$ *power spectral density matrix* Φ. *If* Φ *is positive definite for at least* $n_\theta = n_a + n_b + 1$ *frequencies* ω_n, *where* $-\pi < \omega_n \leq \pi$ *then*

$$\int_{-\pi}^{\pi} \Delta X(e^{j\omega}, \theta)^T \Phi(\omega) \Delta X(e^{-j\omega}, \theta) d\omega = 0 \implies \Delta X_k(q, \theta) = 0$$

for $k = 1, \ldots, d$.

Proof: Consider the expression for the kth entry of $\Delta X(q, \theta)$:

$$\begin{aligned}
\Delta X_k(q,\theta) &= L_k(q,\theta)\Big(\frac{B_k^0(q)}{A_k^0(q)} - \frac{B_k(q,\theta)}{A_k(q,\theta)}\Big) \\
&= L_k(q,\theta)\frac{B_k^0(q)A_k(q,\theta) - B_k(q,\theta)A_k^0(q)}{A_k^0(q)A_k(q,\theta)} \\
&= \Delta P_k(\theta) K_k(q,\theta)
\end{aligned}$$

where B_k^0 and $B_k(\theta)$ are polynomials in q^{-1}, A_k^0 and $A_k(\theta)$ are monic polynomials in q^{-1}, and

$$\begin{aligned}
\Delta P_k(q,\theta) &= B_k^0(q)A_k(q,\theta) - B_k(q,\theta)A_k^0(q), \\
K_k(q,\theta) &= L_k(q,\theta)\frac{1}{A_k^0(q)A_k(q,\theta)}.
\end{aligned}$$

Note that $\Delta P_k(q, \theta)$ is a polynomial of order n_θ.

The implication in the statement of the lemma can now be expressed as:

$$\int_{-\pi}^{\pi} \Delta P(e^{j\omega}, \theta) K(e^{j\omega}, \theta) \Phi(\omega) K^H(e^{j\omega}, \theta) \Delta P^H(e^{j\omega}, \theta) \mathrm{d}\omega = 0,$$

where $(\cdot)^H$ denotes conjugate transpose,

$$\Delta P = [\Delta P_1 \ \cdots \ \Delta P_d] \text{ and } K = \mathrm{diag}\big(K_1, \ldots, K_d\big).$$

Since the term in the integral is nonnegative for all θ, the only way that the integral can equal zero is if the term in the integral is zero for every omega, i.e.:

$$\Delta P(e^{j\omega}, \theta) K(e^{j\omega}, \theta) \Phi(\omega) K^T(e^{-j\omega}, \theta) \Delta P^T(e^{-j\omega}, \theta) = 0$$

for all $\omega \in [-\pi, \pi)$. Because Φ is positive definite at n_θ frequencies, and $K(\theta)$ is full rank for all ω and θ (since K_k is a monic transfer function for $k = 1, \ldots, d$), it follows that $K(e^{j\omega}, \theta)\Phi(\omega)K^T(e^{-j\omega}, \theta)$ is positive definite at n_θ frequencies. However, $\Delta P(q)$ is a vector of polynomials of degree n_θ. By Property 6 in Section 5.4 of Söderström & Stoica (1989a) this implies that $\Delta P(q) = 0$. From the definition of $\Delta P(\theta, \theta)$ this implies that $X_k(\theta) = X_k^0$ for all $k = 1, \ldots, d$.

The proof of Proposition 5.15 now proceeds:

Proof: First consider Case 1 (Conditions (a) and (b) hold for all η). Since the noise model is independently parameterized from the module models, let η denote

the parameters associated with the noise model, and let θ denote the parameters associated with the modules.

From (5.11) and (5.5) the asymptotic objective function is

$$\bar{V}(\theta,\eta)=\bar{\mathbb{E}}\Big[\Big(H_j^{-1}(q,\eta)\Big(w_j(t)-\sum_{k\in\mathcal{D}_j}G_{jk}(q,\theta)w_k^{(\mathcal{X})}(t)-\sum_{k\in\mathcal{P}_j}F_{jk}(q,\theta)r_k(t)\Big)\Big)^2\Big]$$

By (5.14) and (5.15) w_j can be expressed in terms of w_k, $k\in\mathcal{D}_j$, r_k, $k\in\mathcal{P}_j$ and a residual, $p_j(t,\mathcal{D}_j)$ resulting in:

$$\bar{V}(\theta,\eta)=\bar{\mathbb{E}}\Big[\Big(H_j^{-1}(q,\eta)\Big(\sum_{k\in\mathcal{D}_j}\Delta G_{jk}(q,\theta,\mathcal{D}_j)w_k^{(\mathcal{X})}(t)+\sum_{k\in\mathcal{P}_j}\Delta F_{jk}(q,\theta,\mathcal{D}_j)r_k(t)+p_j(t,\mathcal{D}_j)\Big)\Big)^2\Big] \quad (5.24)$$

where $\Delta G_{jk}(q,\theta,\mathcal{D}_j)=\breve{G}^0_{jk}(q,\mathcal{D}_j)-G_{jk}(q,\theta)$, and $\Delta F_{jk}(q,\theta,\mathcal{D}_j)=\breve{F}^0_{jk}(q,\mathcal{D}_j)-F_{jk}(q,\theta)$. By Conditions (a) and (b), the term $H_j^{-1}(q,\eta)p_j(t,\mathcal{D}_j)$ is uncorrelated to the other two terms in (5.24), resulting in:

$$\bar{V}(\theta,\eta)=\bar{\mathbb{E}}\Big[\Big(H_j^{-1}(q,\eta)\Big(\sum_{k\in\mathcal{D}_j}\Delta G_{jk}(q,\theta,\mathcal{D}_j)w_k^{(\mathcal{X})}(t)+\sum_{k\in\mathcal{P}_j}\Delta F_{jk}(q,\theta,\mathcal{D}_j)r_k(t)\Big)\Big)^2\Big]+\bar{\mathbb{E}}\Big[\Big(H_j^{-1}(q,\eta)p_j(t,\mathcal{D}_j)\Big)^2\Big]. \quad (5.25)$$

By Parseval's theorem (5.25) can be expressed as:

$$\bar{V}(\theta,\eta)=\frac{1}{2\pi}\int_{-\pi}^{\pi}\Delta X(e^{j\omega},\theta,\eta)\Phi(\omega)\Delta X^T(e^{-j\omega},\theta,\eta)+H_j^{-1}(e^{j\omega},\eta)\Phi_{p_j}(\omega)H_j^{-1}(e^{-j\omega},\eta)\mathrm{d}\omega \quad (5.26)$$

where

$$\Delta X=H_j^{-1}\begin{bmatrix}\Delta G_{jk_1} & \cdots & \Delta G_{jk_n} & \Delta F_{jm_1} & \cdots & \Delta F_{jm_\ell}\end{bmatrix},$$
$$\Phi(\omega)=\begin{bmatrix}\Phi_{w_\mathcal{D}}(\omega) & \Phi_{w_\mathcal{D}r_\mathcal{P}}(\omega)\\ \Phi_{r_\mathcal{P}w_\mathcal{D}}(\omega) & \Phi_{r_\mathcal{P}}(\omega)\end{bmatrix},$$

where $k_*\in\mathcal{D}_j$, $m_*\in\mathcal{P}_j$ and $\Phi_{**}(\omega)$ are the (cross) power spectral densities of the denoted variables.

Note that second first term of (5.26) is a function of only η. Moreover both terms are positive for any θ and η. Since the parameterization is chosen flexible enough, the first term can equal 0 for particular choices of θ.

By assumption the power spectral density of $[w_j\ w_{k_1}\ \cdots\ w_{k_n}\ r_{m_1}\ \cdots\ r_{m_\ell}]$ is positive definite for a sufficiently large number of frequencies. Since p_j is uncorrelated to the predictor inputs, this implies that Φ in (5.26) is positive definite for a sufficiently large number of frequencies. Thus, all the conditions of Lemma 5.38

hold, and consequently when the first term in (5.26) is zero, this implies that ΔX in (5.26) is zero. It follows that at the minimum of $\bar{V}$, $G_{jk}(q,\theta^*) = \breve{G}_{jk}(q,\mathcal{D}_j)$, $k \in \mathcal{D}_j$, $F_{jk}(q,\theta^*) = \breve{G}_{jk}(q,\mathcal{D}_j)$, $k \in \mathcal{P}_j$, as desired. This concludes the proof for Case 1.

Consider now the proof for Case 2 (Conditions (a) and (b) hold for only η^* and $H_j^{-1}(\eta^*)p_j$ is white). Since the noise model is independently parameterized from the module models, let η denote the parameters associated with the noise model, and let θ denote the parameters associated with the modules.

For notational simplicity, let $H_j^{-1}(q,\eta^*)p_j(t,\mathcal{D}_j)$ be denoted as $s_j(t,\mathcal{D}_j)$. The reasoning will be split into two steps:

1. Show that if Conditions (a) and (b) hold at η^*, then the following bound on the objective function holds:

$$\bar{V}(\theta) \geq \bar{\mathbb{E}}\big[\big(H_j^{-1}(q,\eta^*)p_j(t,\mathcal{D}_j)\big)^2\big]. \tag{5.27}$$

2. Show that when equality holds it implies that $G_{jk}(q,\theta) = \breve{G}^0_{jk}(q,\mathcal{D}_j)$, $k \in \mathcal{D}_j$, and $F_{jk}(q,\theta) = \breve{F}^0_{jk}(q,\mathcal{D}_j)$, $k \in \mathcal{P}_j$.

Step 1. If (5.24) is evaluated at η^*, the third term is equal to s_j.

By Conditions (a) and (b) s_j is uncorrelated to the first two terms in the expression. Moreover, since s_j is white, it is also uncorrelated to delayed versions of itself which means that $\mathbb{E}[\Delta H_j(q,\eta)s_j(t)\cdot s_j(t)] = 0$ where $\Delta H_j(q,\eta,\mathcal{D}_j) = H_j(q,\eta^*) - H_j(q,\eta)$ (the expression holds since H_j is monic, and thus ΔH_j has a delay).

Using this fact to simplify (5.24) results in

$$\bar{V}(\theta,\eta) = \bar{\mathbb{E}}\Big[s_j^2(\mathcal{D}_j)\Big] + \bar{\mathbb{E}}\Big[H_j^{-1}(\eta)\Big(\sum_{k\in\mathcal{D}_j}\Delta G_{jk}(\theta,\mathcal{D}_j)w_k^{(x)} + \sum_{k\in\mathcal{P}_j}\Delta F_{jk}(\theta,\mathcal{D}_j)r_k + \Delta H_j(\eta,\mathcal{D}_j)s_j(\mathcal{D}_j)\Big)^2\Big]. \tag{5.28}$$

The first term of $\bar{V}(\theta,\eta)$ is not a function of θ or η, proving that $\bar{V}(\theta,\eta) \geq \bar{\mathbb{E}}\Big[s_j^2(t,\mathcal{D}_j)\Big]$ as desired.

Step 2. Now it is shown that

$$\bar{V}(\theta,\eta) = \bar{\mathbb{E}}\Big[s_j^2(t,\mathcal{D}_j)\Big] \Rightarrow \begin{cases} G_{jk}(q,\theta) = \breve{G}^0_{jk}(q,\mathcal{D}_j), k \in \mathcal{D}_j \\ F_{jk}(q,\theta) = \breve{F}^0_{jk}(q,\mathcal{D}_j), k \in \mathcal{P}_j \\ H_j(q,\eta) = H_j(q,\eta^*) \end{cases}$$

Consider the equation $\bar{V}(\theta,\eta) = \bar{\mathbb{E}}\Big[s_j^2(t,\mathcal{D}_j)\Big]$. From (5.28) using Parseval's theorem, this results in:

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\Delta X(e^{j\omega},\theta)\Phi(\omega)\Delta X^T(e^{-j\omega},\theta)\mathrm{d}\omega = 0, \tag{5.29}$$

for all $\omega \in [-\pi, \pi]$, where

$$\Delta X = H_j^{-1}\big[\Delta G_{jk_1} \;\; \cdots \;\; \Delta G_{jk_n} \;\; \Delta F_{jm_1} \;\; \cdots \;\; \Delta F_{jm_\ell} \;\; \Delta H_j\big],$$
$$\Phi(\omega) = \begin{bmatrix} \Phi_{w_{\mathcal{D}}}(\omega) & \Phi_{w_{\mathcal{D}} r_{\mathcal{P}}}(\omega) & \Phi_{w_{\mathcal{D}} s_j}(\omega) \\ \Phi_{r_{\mathcal{P}} w_{\mathcal{D}}}(\omega) & \Phi_{r_{\mathcal{P}}}(\omega) & \Phi_{r_{\mathcal{P}} s_j}(\omega) \\ \Phi_{s_j w_{\mathcal{D}}}(\omega) & \Phi_{s_j r_{\mathcal{P}}}(\omega) & \Phi_{s_j}(\omega) \end{bmatrix}, \tag{5.30}$$

where $k_* \in \mathcal{D}_j$, $m_* \in \mathcal{P}_j$ and $\Phi_{**}(\omega)$ are the (cross) power spectral densities of the denoted variables. Recall from (5.15) that w_j can be expressed in terms of $w_k^{(\mathcal{X})}$, $k \in \mathcal{D}_j$, r_k, $k \in \mathcal{P}_j$ and p_j. By rearanging (5.15) an expression for s_j is

$$s_j = H_j^{0^{-1}}\Big(w_j - \sum_{k\in\mathcal{D}_j} \breve{G}^0_{jk} w_k^{(\mathcal{X})} - \sum_{k\in\mathcal{P}_j} \breve{F}^0_{jk} r_k\Big)$$

Consequently, (5.30) can be expressed as $J\Phi_w J^H$, where

$$J = \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ -\breve{G}^0_{j\mathcal{D}} & -\breve{F}^0_{j\mathcal{D}} & 1 \end{bmatrix},$$

Φ_w is the power spectral density of $[w_{k_1} \;\cdots\; w_{k_n} \; r_{m_1} \;\cdots\; r_{m_\ell} \; w_j]$, and $(\cdot)^H$ denotes conjugate transpose. Because J is full rank for all ω, and Φ_w is full rank for at least n_θ frequencies (by the statement of the theorem) it follows that Φ in (5.29) is full rank for at least n_θ frequencies. Because $\Phi(\omega)$ is positive definite for at least n_θ frequencies, and the parameterization is chosen flexible enough, it follows from Lemma 5.38 that $\Delta X = 0$. By the definition of ΔX it follows that (5.29) implies $G_{jk}(q,\theta^*) = \breve{G}_{jk}(q,\mathcal{D}_j)$, $k \in \mathcal{D}_j$, $F_{jk}(q,\theta^*) = \breve{G}_{jk}(q,\mathcal{D}_j)$, $k \in \mathcal{P}_j$, and $H_j(q,\theta^*) = H_j(q,\eta^*)$ as desired.

5.9.7 Proof of Proposition 5.16

Proof: The proof proceeds by showing that Conditions (a) and (b) hold at η^*, and that $H_j^{-1}(\eta^*)p_j$ is white noise. By Condition (d), p_j is not a function of any r terms, and thus from (5.14) it follows that $p_j = \breve{v}_j$. Recall from (5.7) that the equation defining the immersed network is $w = \breve{G}^0 w + \breve{F}^0 r + \breve{v}$ where $w = [w_j \; w_{\mathcal{D}}]^T$, $r = [r_j \; r_{\mathcal{D}} \; r_{\mathcal{Z}}]^T$ and $\breve{v}$ is defined in (5.9). Consequently, w_k can be expressed as

$$w_k = \breve{\mathcal{G}}^0_{kj}(\breve{v}_j + r_j + \breve{F}^0_{j\mathcal{Z}} r_{\mathcal{Z}}) + \sum_{n\in\mathcal{D}_j} \breve{\mathcal{G}}^0_{kn}(\breve{v}_n + r_n + \breve{F}^0_{n\mathcal{Z}} r_{\mathcal{Z}})$$

where $\breve{\mathcal{G}}^0_{jk}$ denotes the (j,k) entry of $(I - \breve{G}^0)^{-1}$. Using this expression for w_k, Condition (a) of Proposition 5.15 can be expressed as:

$$\bar{\mathbb{E}}[H_j^{-1}(q,\eta^*)p_j(t)\cdot\Delta G_{jk}(q,\mathcal{D}_j,\theta)w_k(t)] = \bar{\mathbb{E}}\Big[H_j^{-1}(q,\eta^*)\breve{v}_j(t)\cdot \\ \cdot\,\Delta G_{jk}(q,\mathcal{D}_j,\theta)\sum_{n\in\mathcal{D}_j\cup\{j\}}\breve{\mathcal{G}}^0_{kn}(q)\big(\breve{v}_n(t) + r_n(t) + \breve{F}^0_{n\mathcal{Z}}(q)r_{\mathcal{Z}}(t)\big)\Big].$$

By Assumption 2.19 every v_k is uncorrelated to every external variable. Moreover, by Condition (a) $\breve{v}_j$ is uncorrelated to the other noise terms in the immersed network, and so the above equation can be simplified:

$$\bar{\mathbb{E}}[H_j^{-1}(q,\eta^*)p_j(t)\cdot\Delta G_{jk}(q,\mathcal{D}_j,\theta)w_k(t)] \\ = \bar{\mathbb{E}}[H_j^{-1}(q,\eta^*)\breve{v}_j(t)\cdot\Delta G_{jk}(q,\mathcal{D}_j,\theta)\breve{\mathcal{G}}^0_{kj}(q)\breve{v}_j(t)] \quad (5.31)$$

By Lemma 4.1 (in Appendix 5.9.4) the transfer function $\breve{\mathcal{G}}^0_{kj}$ has a delay if every path (in the immersed network) from w_j to w_k has a delay. It follows by Condition (b) that either $\breve{\mathcal{G}}^0_{kj}$ or $\breve{G}^0_{jk}$ (or both) has a delay. By Condition (c) it follows that either $\breve{\mathcal{G}}^0_{kj}$ or $\Delta G_{jk}(q,\mathcal{D}_j,\theta)$ (or both) has a delay. The result is that $\Delta G_{jk}(q,\mathcal{D}_j,\theta)\breve{\mathcal{G}}^0_{kj}\breve{v}_j$ is a function of only delayed versions of $\breve{v}_j$ (and thus delayed versions of $\breve{e}_j$, where $\breve{e}_j$ is the whitened version of $\breve{v}_j$ as defined in (5.10)). Thus it follows that

$$\bar{\mathbb{E}}[H_j^{-1}(q,\eta^*)p_j(t)\cdot\Delta G_{jk}(q,\mathcal{D}_j,\theta)w_k(t)] = \bar{\mathbb{E}}[\breve{e}_j(t)\cdot\Delta G_{jk}(q,\mathcal{D}_j,\theta)\breve{\mathcal{G}}^0_{kj}(q)\breve{v}_j(t)] = 0$$

which means that the Condition (a) of Proposition 5.15 holds.

Since $p_j = \breve{v}_j$, and by Assumption 2.19, all v's are uncorrelated to all r, it follows that Condition (b) holds as well.

5.9.8 Proof of Proposition 5.19

Proof: The following reasoning will show that $\bar{\mathbb{E}}[\breve{v}_j(t)\cdot\breve{v}_k(t-\tau)] = 0$ for all τ. From (5.9),

$$\bar{\mathbb{E}}[\breve{v}_j(t)\cdot\breve{v}_k(t-\tau)] = \bar{\mathbb{E}}[\big(v_j(t)+\breve{F}^0_{j\mathcal{Z}}(q)v_{\mathcal{Z}}(t)\big)\cdot\big(v_k(t-\tau)+\breve{F}^0_{k\mathcal{Z}}(q)v_{\mathcal{Z}}(t-\tau)\big)]. \quad (5.32)$$

Consider the following three facts. First, by Condition (a), v_j is uncorrelated to all v_k, $k\in\mathcal{D}_j$. Secondly,

$$\bar{\mathbb{E}}[v_j(t)\cdot\breve{F}^0_{kn}(q)v_n(t-\tau)] = 0, \forall\tau, \text{ and } \forall n\in\mathcal{Z}_j \quad (5.33)$$

by the following reasoning. Either one of the conditions holds:

- There is a path from v_n, $n\in\mathcal{Z}_j$ to w_k that passes only through nodes w_k, $k\in\mathcal{Z}_j$. In this case, by Condition (a) v_j is uncorrelated to v_n.
- There is no path from v_n, $n\in\mathcal{Z}_j$ to w_k. In this case, by Lemma 5.10, $\breve{F}^0_{kn}$ is zero. Consequently, $\bar{\mathbb{E}}[v_j(t)\cdot\breve{F}_{kn}(q)v_n(t)] = 0$.

Thirdly, by the same reasoning and by Condition (b), $\bar{\mathbb{E}}[v_k(t)\cdot F_{j\mathcal{Z}}(q)v_{\mathcal{Z}}(t-\tau)] = 0$ for all τ. Consequently, (5.32) can be simplified to:

$$\bar{\mathbb{E}}[\breve{v}_j(t)\breve{v}_k(t-\tau)] = \bar{\mathbb{E}}[\breve{F}^0_{j\mathcal{Z}}(q)v_{\mathcal{Z}}(t)\cdot\breve{F}^0_{k\mathcal{Z}}(q)v_{\mathcal{Z}}(t-\tau)].$$

By Parseval's Theorem this equation can be expressed as

$$\bar{\mathbb{E}}[\breve{v}_j(t)\breve{v}_k(t-\tau)] = \frac{1}{2\pi}\int_{-\pi}^{\pi}\breve{F}^0_{j\mathcal{Z}}(e^{j\omega})\Phi_{v_{\mathcal{Z}}}(\omega)\breve{F}^{0^T}_{k\mathcal{Z}}(e^{-j\omega})e^{j\omega\tau}\mathrm{d}\omega.$$

By Condition (c), $\Phi_{v_{\mathcal{Z}}}$ is diagonal, and so

$$\bar{\mathbb{E}}[\breve{v}_j(t)\breve{v}_k(t-\tau)]=\frac{1}{2\pi}\int_{-\pi}^{\pi}\sum_{\ell\in\mathcal{Z}_j}\breve{F}_{j\ell}(e^{j\omega})\breve{F}_{k\ell}(e^{-j\omega})e^{j\tau\omega}\phi_\ell(\omega)\mathrm{d}\omega$$

where ϕ_ℓ is the power spectral density of v_ℓ. By Lemma 5.10 the transfer function $\breve{F}^0_{jk}$ is zero if there is no path from v_k to w_j that passes only through nodes w_k, $k \in \mathcal{Z}_j$. Consequently, by Condition (d) for each ℓ, $\ell \in \mathcal{Z}_j$, either $\breve{F}^0_{j\ell}$ or $\breve{F}^0_{k\ell}$ (or both) are equal to zero. Consequently, $\bar{\mathbb{E}}[\breve{v}_j(t)\breve{v}_k(t-\tau)]$ is equal to zero for al τ, and for all $k \in \mathcal{D}_j$.

5.9.9 Proof of Proposition 5.25

Proof: The proof proceeds by showing that Case 1 of Proposition 5.15 holds. The predictor inputs $w_k^{(\mathcal{T}_j)}$, $k \in \mathcal{D}_j$ and r_k, $k \in \mathcal{P}_j$ are functions of all r_k, $k \in \{\mathcal{T}_j\}$, except those r_k for which there is no path r_k to w_j (the projection onto this external variable is zero in this case). Thus it is sufficient to show that the optimal output error residual of w_j is not correlated to these r's. From (5.14) p_j is equal to

$$\begin{aligned}p_j(t,\mathcal{D}_j) =& \breve{F}^0_{jj}(q,\mathcal{D}_j)r_j(t) + \breve{v}_j(t) + \sum_{k\in(\mathcal{Z}_j\cap\mathcal{R}_j)\setminus\mathcal{P}_j}\breve{F}^0_{jk}(q,\mathcal{D}_j)r_k(t)\\ &+ \sum_{k\in\mathcal{D}_j}\breve{G}^0_{jk}(q,\mathcal{D}_j)w_k^{(\perp\mathcal{T}_j)}(t).\end{aligned} \tag{5.34}$$

By Assumption 2.19 all r's are uncorrelated to all v's. Thus, only the r terms in p_j could cause a correlation between p_j and the predictor inputs. In particular, it must be shown that p_j is not a function of any r_k, $k \in \mathcal{T}_j$.

Split the variables in $\mathcal{T}_j$ into two categories: the r_k's for which at least one path from r_k to w_j passes only through nodes in $\mathcal{Z}_j$, and the r_k's for which all paths from r_k to w_j pass through at least one node w_k, $k \in \mathcal{D}_j$. By construction, all r_k's that are in the first category are in $\mathcal{P}_j$. Since no variable $r_k \in \mathcal{P}_j$ appears in p_j (see (5.34) none of the variables in the first category appear in the expression for p_j.

By Lemma 5.10 it follows that for all r_k in the second category $\breve{F}^0_{jk}$ is zero. Thus, from (5.34) it follows that no r_k term in the first category will appear in the expression for p_j.

Thus, p_j is not a function of any r_k, $k \in \mathcal{T}_j$. Consequently, p_j is uncorrelated to the predictor inputs, and the conditions of Proposition 5.15 are satisfied.

Lastly, to satisfy all the conditions of Proposition 5.15 we must show that the power spectral density Φ of $[w_j\ w_{k_1}^{(\mathcal{T}_j)}\ \cdots\ w_{k_n}^{(\mathcal{T}_j)}\ r_{m_1}\ \cdots\ r_{m_\ell}]$ is positive definite for at least n_θ frequencies. By (5.15) p_j can be expressed as a function of $w_k^{(\mathcal{T}_j)}$), $k \in \mathcal{D}_j$, and r_k, $k \in \mathcal{P}_j$ and p_j. It has allready been shown that p_j is uncorrelated to all the predictor inputs. Consequently, the power spectral density Φ is equal to

$$\Phi = \begin{bmatrix}1 & [-\breve{G}^0_{j\mathcal{D}} & -\breve{F}^0_{j\mathcal{D}}]\\ 0 & & I\end{bmatrix}\begin{bmatrix}\phi_p & 0\\ 0 & \Phi_w\end{bmatrix}\begin{bmatrix}1 & [-\breve{G}^0_{j\mathcal{D}} & -\breve{F}^0_{j\mathcal{D}}]\\ 0 & & I\end{bmatrix}^H$$

where ϕ_p is the power spectral density of p_j and Φ_w is the power spectral density of

$$[w_{k_1}^{(\mathcal{T}_j)} \ \cdots \ w_{k_n}^{(\mathcal{T}_j)} \ r_{m_1} \ \cdots \ r_{m_\ell}]^T$$

(which is positive definite at n_θ frequencies). Because the first (and last) matrices are full rank for all ω it follows that Φ is full rank for at least n_θ frequencies. Consequently all the conditions of Case 1 of Proposition 5.15 are satisfied.

5.9.10 Proof of Remark 5.27

Proposition 5.39 *Consider a dynamic network that satisfies Assumption 2.19. Consider the situation of Proposition 5.25. Suppose that Conditions (a) to (c) are satisfied. Consistent estimates of $\breve{G}_{ji}^0$ are obtained using the Two-Stage Method, with w_j as a predictor input (i.e. j is an element of $\mathcal{D}_j$) if and only if r_j is present*

Proof: (Sufficiency). We show that if r_j is present, then $V_j(\theta)$ has a unique minimum. Let $\{w_k\}$, $k \in \mathcal{D}_j$ and $\{r_k\}$, $k \in \mathcal{P}_j$ denote the sets of internal and external variables respectively that are chosen as predictor inputs. Let $\{r_k\}$ $k \in \mathcal{T}_j$ denote the set of external variables that are projected onto. Suppose that $j \in \mathcal{D}_j$, i.e. w_j is selected as a predictor input. Then the expression for the identification criterion is:

$$\begin{aligned}\bar{V}_j(\theta) &= \bar{E}[\varepsilon_j^2(t,\theta)] \\ &= \bar{E}\Big[\Big(w_j - \sum_{k\in\mathcal{D}_j} G_{jk}(q,\theta) w_k^{(\mathcal{T}_j)} - \sum_{k\in\mathcal{P}_j} F_{jk}(q,\theta) r_k - r_j\Big)^2\Big]. \end{aligned} \tag{5.35}$$

An important feature of this proposition is that there is a unique representation of w_j with $F_{jj} = 1$. Consider Algorithm 3 for the construction of the immersed network. By Propositions 1 and 2 the immersed network is unique. Consider now removing Step 3 of Algorithm 3. In Step 3 any self loops around w_j are removed. If this step is not applied then the result is an expression for w_j in terms of w_j and all the other selected variables in the immersed network. Moreover, the transfer function F_{jj} in this network is 1. Since applying Step 3 is not essential for the uniqueness result, it follows that w_j can be uniquely expressed in terms of $\{w_k\}$, $k \in \mathcal{D}_j$, where j is an element of $\mathcal{D}_j$ and $F_{jj} = 1$:

$$\begin{aligned} w_j(t) = \sum_{k\in\mathcal{D}_j\setminus\{j\}} \breve{G}_{jk}^0(q,\mathcal{D}_j) w_k(t) + \breve{G}_{jj}^0(q,\mathcal{D}_j) w_j(t) + \sum_{k\in\mathcal{P}_j} \breve{F}_{jk}^0(q,\mathcal{D}_j) r_k(t) \\ + r_j(t) + \breve{v}_j(t). \end{aligned}$$

where the $\breve{\cdot}$ notation here has been used to denote the dynamics of the immersed network with the self-loops removed. Consequently, (5.35) can be expressed as

$$\bar{V}(\theta) = \bar{E}\Big[\Big(\sum_{k\in\mathcal{D}_j} \Delta\breve{G}_{jk}(\theta) w_k^{(\mathcal{T}_j)} + \sum_{k\in\mathcal{P}_j} \Delta\breve{F}_{jk}(\theta) r_k + p_j\Big)^2\Big]$$

where p_j is the optimal output error residual:

$$p_j = \sum_{k \in \mathcal{D}_j} \breve{G}^0_{jk} w_k^{(\perp \mathcal{T}_j)} + \breve{v}_j \tag{5.36}$$

and $\Delta\breve{G}_{jk}(\theta) = \breve{G}^0_{jk} - G_{jk}(\theta)$, and $\Delta\breve{F}_{jk}(\theta) = \breve{F}^0_{jk} - F_{jk}(\theta)$. By Condition (a) p_j is uncorrelated to all $\{r_k\}$, $k \in \mathcal{T}_j$. Thus, $\bar{V}$ can be expressed as:

$$\bar{V}(\theta) = \bar{E}\Big[\Big(\sum_{k \in \mathcal{D}_j} \Delta\breve{G}_{jk}(\theta) w_k^{(\mathcal{T}_j)} + \sum_{k \in \mathcal{P}_j} \Delta\breve{F}_{jk}(\theta) r_k\Big)^2\Big] + \bar{E}\big[(p_j)^2\big]. \tag{5.37}$$

It follows from (5.37) that $\bar{V}(\theta) \geq \bar{E}\big[(p_j)^2\big]$.

The remainder of the proof proceeds by showing that $\bar{V}_j(\theta) = \bar{E}\big[(p_j)^2\big]$ implies that $\Delta\breve{G}_{jk}(\theta) = 0$ for all $k \in \mathcal{D}_j$, and $\Delta\breve{F}_{jk}(\theta) = 0$ for all $k \in \mathcal{P}_j$. This reasoning can be found in the proof of Proposition 5.15 and so will not be repeated here.

(Necessity). We show that if r_j is not present, then $\bar{V}_j(\theta)$ does not have a unique global minimum (and so consistent estimates are not possible). Consider first, two ways of expressing w_j:

$$w_j = \sum_{k \in \mathcal{D}_j \setminus \{j\}} \breve{G}^0_{jk} w_k + \breve{G}_{jj} w_j + \sum_{k \in \mathcal{P}_j} \breve{F}_{jk} r_k + \breve{v}_j \tag{5.38}$$

and

$$w_j = \sum_{k \in \mathcal{D}_j \setminus \{j\}} \frac{\breve{G}^0_{jk}}{1 - \breve{G}^0_{jj}} w_k + \sum_{k \in \mathcal{P}_j} \frac{\breve{F}_{jk}}{1 - \breve{G}^0_{jj}} r_k + \frac{1}{1 - \breve{G}^0_{jj}} \breve{v}_j. \tag{5.39}$$

where in the second equality w_j has been eliminated from the expression. Note that r_j does not appear in the expressions, by assumption.

Consider now, two models in the model set. The models are based on (5.38) and (5.39) respectively. Let *Model 1*, denoted M_1, be defined as:

$$G_{jk}(q, \theta_1) = \breve{G}^0_{jk}(q, \mathcal{D}_j), \text{ for all } k \in \mathcal{D}_j \tag{5.40}$$

where the dynamics $\breve{G}^0_{jk}(q, \mathcal{D}_j)$ are the same as in (5.38). *Model 2*, denoted M_2, is defined as:

$$G_{jk}(q, \theta_2) = \frac{\breve{G}^0_{jk}(q, \mathcal{D}_j)}{1 - \breve{G}^0_{jj}(q, \mathcal{D}_j)}, \text{ for all } k \in \mathcal{D}_j \setminus \{j\}. \tag{5.41}$$

where the transfer functions $\breve{G}^0_{jk}(q, \mathcal{D}_j)$ in (5.41) are the same as those in (5.40).

Note that when r_j is not present, the expression for the optimal output error residual is actually the same as that in (5.36), since r_j does not appear in (5.36). Thus, the optimal output error residual for Model 1 is:

$$p_j(\theta_1) = \sum_{k \in \mathcal{D}_j} \breve{G}^0_{jk}(q, \mathcal{D}_j) w_k^{(\perp \mathcal{T}_j)} + \breve{v}_j. \tag{5.42}$$

Similarly, the optimal output error residual for Model 2 is:

$$p_j(\theta_2) = \sum_{k\in\mathcal{D}_j\setminus\{j\}} \frac{\breve{G}^0_{jk}(q,\mathcal{D}_j)}{1-\breve{G}_{jj}(q,\mathcal{D}_j)} w_k^{(\perp\mathcal{T}_j)} + \frac{1}{1-\breve{G}_{jj}(q,\mathcal{D}_j)}\breve{v}_j. \tag{5.43}$$

where, again, the transfer functions in (5.43) are the same as those in (5.42). In the remaining text, we will show that in fact (5.42) and (5.43) are the same.

The term $\breve{G}^0_{jj} w_j^{(\perp\mathcal{T}_j)}$ will be removed from (5.42). The expression for $w_j^{(\perp\mathcal{T}_j)}$ in terms of $\{w_j^{(\perp\mathcal{T}_j)}\}$, $k \in \mathcal{D}_j \setminus \{j\}$ is obtained by projecting w_j in (5.39) into all singal s that are uncorrelated to $\{r_k\}$, $k \in \mathcal{T}_j$:

$$w_j^{(\perp\mathcal{T}_j)} = \sum_{k\in\mathcal{D}_j\setminus\{j\}} \frac{\breve{G}^0_{jk}}{1-\breve{G}^0_{jj}} w_k^{(\perp\mathcal{T}_j)} + \frac{1}{1-\breve{G}^0_{jj}}\breve{v}_j. \tag{5.44}$$

Substituting (5.44) into the expression for the optimal output error residual of Model 1, (5.42) results in:

$$\begin{aligned} p_j(\theta_1) &= \sum_{k\in\mathcal{D}_j\setminus\{j\}} \Big(1+\frac{\breve{G}^0_{jj}}{1-\breve{G}^0_{jj}}\Big)\breve{G}^0_{jk} w_k^{(\perp\mathcal{T}_j)} + \Big(1+\frac{\breve{G}^0_{jj}}{1-\breve{G}^0_{jj}}\Big)\breve{v}_j \\ &= \sum_{k\in\mathcal{D}_j\setminus\{j\}} \frac{\breve{G}^0_{jk}}{1-\breve{G}^0_{jj}} w_k^{(\perp\mathcal{T}_j)} + \frac{1}{1-\breve{G}^0_{jj}}\breve{v}_j \end{aligned} \tag{5.45}$$

which is the same expression as the optimal output error residual of Model 2. Thus, both models 1 and 2 have the same optimal output error residual. Consequently, both models attain a global minimum of $\bar{V}(\theta)$. Thus, we have shown that in the case that r_j is not present, $\bar{V}_j(\theta)$ does not have a unique global minimum. □

Chapter 6

DEALING WITH SENSOR NOISE

In this chapter we consider the identification of a module that is embedded in a dynamic network using noisy measurements of the internal variables of the network. This is an extension of the errors-in-variables (EIV) identification framework to the case of dynamic networks. The consequence of measuring the variables with sensor noise is that some of the prediction error identification methods no longer result in consistent estimates. The method proposed in this chapter is based on a combination of the instrumental variable philosophy and the Direct Method of closed-loop prediction error identification. The results regarding the predictor input selection of Chapter 5 also apply to the method developed in this chapter. We also present a method that can be used to validate the identified model.[1]

6.1 INTRODUCTION

IN THE METHODS considered so far in the thesis, it was assumed that noise-free measurements of the internal variables are available. Of course, in practice this is not a very realistic assumption, especially since the internal variables are often assumed to be measured using sensors. Every sensor measures a variable with noise. In this chapter methods are presented that use this noisy data to infer a model of a particular module of interest that is embedded in a dynamic network. This chapter represents a further relaxation of the assumptions that need to be made about the system under investigation.

In this chapter we consider a very general framework where there may or may not be known external excitation present, there is both (correlated) process noise and (correlated) sensor noise present in the data collected from the network, and not all internal variables of the network are measurable. We make the assumption that the interconnection structure of the network is known. Including the possibility of sensor noise (in addition to process noise) is not as trivial as it may seem. In many identification methods, the inputs are assumed to be measured without noise (Ljung, 1999). Moreover, if there is sensor noise on the inputs, the methods do not lead to

[1] The material contained in this chapter is based on Dankers et al. (2014c,b).

consistent estimates.

Specifically, we address the following question: under what conditions is it possible to consistently identify a particular module embedded in a dynamic network when only noisy measurements of a subset of the internal variables of the network are available? This is an extension of the so-called Errors-in-Variables framework to the case of dynamic networks.

The open loop EIV problem was briefly presented in Section 3.5 of Chapter 3. In an open-loop setting, either prior knowledge about the system or a controlled experimental setup is required to ensure consistent estimates (Söderström, 2007, 2012). Examples of controlled experimental setups are presented in Schoukens et al. (1997) and Söderström & Hong (2005) where it is shown that using periodic excitation or repeated experiments respectively it is possible to consistently estimate the plant in an open loop EIV setting (without any additional prior knowledge). The method proposed in Söderström & Hong (2005) to deal with the sensor noise is based on an Instrumental Variable (IV) method. The closed-loop EIV problem has been addressed in Pintelon & Schoukens (2012b); Söderström et al. (2013). In both these reference there is an external variable present that is known noise free. In this chapter we do not make that assumption.

In this chapter we also consider IV based approaches, and in a roughly similar vein to Söderström & Hong (2005), instead of using repeated experiments, we show that additional (noisy) measurements generated by the dynamic network can be used to deal with the sensor noise. It is the unique possibility of measuring extra variables in the network that is crucial for dealing with the sensor noise. The extra measurement(s) are used as instrumental variables. We present a method so that any extra measurements that are somehow correlated to the output of the module of interest can be used as instrumental variables, irrespective of their location in the network. However, due to the (possible) presence of loops the process noise must also be specifically dealt with.

As discussed at length in this thesis, the problem with process noise in the presence of loops in the interconnection structure of the data generating system is that the predictor inputs are correlated to the process noise affecting the output. One closed-loop identification method is the Basic Closed-Loop Instrumental Variable (BCLIV) Method (Gilson & Van den Hof, 2005) (see also Section 3.4 of Chapter 3). We show that this method can be easily extended so that it is possible to consistently identify modules embedded in dynamic networks. Interestingly, we show that this method can be used even in the presence of sensor noise. However, for this method, not all additional measurements generated by the dynamic network are candidate instrumental variables. Depending on the measurements available to the user, and the interconnection structure of the network, it may be that there does not exist any candidate instrumental variables, which means that in this case this method cannot be used. Thus, an alternative method must be used, which is presented in the second part of this chapter.

The second method proposed in this chapter is based on a combination of the Instrumental Variable method and the Direct closed-loop Prediction Error method. The problems caused by the process noise are dealt with using the reasoning of the direct closed-loop method, and the problems caused by the sensor noise are dealt

with using an instrumental variable reasoning. In this method all measured variables that are not used to construct the predictor are candidate instrumental variables. The method can be cast as a generalization of the IV method using a one-step-ahead predictor model with a Box-Jenkins model structure. This method can also be cast as a generalization of the first method proposed in this chapter.

The chapter is structured as follows. In Section 6.2 the dynamic network framework is briefly summarized for convenience, and the Basic Closed-Loop Instrumental Variable (BCLIV) method is briefly presented in such a way that the extension to dynamic networks is fairly straight-forward. The first method (a straightforward extension of the BCLIV method) is presented in Section 6.3, the second one (a combination of the IV and Direct Method) is presented in Section 6.4. In Section 6.5 the conditions on the required measurements are further relaxed (i.e. the predictor input selection reasoning of Chapter 5 is applied to the methods of this chapter). In Section 6.6 a practical implementation of the second method is proposed. In Section 6.7 a method is presented to validate the obtained model.

6.2 BACKGROUND

For convenience, in this section we briefly present the data generating system and the basic closed-loop Instrumental Variable (IV) Method. For more information see Chapters 2 and 3 respectively. In addition the concept of persistently exciting data is presented in some detail.

6.2.1 Dynamic Networks

Each internal variable is defined by:

$$w_j(t) = \sum_{k \in \mathcal{N}_j} G^0_{jk}(q) w_k(t) + r_j(t) + v_j(t) \tag{6.1}$$

where G^0_{jk}, $k \in \mathcal{N}_j$ is a proper rational transfer function, and,

- $\mathcal{N}_j$ is the set of indices of internal variables with direct causal connections to w_j, i.e. $k \in \mathcal{N}_j$ iff $G^0_{jk} \neq 0$;

- v_j is *process noise*, that is modeled as a realization of a stationary stochastic process with rational spectral density: $v_j = H^0_j(q)e_j$ where e_j is a white noise process, and H^0_j is a monic, stable, minimum phase transfer function;

- r_j is an *external variable* that is known to the user, and may be manipulated by the user.

It may be that the noise and/or external variables are not present at some nodes. The data generating system is defined as:

$$\begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_L \end{bmatrix} = \begin{bmatrix} 0 & G^0_{12} & \cdots & G^0_{1L} \\ G^0_{21} & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & G^0_{L-1\ L} \\ G^0_{L1} & \cdots & G^0_{L\ L-1} & 0 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_L \end{bmatrix} + \begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_L \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_L \end{bmatrix},$$

where G^0_{jk} is non-zero if and only if $k \in \mathcal{N}_j$ for row j, and v_k (or r_k) is zero if it is not present. Using an obvious notation this results in the matrix equation:

$$w = G^0 w + r + v. \tag{6.2}$$

Each internal variable is measured with some measurement or sensor error:

$$\tilde{w}_k(t) = w_k(t) + s_k(t), k = 1, \ldots, L$$

where $\tilde{w}_k$ denotes the measurement of w_k, and s_k is the *sensor noise*, which is represented by a stationary stochastic process with rational spectral density (s_k is not necessarily white noise).

6.2.2 Closed Loop Instrumental Variable Methods

The Basic Closed-Loop Instrumental Variable (BCLIV) method is presented in Section 3.4 of Chapter 3. Here, we present the method in a different way. The different approach in this section will allow for a straight-forward extension of the closed-loop method to the situation of identification in dynamic networks.

The Instrumental Variable method can be seen as a generalization of the Least Squares Method (Söderström & Stoica, 1983). The mechanism that ensures that the closed-loop IV methods result in consistent estimates is to correlate the input and output with a so-called instrumental variable. As long as the instrumental variable is uncorrelated to the process noise, consistent estimates are possible. In the following text, the Basic Closed-Loop Instrumental Variable (BCLIV) method of Gilson & Van den Hof (2005) is presented, however is a form that highlights the underlying mechanism of the BCLIV that ensures consistent estimates of the plant. In particular, it is shown that the underlying mechanism of interest is the correlation between the prediction error and the instrumental variable.

A closed-loop data generating system is:

$$w_2 = G^0_{21} w_1 + v_2, \tag{6.3a}$$

$$w_1 = G^0_{12} w_2 + r_1. \tag{6.3b}$$

Suppose (for now!) that there is no sensor noise. The objective is to obtain a consistent estimate of G^0_{21}. Consider an ARX model structure, i.e. the module transfer function $G_{21}(\theta)$ is parameterized as (Ljung, 1999):

$$G_{21}(\theta) = \frac{B_{21}(\theta)}{A_{21}(\theta)} = \frac{q^{-n_k}(b^{21}_0 + \cdots + b^{21}_{n_b} q^{-n_b})}{1 + a^{21}_1 q^{-1} + \cdots + a^{21}_{n_a} q^{-n_a}} \tag{6.4}$$

and the noise model is parameterized as $H_2(\theta) = \frac{1}{A_{21}(\theta)}$. Note that A_{21} is a polynomial of order n_a and B_{21} is a polynomial of order n_b. The parameter vector θ is

$$\theta = [a_1^{21} \ \cdots \ a_{n_a}^{21} \ b_0^{21} \ \cdots \ b_{n_b}^{21}]^T. \tag{6.5}$$

From (3.4) the prediction error is

$$\varepsilon_2(t,\theta) = A_{21}(q,\theta)w_2(t) - B_{21}(q,\theta)w_1(t) \tag{6.6}$$

which can be expressed as

$$\varepsilon_2(t,\theta) = w_2(t) - \phi^T(t)\theta \tag{6.7}$$

where

$$\phi^T(t) = \big[-w_2(t-1) \quad \cdots \quad -w_2(t-n_a) \quad w_1(t) \quad \cdots \quad w_1(t-n_b)\big]. \tag{6.8}$$

Let z denote the variable chosen as the *instrumental variable*. In the BCLIV method, r_1 is chosen to be the instrumental variable. The mechanism that forms the foundation of the BCLIV method is presented in the following proposition.

Proposition 6.1 (BCLIV) *Consider a closed-loop system (6.3) that satisfies Assumption 2.19. Consider the prediction error (6.7). Let the instrumental variable $z = r_1$. The equivalence relation*

$$\Big\{\bar{\mathbb{E}}[\varepsilon(t,\theta)z(t-\tau)] = 0, \ \textit{for } \tau = 0,\dots,n\Big\} \iff \Big\{G_{21}(q,\theta) = G_{21}^0(q)\Big\}.$$

holds for any finite $n \geq n_a + n_b + 1$ if the following conditions are satisfied:

(a) $\bar{\mathbb{E}}\big[\phi(t)\cdot[z(t) \ \cdots \ z(t-n_a-n_b)]\big]$ *is nonsingular,*

(b) $\bar{\mathbb{E}}[v_2(t)\cdot z(t-\tau)] = 0, \ \forall \tau \geq 0.$

(c) The parameterization is chosen flexible enough, i.e. there exists a θ such that $G_{21}(q,\theta) = G_{21}^0(q)$. □

The proof is straightforward and follows almost directly from the reasoning presented both in Section 3.4 and Gilson & Van den Hof (2005). The main point is that the cross correlation between the instrumental variable and the prediction error is zero if and only if $G_{21}(q,\theta) = G_{21}^0(q)$ (as long as the conditions hold). This implies that if a parameter vector $\hat{\theta}$ can be found that satisfies the set of equations $R_{\varepsilon z}(\tau)$, $\tau = 0,1,\dots,n$ then $\hat{\theta}$ in fact characterizes a consistent estimate of G_{21}^0. This leads to the following algorithm.

Algorithm 6.2 *Objective: obtain an estimate of G_{21}^0.*

1. Choose r_1 as the instrumental variable. *Let $z = r_1$.*

2. Choose an ARX model structure and construct the prediction error (6.7).

3. Find a solution to the set of equations

$$\frac{1}{N}\sum_{t=0}^{N-1}\varepsilon_2(t,\theta)z(t-\tau)=0,\ \tau=0,\ldots,n_a+n_b \tag{6.9}$$

From Proposition 6.1 it follows that Algorithm 6.2 results in a consistent estimate of G_{21}^0. Note that in Step 3 of Algorithm 6.2 a solution to an approximation of $R_{\varepsilon z}(\tau)$ is obtained. The bigger N is, the better the approximation. Secondly, the solution to the set of equations (6.9) can be found by linear regression. This follows from (6.7) where it appears that (for the ARX model structure) the prediction error is linear in θ.

6.2.3 Persistently Exciting Data

Condition (a) of Proposition 6.1 has a very nice interpretation in terms of *persistence of excitation* of the data. In any identification method the (input) data must be persistently exciting of sufficiently high order to ensure a unique estimate (Ljung, 1999). In this section, first we provide the definition of persistence of excitation, and then it is shown that Condition (a) holds if the external variable r_1 is persistently exciting of sufficiently high order. This provides insight into which mechanisms ensure Condition (a) holds. The formalization presented in this section is used in the extension to the dynamic network case where, due to the presence of many different variables and complex interconnection structures the insight offered by the concept of persistence of excitation becomes increasingly useful. We employ the setting of Söderström & Stoica (1983, 1989a).

Consider first the classical definition of persistence of excitation (Söderström & Stoica, 1989a; Ljung, 1999).

Definition 6.3 *The vector of (internal and/or external) variables u is* persistently exciting of order n *if*

$$\bar{R}_u=\begin{bmatrix} R_u(0) & \cdots & R_u(n-1)\\ \vdots & & \vdots\\ R_u(n-1) & \cdots & R_u(0)\end{bmatrix}$$

is positive definite, where $R_u(\tau)$ is the auto-correlation of the vector u as defined in Section 3.3.1 of Chapter 3.

The concept of persistence of excitation imposes a constraint that the data must satisfy. For the BCLIV method presented in Section 6.2.2 the external variable r_1 must be persistently exciting of sufficiently high order to ensure that the matrix in Condition (a) of Proposition 6.1 is full rank. This is formalized by the following lemma from Söderström & Stoica (1983, 1989a).

Lemma 6.4 *Consider a closed-loop system (6.3) that satisfies Assumption 2.19. Consider Algorithm 6.2. Consider the matrix*

$$R = \bar{\mathbb{E}}\Big[\phi(t)\begin{bmatrix} z(t) & z(t-1) & \cdots & z(t-n_z)\end{bmatrix}\Big]$$

where z is the instrumental variable, and $\phi(t)$ is defined in (6.8). The matrix R generically[2] *has rank n_a+n_b+1 (i.e. full row rank) if the following conditions hold:*

(a) $n_z \geq n_a + n_b + 1$.

(b) r_1 *is (at least) persistently exciting of order* n_a+n_b+1.

(c) The model orders n_a and n_b are equal to the orders n_a^0 and n_b^0 that define G_{21}^0. □

For a proof see Lemma 4.1 and Theorem 4.1 in Söderström & Stoica (1983). The main point of Lemma 6.4 is to provide an interpretation of Condition (a) of Proposition 6.1. It provides the connection between persistence of excitation of the data and consistency of the estimate for IV methods. For a discussion on why the result only holds generically see Söderström & Stoica (1983, 1989a).

Condition (a) of Lemma 6.4 ensures that the matrix R has at least $n_a + n_b + 1$ columns. Clearly this is necessary in order for R to have rank n_a+n_b+1. Condition (c) of Lemma 6.4 ensures that there is a unique θ that represents G_{21}^0 (i.e. the model is *identifiable* in the sense that $G_{21}(\theta) = G_{21}^0 \iff \theta = \theta_0$).

The persistence of excitation of a variable can also be characterized in the frequency domain. Consider the following two propositions.

Proposition 6.5 *Let z be a scalar variable. The variable z is persistently exciting of order n if and only if the spectral density, Φ_z is non-zero for at least n distinct frequencies $\omega \in (-\pi, \pi]$.* □

For a proof see Result A1.3 in Söderström & Stoica (1983).

Proposition 6.6 *Let z be a vector of variables. If the spectral density matrix is full rank for at least n distinct frequencies $\omega \in (-\pi, \pi]$ then z is persistently exciting of order n.* □

For a proof see Result A1.1 in Söderström & Stoica (1983). Interestingly, in the case where z is a vector the frequency domain condition implies that z is persistently exciting, but not the other way around. Thus, there exist vectors of variables that are persistently exciting of order n such that the spectral density is not full rank for at least n distinct frequencies. For example

$$z(t) = [\sin(\omega_1 t)\ \ \sin(\omega_2 t)]$$

is persistently exciting of order 2, but Φ_z is rank deficient at all frequencies.

In the following section a method is presented for identification in networks that is a straightforward extension of the BCLIV method.

[2] A statement is generically true if the statement is true with probability one (Söderström & Stoica, 1989a).

6.3 EXTENSION OF BCLIV METHOD TO DYNAMIC NETWORKS AND SENSOR NOISE

Recall, the objective considered in this chapter is to obtain an estimate of a particular module, G_{ji}^0, embedded in a dynamic network using noisy measurements of the internal variables. In this section a straight-forward extension of the BCLIV method is presented. The extension focuses on three aspects:

- We generalize the method so that it is able to identify a particular module embedded in a dynamic network, not just a closed-loop data generating system.
- We consider the situation that all measured variables can be subject to sensor noise.
- Rather than the classical case where only external variables are considered as candidate instrumental variables, we consider both internal and external variables as candidate instrumental variables.

A main theme in this chapter is that for a dynamic network, there are many different variables present that can serve as potential instrumental variables. For instance, one can choose between several external and internal variables. In this chapter we consider any measured or known variable that is not $\tilde{w}_j$ or a predictor input as a potential instrumental variable. In otherwords, the set of candidate instrumental variables is $\tilde{w}_\ell, \ell \in \{1,\dots,L\} \setminus \{\mathcal{D}_j \cup \{j\}\}$ and r_ℓ, $\ell \in \mathcal{R}$. Let $\mathcal{X}_j$ and $\mathcal{I}_j$ denote the set of indices of external and internal variables respectively chosen as instrumental variables (i.e. r_ℓ is an instrumental variable iff $\ell \in \mathcal{X}_j$ and $\tilde{w}_\ell$ is an instrumental variable iff $\ell \in \mathcal{I}_j$). Since predictor inputs and $\tilde{w}_j$ are not considered as allowable instrumental variables it must be that $\mathcal{I}_j \cap \{\mathcal{D}_j \cup \{j\}\} = \emptyset$.

The variables that are selected as instrumental variables are placed in a vector of instrumental variables, denoted z. Three methods for constructing z are suggested below.

- Choose one or more external variables, $r_{\ell_1},\dots,r_{\ell_n}$, as instrumental variables, resulting in
$$z(t) = [r_{\ell_1}(t) \ \cdots \ r_{\ell_n}(t)]^T \tag{6.10}$$
and $\mathcal{X}_j = \{\ell_1,\dots,\ell_n\}$, $\mathcal{I}_j = \emptyset$.
- Choose one or more measurements of internal variables, $\tilde{w}_{\ell_1},\dots,\tilde{w}_{\ell_n}$, as instrumental variables:
$$z(t) = [\tilde{w}_{\ell_1}(t) \ \cdots \ \tilde{w}_{\ell_n}(t)]^T \tag{6.11}$$
and $\mathcal{X}_j = \emptyset$, $\mathcal{I}_j = \{\ell_1,\dots,\ell_n\}$.
- Choose sums of measured internal variables (or external variables, or a combination of both), $\tilde{w}_{\ell_{11}} + \cdots + \tilde{w}_{\ell_{1n}}$, …, $\tilde{w}_{\ell_{d1}} + \cdots + \tilde{w}_{\ell_{dn}}$. In this case
$$z(t) = \Big[\sum_{m=1}^{n} \tilde{w}_{\ell_{1m}}(t) \ \cdots \ \sum_{m=0}^{n} \tilde{w}_{\ell_{dm}}(t) \Big]^T \tag{6.12}$$

and $\mathcal{X}_j = \emptyset$, $\mathcal{I}_j = \{\ell_{11}, \ldots, \ell_{1n}, \ldots, \ell_{d1}, \ldots, \ell_{dn}\}$.

Of course, combinations of these cases can also be used. Which method is used to choose the instrumental variables will depend on which variables are available (measured), and which choice ensures that z is persistently exciting of sufficiently high order.

In any instrumental variable method, it is essential that the instrumental variables and the predictor inputs are correlated. In the BLCIV method, the instrumental variable was chosen to be r_1, which is correlated to both w_1 and w_2. In the case of dynamic networks it is not automatically guaranteed that a candidate internal variable is correlated to (one or more of) the predictor inputs and/or w_j. The following lemma presents graphical conditions to check wether two variables are correlated.

Lemma 6.7 *Consider a dynamic network as defined in Section 6.2.1 that satisfies Assumption 2.19. Let z_ℓ be an internal or external variable. Then z_ℓ and w_k are not correlated if the following three conditions hold:*

(a) There is no path from z_ℓ to w_k.

(b) There is no path from w_k to z_ℓ.

(c) There is no variable w_p, $p \notin \mathcal{D}_j \cup \mathcal{I}_j \cup \{j\}$ such that there are paths from w_p to both z_ℓ and to w_k. □

The proof can be found in Appendix 6.9.1. This lemma can guide the user to choose appropriate instrumental variables that are correlated to the predictor inputs.

Note that the converse statement of Lemma 6.7 does not guarantee that z_ℓ and w_k are correlated, i.e. if at least one of the conditions of Lemma 6.7 does not hold, it does not necessarily mean that z_ℓ and w_k are correlated. For instance, suppose there are two paths from z_ℓ to w_k (Condition (a) does not hold). Suppose that the dynamics of paths 1 and 2 are G_1^0 and G_2^0 respectively. If $G_1^0 = -G_2^0$, then z_ℓ and w_k are uncorrelated, even though there is a path from z_ℓ to w_k. On the other hand, these situations can be considered to be quite rare, and so if any one (or more) of the conditions of Lemma 6.7 holds then, most likely, z_ℓ and w_k are correlated.

6.3.1 Generalization of BCLIV Method

As in Section 6.2.2 first a mechanism that forms the foundation of the method is presented. Then an algorithm is proposed to exploit this mechanism in order to obtain consistent estimates of a module embedded in a network.

The main modification that must be made to the BCLIV algorithm to be able to use it in dynamic networks is to move to a multiple input, single output (MISO) ARX model structure. For a MISO ARX model structure, the modules and noise model are parameterized as:

$$G_{jk}(q,\theta) = \frac{B_{jk}(q,\theta)}{A_j(q,\theta)}, \text{ and } H_j(q,\theta) = \frac{1}{A_j(q,\theta)}, \tag{6.13}$$

for all $k \in \mathcal{D}_j$, where

$$B_{jk}(q,\theta) = q^{-n_k^{jk}}(b_0^{jk} + b_1^{jk}q^{-1} + \cdots + b_{n_b^{jk}}^{jk} q^{n_b^{jk}}),$$
$$A_j(q,\theta) = 1 + a_1 q^{-1} + \cdots + a_{n_a} q^{-n_a},$$

Note that all modules G_{jk}, $k \in \mathcal{D}_j$ have the same denominator, and that $B_{jk}(\theta)$ is a polynomial of order n_b^{jk} and $A_j(\theta)$ is a polynomial of order n_a. For notational convenience, in the remainder of this chapter, all polynomials $B_{jk}(\theta)$ will be assumed to be of the same order, denoted n_b. Let each module G_{jk}^0, $k \in \mathcal{D}_j$ be expressed as $\frac{B_{jk}^0}{A_{jk}^0}$. Then, from (6.1), w_j can be expressed using transfer functions with a common denominator as follows:

$$w_j(t) = \frac{1}{\breve{A}_j^0(q)} \sum_{k \in \mathcal{N}_j} \breve{B}_{jk}^0(q) w_k(t) + v_j(t) \tag{6.14}$$

where

$$\breve{A}_j^0(q) = \prod_{n \in \mathcal{N}_j} A_{jn}^0(q) \text{ and } \breve{B}_{jk}^0(q) = B_{jk}^0(q) \prod_{n \in \mathcal{N}_j \setminus k} A_{jn}^0(q).$$

From (6.13) and (3.4), the prediction error is:

$$\begin{aligned}\varepsilon_j(\theta) &= A_j(q,\theta)\tilde{w}_j(t) - \sum_{k \in \mathcal{D}_j} B_{jk}(q,\theta)\tilde{w}_k(t) \\ &= \tilde{w}_j(t) - \begin{bmatrix}\tilde{\phi}_{k_1}^T(t) & \cdots & \tilde{\phi}_{k_n}^T(t) & \tilde{\phi}_j^T(t)\end{bmatrix}\theta \\ &= \tilde{w}_j - \tilde{\phi}^T(t)\theta. \end{aligned} \tag{6.15}$$

where $\tilde{\phi}_{k_i}^T(t) = [\tilde{w}_{k_i}(t) \ \cdots \ \tilde{w}_{k_i}(t-n_b)]$, $\tilde{\phi}_j^T(t) = [-\tilde{w}_j(t-1) \ \cdots \ -\tilde{w}_j(t-n_a)]$ and θ is a vector of parameters defined analogously to (6.5).

The result of Proposition 6.1 can now be extended to the case of dynamic networks where only noisy measurements of the internal variables are available.

Proposition 6.8 *Consider a dynamic network as defined in Section 6.2.1 that satisfies Assumption 2.19. Consider the prediction error (6.15). Choose the set of predictor inputs, $\{w_{k_1}, \ldots, w_{k_d}\}$, such that $\{k_1, \ldots, k_d\} = \mathcal{N}_j$ (i.e. $\mathcal{D}_j = \mathcal{N}_j$). Let $d = card(\mathcal{D}_j)$. Choose sets $\mathcal{I}_j$ and $\mathcal{X}_j$ of instrumental variables according to the methods of (6.10) - (6.12) such that $\mathcal{I}_j \cap \{\mathcal{D}_j \cup \{j\}\} = \emptyset$. The equivalence relation*

$$\Big\{R_{\varepsilon z}(\tau) = 0, \ \textit{for } \tau = 0, \ldots n_z\Big\} \iff \Big\{G_{jk}(q,\theta) = G_{jk}^0(q), \forall k \in \mathcal{D}_j\Big\} \tag{6.16}$$

holds for any finite $n_z \geq \lceil (n_a + dn_b)/length\big(z(t)\big) \rceil$ if the following conditions are satisfied:[3]

(a) If v_j is present, then there is no path from w_j to any w_ℓ, $\ell \in \mathcal{I}_j$

[3] $\lceil x \rceil$ denotes the ceiling function, i.e. the smallest integer that is larger than or equal to x

(b) The $\big(dn_b + n_a\big) \times \big(n_z \mathit{length}(z)\big)$ *matrix*

$$\bar{R} = \bar{\mathbb{E}}\Big[\tilde{\phi}(t)[z^T(t) \ \cdots \ z^T(t-n_z)]\Big]$$

is full row rank, where $\tilde{\phi}(t)$ *is defined in (6.15).*

(c) Each sensor noise s_ℓ, $\ell \in \mathcal{I}_j$ *is uncorrelated to all* s_k, $k \in \mathcal{D}_j$.

(d) If v_j *is present, then it is uncorrelated to all* v_m *with a path to* w_j.

(e) The parameterization is flexible enough, i.e. there exists a θ *such that* $G_{jk}(q,\theta) = G^0_{jk}(q)$, $\forall k \in \mathcal{D}_j$. □

The proof can be found in Appendix 6.9.2. Most importantly, the presence of sensor noise does not affect the validity implication (6.16) (as long as Condition (c) holds). Condition (a) puts a restriction on which internal variables are candidate instrumental variables. For example, the candidate instrumental variables cannot be part of any loop that passes through w_j. Note that no similar condition is explicitly stated for the external variables chosen as instrumental variables. This is because, by definition, there is no path from v_j to any r_ℓ.

As was done when analyzing the BCLIV method in Section 6.2.2, Condition (b) of Proposition 6.8 can be further analyzed using the concept of persistence of excitation. Suppose that none of the chosen instrumental variables satisfy all conditions of Lemma 6.7. This alone does not guarantee that the matrix $\bar{R}$ of Condition (b) of Propostion 6.8 is full rank. For the condition to hold, it additionally must be that the vector z of chosen instrumental variables is persistently exciting of sufficiently high order. This is formalized in the following lemma, which is the network counterpart to Lemma 6.4.

Lemma 6.9 *Consider the situation of Proposition 6.8 and*

$$\bar{R} = \bar{\mathbb{E}}\Big[\tilde{\phi}(t)\begin{bmatrix} z^T(t) & z^T(t-1) & \cdots & z^T(t-n_z)\end{bmatrix}\Big]$$

where z *is the vector of instrumental variables, and* $\tilde{\phi}(t)$ *is defined in (6.8). Let* n_θ *denote the size of the vector* $\phi(t)$. *The matrix* $\bar{R}$ *generically has rank* n_θ *(i.e. full row rank) if the following conditions hold:*

(a) $n_z \cdot \mathit{length}\big(z(t)\big) \geq n_\theta$.

(b) z *is persistently exciting of order* $\lceil n_\theta / \mathit{length}\big(z(t)\big)\rceil$.

(c) The parameter vector θ *such that* $G_{jk}(q,\theta) = G^0_{jk}$, $\forall k \in \mathcal{D}_j$ *is unique.*

(d) No instrumental variable satisfies all the conditions of Lemma 6.7 for all w_k, $k \in \mathcal{D}_j \cup \{j\}$.

The proof follows the same reasoning as that of Lemma 6.4. The main point of Lemma 6.9 is that as long as the instruments are correlated to the predictor inputs and w_j, and are persistently exciting of sufficiently high order, then Condition

(b) of Proposition 6.8 (generically) holds. There is no explicit restriction on the number of instrumental variables, as long as the chosen z is persistently exciting of sufficiently high order. However, if only one internal variable is selected as the intrumental variable, then by Condition (b) of Lemma 6.9 z must be persistently exciting of order n_θ. Whereas, if two internal variables are selected as instrumental variables then by the same condition z need only be persistenly exciting of order $n_\theta/2$. Thus Condition (b) may implicitly place a restriction on the required number of intrumental variables.

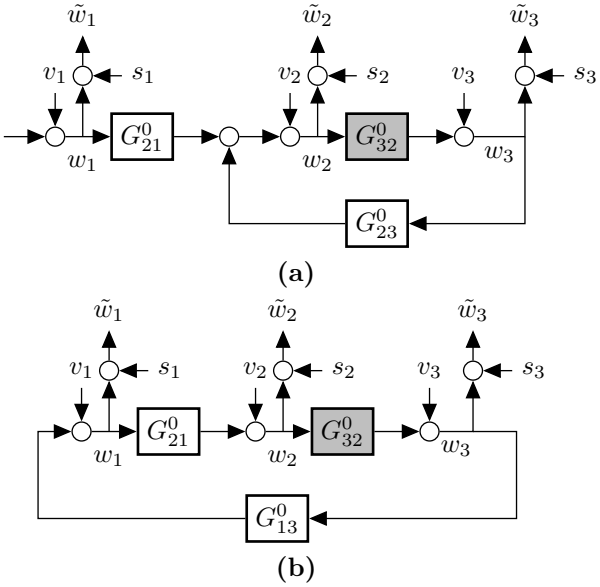

Figure 6.1: Closed loop data generating systems

Example 6.10 *Consider the data generating system shown in Fig. 6.1a. Suppose that the objective is to obtain a consistent estimate of G^0_{32}. Thus, $\{j\} = \{3\}$, and $\mathcal{N}_3 = \{2\}$. There is only one variable left as a candidate instrumental variable (since it must be that $\{\mathcal{D}_j \cup \{j\}\} \cap \mathcal{I}_j = \emptyset$), i.e. $\tilde{w}_1$ must be chosen as the instrumental variable. Since there is no path from w_3 to w_1, Condition (a) of Proposition 6.8 holds. Moreover, Condition (b) of Proposition 6.8 generically holds because the instrumental variable is persistently exciting of sufficiently high order (since v_1 is white noise) and because there is a path from w_1 to both w_2 and w_3 (i.e. there is a path from the instrumental variable to the predictor inputs and w_j). If the remaining conditions of Proposition 6.8 hold, then the implication (6.16) holds.* □

Example 6.11 *Consider the data generating system shown in Fig. 6.1b. Suppose that the objective is to obtain a consistent estimate of G^0_{32}. Thus, choose w_2 as the predictor input, and w_3 is the variable to be predicted. This leaves w_1 as the only candidate instrumental variable. In this case it is not possible to satisfy Condition*

(a) of Proposition 6.8 because v_j is present, and there is a path from the variable to be predicted w_3 to the instrumental variable w_1. Consequently, it is not possible to guarantee that the implication (6.16). □

The following algorithm shows how the implication of Proposition 6.8 can be exploited to obtain an estimate of a module embedded in a dynamic network.

Algorithm 6.12 *Objective: obtain an estimate of G^0_{jk}.*

1. *Choose the set of predictor inputs $\{w_{k_n}, k_n \in \mathcal{N}_j\}$. (i.e. $\mathcal{D}_j = \mathcal{N}_j$).*
2. *Choose the sets $\mathcal{I}_j$ and $\mathcal{X}_j$ of instrumental variables. Construct z, the vector of instrumental variables.*
3. *Choose an ARX model structure and construct the prediction error (6.15).*
4. *Find a solution, $\hat{\theta}_N$ to the set of equations*

$$\frac{1}{N}\sum_{t=0}^{N-1} \varepsilon_j(t,\theta) z^T(t-\tau) = 0, \text{ for } \tau = 0, \ldots, n_z, \tag{6.17}$$

where $n_z \cdot \mathit{length}\big(z(t)\big) \geq n_a + dn_b$.

This algorithm is similar to that of the BCLIV method (Algorithm 6.2). Only Steps 1 and 2 are more involved due to the increased complexity of a network vs. a closed loop. Let $\hat{R}_{\varepsilon z}(\tau)$ denote the function in (6.17). Under weak general convergence conditions of the Prediction-Error Identification methods (Ljung, 1999) it follows that

$$\mathbb{E}[\hat{R}_{\varepsilon z}(\tau)] \to R_{\varepsilon z}(\tau) \text{ as } N \to \infty \tag{6.18}$$

and that the solution to (6.17), denoted $\hat{\theta}_N$ tends to θ_0 as $N \to \infty$. Thus, the estimates of G^0_{jk}, $k \in \mathcal{N}_j$ obtained by Algorithm 6.12 are consistent if the conditions presented in Proposition 6.8 are satisfied. Note that all the conditions are a priori checkable, except Conditions (c) and (d). In Step 4 of Algorithm 6.12 $\hat{\theta}_N$ can be obtained by linear regression. This follows from (6.17) which is affine in θ.

In the following section this method is generalized so that it can be used in the situation where there is a path from w_j to one or more instrumental variables (as was the case in Example 6.11).

6.4 GENERALIZED INSTRUMENTAL VARIABLE APPROACH

In the previous section the set of candidate instrumental variables is restricted by Condition (a) of Proposition 6.8, i.e. it is only allowed to choose instrumental variables for which there is no path from w_j to the instrumental variable. As illustrated in Example 6.11 this can be a restrictive condition. In this section a method is proposed for which all external variables and all internal variables w_ℓ, $\ell \notin \mathcal{D}_j \cup \{j\}$ are candidate instrumental variables.

The key difference in this method is that a Box-Jenkins model structure is used instead of an ARX model structure. In this sense the method presented here is a generalization of the classical IV methods. This change is in line with closed-loop identification reasoning where it is well known that for direct methods, consistent estimates are possible if the process noise is correctly modeled (Forssell & Ljung, 1999). The price for the increased applicability is that the estimates of G^0_{jk} $k \in \mathcal{N}_j$ can no longer be obtained by solving a linear regression problem.

The main reason that a path from w_j to the instrumental variable w_i causes a problem is because then the projections of the predictor inputs onto the instrumental variable(s) are correlated to the output noise. This is equivalent to the closed-loop identification problem where the plant input is correlated to the output noise. From the closed-loop identification literature, there are several methods to deal with this correlation that is induced by feedback (Forssell & Ljung, 1999; Van den Hof et al., 2013). One method, called the Direct Method, deals with the problem by exactly modeling the noise. In the following text it is shown that this idea can be extended to the IV framework, so that all (measured) internal variables $\tilde{w}_\ell$ $\ell \in \{1, \ldots, L\} \setminus \{\mathcal{D}_j \cup \{j\}\}$ are candidate instrumental variables. Note that the idea is to exactly model the process noise term v_j, and not the sensor noise (or a sum of the two). The sensor noise is dealt with using the instrumental variable mechanism.

To exactly model the noise, a Box-Jenkins model structure is required. This amounts to the parameterization:

$$G_{jk}(q,\theta) = \frac{B_{jk}(q,\theta)}{F_{jk}(q,\theta)}, k \in \mathcal{D}_j \text{ and } H_j(q,\theta) = \frac{C_j(q,\theta)}{D_j(q,\theta)}, \tag{6.19}$$

where $F_{jk}(\theta)$, $B_{jk}(\theta)$, $C_j(\theta)$, $D_j(\theta)$ are polynomials in q^{-1} of orders n_f^{jk}, n_b^{jk}, n_c and n_d respectively. For notational convenience all transfer functions $G_{jk}(q,\theta)$ will be assumed to be of the same orders, denoted n_f and n_b.

In the following proposition it is shown that by changing the model structure, the fundamental mechanism on which the IV methods are based, holds for the set of candidate instrumental variables w_ℓ, $\ell \in \{1, \ldots, L\} \setminus \{\mathcal{D}_j \cup \{j\}\}$.

Proposition 6.13 *Consider a dynamic network as defined in Section 6.2.1 that satisfies Assumption 2.19. Consider the prediction error (3.4) and model structure (6.19). Choose the set of predictor inputs such that $\mathcal{D}_j = \mathcal{N}_j$. Choose the sets $\mathcal{I}_j$ and $\mathcal{X}_j$ of instrumental variables according to the methods of (6.10) - (6.12) such that $\mathcal{I}_j \cap \{\mathcal{D}_j \cup \{j\}\} = \emptyset$). Let z denote the vector of instrumental variables. The equivalence relation*

$$\left\{R_{\varepsilon z}(\tau) = 0, \forall \tau \geq 0\right\} \iff \left\{\begin{matrix} G_{jk}(q,\theta) = G^0_{jk}(q) \forall k \in \mathcal{D}_j, \\ H_j(q,\theta) = H^0_j(q) \end{matrix}\right\} \tag{6.20}$$

holds if the following conditions are satisfied:

(a) Every instrumental variable $\tilde{w}_\ell$, $\ell \in \mathcal{I}_j$ is a function of only delayed versions of w_j.

(b) Let $d = card(\mathcal{D}_j)$. *Let* $n_z = \lceil (d+1)/length\big(z(t)\big) \rceil$. *Let* $n_g = \max(n_b + n_f, n_c + n_d)$. *Let*

$$w(t) = [w_{k_1}(t) \ \cdots \ w_{k_d}(t) \ w_j(t)]^T, \{k_1, \ldots, k_d\} = \mathcal{D}_j$$
$$z'(t) = [z^T(t) \ z^T(t - n_g - 1) \ \cdots \ z^T(t - n_z n_g - 1)]^T$$

The cross power spectral density $\Phi_{wz'}$ *is full row rank for at least* n_g *distinct frequencies* $\omega \in (-\pi, \pi]$.

(c) Every sensor noise variable s_k, $k \in \mathcal{D}_j \cup \{j\}$ *is uncorrelated to every* s_ℓ, $\ell \in \mathcal{I}_j$.

(d) The process noise variable v_j *is uncorrelated to all* v_k *with a path to* w_j.

(e) The parameterization is chosen flexible enough, i.e. there exists a parameter θ *such that* $G_{jk}(q, \theta) = G^0_{jk}(q)$, $\forall k \in \mathcal{D}_j$, *and* $H_j(q, \theta) = H^0_j(q)$. □

The proof can be found in Appendix 6.9.3. Condition (a) of Proposition 6.13 can be satisfied in two ways. First, if there is a delay in the path from w_j to the instrumental variable w_ℓ, then w_ℓ is only a function of delayed versions of w_j. Secondly, instead of using $w_\ell(t)$ as an instrumental variable, it is also possible to use a delayed version of w_ℓ, i.e. $w_\ell(t-1)$, as an instrumental variable. In this way Condition (a) can be satisfied.

By Condition (e) the process noise must be exactly modelled. This condition is a signature of the Direct closed-loop method Forssell & Ljung (1999); Van den Hof et al. (2013). This is why we can think of the mechanism proposed in Propostion 6.13 as a hybrid between the Direct closed-loop method and an instrumental variable method. Recall that in Proposition 6.8 exact noise modeling was not required.

In Proposition 6.13 Condition (b) is a condition on the data. For the sake of argument, suppose that z' and w are the same length.[4] Again, it can be interpreted in terms of persistence of excitation. A necessary condition for Condition (b) to hold is that no instrumental variable satisfies all the conditions of Lemma 6.7 for all w_k, $k \in \mathcal{D}_j \cup \{j\}$. Consequently, the vector w is a function of z' and thus

$$\Phi_{wz'}(\omega) = K(e^{j\omega})\Phi_{z'}(\omega).$$

Suppose that $\det(K)$ has no zeros on the unit circle. Then, if $\Phi_{z'}$ is full rank for at least n distinct frequencies, $\Phi_{wz'}$ will be as well. By Proposition 6.6 if $\Phi_{z'}$ is full rank for at least n distinct frequencies, then z' is persistently exciting of order n. Thus, we can link Condition (b) of Proposition 6.13 to the idea of a persistently exciting vector of instrumental variables.

The following examples illustrate the result.

Example 6.14 *Consider again the situation of Example 6.11. Suppose that there is a delay in* G^0_{13}. *Choose,* $\{j\} = \{3\}$, $\mathcal{N}_2 = \{2\}$. *Choose* w_1 *as the instrumental variable, i.e.* $z(t) = \tilde{w}_1(t)$, $\mathcal{I}_j = \{1\}$, *and* $\mathcal{X}_j = \emptyset$. *Condition (a) is satisfied due*

[4] the vector z' of (delayed versions of) instrumental variables can simply be truncated so that z' and w are the same length since z is a user constructed vector

to the delay in G^0_{13}. By Lemma 6.7, since there is a path from w_1 to both w_2 and w_3 the necessary conditions for Condition (b) to hold are satisfied. If the remaining conditions of Proposition 6.13 are satisfied, then the implication (6.20) holds.

The important point, is that for this data generating system, the implication (6.16) of Proposition 6.8 did not hold because it was not possible to satisfy Condition (a) of Proposition 6.8. □

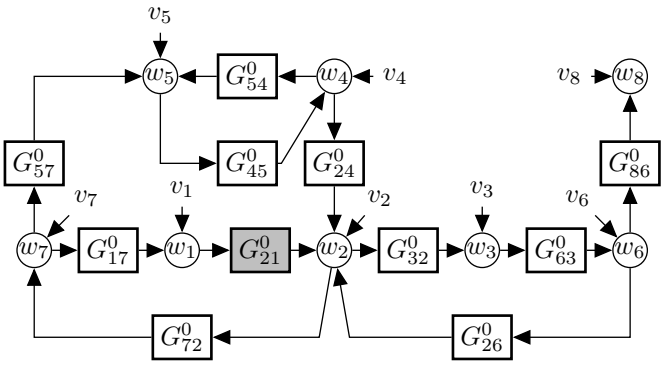

Figure 6.2: Example of a dynamic network. The sensor noise is not shown in the figure, but it is still assumed that in the available data set, each internal variable is measured with sensor noise. The labels of the w_i's have been placed inside the summations indicating that the output of the sum is w_i.

Example 6.15 *Consider the network shown in Fig. 6.2. Suppose that the objective is to obtain a consistent estimate of G^0_{21}. Thus, $\{j\} = \{2\}$, and $\mathcal{N}_2 = \{1, 4, 6\}$. Now choose the instrumental variables. One option is to choose 4 distinct internal variables. A possible choice for the set of instrumental variables is $\tilde{w}_3(t)$, $\tilde{w}_5(t)$, $\tilde{w}_7(t)$ and $\tilde{w}_8(t)$ (i.e. $\mathcal{I}_2 = \{3, 5, 7, 8\}$). In this case if $z(t) = [\tilde{w}_3(t)\ \tilde{w}_5(t)\ \tilde{w}_7(t)\ \tilde{w}_8(t)]^T$ is persistently exciting of sufficiently high order, and the remaining conditions of Proposition 6.13 hold, then the implication (6.20) holds.*

Another option for choosing the instrumental variables is to only use $\tilde{w}_7$. In this case if $z'(t) = [\tilde{w}_7(t)\ \tilde{w}_7(t - n_g - 1)\ \tilde{w}_7(t - 2n_g - 1)\ \tilde{w}_7(t - 3n_g - 1)]$ is persistently exciting of sufficiently high order, and the remaining conditions of Proposition 6.13 hold, then the implication (6.20) holds. Other options for choosing z are also possible depending on the persistence of excitation of z. □

In the following algorithm the implication of Proposition 6.13 is exploited to construct a method to obtain an estimate of a module embedded in a dynamic network.

Algorithm 6.16 *Objective: obtain an estimate of G^0_{jk}.*

1. Choose the set of predictor inputs as $\mathcal{D}_j = \mathcal{N}_j$.

2. *Choose the set $\mathcal{I}_j$ and construct the vector in instrumental variables, z.*

3. *Choose a Box-Jenkins model structure, (6.19), and construct the prediction error (3.4).*

4. *Find a solution to the set of equations*

$$\frac{1}{N}\sum_{t=0}^{N-1}\varepsilon(t,\theta)z(t-\tau)=0,\ \textit{for}\ \tau=1,\dots,n. \tag{6.21}$$

By Proposition 6.13 and the reasoning of (6.18) it follows that the estimate obtained using Algorithm 6.16 is consistent, as long as all the conditions of the proposition are satisfied.

In the following section the choice $\mathcal{D}_j=\mathcal{N}_j$ is relaxed.

6.5 PREDICTOR INPUT SELECTION

In this chapter thus far, the required set of predictor inputs has not been a user choice. In the reasoning thus far, in order to identify a particular module G^0_{ji}, the internal variables with direct connections to w_j, i.e. $\tilde{w}_k$, $k\in\mathcal{N}_j$ must be used as the predictor inputs. From the results of Chapter 5 we know that this is an overly restrictive requirement.

In this section, we show that the notion of predictor input selection developed in Chapter 5 applies almost verbatim to the methods developed in this chapter.

Recall, that in Chapter 5 conditions are derived that the set of predictor inputs must satisfy in order to ensure that it is possible to consistently identify a module of interest, G^0_{ji}. The cost of the increased flexibility in the choice of predictor inputs is that instead of consistently estimating G^0_{jk}, $k\in\mathcal{N}_j$, only the module of interest G^0_{ji} is consistently estimated.

For conviniece, one of the main results of Chapter 5 is repeated here. In Chapter 5 it is shown that in order to consistently identify the module G^0_{jk} using the Direct Method, the set of predictor inputs must have the following property.

Property 6.17 *Consider the internal variables w_i, w_j and the set of indices of predictor inputs, w_k, $k\in\mathcal{D}_j$. Let $\mathcal{D}_j$ satisfy the following conditions:*

(a) $i\in\mathcal{D}_j$, $j\notin\mathcal{D}_j$,

(b) every loop from w_j to w_j passes through a w_k, $k\in\mathcal{D}_j$,

(c) every path from w_i to w_j, excluding the path G^0_{ji}, passes through a w_k, $k\in\mathcal{D}_j$,

(d) there are no confounding variables in the modeling setup[5]. □

[5]See Section 5.4.1 for the definition of a *confounding variable*.

6.5.1 Predictor Input Selection - Extended BCLIV

Consider the following generalization of Proposition 6.8.

Proposition 6.18 *Consider a dynamic network as defined in Section 6.2.1 that satisfies Assumption 2.19. Consider the MISO ARX model structure (6.13) and the prediction error (6.15). Let G_{ji}^0 denote the module of interest. Choose $\mathcal{D}_j$ such that it has Property 6.17. Let $d = card(\mathcal{D}_j)$. The implication*

$$\Big\{R_{\varepsilon z}(\tau)=0,\ \textit{for } \tau=0,\dots n\Big\} \implies \Big\{G_{ji}(q,\theta)=G_{ji}^0(q)\Big\}$$

holds for any finite $n \geq \lceil (n_a + dn_b)/length\big(z(t)\big)\rceil$ if the following conditions are satisfied:

(a) Conditions (a) - (c) of Proposition 6.8 hold.

(b) All process noise variables are uncorrelated to each other.

(c) The parameterization is chosen flexible enough, i.e. there exists a θ such that $R_{\epsilon z}(\tau,\theta)=0$.

Remark 6.19 *Condition (b) can be somewhat relaxed. See Section 5.4.1 of Chapter 5 for details.*

In Algorithm 6.12 only Step 1 is changed. In this case $\mathcal{D}_j$ must be chosen such that it has Property 6.17. Thus, by applying Algorithm 6.12 with a set $\mathcal{D}_j$ that has Property 6.17 a consistent estimate of G_{ji}^0 is obtained. In this case, we no longer make a statement about the remaining transfer functions $G_{jk}(q,\theta)$, $k \in \mathcal{D}_j \setminus \{i\}$ that need to be estimated. For an interpretation of these remaining transfer functions see Section 5.3.1 on the *immersed* network.

6.5.2 Predictor Input Selection - Generalized IV

Similarly, the method of Section 6.4 can be generalized to allow for a more flexible choice of predictor inputs.

Proposition 6.20 *Consider a dynamic network as defined in Section 6.2.1 that satisfies Assumption 2.19. Consider the BJ model structure (6.19) and the prediction error (3.4). Let G_{ji}^0 denote the module of interest. Choose $\mathcal{D}_j$ such that it has Property 6.17. The implication*

$$\Big\{R_{\varepsilon z}(\tau)=0, \forall \tau \geq 0\Big\} \implies \Big\{G_{ji}(q,\theta)=G_{ji}^0(q)\Big\} \tag{6.22}$$

holds if the following conditions are satisfied:

(a) Conditions (a) - (d) of Proposition 6.13 hold.

(b) All process noise variables are uncorrelated to each other.

(c) The parameterization is chosen flexible enough, i.e. there exists a θ such that $R_{\epsilon z}(\tau,\theta) = 0$.

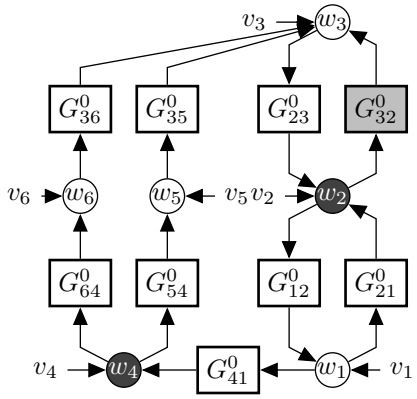

Figure 6.3: Network that is analyzed in Example 6.21. For clarity the sensor noise is not shown, although it is still assumed to be present.

Predictor input selection is illustrated in the following example. The example illustrates the additional flexibility that is allowed in choosing the set $\mathcal{D}_j$.

Example 6.21 *Consider the network shown in Fig. 6.3. Suppose that the objective is to obtain an estimate of G^0_{32}. First we must choose which internal variables to include as predictor inputs, i.e. we must choose $\mathcal{D}_3$ such that it has Property 6.17. By Condition (a) of Property 6.17 $\tilde{w}_2$ must be included as a predictor input. Next, we must check all loops from w_3 to w_3. All such loops pass through w_2, which is allready chosen as a predictor input, so Condition (b) of Property 6.17 is satisfied. Next, we check all paths from w_2 to w_3:*

$$w_2 \to w_1 \to w_4 \to w_5 \to w_3$$
$$w_2 \to w_1 \to w_4 \to w_6 \to w_3$$

for instance. It can be seen that all paths from w_2 to w_3 (not including G^0_{32}) pass through w_4. Thus Condition (c) is satisfied if we include $\tilde{w}_4$ as a predictor input. For $\mathcal{D}_3 = \{2, 4\}$ there are no confounding variables, and so it satisfies all the conditions of Property 6.17. Note that this is not the only choice of $\mathcal{D}_3$ that has Property 6.17.

For this choice of $\mathcal{D}_j$, the candidate instrumental variables are $\{\tilde{w}_1, \tilde{w}_5, \tilde{w}_6\}$. For all these candidates there is a path from w_j to the candidate. Thus, Proposition 6.18 does not apply and we have to defer to Proposition 6.20. If measurements of all the candidate instrumental variables are available, we could choose to use them all, i.e. $\mathcal{I}_j = \{1, 5, 6\}$. Alternatively, if using the smallest number of measurements is desirable, only one of them could be selected as the instrumental variable.

If the remaining conditions of Proposition 6.20 are satisfied, then the implication (6.22) holds. □

6.6 IMPLEMENTATION OF ALGORITHMS

In this section we present a method to implement Algorithm 6.16 An attractive feature of the classical IV methods is that the estimates can be obtained by solving a linear regression problem. When making the move to a BJ model structure, as in the method proposed in Section 6.4, this property is lost. In this section an implementation of the method presented in Section 6.4 is presented.

We show that standard tools for identifying Box-Jenkins models can be used to obtain an estimate of the solution to (6.21). Recall from Section 6.4 that we are interested in finding θ such that

$$R_{\varepsilon z}(\tau, \theta) = 0 \text{ for } \tau = 0, \ldots, n_z.$$

This is equivalent to finding θ such that

$$\sum_{\tau=0}^{n_z} R_{\varepsilon z}^2(\tau, \theta) = 0. \tag{6.23}$$

Since (6.23) is nonnegative for all θ, finding θ such that (6.23) holds is equivalent to finding θ such that

$$\hat{\theta} = \arg\min_{\theta} \sum_{\tau=0}^{n_z} R_{\varepsilon z}^2(\tau, \theta). \tag{6.24}$$

Note that (6.24) is a standard sum of squared errors objective function. Now, consider the expression for $R_{\varepsilon z}(\tau)$:

$$\begin{aligned} R_{\varepsilon z}(\tau) &= \bar{\mathbb{E}}\left[\left(H_j^{-1}(\theta)\left(w_j(t) - \sum_{k \in \mathcal{D}_j} G_{jk}(\theta) w_k(t)\right)\right) z(t-\tau)\right] \\ &= H_j^{-1}(q, \theta)\left(R_{w_j z}(\tau) - \sum_{k \in \mathcal{D}_j} G_{jk}(q, \theta) R_{w_k z}(\tau)\right). \end{aligned} \tag{6.25}$$

The point is that (6.25) has the same form as the prediction error using a Box-Jenkins model structure (see (3.4)), where the "output" is $\hat{R}_{w_j z}(\tau)$ and the predictor "inputs" are $\hat{R}_{w_k z}(\tau)$, $k \in \mathcal{D}_j$. In practice $R_{w_j z}(\tau)$ and $R_{w_k z}(\tau)$ cannot be exactly computed. However, $R_{w_j z}(\tau)$ for instance can be approximated as:

$$\hat{R}_{w_j z}(\tau) = \frac{1}{N - \tau} \sum_{t=\tau}^{N} w_j(t) z(t - \tau).$$

Thus, we can compute $\hat{R}_{w_j z}(\tau)$ and $\hat{R}_{w_k z}(\tau)$ for $\tau = 0, \ldots, n_z$, $k \in \mathcal{D}_j$ resulting in a data set. Now standard identification tools (such as the `bj` function in the MATLAB identification toolbox) can be used to find θ.

Example 6.22 *Consider the system shown in Fig. 6.4. The objective is to obtain an estimate of G_{21}^0 using $\tilde{w}_1$, $\tilde{w}_2$ and $\tilde{w}_3$. Thus, the output is w_2, and the predictor input is $\tilde{w}_1$. This leaves $\tilde{w}_3$ as the only choice for instrumental variable. In this*

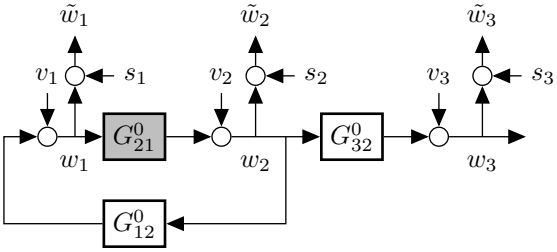

Figure 6.4: Data generating system considered in Example 6.22

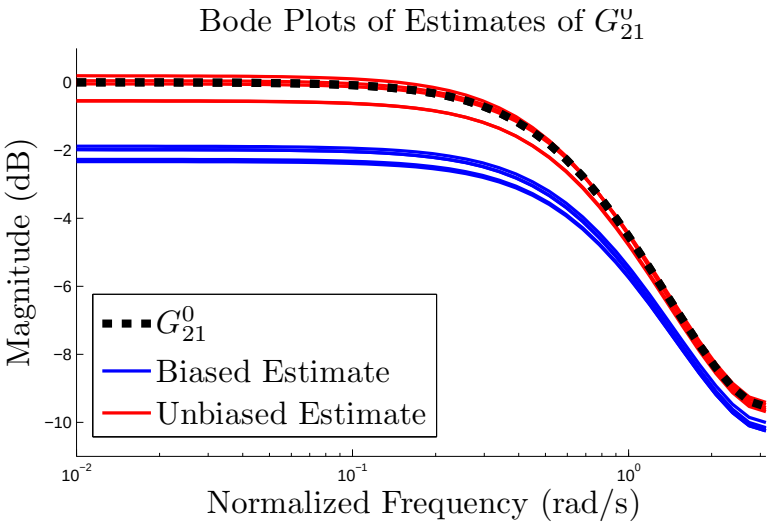

Figure 6.5: Frequency responses related to the system of Fig. 6.4: G_{21}^0 (dashed), five realizations of estimated frequency responses using the Direct Method (blue) and the generalized IV Method (red).

case Algorithm 6.12 does not apply since there is a path from w_2 to the instrumental variable w_3. Thus, we use Algorithm 6.16. All the noise variables v_k and s_k, $k = 1, 2, 3$ are simulated as sequences of low-pass filtered white noise. 5000 data points are simulated. Results are shown in Fig. 6.5. The blue lines denote estimates that are obtained by ignoring the presence of sensor noise and applying the Direct Method of Chapter 4. Clearly these estimates are biased. The red lines denote estimates obtained using the implementation of Algorithm 6.16 presented in this section with $n_z = 1000$. The estimates appear consistent, as expected. □

6.7 MODEL VALIDATION

Once a model is obtained, it is possible to express how confident one is that the obtained model is in fact the model that generated the data. Presumably, $\hat{R}_{\varepsilon z}(\tau, \hat{\theta}_N)$ is small for all $\tau \geq 0$. However, how can one be sure that it is small enough to be considered "very near to zero"? If the variance of $\hat{R}_{\varepsilon z}(\tau, \hat{\theta}_N)$ is known, then it is possible to say that $\hat{R}_{\varepsilon z}(\tau, \hat{\theta}_N)$ is zero with probability p. Then, by the implications (6.16) and (6.20), it follows that it is possible to address the quality of the estimate $G_{jk}(q, \hat{\theta})$.

The steps shown in Söderström & Stoica (1990, 1989a); Ljung (1999) can be closely followed in order to obtain the variance of $\hat{R}^N_{\varepsilon z}(\tau, \hat{\theta}_N)$. The result is that

$$\sqrt{N}\hat{R}^N_{\varepsilon z}(\tau, \hat{\theta}_N) \in \text{As}\mathcal{N}(0, P)$$

where $\text{As}\mathcal{N}(0, P)$ means that as $N \to \infty$ the distribution of $\sqrt{N}\hat{R}^N_{\varepsilon z}(\tau, \hat{\theta}_N)$ tends to a normal distribution with zero mean and variance P, where (Ljung, 1999):

$$P = \sum_{\tau=-\infty}^{\infty} R_\varepsilon(\tau) R_z(\tau).$$

Let n_α denote the α level of the $\mathcal{N}(0, 1)$ distribution. Then it is possible to check if (Ljung, 1999)

$$\left|\hat{R}^N_{\varepsilon z}(\tau, \hat{\theta})\right| \leq \sqrt{\frac{P}{N}} n_\alpha.$$

If the inequality holds, then the obtained model is the correct model with probability α.

Example 6.23 *Consider the same situation as in Example 6.22. Consider a confidence level of* 95*%. The results are shown in Fig. 6.6. From the figure, one can conclude with 95% confidence that there is no evidence in the data that the model is wrong.* □

6.8 SUMMARY

In this chapter a novel method is presented to obtain consistent estimates of a module G^0_{ji} embedded in a dynamic network using noisy measurements of internal variables. The method is based on IV reasoning. Any variable (external or internal) that is not a predictor input or w_j is a candidate instrumental variable. Thus, as long as a variable other than w_k, $k \in \mathcal{D}_j$ and w_j is measurable it is possible to consistently identify G^0_{ji}. Even if all variables are measured with sensor noise! The result becomes even more applicable when it is coupled with the ideas of predictor input selection. Then the predictor inputs need only be chosen such that $\mathcal{D}_j$ has Property 6.17. Thus the experimenter has a relatively flexible choice of which variables to measure in the network. The experimenter could choose to use the cheapest measurements, the least number of measured variables, etc.

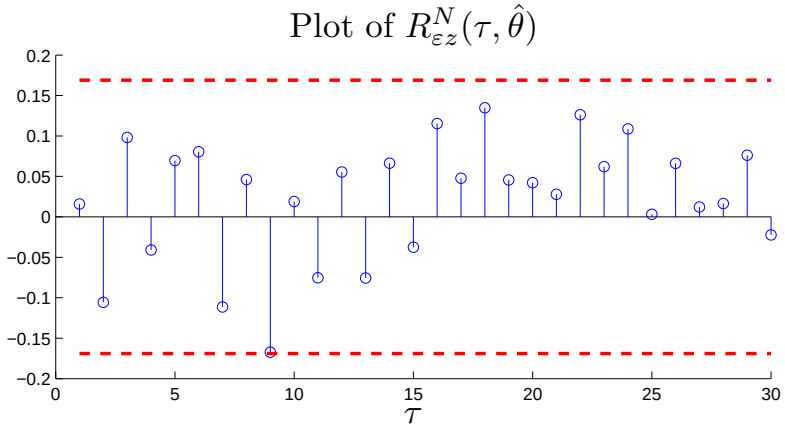

Figure 6.6: Results of the validation test presented in Section 6.7 for the setup considered in Examples 6.22 and 6.23. The red dashed lines represent the 95% confidence intervals.

Since the process noise is modeled in the method of Section 6.4, the variance of the obtained estimate is likely to be relatively close to the statistical optimum. However, more research is needed in this area.

6.9 APPENDIX

6.9.1 Proof of Lemma 6.7

Proof: The proof proceeds by considering $z_\ell = w_\ell$ (the proof for $z_\ell = r_\ell$ is analogous). First both w_ℓ and w_k are expressed in terms of process noise variables. Then Lemma 4.1 is used to prove the result. Using the notation of Lemma 4.1, w_ℓ and w_m can be expressed in terms of only process noise terms:

$$w_\ell(t) = \sum_{n=1}^{L} \mathcal{G}^0_{\ell n}(q) v_n(t) \text{ and } w_k(t) = \sum_{n=1}^{L} \mathcal{G}^0_{kn}(q) v_n(t).$$

Consequently the cross power spectral density $\Phi_{w_\ell w_k}$ is

$$\Phi_{w_\ell w_k}(\omega) = \sum_{\substack{n=1 \\ n \neq \ell, k}}^{L} \mathcal{G}^0_{\ell n}(e^{j\omega}) \Phi_{v_n}(\omega) \mathcal{G}^0_{kn}(e^{-j\omega}) +$$
$$\mathcal{G}^0_{\ell k}(e^{j\omega}) \Phi_{v_k}(\omega) \mathcal{G}^0_{kk}(e^{-j\omega}) + \mathcal{G}^0_{\ell \ell}(e^{j\omega}) \Phi_{v_\ell}(\omega) \mathcal{G}^0_{k\ell}(e^{-j\omega})$$

Suppose that none of the Conditions of Lemma 6.7 hold. By Lemma 4.1 and Condition (a), $\mathcal{G}^0_{k\ell}$ is zero. Thus the third term of $\Phi_{w_\ell w_k}(z)$ is zero. Similarly, by Condition (b) the second term is zero. By Condition (c) for each $n \in \{1, \ldots, L\} \setminus \{k, \ell\}$ either $\mathcal{G}^0_{\ell n}$ or $\mathcal{G}^0_{kn}$ is zero. Thus the first term of $\Phi_{w_\ell w_m}(z)$ is zero. Consequently, if none of the conditions hold, w_k and w_ℓ are uncorrelated. □

6.9.2 Proof of Proposition 6.8

Proof: The proof proceeds by first deriving a simplified expression for $R_{\varepsilon z}(\tau)$ and then showing that this expression equals 0 for $\tau = 0, \ldots, n$ if and only if $\theta = \theta^0$. Using (6.15) $R_{\varepsilon z}(\tau)$ can be expressed as

$$\bar{\mathbb{E}}[\varepsilon(t)z(t-\tau)] = \bar{\mathbb{E}}\Big[\big(\tilde{w}_j(t) - \theta^T\tilde{\phi}(t)\big)z(t-\tau)\Big]$$

Both the predictor inputs and the instrumental variable have a component that is due to the sensor noise. However, By Condition (c) both these components can be removed from the expression of $R_{\varepsilon z}(\tau)$:

$$\begin{aligned} R_{\varepsilon z}(\tau) &= \bar{\mathbb{E}}\Big[\big(w_j(t) + s_j(t) - \theta^T\phi(t) - \theta^T\phi_s(t)\big) \\ &\qquad\qquad \cdot \big(z(t-\tau) + z_s(t-\tau)\big)\Big] \\ &= \bar{\mathbb{E}}\Big[\big(w_j(t) - \theta^T\phi(t)\big)z(t-\tau)\Big] \end{aligned}$$

where

$$\phi_s^T(t) = \begin{bmatrix} s_{d_1}(t) & \cdots & s_{d_1}(t-n_b) & \cdots & s_j(t-1) & \cdots & s_j(t-n_a) \end{bmatrix}$$

and, similarly, $z_s(t)$ is a vector of all the measurement noise terms associated with the instrumental variables. From (6.14) $w_j(t)$ can be expressed as:

$$w_j(t) = \theta_0^T\phi(t) + A_j^0(q)v_j(t)$$

where $\theta^0 = [\breve{b}^0_{jk_1} \ \cdots \ \breve{b}^0_{jk_d} \ a_j^0]$ where $\breve{b}^0_{jk_i}$ is a vector of the coefficients of $\breve{B}^0_{jk_i}$, $k_i \in \mathcal{N}_j$ and a_j^0 is a vector of the coefficients of A_j^0. Using this expression for w_j in $R_{\varepsilon z}(\tau)$:

$$\begin{aligned} R_{\varepsilon z}(\tau) &= \bar{\mathbb{E}}\Big[\big(\theta_0^T\phi(t) + A_j^0(q)v_j(t) - \theta^T\phi(t)\big)z(t-\tau)\Big] \\ &= \bar{\mathbb{E}}\Big[\big(\Delta\theta\phi(t) + A_j^0(q)v_j(t)\big)z(t-\tau)\Big] \end{aligned} \tag{6.26}$$

where $\Delta\theta = \theta_0 - \theta$.

Condition (a) states that there is no path from any predictor input to any variable chosen as an instrument. This implies that each w_ℓ, $\ell \in \mathcal{I}_j$ is not a function of v_j. This statement can be proved using Lemma 4.1 as follows. First, using the notation of Lemma 4.1, express w_ℓ in terms of v:

$$w_i = \sum_{k=1}^{L} \mathcal{G}^0_{nk}v_k.$$

Since there is no path from w_j to w_ℓ, by Lemma 4.1 $\mathcal{G}^0_{\ell j}$ is zero. Thus, w_ℓ is not a function of v_j. Consequently, by Condition (d) w_ℓ and v_j are uncorrelated. This leads to the following simplification of (6.26):

$$R_{\varepsilon z}(\tau) = \bar{\mathbb{E}}\Big[\Delta\theta\phi(t)z(t-\tau)\Big] \tag{6.27}$$

This is the final expression for $R_{\varepsilon z}(\tau)$.

It follows immediately from (6.27) that if $\theta = \theta_0$ then $R_{\varepsilon z}(\tau) = 0$ for all $\tau \geq 0$.

It remains to be shown that if $R_{\varepsilon z}(\tau) = 0$ for all $\tau = 0, \ldots, n$ for any finite $n \geq \lceil (n_a + dn_b)/\text{length}(z(t)) \rceil$ then $\theta = \theta_0$. Consider the set of equations:

$$[R_{\varepsilon z}(0)\ R_{\varepsilon z}(1)\ \cdots\ R_{\varepsilon z}(n)] = 0$$

Then, using (6.27), it follows that

$$\Delta\theta[R_{\phi z}(0)\ \cdots\ R_{\phi z}(\breve{n})\ \cdots\ R_{\phi z}(n)] = 0. \tag{6.28}$$

The matrix $[R_{\phi z}(0)\ R_{\phi z}(1)\ \cdots\ R_{\phi z}(\breve{n})]$ is either square or has more columns than rows. By Condition (b) it is full row rank. Consequently, the only solution to the equation is $\Delta\theta = 0$. This proves the result. □

6.9.3 Proof of Proposition 6.13

Proof: First simplified expressions for ε_j and z are derived in order to arrive at a simple expression for $R_{\varepsilon z}(\tau)$. Then it is shown that this expression equals zero for all $\tau \geq 0$ iff $G_{jk}(\theta) = G^0_{jk}$.

Consider first an expression for the prediction error. Substitute the expressions for $\tilde{w}_j$ and $\tilde{w}_k$ into (3.4):

$$\begin{aligned}\varepsilon_j(\theta) &= H_j^{-1}(\theta)\Big(\sum_{k\in\mathcal{N}_j} G^0_{jk} w_k + v_j + s_j - G_{jk}(\theta)(w_k + s_k)\Big)\\ &= H_j^{-1}(\theta)\sum_{k\in\mathcal{N}_j}\Delta G_{jk}(\theta) w_k + \Delta H_j(\theta) v_j + e_j\\ &\qquad + H_j^{-1}(\theta)\Big(s_j - \sum_{k\in\mathcal{N}_j} G_{jk}(\theta) s_k\Big)\end{aligned} \tag{6.29}$$

with $\Delta G_{jk}(\theta) = G^0_{jk} - G_{jk}(\theta)$ and $\Delta H_j(\theta) = H_j^{-1}(\theta) - H_j^{0^{-1}}$.

Now consider the expression for the instrumental vector:

$$\begin{aligned}z(t) &= \begin{bmatrix}\tilde{w}_{\ell_1}(t) & \cdots & \tilde{w}_{\ell_n}(t)\end{bmatrix}\\ &= \begin{bmatrix}w_{\ell_1}(t) + s_{\ell_1}(t) & \cdots & w_{\ell_n}(t) + s_{\ell_n}(t)\end{bmatrix}.\end{aligned} \tag{6.30}$$

In the following text, an expression for $R_{\varepsilon z}(\tau)$ is derived using (6.29) and (6.30) that is valid for all $\tau \geq 0$. Subsequently, this expression is used to prove the proposition.

No measurement chosen as an instrumental variable can be a predictor input ($\mathcal{N}_j \cap \mathcal{I}_j = \emptyset$ by the statement of the proposition). Thus, no s_ℓ that appears in the instrumental variable vector z (6.30), will appear in the expression for ε_j, (6.29). By Condition (c) each s_k, $k \in \mathcal{D}_j$ is uncorrelated to all s_ℓ, $\ell \in \mathcal{I}_j$, resulting in:

$$\begin{aligned}\bar{\mathbb{E}}[\varepsilon_j(t,\theta)\cdot z(t-\tau)] = \bar{\mathbb{E}}\Big[\Big(H_j^{-1}(q,\theta)\sum_{k\in\mathcal{N}_j}\Delta G_{jk}(q,\theta) w_k(t)\\ + \Delta H_j(q,\theta) v_j(t) + e_j(t)\Big)[w_{\ell_1}(t-\tau)\ \cdots\ w_{\ell_n}(t-\tau)]\Big]\end{aligned} \tag{6.31}$$

By Condition (a) each instrumental variable is a function of only delayed versions of v_j (and thus delayed versions of e_j), resulting in the following simplification

$$\bar{\mathbb{E}}[\varepsilon_j(t,\theta)\cdot z(t-\tau)] = \bar{\mathbb{E}}\Big[\Big(H_j^{-1}(q,\theta)\sum_{k\in\mathcal{N}_j}\Delta G_{jk}(q,\theta)w_k(t) + \Delta H_j(q,\theta)v_j(t)\Big)\big[w_{\ell_1}(t-\tau) \quad \cdots \quad w_{\ell_n}(t-\tau)\big]\Big] \tag{6.32}$$

which holds for all $\tau \geq 0$. Using a vector notation (6.32) can be expressed as:

$$R_{\varepsilon z}(\tau) = \bar{\mathbb{E}}\Big[\Delta X(q,\theta)^T \begin{bmatrix} w_{k_1}(t) \\ \vdots \\ w_{k_d}(t) \\ v_j(t) \end{bmatrix} \big[w_{\ell_1}(t-\tau) \quad \cdots \quad w_{\ell_n}(t-\tau)\big]\Big]$$

where $$\Delta X(q,\theta)^T = \Big[\tfrac{\Delta G_{jk_1}(q,\theta)}{H_j^{-1}(q,\theta)} \quad \cdots \quad \tfrac{\Delta G_{jk_d}(q,\theta)}{H_j^{-1}(q,\theta)} \quad \Delta H_j(q,\theta)\Big]$$

and $\{k_1,\ldots,k_d\} = \mathcal{D}_j$. The variable v_j can be expressed in terms of internal variables as:

$$v_j = w_j - \sum_{k\in\mathcal{N}_j} G_{jk}^0(q)w_k$$

and so

$$\begin{bmatrix} w_{k_1}(t) \\ \vdots \\ w_{k_d}(t) \\ v_j(t) \end{bmatrix} = \begin{bmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ -G_{jk_1}^0(q) & \cdots & -G_{jk_d}^0(q) & 1 \end{bmatrix} \begin{bmatrix} w_{k_1}(t) \\ \vdots \\ w_{k_d}(t) \\ w_j(t) \end{bmatrix} \tag{6.33}$$

Denote the matrix in (6.33) as $J^0(q)$. Using this notation,

$$R_{\varepsilon z}(\tau) = \bar{\mathbb{E}}\Big[\Delta X(q,\theta)^T J^0(q) w(t) \cdot [w_{\ell_1}(t-\tau)\cdots w_{\ell_n}(t-\tau)]\Big] \tag{6.34}$$

where $w(t) = [w_{k_1}(t) \ \cdots \ w_{k_n}(t) \ w_j(t)]^T$. Note that (6.34) is valid for all $\tau \geq 0$.

Now, first consider the 'if' statement. It is shown that if $G_{jk}(q,\theta) = G_{jk}^0$, for all $k \in \mathcal{N}_j$ and $H_j(q,\theta) = H_j^0$, then $R_{\varepsilon z}(\tau) = 0$ for all finite $\tau \geq 0$. Let θ_0 denote this particular parameter vector (such a parameter vector is guaranteed to exist by Condition (e)). Clearly, $\Delta G_{jk}(\theta_0) = 0$ and $\Delta H_j(\theta_0) = 0$. Thus, from (6.34),

$$\bar{\mathbb{E}}[\varepsilon_j(t,\theta_0)\cdot z(t-\tau)] = 0, \quad \text{for all } \tau \geq 0.$$

Now consider the 'only if' statement. It must be shown that if $R_{\varepsilon z}(\tau) = 0$, for all $\tau \geq 0$ then $G_{jk}(\theta) = G_{jk}^0$, for all $k \in \mathcal{N}_j$ and $H_j(\theta) = H_j^0$. If z has fewer elements than w, then increase the size of z by appending delayed versions of z as is done in Condition (b), i.e. construct:

$$z'(t) = [z(t) \ z(t-n_g-1) \ \cdots \ z(t-n_z n_g - 1)].$$

Thus, w and z' are either of equal length, or z' has slightly more elements. Since $R_{\varepsilon z}(\tau) = 0$, for all $\tau \geq 0$ it follows that $R_{\varepsilon z'}(\tau) = 0$, for all $\tau \geq 0$ also. Consequently the following equation holds

$$R_{\varepsilon z'}(\tau) R^T_{\varepsilon z'}(-\tau) = 0, \ \forall \tau.$$

Thus the following equation also holds

$$\sum_{\tau=-\infty}^{\infty} R_{\varepsilon z'}(\tau) R^T_{\varepsilon z'}(-\tau) = 0.$$

Take the Fourier Transform of this equation. Using the time reversal property and Parseval's Theorem results in

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \Phi_{\varepsilon z'}(e^{j\omega}) \Phi^T_{\varepsilon z'}(e^{-j\omega}) \mathrm{d}\omega = 0. \tag{6.35}$$

Substitute (6.34) into (6.35):

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \Delta X(e^{j\omega}, \theta) J^0(e^{j\omega}) \Phi_{wz'}(e^{j\omega}) \cdot \Phi^T_{wz'}(e^{-j\omega}) J^{0^T}(e^{-j\omega}) \Delta X^T(e^{-j\omega}, \theta) \mathrm{d}\omega = 0. \tag{6.36}$$

In the following text, an expression for $\Delta X(e^{j\omega})$ is derived such that the parameters appear in a vector that is not a function of ω.

Consider the expression for the dth entry of $\Delta X(q, \theta)$, where $d \leq \mathrm{card}(\mathcal{D}_j)$:

$$\begin{aligned} H_j^{-1}(q,\theta) \Delta G_{jk}(q,\theta) &= \frac{D_j(q,\theta)}{C_j(q,\theta)} \Big(\frac{B^0_{jk}(q)}{F^0_{jk}(q)} - \frac{B_{jk}(q,\theta)}{F_{jk}(q,\theta)} \Big) \\ &= \frac{D_j(q,\theta)}{C_j(q,\theta)} \frac{B^0_{jd}(q) F_{jk}(q,\theta) - B_{jk}(q,\theta) F^0_{jk}(q)}{F^0_{jk}(q) F_{jk}(q,\theta)} \\ &= \Delta P_{jk}(\theta) K_{jk}(q,\theta) \end{aligned} \tag{6.37}$$

where

$$\begin{aligned} \Delta P_{jk}(q,\theta) &= B^0_{jk}(q) F_{jk}(q,\theta) - B_{jk}(q,\theta) F^0_{jk}(q), \\ K_{jk}(q,\theta) &= \frac{D_j(q,\theta)}{C_j(q,\theta) F^0_{jk}(q) F_{jk}(q,\theta)}. \end{aligned}$$

Note that $\Delta P_{jk}(q,\theta)$ is a polynomial of order $n_f + n_b$ (or less). Similarly, the last entry of $\Delta X(q,\theta)$ can be expressed as:

$$\begin{aligned} \Delta H_j(q,\theta) &= \frac{D_j(q,\theta)}{C_j(q,\theta)} - \frac{D^0_j(q)}{C^0_j(q)} \\ &= \Big(D_j(q,\theta) C^0_j(q) - D^0_j(q) C_j(q,\theta) \Big) \frac{1}{C_j(q,\theta) C^0_j(q)} \\ &= \Delta P_j(q,\theta) K_j(q,\theta). \end{aligned} \tag{6.38}$$

where

$$\Delta P_j(q,\theta) = D_j(q,\theta)C_j^0(q) - D_j^0(q)C_j(q,\theta),$$
$$K_j(q,\theta) = \frac{1}{C_j(q,\theta)C_j^0(q)}.$$

Note the $\Delta P_j(q,\theta)$ is a polynomial of order $n_c + n_d$ (or less). For notational convenience suppose that $n_h \leq n_g$.

The next step is to plug the expressions for $\Delta X(e^{j\omega},\theta)$, (6.37) and (6.38) into (6.36). Let

$$\Delta P(e^{j\omega},\theta) = [\Delta P_{jk_1}(e^{j\omega},\theta) \ \cdots \ \Delta P_{jk_d}(e^{j\omega},\theta) \ \Delta P_j(e^{j\omega},\theta)]$$

where $\{k_1,\ldots,k_d\} = \mathcal{D}_j$, then

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \Delta P(e^{j\omega},\theta)K(e^{j\omega},\theta)J^0(e^{j\omega})\Phi_{wz}(\omega)\Phi_{wz}^H(\omega)$$
$$\cdot J^{0^T}(e^{-j\omega})K^T(e^{-j\omega},\theta)\Delta P^T(e^{-j\omega},\theta)\mathrm{d}\omega = 0,$$

where superscript H denotes conjugate transpose, and

$$K(\theta) = \mathrm{diag}\big(K_{jd_1}(\theta),\ldots,K_{jd_n}(\theta),K_j(\theta)\big)$$

where the argument $e^{j\omega}$ has been dropped for notational clarity, $\mathrm{diag}(\cdot)$ denotes a diagonal matrix with the arguments on the diagonal. Since the term in the integral is nonnegative for all theta, the only way that the integral can equal zero is if the term in the integral is zero for all omega, i.e.:

$$\Delta P(e^{j\omega},\theta)\Upsilon(\omega)\Delta P^T(e^{-j\omega},\theta) = 0 \tag{6.39}$$

for all $\omega \in [-\pi,\pi)$ where

$$\Upsilon = K(\theta)J^0\Phi_{wz}\Phi_{wz}^H J^{0^H}K^H(\theta).$$

By Condition (b) Φ_{wz} is full rank at n_g distinct frequencies. Note that J^0 and $K(\theta)$ are full rank for all ω, and θ. The result is that that Υ is positive definite for at least n_g+1 frequencies. However, $\Delta P(q)$ is a (vector) polynomial of degree n_g only. This implies that $\Delta P(q) = 0$ (Söderström & Stoica, 1983).

From the definition of $\Delta p(\theta)$ this implies that $G_{jd}(q,\theta) = G_{jd}^0(q)$, for all $d \in \mathcal{D}_j$ and $H_j(q,\theta) = H_j^0(q)$. □

Chapter 7

CONTINUOUS-TIME IDENTIFICATION IN DYNAMIC NETWORKS

In the thesis thus far the object to be identified has been a discrete-time dynamic network model. In order to study the consistency of the proposed methods the data generating system was assumed to be a discrete-time dynamic network model. However, in practice the data generating system is often in continuous-time. In the first part of this chapter we investigate the relationship between a discrete-time dynamic network and its continuous-time counterpart. The two features that are investigated are the effect of (not having) anti-aliasing filters in the sensors, and the effect of the intersample behavior on the presence of algebraic loops in the discrete-time representation of the continous-time data generating system. The intent is to establish conditions under which it is possible to use the identified discrete-time module to estimate the corresponding continuous-time module. In the second part of this chapter a method is presented to directly identify continuous-time transfer functions embedded in a continuous-time dynamic network model. The method is a continuous-time version of the method presented in Section 6.3. As such, it is also a generalization of the closed-loop continuous-time instrumental variable methods to the case of identification in dynamic networks. [1]

7.1 INTRODUCTION

MANY PHYSICAL SYSTEMS are naturally modeled in continuous-time (i.e. in terms of (partial) differential equations) rather than discrete-time. Moreover, the continuous-time transfer functions are directly related to physical properties of the system under investigation, such as resistances, capacitances, permeabilities of materials, diameters of pipes, etc. Thus, by identifying a continuous-time transfer function embedded in the continuous-time dynamic network, estimates of the physical properties of the system are obtained. In this chapter we consider the question: under what conditions can a consistent estimate of a continuous-time transfer

[1] The material contained in this chapter is based on Dankers et al. (2014a).

function be obtained when it is embedded in a dynamic network? The underlying objective is to obtain estimates of physical parameters of a physical system.

This chapter is split into two parts. In the first part an *indirect* continuous-time identification approach is investigated, and the second part a *direct* continuous-time identification approach is investigated.

In the first part the relationship between a continuous-time dynamic network model and its discretized counterpart is investigated. In this part of the chapter we address the question: under which conditions can a discrete-time module be identified that is an accurate representation of its continuous-time counterpart? The purpose of the discrete-time estimate is to convert it into a continuous-time transfer function from which the physical parameters of the system can be estimated. This is referred to as the *indirect continuous-time identification approach* (Garnier & Wang, 2008).

Two effects of discretization are investigated in this chapter. The effect of discretization on the interconnection structure of the discrete-time representation, and the effect of discretization on the presence of algebraic loops in the discrete-time representation. It is well known that aliasing can cause discrepancies between the dynamics of a discrete-time transfer function and its continuous-time counterpart. In this chapter, we investigate the effect of aliasing from a dynamic network point of view, where the key question is: under what conditions do the discrete-time and continuous-time dynamic networks describing a system have the same interconnection structure? Discretizing a continuous-time system can also affect the presence of direct feed-through terms in the discrete-time module transfer functions (Laurijsse, 2014). In particular it is the intersample behavior of the external variables that affects the presence of feed-though terms in the discrete-time network. Since the Direct and Joint IO methods are sensitive to the absence of algebraic loops in the discrete-time data generating system (see Chapters 4 and 6) this phenomenon merits our attention.

In the second part of this chapter, we consider directly identifying a continuous-time model (thus skipping the intermediate step of first identifying a discrete-time model which is subsequently converted to a continuous-time model). There are several advantages to such an approach (Garnier & Wang, 2008; Rao & Unbehauen, 2006):

- **Non-uniformly sampled data.** In certain applications such as in biological systems, economics, and environmental science (Young & Garnier, 2006), it may be difficult to obtain uniformly spaced data. Because all these applications also deal extensively with dynamic networks, this is a relevant reason to consider the direct continuous-time methods presented in this chapter. In this situation standard discrete-time techniques cannot be applied because discrete-time transfer functions are not defined for the situation where the sampling rate is non-uniform, whereas continuous-time transfer functions are independent of the sampling rate.

- **Sensitivity to high sampling rates.** As the sampling rate increases, the poles of a discrete-time transfer function tend towards the unit circle. The optimization routines used to obtain the estimates of a discrete-time system

have difficulty with poles near the unit circle because any small change in a pole location could result in an unstable model. This leads to estimates with high variance. Again, since the continuous-time transfer functions are independent of the sampling rate they do not suffer from this affliction.

- **Stiff systems.** A stiff system is a system with some very fast dynamics and some very slow dynamics. Again, since the discrete-time transfer function is dependent on the sampling rate, it becomes difficult to capture both the fast and the slow dynamics.

It is in part due to these advantages that *direct continuous-time identification* is becoming a more prominent topic in the identification literature (see for instance Garnier et al. (2003); Larsson et al. (2006); Gillberg & Ljung (2009); Pintelon & Schoukens (2012b)). This leads to a second question which is addressed in the second part of this chapter: under what conditions can a continuous-time transfer function embedded in a continuous-time dynamic network be directly identified (i.e. without first identifying a discrete-time transfer function)? The method that is presented is a continuous-time version of the extension of the (discrete-time) Basic Closed-Loop Instrumental Variable Method of Section 6.3 in Chapter 6. It can also be considered as an extension of the Continuous-Time Basic Closed-Loop Instrumental Variable method of Gilson et al. (2008) to the situation of identification in dynamic networks.

The chapter proceeds as follows. In Section 7.2 we briefly define a continuous-time dynamic network model, then in Section 7.3 we discuss the indirect continuous-time identification approach applied to dynamic networks, and in Section 7.4 we present an instrumental variable method for direct continuous-time dynamic network identification.

7.2 CONTINUOUS-TIME DYNAMIC NETWORKS

In this section a continuous-time dynamic network model is briefly presented. Analogous to the discrete-time dynamic network model, each internal variable is assumed to be such that it can be written as:

$$w_j(t) = \sum_{k \in \mathcal{N}_j} G_{jk}(p) w_k(t) + r_j(t) + v_j(t)$$

where t is now a continuous-time variable, $G_{jk}(s)$ a continuous-time transfer function, and p is the differential operator, i.e. $pu(t) = \frac{\mathrm{d}}{\mathrm{d}t}u(t)$. The set $\mathcal{N}_j$ has the usual definition, and r_j and v_j represent the external variables and the process noise respectively, as usual. The process noise is modeled as a continuous-time stochastic process with rational power spectral density. Thus the data generating system in

this chapter has the form:

$$\begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_L \end{bmatrix} = \begin{bmatrix} 0 & G_{12}^0 & \cdots & G_{1L}^0 \\ G_{21}^0 & 0 & \ddots & G_{2L}^0 \\ \vdots & \ddots & \ddots & \vdots \\ G_{L1}^0 & G_{L2}^0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_L \end{bmatrix} + \begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_L \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_L \end{bmatrix}$$
$$= G^0(p)w(t) + r(t) + v(t). \tag{7.1}$$

We assume that the continuous-time data generating system (7.1) is well-posed (see Definition 2.11). We suppose that measurements of the internal variables are obtained using *data aquisition systems* (DAQs) or *analog to digital converters* (ADC)s. An analog to digital converter consists of four components: a sensor, an anti-aliasing filter, a sampler and a digitizer (Fadali & Visioli, 2013). A sensor continuously transforms a physical parameter to a form such that it can be sampled. Anti-aliasing filters remove the high frequency content from a signal *before* it is sampled. The sampling operation involves recording a variable every T seconds. The digitizer converts the output of the sampler into a number. Not all DAQs are equipped with anti-aliasing filters. Adopting the notation of Fadali & Visioli (2013), a sampled and digitized version of a continuous-time variable will be denoted with a superscript $\star$. For instance, the sampled and digitized version of the continuous-time variable $r_k(t)$ is

$$r_k^\star(n) = r_k(nT), \quad n = 0, 1, 2, \ldots. \tag{7.2}$$

where n is used to denote discrete-time, and T denotes the sampling period. We assume that every DAQ measures with some error which we call *sensor noise*. The (discrete-time) output of a DAQ is then denoted:

$$\tilde{w}_k^{aa\star}(n) = w_k^{aa}(nT) + s_k(n), \quad n = 1, 2, \ldots \tag{7.3}$$

where $w_k^{aa}(t) = G_{aa,k}(p)w_k(t)$ where $G_{aa,k}(s)$ is the transfer function of the anti-aliasing filter, and s_k is the sensor noise. A superscript $\star$ denotes a discrete-time signal, and $\tilde{w}_k$ denotes that w_k is measured with (sensor) noise. If anti-aliasing filters are not present in the sensor, then $G_{aa,k}(s) = 1$. The sensor noise is modeled as a discrete-time stochastic process with rational power spectral density. It can be shown that the sensor noise can be assumed to be discrete-time without loss of generality (i.e. even if it is really a continuous-time process, after sampling the noise can be approximated arbitrarily well by a discrete-time filtered white noise process) (Åström, 1970; Pintelon & Schoukens, 2012b). Equations (7.1) and (7.3) define the continuous-time data generating system.

Often the external variables, r_k, are digital signals that have been converted into analog signals by a *digital to analog converter* (DAC). A common DAC is equipped with a *zero-order-hold* circuit, which means that the output of the DAC is held constant in between sampling instants (Fadali & Visioli, 2013). For example the continuous-time signal $r_k(t)$ that is the output of a DAC equipped with a *zero-*

order-hold is:

$$r_k(t) = \sum_{n=0}^{\infty} r_k^\star(n) g_h(t - nT), \text{ with } g_h(t) = \begin{cases} 1, & 0 \leq t < T \\ 0, & \text{otherwise} \end{cases} \tag{7.4}$$

In the following text, it will be useful to use the $\text{rect}(t)$ function, defined as:

$$\text{rect}(t) = \begin{cases} 1, & 0 \leq t < T \\ 0, & \text{otherwise} \end{cases}.$$

In other words, $\text{rect}(t)$ is a rectangular pulse of length T. The transfer function of $\text{rect}(t)$ is (Fadali & Visioli, 2013):

$$\text{Rect}(s) = \frac{1 - e^{sT}}{s}.$$

Using this notation, the output of a DAC equipped with a zero-order-hold circuit, i.e. (7.4) can be expressed as

$$r_k(t) = \sum_{n=0}^{\infty} r_k^\star(n) \text{rect}(t - nT).$$

It is well known that the effect of sampling is not trivial. Next we consider the effect of sampling in dynamic networks.

7.3 INDIRECT CONTINUOUS-TIME IDENTIFICATION

In the indirect approach, first a discrete-time transfer function is estimated from the sampled data. Then a continuous-time transfer function is constructed based on the discrete-time transfer function. In order to be able to (easily) obtain an estimate of the continuous-time transfer function $G_{ji}(s)^0$ from the transfer function $G_{ji}^0(z)$ estimated from the (sampled) data, it is essential that $G_{ji}(z)$ is an accurate representation of the dynamics of G_{ji}^0. Thus, it is essential to understand the effect of discretizing (7.1). In this section we investigate two effects: first we investigate the effect of aliasing on the interconnection structure of the discrete-time representation of the continuous-time data generating system, and secondly we investigate the effect of the intersample behavior on the presence of direct feed-though terms in the discrete-time dynamic network. The results of effect of the intersample behavior on the presence of direct feed-through terms is based on the work of Laurijsse (2014).

7.3.1 Effect of Discretization on the Interconnection Structure

In this section we illustrate that when using the indirect approach it is critical that either the ADCs are equipped with anti-aliasing filters or the sampling rate is much faster than the bandwidth of the system. This may seem like a rather obvious result. It is well known in sampling theory that aliasing causes a change

in the dynamics of the discrete-time description with respect to the continuous-time version. However, in this section, we show that not only does aliasing cause a change in the dynamics of the discrete-time representation, but aliasing also causes the interconnection structure of the discrete-time dynamic network to be different from the interconnection structure of the underlying continuous-time data generating system. This is a phenomenon that is a side-effect of aliasing that is unique to dynamic networks.

From the perspective of identification in dynamic networks this can have unexpected consequences. Suppose that anti-aliasing filters are not used to take the measurements. Then, although the interconnection structure of the continuous-time data generating system is known, this interconnection structure is not the same as that of the equivalent discrete-time dynamic network. Thus, if the interconnection structure of the continuous-time data generating system is imposed on the discrete-time representation, a bias will result. This bias is in addition to the usual bias that results from aliasing. Moreover, this bias is unique to identification in dynamic networks (i.e. it does not appear in the classical open and closed-loop setups).

The following setup is used to illustrate the difference between the discrete-time and continuous-time representations of the data. Consider a network of the form (7.1) with no process noise present and all external variables $r_1, \ldots, r_L$ present. Suppose that the external variables are discrete-time signals that have been converted into continuous-time signals by DACs. An example of such a set up is shown in Figure 7.1.

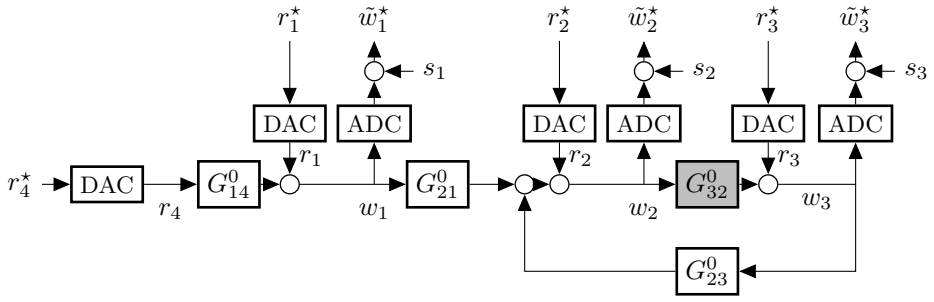

Figure 7.1: Closed loop data generating system with no process noise and continuous-time external variables that are constructed from discrete-time external variables (using DACs).

Since the given data set is completely in discrete-time, i.e. the data set consists of $\{w_1^\star, w_2^\star, w_3^\star, w_4^\star\}$ and $\{r_1^\star, r_2^\star, r_3^\star, r_4^\star\}$ it is possible to exactly represent the system as a discrete-time dynamic network:

$$w^\star(n) = G^\star(q)w^\star(n) + r^\star(n) \tag{7.5}$$

where $G^\star(z)$ is a matrix of discrete-time transfer functions. The question is addressed in this subsection is: how is $G^\star(z)$ of (7.5) related to $G(s)$ of (7.1)?

In the following reasoning, we show that due to aliasing, the elements $G_{jk}^\star(z)$ of $G^\star(z)$ are not simply the discrete versions of their continuous-time counterparts

$G_{jk}(s)$. In fact, we show that the discrete-time network (7.5) may not even have the same interconnection structure as the continuous-time network (7.1). From (7.1) we have

$$w(t) = \big(I - G(p)\big)^{-1} r(t) = F(p) r(t) \tag{7.6}$$

where $F(p) = (I - G(p))^{-1}$. Suppose that each DAC is equipped with a zero-order-hold circuit, then from (7.4), the continuous-time representation of the external variable $r_k^\star(n)$ is

$$\begin{aligned} r_k(t) &= \sum_{m=0}^{\infty} r_k^\star(m)\mathrm{rect}(t - T) \\ &= \sum_{m=0}^{\infty} r_k^\star(m)\mathrm{rect}(t) * \delta(t - mT) \end{aligned}$$

where $*$ denotes convolution, and $\delta(t)$ is an impulse function. Consequently, (7.6) can be expressed as:

$$\begin{bmatrix} w_1(t) \\ \vdots \\ w_L(t) \end{bmatrix} = f(t) * \begin{bmatrix} \mathrm{rect}(t) & & \\ & \ddots & \\ & & \mathrm{rect}(t) \end{bmatrix} * \begin{bmatrix} \sum_{m=0}^{\infty} r_1^\star(m) * \delta(t - mT) \\ \vdots \\ \sum_{m=0}^{\infty} r_L^\star(m) * \delta(t - mT) \end{bmatrix}, \tag{7.7}$$

where $f(t) = \mathcal{L}^{-1}[F(s)]$ where $\mathcal{L}^{-1}[\,\cdot\,]$ denotes the inverse Laplace transform. Let $f_c(t) = \mathcal{L}^{-1}[F(s)\mathrm{diag}(\mathrm{Rect}(s), \ldots, \mathrm{Rect}(s))]$. Then (7.7) can be expressed as:

$$\begin{aligned} w(t) &= f_c(t) * \begin{bmatrix} \sum_{m=0}^{\infty} r_1^\star(m) * \delta(t - mT) \\ \vdots \\ \sum_{m=0}^{\infty} r_L^\star(m) * \delta(t - mT) \end{bmatrix} \\ &= \int_0^t f_c(t - \tau) \begin{bmatrix} \sum_{m=0}^{\infty} r_1^\star(m) * \delta(\tau - mT) \\ \vdots \\ \sum_{m=0}^{\infty} r_L^\star(m) * \delta(\tau - mT) \end{bmatrix} \mathrm{d}\tau \\ &= \sum_{m=0}^{\infty} f_c(t - mT) r^\star(m), \end{aligned} \tag{7.8}$$

where $w(t) = [w_1(t) \ \cdots \ w_L(t)]^T$ and $r^\star(m) = [r_1^\star(m) \ \cdots \ r_L^\star(m)]^T$. From (7.8) by sampling and digitizing $w(t)$ according to (7.2) the discrete-time representation of $w(t)$ is

$$w^\star(n) = \sum_{m=0}^{\infty} f_c^\star(n - m) r^\star(m), \tag{7.9}$$

where $f_c^\star(n) = \big(f_c(t)\big)^\star = \big(\mathcal{L}^{-1}[F_c(s)]\big)^\star$. Interestingly, (7.9) is a discrete-time convolution, an thus using the q notation it can be expressed as:

$$w^\star(n) = F_c^\star(q) r^\star(n). \tag{7.10}$$

where $F_c^\star(q) = \mathcal{Z}[f_c^\star(n)] = \mathcal{Z}\big[\big(\mathcal{L}^{-1}[F_c(s)]\big)^\star\big]$. The equation (7.10) is the discrete-time representation of (7.6) and the relationship between the discrete-time dynamics $F_c^\star(z)$ in (7.10) and the continuous-time dynamics $F(s)$ in (7.6) is:

$$F_c^\star(z) = \mathcal{Z}\Big[\Big(\mathcal{L}^{-1}\big[F(s)\mathrm{diag}\big(\mathrm{Rect}(s),\ldots,\mathrm{Rect}(s)\big)\big]\Big)^\star\Big].$$

Finally, the discrete-time dynamic network equations can be obtained from (7.10). The objective is to obtain an expression of the form $w^\star = G^\star w^\star + r^\star$ (where $G^\star$ has zeros on the diagonal) from the expression (7.10). Consider the following steps:

$$F_c^{\star^{-1}}(q)w^\star(n) = r^\star(n). \tag{7.11}$$

In order to ensure that $G^\star$ has zeros on the diagonal, first set the main diagonal of the matrix on the left in (7.11) to all ones by multiplying by $D^\star$, where $D^\star$ is a diagonal matrix of all the diagonal elements of $F_c^\star$:

$$D^\star(q)F_c^{\star^{-1}}(q)w^\star(n) = D^\star(q)r^\star(n). \tag{7.12}$$

In (7.12) the diagonal elements of $D^\star(q)F^{\star^{-1}}(q)$ are all one, and so it can be expressed as $I + (D^\star(q)F_c^{\star^{-1}}(q) - I)$ where the second matrix in the sum has zeros on the diagonal. Thus, (7.12) can be expressed as:

$$w^\star(n) = (I - D^\star(q)F_c^{\star^{-1}}(q))w^\star(n) + D^\star(q)r^\star(n). \tag{7.13}$$

Finally, let $G^\star(q) = (I - D^\star(q)F_c^{\star^{-1}}(q))$. Then (7.13) can be expressed as

$$w^\star(n) = G^\star(q)w^\star(n) + D^\star(q)r^\star(n) \tag{7.14}$$

which is an equation that represents a discrete-time dynamic network. Equation (7.14) describes the discrete-time representation of the continuous-time data generating system (7.1). This expression can now be used to investigate the effect of sampling on the interconnection structure of the discrete-time dynamic network. As a summary of the previous derivation, the key equations relating the continuous-time and discrete-time dynamic networks are:

$$F_c^\star(z) = \mathcal{Z}\Big[\Big(\mathcal{L}^{-1}\big[F(s)\mathrm{diag}\big(\mathrm{Rect}(s),\ldots,\mathrm{Rect}(s)\big)\big]\Big)^\star\Big] \tag{7.15a}$$

$$G^\star(z) = I - D^\star(z)F_c^{\star^{-1}}(z) \tag{7.15b}$$

$$G(s) = I - F^{-1}(s) \tag{7.15c}$$

From (7.15b) and (7.15c) to obtain $G(s)$ and $G^\star(z)$ from $F(s)$ and $F^\star(z)$ respectively, involves simply algebraic manipulations. Thus, the main effect of sampling is encoded into the first relationship (7.15a).

An important feature of the continuous-time to discrete-time transformation by sampling and digitizing, i.e. the operation $\mathcal{Z}[(\mathcal{L}^{-1}[\ \cdot\ ])^\star]$ is that a product (or quotient) of two continuous-time transfer functions (say $G_{jk}(s)$ and $G_{k\ell}(s)$) cannot be separated/factored in the discrete-time domain, i.e.

$$\mathcal{Z}\big[\big(\mathcal{L}^{-1}[G_{jk}(s)G_{k\ell}(s)]\big)^\star\big] \neq \mathcal{Z}\big[\big(\mathcal{L}^{-1}[G_{jk}(s)]\big)^\star\big] \cdot \mathcal{Z}\big[\big(\mathcal{L}^{-1}[G_{k\ell}(s)]\big)^\star\big]. \tag{7.16}$$

The reason is due to *aliasing*. The effect of aliasing is easier to see in frequency domain (Pintelon & Schoukens, 2012b):

$$\begin{aligned}\mathcal{F}\big[\big(\mathcal{L}^{-1}[G_{jk}(s)G_{k\ell}(s)]\big)^\star\big] &= \Big(G_{jk}(j\omega)G_{k\ell}(j\omega)\Big) * \Big(\frac{1}{T}\sum_{n=-\infty}^{\infty}\delta(\omega - n\omega_s)\Big)\\ &= G_{jk}(j\omega)G_{k\ell}(j\omega) + \underbrace{\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty} G_{jk}(j\omega - j\omega_s n)G_{k\ell}(j\omega - j\omega_s n)}_{\text{component due to aliasing}}, \end{aligned} \tag{7.17}$$

where $\mathcal{F}[\,\cdot\,]$ is the Discrete-Time Fourier Transform and $\omega_s = 2\pi/T$ is the sampling frequency. From (7.17) if $G_{jk}(j\omega)G_{k\ell}(j\omega)$ is non-zero for frequencies greater than ω_s then the component due to aliasing in (7.17) will be non-zero (i.e. the shifted versions of the Fourier Transform of the unsampled signals overlap).

The matrix $F(s) = (I - G(s))^{-1}$ of course contains many products of transfer functions. Due to the fact that products of transfer functions can not be "split" after sampling, the operations in (7.15b) to obtain $G^\star(z)$ from $F_c^\star(z)$ do not "undo" the operations in (7.6) where $F(s)$ is obtained from $G(s)$. The result is that, in general, the (j,k)th element of $G^\star(z)$, i.e. $G_{jk}^\star(z)$, is not simply a discretized version of $G_{jk}(s)$, i.e.

$$G_{jk}^\star(z) \neq \mathcal{Z}\big[\big(\mathcal{L}^{-1}[G_{jk}(s)]\big)^\star\big].$$

In fact, the discrete-time dynamic network may not even have the same interconnection structure as its continuous-time counterpart. This is illustrated in the following example.

Example 7.1 *Consider the system shown in Fig. 7.1. Suppose that the ADCs are not equipped with anti-aliasing filters. Suppose that all the DACs are equipped with zero-order-hold circuits. The continuous-time dynamic network is*

$$\begin{bmatrix} w_1(t)\\ w_2(t)\\ w_3(t)\\ w_4(t)\end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & G_{14}(s)\\ G_{21}(s) & 0 & G_{23}(s) & 0\\ 0 & G_{32}(s) & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}\begin{bmatrix} w_1(t)\\ w_2(t)\\ w_3(t)\\ w_4(t)\end{bmatrix} + \begin{bmatrix} r_1(t)\\ r_2(t)\\ r_3(t)\\ r_4(t)\end{bmatrix} \tag{7.18}$$

By following the steps shown in (7.7) to (7.14) the resulting discrete-time representation of the system is:

$$\begin{bmatrix} w_1^\star(n)\\ w_2^\star(n)\\ w_3^\star(n)\\ w_4^\star(n)\end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & G_{14}(q)\\ G_{21}(q) & 0 & G_{23}(q) & G_{24}(q)\\ G_{31}(q) & G_{32}(q) & 0 & G_{34}(q)\\ 0 & 0 & 0 & 0\end{bmatrix}\begin{bmatrix} w_1^\star(n)\\ w_2^\star(n)\\ w_3^\star(n)\\ w_4^\star(n)\end{bmatrix} + \begin{bmatrix} r_1^\star(n)\\ D_2(q)r_2^\star(n)\\ D_3(q)r_3^\star(n)\\ r_4^\star(n)\end{bmatrix} \tag{7.19}$$

Note that the resulting interconnection structure of the discrete-time network is not the same as that of the continuous-time network. The expression for the transfer

function $G_{23}(z)$ is equal to

$$G_{23}(z) = \frac{\mathcal{Z}\Big[\big(\mathcal{L}^{-1}[G_{23}(s)Rect(s)S(s)]\big)^{\star}\Big]}{\mathcal{Z}\Big[\big(\mathcal{L}^{-1}[S(s)]\big)^{\star}\Big]}$$

where $S(s) = (\frac{1}{1-G_{23}(s)G_{32}(s)})Rect(s)$. If (7.16) were to hold with equality, then the product in the numerator could be "split" resulting in a cancellation with the denominator, and $G_{23}(z)$ would simply be the discrete-time version of $G_{23}(s)$. □

Remark 7.2 *We have shown that the interconnection structure of the continuous-time and corresponding discrete-time dynamic network models are (in general) not the same. In the framework that we consider in this thesis, the interconnection structure of the continuous-time dynamic network model is known, thus, one could simply impose this known interconnection structure on the discrete-time model. Consider the case illustrated in Example 7.1. Suppose that one is attempting to identify G_{23}. Based on the discrete-time interconnection structure, we would need to include w_1, w_3 and w_4 as predictor inputs (see (7.19)). However, based on the continuous-time interconnection structure, we only need to include w_1 and w_3 as predictor inputs (see (7.18)). Thus, imposing the continuous-time interconnection structure on the discrete-time system is equivalent to ignoring the effect of w_4. Because both w_1 and w_3 are correlated to w_4, this will result in additional bias. Further investigation is needed to obtain an exact expression for this additional bias term.*

There are two important cases when the elements of $G^{\star}(z)$ are (approximately) the discrete versions of the corresponding elements of $G(s)$: when the sampling rate is fast enough such that the system does not have significant frequency content at frequencies greater than half the sampling frequency (i.e. the Nyquist frequency) and/or when anti-aliasing filters are used to take the measurements.

First consider the effect of a fast sampling rate. Suppose that all the dynamics in the network are low-pass. By increasing the sampling rate, the aliasing terms in (7.17) become smaller with the result that

$$\mathcal{Z}\big[\big(\mathcal{L}^{-1}[G_{jk}(s)G_{k\ell}(s)]\big)^{\star}\big] \approx \mathcal{Z}\big[\big(\mathcal{L}^{-1}[G_{jk}(s)]\big)^{\star}\big] \cdot \mathcal{Z}\big[\big(\mathcal{L}^{-1}[G_{k\ell}(s)]\big)^{\star}\big]. \tag{7.20}$$

Since all the products of transfer functions can be split into separate components, the operations in (7.15b) to obtain $G^{\star}(z)$ from $F^{\star}(z)$ "undo" the operations in (7.6) where $F(s)$ is obtained from $G(s)$ with the result that each entry of $G^{\star}(z)$ is (approximately) a discrete-time version of the corresponding entry of $G(s)$.

If the external variables and the process noise variables only have power in the frequency range $[-\omega_s/2, \omega_s/2]$ then the dynamics in (7.17) that result in aliasing are not excited. Thus, the approximation (7.20) can be quite good, with again, the result that each entry of $G^{\star}(z)$ is (approximately) a discrete-time version of the corresponding entry of $G(s)$. Note however, that requiring external variables to have zero frequency content outside of the frequency range $[-\omega_s/2, \omega_s/2]$ the DACs cannot be equipped with zero-order-holds since this causes the continuous time external variable to have frequency content over all frequencies. Thus, if the external

variables are injected into the system using DACs, it may not be possible to satisfy the condition that they have no frequency content outside the range $[-\omega_s/2, \omega_s/2]$, depending on the DACs.

A second alternative is that anti-aliasing filters are used to take the measurements, i.e. before sampling, the internal variables are lowpass filtered. The consequence is that all frequencies outside of $[-\omega_s/2, \omega_s/2]$ in (7.17) are filtered out. The result is that

$$\mathcal{Z}\big[\big(\mathcal{L}^{-1}[G_{aa}(s)G_{jk}(s)G_{k\ell}(s)]\big)^\star\big] \approx \mathcal{Z}\big[\big(\mathcal{L}^{-1}[G_{aa}(s)G_{jk}(s)]\big)^\star\big]\cdot\mathcal{Z}\big[\big(\mathcal{L}^{-1}[G_{aa}(s)G_{k\ell}(s)]\big)^\star\big] \quad (7.21)$$

where $G_{aa}(s)$ is the transfer function of the anti-aliasing filter. This setup is suggested by Pintelon et al. (2008); Pintelon & Schoukens (2012b) which they call a *band-limited setup*. Note with this approach, the dynamics of the anti-aliasing filter become part of the equations. Suppose that each ADC is equipped with an anti-aliasing filter. The expression for the discrete-time internal variables is then:

$$w^\star(n) = \Big(A(p)\big(I - G(p)\big)^{-1}r(t)\Big)^\star \quad (7.22)$$

where A is a diagonal matrix with the anti-aliasing filter of the kth ADC in the (k,k)th entry. In the following example it is shown how the dynamics of the anti-aliasing filters become part of the equations.

Example 7.3 *Consider the same setup as in Example 7.1. Suppose that each ADC is equipped with an anti-aliasing filter. From (7.22) and the steps shown in equations (7.11) to (7.14) the expression for $G_{23}(z)$ is:*

$$G_{23}(z) = \frac{\mathcal{Z}\Big[\big(\mathcal{L}^{-1}[A_{22}(s)G_{23}(s)Rect(s)S(s)]\big)^\star\Big]}{\mathcal{Z}\Big[\big(\mathcal{L}^{-1}[A_{33}(s)S(s)]\big)^\star\Big]} \quad (7.23)$$

where $S(s) = (\frac{1}{1-G_{23}(s)G_{32}(s)})Rect(s)$, and $A_{22}(s)$ and $A_{33}(s)$ are the anti-aliasing filters on the 2nd and 3rd ADCs respectively. Because the anti-aliasing filters remove all the high-frequency content the approximation (7.21) holds. The result is that from (7.23) the expression for $G_{23}(z)$ is (approximately):

$$G_{23}(z) = \frac{\mathcal{Z}\Big[\big(\mathcal{L}^{-1}[A_{22}(s)]\big)^\star\Big]}{\mathcal{Z}\Big[\big(\mathcal{L}^{-1}[A_{33}(s)]\big)^\star\Big]}\mathcal{Z}\Big[\big(\mathcal{L}^{-1}[G_{23}(s)]\big)^\star\Big].$$

Thus, if the anti-aliasing filters both have the same dynamics, or are calibrated so that they are the same, then the first term in the expression is simply 1, with the result that $G_{23}(z)$ is a discrete version of $G_{23}(s)$.

In general, if the anti-aliasing filters are all the same, then their transfer functions will cancel out of the expression for G_{ji}. For certain open loop or closed loop systems, this can be relatively simple (called a relative callibration Pintelon & Schoukens (2012b)). However, for certain dynamic networks this could be difficult.

7.3.2 Effect of Discretization on the Presence of Delays

A second feature of discretization that is particularly relevant in a dynamic network setting is the effect of discretization on the feed-through terms of the discrete-time dynamic network. In Chapters 4 to 6 considerable emphasis was placed on the presence/absence of algebraic loops in the data generating system. In particular, it was shown that for the Direct and Joint IO methods the discrete-time data generating system should not have algebraic loops in order to be able to guarantee consistent estimates. However, the Two-Stage and Instrumental Variable based methods do not place this restriction on the discrete-time data generating system. In this section we show that the inter-sample behavior of the external variables and noise variables affects the presence of direct feed-through terms in the discrete-time representation of the continuous-time system.

The main reasoning is based on a result from Laurijsse (2014). The result is that the direct feed-through term of a discrete-time transfer function which represents an under lying continuous-time data generating system depends on the inter-sample behavior of the input to the transfer function. In particular, consider the following equation:

$$w_k(t) = F_{kk}(p)r_k(t), \tag{7.24}$$

with the following discrete-time representation:

$$w_k^\star(n) = F_{kk}^\star(q)r_k^\star(n). \tag{7.25}$$

Consider the direct feed-through term of $F_{kk}^\star(z)$, i.e. $f_{kk}^\star(0)$. The result of Laurijsse (2014) says that the whether $f_{kk}^\star(0)$ is zero or not depends (in part) on the inter-sample behavior of the continuous-time variable $r_k(t)$ in (7.24). Formally, consider the following proposition is taken from Laurijsse (2014).

Proposition 7.4 *Consider a continuous-time causal SISO system of (7.24). Consider the discrete-time representation of the system, (7.25). The direct feed-through term $f_{kk}^\star(0)$ is dependent on the inter-sample behavior of $r(t)$:*

- *For a piece-wise constant (zero-order-hold) $r(t)$,*

$$f_{kk}^\star(0) = f_{kk}(0).$$

- *For a piece-wise linear (first-order-hold) $r(t)$,*

$$f_{kk}^\star(0) = \int_0^T \frac{T-\tau}{T} f_{kk}(\tau)d\tau.$$

- *For a band-limited $r(t)$ (i.e. $r(t)$ has no power above the frequency $w_s = \frac{\pi}{T}$),*

$$f_{kk}^\star(0) = \int_0^T sinc\Big(\frac{\pi\tau}{T}\Big) f_{kk}(\tau)d\tau.\square$$

For a proof see Laurijsse (2014).

In the previous section, we made the assumption that every internal variable in the network was exclusively driven by a zero-order-hold signal. Thus, by Proposition 7.4 the delay structure of $F(s)$ in (7.6) is the same as that of $F^\star(z)$ in (7.10). However, for a realistic continuous-time data generating system, there will be continuous-time noise variables $v_k(t)$ that excite the system. A continuous-time noise source will (most likely) not be piece-wise constant. A more realistic assumption is that the noise is band-limited. As a consequence, by Proposition 7.4 all non-zero entries of the matrix $F^\star(z)$ will have direct feed-through terms. Since $F^\star = (I - G^\star)^{-1}$, it follows that the discrete-time dynamic network will have algebraic loops.

Based on the results of Chapters 4 to 6 the Two-Stage and Instrumental Variable-based methods do not require that the (discrete-time) data generating system is free of algebraic loops. Thus these methods can be used to identify a module embedded in a discrete-time representation of a continous-time dynamic network. From the results of Chapters 4 and 6 this does mean however, that either at least one external variable needs to be present, or measurements of extra variables in the network are required.

The Direct and Joint IO methods cannot be used since they both rely on the absence of algebraic loops in the discrete-time data generating system. A potential solution to this problem is proposed in Schoukens & Relan (2014) where a model structure is chosen with every transfer function parameterized with a delay. The result is a biased estimate of the transfer function of interest. However, in Schoukens & Relan (2014) it is shown that this bias can be made small by faster sampling.

7.3.3 Discussion

By the results of Sections 7.3.1 and 7.3.2 the indirect continuous-time identification approach will work well if (1) ADC equipped with anti-aliasing filters are used to take the measurements or, if all the dynamics in the network are low pass and a high sampling rate is used and (2) the Two-Stage Method or an IV based method are used to identify the module embedded in the discrete-time dynamic network (that represents the continuous-time dynamic network). In this case we have shown that the discrete-time dynamics $G_{ji}(z)$ between $w_i^\star$ and $w_j^\star$ are directly related to $G_{ji}(s)$. Moreover, by using the Two-Stage or IV based methods the algebraic loops present in the discrete-time network do not pose a problem.

The analysis of Section 7.3.1 was based on a continuous-time dynamic network with only external variables present (no process noise). Further research is required to extend the analysis to the more general situation where noise is also exciting the network. However, the end result is expected to be the same. In the literature there are quite a few publications that deal with continuous-time noise modeling (see Pintelon et al. (2006); Larsson et al. (2008); Gillberg & Ljung (2009) for instance). It is likely that the result we seek is based on the reasoning in these papers.

In certain situations the required assumptions for the indirect continuous-time identification approach are not satisfied. For instance in applications such as biological or economic systems it may not be possible to use anti-aliasing filters.

7.4 DIRECT CONTINUOUS-TIME IDENTIFICATION IN NETWORKS

Similar to the other methods presented in this thesis, here we extend continuous-time closed-loop identification methods to the case of dynamic networks. In the literature there are several continuous-time closed-loop methods. Gilson et al. (2008); Cheng & Wang (2011) present an instrumental variable (IV) method for direct continuous-time identification. However, they do not allow for the possibility of sensor noise. In Pintelon et al. (2008) a frequency domain approach is proposed to identify continuous-time closed-loop systems, with the requirement that an external reference signal that is periodic is present. In this case they allow for sensor noise on all variables, except the reference.

In this section the so called Basic CLIVC method of Gilson et al. (2008) is extended such that it can be used (a) in the presence of sensor noise, (b) for identification in interconnection structures more complex than a closed loop. The method we develop in this section is the continuous-time counterpart to that presented in Section 6.3 in Chapter 6.

The objective is to identify a transfer function $G_{ji}^0(s)$ embedded in a continuous-time dynamic network. First, a set of internal variables must be chosen to form the data set. The internal variable w_j is chosen as the 'output'. For now, choose the set of 'inputs' as all internal variables that have a direct causal connection to w_j (i.e. choose w_k, $k \in \mathcal{N}_j$ as inputs). The transfer functions to be identified are parameterized as rational functions in p:

$$G_{jk}(p,\theta) = \frac{B_{jk}(p,\theta)}{A_{jk}(p,\theta)}, \tag{7.26}$$

for all $k \in \mathcal{N}_j$, where B_{jk} and A_{jk} are polynomials in p:

$$\begin{aligned} B_{jk}(p,\theta) &= b_0^{jk} p^{n_b} + b_1^{jk} p^{n_b-1} + \cdots + b_{n_b}^{jk}, \\ A_{jk}(p,\theta) &= p^{n_a} + a_1^{jk} p^{n_a-1} + \cdots + a_{n_a}^{jk}. \end{aligned}$$

where the parameters are a_n^{jk}, $n = 1, \ldots, n_a$, $k \in \mathcal{N}_j$ and b_n^{jk}, $n = 0, \ldots, n_b$, $k \in \mathcal{N}_j$, i.e.

$$\theta = [a_1^j \ \cdots \ a_{n_a}^j \ b_0^{jk_1} \ \cdots \ b_{n_b}^{jk_1} \ \cdots \ b_0^{jk_d} \ \cdots \ b_{n_b}^{jk_d}]^T$$

where $\{k_1, \ldots, k_d\} = \mathcal{N}_j$. For notational convenience, we assume all polynomials $B_{jk}(\theta)$ and $A_{jk}(\theta)$, $k \in \mathcal{N}_j$ have the same orders, denoted n_b, and n_a respectively. The internal variable w_j can be expressed as

$$\begin{aligned} w_j(t) &= \sum_{k \in \mathcal{N}_j} G_{jk}(p)^0 w_k(t) + v_j(t) \\ &= \frac{1}{\breve{A}_j^0(p)} \sum_{k \in \mathcal{N}_j} \breve{B}_{jk}^0(p) w_k(t) + v_j(t) \end{aligned} \tag{7.27}$$

where

$$\breve{A}_j^0(p) = \prod_{n\in\mathcal{N}_j} A_{jn}^0(p) \text{ and}$$
$$\breve{B}_{jk}^0(p) = \prod_{n\in\mathcal{N}_j\setminus k} B_{jk}^0(p)A_{jn}^0(p).$$

Consequently, the following differential equation holds:

$$\breve{A}_j^0(p)w_j(t) = \sum_{k\in\mathcal{N}_j} \breve{B}_{jk}^0(p)w_k(t) + \breve{A}_j^0(p)v_j(t). \tag{7.28}$$

Moreover, (7.28) holds at each time instant $t = 0, T, 2T, \ldots$. Recall from (7.3) that each internal variable is measured at regular time instants $t = 0, T, 2T, \ldots$. Suppose that the ADCs are not equipped with anti-aliasing filters, i.e. G_{aa} in (7.3) is 1. Thus, the differential equation relating the measured values of the internal variables $\tilde{w}_j$ and $\tilde{w}_k$, $k \in \mathcal{N}_j$ at times $t = 0, T, 2T, \ldots$ is

$$\begin{aligned}\breve{A}_j^0(p)\tilde{w}_j(t_n) &= \sum_{k\in\mathcal{N}_j} \breve{B}_{jk}^0(p)\big(\tilde{w}_k(t_n) - s_k(t_n)\big) + \breve{A}_j^0(p)\big(v_j(t_n) + s_j(t_n)\big)\\ &= \sum_{k\in\mathcal{N}_j} \breve{B}_{jk}^0(p)\tilde{w}_k(t_n) + \breve{v}_j(t_n)\end{aligned} \tag{7.29}$$

where t_n denotes the time instant $t = nT$, $n = 0, 1, \ldots$ and

$$\breve{v}_j(t_n) = \sum_{k\in\mathcal{N}_j} -\breve{B}_{jk}^0(p)s_k(t_n) + \breve{A}_j^0(p)\big(v_j(t_n) + s_j(t_n)\big). \tag{7.30}$$

The notation in (7.29) and (7.30) is perhaps a little sloppy because the continuous-time operator p cannot be applied to a discrete-time sequence. However, we adopt the notation from (Gilson et al. (2008)). The expression (7.29) can be rearranged to collect all coefficients on the right-hand side:

$$\begin{aligned}\tilde{w}_j^{(n_a)}(t_n) &= \sum_{k\in\mathcal{N}_j} \breve{B}_{jk}^0(p)\tilde{w}_k(t_n) + \big(1 - \breve{A}_j^0(p)\big)\tilde{w}_j(t_n) + \breve{v}_j(t_n)\\ &= \tilde{\phi}_j^T(t_n)\breve{\theta}^0 + \breve{v}_j(t_n)\end{aligned} \tag{7.31}$$

where superscript (m) denotes the mth order derivative, $\breve{\theta}^0$ denotes the vector of coefficients of the data generating system, i.e.

$$\breve{\theta}^0 = [\breve{a}_1^{j^0} \ \cdots \ \breve{a}_{\check{n}_a}^{j^0} \ \breve{b}_0^{jk_1{}^0} \ \cdots \ \breve{b}_{\check{n}_b}^{jk_1{}^0} \ \cdots \ \breve{b}_0^{jk_d{}^0} \ \cdots \ \breve{b}_{\check{n}_b}^{jk_d{}^0}]^T \tag{7.32}$$

where $\{k_1, \ldots, k_d\} = \mathcal{N}_j$, and

$$\tilde{\phi}_j^T(t_n) = \big[-\tilde{w}_j^{(n_a-1)}(t_n) \cdots -\tilde{w}_j(t_n) \ \tilde{w}_{k_1}^{(n_b)}(t_n) \cdots \tilde{w}_{k_1}(t_n) \cdots \ \tilde{w}_{k_d}^{(n_b)}(t_n) \ \cdots \ \tilde{w}_{k_d}(t_n)\big] \tag{7.33}$$

where $\{k_1, \ldots, k_d\} = \mathcal{N}_j$. Note that if $\breve{\theta}$ is known, it is possible to calculate the value of θ in (7.26).

The IV estimate of $\breve{\theta}^0$ is the solution to

$$\hat{\theta}_{IV} = \operatorname*{sol}_{\breve{\theta}} \left\{ \frac{1}{N} \sum_{n=0}^{N-1} Z(t_n) \big(\tilde{w}_j^{(n_a)}(t_n) - \tilde{\phi}_j^T(t_n) \breve{\theta} \big) = 0 \right\}, \tag{7.34}$$

where $Z(t_n) = [z^{(n_z)}(t_n)\ z^{(n_z-1)}(t_n)\ \cdots\ z(t_n)]^T$ and $z(t_n)$ is a vector of instrumental variables. The choice of the instrumental variables is critical with respect to the consistency of the estimates. In this chapter we consider all external and (measurements of) internal variables, except $\tilde{w}_k$, $k \in \mathcal{N}_j \cup \{j\}$ to be potential candidate instrumental variables. Let $\mathcal{I}_j$ and $\mathcal{X}_j$ denote the sets of indices of internal and external variables respectively chosen as instrumental variables. Then,

$$z(t_n) = [r_{\ell_1}(t_n)\ \cdots\ r_{\ell_n}(t_n)\ \tilde{w}_{m_1}(t_n)\ \cdots\ \tilde{w}_{m_n}(t_n)]^T \tag{7.35}$$

where $\mathcal{X}_j = \{\ell_1, \ldots, \ell_n\}$ and $\mathcal{I}_j = \{m_1, \ldots, m_n\}$. The parameter estimate is

$$\hat{\theta}_{IV} = \Big(\frac{1}{N} \sum_{n=0}^{N-1} Z(t_n) \tilde{\phi}_j^T(t_n) \Big)^- \Big(\sum_{n=0}^{N-1} Z(t_n) w_j^{(n_a)}(t_n) \Big). \tag{7.36}$$

An expression of $\hat{\theta}_{IV}$ in terms of $\breve{\theta}^0$ can be obtained by substituting (7.31) into (7.36):

$$\hat{\theta}_{IV} = \breve{\theta}^0 + \Big(\frac{1}{N} \sum_{n=0}^{N-1} Z(t_n) \tilde{\phi}_j^T(t_n) \Big)^{-1} \Big(\sum_{n=0}^{N-1} Z(t) \breve{v}_j^{(n_a)}(t_n) \Big). \tag{7.37}$$

Thus, by starndard IV analysis, $\hat{\theta}_{IV} \to \breve{\theta}^0$ as $N \to \infty$ with probability 1, i.e. it is *consistent*, if the following conditions hold:

(a) $\bar{\mathbb{E}}\big[Z(t_n)\tilde{\phi}_j^T(t_n)\big]$ has full column rank.

(b) $\bar{\mathbb{E}}[Z(t_n)\breve{v}_j(t_n)] = 0$.

It is clear that for Condition (a) to hold, the instrumental variables must be correlated to the internal variables w_k, $k \in \mathcal{N}_j \cup \{j\}$ that make up $\tilde{\phi}_j(t)$. Recall Lemma 6.7 in Chapter 6 where conditions were presented that ensure two internal variables are correlated.

The second condition, i.e. that $\bar{\mathbb{E}}[Z(t_n)\breve{v}(t_n)] = 0$ also has a nice interpretation in terms of paths in a dynamic network. Recall the reasoning in Section 6.3. The following reasoning is entirely equivalent. Suppose $\tilde{w}_\ell$ is chosen as an instrumental variable. If there is no path from w_j to w_ℓ, then w_ℓ is not a function of v_j. If, in addition s_ℓ is uncorrelated to s_j and s_k, $k \in \mathcal{N}_j$, then the condition holds. Thus this condition places a restriction on the set of candidate instrumental variables. Note that every external variable is still a candidate instrumental variable, since, by definition there is no path from v_j to any r_ℓ. The reasoning is summarized in the following proposition which is the continuous-time counterpart to Proposition 6.8 of Chapter 6.

Proposition 7.5 *Consider a dynamic network as defined in (7.1). Choose the sets $\mathcal{I}_j$ and $\mathcal{X}_j$ of instrumental variables such that $\mathcal{I}_j \cap \{\mathcal{N}_j \cup \{j\}\} = \emptyset$. Consider the estimate $\hat{\theta}_{IV}$ of (7.34) where $n_z \geq \lceil \text{length}(\tilde{\phi}_j(t))/\text{length}(z(t)) \rceil$. The estimate $\hat{\theta}_{IV}$ is consistent if the following conditions are satisfied:*

(a) If v_j is present, there is no path from w_j to any w_ℓ, $\ell \in \mathcal{I}_j$

(b) The matrix $\bar{\mathbb{E}}\Big[Z(t_n)\tilde{\phi}_j(t_n)]\Big]$ has full column rank.

(c) Each sensor noise s_ℓ, $\ell \in \mathcal{I}_j$ is uncorrelated to all s_k, $k \in \mathcal{N}_j$.

(d) If v_j is present, then it is uncorrelated to all v_m with a path to w_j.

(e) The parameterization is flexible enough, i.e. there exists a θ such that $G_{jk}(s,\theta) = G_{jk}(s)$, $\forall k \in \mathcal{N}_j$. □

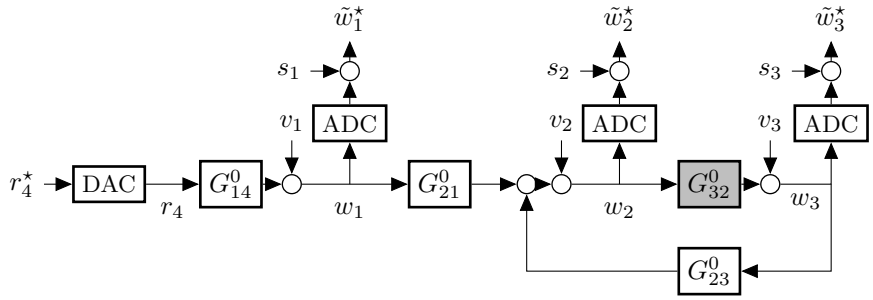

Figure 7.2: Closed loop data generating system

Example 7.6 *Consider the data generating system shown in Fig. 7.2. Suppose that the objective is to obtain a consistent estimate of G_{32}^0. Thus, $\{j\} = \{3\}$, and $\mathcal{N}_3 = \{2\}$. Suppose that we choose $\tilde{w}_1$ as the instrumental variable ($\mathcal{I}_3 = \{1\}$, and $\mathcal{X}_3 = \emptyset$). This choice satisfies the requirement $\{\mathcal{N}_3 \cup \{3\}\} \cap \mathcal{I}_3 = \emptyset$. Since there is no path from w_3 to w_1, Condition (a) of Proposition 7.5 holds.*

By Lemma 6.7 the necessary conditions for Condition (b) of Proposition 7.5 to hold are satisfied since there is a path from w_1 to both w_2 and w_3. Thus, if the remaining conditions of Proposition 7.5 hold, then consistent estimates of G_{jk}^0, $k \in \mathcal{N}_j$ are obtained by solving (7.34). □

By the exact same reasoning as in Section 6.5 of Chapter 6 it is possible to incorporate the predictor input selection methods of Chapter 5 to the situation presented here. The reasoning is not repeated here.

In order to calculate $\hat{\theta}_{IV}$ in (7.34) it is necessary to calculate the derivatives of w_k $k \in \mathcal{N}_j \cup \{j\}$ and the instrumental variables. A standard approach in the

continuous-time identification literature is to approximate the derivative as (Garnier et al., 2003)

$$w_k^{(m)}(t_n) = p^n w_k(t_n) \approx \frac{p^m}{(p+\lambda)^{m_s}} w_k(t_n) \tag{7.38}$$

where $m_s \geq m$. The parameter λ determines the bandwidth of the approximation. Within the bandwidth determined by λ, the filter $\frac{p^m}{(p+\lambda)^{m_s}}$ is equal to the derivative operator. In the literature this is referred to as the *state-variable filter* method (Garnier & Wang, 2008; Garnier et al., 2003). Other methods for approximating the derivative exist (see Garnier et al. (2003) for instance). For many of the methods the approximation of the derivative improves with increasing sampling rate and low pass dynamics (Van Hamme et al., 1991; Marelli & Fu, 2010).Thus, it may be beneficial to use the fastest sampling rate possible when applying the method presented in this section in practice.

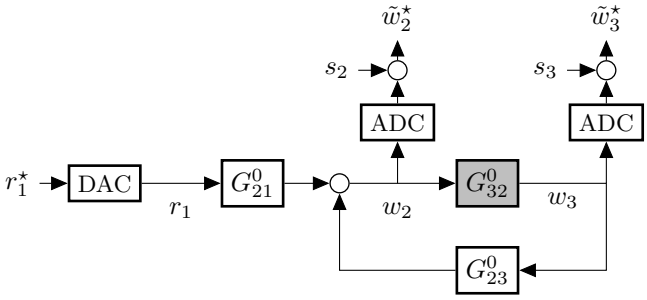

Figure 7.3: Closed loop data generating system considered in Example 7.7

Example 7.7 *Consider the network shown in Figure 7.3. Note that sensor noise is present on both measured variables, and no process noise is present. This is not because the methods cannot handle process noise, but simply to be inline with the reasoning of Section 7.3.1. Both the indirect and direct approaches have been performed using a data set consisting of* 10000 *data points sampled at* 100*Hz using ADCs not equipped with anti-aliasing filters. In this simulation only a little sensor noise is added in order to highlight the bias of the estimates. A bode plot of the dynamics of the continuous-time $G_{32}(s)$ is shown in the bottom plot of Fig. 7.4 in grey. The corresponding discrete-time transfer function $G_{32}(z)$ calculated using the reasoning in Section 7.3 is shown in the top plot in grey. The difference in dynamics between the two plots is because of high-frequency content in other parts of the network. In the top plot the discrete-time IV method of Section 6.3 is used to estimate $G_{32}(z)$ (i.e. the discrete-time counterpart to the continuous-time method described in this section). The estimates are shown in blue. In the bottom plot the continuous-time IV method of this paper is used to estimate $G_{32}(s)$. From the figure it is clear that the dynamics of $G_{23}(z)$ and $G_{23}(s)$ do not match (due to aliasing) and so it is not possible to reconstruct a good estimate of $G_{32}(s)$ from $G_{32}(z, \theta)$.*

Also note that the estimate of $G_{23}(s)$ is biased. This is because due to the fact that when the sampling rate is not fast enough compared to the bandwidth of the signal, the state-variable-filter method for calculating the derivative is not very accurate.

In Figure 7.5 the same experiment is repeated but with a sampling rate of $250Hz$. In this case more sensor noise is added to the system in order to get an idea of the variances of the two approaches. In this case it both the direct and indirect continuous-time identification approaches result in consistent estimates of $G^0_{23}(s)$. □

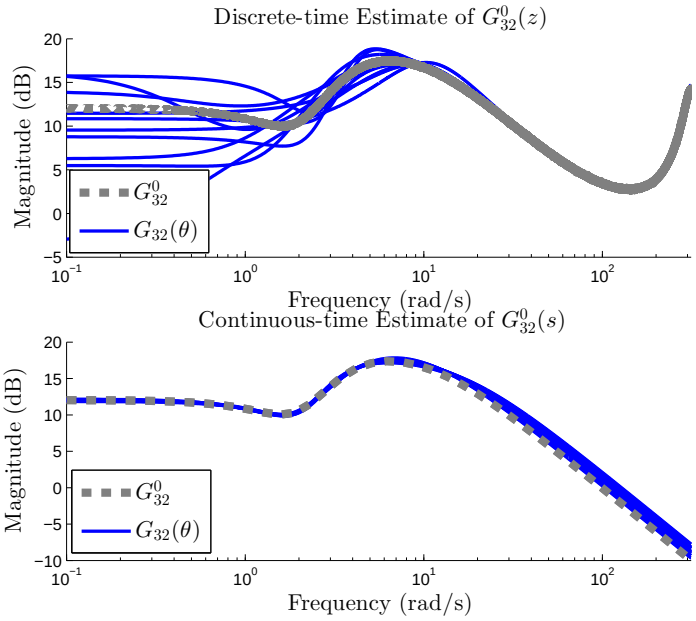

Figure 7.4: Results of direct and indirect continous-time identification approaches for the continuous-time system shown in Fig. 7.3 with a sampling frequency of 100Hz.

Remark 7.8 *Another advantage of the proposed direct continuous-time identification method presented in this chapter is that it is easily modified so that it is possible to identify non-rational transfer functions, such as transfer functions in $\sqrt{s}$. This is a useful feature because diffusive phenomena such as heat conduction, viscous flow, flow through porous media, and transmission lines are modeled more accurately using fractional order models Oldham & Spanier (1974); Caponetto et al. (2010); Herrmann (2011). There already exist some identification methods to identify transfer functions in $\sqrt{s}$ (Pintelon et al., 2005; Monteyne et al., 2011, 2014). However, these are currently formulated as open-loop methods, and the presence of sensor noise is not dealt with in any of these methods. The method presented in this chapter can be easily extended in order to identify transfer functions that are*

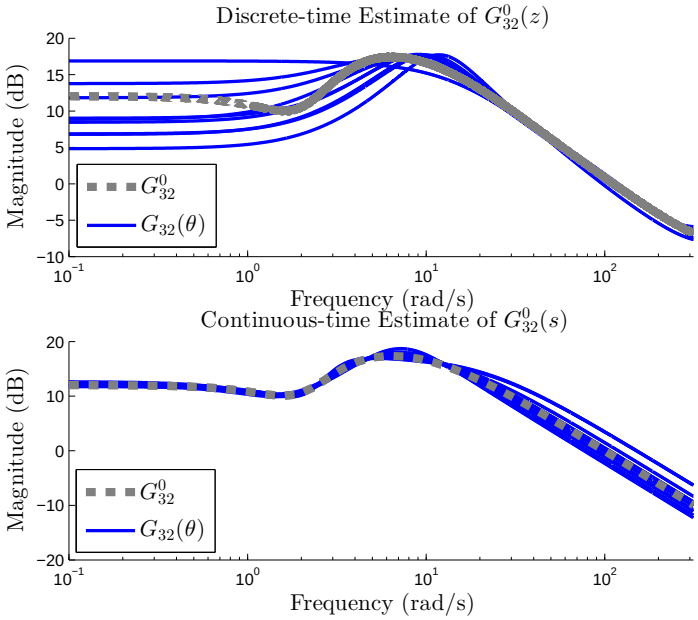

Figure 7.5: Results of direct and indirect continous-time identification approaches for the continuous-time system shown in Fig. 7.3 with a sampling frequency of 250Hz.

functions of $\sqrt{s}$. Essentially the only difference is to replace the expression for a derivative in (7.38) with an approximation for a fractional derivative.

7.5 SUMMARY

In the first part of this chapter we investigated the indirect continuous-time identification approach. We showed that if anti-aliasing filters and/or fast enough sampling rate are used to take the measurements then each transfer function in the resulting discrete-time dynamic network is directly related to its continuous-time counterpart. Some more work is required in order to obtain an exact expression for the bias, in addition the analysis presented in this chapter did not account for the presence of process noise. Secondly we showed that the inter-sample behavior of the external variables affects the presence of direct feed-through terms in the discrete-time representation of the continuous-time data generating system. Consequently, there are algebraic loops present in the discrete-time representation, which means that only the Two-Stage or IV-based methods can be used to consistently identify the discrete-time dynamics. In the second part of the chapter, we considered the objective of directly obtaining an estimate of $G_{ji}(s)$ embedded in a dynamic network. We present a direct continuous-time IV method to consistently estimate a transfer function embedded in a dynamic network.

Chapter 8

CASE STUDY - WELL TEST ANALYSIS

In this chapter the tools that have been developed throughout the thesis are applied to a practical situation. The idea is to use measurements of pressure and flow rate in a well bore to estimate the thickness and permeability of an oil reservoir.[1]

8.1 INTRODUCTION

WELL TEST ANALYSIS is a standard procedure in reservoir engineering to extract information about dynamic properties and geological features of an underground hydrocarbon reservoir from flow and pressure measurements. In particular Well Test Analysis is used to obtain estimates of the average permeability and reservoir thickness. The test involves producing from a well based on a sequence of planned wellhead flow rates and continuously recording the pressure and flow rate at the bottom hole of the well. Conventionally, in the analysis phase either the step or impulse response of the reservoir is calculated by deconvolution (Gringarten, 2008). Then the impulse response (or step response) is used to estimate the physical parameters of the reservoir (Gringarten, 2008).

Mansoori et al. (2013) proposed a new framework for the Well Test Analysis in which first a dynamical model is identified based on the measurements by using suitable system identification techniques. Then the identified model is used to estimate the physical parameters of the reservoir by comparing the estimated transfer functions to a physics based model.

The chapter proceeds as follows. In Section 8.2 it is shown how a well and reservoir can be modeled using the procedure suggested in Chapter 2 using bilaterally coupled systems. In Section 8.3 the EIV instrumental variable method of Section 6.3 is applied to the data generating system considered here in order to identify the dynamics of the reservoir.

[1] The material presented in this chapter is based on joint work with Mehdi Mansoori. The results contained in this chapter are based on Mansoori et al. (2014)

8.2 PHYSICS BASED MODELING OF THE PRODUCTION SYSTEM

We consider a single vertical well connected to a homogeneous reservoir as shown in Figure 8.1. The production system is comprised of a convective flow in the well bore and diffusive flow in the reservoir that interact with each other at the bottom hole of the well.

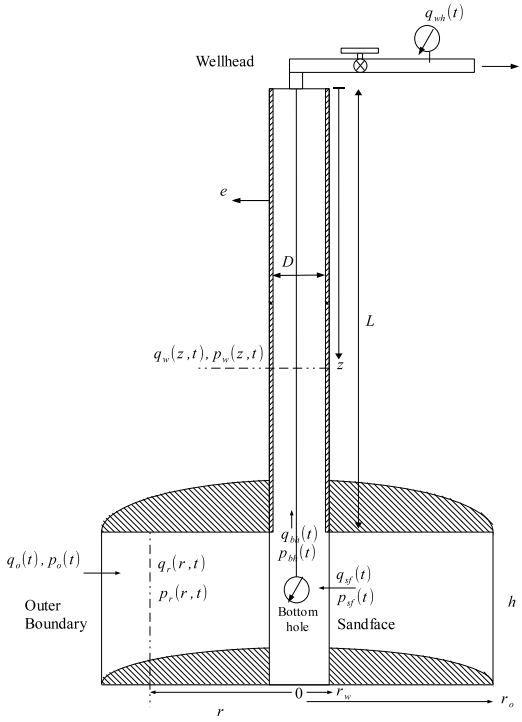

Figure 8.1: A cylindrical homogeneous reservoir with a vertical well. This figure was created by Mehdi Mansoori (Mansoori et al., 2014)

The reservoir and well bore shown in Figure 8.1 are modeled as two bilaterally coupled subsystems. First the equations for the well bore and then the reservoir are presented. The following derivation of the equations for the wellbore and the reservoir was done by Mehdi Mansoori (Mansoori et al., 2014).

8.2.1 Modeling the Well Bore

Consider modeling the well bore. The flow in the well bore is governed by the so called *Water-Hammer equations* if the fluid flow velocity is much smaller that the velocity of sound in the fluid (Chaudhry, 1987). Thus the equations governing the

flow rate, $q(z,t)$, and pressure, $p(z,t)$, at depth z and time t are

$$\frac{\rho a^2}{A}\frac{\partial q(z,t)}{\partial z} + \frac{\partial p(z,t)}{\partial t} = 0, \tag{8.1}$$

$$\frac{A}{\rho}\frac{\partial p(z,t)}{\partial z} + \frac{\partial q(z,t)}{\partial t} + Rq(z,t) - Ag = 0, \tag{8.2}$$

where g (9.81 m/s^{-2}) is the acceleration due to gravity; A (m^2) is the cross sectional area of the well; $a^2 = K/[\rho + KD\rho(eE)^{-1}]$ (m/s) is the velocity of the Water-Hammer wave; $R = 32\nu/D^2$ (s^{-1}) is the laminar flow friction effect;, and the remaining parameters are defined in Table 8.1.

Table 8.1: Well bore and Reservoir Properties

Model Parameters	Parameter Values
Reservoir boundary, (r_e)	3000 m
Well radius, (r_w)	0.1 m
Reservoir height, (H)	50 m
Pipe internal diameter, (D)	0.1 m
pipe wall thickness,(e)	16×10^{-3}m
Well length, (L)	2000 m
Permeability of rock, (k)	2×10^{-13}m^2
Porosity of rock, (ϕ)	0.2
Viscosity of fluid, (μ)	0.01 Pa.s
Total compressibility, (C_t)	7.25×10^{-9} Pa^{-1}
bulk modulus elasticity of the fluid, (K)	1.5×10^{9} Pa
kinematic viscosity of the fluid,(ν)	1.11×10^{-5} m^2 s^{-1}
Density of fluid, (ρ)	900 Kgm^{-3}
Young's Modulus of elasticity, (E)	200×10^{9}Pa

Solving the equations in the Laplace domain leads to a hyperbolic equation that needs two boundary conditions. At each side of the well bore only one of the variables can be the boundary condition, therefore in accordance with the well testing configuration the flow rate at the well head and the pressure at the bottom hole (denoted $q_{wh}(t)$ and $p_{bh}(t)$ respectively) are taken as the boundary conditions. This results in the following equations:

$$\begin{bmatrix}\mathcal{P}_{wh}(s)\\ \mathcal{Q}_{bh}(s)\end{bmatrix} = \begin{bmatrix}W_{11} & W_{12}\\ W_{21} & W_{22}\end{bmatrix}\begin{bmatrix}\mathcal{Q}_{wh}(s)\\ \mathcal{P}_{bh}(s)\end{bmatrix}, \tag{8.3}$$

where $\mathcal{P}_{wh}(s)$ and $\mathcal{P}_{bh}(s)$ are the Laplace transforms of the pressure at the well head and bottom hole respectively, $\mathcal{Q}_{wh}(s)$ and $\mathcal{Q}_{bh}(s)$ and the Laplace transforms of the flow rate at the well head and bottom hole respectively, and

$$W_{11} = \frac{\rho\mu a^2}{sA}\tanh\mu L \qquad W_{12} = \frac{1}{\cosh\mu L}$$

$$W_{21} = \frac{1}{\cosh\mu L} \qquad W_{22} = -\frac{sA}{\rho\mu a^2}\tanh\mu L$$

This is the model of the well.

8.2.2 Modeling the Reservoir

Now consider modeling the reservoir. A reservoir is porous rock filled with fluid. The reservoir is modeled as a cylinder, with fluid flowing radially toward the well bore. The outer edge of the reservoir is called the *outer boundary*. The intersection of the well bore and the reservoir is called the *sand face*. In this situation, the radial flow rate $q(r,t)$ and pressure $p(r,t)$ in the reservoir at radial distance r from the symmetry axis, satisfy the diffusivity equation and Darcy's Law

$$\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial p(r,t)}{\partial r} = \frac{1}{\eta}\frac{\partial p(r,t)}{\partial t} \tag{8.4}$$

$$q(r,t) = -\frac{2\pi rkh}{\mu}\frac{\partial p(r,t)(t)}{\partial r} \tag{8.5}$$

with $\eta = \phi\mu c_t/k$ (s^{-1}) is the hydraulic diffusivity; and the remaining parameters are defined in Table 8.1.

Solution of the elliptic diffusivity equation in the Laplace domain requires two boundary conditions which are chosen to be the flow rate at the sand face and the pressure at the outer boundary (denoted $q_{sf}(t)$ and $p_o(t)$ respectively). This leads to

$$\begin{bmatrix}\mathcal{P}_{sf}(s)\\ \mathcal{Q}_o(s)\end{bmatrix} = \begin{bmatrix}R_{11} & R_{12}\\ R_{21} & R_{22}\end{bmatrix}\begin{bmatrix}\mathcal{Q}_{sf}(s)\\ \mathcal{P}_o(s)\end{bmatrix} \tag{8.6}$$

where

$$R_{11} = \frac{\mu}{2\pi khr_w\sqrt{\frac{s}{\eta}}}\frac{I_{0e}K_{0w} - I_{0w}K_{0e}}{I_{0e}K_{1w} + I_{1w}K_{0e}}, \tag{8.7}$$

$$R_{12} = \frac{I_{0w}K_{1w} + I_{1w}K_{0w}}{I_{0e}K_{1w} + I_{1w}K_{0e}}, \tag{8.8}$$

$$R_{21} = \frac{r_e}{r_w}\frac{I_{1e}K_{0e} - I_{0e}K_{1e}}{I_{0e}K_{1w} + I_{1w}K_{0e}}, \tag{8.9}$$

$$R_{22} = \frac{2\pi kh}{\mu}r_e\sqrt{\frac{s}{\eta}}\frac{I_{1w}K_{1e} - I_{1e}K_{1w}}{I_{0e}K_{1w} + I_{1w}K_{0e}}. \tag{8.10}$$

where I and K are modified Bessel functions of the first and second kind and $I_{ij} = I_i(r_j\sqrt{\frac{s}{\eta}})$ and $K_{ij} = K_i(r_j\sqrt{\frac{s}{\eta}})$.

The complete production system model is obtained by concatenating of the bottom hole side of the well bore model to the sand face side of the reservoir model by coupling $q_{bh}(t)$ to $q_{sf}(t)$ and $p_{sf}(t)$ to $p_{bh}(t)$ as shown in Figure 8.2.

Recall that the purpose of the well test analysis is to obtain estimates of the average permeability, k, and the reservoir thickness, h. From a well test analysis point of view, the parameters are conventionally estimated from R_{11}. Thus, we are interested in obtaining an estimate of R_{11}.

To this end, we isolate the parts of the network shown in Fig. 8.2 that are relevant for the identification of R_{11} into the structure that is given in Fig. 8.3. We have removed R_{12} from the network because it has a very small gain.

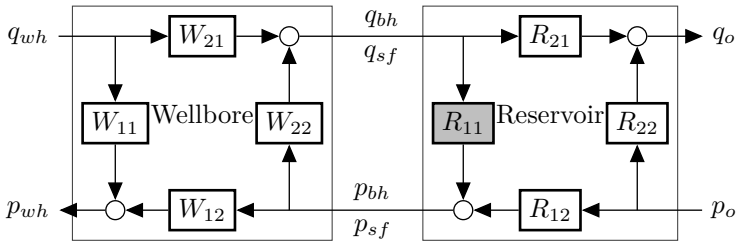

Figure 8.2: Bilaterally coupled reservoir and well bore model.

8.2.3 Data generating system

Formulating the corresponding system identification problem for the well test analysis requires using measurement devices in the correct positions in the model. In a typical production setup, (see Figure 8.1), the wellhead flow rate and the bottom hole pressure and flow rate are measured. Both process noise and sensor noise are present in the data. The process noise is due to phenomena such as turbulence, sudden well bore reservoir condition change, two phase flow occurrence, etc. It is assumed that the well head measurement does not have process noise. In Figure 8.3 the model is shown with all measurement devices and noise terms.

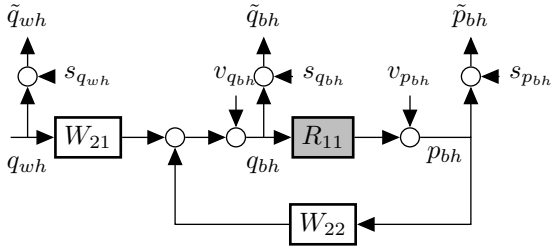

Figure 8.3: Bilaterally coupled reservoir/well bore with measurements.

The objective is to estimate the transfer function R_{11} shown in Figure 8.3. By estimating this transfer function, it is possible to infer the average permeability and thickness of the reservoir at the well location based on the expression (8.7).

Because there is sensor noise present in the data generating system, one of the methods of Chapter 6 must be used. Because the flow rate at the well head, q_{wh} is also measured in addition to the bottom hole pressure and flow rate (p_{bh} and q_{bh}) respectively), this variable can be used as an instrumental variable. From Figure 8.3, it is clear that there are no paths from $v_{p_{bh}}$ to the instrumental variable q_{wh}. Consequently, the method of Section 6.3 can be used to consistently identify the transfer function R_{11} as shown in Figure 8.3.

Then, this estimate of R_{11} is used for physical parameter estimation by comparing it with the physics based model of the reservoir, (8.7). Let $\beta = [k\ h]$ denote a vector

of the physical parameters in R_{11} that are to be estimated. The identified model $R_{11}(z,\theta)$ and the physics based model $R_{11}(s,\beta)$ can be compared in the frequency domain:

$$R_{11}(\omega,\boldsymbol{\beta}) \approx R_{11}(e^{j\omega},\theta) \tag{8.11}$$

The difference of the models can be evaluated in a frequency range of interest $[\omega_0,\omega_L]$. Then the physical parameters can be estimated as

$$\boldsymbol{\beta} = \arg\min_{\beta} \frac{1}{L}\sum_{l=1}^{L} \big\| R_{11}(\omega_l,\boldsymbol{\beta}) - R_{11}(e^{j\omega_l},\theta) \big\| W(\omega_l) \tag{8.12}$$

where $W(\omega_l)$ is a user defined weighting function.

8.3 RESULTS AND DISCUSSION

To simulate the system, the discrete approximation of transfer functions in (8.3) and (8.6) are used. The system is excited with a PRBS signal with the clock parameter of 1000 seconds; i.e. the surface choke to be in one state for at least for this period of time. Practically it is not feasible to change the surface choke setting too often. White noise signals are added to q_{wh}, q_{bh} and p_{bh} with SNR of 40, 30 and 35. The SNR of q_{bh} is conventionally considered to be the lowest Von Schroeter et al. (2004). Low pass frequency noise is added to q_{bh} and p_{bh}. A reasonable duration for a well test is one day. The sampling time is $T_s = 1$ second. Thus a reasonable data length is $N = 50000$. The physical parameters for the well bore and reservoir are listed in table 8.1. Both the non-parametric and IV methods are applied to the simulated data set. The results are shown in Fig. 8.4.

The high variance of the non-parametric estimate at high frequency is due to the fact that the system is not excited at these frequencies. For the IV estimate the data was resampled by a factor of 9 in order to remove the higher frequency content from the data.

The reservoir module R_{11} has a diffusive behavior which results in a low pass frequency response. The identified model captures this behavior for $\omega = [1\times10^{-4} - 1\times10^{-1}]$ rad/sec. The estimation results for the physical parameters, k and h, are listed in Table 8.2.

Table 8.2: Single Parameter Estimation

Parameters	True Values	Estimated Values
Permeability, (k)	200	182.3 mD
Reservoir thickness. (h)	50	45.4 m

8.4 SUMMARY

In this chapter a first principles model of a vertical well and cylindrical reservoir was developed. The objective is to obtain an estimate of the reservoir thickness

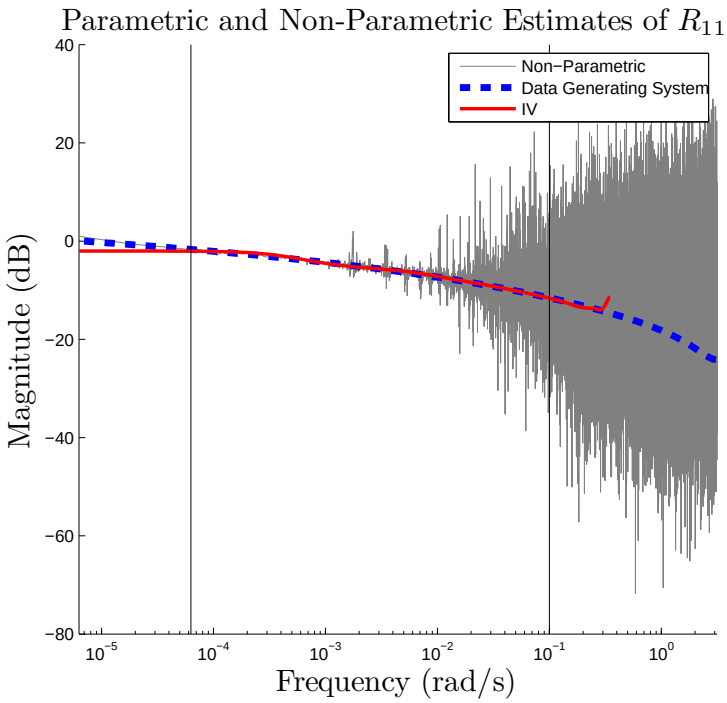

Figure 8.4: Parametric and non-parametric estimates of R_{11}

and permeability. From the first principles model, it appears that if an estimate of the transfer function R_{11} can be obtained, the reservoir thickness and permeability can be estimated. Thus the methods developed in this thesis were applied in order to obtain an estimate of the transfer function R_{11}. Since measurement noise was supposed to be present in the data, the methods of Chapter 6 were used. Using these methods a reasonable estimate of the parameters of interest were obtained.

Chapter 9

CONCLUSIONS

In this chapter some concluding remarks are made regarding the research presented in this thesis. In addition some thoughts for future work are suggested.

9.1 SUMMARY OF THE THESIS

IN THIS THESIS we have attempted to lay the foundation for identification in dynamic networks. We have shown that many physical phenomena can be modeled as dynamic networks. Moreover, we have shown that there is a strong motivation to identify particular transfer functions embedded in the dynamic networks. Examples include distributed control, power systems, and reservoir engineering.

In this thesis closed-loop identification tools are extended so that they can be used to consistently identify a particular transfer function that is embedded in a dynamic network. Various types of conditions have been considered such as

- presence and correlation of process noise (all chapters),
- presence of external variables that can be manipulated by the experimenter (Chapter 4),
- presence of known dynamics in the network, such as known controllers (Chapter 4),
- measurability/availability of the internal variables (Chapter 5), and
- presence of measurement noise (Chapter 6).

The concept of identifying a particular module embedded in a dynamic network has opened the door to a vast number of possibilities of estimating physical parameters of complex systems. The case presented in Chapter 8 is one example. In these types of applications the entire identification procedure becomes like a sensor, i.e. the objective of the identification procedure is a tool to infer the value of a physical parameter that is not measurable (such as the thickness of an oil reservoir). Often

the physics based models that are used in these types of applications are continuous-time in nature. The relationship between a continuous-time data generating system and a discrete-time representation thereof has been investigated in Chapter 7.

This thesis is only a first step in developing a theory for the identification in dynamic networks. There are still many different aspects where more work needs to be done in order to form a complete picture of identification in dynamic networks.

9.2 SUGGESTIONS FOR FUTURE WORK

In this section we present some areas where further work is required.

9.2.1 Conditions on the Informativity of the Data

In practice there are often constraints as to how exciting a signal can be. For instance it may not be possible to continuously open and close a value on a pipe, or power system operators do not like injecting large disturbances into the grid for fear of instability. Throughout this thesis a rather severe condition has been imposed on the excitation on the data. For the majority of the thesis (with the exception of some of the results of Chapter 6) the condition on the data was that the power spectral density of the predictor inputs should be positive definite for all $\omega \in (-\pi, \pi]$. It is well known that this is not a necessary condition in order to obtain consistent estimates (Gevers et al., 2009a,b). In particular for the Two-Stage Method this is a very severe condition.

It would be very useful to determine what the minimum conditions on the data should be. This will require an in depth investigation into the persistence of excitation of multi-variable signals.

9.2.2 Structure Detection

Another tantalizing objective is to be able to detect the structure of a dynamic network, given only a set of measurements obtained from the system. This could lead to new insights in many different fields of science including biology and chemistry. Being able to detect the causal relationships between a set of measured variables could give a biologist insight into the process that a cell uses ot transforms sugar into energy. Although this objective is not specifically dealt with in this thesis, tools have been developed that could be used in an attempt to detect the structure of a dynamic network. In fact, many of the propositions regarding the identification of a module in a dynamic network can be straightforwardly extended to the structure detection problem. This is because structure detection is simply the consistent estimation of transfer functions that are 0.

The difficulty lies in the numerical implementation of the results of the propositions. Special parameterizations may have to be considered, and regularization terms may need to be added to the objective function in order to attract certain parameters to exactly zero. This is an active field of research with many different options such as LASSO, Ridge Regression, and Sparseva to name a few.

9.2.3 Variance Expressions for the Obtained Estimates

In this thesis we have exclusively focused on the consistency of the estimated transfer functions. However, the variance of the estimate is equally important. Some work has already been done on this topic (Wahlberg et al., 2009; Everitt et al., 2013, 2014; Gunes et al., 2014). In these papers it is shown that using extra measurements can reduce the variance of the estimated transfer function. In Wahlberg et al. (2009); Everitt et al. (2013, 2014) only a cascaded network is considered. However, in Gunes et al. (2014) it is shown that the results hold for the case of more complex interconnection structures as well. All of the above mentioned papers consider a framework with only sensor noise present, and no process noise. These results should be extended to the more general framework considered in this thesis.

9.2.4 Checkability of The Conditions

Another area where further study would be interesting is in the checkability of the conditions. Many of the propositions in this thesis state that consistent estimates of G_{ji}^0 are possible if v_j is uncorrelated to all other noise sources, or if there is no sensor noise, or if there are no confounding variables, etc. However, given a data set, it is hard to know if these conditions hold. It would be very useful to have a test to determine if there is (significant) measurement noise present, or if there are confounding variables present.

9.2.5 Extension to Non-linear Systems

In this thesis only linear data generating systems are considered. Moving to non-linear systems will greatly increase the number of potential applications of identification in dynamic networks. In addition, it could result in more accurate estimates.

As a first step it would be interesting to consider that case where some blocks are allowed to be non-linear, however, the module of interest is still linear. Which tools could be used in order to consistently estimate the linear dynamics embedded in a non-linear dynamic network? For a closed-loop system it is possible to show that even when the feedback is nonlinear, it is still possible to consistently identify a linear plant using standard identification tools (Pintelon & Schoukens, 2012a, 2013). It would be very interesting and relevant to extend this result to the case of identification in dynamic networks.

A second incremental step that could be taken is to move to networks of interconnected *linear parameter varying* systems. This is a situation that occurs when modeling physical systems where temperature acts as a *modulator* for various modules embedded in a network.

It is clear that much work still needs to be done. However, the identification in networks holds many opportunities for interesting contributions in many areas of science.

Bibliography

Adebayo, J., Southwick, T., Chetty, V., Yeung, E., Yuan, Y., Gonçalves, J., Grose, J., Prince, J., Stan, G. B., & Warnick, S. (2012). Dynamical structure function identifiability conditions enabling signal structure reconstruction. In *Proceedings of the 51st IEEE Conference on Decision and Control (CDC)* (pp. 4635 – 4641). Maui, HI, USA.

Agüero, J., & Goodwin, G. C. (2008). Identifiability of errors in variables dynamic systems. *Automatica*, *44*, 371–382.

Agüero, J. C., & Goodwin, G. C. (2006). Identifiablity of errors-in-variables dynamic systems. In *Proceedings of the 14th IFAC symposium on System Identification* (pp. 196–201). Newcastle, Australia.

Ali, M., Ali, A., Abbas, H., & Werner, H. (2011a). Identification of Box-Jenkins models for parameter-varying spatially interconnected systems. In *Proceedings of the American Control Conference (ACC)* (pp. 145–150). San Francisco, CA, USA.

Ali, M., Ali, A., Chughtai, S., & Werner, H. (2011b). Consistent identification of spatially interconnected systems. In *Proceedings of the American Control Conference (ACC)* (pp. 3583–3588). San Francisco, CA, USA.

Ali, M., Chughtai, S. S., & Werner, H. (2009). Identification of spatially interconnected systems. In *Proceedings of the 48th IEEE Conference on Decision and Control (CDC), held jointly with the 28th Chinese Control Conference (CCC)* (pp. 7163–7168). Shanghai, China.

Ali, M., Popov, A., Werner, H., & Abbas, H. (2011c). Identification of distributed systems with identical subsystems. In *Proceedings of the 18th IFAC World Congress* (pp. 5633–5638). Milan, Italy.

Anderson, B. D. O. (1967). Algebraic solution to the spectral factorization problem. *IEEE Transactions on Automatic Control*, *12*, 410–414.

Anderson, B. D. O. (1973). Algebraic properties of minimal degree spectral factors. *Automatica*, *9*, 491–500.

Anderson, B. D. O. (1974). Recursive algorithm for spectral factorization. *IEEE Transactions on Circuits and Systems*, *21*, 742–750.

Anderson, B. D. O. (1975). Corrections to algebraic properties of minimal degree spectral factors. *Automatica*, *11*, 321–322.

Anderson, B. D. O., & Gevers, M. R. (1982). Identifiability of linear stochastic systems operating under linear feedback. *Automatica*, *18*, 195–213.

Annoni, J., Seiler, P., Johnson, K., Fleming, P., & Gebraad, P. (2014). Evaluating wake models for wind farm control. In *Proceedings of the American Control Conference (ACC)*. Portland, OR, USA. Accepted.

Araki, M., & Saeki, M. (1983). A quantitative condition for the well-posedness of interconnected dynamical systems. *IEEE Transactions on Automatic Control*, *28*, 625–637.

Aravkin, A., Burke, J. V., Chiuso, A., & Pillonetto, G. (2011). Convex vs nonconvex approaches for sparse estimation: Lasso, multiple kernel learning and hyperparameter lasso. In *Proceedings of the 50th IEEE Conference on Decision and Control (CDC) and the European Control Conference (ECC)* (pp. 156–161). Orlando, FL, USA.

Aravkin, A., Burke, J. V., Chiuso, A., & Pillonetto, G. (2012). On the MSE properties of empirical Bayes methods for sparse estimation. In *Proceedings of the 16 IFAC Symposium on System Identification* (pp. 965–970). Brussels, Belgium.

Åström, K. J. (1970). *Introduction to Stochastic Control Theory*. New York and London: Academic Press.

Bazanella, A. S., Gevers, M., & Mišković, L. (2010). Closed-loop identification of MIMO systems: A new look at identifiability and experiment design. *European Journal of Control*, *3*, 228–239.

Bottegal, G., & Picci, G. (2013). Modeling random flocks through generalized factor analysis. In *Proceedings of the European Control Conference (ECC)* (pp. 2421–2426). Zürich, Switzerland.

Bottegal, G., & Picci, G. (2014). Analysis and identification of complex stochastic systems admitting flocking structure. In *Proceedings of 19th IFAC World Congress*. Cape Town, South Africa. Accepted.

Boyd, S., El Ghaoui, L., Feron, E., & Balakrishnan, V. (1994). *Linear Matrix Inequalities in System and Control Theory* volume 15 of *Studies in Applied Mathematics*. Philadelphia, PA: SIAM.

Caines, P. E. (1976). Weak and strong feedback-free processes. *IEEE Transactions on Automatic Control*, *21*, 737–739.

Caines, P. E., & Chan, C. W. (1975). Feedback between stationary stochastic processes. *IEEE Transactions on Automatic Control*, *20*, 498–508.

Caponetto, R., Dongola, G., Fortuna, L., & Petráš, I. (2010). *Fractional Order Systems: Modeling and Control Applications* volume 72 of *Nonlinear Science*. World Scientific Publishing Co. Pte. Ltd.

Chaudhry, M. (1987). *Applied Hydraulic Transients.* (2nd ed.). New York: Van Nostrand Reinhold Co.

Cheng, C., & Wang, J. (2011). Identification of continuous-time systems in closed-loop operation subject to simple reference changes. In *Proceedings of the International Symposium on Advanced Control of Industrial Processes (ADCONIP)* (pp. 363–368). Hangzhou, China.

Chiuso, A. (2007). The role of vector autoregressive modeling in predictor-based subspace identification. *Automatica*, *43*, 1034 – 1048.

Chiuso, A., & Picci, G. (2005). Consistency analysis of some closed-loop subspace identification methods. *Automatica*, *41*, 377–391.

Chuiso, A., & Picci, G. (2003). Constructing the state of random processes with feedback. In *Proceedings of the 13th IFAC Symposium on System Identification* (pp. 881–886). Rotterdam, The Netherlands.

Chuiso, A., & Pillonetto, G. (2012). A Bayesian approach to sparse dynamic network identification. *Automatica*, *48*, 1553–1565.

Dake, L. P. (1983). *Fundamentals of reservoir engineering.* Elsevier.

D'Andrea, R., & Dullerud, G. E. (2003). Distributed control design for spacially interconnected systems. *IEEE Transactions on Automatic Control*, *48*, 1478–1495.

Dankers, A., Van den Hof, P. M. J., & Bombois, X. (2014a). An instrumental variable method for continuous-time identification in dynamic networks. In *Proceedings of 53th Conference on Decision and Control (CDC).* Los Angeles, CA, USA. Submitted.

Dankers, A., Van den Hof, P. M. J., Bombois, X., & Heuberger, P. S. C. (2013a). Predictor input selection for two stage identification in dynamic networks. In *Proceedings of the European Control Conference (ECC)* (pp. 1422–1427). Zürich, Switzerland.

Dankers, A., Van den Hof, P. M. J., Bombois, X., & Heuberger, P. S. C. (2014b). Errors in variables identification in dynamic networks - consistency results for an instrumental variable approach. *Automatica*, . Submitted.

Dankers, A., Van den Hof, P. M. J., Bombois, X., & Heuberger, P. S. C. (2014c). Errors in variables identification in dynamic networks by an instrumental variable approach. In *Proceedings of 19th IFAC World Congress.* Cape Town, South Africa. Accepted.

Dankers, A., Van den Hof, P. M. J., & Heuberger, P. S. C. (2013b). Predictor input selection for direct identification in dynamic networks. In *Proceedings of the 52nd IEEE Conference on Decision and Control (CDC)* (pp. 4541–4546). Florence, Italy.

Dankers, A., Van den Hof, P. M. J., Heuberger, P. S. C., & Bombois, X. (2012a). Dynamic network identification using the direct prediction error method. In *Proceedings of the 51st IEEE Conference on Decision and Control (CDC)* (pp. 901–906).

Dankers, A., Van den Hof, P. M. J., Heuberger, P. S. C., & Bombois, X. (2012b). Dynamic network structure identification with prediction error methods - basic examples. In *Proceedings of 16th IFAC Symposium on System Identification* (p. 876–881). Brussels, Belgium.

Dankers, A., Van den Hof, P. M. J., Heuberger, P. S. C., & Bombois, X. (2014d). Identification of dynamic models in complex networks with prediction error methods - predictor input selection. *IEEE Transactions on Automatic Control*, . Submitted.

de Callafon, R. A., Van den Hof, P. M. J., & Steinbuch, M. (1993). Control relevant identification of a compact disc pick-up mechanism. In *Proceedings of the 32nd IEEE Conference on Decision and Control (CDC)* (pp. 2050–2055 vol.3). San Antonio, TX, USA.

Deistler, M. (1986). Linear dynamic errors-in-variables models. *Journal of Applied Probability*, *23*, pp. 23–39.

Deo, N. (1974). *Graph Theory with Applications to Engineering and Computer Science*. Connaught Circus, New Delhi: Prentice Hall of India.

Diestel, R. (1997). *Graph Theory*. Number 173 in Graduate Texts in Mathematics. Springer-Verlag.

Dosiek, L., & Pierre, J. W. (2013). Estimating electromechanical modes and mode shapes using the multichannel ARMAX model. *IEEE Transactions on Power Systems*, *28*, 1950–1959.

Duindam, V., Macchelli, A., Stramigioli, S., & Bruyninckx, H. (2009). *Modeling and Control of Complex Physical Systems: The Port-Hamiltonian Approach*. Berlin Heidelberg: Springer-Verlag.

Everitt, N., Hjalmarsson, H., & Rojas, C. (2013). A geometric approach to variance analysis of cascaded systems. In *Proceedings of the 52nd IEEE Conference on Decision and Control* (pp. 6496 – 6501). Florence, Italy.

Everitt, N., Hjalmarsson, H., & Rojas, C. (2014). Variance results for parallel cascade serial systems. In *Proceedings of 19th IFAC World Congress*. Cape Town, South Africa. Accepted.

Fadali, M. S., & Visioli, A. (2013). *Digital Control Engineering*. (2nd ed.). Waltham, MA, USA: Academic Press.

Forssell, U., & Ljung, L. (1999). Closed-loop identification revisited. *Automatica*, *35*, 1215–1241.

Forssell, U., & Ljung, L. (2000). Identification of unstable systems using output error and box-jenkins model structures. *IEEE Transactions on Automatic Control*, *45*, 137 – 141.

Fraanje, R., & Verhaegen, M. (2005). A spatial canonical approach to multidimensional state-space identification for distributed parameter systems. In *Proceedings of the Fourth International Workshop on Multidimensional Systems* (pp. 217–222).

Friedman, J., Hastie, T., & Tibshirani, R. (2010). Applications of the lasso and grouped lasso to the estimation of sparse graphical models. Unpublished.

Garnier, H., Mensler, M., & Richard, A. (2003). Continuous-time model identification from sampled data: Implementation issues and performance evaluation. *International Journal of Control*, *76*, 1337–1357.

Garnier, H., & Wang, L. (Eds.) (2008). *Identification of Continuous-Time Models from Sampled Data*. Advances in Industrial Control. London: Springer-Verlag.

Gevers, M., Bazanella, A. S., Bombois, X., & Mišković, L. (2009a). Identification and the information matrix: How to get just sufficiently rich? *IEEE Transactions on Automatic Control*, *54*, 2828–2840.

Gevers, M., Brazanella, A., & Bombois, X. (2009b). Connecting informative experiments, information matrix and the minima of a prediction error identification criterion. In *Proceedings of 15th IFAC symposium on System Identification (SYSID2009)* (pp. 675–680). Saint-Malo, France.

Gevers, M., Mišković, L., Bonvin, D., & Karimi, A. (2006). Identification of multi-input systems: variance analysis and input design issues. *Automatica*, *42*, 559 – 572.

Gevers, M. R., & Anderson, B. D. O. (1981). Representing of jointly stationary stochastic feedback processes. *International Journal of Control*, *33*, 777–809.

Gevers, M. R., & Anderson, B. D. O. (1982). On jointly stationary feedback-free stochastic processes. *IEEE Transactions on Automatic Control*, *27*, 431–436.

Gillberg, J., & Ljung, L. (2009). Frequency-domain identification of continuous-time ARMA models from sampled data. *Automatica*, *45*, 1371 – 1378.

Gilson, M., & Garnier, H. (2003). Continuous-time model identification of systems operating in closed-loop. In *Proceedings of the 13th IFAC Symposium on System Identification* (pp. 405–410). Rotterdam, The Netherlands.

Gilson, M., Garnier, H., Young, P., & Van den Hof, P. M. J. (2008). Optimal instrumental variable methods for closed-loop continuous time model identification. In H. Garnier, & L. Wang (Eds.), *Identification of Continuous-Time Models from Sampled Data* Advances in Industrial Control chapter 5. (pp. 133–160). London: Springer-Verlag.

Gilson, M., Garnier, H., Young, P. C., & Van den Hof, P. M. J. (2009). Refined instrumental variable methods for closed-loop system identification. In *Proceedings of the 15th IFAC Symposium on System Identification* (pp. 284–289). Saint-Malo, France.

Gilson, M., Garnier, H., Young, P. C., & Van den Hof, P. M. J. (2011). Optimal instrumental variable method for closed-loop identification. *IET control theory & applications*, *5*, 1147–1154.

Gilson, M., & Van den Hof, P. M. J. (2005). Instrumental variable methods for closed-loop system identification. *Automatica*, *41*, 241–249.

Gonçalves, J., & Warnick, S. (2008). Necessary and sufficient conditions for dynamical structure reconstruction of LTI networks. *IEEE Transactions on Automatic Control*, *53*, 1670–1674.

Granger, C. W. J. (1969). Investigating causal relations by econometric models and cross-spectral methods. *Econometrica*, *37*, 424–438.

Granger, C. W. J. (1980). Testing for causality; a personal viewpoint. *Journal for Economic Dynamics and Control*, (pp. 329–352).

Gringarten, A. (2008). From straight lines to deconvolution: The evolution of the state of the art in well test analysis. *SPE Reservoir Evaluation Engineering*, *11*, 41–62.

Gudi, R. D., & Rawlings, J. B. (2006). Identification for decentralized model predictive control. *AIChE Journal*, *52*, 2198–2210.

Gunes, B., Dankers, A., & den Hof, P. M. J. V. (2014). Variance reduction for identification in dynamic networks. In *Proceedings of 19th IFAC World Congress*. Cape Town, South Africa. Accepted.

Haber, A., Fraanje, P. R., & Verhaegen, M. (2011). Subspace identification of spacially distributed systems. In *Proceedings of the 18th IFAC World Congress* (pp. 6499–6504). Milan, Italy.

Haber, A., & Verhaegen, M. (2012). Identification of spatially distributed discrete-time state-space models. In *Proceedings of the 16th IFAC Symposium on System Identification* (pp. 410–415). Brussels, Belgium.

Haber, A., & Verhaegen, M. (2013). Moving horizon estimation for large-scale interconnected systems. *IEEE Transactions on Automatic Control*, *58*, 2834–2847.

Hayden, D. P., Yuan, Y., & Gona lves, J. M. (2014a). Network reconstruction from intrinsic noise. In *Proceedings of the American Control Conference (ACC)*. Portland, USA. Accepted.

Hayden, D. P., Yuan, Y., & Gona lves, J. M. (2014b). Network reconstruction from intrinsic noise: Non-minimum-phase systems. In *Proceedings of 19th IFAC World Congress*. Cape Town, South Africa. Accepted.

Herrmann, R. (2011). *Fractional Calculus: An Introduction For Physicists.* World Scientific Publishing Co. Pte. Ltd.

Heuberger, P. S. C., Hof, P. M. J. V. D., & Whalberg, B. (Eds.) (2005). *Modelling and Identification with Rational Orthogonal Basis Functions.* London: Springer-Verlag.

Hinnen, K., Verhaegen, M., & Doelman, N. (2005). Robust spectral factor approximation of discrete-time frequency domain power spectra. *Automatica*, *41*, 1791–1798.

Innocenti, G., & Materassi, D. (2009). Topological properties in identification and modeling techniques. In A. Chiuso, L. Fortuna, M. Frasca, A. Rizzo, L. Schenato, & S. Zampieri (Eds.), *Modelling, Estimation and Control of Networked Complex Systems* Understanding Complex Systems (pp. 161–175). Springer Berlin Heidelberg.

Johnson, K. E., & Thomas, N. (2009). Wind farm control: Addressing the aerodynamic interaction among wind turbines. In *Proceedings of the American Control Conference (ACC)* (pp. 2104–2109). St. Louis, MO, USA.

Julius, A., Zavlanos, M., Boyd, S., & Pappas, G. J. (2009). Genetic network identification using convex programming. *IET Systems Biology*, *3*, 155–166.

Kanevsky, A. (1993). Finding all minimum-size separating vertex sets in a graph. *Networks*, *23*, 533–541.

Kundur, P. (1994). *Power System Stability and Control.* EPRI Power System Engineering Series. McGraw-Hill Inc.

Langston, M. A., & Plaut, B. C. (1998). On algorithmic applications of the immersion order. *Discrete Mathematics*, *182*, 191–196.

Larsson, E. K., Mossberg, M., & Soderstrom, T. (2006). An overview of important practical aspects of continuous-time arma system identification. *Circuits, Systems and Signal Processing*, *25*, 17–46.

Larsson, E. K., Mossberg, M., & Söderström, T. (2008). Estimation of continuous-time stochastic system parameters. In H. Garnier, & L. Wang (Eds.), *Identification of Continuous-Time Models from Sampled Data* Advances in Industrial Control chapter 2. (pp. 31–63). London: Springer-Verlag.

Laurijsse, H. D. J. (2014). *System Identification of Individual Control Area Dynamics in Interconnected Power Systems.* Master's thesis Eindhoven University of Technology.

Leskens, M., & Van den Hof, P. M. J. (2007). Closed-loop identification of multivariable processes with part of the inputs controlled. *International journal of Control*, *80*, 1552–1561.

Lindquist, A., & Picci, G. (1996). Geometric methods for state space identification. In *In Identification, Adaptation, Learning - The Science of Learning Models from Data, NATO ASI Series F* (pp. 1–69). Springer.

Ljung, L. (1999). *System Identification. Theory for the User*. (2nd ed.). Prentice Hall.

Ljung, L., & McKelvey, T. (1996). Subspace identification from closed loop data. *Signal Processing*, *52*, 209 – 215.

Mansoori, M., Dankers, A., & Van den Hof, P. M. J. (2014). Errors-in-variables identification in bilaterally coupled systems with application to oil well testing. In *Proceedings of 19th IFAC World Congress*. Cape Town, South Africa. Accepted.

Mansoori, M., Van den Hof, P. M. J., Jansen, J. D., & Rashtchian, D. (2013). *Identification of Reservoir Characteristics from Generalized Welltest Data*. Technical Report DCSC.

Marelli, D., & Fu, M. (2010). A continuous-time linear system identification method for slowly sampled data. *IEEE Transactions on Signal Processing*, *58*, 2521–2533.

Mari, J., & Stoica, P. (2000). Vector ARMA estimation: A reliable subspace approach. *IEEE Transactions on Signal Processing*, *48*, 2092–2104.

Marques, V. M., Munaro, C. J., & Shah, S. L. (2013). Data-based causality detection from a system identification perspective. In *Proceedings of the European Control Conference (ECC)* (pp. 2453–2458). Zürich, Switzerland.

Mason, S. J. (1953). Feedback theory - some properties of signal flow graphs. *Proceedings of The IRE*, (pp. 1144–1156).

Mason, S. J. (1956). Feedback theory - further properties of signal flow graphs. *Proceedings of The IRE*, (pp. 920–926).

Massioni, P., & Verhaegen, M. (2008). Subspace identification of circulant systems. *Automatica*, *44*, 2825 – 2833.

Massioni, P., & Verhaegen, M. (2009). Subspace identification of distributed, decomposable systems. In *Proceedings of the 48th IEEE Conference on Decision and Control (CDC)* (pp. 3364 – 3369). Shanghai, China.

Materassi, D., & Innocenti, G. (2010). Topological identification in networks of dynamical systems. *IEEE Transactions on Automatic Control*, *55*, 1860–1871.

Materassi, D., & Salapaka, M. V. (2012). On the problem of reconstructing an unknown topology via locality properties of the Wiener filter. *IEEE Transactions on Automatic Control*, *57*, 1765–1777.

Materassi, D., Salapaka, M. V., & Giarré, L. (2011). Relations between structure and estimators in networks of dynamical systems. In *Proceedings of the 50th IEEE Conference on Decision and Control (CDC)* (pp. 162–167). Orlando, USA.

Monteyne, G., Javed, S., & Vandersteen, G. (2014). Heat transfer in a borehole heat exchanger: Frequency domain modeling. *International Journal of Heat and Mass Transfer*, *69*, 129 – 139.

Monteyne, G., Vandersteen, G., Verhelst, C., & Helsen, L. (2011). On the use of Laplace and Warburg variables for heat diffusion modeling. In *Proceedings of the 10th International Conference on Environment and Electrical Engineering (EEEIC)* (pp. 1–4). Rome, Italy.

Mooij, J. M., Janzig, D., Heskes, T., & Schölkopf, B. (2011). On causal discovery with cyclic additive noise models. In *Advances in Neural Information Processing Systems 24 (NIPS)* (pp. 639–647). Granada, Spain.

Ng, T. S., Goodwin, G. C., & Anderson, B. D. O. (1977). Identifiability of MIMO linear dynamic systems operating in closed loop. *Automatica*, *13*, 477–485.

Oldham, K. B., & Spanier, J. (1974). *The Fractional Calculus: Theory and Applications of Differentiation and Integration to Arbitrary Order* volume 111 of *Mathematics in Science and Engineering*. Academic Press.

O'Neil, P. (2007). *Advanced Engineering Mathematics*. (6th ed.). Thompson Canada.

Pearl, J. (1988). *Probabilistic Reasoning in Intelligent Systems: Networks of Plausible Inference*. San Mateo, California: Morgan Kaufmann Publishers, Inc.

Pearl, J. (2000). *Causality: Models, Reasoning, and Inference*. 40, West 20th Street, New York: Cambridge University Press.

Pearl, J. (2009). Causal inference in statistics: An overview. *Statistics Surveys*, *3*, 96–146.

Pierre, J. W., Trudnowski, D., Donnelly, M., Zhou, N., Tuffner, F. K., & Dosiek, L. (2012). Overview of system identification for power systems from measured responses. In *Proceedings of the 16th IFAC Symposium on System Identification* (pp. 989–1000). Brussels, Belgium.

Pintelon, R., & Schoukens, J. (2006). Box - Jenkins identification revisited - part I: Theory. *Automatica*, *42*, 63 – 75.

Pintelon, R., & Schoukens, J. (2012a). The best linear approximation of nonlinear systems operating in feedback. In *Proceedings of the IEEE International Instrumentation and Measurement Technology Conference (I2MTC)* (pp. 2092–2097). Graz, Austria.

Pintelon, R., & Schoukens, J. (2012b). *System Identification, A Frequency Domain Approach*. (2nd ed.). Hoboken, New Jersey, USA: IEEE Press, John Wiley and Sons, Inc.

Pintelon, R., & Schoukens, J. (2013). FRF measurement of nonlinear systems operating in closed loop. *IEEE Transactions on Instrumentation and Measurement*, *62*, 1334–1345.

Pintelon, R., Schoukens, J., & Guillaume, P. (2006). Continuous-time noise modeling from sampled data. *Instrumentation and Measurement, IEEE Transactions on*, *55*, 2253–2258.

Pintelon, R., Schoukens, J., Pauwels, L., & Van Gheem, E. (2005). Diffusion systems: stability, modeling, and identification. *IEEE Transactions on Instrumentation and Measurement*, *54*, 2061–2067.

Pintelon, R., Schoukens, J., & Rolain, Y. (2008). Frequency-domain approach to continuous-time system identification: Some practical aspects. In H. Garnier, & L. Wang (Eds.), *Identification of Continuous-Time Models from Sampled Data* Advances in Industrial Control chapter 8. (pp. 215–248). London: Springer-Verlag.

Polderman, J. W., & Willems, J. C. (1991). *Introduction to Mathematical Systems Theory, A Behavioral Approach*. Heidelberg: Springer.

Proakis, J. G., & Salehi, M. (2001). *Communication Systems Engineering*. (2nd ed.). Pearson Education.

Rao, G. P., & Unbehauen, H. (2006). Identification of continuous-time systems. *Control Theory and Applications, IEE Proceedings -*, *153*, 185–220.

Ren, W., & Beard, R. (2008). *Distributed Consensus in Multi-vehicle Cooperative Control: Theory and Applications*. Communication and Control Engineering. London: Springer-Verlag.

Russell, B. (1912). On the notion of cause. *Proceedings of the Aristotelian Society*, *13*, pp. 1–26.

Sanandaji, B. M., Vincent, T. L., & Wakin, M. B. (2011). Exact topology identification of large-scale interconnected dynamical systems from compressive observations. In *Proceedings of the American Control Conference (ACC)* (pp. 649–656). San Francisco, CA, USA.

Sanandaji, B. M., Vincent, T. L., & Wakin, M. B. (2012). A review of sufficient conditions for structure identification in interconnected systems. In *Proceedings of the 16th IFAC Symposium on System Identification* (pp. 1623–1628). Brussels, Belgium.

Sarwar, A., Voulgaris, P. G., & Salapaka, S. M. (2010). System identification of spatiotemporally invariant systems. In *Proceedings of the American Control Conference (ACC)* (pp. 2947–2952). Baltimore, MD, USA.

Scherer, C. (2001). *Theory of Robust Control*. Delft University of Technology.

Schoukens, J., Pintelon, R., Vandersteen, G., & Guillaume, G. (1997). Frequency-domain system identification using non-parametric noise models estimated from a small number of data sets. *Automatica*, *33*, 1073–1086.

Schoukens, J., & Relan, R. (2014). Error analysis of discrete-time models with a direct term for band-limited measurements. In *Proceedings of the 33rd Benelux Meeting on Systems and Control* (p. 76). Heijen, The Netherlands.

Seneviratne, A., & Solo, V. (2012a). Topology identification of a sparse dynamic network. In *Proceedings of the 51st IEEE Conference on Decision and Control (CDC)* (pp. 1518–1523). Maui, HI, USA.

Seneviratne, A. J., & Solo, V. (2012b). Sparse coloured system identification with guaranteed stability. In *Proceedings of the 51st IEEE Conference onDecision and Control (CDC)* (pp. 2826–2831). Maui, HI, USA.

Shearer, J. L., Murphy, A. T., & Richardson, H. H. (1971). *Introduction to System Dynamics*. Addison-Wesley Publishing Company.

Söderström, T. (2007). Errors-in-variables methods in system identification. *Automatica*, *43*, 939–958.

Söderström, T. (2011). A generalized instrumental variable estimation method for errors-in-variables identification problems. *Automatica*, *47*, 1656–1666.

Söderström, T. (2012). System identification for the errors-in-variables problem. *Transactions of the Institute of Measurement and Control*, *34*, 780–792.

Söderström, T., & Hong, M. (2005). Identification of dynamic errors-in-variables systems with periodic data. In *Proceedings of the 16th IFAC World Congress* (pp. 809–814). Prague, Czech Republic.

Söderström, T., & Stoica, P. (1983). *Instrumental Variable Methods for System Identification*. LNCIS. Berlin: Springer.

Söderström, T., & Stoica, P. (1989a). *System Identification*. Systems and Control Engineering. Hertfordshire, UK: Prentice Hall.

Söderström, T., & Stoica, P. (1989b). *System Indentification*. Hemel Hempstead, U.K.: Prentice-Hall Internatinal.

Söderström, T., & Stoica, P. (1990). On covariance function tests used in system identification. *Automatica*, *26*, 125–133.

Söderström, T., Stoica, P., & Trulsson, E. (1988). Instrumental variable methods for closed loop systems. In *Proceedings of the 10th IFAC World Congress* (pp. 363–368). Munich, Germany.

Söderström, T., Wang, L., Pintelon, R., & Schoukens, J. (2013). Can errors-in-variables systems be identified from closed-loop experiments. *Automatica*, *49*, 681–684.

Soleimanzadeh, M., & Wisniewski, R. (2011). Controller design for a wind farm, considering both power and load aspects. *Mechatronics*, *21*, 720 – 727.

Takahashi, Y., Rabins, M. J., & Auslander, D. M. (1970). *Control and Dynamic Systems*. Addison-Wesley Publishing Company.

Tan, V. Y. F., & Willsky, A. S. (2011). Sample complexity for topology estimation in networks of LTI systems. In *Proceedings of the 50th IEEE Conference on Decision and Control (CDC) and European Control Conference (ECC)* (pp. 187–192). Orlando, FL, USA.

Tibshirani, R. (1994). Regression shrinkage and selection via the lasso. *Journal of the Royal Statistical Society, Series B*, *58*, 267–288.

Torres, P. (2014). *Linear State-Space Identification of Interconnected Systems: A Structured Approach.* Ph.D. thesis Delft University of Technology.

Torres, P., van Wingerden, J., & Verhaegen, M. (2014a). Hierarchical PO-MOESP subspace identification for directed acyclic graphs. *International Journal of Control*, . Accepted.

Torres, P., van Wingerden, J., & Verhaegen, M. (2014b). Output-error identification of large scale 1-D spatially varying interconnected systems. *IEEE Transactions on Automatic Control*, . Accepted.

Van den Hof, P. (2006). *System Identification.* Delft University of Technology.

Van den Hof, P. M. J. (1998). Closed-loop issues in system identification. *Annual Reviews in Control*, *22*, 173–186.

Van den Hof, P. M. J., Dankers, A., Heuberger, P. S. C., & Bombois, X. (2012). Identification in dynamic networks with known interconnection topology. In *Proceedings of the 51st IEEE Conference on Decision and Control (CDC)* (pp. 895–900).

Van den Hof, P. M. J., Dankers, A., Heuberger, P. S. C., & Bombois, X. (2013). Identification of dynamic models in complex networks with prediction error methods - basic methods for consistent module estimates. *Automatica*, *49*, 2994–3006.

Van den Hof, P. M. J., & de Callafon, R. A. (1996). Multivariable closed-loop identification: from indirect identification to dual-Youla parametrization. In *Proceedings of the 35th IEEE Conference on Decision and Control (CDC)* (pp. 1397–1402). Kobe, Japan volume 2.

Van den Hof, P. M. J., de Vries, D. K., & Schoen, P. (1992). Delay structure conditions for identifiablility of closed loop systems. *Automatica*, *28*, 1047–1050.

Van den Hof, P. M. J., & Schrama, R. (1993). An indirect method for transfer function estimation from closed loop data. *Automatica*, *29*, 1523–1527.

Van den Hof, P. M. J., Schrama, R. J. P., de Callafon, R. A., & Bosgra, O. H. (1995). Identification of normalised coprime plant factors from closed-loop experimental data. *European Journal of Control*, *1*, 62–74.

Van der Veen, G., Van Wingerden, J.-W., Bergamasco, M., Lovera, M., & Verhaegen, M. (2013). Closed-loop subspace identification methods: an overview. *Control Theory Applications, IET*, *7*, 1339–1358.

Van der Veen, G., Van Wingerden, J.-W., & Verhaegen, M. (2010). Closed-loop MOESP subspace model identification with parametrisable disturbances. In *Proceedings of the 49th IEEE Conference on Decision and Control (CDC)* (pp. 2813–2818). Atlanta, GA, USA.

Van Donkelaar, E. T., & Van den Hof, P. M. J. (2000). Analysis of closed-loop identification with a tailor-made parameterization. *European Journal of Control*, *6*, 54 – 62.

Van Hamme, H., Pintelon, R., & Schoukens, J. (1991). Identification and discrete time modeling of continuous time systems: a general framework. In N. K. Sinha, & G. P. Rao (Eds.), *Identification of Continuous-Time Systems: Methodology and Computer Implementation* chapter 2. (pp. 17–77). Dordrecht, The Netherlands: Kluwer Academic Publishers.

Van Overschee, P., De Moor, B., Dehandschutter, W., & Swevers, J. (1997). A subspace algorithm for the identification of discrete time frequency domain power spectra. *Automatica*, *33*, 2147–2157.

Van Wingerden, J.-W., & Torres, P. (2012). Identification of spatially interconnected systems using a sequentially semi-separable gradient method. In *Proceedings of the 16th IFAC Symposium on System Identification* (pp. 173–178). Brussels, Belgium.

Verhaegen, M. (1993). Application of a subspace model identification technique to identify LTI systems operating in closed loop. *Automatica*, *29*, 1027–1040.

Vidyasagar, M. (1980). On the well-posedness of large scale interconnected systems. *IEEE Transactions on Automatic Control*, *25*, 413–421.

Virag, A., Jokic, A., Van den Bosch, P. P. J., & Van den Hof, P. M. J. (2013). Using market schedules to improve secondary control design. In *Proceedings of the 10th International Conference on the European Energy Market (EEM)* (pp. 1–8). Stockholm, Sweden.

Vitacco, W. R., & Michel, A. N. (1977). Qualitative analysis of interconnected dynamical systems with algebraic loops: Well-posedness and stability. *IEEE Transactions on Circuits and Systems*, *24*, 625–637.

Von Schroeter, T., Hollaender, F., & Gringarten, A. (2004). Deconvolution of well-test data as a nonlinear total least-squares problem. *SPE Journal*, *9*, 375–390.

Wahlberg, B., Hjalmarsson, H., & Mårtensson, J. (2009). Variance results for identification of cascade systems. *Automatica*, *45*, 1443 – 1448.

Willems, J. C. (1971). *The Analysis of Feedback Systems*. Number 62 in Research Monograph. Cambridge, Massachusetts: The M.I.T. Press.

Willems, J. C. (1997). On interconnections, control, and feedback. *IEEE Transactions on Automatic Control*, *42*, 326–399.

Willems, J. C. (2007). The behavioural approach to open and interconnected systems. *IEEE Control Systems Magazine*, (pp. 46–99).

Willems, J. C. (2008). Modelling interconnected systems. In *Proceedings of the 3rd ISCCSP* (pp. 421–424). St. Julian's, Malta.

Willems, J. C. (2010). Terminals and ports. *IEEE Circuits and Systems Magazine*, (pp. 8–26).

Wu, C., Lu, C., & Han, Y. (2012). Closed-loop identification of power system based on ambient data. *Mathematical Problems in Engineering*, *2012*.

Yeung, E., Gonçalves, J., Sandberg, H., & Warnick, S. (2010). Representing structure in linear interconnected dynamical systems. In *Proceedings of 49th IEEE Conference on Decision and Control (CDC)* (pp. 6010–6015). Atlanta, GA, USA.

Yeung, E., Goncalves, J., Sandberg, H., & Warnick, S. (2011). Mathematical relationships between representations of structure in linear interconnected dynamical systems. In *Proceedings of the American Control Conference (ACC)* (pp. 4348–4353). San Francisco, CA, USA.

Youla, D. C. (1961). On the factorization of rational matrices. *IRE Transactions on Information Theory*, *7*, 172–189.

Young, P., & Garnier, H. (2006). Identification and estimation of continuous-time, data-based mechanistic (dbm) models for environmental systems. *Environmental Modelling and Software*, *21*, 1055 – 1072.

Yuan, Y., Stan, G., Warnick, S., & Goncalves, J. (2011). Robust dynamical network structure reconstruction. *Automatica*, *47*, 1230 – 1235.

Yuan, Y., Stan, G.-B., Warnick, S., & Gonçalves, J. (2009). Minimal dynamical structure realisations with application to network reconstruction from data. In *Proceedings of 48th IEEE Conference on Decision and Control (CDC)* (pp. 4808–4813). Shanghai, China.

Yuan, Y., Stan, G.-B., Warnick, S., & Gonçalves, J. (2010). Robust dynamical network reconstruction. In *Proceedings of 49th IEEE Conference on Decision and Control (CDC)* (pp. 810–815). Atlanta, USA.

Zames, G. (1964). Realizability conditions for nonlinear feedback systems. *IEEE Transactions on Circuit Theory*, *11*, 186–194.

Zhou, K., Doyle, J. C., & Glover, K. (1996). *Robust and Optimal Control*. Englewood Cliffs, New Jersey: Prentice Hall.

Zhu, Y.-C., & Stoorvogel, A. A. (1992). Closed loop identification of coprime factors. In *Proceedings of the 31st IEEE Conference on Decision and Control (CDC)* (pp. 453–454). Tucson, AZ, USA.

SUMMARY

System Identification in Dyanmic Networks
A.G. Dankers

Systems in engineering such as power systems, telecommunication systems, and distributed control systems are becoming more complex and interconnected. Many of these systems are part of the foundation of modern society and their seamless operation is paramount. However, the increasing complexity and size of the systems poses real engineering challenges (in maintaining stability of the electrical power grid, increasing data throughput of telecommunication networks, etc.). These systems cannot be operated, designed, and maintained without the help of models.

Tools from system identification are well suited to construct models using measurements obtained from a system. However, the field of system identification is primarily focused on identifying open and closed-loop systems. Recently, there has been a move to considering more complex interconnection structures. The literature on identification and dynamic networks can be split into two categories based on whether the interconnection structure of the network is assumed to be known or not. In the latter the objective is to detect the topology of the network, whereas in the former the objective is to identify the dynamical transfers in the network.

A variety of approaches have been presented in the literature in order to detect the topology of a network based on a given data set. The main features that these approaches have in common is that all internal variables in the network are assumed to be known, each internal variable is driven by an independent stochastic variable, and often is it assumed that all transfer functions in the network are strictly proper. Under these conditions it is shown that topology detection is possible.

The literature on identification of dynamic networks with known interconnection structure is dominated by identification of *spatially distributed systems* (systems where each node is connected only to its direct neighbors). In this literature there is a strong emphasis on developing numerically fast algorithms.

In this thesis our approach is to use the known interconnection structure in order to relax the assumptions imposed on the data generating system such as the requirement that each variable is measured, and that each variable is driven by an independent noise source. In addition, we do not limit the interconnection to a spatially distributed topology. Our approach is to focus on identifying a single module embedded in the network, resulting in a local approach where only a small number of variables need to be measured in order to identify the object of interest.

The resulting algorithms can be applied to small to medium scale networks, or to large networks with sparse interconnection structures. Our approach is based on extending closed-loop system identification methods to develop to the situation of dynamic networks. One advantage of this approach is that we suppose that both noise and known user defined signals (called reference signals in the closed loop literature) can drive or excite the network. The user defined signals can be very beneficial since they can be used to further relax the assumptions made about the noise in the system.

The main objective of this thesis is to investigate under which conditions it is possible to consistently identify a module embedded in a dynamic network. The thesis is structured in such a way that the results become more applicable to real systems as the chapters progress.

At first several closed-loop prediction error methods, namely the Direct, Two Stage, and Joint IO methods, are extended to the situation of dynamic networks. We show that for the Direct Method consistent estimates of a module embedded in a dynamic network are possible if there is a delay in every loop; all variables with a direct connection to the output are measured, the process noise affecting the output is uncorrelated to the other noise variables in the network; the process noise is exactly modeled; and there is no sensor noise present. Clearly, there are quite a few conditions that need to be satisfied, and some of the conditions are not checkable. The Two-Stage method makes explicit use of external variables that are present in the network. This leads to less conditions: using the Two Stage method consistent estimates are possible if there are external variables present in the network that are informative and every variable with a direct connection to the output is measured.

Subsequently, we analyze the flexibility that exists in which selection of measured variables leads to consistent identification of the module of interest. We derive conditions that the set of measured variables need to satisfy in order to guarantee that a particular module of interest can be estimated consistently. The conditions can be used to determine if is it possible to avoid measuring variables that are expensive, difficult or unsafe to measure. Or if sensors are expensive, then these conditions can be used to design a sensor placement scheme with a small number of sensors.

Subsequently, methods are developed such that consistent estimates are possible even in the situation where sensor noise is present in the measurements. The methods are based on a combination of the Instrumental Variable and Direct Method style of reasoning. This is a significant step in rendering the methods more practical because internal variables are always measured using sensors, and sensors inherently do not make perfect measurements.

Finally, a preliminary analysis is made on how the proposed methods work on continuous-time systems. This is an important step because many systems in practice are inherently of a continuous-time nature.

There are still many interesting challenges ahead. Among other things, the analysis of continuous time systems needs to be completed, the variance of the methods needs to analyzed, and the effects of nonlinearities in the systems needs to be investigated.

SAMENVATTING

Systeemidentificatie in Dynamische Netwerken
A.G. Dankers

Veel technologische systemen zoals elektrische netwerken, telecommunicatiesystemen, en gedistribueerde regelsystemen, worden gekenmerkt door groeiende complexiteit en daarnaast zijn deze systemen in toenemende mate onderling verbonden. De moderne maatschappij is in sterke mate afhankelijk van deze systemen en daarom is het van groot belang dat deze systemen zo probleemloos mogelijk functioneren. De toenemende complexiteit en omvang van deze systemen leiden tot grote technologische uitdagingen (bijvoorbeeld in het handhaven van de stabiliteit van elektrische netwerken, en het verwerken van de sterk groeiende datastromen in telecommunicatiesystemen). Het gebruik van (wiskundige) modellen is van intrinsiek belang voor het ontwerp, gebruik en onderhoud van deze systemen.

Methoden uit de systeemidentificatie zijn uitermate geschikt om modellen te ontwikkelen op basis van meetdata van een systeem. Dit vakgebied richt zich echter voornamelijk op het identificeren van open en gesloten lus systemen. In de laatste jaren zien we een trend om de aandacht ook te richten op meer complexe en onderling gekoppelde structuren. De literatuur over identificatie en dynamische netwerken kan in twee categorieën verdeeld worden, gebaseerd op exacte/ontbrekende kennis over de structuur van het netwerk. In het laatste geval is het doel om de topologie van het netwerk te bepalen, terwijl men zich in de eerste situatie richt op het identificeren van dynamische overdrachtsfuncties in het netwerk.

In de literatuur zijn er een reeks benaderingen voor topologiedetectie van netwerken. De gemeenschappelijke kenmerken van deze benaderingen zijn dat alle variabelen in het netwerk bekend worden verondersteld, dat elke variabele gestuurd wordt door een onafhankelijke stochastische variabele, en in de meeste gevallen dat alle overdracht functies causaal zijn. Onder deze aannamen is topologiedetectie mogelijk.

De literatuur over de identificatie van dynamische netwerken met een bekende verbindingsstructuur wordt gedomineerd door ruimtelijk verdeelde systemen (systemen waar elke knooppunt alleen verbonden is met zijn directe buren). In deze literatuur ligt een nadruk op het ontwikkelen van efficiënte numerieke algoritmes.

In dit proefschrift wordt een benadering voorgesteld waarbij exacte kennis van de verbindingsstructuur wordt gebruikt om een aantal aannamen te versoepelen, zoals de eis dat elke variabele gemeten wordt en aangedreven wordt door een onafhankelijke ruisbron. Bovendien word de verbindingsstructuur niet beperkt tot een ruimtelijk

verdeelde topologie. In onze benadering wordt de nadruk gelegd op het identificeren van één enkele module in het netwerk, met als resultaat een lokale aanpak waarbij slechts een klein aantal variabelen gemeten hoeft te worden. De resulterende algoritmes kunnen toegepast worden op kleine tot middelgrote netwerken, dan wel op grote netwerken met een ijle verbindingsstructuur. De benadering is gebaseerd op het uitbreiden van gesloten lus identificatiemethoden naar de situatie van dynamische netwerken. Een voordeel van deze benadering is dat we veronderstellen dat zowel ruissignalen als door de gebruiker gedefinieerde signalen (referentiesignalen) het netwerk kunnen aandrijven. Een voordeel van deze signalen is dat ze kunnen gebruikt worden om de aannamen over de ruis in het systeem te versoepelen.

Het doel van dit proefschrift is om te onderzoeken onder welke omstandigheden het mogelijk is om een module in een dynamisch netwerk op een consistente manier te identificeren.

Allereerst worden verschillende gesloten lus voorspellingsfout (*prediction error*)-methoden uitgebreid om toegepast te kunnen worden op dynamische netwerken, waarbij met name de direct, two-stage, en joint io methoden bekeken worden. We laten zien dat met de direct methode consistente schattingen van modules in het dynamisch netwerk mogelijk zijn, indien er vertraging is in elke lus; alle variabelen met een directe verbinding met de uitgang gemeten worden; er geen correlatie is tussen de procesruis van de uitgang en de andere ruisvariabelen in het netwerk; de procesruis exact gemodelleerd wordt; en er geen sensorruis aanwezig is. Er zijn dus veel condities die vervuld moeten worden, en sommige van deze condities kunnen niet gecontroleerd worden. De two-stage methode maakt expliciet gebruik van de externe variabelen die aanwezig zijn in het netwerk. Dit resulteert in een kleiner aantal condities. Het gebruik van de two-stage methode leidt tot consistente schattingen als er externe variabelen aanwezig zijn die informatief zijn, en elke variabele met een directe verbinding met de uitgang gemeten wordt.

Vervolgens analyseren wij de flexibiliteit die bestaat in de keuze van gemeten variabelen, en ontwikkelen condities waaraan een selectie van metingen moet voldoen om te garanderen dat de betreffende module consistent geschat kan worden. Deze condities kunnen worden gebruikt om te bepalen of het mogelijk is situaties te vermijden, waarin metingen duur, moeilijk of onveilig zijn. In de situatie waarin sensoren duur zijn geven deze condities de mogelijkheid om de sensoren zodanig te plaatsen dat zo weinig mogelijk sensoren nodig zijn.

In de volgende stap worden methoden ontwikkeld die consistente schattingen ook mogelijk maken in de situatie waarbij wel sensorruis aanwezig is in de metingen. Dit is een belangrijke uitbreiding om de aanpak meer praktisch toepasbaar te maken omdat interne variabelen altijd met sensoren gemeten worden, waarbij in de praktijk altijd sprake is van meetfouten.

Ten slotte wordt een voorlopige analyse gemaakt van de toepassing van de voorgestelde methoden op continue tijd systemen. Dit is een belangrijk stap omdat in de praktijk veel systemen van nature continue tijd systemen zijn.

Er blijven echter nog veel uitdagingen. Onder andere moet de analyse voor continue tijd systemen nog verder ontwikkeld worden, moet de variantie van de resulterende schattingen worden geanalyseerd, en moeten de effecten van niet lineaire modules in het netwerk onderzocht worden.

CURRICULUM VITAE

Arne Dankers was born on June 1 1980 in Ottawa, Canada. He obtained his B.Sc. and M.Sc. degrees in Electrical and Computer Engineering at the University of Calgary. During his M.Sc. he worked on system identification using orthogonal basis functions under the supervision of Dr. David Westwick. The research was aimed at determining the optimal pole locations for the orthogonal basis functions.

In April 2010, he started his Ph.D. at the Delft Center for Systems and Control. During his Ph.D. he worked on system identification in dynamic networks under the supervision of Prof. Paul van den Hof, Dr. Peter Heuberger and Dr. Xavier Bombois. The primary focus of this research was to determine under what conditions it is possible to consistently estimate a transfer function embedded in a dynamic network. During his Ph.D. he received the DISC certificate for completing the graduate course program offered by the Dutch Institute for Systems and Control. he also received the Best Student Paper Award at the 2013 European Control Conference for the paper *Predictor Input Selection for Two-Stage Identification in Dynamic Networks*.

His main research interests are system identification, closed-loop identification, errors-in-variables identification, and orthogonal basis functions.