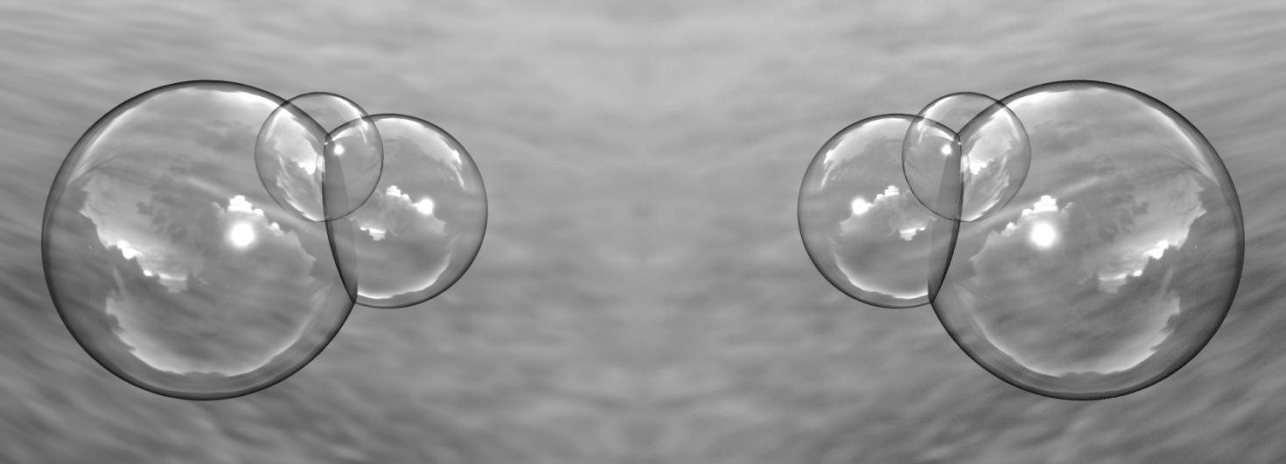
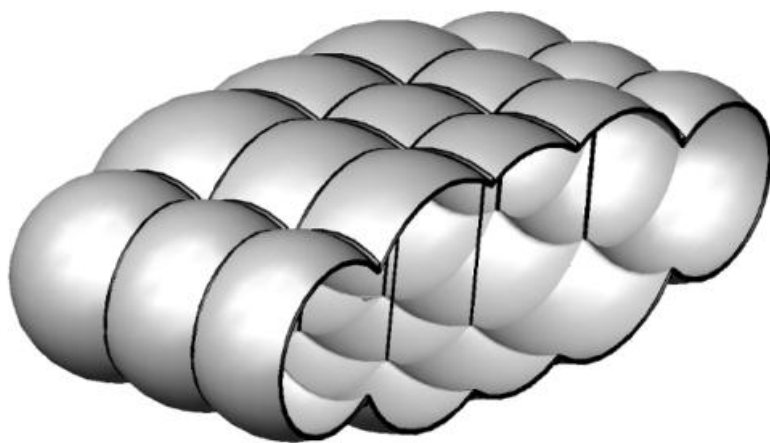


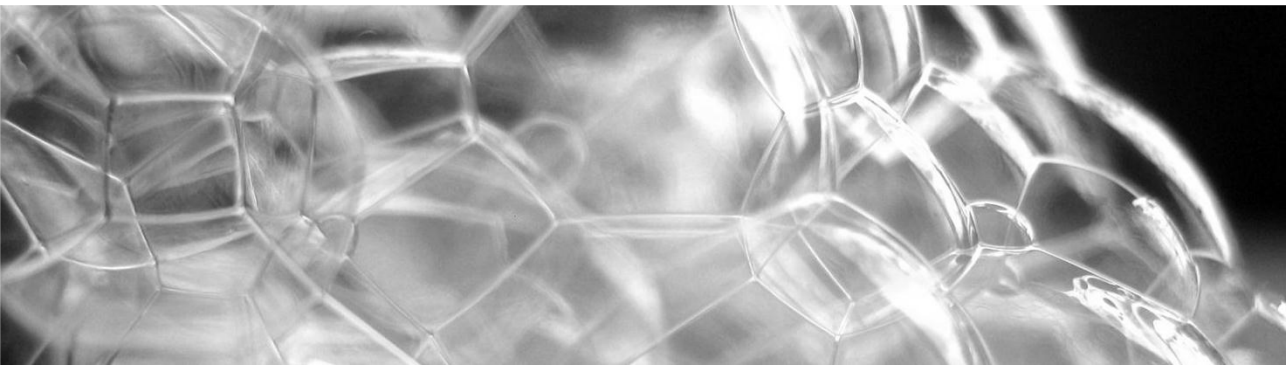


## **Conformable Pressurized Structures**

Design & Analysis



François Geuskens





# **Conformable Pressurized Structures:**

## Design and Analysis

### **Proefschrift**

ter verkrijging van de graad van doctor  
aan de Technische Universiteit Delft,  
op gezag van de Rector Magnificus prof.ir. K.C.A.M. Luyben,  
voorzitter van het College voor Promoties,  
in het openbaar te verdedigen op vrijdag 2 November 2012 om 10 uur

door

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*Samenstelling promotiecommissie:*

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I had a strange start to this PhD research. At the end of my Master's project, I was asked to do research for CleanEra. CleanEra's mission was to develop out-of-the-box technologies that enable a drastic reduction of the environmental impact of future aircraft. My chair's interest in this project was to investigate non-cylindrical pressure cabins for Blended Wing Bodies. I was interviewed by my promotor Adriaan Beukers and co-promoter Sotiris Koussios. Adriaan tried to inspire me by saying that I should first investigate the mechanics of soap bubbles. I could not think of a more boring suggestion. As a result, I received a letter one week later saying that I was rejected due to motivational issues.

A few weeks later, I met the former Dean, Ben Droste, who was surprised that I did not get the job. Ben Droste noticed that I would be interested in a job that gave me more freedom, diversity and influence. Adriaan picked up on that idea, changed the conditions of the research and as a result, I accepted a fantastic job! I was interim project leader of CleanEra, Lecturer of the Master's Course 'Design of Composite Structures', PhD researcher and was executing multiple activities with respect to project management and educational activities. I therefore want to thank Ben Droste for the conversation that initiated my career at the TU Delft.

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# SUMMARY

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There are many applications where volume needs to be pressurised within a geometrical space for which conventional pressure vessels do not provide suitable solutions. Applications are for example found in pressure cabins for Blended Wing Body Aircraft and conformable pressure vessels for an incompressible medium that has to be stored in a pressurised environment, specifically liquid gasses (e.g. propane) or cryogenic applications. These applications demand conformable pressurised structures but up until now, there are no conformable pressurised structures with full spatial freedom, made from high specific strength materials and optimal structural efficiency realised yet.

The aim of this thesis is to provide understanding in the analysis and structural design of conformable pressurized structures. This thesis will enable the development of conformable pressure cabins and pressure vessels.

From a structural topology point of view, efficient structures are in-plane, rather than out-of-plane, loaded structures. In this thesis, it is shown by means of linear membrane analysis that any arbitrary combination of intersecting axi-symmetrical membrane structures will always carry the pressurization loads via in-plane stresses provided that there is a proper in-plane loaded reinforcement at the surface of intersection. The most well-known example of a structure like this is a cluster of soap bubbles. In this example, the axi-symmetrical membrane structure is a sphere and the intersecting walls are the in-plane loaded reinforcements. This type of conformable pressure vessel is therefore called the multi-bubble. Simple shapes like intersecting spheres, intersecting cylinders or intersecting toroids are easy to analyse but the analysis becomes complex when different types of axi-symmetric elements are intersecting each other. This problem has been tackled by the visual interpretation of the linear membrane theory. To solve for loads and forces in the multi-bubble, it is shown that the solution simply depends on pressure and geometric variables.

The solution is only valid however when the shell has the optimal stiffness distribution. The optimal stiffness distribution causes the inflated shape to be

identical to the original shape, only the size is different. The optimal stiffness distribution is derived in this thesis. Manufacturing restrictions and cost limitations might cause the optimal stiffness distribution not to be economically feasible. A non-optimal stiffness distribution causes a nonlinear membrane response. A modified linear membrane solution for orthotropic membranes is derived in this thesis to solve for forces and deformations in non-optimal membranes. Furthermore, the understanding of the membrane response of pressurised structures also enables the formulation of a tailored stiffness distribution in order to control/eliminate deformations in particular directions. This can ease the integration of the pressurized structure in the surrounding structure.

The achievements of this research can be summarized into:

- ✓ Provision of a deeper understanding of the shape of (conformable) pressurized structures
- ✓ Development of the method to analyse the forces at the surface of intersection of the intersecting membrane structures
- ✓ Definition of the optimal stiffness distribution in order to derive the desired membrane response.
- ✓ Formulation of a tailored stiffness distribution in order to control/eliminate deformations in particular directions. This can ease the integration of the pressurized structure in the surrounding structure.
- ✓ Formulated the orthotropic version of the modified linear membrane theory in order to analyse the nonlinear membrane response in case the membrane's stiffness distribution differs from the optimal distribution.
- ✓ Presentation of conceptual ideas of conformable pressurized structures applied to pressure tanks and pressure cabins.

# SAMENVATTING

Er zijn veel toepassingen waar volume onder druk moet worden gezet binnen een geometrische ruimte waarvoor conventionele drukvaten geen geschikte oplossing bieden. Toepassingen zijn bijvoorbeeld te vinden in drukcabines van 'Blended Wing Body' vliegtuigen en gelijkvormige drukvaten waarvoor een onsamendrukbaar medium moet worden opgeslagen in een omgeving die onder druk staat. Deze media bestaan voornamelijk uit vloeibare gassen (bijvoorbeeld LPG) of cryogene toepassingen. Deze toepassingen vereisen gelijkvormige onder druk staande structuren. Er zijn echter nog geen gelijkvormige onder druk staande structuren gecreëerd met volledige ruimtelijke vrijheid, waarvan de constructie gemaakt is van materialen met hoge specifieke sterkte en waarbij een optimale structurele efficiëntie gerealiseerd is.

Het doel van dit proefschrift is om inzicht te geven in de analyse en het constructief ontwerp van gelijkvormige onder druk staande structuren. Dit proefschrift zal de ontwikkeling van de gelijkvormige druk cabines en drukvaten mogelijk maken.

Vanuit een constructief topologisch oogpunt zijn efficiënte structuren in het vlak, in plaats van uit het vlak belaste structuren. In dit proefschrift wordt aangetoond met behulp van lineaire membraan analyse dat elke willekeurige combinatie van elkaar doorsnijdende axi-symmetrische membranen altijd de drukbelastingen in het vlak dragen, op voorwaarde dat er een in het vlak geladen versterking is aangebracht in het snijvlak. Een bekend voorbeeld van deze structuur is een groep zeepbellen. In dit voorbeeld is de axi-symmetrische membraanstructuur een bol en de gemeenschappelijke wand is de in het vlak geladen versterking. Dit type gelijkvormig drukvat wordt daarom ook de multi-bubble genoemd. Eenvoudige vormen zoals snijdende bollen, snijdende cilinders of snijdende torussen zijn gemakkelijk te analyseren, maar de analyse wordt snel complex als verschillende soorten axi-symmetrische elementen elkaar willekeurig gaan snijden. Dit probleem is aangepakt door de lineaire membraan theorie visueel te interpreteren. Het wordt aangetoond dat de oplossing voor de belastingen in een multi-bubble lineair afhankelijk is van de druk en eenvoudig te bepalen is door middel van geometrische variabelen.

De oplossing is echter alleen geldig wanneer de schaal (membraan) de optimale stijfheidsverdeling bezit. De optimale stijfheidsverdeling zorgt ervoor dat de vorm

van de opgeblazen structuur identiek is aan de originele structuur, alleen het formaat verschilt. De optimale stijfheidsverdeling wordt afgeleid in dit proefschrift. Productie beperkingen en kostprijs beperkingen kunnen ertoe leiden dat een optimale stijfheidsverdeling niet economisch haalbaar is. Een niet-optimale verdeling van de stijfheid zorgt voor een niet-lineair membraan gedrag. Een aangepaste lineaire membraan oplossing voor orthotrope membranen wordt afgeleid in dit proefschrift om de belastingen en vervormingen in niet-optimale membranen op te lossen. Inzicht in het vervormingsgedrag van het membraan maakt het ook mogelijk om een stijfheidsverdeling te formuleren waarbij de vervormingen in bepaalde richtingen gecontroleerd of geëlimineerd worden. Dit kan de integratie van de onder druk gebrachte constructie met de omringende constructie vergemakkelijken.

Tot slot kunnen de resultaten van dit onderzoek als volgt worden samengevat:

- ✓ Het verstrekken van een dieper inzicht in de vorm van (gelijkvormige) onder druk gebrachte structuren
- ✓ Ontwikkeling van de methodiek om de krachten te analyseren op het snijvlak van de elkaar snijdende membraanstructuren
- ✓ Definitie en afleiding van de optimale stijfheidsverdeling om het gewenste membraan gedrag te bepalen.
- ✓ Definitie en afleiding van een stijfheidsverdeling om de vervormingen in bepaalde richtingen te controleren of te elimineren. Dit kan de integratie van de onder druk gebrachte constructie met de omringende constructie vergemakkelijken.
- ✓ Het afleiden van de orthotrope versie van de gemodificeerde lineaire membraan theorie om het niet-lineaire membraan gedrag te analyseren in het geval de stijfheidsverdeling verschilt van de optimale stijfheidsverdeling.
- ✓ Presentatie van conceptuele ideeën van vervormbare onder druk staande constructies voor de toepassing in druk cabines en drukvaten.

# NOMENCLATURE

|                    |   |                                                                                   |
|--------------------|---|-----------------------------------------------------------------------------------|
| $A$                | = | cross-sectional area of area element                                              |
|                    |   | Extensional stiffness coefficient                                                 |
| $A, B, C, D$       | = | Stiffness terms                                                                   |
| $A_c$              | = | Circumferential area                                                              |
| $A_f$              | = | deformed area element                                                             |
| $[A]$              | = | In-plane extensional stiffness matrix                                             |
| $A_t$              | = | 'Reinforcement area' perpendicular to the line of intersection                    |
| $A_u$              | = | Surface area corresponding with the intersecting surface                          |
| $A_v$              | = | Distinctive surface area perpendicular to $A_u$ and $A_t$                         |
| $a$                | = | major axis ellipse                                                                |
|                    |   | configuration parameter torus                                                     |
| $a_i, b_i$         | = | Coefficients of minimizing series                                                 |
| $b$                | = | minor axis ellipse                                                                |
| $c, s, t$          | = | Indices to denote the type of membrane element: (cylinder, sphere, torus)         |
| $C$                | = | Integration constant                                                              |
| $D$                | = | flexural stiffness coefficients                                                   |
| $[D]$              | = | flexural stiffness matrix                                                         |
| $d...$             | = | differential (denotation of infinite small element)                               |
| $E$                | = | Young's modulus                                                                   |
| $e$                | = | second order strain displacement                                                  |
| $F$                | = | Modified Elastic potential energy functional                                      |
| $F_R$              | = | resultant membrane force                                                          |
| $F_u$              | = | Force induced by the pressure working on area $A_u$                               |
| $F_v$              | = | Force induced by the pressure working on area $A_v$                               |
| $F_x$              | = | $X$ component of the resultant membrane force $F_R$                               |
|                    |   | Membrane force in x-direction                                                     |
| $F_y$              | = | $Y$ component of the resultant membrane force $F_R$                               |
|                    |   | Membrane force in y-direction                                                     |
| $F_\phi$           | = | resultant membrane force at the intersecting surface (membrane force in the wall) |
| $i, j$             | = | Integer indices                                                                   |
| $k_\phi, k_\theta$ | = | Stiffness parameters in meridional and circumferential direction                  |

|              |   |                                                                             |
|--------------|---|-----------------------------------------------------------------------------|
| $L$          | = | Length                                                                      |
| $m$          | = | mass of the load carrying elements in the pressure vessel                   |
| $m, c$       | = | Indices to denote the <u>m</u> eridional or <u>c</u> ircumferential element |
| $M$          | = | Bending moment                                                              |
| $n$          | = | number of terms in minimizing series                                        |
| $p$          | = | pressure                                                                    |
| $r_0$        | = | circumferential radius                                                      |
| $R$          | = | radius                                                                      |
|              |   | Radial                                                                      |
| $R, \phi, z$ | = | Geodetic coordinates                                                        |
| $R_m$        | = | radius of curvature of the meridian                                         |
| $R_p$        | = | radius of the parallel                                                      |
| $N_x$        | = | axial membrane force                                                        |
| $N_\phi$     | = | meridional force                                                            |
| $N_\theta$   | = | circumferential/hoop force                                                  |
| $Q$          | = | Shear force                                                                 |
| $s$          | = | Arc length                                                                  |
|              |   | Stiffness coefficient                                                       |
| $S$          | = | Stiffness matrix                                                            |
| $S_m$        | = | meridional surface element                                                  |
| $t$          | = | membrane thickness                                                          |
| $T$          | = | tensile load in reinforcement / reinforcement ring                          |
| $u, w$       | = | displacement in $x$ (horizontal) and $z$ (vertical) direction respectively  |
| $U$          | = | Strain energy                                                               |
|              |   | transversal component of load $T$                                           |
| $V$          | = | axial component of load $T$                                                 |
|              |   | volume pressure vessel                                                      |
| $W$          | = | Elastic potential energy                                                    |
| $x, y, z$    | = | Cartesian coordinates                                                       |
| $X, Y$       | = | Forces in respectively $x$ and $y$ -direction                               |

|                         |   |                                                      |
|-------------------------|---|------------------------------------------------------|
| '                       | = | prime (denotation of differentiated variable)        |
| $\alpha$                | = | Fibre angle                                          |
| $\delta$                | = | Fraction of the stiffness ratio                      |
| $\varepsilon$           | = | strain                                               |
| $\phi, \varphi, \alpha$ | = | meridional angles                                    |
| $\theta$                | = | Angle on the parallel                                |
| $\theta, \beta$         | = | Shell rotation                                       |
| $\sigma$                | = | Normal plane stress, indices denote the direction    |
| $\bar{\sigma}_{1,2}$    | = | Allowable (plane) stress in the 1/2 direction        |
| $\rho$                  | = | density material pressure vessel                     |
| $\psi$                  | = | midsurface rotation in $\phi$ -direction             |
| $\chi$                  | = | Ratio of the forces in hoop and meridional direction |
|                         |   | Curvature                                            |
| $\nu$                   | = | Poisson's Ratio                                      |
| $\Delta$                | = | Displacement normal to undeformed midsurface         |
| $\Delta_t$              | = | Displacement tangential to undeformed midsurface     |
| $\Delta_f$              | = | Total displacement of membrane element               |
| $/xx/$                  | = | distance between two points                          |
| $[xxx]$                 | = | area of the triangle governed by three points        |
| $[xxxx]$                | = | area of the quadrangle governed by four points       |



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# CHAPTER 1

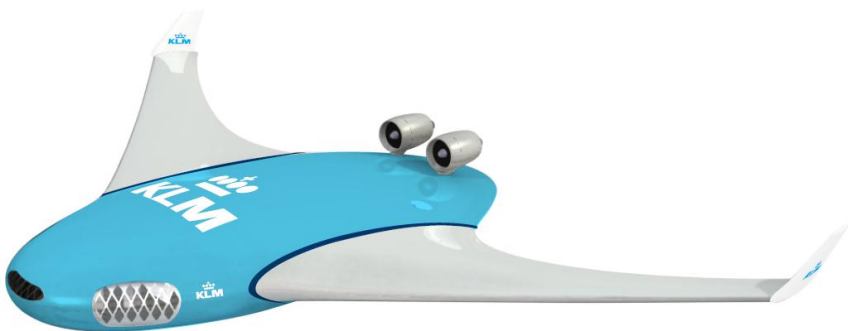
## The need for efficient pressurised space

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### 1.1 Relevance of this research and state of the art

Pressurized structures are generally axi-symmetric shells but there are applications where volume needs to be pressurised within a geometrical space for which axi-symmetrical pressurised structures do not provide suitable solutions. Of particular interest are:

- 1) Pressure vessels that are required to provide usable space, e.g. a structurally efficient pressure cabin for Blended Wing Body aircraft, shown in figure 1.1
- 2) Applications for which the medium inside the pressure vessel cannot be compressed because it is in a liquid state, e.g. cryogenic pressure tanks or propane tanks. For these applications, the designer wants to make the best possible use of the available space due to the incompressibility of the medium. Consider for example a LPG-tank; a flat pressurised tank between the wheels is more favourable than a pressure vessel in the hood.
- 3) Applications for which the shape of the pressurized structure fulfils an additional functionality. Inflatable wings for example need to preserve their aerodynamic shape and sustain the pressurisation loads.



**Fig. 1.1: Impression of a Blended Wing Body Aircraft**

Conformable pressurised structures provide solutions for these applications. The understanding in conformable pressurised structures is not fully developed yet and only intersecting cylinders and spheres have been analysed until now [2, 5, 15, 16, 20, 24, 25, 31, 36].

The conformable pressurised structures we are familiar with consist of intersecting cylinders. The reason for this is mainly related to manufacturability: the membranes or shells are singly curved and the analysis of forces and deformations is easy to comprehend.

The most known applications are low-tech applications such as air mattresses and bouncy castles. Structural efficiency is not the design driver for these applications and they are therefore not considered in this dissertation.

More high-tech applications are found in inflatable wings [Breuer, 2009]. These conformable inflatable beams, figure 1.2, are usually made out of stitched fabrics that are coated or equipped with internal inflatable bladders. The wings are generally straight because variable chords and thicknesses complicate the manufacturability.



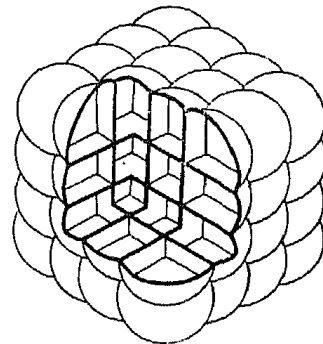
**Fig. 1.2: The inflatable wing of the pumpolino, made by prospective concepts**

Another well-known example is the double bubble fuselage that was used in the Boeing 377, figure 1.3. The usability of the space of the double bubble configuration was better with respect to the design requirements that were set for the Boeing 377. The better usability of the space helped to reduce the drag weight of the aircraft due to the smaller frontal area and surface area of the shell.

Intersecting metallic spherical tanks have been investigated by [Jackson, 1944], [Bert, 1962] and [Komarov, 1970], figure 1.4. The configuration was inspired by a cluster of soap bubbles. Soap film, like metal, is an isotropic material and an efficient pressurised structure was created by mimicking a cluster of soap bubbles. The intersecting spherical metal tank was an interesting concept for large pressurized structures since highly loaded, weight critical structures were made of metals before the 1970's. This changed with the advent of composite materials. The anisotropic material properties of composites made it possible to make all axis-symmetric pressure vessels structurally efficient. Furthermore, the manufacturability of intersecting spherical tanks was an issue because metal spherical caps require excessive forming while welding the spherical shells together is an elaborate job. The consequence of this is that there have never been commercial applications of intersecting spherical pressure tanks.



**Fig. 1.3: Cross-section of the fuselage of the Boeing 377**



**Fig. 1.4: Multi-spherical structure from [Komarov, 1970]**



**Fig. 1.5: Conformable pressure tank from ATK**

Conformable tanks for commercial use have been recently developed by an Australian Company, Propane Performance Industries (PPI), and an American company, ATK, figure 1.5. The tanks from PPI are developed because there is a market need for flat tanks to store Liquid Petrol Gas (LPG). The current alternatives are bulky cylinders or toroids that take up more space for the same amount of LPG. The conformable tanks from both companies are optimized with respect to the limitations of their manufacturing processes. These limitations lead to a restricted spatial freedom and the structural efficiency is less than optimal. No conformable pressure vessels with full spatial freedom, made from high specific strength materials and optimal structural efficiency have yet been realised.

## **1.2 Thesis aim and objectives**

The initial aim of this research is to improve the understanding in the analysis and structural design of conformable pressurized structures. This general objective is divided in several individual objectives:

- To provide a thorough understanding of the shape of (conformable) pressurised structures.
- To develop a method to assess the forces in conformable pressurised structures.
- To assess the deformations in (conformable) pressurised structures, taking into account the elasticity of the material.
- To define the stiffness distribution in order to obtain the desired membrane response.
- To present conceptual ideas of conformable pressurised structures applied to pressure vessels and pressure cabins.

## **1.3 Thesis Outline**

Every chapter in this thesis has its individual introduction and/or literature review in order to improve the understanding of the relevant section. The thesis is divided into 8 chapters.

Chapters 2, 3, 4 and 5 focus on the analysis of axi-symmetric shells and are relevant for the analysis and design of conformable pressurized structures. Chapter 2 involves a general introduction in the analysis of pressure vessels, addresses the first objective of this thesis and is essential for the understanding of the following chapters in this thesis. Chapter 3 is concerned with the membrane response in pressurised membrane structures. This chapter addresses the assessment of the deformations and it defines the stiffness distribution in order to control or manipulate the deformations in pressurised structures. Chapter 4 discusses the shell response when two pressurised shells of different geometries are joined to each other. The mathematics behind the interaction of these two shells is very complex but a simplified method that yields good solutions is presented. The interpretation of this method shows that solutions for thin-walled interacting shells can in most cases easily be predicted. Chapter 5 discusses the optimal structural efficiency for axi-symmetric pressurized structures and the required stiffness requirements to obtain the optimal structural efficiency.

Chapter 6 presents the method to assess the forces in conformable pressure vessels. Finally, a derivation for the optimal structural efficiency for conformable pressure vessels is given.

Chapter 7 discusses the structural concept of a conformable pressure vessel. The prime focus is to discuss the structural configuration. The manufacturability and feasibility of the different structural concepts is mentioned briefly.

Finally, chapter 8 gives a brief introduction of the Blended Wing Body Aircraft, followed by an overview of the possible structural concepts for the pressure cabin.



# CHAPTER 2

## INTRODUCTION TO THE THEORY OF PRESSURIZED SHELLS

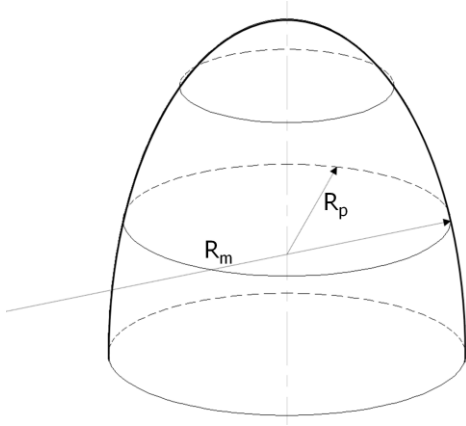
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Thin-walled pressure vessels are generally built as shells of revolution, they have to resist an internal pressure  $p$ , constant and perpendicular to the surface. This chapter gives a background in the analysis of pressure vessels which is necessary for the understanding of subsequent chapters in this thesis. Section 2.1 explains why pressure vessels have their particular shape. Section 2.2 describes the geometry of the shell surface which is needed to analyse the forces in the shell element in section 2.3. Finally, a visual interpretation of the forces in the shell element is given in section 2.4. The content in this chapter allows the reader to understand the entire dissertation without the need of further background reading.

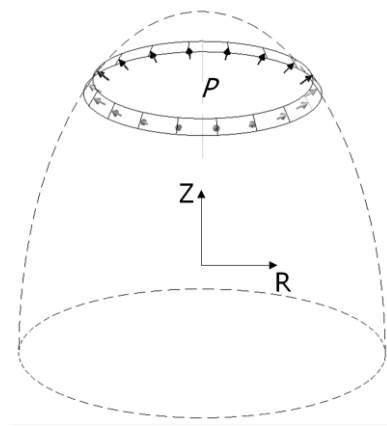
### **2.1 Thin-walled axi-symmetric pressurized shells**

The surface of an axi-symmetric shell as shown in figure 2.1 is characterized by two principal radii of curvature; the radius of curvature of the meridian  $R_m$  and the radius of the parallel circle  $R_p$  (measured on a normal to the meridian between its intersection with the axis of rotation and the middle surface).

Shells of revolution have the property that under axi-symmetric loading, the stresses act tangentially to the curved shell. This is clarified in figure 2.2 where a small ring (small segment cut-out by two parallel circles) of a shell of revolution is subjected to a uniform distributed load working perpendicular to the surface of the ring. The ring is circular since the shell is axi-symmetric and the resultant circumferential forces are always acting tangentially to the wall. The circumferential forces are also referred as hoop forces. The meridional forces on the shell are also acting tangentially as long as there is a smooth (tangential) transition between the ring and the cap that closes the ring.



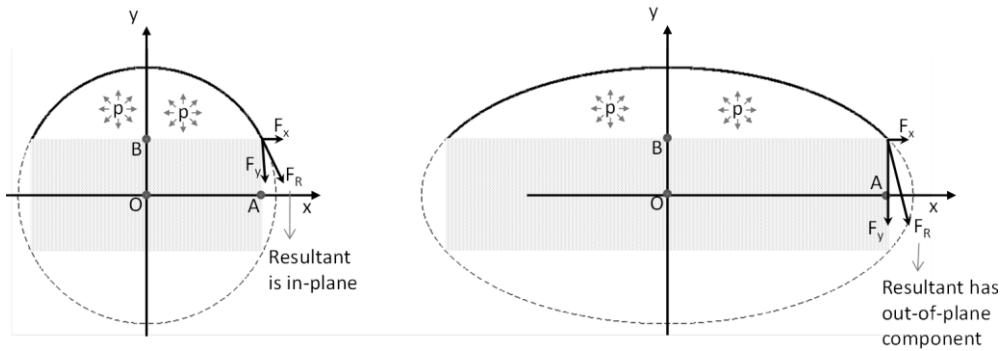
**Fig. 2.1: Principal curvatures in a shell of revolution**



**Fig. 2.2: pressurized segment of an axisymmetric shell**

The pressure causes in-plane internal forces in shells of revolution because the internal stresses act tangentially to the curved shell and the meridional and circumferential forces are therefore referred to as membrane forces.

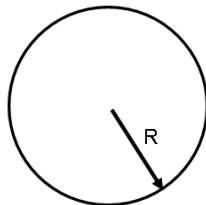
Deviating from the circular contour introduces bending stresses in the wall and reduces the structural efficiency considerably. The pressurization loads acting against the surface of the ring can be visualized by cutting a segment from the ring (figure 2.3). The pressurisation load acting on the edge of the segment can be visualised by projecting the pressurisation loads on the  $x$ - and the  $y$ -axis. The projected loads on the  $x$ -axis [O-A] are counteracted by the vertical membrane force  $F_y$  and the projected load on the  $y$ -axis [O-B] is counteracted by the horizontal membrane force  $F_x$ . The resultant membrane force  $F_R$  is always applied tangentially to the shell wall when the cross-section is circular. Different cross-sections such as an unsupported elliptic one have an out-of-plane component in the resultant force and this introduces bending moments in the skin. The analysis of the bending moment in pressurised cylinders of an arbitrary smooth profile, having two axes of symmetry, has been developed in polar co-ordinates by Holland [Holland, 1982].



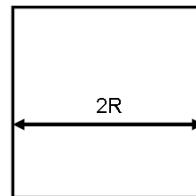
**Fig. 2.3: Load visualization on the edge of the upper segment of a pressurized shell with a) circular circumference and b) elliptical circumference**

Figure 2.4 illustrates the difference between a cross-section carrying pressurization loads solely by membrane stresses (circular cross-section) and a structure stress that carries the pressurization loads primarily via a bending moment (square cross-section). The square cross-section also carries in-plane stresses with an order of magnitude of  $PR/t$ . The maximum stress levels vary linearly with the diameter of the circular tube and more or less quadratically with the width of the square tube. The circular tube requires a much thinner wall to carry the pressurization loads. This supports the claim that from a structural topology point of view, structures subjected solely to in-plane loading are more efficient than out-of-plane loaded structures hence the energy required for the membrane to sustain pressurization is a lot lower for in-plane loaded structures. Cross-sections that deviate from the circular shape such as an elliptical one combine both in-plane stresses with bending stresses and also require more energy to sustain pressurization loads.

$$\sigma = \text{order}[PR/t] \quad \sigma = \text{order}[P(R/t)^2]$$



(a)



(b)

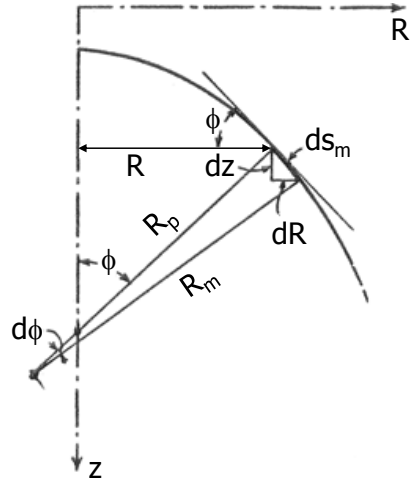
**Fig. 2.4: Order of magnitude of the stress in a (a) circular tube and a (b) rectangular tube**

All (also conformable) structurally efficient pressure vessels consist of circular cross-sections and are therefore only subjected to in-plane loading when pressurized.

The role of the meridional curvature has still not been mentioned. What makes a soap bubble conform to a spherical shape and not a cylindrical one? We need the ability to analyse the membrane forces in pressurised axi-symmetric shells in order to answer this question. This analysis is explained in the next two sections.

## 2.2 Shell geometry

The geometry of a membrane element needs to be defined in order to set-up the equations of equilibrium that later on enable the analysis of the mechanical response of the membrane. The geometry is described by geodetic coordinates, shown in figure 2.5. The geodetic coordinate system is a local reference system that consists of three directions,  $\theta$  (parallel),  $\phi$  (meridian) and  $R$  (radial). Additionally, the parameters  $R_m$  and  $R_p$  are required to define the equilibrium conditions of the membrane element.



**Fig. 2.5: Meridian of a Shell of revolution**

For the distance of a point on the shell with the axis of rotation ( $R$ ) we have:

$$R = R_p \sin \phi \quad (2.1)$$

The line element  $ds_m$  of the meridian is defined as:

$$ds_m = R_m d\phi \quad (2.2)$$

And since,

$$dR = ds_m \cos \phi, \quad dz = ds_m \sin \phi \quad (2.3a, b)$$

We have the relations

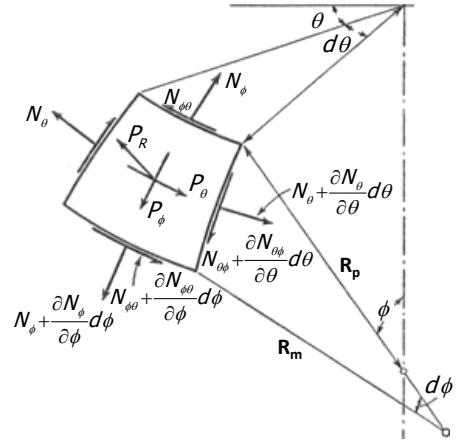
$$\frac{dR}{d\phi} = R_m \cos \phi, \quad \frac{dz}{d\phi} = R_m \sin \phi \quad (2.4a, b)$$

From eqs. (2.1) and (2.4a) we obtain:

$$\frac{1}{R} \frac{dR}{d\phi} = \frac{R_m}{R_p} \cot \phi \quad (2.5)$$

## 2.3 Equilibrium of the Membrane Element

The membrane element ( $Rd\theta \times R_m d\phi$ ) is cut out by two meridians and two parallel circles, shown in figure 2.6. The external loads are denoted by  $P$ , defined as force per unit area. The shell forces are denoted by  $N$ , defined as force per unit length. The conditions of its equilibrium will furnish three equations, just enough to determine the three unknown forces: the meridional force  $N_\phi$ , the hoop force  $N_\theta$ , and the shear force  $N_{\phi\theta}$ .



**Fig 2.6: Forces and loads on a membrane element**

The equilibrium of forces in meridional direction gives [Flügge, 1960]:

$$\sum F_\phi = 0 \Rightarrow \frac{\partial N_{\theta\phi}}{\partial \theta} R_m + \frac{\partial}{\partial \phi} (R N_\phi) - N_\theta R_m \cos \phi + p_\phi R R_m = 0 \quad (2.6)$$

The equation for the forces in the direction of the parallel circle gives:

$$\sum F_\theta = 0 \Rightarrow \frac{\partial}{\partial \phi} (R N_{\phi\theta}) + R_m \frac{\partial N_\theta}{\partial \theta} + R_m N_{\theta\phi} \cos \phi + p_\theta R R_m = 0 \quad (2.7)$$

The third equation refers to the forces which are perpendicular to the middle surface of the shell. Equilibrium gives:

$$\sum F_r = 0 \Rightarrow \frac{N_\phi}{R_m} + \frac{N_\theta}{R_p} = p_R \quad (2.8)$$

These equations can be simplified considerably for pressurized axisymmetric shells. The loads  $P_\theta$  and  $P_\phi$  are zero when only pressurization loads are considered. In addition, the applied loads are rotationally symmetric which means that the derivatives with respect to  $\theta$  are zero. These simplifications eliminate eq. (2.7) which describes a torsional force in the shell around its axis and eq. (2.6) simplifies into:

$$\frac{\partial}{\partial \phi} (RN_\phi) = N_\theta R_m \cos \phi \quad (2.9)$$

A first order differential equation for  $N_\phi$  is obtained when eq. (2.8) is solved for  $N_\theta$  and subsequently substituted in eq. (2.9). The meridional force is obtained by integration with respect to  $\phi$ . By making use of 2.3a, changing  $P_R$  into  $p$ , the simplified integral reads:

$$N_\phi = \frac{p}{R_p \sin^2 \phi} \int_0^R R dR \quad (2.10)$$

### 2.3.1 Membrane forces in classical pressure vessels

The integration constant is zero for axi-symmetric shells where the rotation axis corresponds with the line that connects the circular cross-sections of the pressure vessel. This includes cylinders, cones and boiler ends. Boiler ends are characterized with shapes of which the axis of rotation goes through the vertex, such as the spheroid. Shells that comply to this description are denoted as classical pressure vessels.

Making use of eq. (2.1), eq. (2.10) simplifies into:

$$N_\phi = \frac{1}{2} p R_p \quad (2.11)$$

Substitution of eq. (2.11) in eq. (2.8) also leads to a simple expression for the circumferential force:

$$N_{\theta} = pR_p \frac{2R_m - R_p}{2R_m} \quad (2.12)$$

A property that is true for all boiler ends is that the circumferential force equals the meridional force at the vertex where  $\phi=0$ . At the vertex, all meridians meet, and any direction is parallel to one of them and at right angles to another. The curvatures are in both directions the same and therefore also the stress.

Dividing eq. (2.12) by (2.11) gives the ratio of forces. Dividing the force by the thickness gives the membrane stress. The stress and force ratio in classical pressure vessels is therefore defined as:

$$\frac{\sigma_{\theta}}{\sigma_{\phi}} = \frac{N_{\theta}}{N_{\phi}} = 2 - \frac{R_p}{R_m} \quad (2.13)$$

It is evident that from a strength point of view, the sphere is the ideal pressure vessel for isotropic materials, the curvatures are in both directions the same and so are the stresses. Equal bi-axial loading is the most efficient way of loading isotropic materials but spherical pressure vessels are most often not favourable. Reasons are generally related to manufacturability but also usable space. Aerodynamics can be arguments as well when for example we consider pressurized fuselages.

Returning to eq. (2.13), we notice that for cylinders, the stress ratio is two. In this case, an orthotropic material is desired of which the allowable stress ratio is also two. Fibre reinforced materials do not pose the restriction of equal bi-axial loading due to their tailorability; this property leads to increased spatial freedom for pressure vessels that are required to have high structural efficiency.

Regarding the reason why soap bubbles are spherical, soap can be regarded as an isotropic material. Cohesive forces among the liquid molecules in the soap bubble create 'surface tension' which make the layer of the bubble behave like an elastic sheet. In still air, the soap bubble forms almost a perfect sphere. Only the (little) weight of the surface layer slightly distorts the geometry. A more scientific explanation with respect to the shape of the meridian is given in chapter 5, where the concept of elastic energy is introduced.

The subsequent subsections show interesting cases (examples) of the analysis of classical pressure vessels, starting with the pressurised spheroid, followed by the isotenoid pressure vessel, cones & cylinders and boiler ends without discrepancies.

### 2.3.1.1 Membrane forces in the Spheroid

The oblate spheroid is obtained by rotating an ellipse around its minor axis and the prolate spheroid results from rotating an ellipse around the major axis.

The radii of curvature for both spheroids are defined as:

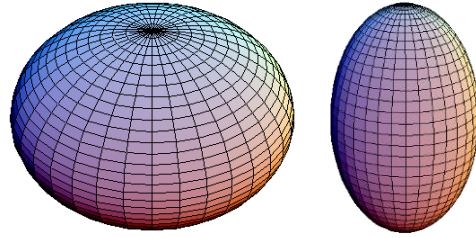


Fig. 2.7: a) Oblate spheroid, b) Prolate spheroid

$$R_p = \frac{a^2}{(a^2 \sin^2 \phi + b^2 \cos^2 \phi)^{1/2}} \quad (2.14)$$

$$R_m = \frac{a^2 b^2}{(a^2 \sin^2 \phi + b^2 \cos^2 \phi)^{3/2}} \quad (2.15)$$

Where  $a$  is defined as the major axis and  $b$  is defined as the minor axis. Introducing these radii in eqs. (2.11) and (2.12) gives the membrane forces. The distribution of the membrane forces for the oblate spheroid is visualized in figure 2.8.

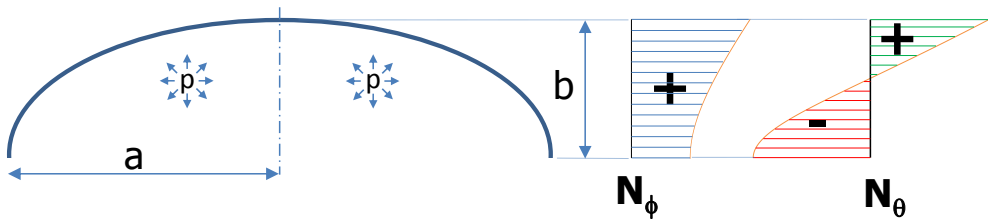


Fig. 2.8: Force distribution in the oblate spheroid

From figure 2.8 it is clear that the circumferential forces in the oblate spheroid can become negative. Equation (2.12) tells us that the circumferential forces turn into compressive forces as soon as  $R_p > 2R_m$  or, for the oblate spheroid, when  $b < a\sqrt{2}$ .

Pressure vessels that are subjected to compressive forces need to be designed for stability (buckling). The compressive in-plane stresses cause out-of-plane failure modes which imply that out-of-plane stiffness is required; this conflicts with membrane theory.

### 2.3.1.2 Membrane forces in the axisymmetric isotensoïd pressure vessel

It was shown in the previous paragraph that hoop forces can get negative for certain pressure vessels. There exists also a geometry for which the hoop forces are zero and the pressurization loads are solely carried via meridional forces. This shape is known as the axisymmetric isotensoïd and differentiates itself in the fact that only load carrying capabilities are required in the meridional direction. The isotensoïd shape was discovered by Taylor who investigated the shape of parachutes in 1919. In this case, the ropes of the parachute were the meridional load carrying reinforcements. From eq. (2.12) we can deduce that the circumferential force equals zero when  $R_p = 2R_m$ . Substituting this in eq. (2.5) gives:

$$\frac{dR}{R} = \frac{1}{2} \cot \phi d\phi \quad (2.16)$$

The shape of the meridian can be found by integrating this differential equation and leads to a solution of elliptic integrals, as shown by [Vasiliev, 2009] and [Lennon, 2002]. The numerical calculation is a lot easier however. The meridian of the isotensoïd is shown in Figure 2.9 alongside the meridian of an ellipse having the same axes.

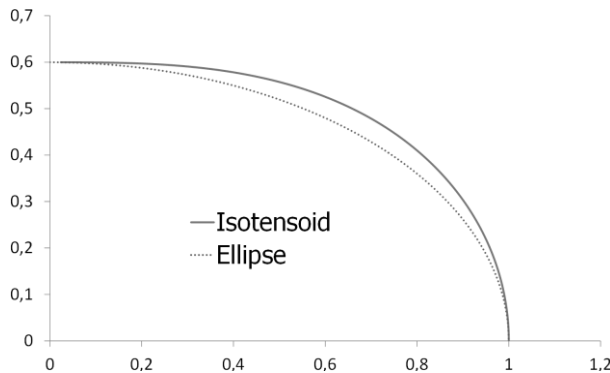


Fig. 2.9: Comparison of the meridian of an isotensoïd with an ellipse

The isotenoid meridian in figure 2.9 was obtained numerically using the euler integration method. The dimensionless radius  $\bar{R}$  ( $=R_\phi/R_{90^\circ}$ ) runs from  $90^\circ$  to  $0^\circ$  and  $d\phi$  is chosen to be constant. From eq. (2.16), we have:  $d\bar{R} = (\bar{R} \cot \phi d\phi)/2$ .

Combining eq. (2.4a) and (2.4b) subsequently gives:  $d\bar{z} = \tan \phi d\bar{R}$ . The dimensionless height  $\bar{z}$  is zero at  $R_{90^\circ}$ .

The isotenoid pressure vessel is interesting from a manufacturing point of view when fibre reinforced materials are chosen since the fibers only need to be wounded around the meridian. However, what seems to be ideal in theory, leads to excessive pile up of the reinforcing fibers near the polar areas. The search for the optimal isotenoid pressure vessel from a manufacturing point of view is not covered in this dissertation but is extensively covered in [Koussios, 2004] and [Zu, 2012].

### 2.3.1.3 Membrane forces in cones and cylinders

The following geometrical relations apply for the cone:

$$R_p = \frac{R}{\sin(\phi)}, \quad R_m = \infty \quad (2.17)$$

The cylinder is a special cone for which  $\phi$  equals  $90^\circ$  and  $R_p$  therefore equals  $R$ .

From eq. (2.11) and (2.12) we obtain:

$$N_\phi = \frac{1}{2} p R_p \quad (2.18)$$

$$N_\theta = 2N_\phi \quad (2.19)$$

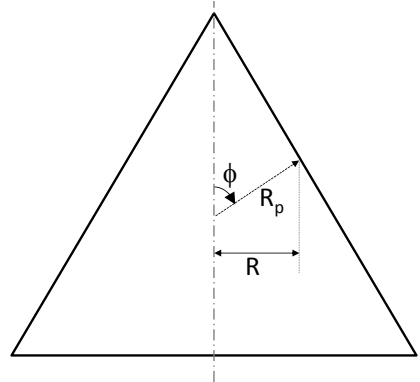


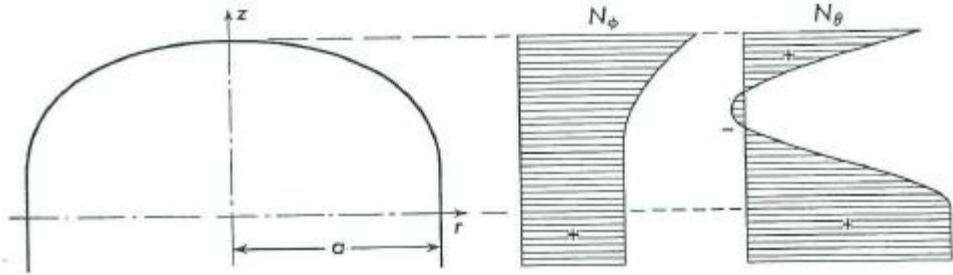
Fig. 2.10: Meridian of a conical shell

### 2.3.1.4 Boiler ends with no discrepancies

When spheroids are used as boiler ends for cylindrical drums, discrepancies appear in the hoop forces at the connection of the spheroid and the drum. Many pressure vessels are manufactured in one shot which implies that the stiffness properties are similar on each end of the connection. In this case, the discrepancies in hoop

forces also cause discrepancies in the deformations. Thick walled, one-shot manufactured pressure vessels with spheroids as boiler ends have bending stresses at the connection of the drum and the boiler. In case the pressure vessel would consist of very thin membranes, the shape of the meridian would change at the location of the connection. More information on the interaction of drums and boiler ends is given in chapter 4.

The discrepancies of the hoop forces can be avoided by choosing another meridian shape. The only requirement is that the meridional radius at the connection equals infinity ( $R_m = \infty$  at  $\phi = 90^\circ$ ). Multiple curves can fulfil this condition of which the cassini oval is one example.



**Fig. 2.11: Boiler End without discontinuity in the hoop forces [figure from Flügge, 1960]**

### 2.3.2 Membrane forces in the pressurized torus

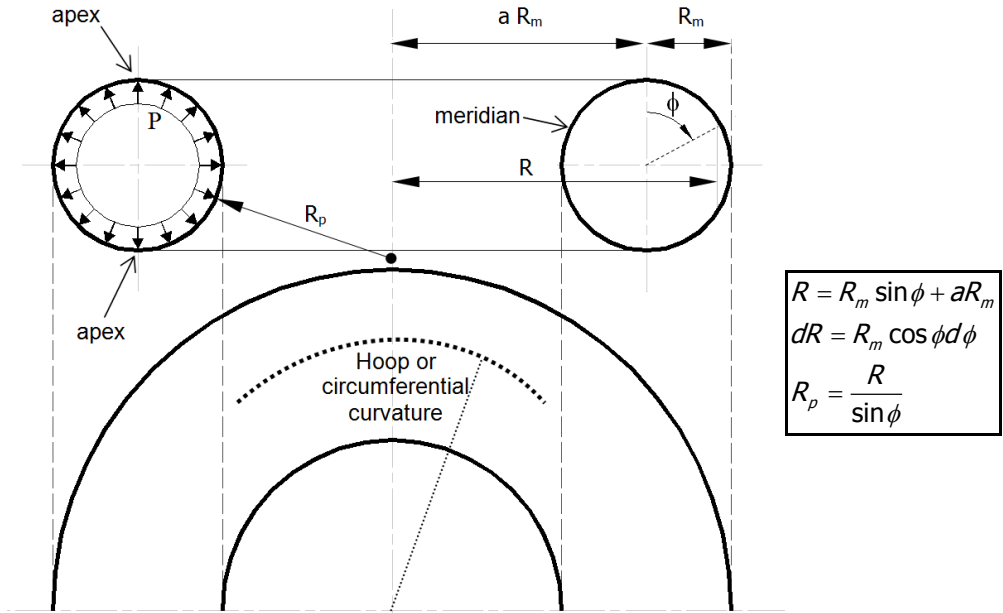
Equation (2.11) and (2.12) are not valid when the axis of rotation does not correspond with the line that connects the circular cross-sections. An example of this is the torus which is generated by the rotation of a closed circle about an axis passing outside. The torus is an important membrane element since it is also an interesting option in conformable pressurized structures. The notation for the toroidal shell is illustrated in figure 2.12.

For this shape, the circular cross-sections are connected by an arc with radius  $aR_m$ , where  $a$  denotes the configuration parameter. Writing eq. (2.10) for the torus gives:

$$N_{\phi,t} = \frac{p}{(R_m \sin \phi + aR_m) \sin \phi} \left[ \int (R_m \sin \phi + aR_m) R_m \cos \phi d\phi + C \right] \quad (2.20)$$

where the index  $t$  refers to the torus. The constant  $C$  is here determined on the requirement that the singularities disappear simultaneously at  $\phi=0$  and  $\phi=\pi$ . This yields:

$$N_{\phi,t} = \frac{pR_m}{2} \left( \frac{2a + \sin \phi}{a + \sin \phi} \right) \text{ and } N_{\theta,t} = \frac{pR_m}{2} \quad (2.21a, b)$$



**Fig. 2.12: Notation torus**

The pressurized torus is a complicated membrane structure. In the next chapter it will be shown that the forces resulting from linear membrane theory can conflict with the displacements; this is especially apparent in the (isotropic) toroidal shell. Equation (2.20) is therefore not sufficient to describe the force distribution in the torus. The reason for this is that linear theory does not take any deformations into account, i.e. it is based on an infinitely stiff membrane.

## 2.4 Visualization of membrane forces

It will be shown in chapter 6 that it is useful to visualize the membrane forces in conformable pressurised structures. Interpretation of the linear membrane theory shows that a visual interpretation can be given to the membrane forces. The forces can be assessed by taking the surfaces perpendicular to the shell surface into consideration (the pressure is working perpendicular to the shell surface).

### 2.4.1 Visualization of the hoop/circumferential forces

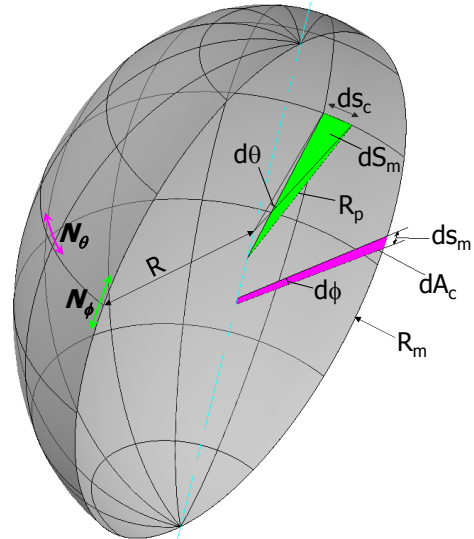
The circumferential force working on a membrane element is found by multiplying the pressure with the ratio of the circumferential area element ( $dA_c$ ) to the arc length of the meridional element ( $ds_m$ ):

$$N_\theta = p \frac{dA_c}{ds_m} \quad (2.22)$$

The circumferential area element is the area enclosed by the arc of the meridional element, the lines perpendicular to edges of the meridional element and the line or arc that connects the centroids of all the circular cross-sections of the shell. For the spheroid, the axis of rotation is the line that connects the centroids of all circular cross-sections and the perpendicular lines are both the local  $R_m$ . (Figure 2.13)

The circumferential area element is obviously a circular sector in case the meridian is circular, which is the case for the sphere and the torus.

The line that connects the centroids of all circular cross-sections is a circle for the torus, shown in figure 2.14. The torus demonstrates that the line that connects the

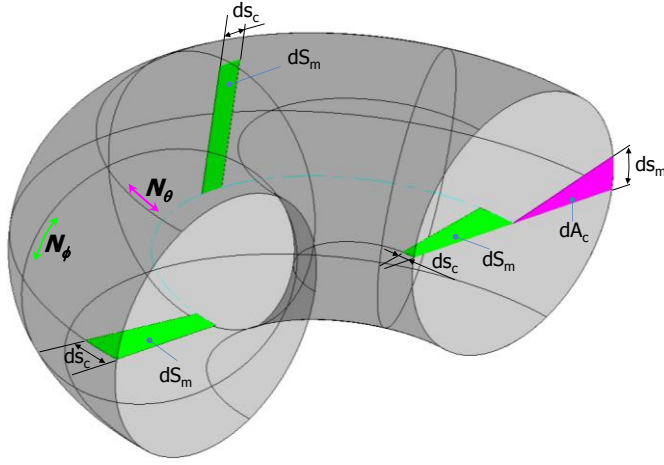


**Fig 2.13: Visualization of distinctive surfaces in a pressurized shell of revolution**

centroids of all cross-sections can deviate from the axis of rotation. The cylinder, sphere and torus with circular cross-section are the only membrane elements where the circumferential force is constant at any location on the meridian and yields:

$$N_{\theta,c} = N_{\theta,s} = N_{\theta,t} = \frac{pR_m}{2} \quad (2.23)$$

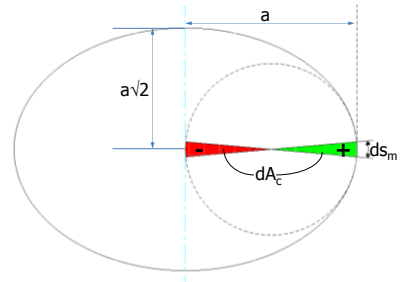
Where the index  $c$ ,  $s$  and  $t$  respectively refer to cylinder, sphere and torus.



**Fig 2.14: Visualization of distinctive surfaces in a pressurized torus**

The visual assessment is a bit harder to interpret for bodies that have a polar axis that is shorter than the diameter of the equatorial circle whose plane bisects it. An example of such a shape is the oblate spheroid (figures 2.7a and 2.15).

In figure 2.15 we see that in the region where  $\phi$  equals  $90^\circ$ , the perpendicular lines intersect in the regions around the equator. For these bodies, the circumferential area element between the intersection and the line that connects the centroid of all cross-sections (in this case the axis of rotation) is given a negative value. A further reduction of the minor axis of the oblate spheroid leads to compressive forces as was demonstrated analytically in 2.3.1.1.



**Fig 2.15: Circumferential area element of an oblate spheroid**

### 2.4.2 Visualization of the meridional forces

The meridional force working on a membrane element is found by multiplying the pressure with the ratio of the meridional surface element ( $dS_m$ ) to the arc length of the circumferential element ( $ds_c$ ) as shown in figures 2.13 and 2.14.

$$N_\phi = p \frac{dS_m}{ds_c} \quad (2.24)$$

The meridional surface element is the surface enclosed by the arc of the circumferential element, the lines perpendicular to the edges of the membrane element and the line or arc that connects the centroids of all the circular cross-sections of the shell. An important remark is that the arc ( $ds_c$ ) is considered with respect to the meridional surface element and not the plane ( $R, \theta$ ). The definition of the arc length of the circumferential element is therefore:

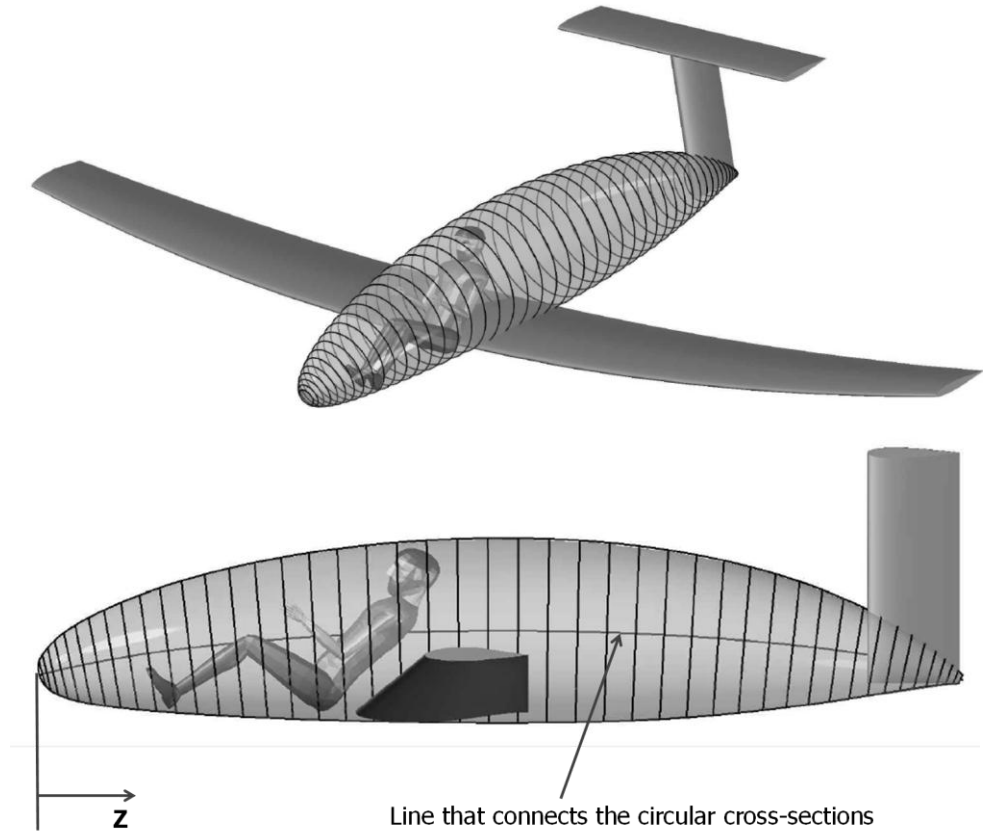
$$ds_c = \frac{Rd\theta}{\sin\phi} = R_p d\theta \quad (2.25)$$

The meridional force for axi-symmetric shells (e.g. the spheroid) is interpreted as the pressure working on a circle with area  $\pi R_p^2/2$  divided by the circumference defined as  $2\pi R_p$ . This explains why the meridional force corresponds with eq. (2.21a). The visual interpretation for the meridional force in the torus is demonstrated in figure 2.14. Due to the varying arc length ( $ds_c$ ) depending on the location on the torus, it is obvious that the meridional force is higher on the inner ring, identical with the meridional force of a cylinder at the apex, and lower on the outer ring.

### 2.4.3 Further comments

Determining the membrane forces graphically is especially useful for pressurized structures that are not axi-symmetric. Conformable pressure vessels are examples of this but also more simple shells such as the pressure fuselage in figure 2.16. The shell in figure 2.16 consists of circular cross-sections and carries the pressure loads by membrane forces only but is not axi-symmetric. For this shell an attempt could be made to formulate the shell as a torus of which the meridional radius and

the configuration parameter ' $a$ ' are a function of  $z$ . Defining the membrane forces graphically is however easier to understand and detailed enough for many engineering applications.



**Fig. 2.16: Example of a pressure fuselage without axis-symmetry**

# CHAPTER 3

## MEMBRANE RESPONSE IN PRESSURIZED SHELLS OF REVOLUTION

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Thin-walled membrane structures are currently the most interesting application for conformable pressurized structures. Both for pressure cabins and cryogenic pressure vessels, pressure differentials are low (in the order of a few bars or less) and the structural efficiency is required to be high which implies that in-plane loading (*read: thin-walled membranes*) is desired and out-of-plane loading (*bending*) generally unwanted. Considering only thin-walled shells, this means that theories are used which are based on the assumption that the shell wall does not resist bending and can only take membrane forces. The previous chapter introduced us to the assessment of the membrane forces in the pressurized membrane. Membranes are made of elastic materials and this chapter explains the membrane response in order to fully understand the forces and displacements in thin-walled pressurized structures.

The first section considers the displacements in the simplest shell theory, linear membrane theory, which is traditionally used to design pressure vessels. Linear membrane theory has however restrictions because it does not take the nonlinear effect associated with the change of the shell geometric parameters in the process of deformation into account. These restrictions are explained in section 3.1, and section 3.2 explains how these restrictions can be bypassed. Section 3.3 shows how the deformations can be tailored to ease for example the integration of a pressure cabin in a Blended Wing Body. Section 3.4 presents the nonlinear equations in order to describe the true nonlinear membrane response. These nonlinear equations need to be solved iteratively and finding an adequate solution is a challenging task. The nonlinear equations can be further simplified and reduced to the quasi-linear form that allows us to find the solution directly. The modified linear membrane theory is the energy formulation of the quasi-linear

membrane theory and is presented in section 3.5 to solve the nonlinear membrane response. The modified linear membrane theory for isotropic membranes was developed by [Tielking, 1971]. The modified linear membrane theory has been modified for this research in order to implement it for the analysis of orthotropic axi-symmetric pressure vessels.

### 3.1 Deformations in linear membrane theory

In linear membrane theory the displacements are assumed to be so small that the force equilibrium equations may be written about an element of the undeformed mid-surface. This implies that the application of linear membrane theory is restricted to membranes that do not noticeably expand.

The linear membrane solution can occasionally reveal peculiarities in the displacement field in pressurized structures. A prime example of this is the deformations in the pressurized torus. A modified membrane theory for these shell structures will be presented later on.

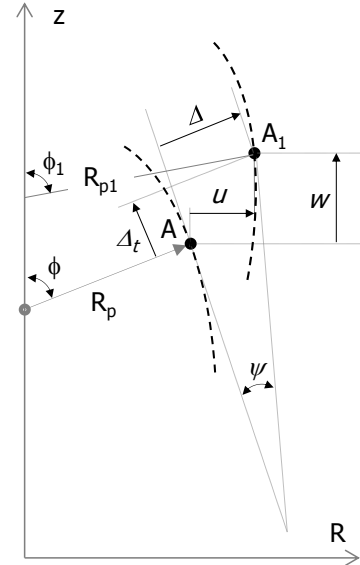
Figure 3.1 shows the displacements of a meridional element where point A represents the initial position and point A<sub>1</sub> represents the final position. Letting  $u$  and  $w$  denote the horizontal and vertical displacements, the strain displacement relations are [Tielking, 1971]:

$$u = R\varepsilon_\theta \quad (3.1)$$

$$w' = \frac{(u' \cos \phi - R_m \varepsilon_\phi)}{\sin \phi} \quad (3.2)$$

Where prime (') denotes differentiation with respect to  $\phi$ .

The strains can also be expressed by a simplified form of the constitutive equations (Appendix A). The simplification is caused by axisymmetry (no shear stress or strain) and by the fact that it is a membrane (only in-plane response is considered).



**Fig. 3.1: Displacements of a meridional element**

For an orthotropic membrane, we have:

$$\begin{bmatrix} N_\phi \\ N_\theta \end{bmatrix} = \begin{bmatrix} A_{\phi\phi} & A_{\phi\theta} \\ A_{\phi\theta} & A_{\theta\theta} \end{bmatrix} \begin{bmatrix} \varepsilon_\phi \\ \varepsilon_\theta \end{bmatrix} \quad (3.3)$$

And vice versa:

$$\begin{bmatrix} \varepsilon_\phi \\ \varepsilon_\theta \end{bmatrix} = \frac{1}{A_{\phi\phi}A_{\theta\theta} - A_{\phi\theta}^2} \begin{bmatrix} A_{\theta\theta} & -A_{\phi\theta} \\ -A_{\phi\theta} & A_{\phi\phi} \end{bmatrix} \begin{bmatrix} N_\phi \\ N_\theta \end{bmatrix} \quad (3.4)$$

The displacement  $u$  is uncoupled from the rest of the shell and can be found just by knowing the local geometry and stiffness properties of the shell. First the forces in the shell are determined (previous chapter), next the circumferential strain is assessed by knowing the stiffness properties in the shell (eq. 3.4) and subsequent substitution in eq. (3.1) reveals the displacement  $u$ .

The vertical displacement is however dependent on the geometry and the properties of the shell (both can be a function of  $\phi$ ) and is found after integrating eq. (3.2):

$$w = \int_{\phi_a}^{\phi_b} \frac{(u' \cos \phi - R_m \varepsilon_\phi)}{\sin \phi} d\phi + C \quad (3.5)$$

The integration constant is found by knowing the boundary conditions.

It suffices to consider only one quarter of the meridian when the deformed meridian has two axes of symmetry. This accounts for classical boiler ends or domes such as the spheroid, cassini domes etc. The integration interval for these shells are:  $\phi_a=0$  and  $\phi_b=\pi/2$ . The boundary conditions for these shells are:

$$u(0) = w(\pi/2) = u'(0) = u'(\pi/2) = w'(0) = w'(\pi/2) = 0 \quad (3.6)$$

The torus for example has only one axis of symmetry for the deformed meridian. In this case the integration would occur in the interval  $\phi_a=-\pi/2$  and  $\phi_b=\pi/2$ . The boundary conditions for axi-symmetric shells with one axis of symmetry are:

$$w(\pm\pi/2) = u'(\pm\pi/2) = w'(\pm\pi/2) = 0 \quad (3.7)$$

The linear membrane deformations for the spheroid and the torus are derived as an example in the next two subsections.

### 3.1.1 Linear membrane deformations in the pressurized orthotropic spheroid

The forces in the spheroid were derived in subsection 2.3.1. Substituting the membrane forces in eq. (3.4) gives:

$$\varepsilon_{\phi} = \frac{pa^2 \left\{ A_{\phi\theta} \left[ b^2 (\cos^2 \phi - 2) + a^2 \sin^2 \phi \right] + A_{\theta\theta} b^2 \right\}}{2b^2 (A_{\phi\phi} A_{\theta\theta} - A_{\phi\theta}^2) \sqrt{a^2 \sin^2 \phi + b^2 \cos^2 \phi}} \quad (3.8)$$

$$\varepsilon_{\theta} = \frac{pa^2 \left\{ A_{\phi\phi} \left[ b^2 (2 - \cos^2 \phi) - a^2 \sin^2 \phi \right] - A_{\phi\theta} b^2 \right\}}{2b^2 (A_{\phi\phi} A_{\theta\theta} - A_{\phi\theta}^2) \sqrt{a^2 \sin^2 \phi + b^2 \cos^2 \phi}} \quad (3.9)$$

The horizontal displacement can now be found by dividing eq. (3.9) by  $R$ . The radius  $R$  is defined by substituting eq. (2.14) into (2.1).

The horizontal displacement  $u$  for the orthotropic pressurized spheroid is therefore given as:

$$u = \frac{pa^4 \sin \phi \left[ A_{\phi\phi} \cos^2 \phi (b^2 - a^2) + A_{\phi\theta} (a^2 - 2b^2) + A_{\theta\theta} b^2 \right]}{2b^2 (A_{\phi\phi} A_{\theta\theta} - A_{\phi\theta}^2) [(a^2 - b^2) \cos^2 \phi - a^2]} \quad (3.10)$$

Differentiating this with respect to  $\phi$  gives  $u'$ . The expression for  $w'$  therefore becomes (eq. 3.2):

$$w' = \frac{-pa^4 \left[ \begin{aligned} &A_{\phi\phi} (a-b)^2 (a+b)^2 \cos^6 \phi \\ &+ (-2A_{\phi\phi} a^4 + A_{\phi\theta} a^2 b^2 + b^4 (2A_{\phi\phi} - A_{\phi\theta})) \cos^4 \phi \\ &+ (A_{\phi\phi} a^4 + 2b^2 a^2 (A_{\phi\phi} - A_{\phi\theta}) - b^4 (4A_{\phi\phi} - 3A_{\phi\theta})) \cos^2 \phi \\ &+ (A_{\phi\theta} a^2 - b^2 (2A_{\phi\theta} - A_{\theta\theta})) b^2 \end{aligned} \right]}{2b^2 (A_{\phi\phi} A_{\theta\theta} - A_{\phi\theta}^2) [(a^2 - b^2) \cos^2 \phi - a^2] \sin \phi} \quad (3.11)$$

Integrating this expression with respect to  $\phi$  gives the vertical displacement  $w$ . There is no analytical solution for  $w$  when  $A_{\phi\phi} \neq A_{\theta\theta}$ . Numerical integration is more convenient since the analytical expression is very elaborate. Results for the oblate and prolate spheroid are shown in Figure 3.2.

The solution in Figure 3.2 is presented with respect to the non-dimensional stiffness parameter  $k$ . The stiffness parameters for the spheroid are defined as:

$$k_{\phi} = \frac{A_{\theta\theta} pa}{A_{\phi\phi} A_{\theta\theta} - A_{\phi\theta}^2} \quad (3.12)$$

$$k_{\theta} = \frac{A_{\phi\phi} pa}{A_{\phi\phi} A_{\theta\theta} - A_{\phi\theta}^2} \quad (3.13)$$

$$\nu_{\phi\theta} = \frac{A_{\phi\theta}}{A_{\theta\theta}} \quad (3.14)$$

Experience has shown that linear membrane theory works well for almost all shells except the pressurized torus which will be investigated in the next section. The remark however needs to be made that experience mainly relies on linear membrane response of thin-walled isotropic shells. Figure 3.3 shows the analysis of a sphere with anisotropic material properties. Notice the strange deformation around the vertex of the shell. Intuitively, this result seems to be incorrect and the validity of linear membrane theory is doubtful in this particular example. This is especially shown by  $w'$  which is supposed to be zero at the vertex. For this example it seems that a different membrane theory is applicable. A modified membrane theory that solves these problems is presented in section 3.4 while section 3.2 explains for which conditions linear membrane theory gives accurate answers. First, the pressurized torus is presented in the next section.

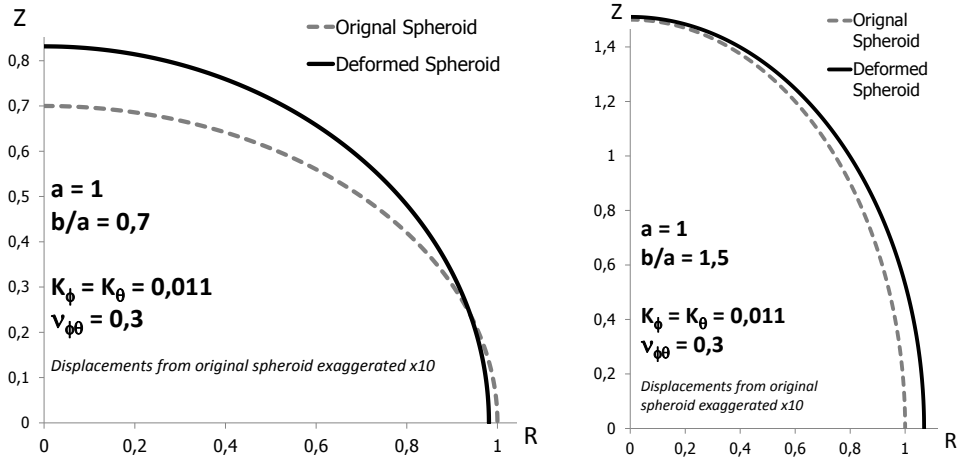


Fig. 3.2: Linear membrane deformations of the a) oblate and b) prolate isotropic spheroid.

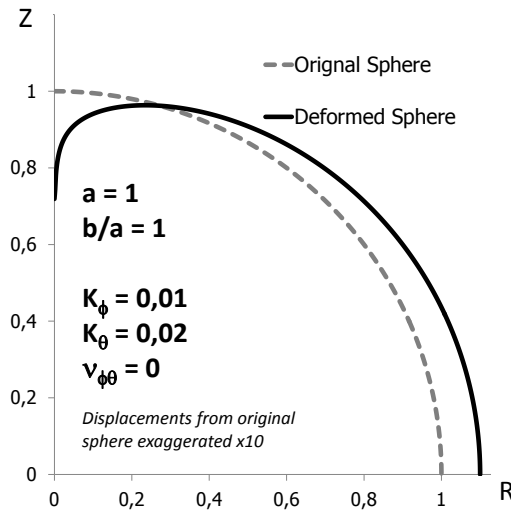


Fig. 3.3: Linear membrane result of the deformation of an anisotropic pressurized sphere

### 3.1.2 Linear membrane deformations in the pressurized torus

The pressurized torus was already introduced in the previous section (figure 2.12 and eq. 2.21). The strains are obtained by substituting eq. (2.21) into eq. (3.4):

$$\varepsilon_{\phi} = \frac{pR_m}{2(A_{\phi\phi}A_{\theta\theta} - A_{\phi\theta}^2)} \left( \frac{A_{\theta\theta}(2a + \sin\phi)}{a + \sin\phi} - A_{\phi\theta} \right) \quad (3.15)$$

$$\varepsilon_{\theta} = \frac{pR_m}{2(A_{\phi\phi}A_{\theta\theta} - A_{\phi\theta}^2)} \left( A_{\phi\theta} - \frac{A_{\phi\theta}(2a + \sin\phi)}{a + \sin\phi} \right) \quad (3.16)$$

Referring to figure 2.12, it is seen that:

$$R = R_m (a + \sin\phi) \quad (3.17)$$

Using eq. (3.1), the horizontal displacement is:

$$u = \frac{pR_m^2}{2(A_{\phi\phi}A_{\theta\theta} - A_{\phi\theta}^2)} [A_{\phi\phi}(a + \sin\phi) - A_{\phi\theta}(2a + \sin\phi)] \quad (3.18)$$

Differentiating this and using eq. (3.2) gives:

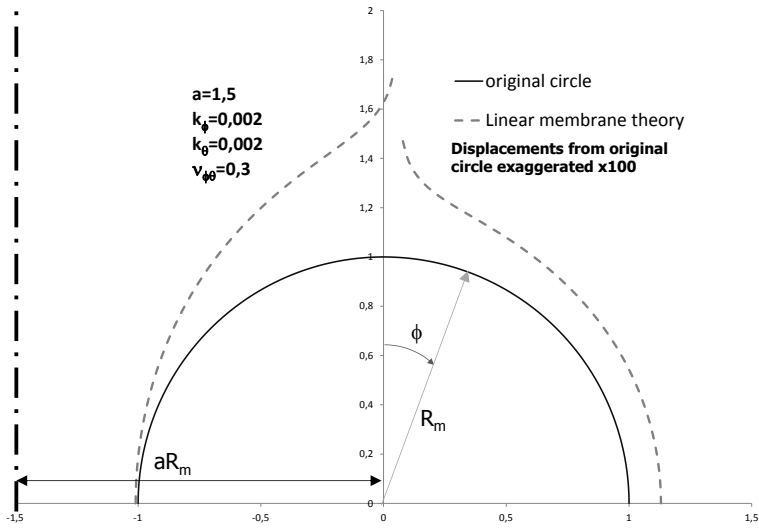
$$w' = \frac{pR_m^2 \left[ (A_{\phi\phi} - A_{\phi\theta})(\cos\phi)^2 + A_{\phi\theta} \right] (a + \sin\phi) - A_{\phi\theta}(2a + \sin\phi)}{2\sin\phi(A_{\phi\phi}A_{\theta\theta} - A_{\phi\theta}^2)(a + \sin\phi)} \quad (3.19)$$

Integration of eq. (3.19) with respect to  $\phi$  results in:

$$w = \frac{pR_m^2}{2(A_{\phi\phi}A_{\theta\theta} - A_{\phi\theta}^2)} \left[ (A_{\phi\phi} - 2A_{\phi\theta}) \ln \left| \tan \frac{\phi}{2} \right| + (A_{\phi\phi} - A_{\phi\theta})(\cos\phi + 1) + \frac{2A_{\phi\theta}}{\sqrt{a^2 - 1}} \arctan \left( \frac{a \tan \frac{\phi}{2} + 1}{\sqrt{a^2 - 1}} \right) \right] + C \quad (3.20)$$

The constant  $C$  corresponds to a rigid body displacement of the shell in the direction of its axis and must be determined by a boundary condition. The boundary condition is such that the displacement  $w$  is zero when  $\phi$  is  $90^\circ$ .

In eq. (3.20), the  $\ln/\tan(\phi/2)/$  is singular at  $\phi=0$  and at  $\phi=\pi$ . This singularity is evident on the deformation curve shown in figure 3.4 which shows the dimensionless displacements, magnified with a factor 100.



**Fig. 3.4: Linear membrane result of the deformation of a pressurized isotropic torus**

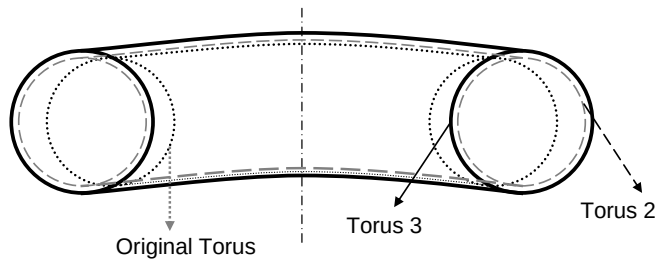
The stiffness parameters used for the assessment of the displacements for the torus are defined as:

$$k_\phi = \frac{A_{\theta\theta} p R_m}{A_{\phi\phi} A_{\theta\theta} - A_{\phi\theta}^2} \quad (3.21)$$

$$k_\theta = \frac{A_{\phi\phi} p R_m}{A_{\phi\phi} A_{\theta\theta} - A_{\phi\theta}^2} \quad (3.22)$$

$$\nu_{\phi\theta} = \frac{A_{\phi\theta}}{A_{\theta\theta}} \quad (3.23)$$

The deformations that result from the linear membrane solution are incompatible with the assumption of a continuous shell with a uniform stiffness and thickness. The cause of this irregularity in the pressurized torus is due to the coupling between the deformations in meridional direction (small circle) and circumferential or hoop direction (figure 2.12). This coupling is present for every doubly curved shell and is for example manifested in the singularities shown in figures 3.3 and 3.4. The coupling for the torus is illustrated in figure 3.5 where the original unpressurized torus is visualized by the dotted line. Suppose this torus is infinitely stiff in the meridional direction and flexible in hoop direction. This would imply that when this torus is inflated, the torus deforms into the shape represented by the gray dashed line (torus 2). Torus 3 represents a torus with similar hoop stiffness as torus 2 but with flexibility in the meridional direction. Superposing the contribution of the deformations due to the meridional stresses leads to a torus that makes the parallel circles (with radius  $R$ , figure 2.12) in the hoop direction become shorter or longer, hence explaining the coupling between the meridional and hoop deformations.



**Fig. 3.5: Illustration of the coupling between the deformations in hoop and meridional direction in a toroidal shell**

The pressurized toroidal membrane with constant thickness and uniform stiffness is an axi-symmetric membrane that, due to the coupling effects, behaves in a non-linear way. The deformation incompatibilities are such that superficially they would seem to require the introduction of bending and shear stresses. Bending and shear stresses would however invalidate the physical concept of an ideal membrane. The thin-walled isotropic pressurized torus was extensively researched in the sixties and it is been proven that an adequate membrane solution exists. The modified membrane solution for orthotropic axisymmetric shells is presented in chapter 3.4.

## 3.2. Conditions for a linear membrane response

The singularities that appear in the isotropic pressurized torus are caused by a constant membrane stiffness and a variable stress distribution (eq. 2.21). It is shown in eqs. (3.15) and (3.16) that the strains are variable. This property disturbs the shape of the torus and changes the stress distribution. The same observation was made for the orthotropic pressurized sphere.

Linear membrane theory is only valid when the shape of the deformed meridian is identical to the shape of the original meridian. This can be achieved by having a variable stiffness distribution where the strains in the hoop direction are equal to the strains in the meridional direction or, in other words, by ensuring that the pressurized structure is in a state of uniform equal bi-axial extension.

Pressurized structures that have a stiffness distribution that allow the structure to be in a state of uniform equal bi-axial extension are from now on denoted as optimal pressurized structures. A thorough explanation why a uniform equal bi-axial extension represents the optimality condition is given in chapter 5.

Setting  $\varepsilon_\phi = \varepsilon_\theta$ , eq. (3.3) gives:

$$\varepsilon_\phi = \varepsilon_\theta = \frac{N_\phi}{A_{\phi\phi} + A_{\phi\theta}} = \frac{N_\theta}{A_{\theta\theta} + A_{\phi\theta}} \quad (3.24)$$

The only shells that do not have variable curvatures are cylindrical and spherical shells. All other shells have variable curvatures and therefore require a variable stiffness distribution in order for linear membrane theory to be exact.

### 3.2.1 The optimal classical pressure vessel

The stiffness distribution for the optimal classical pressure vessel (see previous section) is found by substituting eq. (2.13) in (3.24):

$$\frac{A_{\theta\theta} + A_{\phi\theta}}{A_{\phi\phi} + A_{\phi\theta}} = 2 - \frac{R_\rho}{R_m} \quad (3.25)$$

For these pressure vessels the deformed shape is equal to the original shape multiplied with scale factor  $(1+\varepsilon)$ .

### 3.2.2 The optimal pressurized torus

For the pressurized torus, the optimal stiffness distribution is (eq. 2.21 into eq. 3.24):

$$\frac{A_{\phi\phi} + A_{\phi\theta}}{A_{\theta\theta} + A_{\phi\theta}} = \frac{2a + \sin\phi}{a + \sin\phi} \quad (3.26)$$

The horizontal displacement for the optimal torus is then defined as:

$$u = \frac{pR_m^2 (a + \sin\phi)}{2(A_{\theta\theta} + A_{\phi\theta})} \quad (3.27)$$

The solution for  $w'$  becomes:

$$w' = -\frac{pR_m^2 \sin\phi}{2(A_{\theta\theta} + A_{\phi\theta})} \quad (3.28)$$

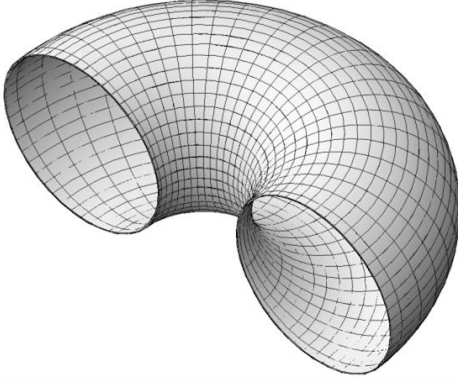
Integration with respect to  $\phi$  leads to:

$$w = \frac{pR_m^2}{2(A_{\theta\theta} + A_{\phi\theta})} \cos(x) \quad (3.29)$$

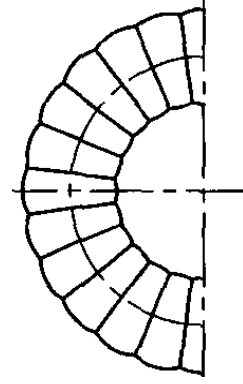
It is clear that the singularity has disappeared and the deformed shape also equals the original shape multiplied with scale factor  $(1+\varepsilon)$ .

Fibre reinforced materials, also referred to as composite materials are very interesting to consider because the stiffness and strength of these materials can be designed by changing the direction and amount of the load-carrying fibres. The tailorability of fibre reinforced materials allows the design of a membrane with a stiffness distribution that corresponds with eq. (3.25) or eq. (3.26). The best way to imagine a toroidal shell that complies with the stiffness distribution of eq. (3.26) is to imagine an orthotropic lay-up with fibres wound in circumferential and

meridional (principal curvature) directions as shown in figure 3.6. Assuming that the stiffness contribution of the matrix is negligible, the optimal stiffness distribution is achieved when the thickness ratio of the fibres in meridional direction with respect to the fibres in circumferential direction at the apex is two. This corresponds with the optimal solution for a cylinder and is of no surprise because the curvatures of the cylindrical shell are identical to the curvatures of the toroidal shell at the apex. Representing the fibres in the meridional direction as circles, we notice that the circles are more closely stacked on the inner side of the torus, which explains the higher meridional membrane stiffness on the inside of the pressurized torus (eq. 3.26). In practice, the variable package results in a higher membrane thickness on the inside.



**Fig. 3.6: Meridional and parallel curves on a toroidal shell**



**Fig. 3.7: Visualization of a torus as an open cell multi-sphere**

Although the lay-up described above is good for imaginative purposes, the manufacturability of such a lay-up is complicated. On top of that, the aforementioned lay-up does not protect the shell against secondary load cases, which have small load magnitudes and thus are not included in the design effort, but could lead to premature failure because the shell has limited strength. An interesting solution from a manufacturing point that also complies to a linear membrane response is a filament overwound (quasi-)isotropic toroidal pressure vessel [NASA Tech briefs, 2005]. The (quasi-) isotropic shell has a constant thickness and the filaments are wound in meridional direction and dimensioned such that  $(A_{\phi\phi} + A_{\theta\theta})_{fil} = (A_{\phi\phi} + A_{\theta\theta})_{iso}$  at the apex. The index *fil* refers to the filaments and the index *iso* refers to the (quasi-)isotropic shell. This solution is closely related to the multispherical torus of Komarov (Figure 3.7).

### 3.3. Tailoring membrane deformations

In chapter 1 it was described that there is a need for conformable pressure vessels because they need to fit in a dedicated geometrical space. For these cases, considerations also need to be made with respect to the integration of the pressurized structure with the surrounding structure. Take for example a segregated pressure cabin in a Blended Wing body as was mentioned in the introduction. Integrating a multi-bubble in an aerodynamic shell requires three degrees of freedom (three translations) at every connection when the deformations of both structures are not compatible with each other. Constraining the structure at the connection is not a viable option because this would cause complex stress and strain distributions. For these situations it might be interesting to tailor the deformations of the pressurized structure. In this section it is chosen to eliminate the deformations in one direction but the methodology presented in this section also allows controlling the deformation differently. Eliminating the deformations completely is physically impossible but an example is shown of a pressurized structure of which the deformation in meridional direction is eliminated. Eliminating the deformation in particular directions can be realized by making sure that the strain in this direction equals zero. Take for example a shell (cylinder) that is not allowed to deform in the meridional (axial) direction. Equation (3.3) gives:

$$\begin{bmatrix} N_\phi \\ N_\theta \end{bmatrix} = \begin{bmatrix} A_{\phi\phi} & A_{\phi\theta} \\ A_{\phi\theta} & A_{\theta\theta} \end{bmatrix} \begin{bmatrix} 0 \\ \varepsilon_\theta \end{bmatrix} \quad (3.30)$$

Equation (3.30) reduces to:

$$\varepsilon_\theta = \frac{N_\theta}{A_{\theta\theta}} = \frac{N_\phi}{A_{\phi\theta}} \quad (3.31)$$

In order to eliminate the meridional or axial displacement, a composite laminate is required for which the following stiffness ratio applies:

$$\frac{N_\theta}{N_\phi} = \frac{A_{\theta\theta}}{A_{\phi\theta}} \quad (3.32)$$

Equation 3.5 shows that setting  $\varepsilon_\phi$  to zero simplifies the analysis of the vertical deformations.

Imagine a lay-up consisting of an angle-ply with orientation  $\pm\alpha$  combined with a hoop overwrap (a ply of  $0^\circ$ ). The coupling terms  $A_{\theta\theta}$  and  $A_{\phi\theta}$  are, according to appendix A, defined as:

$$\begin{aligned} A_{\theta\theta} &= t \left[ x s_{11} + (1-x) (s_{11} \cos^4 \alpha + s_{22} \sin^4 \alpha + 2(s_{12} + 2s_{66}) \cos^2 \alpha \sin^2 \alpha) \right] \\ A_{\phi\theta} &= t \left[ x s_{12} + (1-x) (s_{12} (\cos^4 \alpha + \sin^4 \alpha) + (s_{11} + s_{22} - 4s_{66}) \cos^2 \alpha \sin^2 \alpha) \right] \end{aligned} \quad (3.33)$$

where  $t$  is the total thickness of the membrane,  $x$  is the fraction of the hoop overwrap and the terms  $s_{ij}$  are the terms of the stiffness matrix  $S$ , defined as:

$$S = \begin{bmatrix} s_{11} & s_{12} & s_{16} \\ s_{12} & s_{22} & s_{26} \\ s_{16} & s_{26} & s_{66} \end{bmatrix} \rightarrow \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \begin{bmatrix} s_{11} & s_{12} & s_{16} \\ s_{12} & s_{22} & s_{26} \\ s_{16} & s_{26} & s_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} \quad (3.34)$$

Equation (3.34) represents the notation of Hooke's Law of one ply along the material axes and eq. (3.33) represents the stiffness properties of the membrane ( $\phi, \theta$  - direction).

Suppose that the axial deformation of a pressurized cylinder is eliminated and that the fiber orientations are restricted to  $\pm 45^\circ$  (angle-ply) and  $0^\circ$  (hoop-layer). In addition, suppose that the material applied is a uni-directional carbon fibre reinforced polymer (CFRP) with the following stiffness properties:

$$S = \begin{bmatrix} 150845 & 2900 & 0 \\ 2900 & 10358 & 0 \\ 0 & 0 & 7170 \end{bmatrix}$$

Implementing eqs. (3.32-3.34) reveals that the fraction  $x$  should be approximately 0.12. This means that the hoop overwrap represents 12% of the membrane's thickness and the angle-ply of  $\pm 45^\circ$  represents the remaining 88%. A comparison of the load-carrying capabilities for the same maximum strain levels (*maximum strain criterion*, Appendix A) was made in order to compare the structural efficiency

of the aforementioned laminate with the optimal laminate (which is in the uniform equal bi-axial strain state). The optimal laminate (made from the same material) is obtained with the following lay-up: 60% of the angle-ply with an orientation of  $\pm 45^\circ$  and 40% unidirectional (hoop overwrap). A comparison of the difference in structural efficiency reveals that the membrane without axial displacements is 84% heavier compared to the optimal membrane.

A force ratio (eq. 3.32) of one would for the same stiffness properties imply that the share of the hoop overwrap cannot be more than 3.3% combined with an angle-ply of  $62^\circ$ . This is practically difficult to implement and a combination of two angle plies is more suitable in order to have some protection against secondary load cases. By setting  $x$  in eq. (3.33) to zero, substituting this in eq. (3.32) and solving subsequently for  $\alpha$  gives:

$$\alpha = \pm \text{atan} \left( \frac{1}{2A} \left( \sqrt{2} \cdot \sqrt{A(B + \sqrt{C + D})} \right) \right) \quad (3.35)$$

Where  $A$ ,  $B$  and  $C$  are defined as:

$$A = -s_{22} + \chi s_{12}$$

$$B = 2s_{12} + 4s_{66} + \chi(-s_{11} - s_{22} + 4s_{66})$$

$$C = 16s_{66}^2(1 + \chi)^2 + 16s_{12}s_{66} + 4s_{12}^2(1 - \chi^2) + (\chi s_{11} + \chi s_{22})^2$$

$$D = -8\chi s_{66}(\chi s_{11} + \chi s_{22} + s_{11} + s_{22} - 2s_{12}) - 4s_{22}s_{11}$$

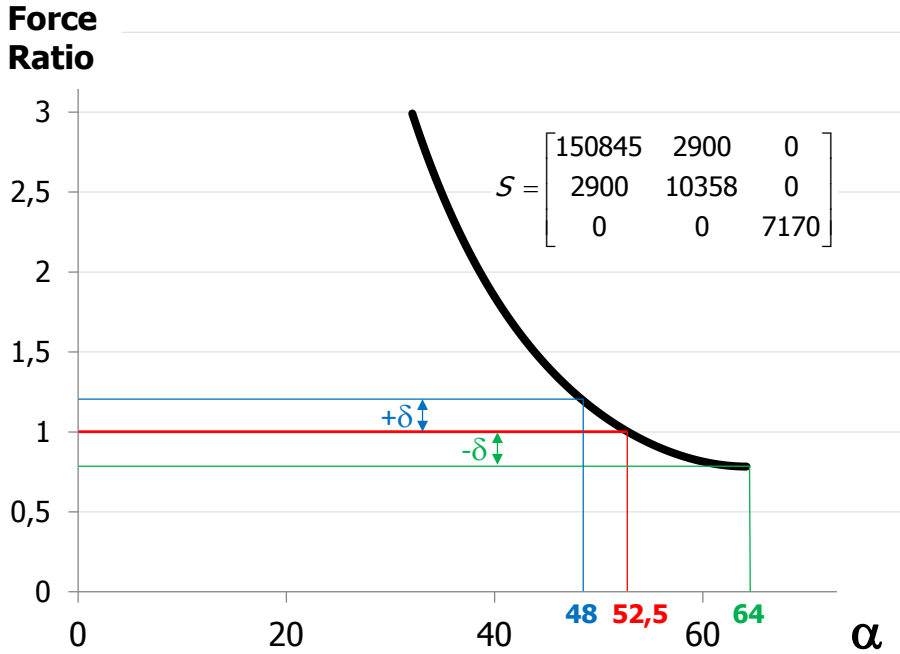
And  $\chi$  is the ratio of the in-plane membrane forces in the hoop and meridional direction, given by:

$$\chi = \frac{N_\theta}{N_\phi} \quad (3.36)$$

Equation (3.35) gives the required angle of the angle-ply to eliminate the deformation in the meridional ( $90^\circ$ -) direction for a given force ratio. A graph can be plotted for every ply with its own stiffness properties (figure 3.8). This graph can also be used to create a lay-up of 2 angle-ply of which the combination eliminates the deformation. The force ratio can also be expressed as:

$$\chi_{\varepsilon=0} = \left( \frac{A_{\theta\theta}}{A_{\phi\theta}} \right)_1 - \delta = \left( \frac{A_{\theta\theta}}{A_{\phi\theta}} \right)_2 + \delta \Rightarrow \chi_{\varepsilon=0} = \frac{\left( \frac{A_{\theta\theta}}{A_{\phi\theta}} \right)_1 + \left( \frac{A_{\theta\theta}}{A_{\phi\theta}} \right)_2}{2} \quad (3.37)$$

As an illustration, for a force ratio of one, the meridional strain (90°-direction) can be eliminated by one angle-ply of 52.5° or by two angle-plys of which 50% has an orientation  $\pm 48^\circ$  and the other 50% has an orientation of  $\pm 64^\circ$ .



**Fig. 3.8: Graph to determine the angles in an angle-ply that eliminates the strain in the 90° direction for a given force ratio  $\chi$ .**

The theory in this section is especially useful for cylindrical pressurized drums because there are no coupling effects between the circumferential and axial deformations.

### 3.4. Nonlinear membrane response

The singularities revealed in figure 3.3 and 3.4 can be resolved by taking the change of the shell's geometric parameters in the process of deformation into consideration. When point A in figure 3.1 of the shell reference surface is compared with point A<sub>1</sub> on the deformed surface, there is the following change with respect to the geometrical parameters:

$$R_1 = R + u = R(1 + \varepsilon_\theta) \quad (3.38)$$

Similarly, the length of the meridional element changes as well:

$$ds_1 = ds(1 + \varepsilon_\phi) \quad (3.39)$$

The principal curvatures before and after the deformation are:

$$\begin{aligned} \frac{1}{R_m} &= \frac{d\phi}{ds} & \frac{1}{R_p} &= \frac{\sin \phi}{R} \\ \frac{1}{R_{m1}} &= \frac{d\phi_1}{ds_1} & \frac{1}{R_{p1}} &= \frac{\sin \phi_1}{R_1} \end{aligned} \quad (3.40a-d)$$

The rotation  $\psi$  is defined as the angle of rotation of the tangent of the shell meridian and is defined as:

$$\phi_1 = \phi + \psi \quad (3.41)$$

Equations (3.40) therefore yield:

$$\frac{1}{R_{m1}} = \frac{1}{1 + \varepsilon_\phi} \left( \frac{1}{R_m} + \psi' \right) \quad (3.42a)$$

$$\frac{1}{R_{p1}} = \frac{\sin(\phi + \psi)}{R(1 + \varepsilon_\theta)} \quad (3.42b)$$

It can be concluded from eqs. (3.42) that the shell deformation cannot be described by linear membrane theory if:

- the strains  $\varepsilon_\theta$  and  $\varepsilon_\phi$  cannot be neglected in comparison with unity
- the rotation angle,  $\psi$ , cannot be neglected in comparison with angle  $\phi$
- the rate of rotation,  $\psi'$ , cannot be neglected in comparison with the initial meridian curvature.

Including this rotation  $\psi$  leads to a change in the geometric equations [Tielking et al., 1971]. The second order strain displacements are now defined as:

$$e_\theta = \varepsilon_\theta \quad (3.43a)$$

$$e_\phi = \varepsilon_\phi + \frac{1}{2}\psi^2 \quad (3.43b)$$

This changes the constitutive equations into:

$$\begin{bmatrix} N_\phi \\ N_\theta \end{bmatrix} = \begin{bmatrix} A_{\phi\phi} & A_{\phi\theta} \\ A_{\phi\theta} & A_{\theta\theta} \end{bmatrix} \begin{bmatrix} e_\phi \\ e_\theta \end{bmatrix} \quad (3.44)$$

And vice versa:

$$\begin{bmatrix} e_\phi \\ e_\theta \end{bmatrix} = \frac{1}{A_{\phi\phi}A_{\theta\theta} - A_{\phi\theta}^2} \begin{bmatrix} A_{\theta\theta} & -A_{\phi\theta} \\ -A_{\phi\theta} & A_{\phi\phi} \end{bmatrix} \begin{bmatrix} N_\phi \\ N_\theta \end{bmatrix} \quad (3.45)$$

The rotation of the meridional element  $\psi$  is defined as [Tielking, 1971]:

$$\psi = \frac{-(u' \sin \phi + w' \cos \phi)}{R_m} \quad (3.46)$$

Where prime (') denotes the differentiation with respect to  $\phi$ .

The nonlinear equations can be solved by first setting the rotation  $\psi$  to zero and integrate the linear set of equations (eqs. 2.9 and 2.10). These solutions lead to second order strains (eqs. 3.45) and displacements respectively (eqs. 3.1, 3.2), including a new value of the rotation (eq. 3.46). This gives the opportunity to

change the geometry (eqs. 3.42) that is again used as a new iteration of the nonlinear analysis. Doing several iterations will finally lead to convergence [Vasiliev, 2009].

### **3.5 The modified membrane theory**

In section 3.2 it was shown that axi-symmetric pressure vessels (e.g. the thin-walled torus) behave like linear membrane structures when the material properties are tailored in a way that the membrane is always in a state of equal bi-axial strain. It is possible in practice however that these tailored properties are not feasible or economically viable from a manufacturing point of view and that shells with different stiffness distribution need to be considered. In that case, a modified membrane theory might be required to solve for stresses and deformations. In contrast to the linear membrane theory considered so far, the modified linear membrane theory allows us to construct the continuous minimum total potential energy solution for the membrane displacements. The modified linear membrane theory is the energy formulation of the quasi-linear theory [Vasiliev, 2008]. The only nonlinear effect that is taken into account in this theory is the change of the shell meridional curvature. The change of  $R$  and  $R_p$  under loading is therefore neglected. The membranes are in this theory described by linear equations but the corresponding solution is nonlinear with respect to pressure. The equations for thin-walled orthotropic pressurized axi-symmetric shells with arbitrary stiffness distribution are derived in this section and solutions are presented for the pressurized torus.

#### **3.5.1 Membrane solutions for the pressurized isotropic torus**

The reason to modify the classical linear membrane theory was mainly created by the absence of an adequate membrane solution for the pressurized isotropic torus. As was mentioned in the previous section, linear membrane theory is from a theoretical point of view only valid when the stiffness distribution matches the force distribution or when the shell is so stiff that it does not noticeably deform. The latter assumption is very vague however and it is difficult to quantify. Stiff

materials are generally associated to be metals or materials with an equivalent stiffness that are loaded in the elastic range. Typical maximum strain levels are considered to be less than 2%. From literature, it may be concluded that the linear membrane theory works well for all shapes built from stiff isotropic materials except the pressurized toroidal membrane. It was also shown that linear membrane theory can be inaccurate from the moment that isotropy is abandoned (see for example the sphere in figure 3.3).

The concept of an ideal membrane (no bending and shear stresses) was extensively researched in the sixties and it was shown that an adequate nonlinear membrane solution for the pressurized torus exists. This was first demonstrated by [Jordan, 1962] who obtained a nonsingular displacement solution for small strains by using the classical constitutive equations and a finite displacement theory formulated by [Reissner, 1950]. [Sanders and Liepins, 1963] re-examined the problem in an alternative way and obtained a solution by asymptotic integration. In both cases, the solution was obtained by assuming that the role of the strains is negligible and that the membrane rotation plays the dominant role in the deformed configuration. A large deformation analysis was carried out by [Chou, 1964], [Kydonieffs 1967], [Kydonieffs and Spencer, 1967], [Li and Steigmann, 1995]], [Papargyri-Pegiou, 1995] and [Papargyri-Pegiou and Stavrakis, 2000]. [Rosettos and Sanders, 1965] obtained an accurate solution for the pressurized toroidal shell where both linear bending and nonlinear membrane action need to be considered simultaneously. [Colbourne and Flügge, 1969] have obtained a singular perturbation solution of Chou's equations for very low values of the load parameter  $k$ . The load parameter is non-dimensional and defined as  $k = pR_m/Et$  where  $p$  is defined as the pressure,  $R_m$  the radius of the meridian as shown in figure 2.12,  $E$  the young's modulus and  $t$  the membrane thickness.

All solutions obtained by the aforementioned authors are applicable for pressurized circular toroids of isotropic materials. The solutions obtained are all characterized by a nonsingular displacement field. They reveal that:

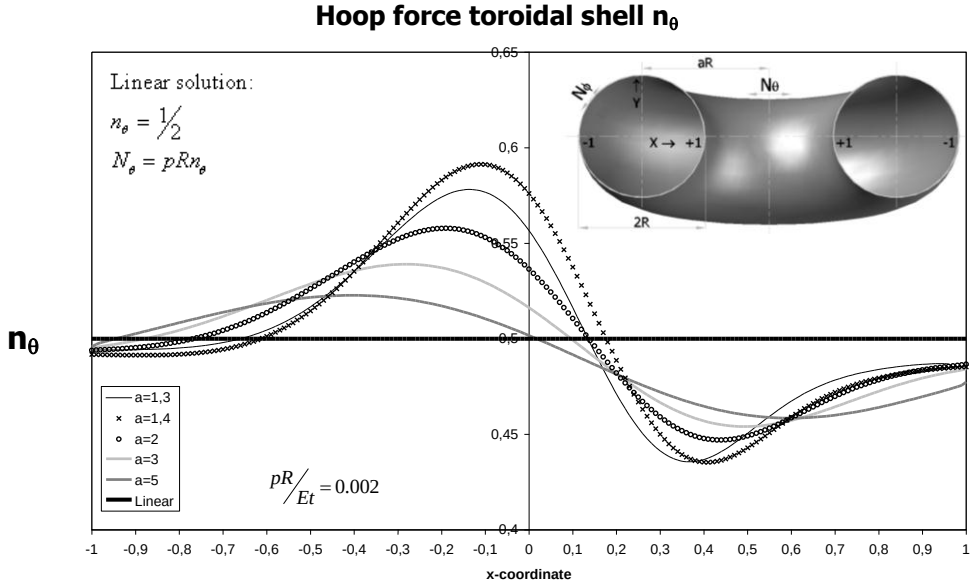
- the pressurized toroidal membrane is inherently nonlinear in the region around the apex (figure 2.12)
- there is very little change in the meridional force  $N_{\phi}$ , whereas the circumferential or hoop force  $N_{\theta}$  differs markedly from the value predicted from linear theory. The non-dimensional hoop force is illustrated in figure

3.9 that shows the hoop force distribution in the thin-walled torus based on the theory of [Jordan, 1962]

The non-dimensional forces in the torus are defined as:

$$n_{\theta,t} = \frac{N_{\theta}}{pR_m} = \frac{1}{2} \quad (3.47)$$

$$n_{\phi,t} = \frac{N_{\phi}}{pR_m} = \frac{2a + \sin \phi}{2(a + \sin \phi)} \quad (3.48)$$



**Fig. 3.9: Circumferential forces in the thin-walled pressurized torus based on the theory of Jordan**

[Tielking, McIvor and Clark, 1971] proposed a modified linear formulation for the internally pressurized isotropic toroid to simplify the solution process and aid the extension to orthotropic materials. The equations in the derivation are linearized by exploiting the insensitivity of the meridional stress resultant to the deformation. These results show very good agreement with previously published nonlinear solutions for a wide range of the load parameter, defined in eqs. (3.21) and (3.22).

[Clark, Budd and Tielking, 1972] analysed an open orthotropic torus to investigate the deformation of a tire. The formulation can easily be used for closed toroids and similarly in the research of [Rosettos and Sanders, 1965], both linear bending and nonlinear membrane action are considered simultaneously.

The aforementioned work can be ascribed to the category of analytic-numerical work. The other category of work deals with Finite Element Analysis (FEA) of which the paper of [Papargyri-Beskou, 2005] and the graduation work of [Vick, 2010] are an example. The work of Papargyri-Beskou deals with large deformations of a pressurized torus and the work of Vick is focused on the optimization of an orthotropic toroidal pressure vessel.

The formulation of [Tielking, 1971] is extended in this section in such a way that the modified membrane theory is applicable to orthotropic axi-symmetric membranes. It seems that the publication of [Clark, Budd and Tielking, 1972] is applicable as well but in this paper the rotation term is excluded from the strain equation, meaning that the analysis does not work for membranes and this paper is therefore only applicable for toroids where the deformation incompatibilities are handled via bending. It is straightforward nowadays to use Finite Element Analysis (FEA) for shells for which bending needs to be included. FEA is difficult to apply for stiff membranes because the coupling of large rotations with small strains makes it difficult to obtain adequate results.

The modified membrane theory gives a better insight in the stresses and deformations in stiff orthotropic (toroidal) membranes and this insight is useful for the analysis of stresses and deformations in the pressurized thin-walled multi-torus.

### **3.5.2 Minimum potential energy of orthotropic thin-walled pressure vessels**

In order to introduce the modified membrane theory for orthotropic pressurized structures, the minimum potential energy needs to be formulated. The minimum potential energy principle for this problem means that the membrane deforms in a way that the strain energy is minimal and equals the elastic potential energy of which both energies are caused by the pressurization.

The strain energy  $U$  of a uniformly loaded, axisymmetric orthotropic membrane per unit thickness is:

$$U = \frac{1}{2} \int_{A_f} (N_\phi e_\phi + N_\theta e_\theta) dA_f \quad (3.49)$$

Where  $e_\phi$  and  $e_\theta$  are the corresponding second order strains and  $A_f$  is the deformed area element.

The elastic potential energy  $W$  for a pressurized membrane is defined as:

$$W = p \int_{A_f} \Delta_f dA_f \quad (3.50)$$

Where  $\Delta_f$  denotes the total displacement of the membrane.

Figure 3.1 shows the displacements of a meridional element. From this, the normal deflection  $\Delta$  and tangential deflection  $\Delta_t$  are (by [Tielking, 1971]) defined as follows:

$$\Delta = u \sin \phi + w \cos \phi \quad (3.51)$$

$$\Delta_t = u \cos \phi - w \sin \phi \quad (3.52)$$

$$\Delta_f = \Delta + \psi \Delta_t \quad (3.53)$$

Relating the displacements to strain, it is noticeable that a change of the circumferential radius ( $R$ ) is only coupled with the circumferential strain while for the meridional arc, the radius  $R_m$  changes by both strain and rotation.

Substituting the constitutive equations (eq. 3.44 in eq. 3.49) gives:

$$U = \frac{1}{2} \int_{A_f} (A_{\phi\phi} e_\phi^2 + A_{\theta\theta} e_\theta^2 + 2A_{\phi\theta} e_\phi e_\theta) dA_f \quad (3.54)$$

Replacing the second order strains by eqs. (3.43) results in:

$$U = \frac{1}{2} \int_{A_f} \left( A_{\phi\phi} \varepsilon_\phi^2 + A_{\theta\theta} \varepsilon_\theta^2 + 2A_{\phi\theta} \varepsilon_\phi \varepsilon_\theta + \psi^2 (A_{\phi\phi} \varepsilon_\phi + A_{\phi\theta} \varepsilon_\theta) + A_{\phi\phi} \frac{\psi^4}{4} \right) dA_f \quad (3.55)$$

This integral represents the strain energy of any uniformly loaded axisymmetric shell. The strains and rotations are very small compared to unity for a stiff shell which implies that the last term in the integral ( $\psi^4$ ) can be neglected. The validity of this assumptions will be shown later on, when the results are presented. Similarly, the area element of the deformed mid-surface is:

$$dA_r = (1 + e_\phi)(1 + e_\theta) dA \approx dA = RR_m d\phi d\theta \quad (3.56)$$

For a full axi-symmetric shell, the area element integration is:

$$\int_A dA = 2\pi \cdot 2 \int_{-\pi/2}^{\pi/2} RR_m d\phi \quad (3.57)$$

The fourth term in the integral (eq. 3.55) represents a nonlinear correction due to mid-surface rotation. This correction term only matters for shells of which the stiffness distribution is different from the stress distribution. Cylinders and spheres with a constant stiffness distribution are the only shells for which a correction term does not apply because the membrane forces are constant along the meridian. In case the membrane forces vary (which is the case for all other axi-symmetric shells), the membrane stiffness needs to vary likewise (see previous section). Taking a closer look at the correction term in eq. (3.55), it shows that the term,  $(A_{\phi\phi}\varepsilon_\phi + A_{\phi\theta}\varepsilon_\theta)$  represents the meridional force  $N_\phi$ . Solutions to the nonlinear toroidal membrane equations [Jordan, 1962], [Sanders and Liepins, 1963], indicate that there is very little deviation in the meridional force  $N_\phi$  given by linear and non-linear theory. Based on these observations, eq. (3.55) simplifies to:

$$U = 2\pi \int_{-\pi/2}^{\pi/2} (A_{\phi\phi}\varepsilon_\phi^2 + A_{\theta\theta}\varepsilon_\theta^2 + 2A_{\phi\theta}\varepsilon_\phi\varepsilon_\theta + N_\phi\psi^2) RR_m d\phi \quad (3.58)$$

where  $N_\phi$  is the meridional force of linear membrane theory. Replacing the strains and rotation by displacements leads to (substitution of eqs. 3.1, 3.2 and 3.46):

$$U = 2\pi \int_{-\pi/2}^{\pi/2} \left( A_{\phi\phi} \left( \frac{u' \cos \phi - w' \sin \phi}{R_m} \right)^2 + 2A_{\phi\theta} \left( \frac{u' \cos \phi - w' \sin \phi}{R_m} \right) \left( \frac{u}{R} \right) + N_{\phi} \left( \frac{u' \sin \phi + w' \cos \phi}{R_m} \right)^2 + A_{\theta\theta} \left( \frac{u}{R} \right)^2 \right) RR_m d\phi \quad (3.59)$$

Similarly, the elastic potential energy written in terms of displacements gives (substitution of eqs. 3.51 - 3.53 and 3.46 into eq. 3.50):

$$W = 4\pi p \int_{-\pi/2}^{\pi/2} \left( (u \sin \phi + w \cos \phi) - \frac{(u' \sin \phi + w' \cos \phi)}{R_m} (u \cos \phi - w \sin \phi) \right) RR_m d\phi \quad (3.60)$$

The modified potential energy functional ( $F$ ) for the thin-walled orthotropic axisymmetric pressure vessel that needs to be minimized is:

$$F = U - W$$

$$= \int_{\phi_a}^{\phi_b} \left( A_{\phi\phi} \frac{R(u' \cos \phi - w' \sin \phi)^2}{R_m} + 2A_{\phi\theta} u(u' \cos \phi - w' \sin \phi) + A_{\theta\theta} R_m \frac{u^2}{R} + N_{\phi} (u' \sin \phi + w' \cos \phi)^2 - 2pRR_m (u \sin \phi + w \cos \phi) + 2pR(u' \sin \phi + w' \cos \phi)(u \cos \phi - w \sin \phi) \right) d\phi \quad (3.61)$$

The integration constants are replaced by  $\phi_a$  and  $\phi_b$  because the energy functional is also applicable for spheroids where the interval  $[0, \pi/2]$  is considered. This functional is the orthotropic version of the functional derived by [Tielking et al., 1971]. The integration intervals were given in section 3.1.

It was shown that the modified potential energy functional possesses a positive definite second variation [Tielking, 1969] which means that the displacements  $u$  and  $w$  which minimize the potential energy generate unique stresses.

The Ritz method is used here to construct finite series approximations of  $u$  and  $w$  that minimize the functional  $F$ . The displacements  $u$  and  $w$  are expressed in trigonometric series that satisfy the geometric boundary conditions given in eqs. (3.6) and (3.7).

For shells with 2 axes of symmetry for the deformed meridian, the trigonometric series for an integration interval  $[0, \pi/2]$  read:

$$u_n = \sum_{i=1}^n a_i \sin[(2i-1)\phi] \quad (3.62)$$

$$w_n = \sum_{i=1}^n b_i \cos[(2i-1)\phi] \quad (3.63)$$

For the torus, the integration interval is  $[-\pi/2, \pi/2]$  and the trigonometric series are:

$$u_n = \sum_{i=0}^n a_i \cos[i(\phi + \pi/2)] \quad (3.64)$$

$$w_n = \sum_{i=1}^n b_i \sin[i(\phi + \pi/2)] \quad (3.65)$$

Substituting eqs. (3.62-3.63) or (3.64-3.65) into the modified potential energy functional (eq. 3.61) leads to a solution where each term consists of a product of two coefficients ( $a_i$  or  $b_i$ ). The Ritz method selects the coefficients which minimize the potential energy by setting:

$$\frac{\partial F}{\partial a_i} = 0 \quad i=(0), 1, \dots, n \quad (3.66)$$

$$\frac{\partial F}{\partial b_i} = 0 \quad i=1, 2, \dots, n \quad (3.67)$$

This reduces the problem to  $2n$  or  $2n+1$  simultaneous linear equations for the classical pressure vessel and torus respectively. These linear equations can be organized in a  $2n \times 2n$  or  $(2n+1) \times (2n+1)$  matrix. The coefficients are subsequently found by taking the row reduced echelon form of this matrix. The values of the coefficients are used to solve for displacements (eqs. 3.62-3.63 or eqs. 3.64-3.65), strains and rotations next (eqs. 3.1, 3.2 and 3.46) and finally the membrane forces (eq. 3.3).

A sensitivity analysis for the required numbers of coefficients and a comparison with previous obtained results ([Tielking, 1971]) by other methods have shown that sufficient accurate results are obtained for  $n=16$ .

Setting up the mathematical formulation is an elaborate job and the modified membrane theory is until now only applied for the orthotropic torus. It shows from literature that the pressurized torus is the only structure of which linear membrane theory shows clear discrepancies between the forces and the displacements in the isotropic toroidal membrane. From a scientific point of view it is interesting to investigate the sphere that is shown in figure 3.3. On one hand it seems odd to use composite over isotropic materials when the use does not improve the structural performance. On the other hand it is imaginable that pressurized structures subjected to multiple load conditions could require a stiffness distribution for which linear membrane theory gives incorrect results.

The only other membrane structure for which the modified membrane theory was used until now was the isotropic spheroid [Tielking, 1967]. The result of this analysis was that very little difference between linear membrane theory and the modified membrane theory was observed.

### 3.5.3. Membrane response in the thin-walled orthotropic torus

Solutions of the analysis are presented in figures. 3.10-3.13 for different values for the configuration and stiffness parameter ( $a$  and  $k$ ). The stiffness parameters are defined as:

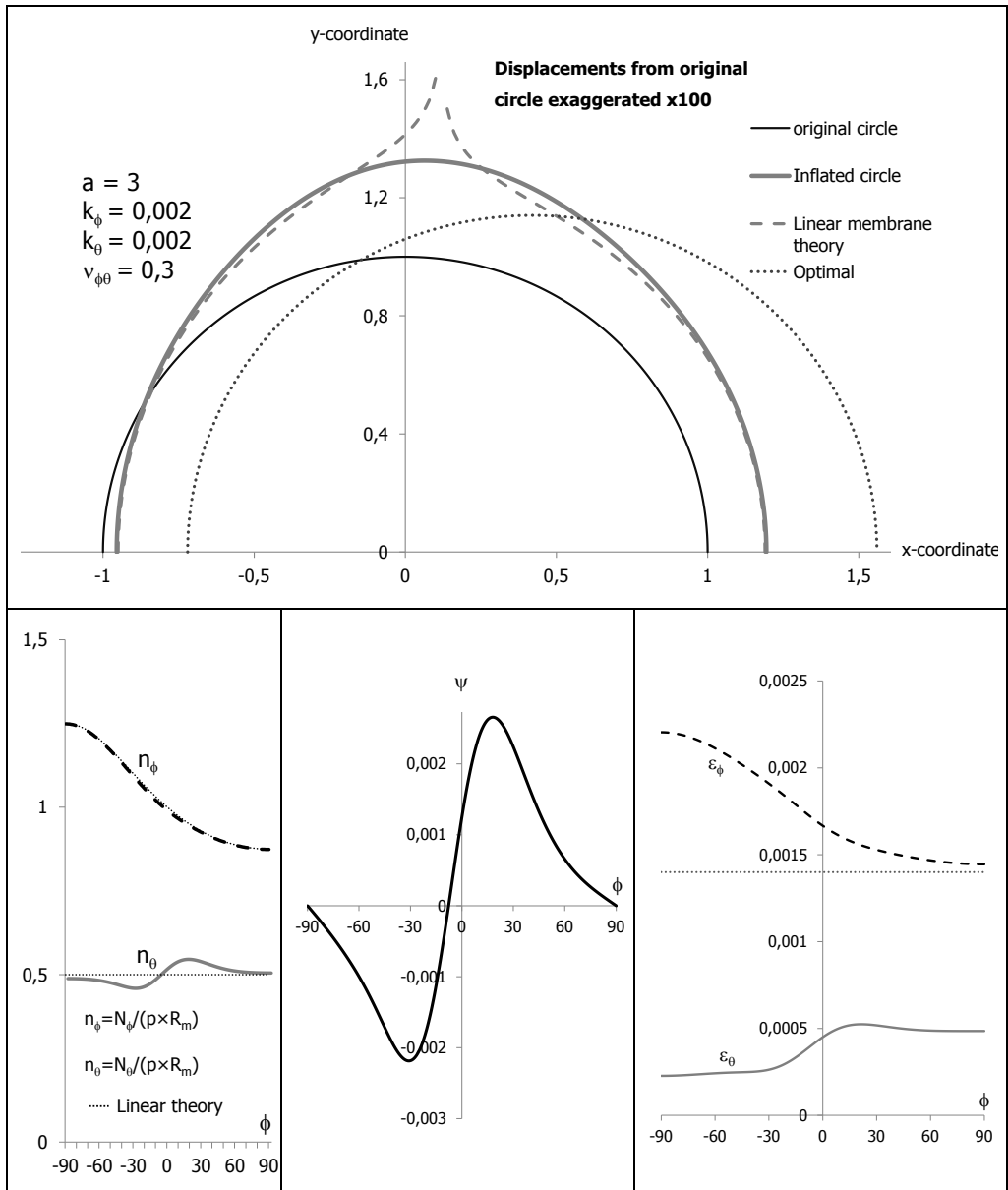
$$k_{\phi} = \frac{A_{\theta\theta} \rho R_m}{A_{\phi\phi} A_{\theta\theta} - A_{\phi\theta}^2} \quad (3.68)$$

$$k_{\theta} = \frac{A_{\phi\phi} \rho R_m}{A_{\phi\phi} A_{\theta\theta} - A_{\phi\theta}^2} \quad (3.69)$$

$$\nu_{\phi\theta} = \frac{A_{\phi\theta}}{A_{\theta\theta}} \quad (3.70)$$

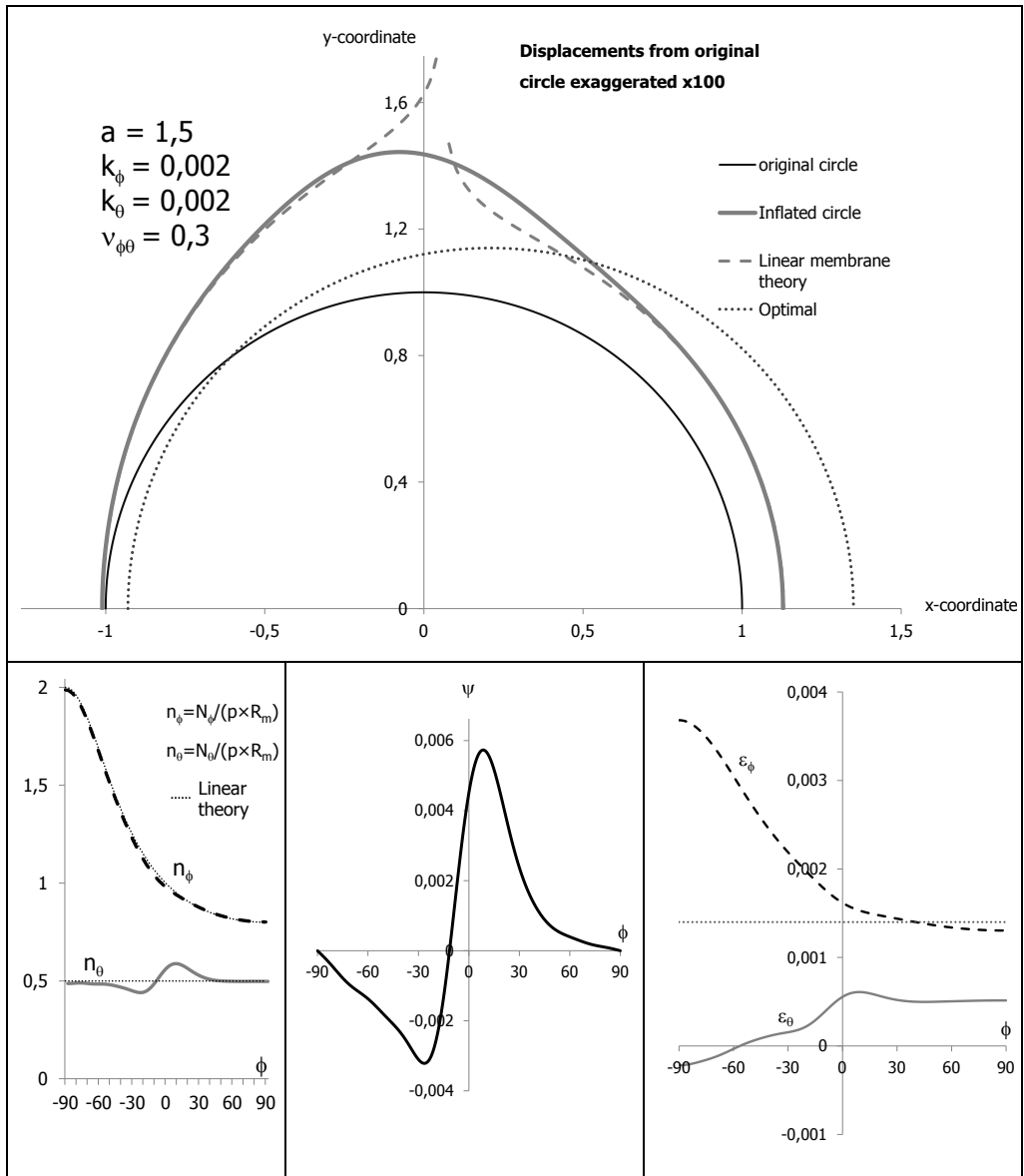
The stiffness parameters are constant in this paper but can also be a function of the angle  $\phi$  which would be the case if a toroidal winder is used. The results in this dissertation are identical to the previous obtained solutions of Tielking when quasi-isotropic material properties (figures 3.10 and 3.11) are used. The nonlinear membrane solutions are plotted together with the optimal membrane solutions in figures 3.10-3.13. From an engineering perspective, these graphs cannot be

compared because the strain levels are very different but it is obvious in figures 3.12-3.13 that the discrepancy at the apex becomes smaller when the stiffness distribution comes closer to the optimal stiffness distribution. It was already mentioned in the introduction that the nonlinearity is strong in the region around the apex. The stiffness in meridional and circumferential direction for the torus in figures 3.12 and 3.13 was chosen such that it comes close to the optimal stiffness distribution that is needed at the apex. The assumption that it is reasonable to neglect the  $\psi^4$  term in eq. (3.55) is valid when we look at the value of magnitude for the rotation. This was also observed by Tielking by verifying that the modified linear membrane theory is valid for a wide range of the loading parameter.



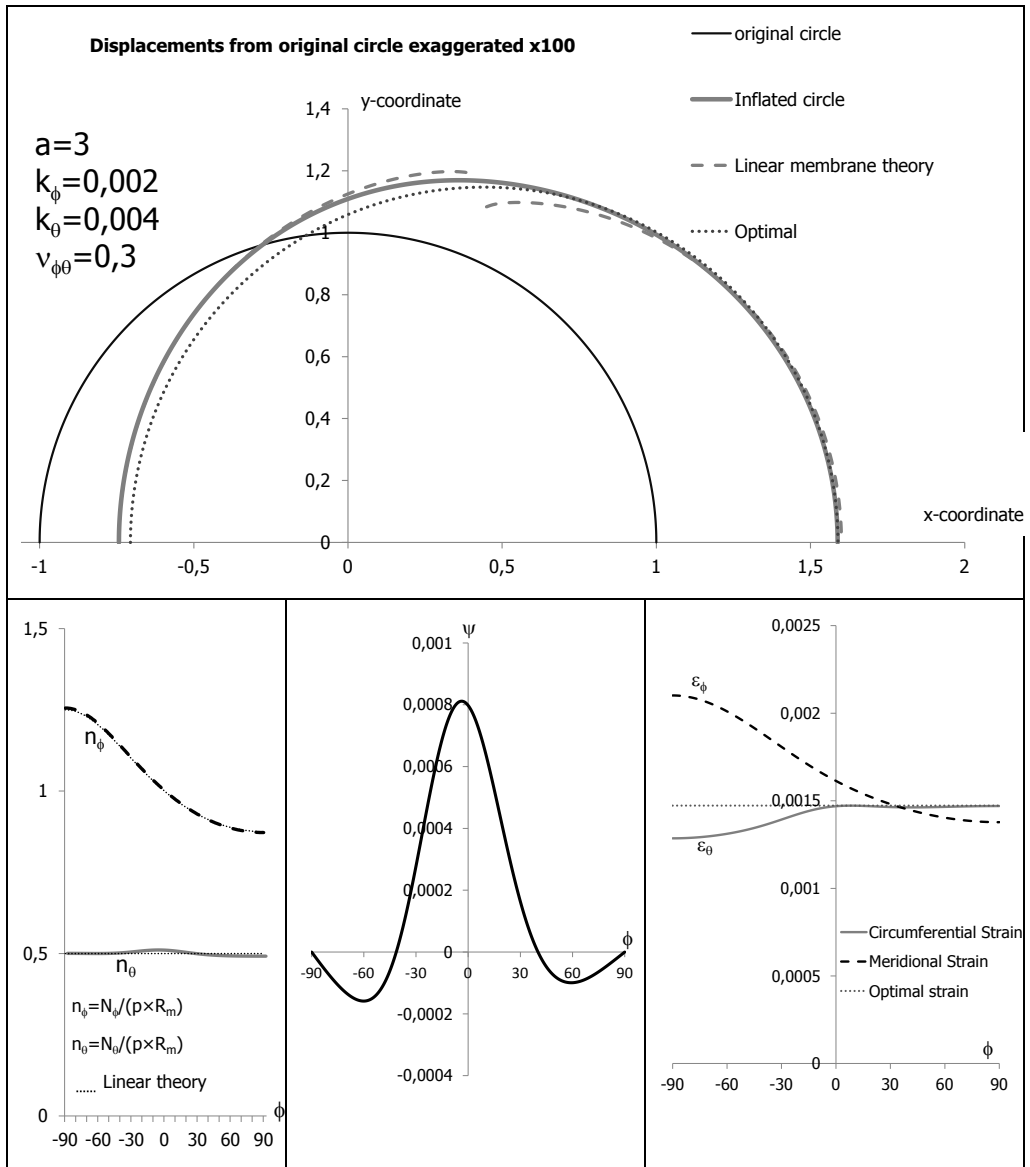
**Fig. 3.10: Modified Linear Membrane analysis of the (quasi-) isotropic pressurized torus with  $a=3$ .**

**a) Deformed shape compared with linear theory and the 'optimal' shell, b) Dimensionless membrane forces, c) Rotation of the membrane, d) Membrane strains**



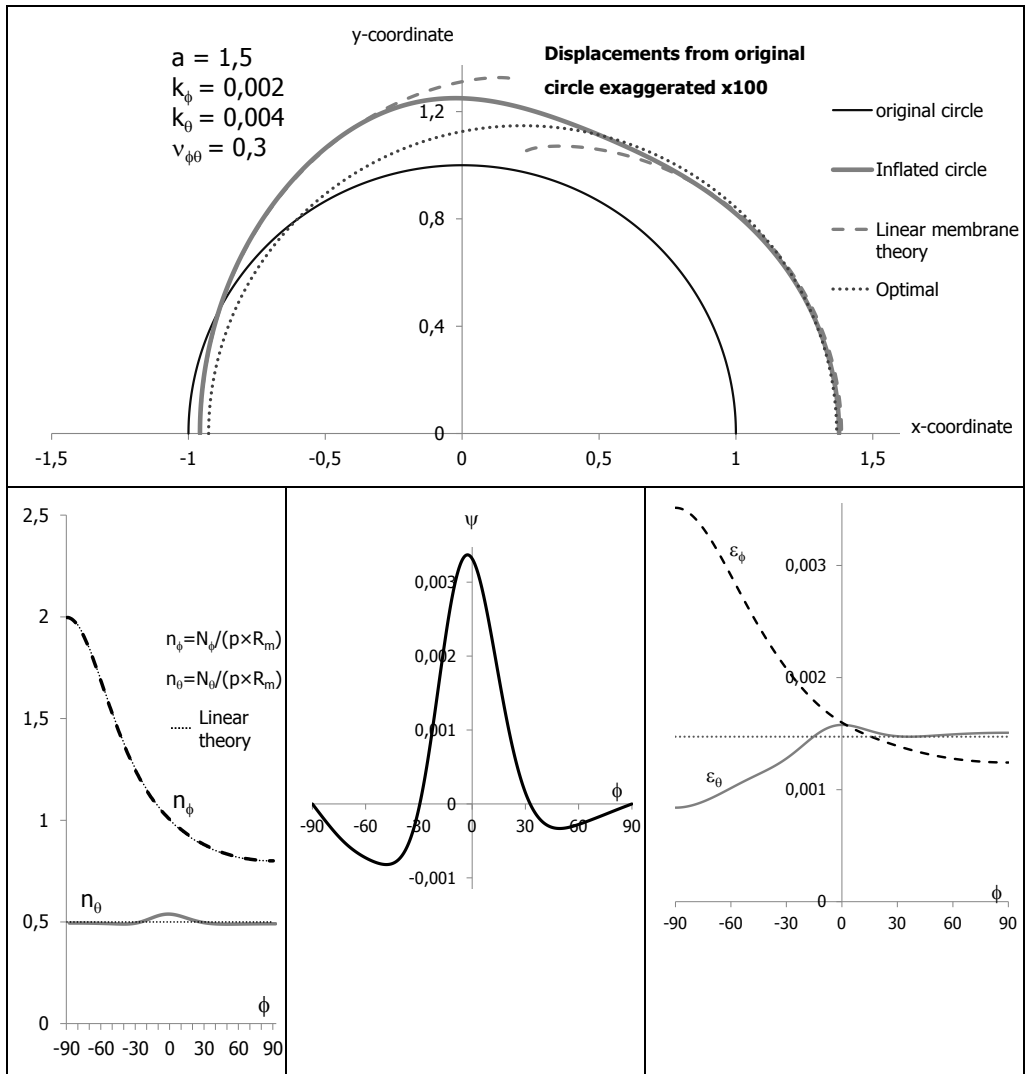
**Fig. 3.11: Modified Linear Membrane analysis of the (quasi-) isotropic pressurized torus with  $a=1.5$ .**

**a) Deformed shape compared with linear theory and the 'optimal' shell, b) Dimensionless membrane forces, c) Rotation of the membrane, d) Membrane strains**



**Fig. 3.12: Modified Linear Membrane analysis of orthotropic pressurized torus with  $a=3$ .**

**a) Deformed shape compared with linear theory and the 'optimal' shell, b) Dimensionless membrane forces, c) Rotation of the membrane, d) Membrane strains**



**Fig. 3.13: Modified Linear Membrane analysis of orthotropic pressurized torus with  $a=1.5$ .**

**a) Deformed shape compared with linear theory and the 'optimal' shell, b) Dimensionless membrane forces, c) Rotation of the membrane, d) Membrane strains**

# CHAPTER 4

## Interaction between shells of various geometries

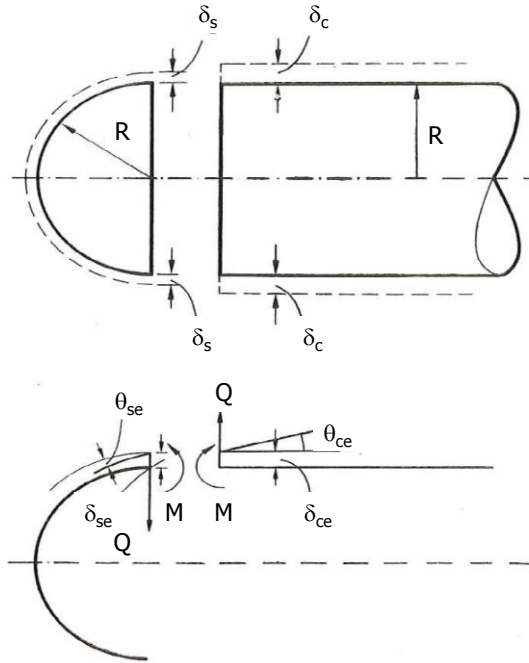
One of the most common problems in pressure vessels is the stress analysis of the shell at the junction of two shells of different geometry.

Consider the simplest well-known problem of the junction of an isotropic cylindrical vessel and an isotropic hemispherical end of the same thickness subjected to internal pressure, shown in figure 4.1. The membrane solution gives:

$$\delta_c = \frac{pR^2}{2Et}(2-\nu)$$

$$\delta_s = \frac{pR^2}{2Et}(1-\nu)$$

For both sphere and cylinder, the membrane solution gives no edge rotations and the deflections are not equal. Most pressure vessels are rigid shells and in order to maintain continuity there must be an equal and opposite shear force  $Q$  and bending moment at the junction.



**Fig. 4.1: Displacements at the junction of shells of different geometry [Gill, 1970]**

Although the stress analysis at the junction did not form a part of the research, it is a problem that also applies to a multi-cylinder that is closed off by a multi-spherical bulkhead.

This chapter presents the complete set of equations for pressure vessels that are also subjected to bending and transverse shear in the shell wall. The equations presented in this chapter are simplified because of rotational symmetry in the loading and the geometry of the shell. The elaborate derivations for shells in general are found in [Flügge, 1960], [Timoshenko, 1959], [Novozhilov, 1964] and [Kraus, 1967].

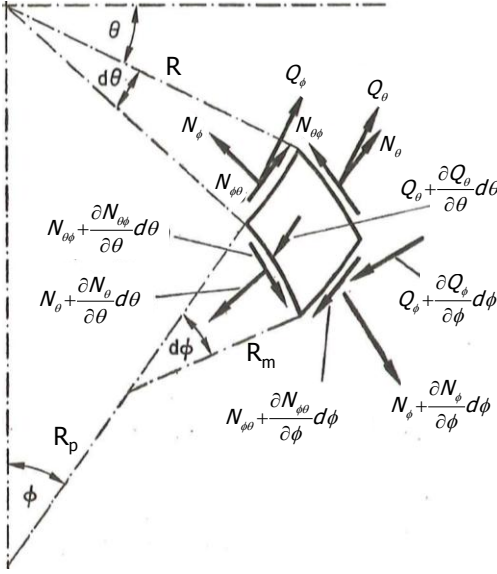
The bending theory works well for most applications in order to resolve the incompatibility at the junction of shell elements. The load factor (eq. 3.12 & 3.13) for thin-walled pressure cabins can however be as small as 0.0005. As an illustration, this resembles a steel cylinder with a thickness of 1 [mm], a radius of 1000 [mm] and an overpressure of 1 bar. This would subsequently give a hoop stress of only 100 MPa which is well within the material allowables of most steel alloys. Many steel alloys would even allow wall thicknesses of less than 0.5 [mm].

For these small values, a membrane solution is sufficient and the contribution of bending action is negligible due to the small out of plane stiffness of the shell. A quasi-linear or modified membrane solution is able to resolve the incompatibility at the junction of membranes of different geometries. This chapter introduces the bending theory for pressurized shells and the method to resolve the incompatibilities at the junction of rigid shells. Insight in these incompatibilities is very helpful to form an opinion about the membrane solution as it will be discussed that there is no added value in the exact solution from an engineering perspective.

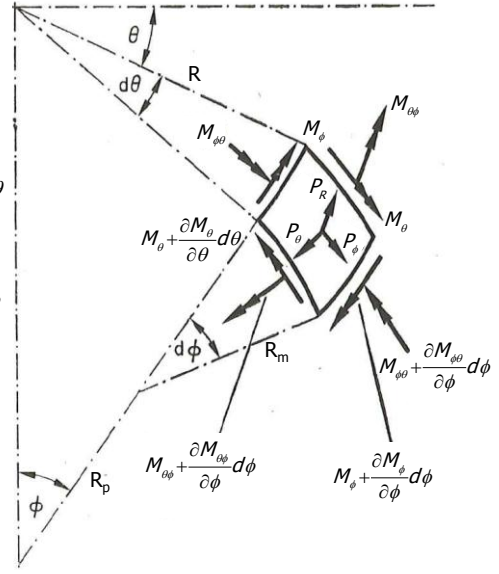
## 4.1 Equilibrium of the shell element

Similar to the derivation in chapter two, we consider an element ( $Rd\theta \times R_m d\phi$ ) cut out by two meridians and two parallel circles. Instead of calling it a membrane element, it is termed a shell element because transverse shear and bending is taken into account as well. Figure 4.2 shows the direct and shear stress resultants in the plane of the surface of the shell and also the transverse shear stress resultants which were not considered in chapter 2.

Figure 4.3 shows the bending and twisting moments on the element.



**Fig. 4.2: Forces on a shell element**



**Fig. 4.3: Moments and external forces on a shell element**

As is usual in thin shell analysis we consider all stresses normal to the shell surface as being negligible since they are small and the equations of equilibrium ignore all changes in shape of the shell due to the loads. The detailed steps for the derivation of the equations of equilibrium are given in [Gill, 1970]. The loading is rotationally symmetric and since  $p_\theta$  equals zero, it follows that  $Q_\theta$ ,  $N_{\theta\phi}$ ,  $N_{\phi\theta}$ ,  $M_{\theta\phi}$  and  $M_{\phi\theta}$  equal zero and  $N_\theta$  and  $M_\theta$  are independent of  $\theta$ . The equations of equilibrium for the shell element take the form:

$$\frac{\partial}{\partial \phi} (RN_\phi) - RQ_\phi - N_\theta R_m \cos \phi + p_\phi RR_m = 0 \quad (4.1a)$$

$$\frac{\partial}{\partial \phi} (RQ_\phi) + R_m N_\theta \sin \phi + RN_\phi - p_R RR_m = 0 \quad (4.1b)$$

$$\frac{\partial}{\partial \phi} (RM_\phi) - R_m M_\theta \cos \phi - Q_\phi RR_m = 0 \quad (4.1c)$$

Eliminating  $N_\theta$  from eq. (4.1) and (4.1b) gives:

$$\frac{\partial}{\partial \phi} (RN_\phi) \sin \phi - RQ_\phi \sin \phi + p_\phi RR_m \sin \phi + RN_\phi \cos \phi + \frac{\partial}{\partial \phi} (RQ_\phi) \cos \phi - RR_m p_R \cos \phi = 0$$

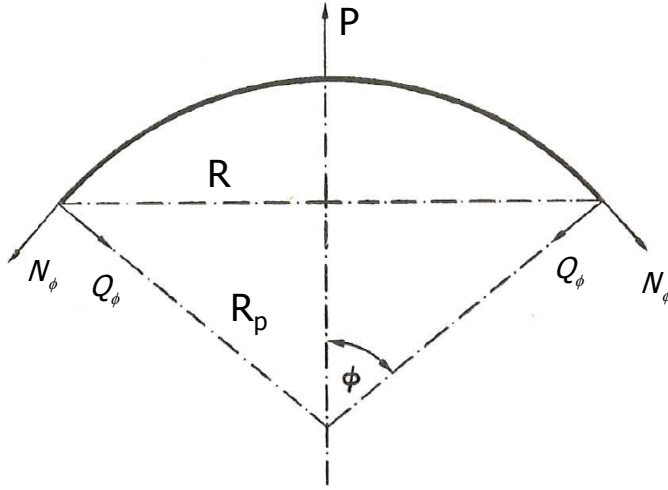
i.e.

$$\frac{\partial}{\partial \phi} (RN_{\phi} \sin \phi) + \frac{\partial}{\partial \phi} (RQ_{\phi} \cos \phi) RR_m (p_{\phi} \sin \phi - p_R \cos \phi) = 0$$

Therefore

$$2\pi (RN_{\phi} \sin \phi + RQ_{\phi} \cos \phi) = 2\pi \int RR_m (p_{\phi} \sin \phi - p_R \cos \phi) d\phi = P \quad (4.1d)$$

where P is the total load on the shell above the parallel circle, figure 4.4.



**Fig. 4.4: Loads on the shell above the parallel circle [Gill, 1970]**

## 4.2 Strains and displacements in shells of revolution

The strain displacement relationships were already given in chapter 3. The strains expressed in terms of the normal and tangential displacements are:

$$\varepsilon_{\theta} = \frac{1}{R} [\Delta_t \cos \phi + \Delta \sin \phi] \quad (4.2a)$$

$$\varepsilon_{\phi} = \frac{1}{R_m} \left[ \frac{\partial \Delta_t}{\partial \phi} + \Delta \right] \quad (4.2b)$$

An approximation of the curvatures taking axi-symmetry into account gives [Kraus, 1967]:

$$\chi_\theta = \frac{\cot \phi}{R_m R_p} \left[ \frac{\partial \Delta}{\partial \phi} - \Delta_t \right] \quad (4.2c)$$

$$\chi_\phi = \frac{1}{R_m} \frac{\partial}{\partial \phi} \left[ \frac{1}{R_m} \frac{\partial \Delta}{\partial \phi} - \frac{\Delta_t}{R_m} \right] \quad (4.2d)$$

Neglecting shear and twisting moments the constitutive equations for a specially orthotropic shell in the simplified form (Appendix A) is defined as:

$$\begin{bmatrix} N_\phi \\ N_\theta \\ M_\phi \\ M_\theta \end{bmatrix} = \begin{bmatrix} A_{\phi\phi} & A_{\phi\theta} & 0 & 0 \\ A_{\phi\theta} & A_{\theta\theta} & 0 & 0 \\ 0 & 0 & D_{\phi\phi} & D_{\phi\theta} \\ 0 & 0 & D_{\phi\theta} & D_{\theta\theta} \end{bmatrix} \begin{bmatrix} \varepsilon_\phi \\ \varepsilon_\theta \\ \chi_\phi \\ \chi_\theta \end{bmatrix} \quad (4.3a)$$

And vice versa:

$$\begin{bmatrix} \varepsilon_\phi \\ \varepsilon_\theta \\ \chi_\phi \\ \chi_\theta \end{bmatrix} = \begin{bmatrix} \frac{A_{\phi\phi}}{A_{\phi\phi}A_{\theta\theta} - A_{\phi\theta}^2} & \frac{A_{\phi\theta}}{A_{\phi\phi}A_{\theta\theta} - A_{\phi\theta}^2} & 0 & 0 \\ \frac{A_{\phi\theta}}{A_{\phi\phi}A_{\theta\theta} - A_{\phi\theta}^2} & \frac{A_{\theta\theta}}{A_{\phi\phi}A_{\theta\theta} - A_{\phi\theta}^2} & 0 & 0 \\ 0 & 0 & \frac{D_{\phi\phi}}{D_{\phi\phi}D_{\theta\theta} - D_{\phi\theta}^2} & \frac{D_{\phi\theta}}{D_{\phi\phi}D_{\theta\theta} - D_{\phi\theta}^2} \\ 0 & 0 & \frac{D_{\phi\theta}}{D_{\phi\phi}D_{\theta\theta} - D_{\phi\theta}^2} & \frac{D_{\theta\theta}}{D_{\phi\phi}D_{\theta\theta} - D_{\phi\theta}^2} \end{bmatrix} \begin{bmatrix} N_\phi \\ N_\theta \\ M_\phi \\ M_\theta \end{bmatrix} \quad (4.3b)$$

We obtain four equations relating the forces & moments and the displacements when eqs. (4.2) are substituted into eq. (4.3a). We also have three equations of equilibrium (4.1a-c), making a total of seven equations for seven unknowns  $N_\phi$ ,  $N_\theta$ ,  $M_\phi$ ,  $M_\theta$ ,  $Q_\phi$ ,  $\Delta$  and  $\Delta_t$  and hence the general problem of the elastic shell of revolution may in principle be solved. A solution would however be of great complexity and a solution is in most cases only found by numerical procedures

[Gill,1970]. Finite element analysis is nowadays more straightforward to obtain solutions for the incompatibilities at the junction of pressure vessels. Analytical derivations for the sphere and the cylinder both made from isotropic materials (figure 4.1) can for example be found in [Gill, 1968]. The next section presents a practical engineering method called the force method which allows the analysis of complicated shells in a relatively short time.

### **4.3 The Force Method [Baker, 1972]**

In the previous chapter, the membrane theory has been highlighted for several structures. Membrane theory has limitations as was shown in the introduction of this chapter. The bending theories are more exact, but generally speaking, they are cumbersome and do often not have analytical solutions.

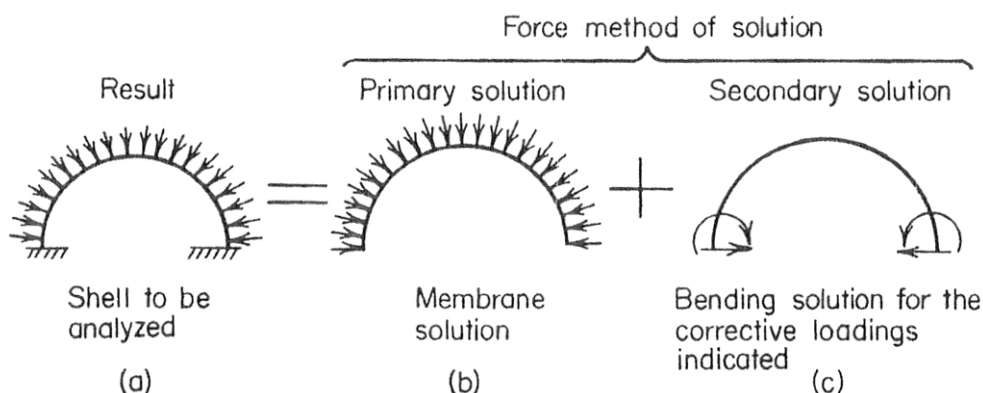
To compare the two theories, assume a non-shallow spherical shell with built-in edges subjected to some axi-symmetrical loading as shown in figure. 4.5. Assume further that two solutions are obtained, one using just membrane theory, another bending theory. When the results of the analysis are compared, the following conclusions can be made:

1. The stresses and deformations are almost identical for all locations of the shell, with the exception of a narrow strip on the shell surface which is adjacent to the boundary. This narrow strip is generally no wider than  $\sqrt{R_p t}$ , where  $R_p$  is the radius of the parallel circle and  $t$  is the thickness of the shell.
2. Except for the strip along the boundary, all bending moments, twisting moments and vertical shears are negligible; this causes the entire solution to be practically identical to the membrane solution.
3. Disturbances along the supporting edge are very significant; however, the local bending and shear decrease rapidly along the meridian and may become negligible outside the narrow strip described in item 1.

These observations lead to a practical engineering method, called the "force method", which eliminates limitations of both theories and makes it possible to analyze complicated shells in a relatively short time.

The shell that needs to be analysed (figure 4.5a) is first analysed by the membrane theory which gives the primary solution (figure 4.5b), then at the

edges, the corrective moments and shears as shown in figure 4.5c (secondary solutions) can be applied to bring the displaced edge of the shell into the position prescribed by boundary conditions.



**Fig. 4.5: The force method procedure [Baker et al., 1972]**

### 4.3.1 Secondary solutions

Secondary solutions are the deformations  $\delta$  and rotations  $\theta$  in a shell as a result of unit shear forces and unit shear moments as shown in figure 4.1. Bending theory is used to obtain the solutions for the unit loadings. The solutions for unit edge loadings can be used with the primary solutions, to obtain discontinuity stresses.

Appendix B presents secondary solutions for isotropic non-shallow spherical, conical and cylindrical shells and circular plates. The equations presented in appendix B can be re-arranged in order to find the loads when the displacements are prescribed.

The equations in appendix B account for isotropic shells. At the moment of writing this dissertation, there is no awareness of the existence of secondary solutions for orthotropic shells.

### 4.3.2. Secondary solutions for edge loading

Figure 4.6 shows the deflections and internal loads due to the edge loadings. Observe that all stresses and deflections due to unit edge loadings disappear for  $\alpha > 20^\circ$  and are negligible for  $\alpha > 10^\circ$ , as is shown in figure 4.6 for a spherical shell.

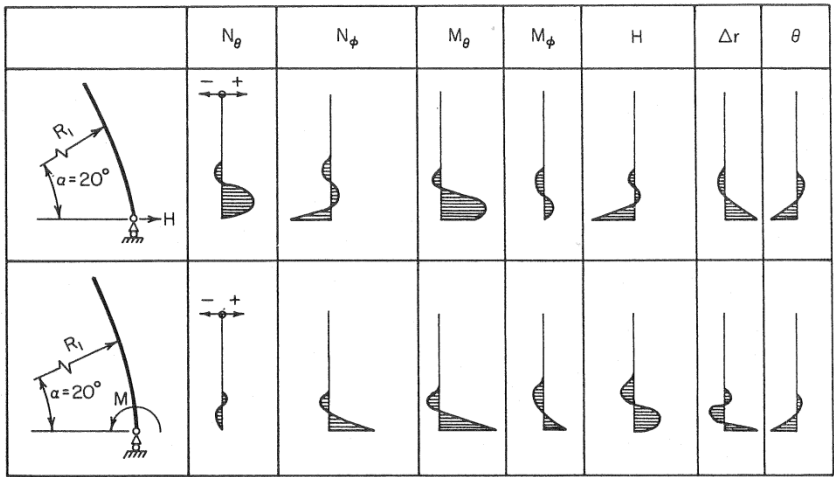


Fig. 4.6 : Edge loading and deformation solutions [Baker et al., 1972]

These observations allow us to analyse shells that are composed of multiple shell elements (figure 4.7):

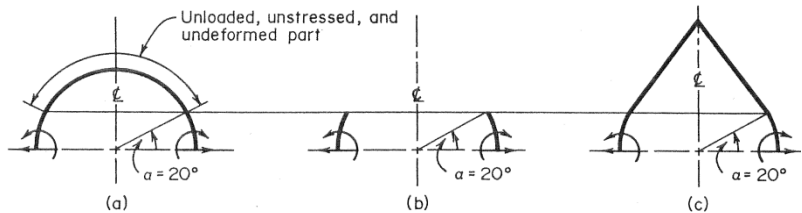


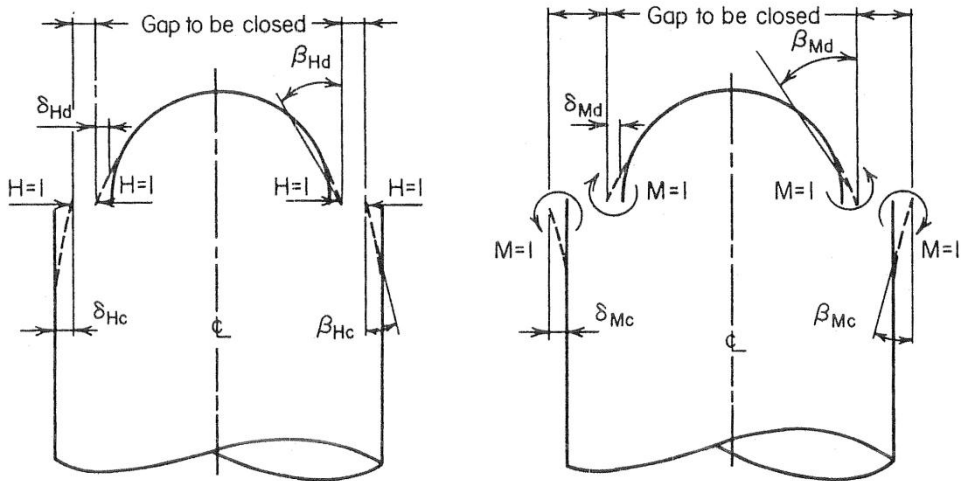
Fig. 4.7 Statically analogous shells [Baker et al., 1972]

Another approximation, known as Geckeler's assumption may be useful: If the thickness of the shell  $t$  is small in comparison with the equatorial radius and is limited by the relation  $R/t > 50$ , the bending stresses at the edge may be determined by cylindrical shell theory. This is shown in chapter 4.4.

### 4.3.3 Interaction between two shell elements

The equations in appendix B are useful when the edges are constrained or when the displacements are prescribed. When we consider a boiler end with a cylindrical drum, as shown in figure 4.8, we have a system of simple members that mechanically interact with each other.

Stresses and deformations introduced by internal pressure can be determined for each part separately, using membrane analysis. The membrane analysis (primary solution) yields the radial displacements  $\Delta r = \delta_c$  and rotation  $\beta_c$  for the cylinder along the discontinuity line and  $\Delta r = \delta_d$  and  $\beta_d$  for the dome along the discontinuity line.



**Fig. 4.8 Unit deformations, unit loadings, and remaining gaps to be closed by corrective loadings M & H [Baker et al., 1972]**

Since the structure is separated into two elements,

$$\delta_c \neq \delta_d$$

$$\beta_c \neq \beta_d$$

Consequently, there exists the discontinuity:

1. In displacement  $\delta_c - \delta_d$
2. In slope  $\beta_c - \beta_d$

To close this gap, unknown forces  $H$  and moments  $M$  will be introduced around the junction to hold the two pieces together. The displacement and rotation of the edge of the cylinder and the dome due to unit values of  $H$  and  $M$  are defined as follows:

The cylinder:  $\delta_{Hc}, \beta_{Hc}$  and  $\delta_{Mc}, \beta_{Mc}$

The dome:  $\delta_{Hd}, \beta_{Hd}$  and  $\delta_{Md}, \beta_{Md}$

These unit deformations and unit loadings at the junctions are presented appendix B.

The sign convention to solve this problem is arbitrary, as long as it is logical and consistent. The following sign convention is used for this illustrative example:

1. Horizontal deflection  $\delta$  is positive outward
2. Shears are positive if they cause deflection outwards
3. Moments are positive if they cause tension on the inside surface of the shell
4. Rotations of the meridian are positive if the rotation is in a clock-wise direction.

To close the gap, the following equations can be written:

$$\begin{aligned}\delta_d + \delta_{Hd}H + \delta_{Md}M &= \delta_c - \delta_{Hc}H + \delta_{Mc}M \\ \beta_d - \beta_{Hd}H - \beta_{Md}M &= \beta_c - \beta_{Hc}H + \beta_{Mc}M\end{aligned}$$

This leads to the following set of equations:

$$\begin{aligned}(\delta_{Hd} + \delta_{Hc})H + (\delta_{Md} - \delta_{Mc})M &= \delta_c - \delta_d \\ (\beta_{Hc} - \beta_{Hd})H + (-\beta_{Md} - \beta_{Mc})M &= \beta_c - \beta_d\end{aligned}\tag{4.4}$$

Where  $H$  and  $M$  are the corrective edge shear and moment necessary to assure continuity of deflection and slope at the junction. All coefficients  $\delta$  and  $\beta$  are known (tabulated in appendix B). Thus, the following can be presented:

$$\begin{aligned}\delta_{Hc} + \delta_{Hd} &= \delta_H \\ \delta_{Mc} + \delta_{Md} &= \delta_M \\ \delta_c - \delta_d &= \delta\end{aligned}\tag{4.5a-c}$$

$$\begin{aligned}\beta_{Hc} - \beta_{Hd} &= \beta_H \\ -\beta_{Mc} - \beta_{Md} &= \beta_M \\ \beta_c - \beta_d &= \beta\end{aligned}\tag{4.6a-c}$$

Finally, eq. (4.6) is reduced to a system of two equations with two unknowns  $H$  and  $M$ :

$$\begin{aligned}\delta_H H + \delta_M M &= \delta \\ \beta_H H + \beta_M M &= \beta\end{aligned}\tag{4.7a-b}$$

The determinants of the above systems are as follows:

$$D = \begin{vmatrix} \delta_H & \delta_M \\ \beta_H & \beta_M \end{vmatrix} \quad D_1 = \begin{vmatrix} \delta & \delta_M \\ \beta & \beta_M \end{vmatrix} \quad D_2 = \begin{vmatrix} \delta_H & \delta \\ \beta_H & \beta \end{vmatrix}\tag{4.8a-c}$$

The statically indeterminate values of  $H$  and  $M$  are now determined:

$$H = \frac{D_1}{D} \quad M = \frac{D_2}{D}\tag{4.9a-b}$$

It is noted that one cut through the shells leads to two algebraic equations with two unknowns.

In addition to  $M$  and  $H$ , there is an axial force due to the reaction of the bulkhead distributed around the junction between the cylinder and the dome but the effect of this force proves in practice to be negligible.

#### 4.3.4 Interaction between shell elements

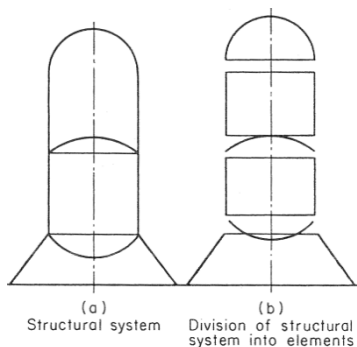
A multi-shell structure consists of simple shell elements, where the segments are connected "in series".

The simplest interaction of a two-shell element was shown in the previous section. If the shell elements are of such geometry that all disturbances due to the corrective loading die out before reaching the opposite edge of the shell element, each junction may be considered separately regardless of the rest of the structure, as is done in the previous section.

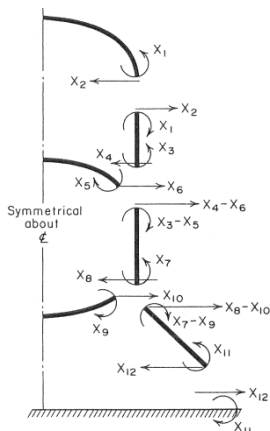
If, however, the length of the shell element is such that the influences due to the corrective loading do not fade out before reaching the opposite edge of the shell element, it is not possible to solve one junction separately and the rest of the structure must be considered. Similarly, a junction of 3 shell elements is also called a multi-shell structure and is regularly found back in conformable pressure vessels. Previously, when considering only one imaginary cut, only two equations with two unknowns were obtained. If  $n$  imaginary cuts are introduced simultaneously,  $2n$  linear equations with  $2n$  unknowns can be obtained.

#### Interaction procedure

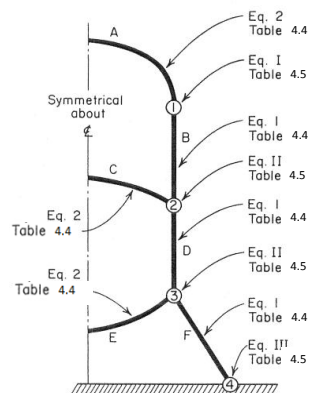
1. Separate the multi-shell structure into single shell elements as is shown in figure 4.9. Each shell element of this type can be designated as a "statically determinate element", as was explained in the membrane theory.
2. Enter at every junction the corrective loadings, considering the values on the right side of figure 4.10 as positive for the purpose of interaction.
3. Depending on the geometry of each shell element, prescribe to each the corresponding set of equilibrium equations as shown in table 4.1. (= Final solution of primary solution and secondary solution).
4. In addition for every junction enter a corresponding set of equations as shown in table 4.2. Consider all boundary conditions where needed (figure 4.11)
5. All these equations reduce to one final set of algebraic equations with as many unknowns as there are equations.
6. Solve the set of equations and find all corrective loadings, which are indicated in figure 4.9 and figure 4.12 by  $X_i$  instead of  $M_i$  and  $H_i$ .



**Fig. 4.9: The Multi-shell separated in single shell elements**



**Fig. 4.10: The corrective loadings at each junction of the multi-shell**



**Fig. 4.11: The set of relations to be used in order to find the statically indeterminate loadings**

| No. | Geometry                            | Equations                                                                                                                                                                                                                     | Matrix                                                                                                                                                                                                                                                                                                   |
|-----|-------------------------------------|-------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| 1   | <p>Shell</p> <p>or</p> <p>Plate</p> | $\begin{bmatrix} \beta_1 \\ \delta_1 \\ \beta_2 \\ \delta_2 \end{bmatrix} = \alpha \begin{bmatrix} M_1 \\ H_1 \\ M_2 \\ H_2 \end{bmatrix} + \begin{bmatrix} \beta_1^0 \\ \delta_1^0 \\ \beta_2^0 \\ \delta_2^0 \end{bmatrix}$ | $\begin{bmatrix} \beta_1^{M_1} & \beta_1^{H_1} & \beta_1^{M_2} & \beta_1^{H_2} \\ \delta_1^{M_1} & \delta_1^{H_1} & \delta_1^{M_2} & \delta_1^{H_2} \\ \beta_2^{M_1} & \beta_2^{H_1} & \beta_2^{M_2} & \beta_2^{H_2} \\ \delta_2^{M_1} & \delta_2^{H_1} & \delta_2^{M_2} & \delta_2^{H_2} \end{bmatrix}$ |
| 2   | <p>Shell</p> <p>or</p> <p>Plate</p> | $\begin{bmatrix} \beta_1 \\ \delta_1 \end{bmatrix} = \alpha \begin{bmatrix} M_1 \\ H_1 \end{bmatrix} + \begin{bmatrix} \beta_1^0 \\ \delta_1^0 \end{bmatrix}$                                                                 | $\begin{bmatrix} \beta_1^{M_1} & \beta_1^{H_1} \\ \delta_1^{M_1} & \delta_1^{H_1} \end{bmatrix}$                                                                                                                                                                                                         |

**Table 4.1 Shell-element equations**

| No. | Junction                             | Equations                                                                                                                                                                                                                                                                                                               |                                  |
|-----|--------------------------------------|-------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|----------------------------------|
| I   | Junction of two shell elements<br>   | $\begin{bmatrix} M_2 \\ H_2 \\ \beta_2 \\ \delta_2 \end{bmatrix} - \begin{bmatrix} M'_2 \\ H'_2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \beta'_2 \\ \delta'_2 \end{bmatrix}$                                                                                                                                |                                  |
| II  | Junction of three shell elements<br> | $\begin{bmatrix} M_2 \\ H_2 \\ \beta_2 \\ \beta_2 \\ \delta_2 \\ \delta_2 \end{bmatrix} - \begin{bmatrix} M'_2 \\ H'_2 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} M''_2 \\ H''_2 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \beta'_2 \\ \beta''_2 \\ \delta'_2 \\ \delta''_2 \end{bmatrix}$ |                                  |
| III | Fixed edge                           | Use formulas I or 2 (Table 6-I) and I or II (this table) considering:                                                                                                                                                                                                                                                   | $\beta_1 = 0 \quad \delta_1 = 0$ |
| IV  | Pinned edge                          |                                                                                                                                                                                                                                                                                                                         | $M_1 = 0 \quad \delta_1 = 0$     |
| V   | Simple support edge                  |                                                                                                                                                                                                                                                                                                                         | $M_1 = 0 \quad H_1 = 0$          |

**Table 4.2 Junction Equations**

| i  | X <sub>1</sub> | X <sub>2</sub> | X <sub>3</sub> | X <sub>4</sub> | X <sub>5</sub> | X <sub>6</sub> | X <sub>7</sub> | X <sub>8</sub> | X <sub>9</sub> | X <sub>10</sub> | X <sub>11</sub> | X <sub>12</sub> | C <sub>10</sub>   |
|----|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|-----------------|-----------------|-----------------|-------------------|
| 1  | ==             | —              | —              | —              |                |                |                |                |                |                 |                 |                 | C <sub>10</sub>   |
| 2  | —              | ==             | —              | —              |                |                |                |                |                |                 |                 |                 | C <sub>20</sub>   |
| 3  | —              | —              | ==             | —              | —              | —              | —              | —              |                |                 |                 |                 | C <sub>30</sub>   |
| 4  | —              | —              | —              | ==             | —              | —              | —              | —              |                |                 |                 |                 | C <sub>40</sub>   |
| 5  |                |                | —              | —              | ==             | —              | —              | —              |                |                 |                 |                 | C <sub>50</sub>   |
| 6  |                |                | —              | —              | —              | ==             | —              | —              |                |                 |                 |                 | C <sub>60</sub>   |
| 7  |                |                | —              | —              | —              | —              | ==             | —              | —              | —               | —               | —               | C <sub>70</sub>   |
| 8  |                |                | —              | —              | —              | —              | —              | ==             | —              | —               | —               | —               | C <sub>80</sub>   |
| 9  |                |                | —              | —              | —              | —              | —              | —              | ==             | —               | —               | —               | C <sub>90</sub>   |
| 10 |                |                |                |                |                |                | —              | —              | —              | ==              | —               | —               | C <sub>10 0</sub> |
| 11 |                |                |                |                |                |                | —              | —              | —              | —               | ==              | —               | C <sub>11 0</sub> |
| 12 |                |                |                |                |                |                | —              | —              | —              | —               | —               | ==              | C <sub>12 0</sub> |

**Fig. 4.12: The matrix of unknowns if the boundary disturbances at opposite boundaries are influencing each other.**

## 4.4 Results & Discussion

The force method is applicable for all types of axi-symmetrical shells but the secondary solutions presented in appendix B are only applicable for isotropic shells. Investigating secondary solutions for orthotropic shells is out of the scope of this work. When the shell is very thin and therefore the load parameter (eq. 3.12 and 3.13) very small, bending action can be neglected and only a membrane solution needs to be considered. In this case, the quasi-linear theory or the modified linear membrane theory can be applied. A solution is found by imposing a displacement boundary condition. Consider for example the spherical shell in figure 4.8, the displacement boundary condition the shell in this case is:  $u(\pi/2)=\delta_H$ . The same is done for the cylinder  $u(s=0)=\delta_H$  and a solution is found when for that displacement  $\delta_H$  that gives a continuity in slope ( $\beta_c=\beta_d$ ). It is noted that the formulation in chapter 3 needs changing when applying it for a cylinder or cone because  $\phi$  is a constant. In this case we have the variable arc-length  $s$  ranging from 0 to length  $L$ . The membrane analysis of the junction of an orthotropic dome with an orthotropic cylindrical drum is presented in [Vasiliev, 2009]. The engineering value of this analysis is however small because the stresses and deformations at the junction of two membrane elements can be figured out by simple reasoning.

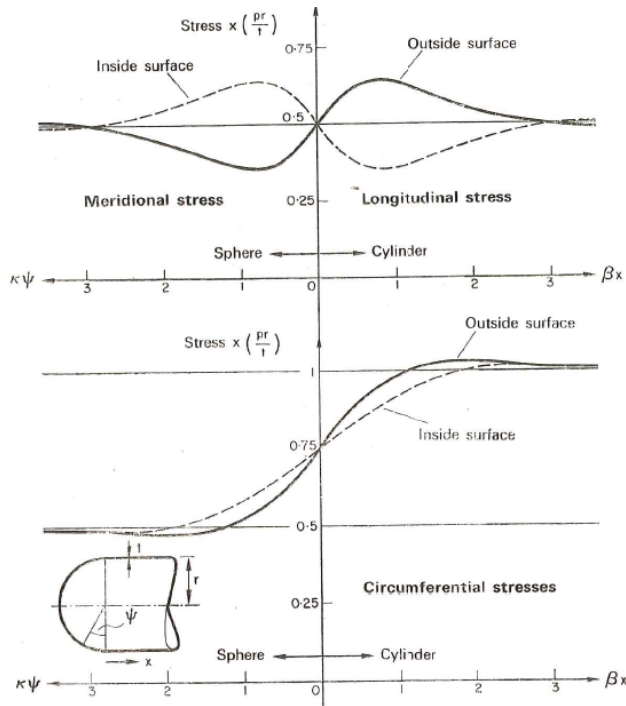
Figure 4.13 [from Gill, 1968] shows the non-dimensional force distribution at the junction of two isotropic shells (sphere and cylinder) of equal thickness. Geckeler's approximation to regard every thin-walled shell as a cylinder is valid because we see that there is point symmetry in the force distribution at the junction of the two shells.

Considering a membrane solution for the example shown in figure 4.13, we can conclude that the radius of the meridional curvature at the junction will gradually change from infinite (cylinder) to the radius of the sphere. Furthermore, the forces in the outside surface will coincide with the forces in the inside surface from the moment the shell becomes a membrane. The meridional / longitudinal force will therefore comply with linear membrane theory. Due to Geckeler's approximation we know that the non-dimensional hoop force will be 0.75 at the junction (for this example). The only thing that is likely to be different is the size of the zone which is indicated by the horizontal axis. The exact distribution can be assessed by means of the quasi-linear theory but from previous observations [Baker, 1972], this zone is a narrow strip and not wider than  $\sqrt{(R_p t)}$ . The analysis of a membrane

solution to predict the size of this zone is not relevant from an engineering point of view. The highest forces are for this case found in the cylinder and the membrane solution predicts a load (force) reduction in the transition zones.

There is not much information available that considers the interaction of orthotropic shells with different geometries. One reason for this is that the tailorability of the composite properties gives the opportunity to reduce problems associated with compatibility of deformations. Considering the research of conformable pressurized structures, thin-walled shells are of main interest which implies that a membrane solution is preferred. The membrane solution will however not give significant changes other than a different size of the transition zone.

A practical conclusion with respect to the junction of different shell elements is that when the compatibility of deformations is not achievable, the designer should try to keep the out-of-plane stiffness ( $D_{\phi\phi}$  &  $D_{\phi\theta}$ ) and the thickness in the transition zone as low as possible.



**Fig. 4.13: Non-dimensional force distribution at the junction of a isotropic sphere and cylinder with equal thickness, [Gill, 1970]**

# CHAPTER 5

## Optimal Pressurized Structures

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One unanswered question that was posed in chapter 2 was why a soap bubble has a spherical shape? The scientific answer relies in the minimum total potential energy principle which was already introduced in chapter 3. This principle asserts that a structure or body shall deform or displace to a position that minimizes the total potential energy. Returning to the spherical pressure vessel, e.g. soap bubble, it is known that the sphere has the lowest possible surface area for a given volume. Combine this with the knowledge that an isotropic material, e.g. soap, is the optimal choice because the stresses in a spherical pressure vessel are in all directions the same and it is not hard to understand why soap bubbles are spherical. Pressurized cylindrical drums would also conform to a spherical shape in case they would be made of hyper-elastic isotropic material. Most pressure vessels are however made of engineering materials that only have small ultimate strain levels and the volumetric change of the pressure vessel is therefore negligible. It suffices therefore to leave the potential energy out of the consideration and only to define the material properties such that the strain energy in the pressurized shell is minimized when we look for the optimal pressure vessel. The optimal pressure vessel was already mentioned in chapter 3 and this chapter will confirm that a pressurized structure is optimal when all the strains in every direction are equal.

The structural efficiency for optimal pressure vessels is defined next, followed by a discussion of the advantages of optimal pressurized structures. The content in this chapter builds on work from [Zu, 2012 and Vasiliev, 2009].

### 5.1 Optimal stiffness distribution

Membranes that are in a state of uniform equal bi-axial extension were in chapter 3.2. already denoted as optimal membranes. A linear membrane response is not the only reason why these membranes are denoted as optimal. The strain energy per unit thickness (see chapter 3.4) for an infinitely small membrane element is:

$$U = \frac{1}{2} (N_{\phi} \varepsilon_{\phi} + N_{\theta} \varepsilon_{\theta}) \quad (5.1)$$

Only the first order strains are considered in eq. (5.1) (no rotation) and shear forces and deformations are left out because they do not apply to axi-symmetric loaded structures.

Substituting the constitutive equations (eqs. 3.4 and 3.36) gives:

$$U = \frac{N_{\phi}^2}{2} \frac{A_{\theta\theta} - 2\chi A_{\phi\theta} + \chi^2 A_{\phi\phi}}{A_{\theta\theta} A_{\phi\phi} - A_{\phi\theta}^2} \quad (5.2)$$

The following invariant equation is now provided for the membrane:

$$A_{\theta\theta} - 2A_{\phi\theta} + A_{\phi\phi} = C \quad (5.3)$$

$C$  is a constant value, determined by the material constants and layer thickness.

The optimal laminate configuration is obtained by minimizing the strain energy density in eq. (5.1), subjected to the equality constrain in eq. (5.3). Introducing this constraint with the aid of the Lagrange multiplier [Vapnyarskii, 2001], the following augmented function should be minimized:

$$f(A_{\phi\phi}, A_{\theta\theta}, A_{\phi\theta}) = \frac{N_{\phi}^2}{2} \frac{A_{\theta\theta} - 2\chi A_{\phi\theta} + \chi^2 A_{\phi\phi}}{A_{\theta\theta} A_{\phi\phi} - A_{\phi\theta}^2} - \beta (A_{\theta\theta} - 2A_{\phi\theta} + A_{\phi\phi} - C) \quad (5.4)$$

where the components of the extensional stiffness ( $A_{\phi\phi}$ ,  $A_{\theta\theta}$ ,  $A_{\phi\theta}$ ) are considered as the design variables. The minimum strain energy is obtained when the following condition is active:

$$\frac{\partial f}{\partial A_{\phi\phi}} = \frac{\partial f}{\partial A_{\theta\theta}} = \frac{\partial f}{\partial A_{\phi\theta}} = 0 \quad (5.5)$$

The solution of eq. (5.5) in conjunction with constraint 5.3 results in the optimal stiffness distribution for the shell of a pressurized structure:

$$\chi = \frac{A_{\theta\theta} + A_{\phi\theta}}{A_{\phi\phi} + A_{\phi\theta}} \quad (5.6)$$

From chapter 3 we know that eq. (5.6) corresponds to the stiffness distribution that results in a membrane in which the strains are the same in all directions. Furthermore, equal strains also implies that there is no shear ( $\gamma=0$ ) which confirms not considering shear deformations. The strain energy is minimized which means that the structural efficiency is maximized. The structural efficiency is discussed in the next section.

## 5.2 Structural efficiency for optimal pressurized shells of revolution

The structural efficiency is defined as the ratio of the mass and the pressurised volume of the pressure vessel. The structural efficiency of an optimal shell (equal strain in all directions) made from an orthotropic material is derived as [Vasiliev, 2003]:

$$\frac{PV}{m} = \frac{\sqrt{\sigma_{\phi}^2 + \sigma_{\theta}^2}}{3\rho} \quad (5.7)$$

The structural efficiency is defined as the ratio of the pressurised volume  $P \cdot V$  and the mass  $m$  of the pressure vessel. The structural efficiency depends on the density ( $\rho$ ) of the material of the pressure vessel, and the allowable meridional and hoop stresses.

The failure criterion that was used to derive eq. (5.7) is a conservative version of the Tsai-Hill failure criterion (appendix A):

$$\left( \frac{\sigma_1}{\sigma_1} \right)^2 + \left( \frac{\sigma_2}{\sigma_2} \right)^2 \leq 1 \quad (5.8)$$

There are no geometrical parameters in eq. (5.8). This means that the structural efficiency is the same for any pressurised shape when the material is in a state of uniform equal bi-axial extension.

An alternative version of Vasiliev's equation is given in chapter 6.

### 5.3 Concluding comments

From a structural point of view, optimal pressurized shells of revolution have multiple benefits:

- Linear membrane theory is always valid which simplifies the analysis of forces and deformations.
- The deformed geometry is easily assessed once the strain  $\varepsilon$  is known. The deformed shape is a scale factor  $(1+\varepsilon)$  larger than the original shape
- The structural efficiency is maximized and is not related to the shape of the pressure vessel.
- Even more important and a logical consequence of the uniform deformation is the disposal of compatibility of deformations when combining different membrane elements, e.g. bulkheads on cylinders and conformable pressure vessels.

It needs to be addressed that in the derivation of the optimal structure, the potential energy was excluded. The aforementioned derivations are therefore only valid for shells for which the deformations are not significant. The derivation for the structural efficiency is also not based on the most reliable failure theory. The outcomes would however not differ much in case different failure theories are used. Moreover, the failure theory in the aforementioned derivation is a conservative one.

# CHAPTER 6

## Introducing the multi-bubble

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The previous four chapters involved the analysis of pressurized shells of revolution. Shells of revolution are however not usable for non-circular pressure cabins and when pressure vessels are considered, in many cases axi-symmetric shells do not take the full advantage of the available volume when liquids need to be confined in a pressurized environment.

Equation (5.7) showed that pressurised gas tanks can be made smaller by increasing the pressure while the weight remains the same. This chapter and the next one will show the complexity of conformable pressure vessels. It will be obvious that the manufacturing of multiple individual bottles is less complicated than the manufacturing of a conformable pressure vessel. This is the reason why the application of the multi-bubble is mainly restricted to;

- a) Cryogenic and liquid gas tanks because the pressurized liquid is an incompressible medium.
- b) Pressure cabins.

A simple example of liquid gas tanks are Liquid Petrol Gas (LPG-) tanks. These tanks still appear as bulky axisymmetric pressure vessels in the boot of the car while a more practical form would be flat to increase available storage space. A solution of this problem is a conformable tank termed the multi-bubble. The reason for calling it a multi-bubble is explained by the strong relationship between a cluster of soap bubbles and the multi-bubble. The multi-bubble is a pressure vessel that has the ability to pressurize a volume with substantial spatial freedom and it carries the pressurization loads via in-plane forces only and is therefore structurally efficient. The multi-bubble is an articulated pressurisable structure that consists of intersecting axi-symmetric membrane elements equipped with walls and/or reinforcements at the intersections in order to ensure structural integrity.

The first known multi-bubble was invented by Jackson in 1944. Inspired by soap bubbles, Jackson patented a multi-spherical tank structure. Figure 6.1a shows the replacement of a classical cylinder by its multi-spherical alternative. The replacement of the cylinder by its multi-spherical equivalent gave a more efficient

use of the isotropic materials (metals) that were the main engineering materials in those days. In figure 6.1 we see that Jackson only considered spheres with equal radii. [Bert, 1962] investigated the intersecting spherical pressure tank for aerospace applications and [Komarov] published in 1970 the analysis of intersecting spherical pressure tanks for bubbles with variable radii.

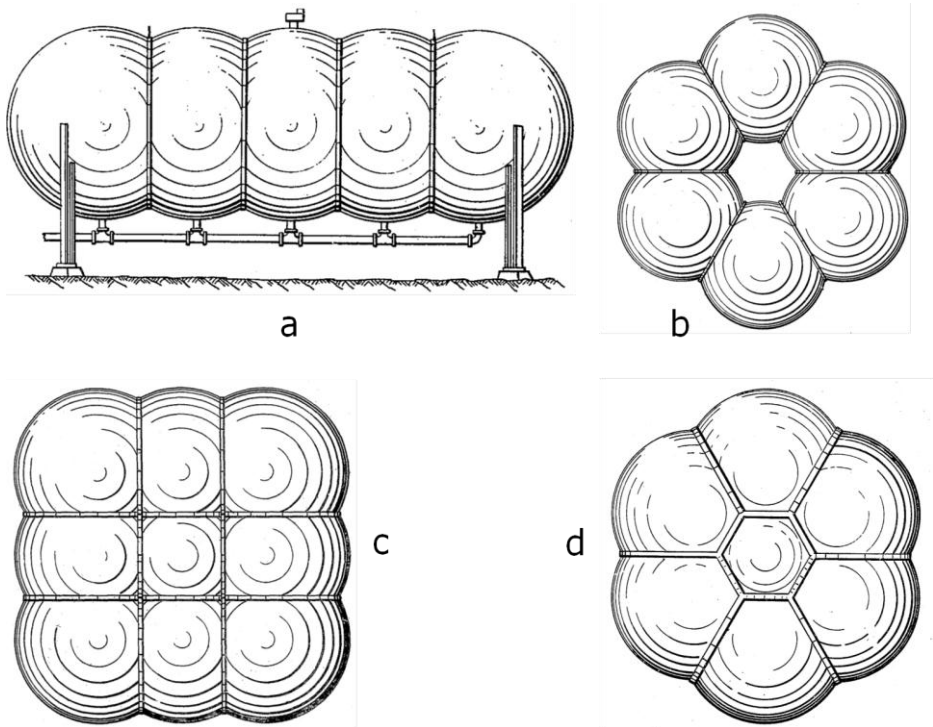
The introduction of fibre reinforced materials caused the intersecting tank structure to fall into oblivion because the tailorability of the fibres gave an increased spatial freedom for pressure vessels, e.g. the cylinder and spheroid, with high structural efficiency.

The first example of a multi-bubble (double bubble) as a pressure cabin was the Boeing 377, also called the stratocruiser. In this case it involved two cylinders that were slid into one another with a floor at the intersection.

Recent developments of the multi-bubble for high-tech applications are:

- fuel tanks for liquid petrol gas (LPG) of which the conformable tanks of PPI and Thiokol are an example (figure 6.3)
- Inflatable wings (figure 6.4) where the multi-bubble preserves the aerodynamic shape while being an inflatable beam at the same time. Prospective Concepts was an example of a company that has made big developments in this field.

The multi-bubble examples that have been presented until now are fairly easy and straight-forward to analyse. At the time of beginning this research effort, complex conformable pressurized structures were not fully understood and conformable pressure vessels with full spatial freedom, made from high specific strength materials and optimal structural efficiency have not been realised. The aim of this dissertation is to investigate the structural concept of a pressure vessel that needs to fit within a prescribed geometrical space. The concept of the multi-bubble pressure vessel is therefore outlined in this chapter.



**Fig. 6.1 Multi-spherical tanks patented by Jackson in 1944**



**Fig. 6.2: Boeing 377 & a cross-section view of the Boeing "Stratocruiser"**



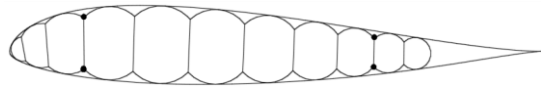
**Fig. 6.3: Conformable tank from [PPI]**



**Fig. 6.4: Conformable prototype tank from [Thiokol]**



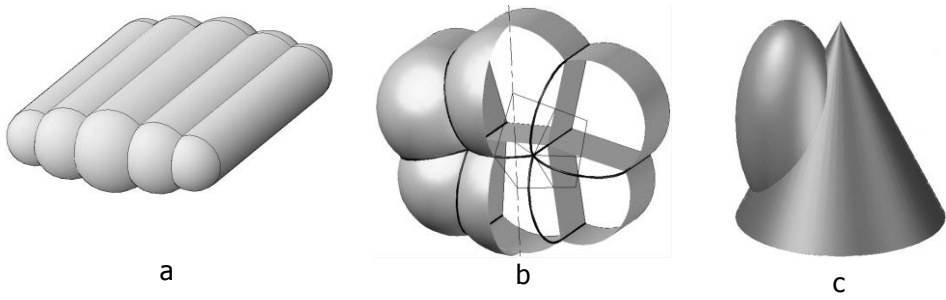
**Fig. 6.5: Inflatable aircraft from [Prospective Concepts]**



**Fig. 6.6: Example of a cross-section of an inflatable wing**

## The multi-bubble concept

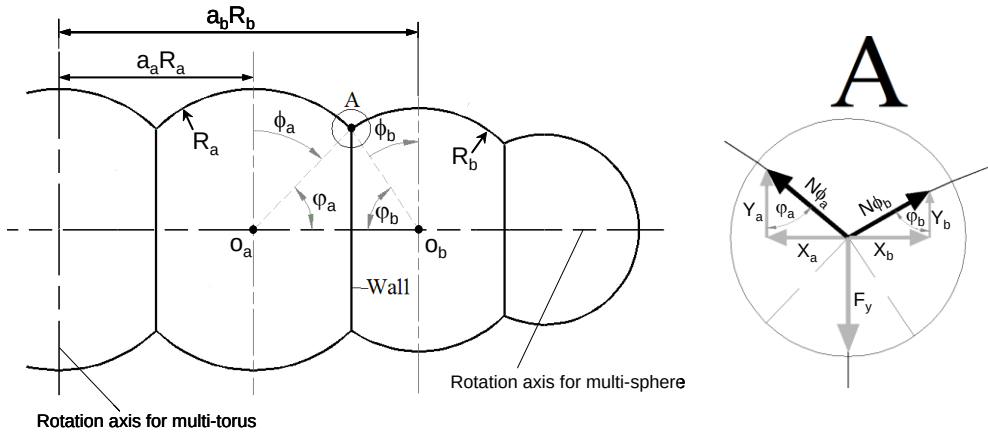
The multi-bubble consists of intersecting axi-symmetric membrane elements. This chapter outlines the analysis of cylindrical, spherical and toroidal membrane elements. Subsequently, the multi-bubble is re-interpreted and it is shown that multi-bubble structures can be reduced to multi-spherical structures which will allow us to analyse all multi-bubble structures. In order to keep the overview, simple configurations will be analysed first and complexity will be added later on to analyse more complicated configurations as shown in figure 6.7.



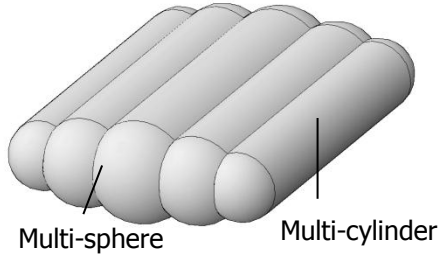
**Fig. 6.7: Overview of simple (a) and more complex (b&c) configurations of the multi-bubble**

## 6.1 The single row multi-bubble

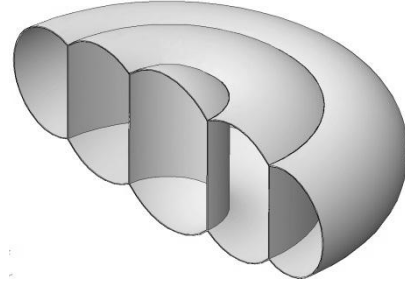
Figure 6.8 shows the cross-section of a single row multi-bubble. Extruding this cross-section gives a multi-cylinder, revolving it around the horizontal axis creates a multi-sphere and revolving it around the vertical axis creates the multi-torus. Examples of these membrane elements are given in figures 6.9 and 6.10.



**Fig. 6.8: Cross-section of the single-row multi-bubble**



**Fig. 6.9: Example of a multi-cylinder combined with a multi-sphere**



**Fig. 6.10: Example of a single-row multi-torus**

The forces need to be balanced at the intersections of the membrane elements. The multi-cylinder is investigated first, next the multi-sphere and the forces in the multi-torus will be investigated subsequently.

### 6.1.1 Analysis of the single row multi-cylinder

The circumferential tension force in the cylindrical membrane is given by:

$$N_{\phi,c} = pR \quad (6.1)$$

where  $R$  is the radius of the cylinder and index "c" refers to the cylinder. It was already mentioned in chapter two that all membrane forces are defined per unit length; multiplying the membrane force with the length gives the total load, while dividing the membrane force with the membrane's thickness gives the average stress in the membrane. The circumferential tension forces are balanced out by a single wall where the different radii meet. (figure 6.8)

The membrane force in the wall ( $F_{\phi,c}$ ) at location A is found by considering the equilibrium of forces in the  $x$ - and  $y$ -direction (figure 6.8b):

$$F_{y,c} = Y_a + Y_b = N_{\phi a} \cos \varphi_a + N_{\phi b} \cos \varphi_b \quad (6.2)$$

$Y_a$  and  $Y_b$  are defined as the components in  $y$ -direction of the circumferential tension force of the cylinders with  $R_a$  and  $R_b$  respectively. The angle  $\varphi$  is defined as the angle between the local  $x$ -axis and the line that runs from the local origin to the intersection.

Rewriting eq. (6.2) shows that a geometric interpretation can be given to the force in the wall (see figure 6.8):

$$F_{y,c} = Y_a + Y_b = p \cdot |O_a O_b| \quad (6.3)$$

where  $|O_a O_b|$  denotes the distance between point  $O_a$  and point  $O_b$ . Giving geometric interpretations to pressurized structures will simplify the analysis considerably later on when more complicated structures are investigated.

The horizontal forces of the neighbouring cylinders ( $X_a$  and  $X_b$ ) are naturally balanced out. The membrane forces  $X_a$  and  $X_b$  are defined as:

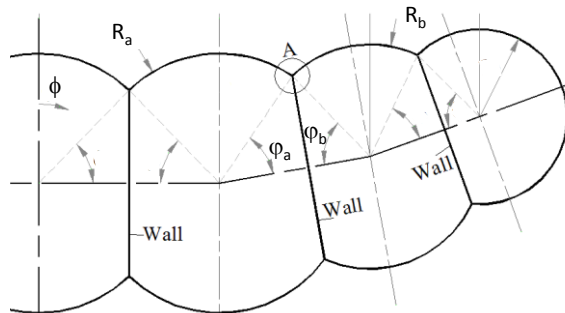
$$\begin{aligned} X_a &= N_{\varphi a} \sin \varphi_a = p R_a \sin \varphi_a \\ X_b &= N_{\varphi b} \sin \varphi_b = p R_b \sin \varphi_b \end{aligned} \quad (6.4)$$

In order to have equilibrium in the x-direction at location  $A$ , the following condition must be met:

$$R_a \sin \varphi_a = R_b \sin \varphi_b \quad (6.5)$$

Looking at this equation from a geometrical point of view, it is clear that eq. (6.4) is met for every cylinder that is merged on to another cylinder as long as the axes of the intersecting cylinders are parallel with respect to each other.

This condition provides considerable flexibility in controlling the geometry of the cross sectional area. Bubbles with different diameters can be connected and the intersecting walls do not need to be vertical (figure 6.11). Several bubbles of the multi-cylinder can be placed higher or lower by rotating these bubbles over the centre of the adjacent bubble as shown in figure 6.11.



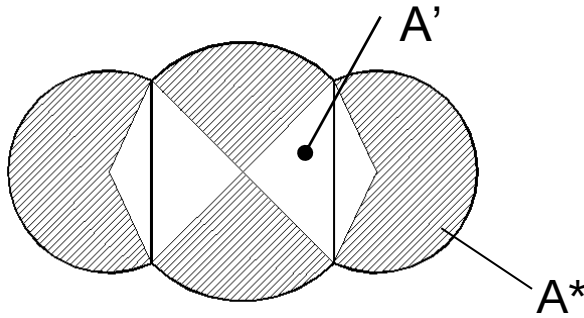
**Fig. 6.11: A variation on the single-row multi-bubble**

The analysis remains applicable because the local reference system is also rotated. The axial force in the multi-cylinder is induced by the meridional forces of the closing boiler end that covers it and is defined as (see chapter 2):

$$N_{z,c} = \frac{pR}{2} \quad (6.6)$$

Multiplying this axial force with the entire arc length of the cylindrical elements and dividing it by the pressure equals an area  $A^*$ , which represents the hatched area in figure 6.12.

$$A^* = \frac{N_{z,c} \cdot R(2\pi - 2\varphi)}{p} = R^2(\pi - \varphi) \quad (6.7)$$



**Fig. 6.12: Equivalent pressure area ( $A^*$ ) induced by the axial forces in the curved membrane of the multi-cylinder**

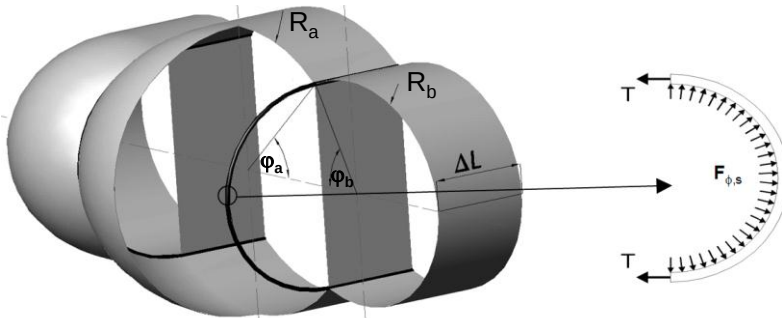
The axial force in the curved membrane does not resist the full load that is exerted by the pressure on the cross-sectional area. The pressure load exerted on the cross-sectional area is partly carried by the curved membrane and partly by the vertical wall or a separate reinforcement depending on the closed or open cell configuration. More explanation on the axial load in the multi-cylinder and the possible configurations of the multi-sphere is provided in the next section.

### 6.1.2 Analysis of the single row multi-sphere

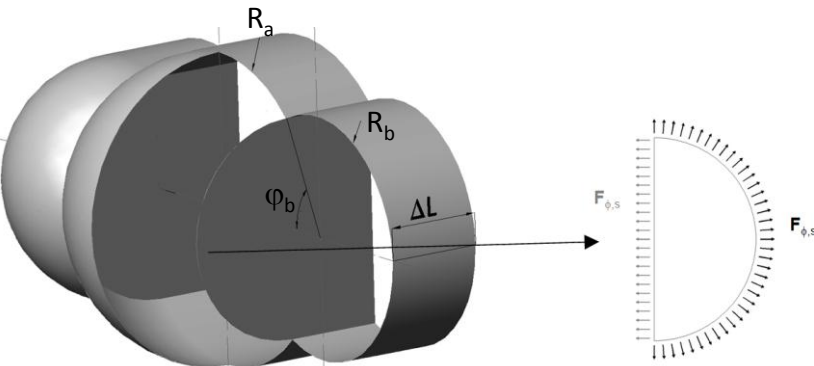
The membrane forces (meridional and circumferential) for the spherical membrane were in chapter two defined as:

$$N_{\phi,s} = N_{\theta,s} = \frac{pR}{2} \quad (6.8)$$

The index  $s$  refers to the sphere. The condition to obtain equilibrium for the multi-cylinder (eq. 6.5) also holds for the intersection of the multi-sphere. The intersection of the multi-sphere in figures 6.13 and 6.14 is represented as a semi-circle with radius  $R_a \sin \varphi_a$ . There are two possible configurations for the multi-sphere; open-cell or closed cell.



**Fig. 6.13 Visualization of the loads working on the reinforcement ring in the open-cell multi-sphere**



**Fig. 6.14 Visualization of the loads working on the wall in the closed-cell multi-sphere**

The force exerted by the intersecting spherical membranes in radial direction (figures 6.13b and 6.14b) is found in the same way as in eq. (6.2):

$$F_{\phi,s} = \frac{pR_a \cos \varphi_a + pR_b \cos \varphi_b}{2} = \frac{F_{y,c}}{2} = p \frac{|O_a O_b|}{2} \quad (6.9)$$

Where  $O_a$  and  $O_b$  correspond with the origins of the bubbles with radius  $R_a$  and  $R_b$  respectively. There are two ways of transferring these forces into the structure:

- Via a reinforcement ring, in case the multi-bubble is an open cell construction. The reinforcement ring is interpreted as a cable that runs over the intersection on the outside surface of the multi-bubble (figure 6.13)
- Via a circular wall in the case where the multi-bubble is a closed cell construction. (figure 6.14)

The choice of the reinforcement depends on the application (open cell vs. closed cell), the structural lay-out of the multi-bubble, and the material of choice. These choices are further addressed in chapter 7.

For the reinforcement ring, the tensile load in the ring (figure 6.13) is defined as:

$$T = R_a \sin \varphi_a F_{\phi,s} \quad (6.10)$$

Equilibrium shows that  $T$  is equal to the induced load of the remaining pressurised cross-sectional area  $A'$  that is illustrated in figure 6.13. The load induced by this area is:

$$\text{Load} = p \cdot A' = p \cdot R_a \sin \varphi_a \times (R_a \cos \varphi_a + R_b \cos \varphi_b) = 2T \quad (6.11)$$

This clarifies the axial load in the multi-cylinder mentioned in the previous section. The axial load in the multi-cylinder is carried in the curved membrane and additional reinforcements in case the multi-sphere is configured as an open cell construction. For this configuration, the vertical wall in the multi-cylinder and the reinforcement are both uni-axially loaded.

For the closed cell construction, structural integrity is obtained via a circular wall at the intersection of the spheres. The reactive force at the intersection is also  $F_{\phi,s}$ . The difference here is the stress distribution. The circular wall in the multi-sphere

is in an equal bi-axial stress state and the stress ratio of the wall in the multi-cylinder is two (eqs. 6.3 and 6.9) for a closed cell configuration. This means that the stress ratio in the membrane elements of the closed cell pressurised multi-cylinder ( $N_{\phi,c}$  vs.  $N_{z,c}$  and  $F_{y,c}$  vs.  $F_{\phi,s}$ ) is exactly the same as the stress ratio for the pressurised cylinder. The same accounts for the closed cell pressurised multi-sphere with respect to the sphere.

The practical implications of the closed cell and the open cell construction will be discussed in chapter 7.

### 6.1.3 Analysis of the single row multi-torus

The multi-torus is a complicated membrane structure because the pressurized toroidal membrane is a structural element that cannot be treated properly by means of linear membrane theory. The stress distributions that result from linear membrane theory do not exhibit any irregularities because linear theory does not take any deformations into account, i.e. it is based on an infinitely stiff membrane. The irregularities in the toroidal membrane have been extensively investigated in chapter 3. The linear membrane solution of the multi-torus is presented next. The subsequent section discusses the methodology to analyze a multi-torus by taking the irregularities into consideration.

#### 6.1.3.1 Linear membrane analysis of the single row multi-torus

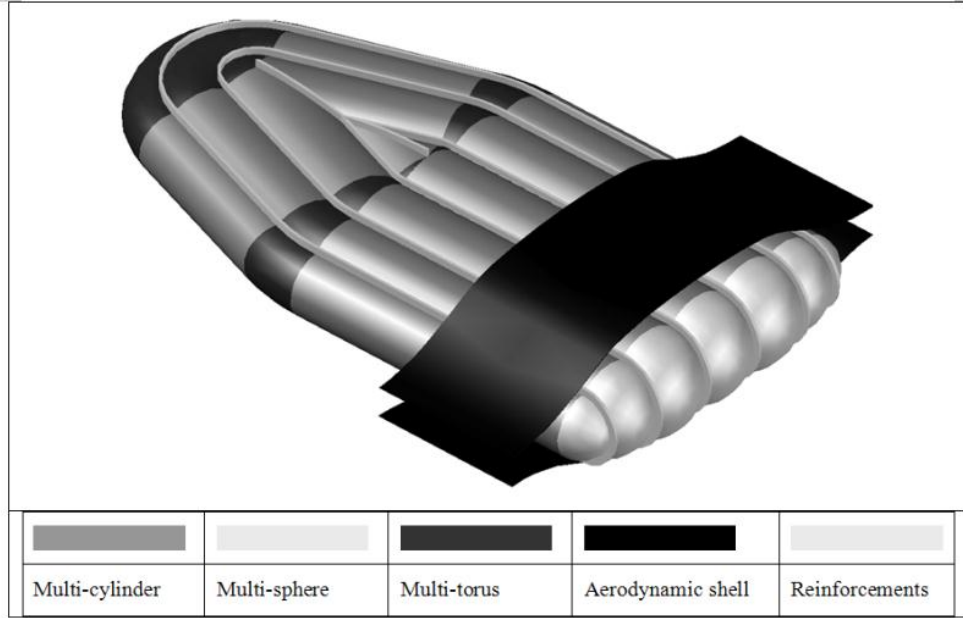
The meridional forces in the pressurized toroidal shell (figure 2.12) with circular cross-section are, according to linear membrane theory defined as [Flügge, 1960]:

$$N_{\phi,t} = \frac{pR_m}{2} \left( \frac{2a + \sin \phi}{a + \sin \phi} \right) \quad (6.12)$$

The index  $t$  refers to the torus. The circumferential forces in the pressurized toroidal shell are defined as:

$$N_{\theta,t} = \frac{pR_m}{2} \quad (6.13)$$

The circumferential forces are according to linear membrane theory exactly the same as the axial and meridional/circumferential forces of the cylinder and the sphere respectively. This provides significant spatial freedom due to the possibility of combining spherical, cylindrical and toroidal membranes together (figure 6.15).



**Fig. 6.15: Possible lay-out of a multi-bubble fuselage**

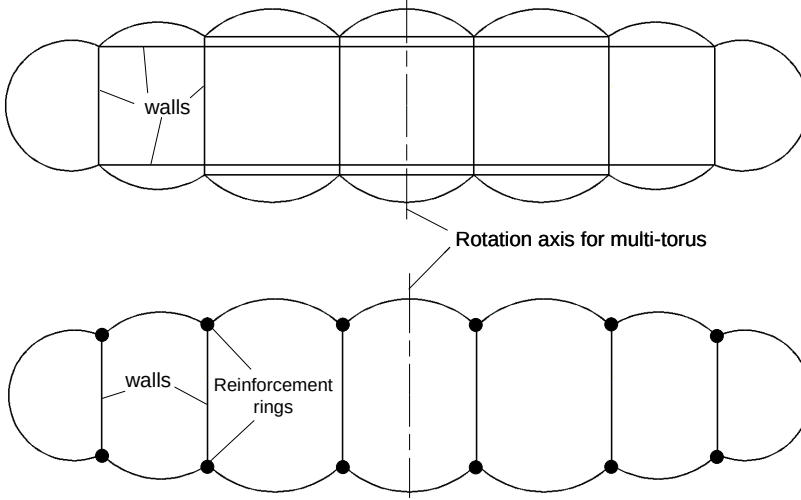
Eqs. (6.12) and (6.13) are valid for all individual toroids of the multi-torus. As with a multi-bubble composed of multi-cylinders and multi-spheres, structural members are needed at the intersections of the toroidal membranes to ensure structural integrity. The tension forces at the intersections where the different toroids meet have to be balanced out by two walls, figure 6.16a or a wall with a reinforcement ring, figure 6.16b.

The forces and loads in the walls and rings can be derived by taking the equilibrium of forces in the  $x$  and  $y$  direction into consideration. Figure 6.8 shows the geometry of the torus in more detail. Examination of the membrane force ( $F_{y,t}$ ) in the wall at location A in figure 6.8 results in:

$$F_{y,t} = Y_a + Y_b = N_{\phi a} \cos \varphi_a + N_{\phi b} \cos \varphi_b \quad (6.14)$$

The force induced due to the components of the meridional forces in  $x$ -direction is defined as:

$$F_{x,t} = N_{\phi b} \cos \phi_b - N_{\phi a} \cos \phi_a \quad (6.15)$$



**Fig. 6.16: Structural concept of the multi-torus with, a) two walls or, b) a ring and a wall at the intersection of the toroidal membranes**

By substituting eq. (6.12) in eqs. (6.14) and (6.15) and taking the following geometrical relationships into account:

$$a_a R_a + R_a \sin \phi_a = a_b R_b + R_b \sin \phi_b \quad (6.16)$$

$$R_b a_b - R_a a_a = R_a \cos \phi_a + R_b \cos \phi_b \quad (6.17)$$

$$\cos \phi_a = \sin \phi_b \quad (6.18a)$$

$$\cos \phi_b = \sin \phi_a \quad (6.18b)$$

We obtain the following general expressions for the force  $F_{y,t}$  and  $F_{x,t}$ :

$$F_{y,t} = \frac{p \left( (R_{ext} a_{ext})^2 - (R_{int} a_{int})^2 \right)}{2 (R_{int} a_{int} + R_{int} \sin \phi_{int})} \quad (6.19)$$

$$F_{x,t} = \frac{pR_{int} \cos(\phi_{int})(R_{ext}a_{ext} - R_{int}a_{int})}{2(R_{int}a_{int} + R_{int} \sin \phi_{int})} \quad (6.20)$$

The subscript *ext* and *int* refer to the bubble which is the furthest from, and closest to, the axis of rotation respectively. These forces will always turn out to be positive or, in other words, result in tensile stresses. Equation (6.19) simply represents a force that is induced by the pressure working on the surface area of a flat ring with outer radius  $R_{ext}a_{ext}$  and inner radius  $R_{int}a_{int}$ .

As was stated, there are two ways of transferring this force into the structure: through a reinforcement ring that is uni-axially loaded or a wall that is bi-axially loaded. The wall is optimal from a structural efficiency point of view in the case where only isotropic materials would be considered. From a practical point of view however, the reinforcement ring is favourable due to the fact that difficult intersections are avoided. A uni-axially loaded wall that carries the load  $F_{y,t}$  and a uniaxially loaded ring that carries  $F_{x,t}$  is preferable to two walls due to simplicity. The magnitude of the tensile load in the ring in figure 6.8 is derived as:

$$T = F_{x,t} (R_a a_a + R_a \sin \phi_a) \quad (6.21)$$

Resulting in:

$$T = \frac{pR_a \cos(\phi_a)(R_b a_b - R_a a_a)}{2} \quad (6.22)$$

Equation (6.21) can be rewritten by making use of the geometrical parameters (eqs. 6.17 and 6.18) into:

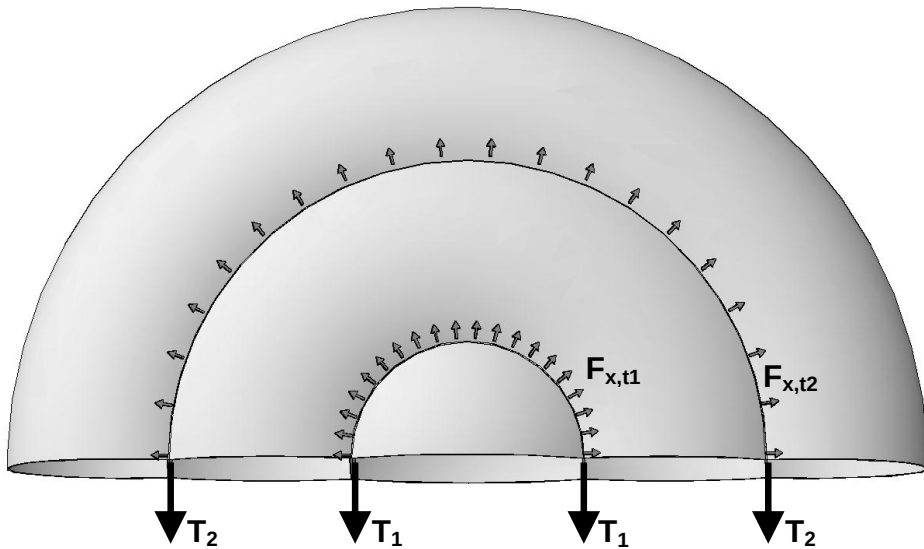
$$T = \frac{pR_a \sin(\varphi_a)(R_a \cos \varphi_a + R_b \cos \varphi_b)}{2} \quad (6.23)$$

Equation (6.23) shows that the tensile load in the reinforcement ring of the multi-torus is identical to the load in the reinforcement ring of the multi-sphere.

Taking a closer look at eq. (6.23), it shows that there is no dependence between the loads in the reinforcement ring and the configuration parameter  $a$ . Similar to the reinforcement ring in the multi-sphere, the tension load  $T$  resists the pressure

loads that is working on the area (triangle) governed by the three points,  $[o_a A o_b]$  in figure 6.8. This equality is the expected result of equilibrium in the membrane structure and there is no difference in using a multi-sphere as a bulkhead for the multi-cylinder or a multi-torus which explains the combination shown in figure 6.15.

The explanation of this result is quite straightforward. Having a closer look at eqs. (6.12) and (6.13), we notice that these equations are also valid for the sphere and the cylinder. The cylinder can be considered as a toroidal shell of which the parameter " $a$ " is infinite. Similarly, the sphere is a toroidal shell for which the parameter " $a$ " is zero. Equation (6.15) can also be applied for the multi-cylinder and the multi-sphere and it was just shown that the load  $T$  in the reinforcement ring (eq. 6.24) is not a function of the parameter  $a$ . The multi-bubble consisting of multi-spheres, multi-cylinders is therefore a collection of multi-toroids.

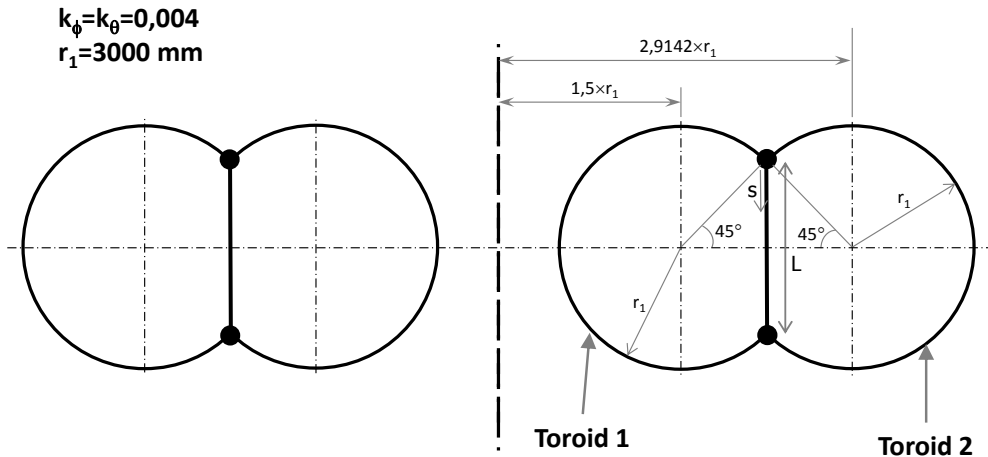


**Fig. 6.17: Forces and loads for the multi-torus equipped with a reinforcement ring and a wall at the intersections**

### 6.1.3.2. Membrane response in the multi-torus

The linear membrane solution of the multi-torus has been presented in the previous section. This analysis is however only valid for a multi-torus that consists of 'optimal' toroidal membranes. How should the stresses and deformations be assessed when the multi-torus is build-up of membranes for which compatibility of deformations is not valid? This question will be discussed in this section by means of an example.

Figure 6.18 shows the geometry of a multi-torus that consists of two isotropic toroids; toroid 1 and toroid 2. It is assumed that the membrane's thickness is constant and that it is made of an isotropic material.



**Fig. 6.18: Geometry multi-torus used for example**

Investigation of the individual toroids by means of the modified linear membrane theory (chapter 3) shows that the displacements at the junction of the two toroidal elements are not compatible. The displacement  $u$  of toroid 1 and toroid 2 is respectively 6.86 mm and 2.83 mm. This discontinuity is obviously not correct with the assumption of a continuous shell but the solution to solve this irregularity is less straightforward than the solution of a simple pressurized torus. The difficulty is found in the interaction of 4 elements; the two toroidal membranes, the vertical

wall and the reinforcement ring. These four elements will interact in such a way that the total potential energy involved by the pressurization is minimal. The methodology that comes to mind is to formulate an energy functional that prescribes the three membrane elements and the reinforcement ring simultaneously of which the displacements are formulated in a series that satisfy the boundary conditions:

$$\begin{aligned}
 u'(-\pi/2)_{toroid1} &= w(-\pi/2)_{toroid1} = u'(\pi/2)_{toroid2} = w(\pi/2)_{toroid2} = w(L/2)_{wall} = 0 \\
 u(\pi/4)_{toroid1} &= u(-\pi/4)_{toroid2} = u(s_0)_{wall} = u_{ring} \\
 w(\pi/4)_{toroid1} &= w(-\pi/4)_{toroid2} = w(s_0)_{wall}
 \end{aligned}$$

Finding a mathematical formulation to incorporate all these elements in a functional is an enormous challenge. The other option is to find the solution iteratively by subjecting the elements at the junction to displacement boundary conditions. The final solution is that displacement that leads to the lowest total potential energy. This method is obviously very elaborate.

The investigation of a modified membrane theory for the multi-torus or a different method to assess the displacements of a multi-torus that has no compatibility of deformations is not part of this dissertation. The only alternative that remains at this moment is to design the multi-torus in such a way that it complies with the membrane solution at the ultimate pressure it is designed for. Discontinuity might still occur at lower loads but these are not of interest due to the fact that the stresses and strains are lower because the pressure is lower. There are three ways to resolve this;

- a) change the geometrical parameters to create compatibility at the junction at the design pressure
- b) change the stiffness parameter of the neighbouring toroid such that it results in compatibility at the junction or
- c) make a combination of the two previous options.

Considering alternative a, compatibility after pressurization is achieved when the geometrical parameter of one (or both) of the toroids is changed. Doing this for example for toroid 2 results in  $a_2=2.9156$ . This leads to complications with respect to joining the two toroids because a gap of 4 mm emerges at the junction when the toroids are unloaded. On top of that, the displacement of the wall (due to the poisson effect) and the reinforcement ring need to be taken into account as well.

Taking the reinforcement ring into account and assume that the size of the ring matches with toroid 1 when it is unloaded, than the displacement ( $u_{ring}$ ) after pressurization is supposed to be 6.86mm.

From strength of materials, we can deduce that:

$$\varepsilon_{ring} = \frac{6.86}{r_1(a_1 + \cos 45)} = \frac{T}{(EA)_{ring}}$$

In this example, we see that  $\varepsilon_{ring}$  is only 0.1% which is very small and therefore means that the extensional stiffness of the ring needs to be very large. A calculation of the required cross-sectional area assuming that steel is used ( $E=210000 \text{ MPa}$ ) reveals that  $A_{ring} = 20683 \text{ mm}^2$ . This is completely out of proportion because a  $k$ -value of 0.004 represents a steel membrane of approx. 0.75mm thickness. It could be argued that the cross-sectional area of the ring can be decreased because the circumferential strains in the ring and in the membranes (figures 3.10 and 3.11) are small. Decreasing the cross-sectional area would unload the ring and increase the circumferential forces in the membranes at the junction. A proper assessment of the stresses and deformations becomes very hard however and it is difficult to say how far one can go in decreasing the cross-sectional area in the ring.

Another route to take is to change the geometrical parameters again. Decreasing the radius of the ring by  $4 \times 6.86 = 27.44 \text{ mm}$  results in a 5 fold decrease in required cross sectional area and results in a multi-torus without irregularities at maximum pressure.

The geometrical parameters of the cylindrical wall can be changed in a similar way in order to obtain compatibility at maximum pressure. As already stated, these measures cause serious complications in the manufacturing. Membrane elements and reinforcements need to be pre-stretched in order to join them and membrane elements can be severely wrinkled at the junction after joining.

Considering alternative b, a change of the stiffness parameter for toroid 1 ( $k_I=0.00173$ ) results in compatibility of deformations at the junction of toroid 1 and toroid 2 and rules out manufacturing complications. Similarly the stiffness of the wall and reinforcement ring can also be tailored for displacement compatibility. To a certain extent it makes sense to thicken the membrane of toroid 1 because

the maximum stresses for toroid 1 (at  $\phi=-90^\circ$ ) is 1.74 times higher than for toroid 2 (at  $\phi=-45^\circ$ ). A compromise could be a combination of the two methods, where a change of the stiffness parameter for toroid 1 ( $k_f=0.0022$ ) would result in identical maximum strain levels (0.4%) while a geometrical parameter  $a=2.9145$  gives compatibility of deformations at maximum pressure.

Considering cylindrical drums and boiler ends, it was shown in chapter 4 that the importance of compatibility of deformations is not an issue as long as the drum and boiler end can be considered as membrane structures.

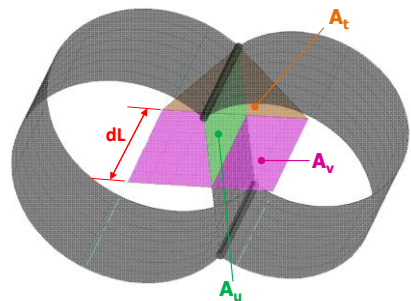
This discussion however needs a more balanced judgement when multi-bubbles are considered. The example presented is a showcase of the importance for a multi-torus to approximate the 'optimal' membrane as much as possible. The optimal multi-torus is easy to analyze, has maximum structural efficiency and there is always compatibility of deformations at every junction.

#### 6.1.4 Visualization of forces and loads in the multi-bubble

Along the same lines described in chapter 2, interpretation of the forces and loads in this chapter shows that the forces and loads in the multi-bubble can be assessed visually by considering the surfaces perpendicular to the shell surface/reinforcement. Figures 6.19, 6.20 and 6.21 shows the distinctive details to give a visual interpretation to the forces and loads at the intersections of the multi-bubble. Only one half is considered in figures 6.19 and 6.21 because there are 2 intersections.

The tensile force components that balance each other out at the intersection of the multi-bubble (eq. 6.4) are found by multiplying the pressure with the ratio of the area  $A_u$  to the (arc-) length  $dL$  or  $ds$ . Visually, it is clear that these forces balance each other out since the area  $A_u$  is a shared surface of both neighbouring elements.

The force in the height direction (eqs. 6.3 and 6.19) of the wall (perpendicular to the (arc-)

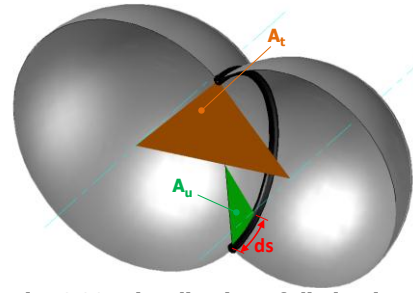


**Fig. 6.19: Visualization of distinctive surfaces at the intersection of the multi-cylinder**

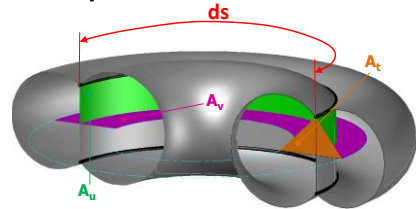
length) is found by multiplying the pressure with the ratio of the area  $A_v$  to the (arc-) length  $dL$  or  $ds$ . This force is not applicable for the multi-sphere. The area  $A_v$  is bounded by the arcs or lines that are the centres of the tubes.

The axial / circumferential load (eq. 6.24) at the intersection is found by multiplying the pressure with the triangular area  $A_t$ . The triangle is formed by the centres of the individual membrane elements and the intersection. The load is carried by a uni-axially loaded reinforcement but can also be carried by the wall in the multi-cylinder and a wall that replaces the reinforcement ring in the multi-sphere. The force in that case is found by multiplying the pressure with the ratio of the area  $A_t$  to half the height of the wall (multi-cylinder) or to the radius of the intersecting circle (multi-sphere).

The contribution of the individual elements to the total loads and forces at the intersection (e.g.  $Y_a$  and  $Y_b$  in eq. 6.2) is easily assessed by considering the contribution on one side of the intersecting surface.



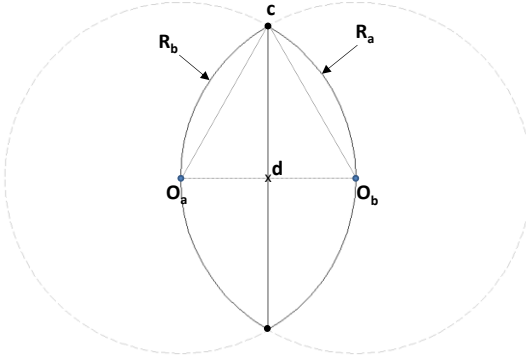
**Fig. 6.20: Visualization of distinctive surfaces at the intersection of the multi-sphere**



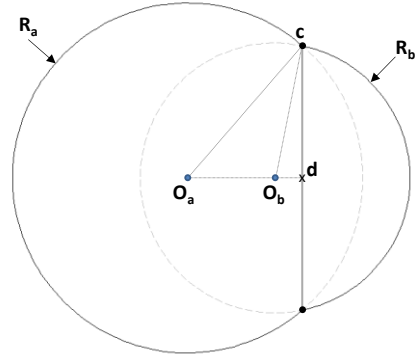
**Fig. 6.21: Visualization of distinctive surfaces at the intersection of the multi-torus**

### 6.1.5 Geometrical constraints in the lay-out of the multi-bubble

The structural members in the multi-bubble shown in the previous subsections were all loaded in tension. A problem arises when the intersecting bubbles have a "pointed intersection". This occurs when the centres of both intersecting membrane elements are located on the other side of the intersection, figure 6.22. Figure 6.23 shows an intersection where one membrane element has its centre located on the other side of the intersection. Pointed intersections lead to intersections that are loaded in compression.



**Fig. 6.22: This pointed intersection leads to compressive loads in the reinforcement(s) of this multi-bubble**



**Fig. 6.23: The bubble with radius  $R_b$  reduces the loads in the reinforcement(s) of this multi-bubble**

The analysis of the multi-bubbles in figures 6.22 and 6.23 is identical to the analysis in the previous sections. The loads and forces at the intersection of figure 6.22 will therefore be negative instead of positive. A multi-cylinder with the cross-section shown in figure 6.22 has a compressive force in the height direction (eq. 6.2) of:

$$F_{y,c} = Y_a + Y_b = -p \cdot |O_a d| - p \cdot |O_b d| = -p \cdot |O_a O_b|$$

Similarly, the axial load would also be compressive and defined as:

$$T = -p \cdot [O_a c d] - p \cdot [O_b c d] = -p \cdot [O_a c O_b] = F_{y,s} \cdot |c d|$$

This negative contribution of the intersection also leads to negative consequences for the structural design of the multi-bubble. First of all, compressive forces require design for stability (buckling) which conflicts with membrane theory. Secondly, imagine a multi-cylinder with positive axial membrane stresses in the curved membrane and negative axial membrane stresses in the reinforcements. Conflicts arise when deformations are taken into account at the intersection of the multi-cylinder. The only solution to counteract this conflict is by having an “infinitely stiff” bulkhead such that the axial stresses over the cross-section of the multi-bubble are uniform. This solution requires bending moments in the bulkhead (e.g. multi-sphere or multi-torus) and would for this part also conflict with membrane theory.

Pointed intersections cannot always be avoided however. The front part of the multi-bubble shown in figure 6.15 is tapered and this tapering locally creates an inevitable pointed intersection in the multi-cylinder. The analysis of the tapered multi-cylinder where the axes of the cylinders are not parallel with respect to each other will be covered in section 6.3.

In figure 6.23 a multi-bubble is shown where the bubble with radius  $R_b$  gives a negative contribution with respect to the loads and forces in generates at the intersection. The total contribution, taking the bubble with radius  $R_a$  into account as well, is however positive and the reinforcement is therefore still loaded in tension.

A multi-cylinder with the cross-section shown in figure 6.23 has a force in the height direction (eq. 6.2) defined as:

$$F_{y,c} = Y_a + Y_b = p \cdot |O_a d| - p \cdot |O_b d| = p \cdot |O_a O_b|$$

Similarly, the axial load gives:

$$T = p \cdot [O_a cd] - p \cdot [O_b cd] = p \cdot [O_a c O_b] = F_{y,s} \cdot |cd|$$

From this section, it can be concluded that intersections loaded in compression (of which figure 6.22 forms an example) should be avoided as much as possible.

## 6.2 The multi-cell multi-bubble

A more complex configuration of the multi-bubble is obtained when all cells have multiple neighbouring cells. Three intersecting bubbles are very common in nature, such as soap-bubbles, while four intersecting bubbles are exceptional. A multi-bubble with three intersecting bubbles is shown in figures 6.24a-b and a multi-bubble with 4 intersecting bubbles is shown in figures 6.25a-d. Figure 6.7b shows the simplest form of a multi-bubble with 4 intersecting bubbles and is obtained by mirroring a single row multi-bubble with a structural reinforcement located at the mirror line. The multi-bubble in figure 6.25 has four intersecting bubbles that are all different. This chapter will show the analysis of loads and forces in the multi-cell multi-bubble of which the multi-row multi-bubble is a simplified version. Common sense tells us that when all cells are in equilibrium with one of their neighbouring cells, the communal intersection of all cells needs to be in equilibrium as well. This statement will be proven in this chapter.

### 6.2.1. Analysis of the multi-cell multi-cylinder

The analysis of the membrane forces in the curved membrane and the wall is identical to the analysis shown in section 6.1.1. The multi-cylinder in figure 6.25 is in equilibrium when the intersection of the four intersecting wall is in equilibrium.

The membrane forces in the curved membrane are defined as:

$$\begin{aligned} N_{\phi, c-a} &= p \cdot R_a & N_{z, c-a} &= p \cdot R_a / 2 \\ N_{\phi, c-b} &= p \cdot R_b & N_{z, c-b} &= p \cdot R_b / 2 \\ N_{\phi, c-c} &= p \cdot R_c & N_{z, c-c} &= p \cdot R_c / 2 \\ N_{\phi, c-d} &= p \cdot R_d & N_{z, c-d} &= p \cdot R_d / 2 \end{aligned} \tag{6.24a-h}$$

It is known from the analysis in section 3.1.1. that the forces in the walls that are induced by the circumferential forces in the cylindrical membranes are defined as:

$$\begin{aligned}
F_{\phi,c_{-jj}} &= p \cdot |O_a O_b| \rightarrow \text{membrane force in wall [z,e]} \\
F_{\phi,c_{-mn}} &= p \cdot |O_b O_c| \rightarrow \text{membrane force in wall [z,f]} \\
F_{\phi,c_{-qr}} &= p \cdot |O_c O_d| \rightarrow \text{membrane force in wall [z,g]}
\end{aligned} \tag{6.25a-c}$$

The index 'jj' refers to the intersection that is enclosed by the angles  $\varphi_i$  and  $\varphi_j$ . The force in the wall of the multi-cylinder is characterized by the distance between the axes of the individual cylinders. Furthermore, it is important to observe that the line that connects these axes is always perpendicular with respect to the wall.

Reformulating this, for the multi-cylinder in figure 6.25, we have:

- a force characterized by the length  $|O_a O_b|$  and perpendicular to the line  $[O_a O_b]$
- a force characterized by the length  $|O_b O_c|$  and perpendicular to the line  $[O_b O_c]$
- a force characterized by the length  $|O_c O_d|$  and perpendicular to the line  $[O_c O_d]$

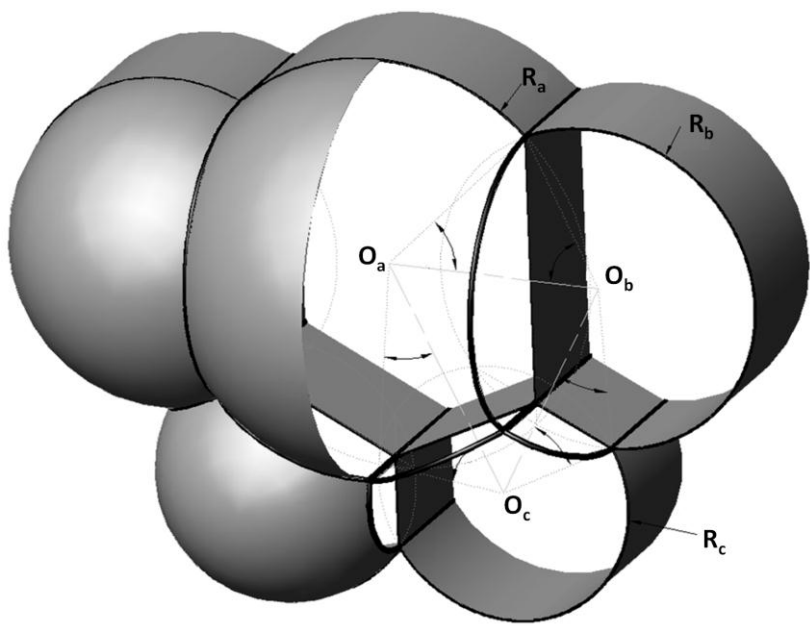
In order for the mutual intersection to be in equilibrium (*the forces in the wall cancel each other out*), the quadrangle  $[O_a O_b O_c O_d]$  needs to be closed, meaning that we need a force characterized by the length  $|O_a O_d|$  and perpendicular to the line  $[O_a O_d]$ .

Hence, the force in the wall  $[z,h]$  becomes:

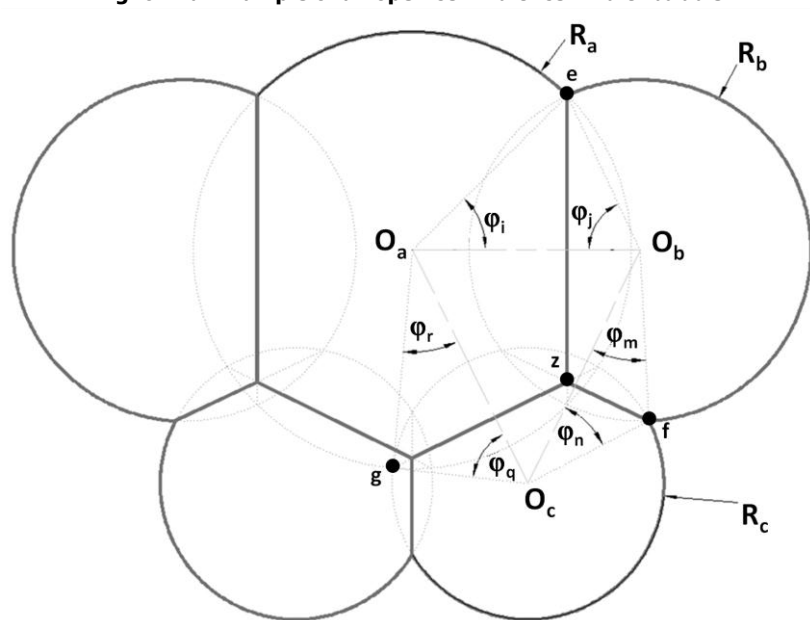
$$F_{\phi,c_{-uv}} = p \cdot |O_a O_d| \tag{6.25d}$$

The analysis is exactly the same for the multi-cylinder in figure 6.24. The mutual intersection is in equilibrium because of the closed triangle  $[O_a O_b O_c]$ . And the force  $F_{\phi,c}$  in the wall is found by multiplying the pressure with the distance between the axes of the intersecting cylinders.

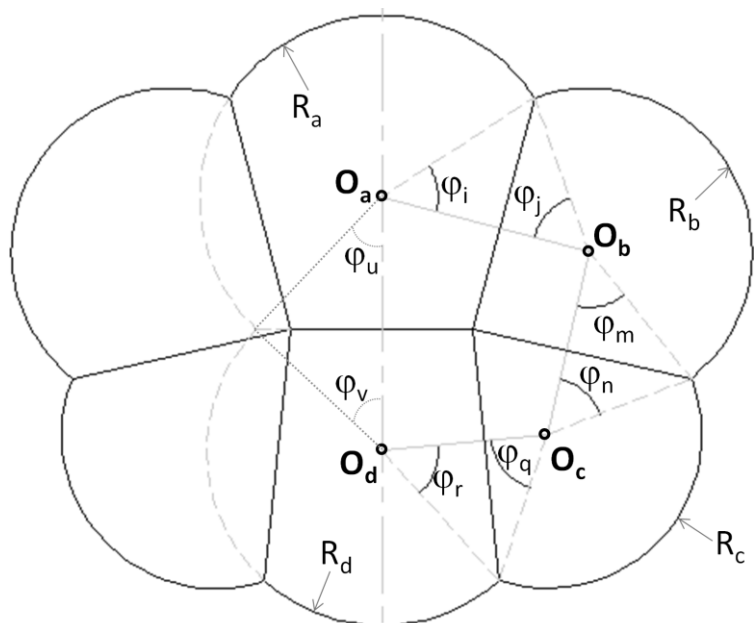
The axial force in the wall depends on the configuration and is  $F_{\phi,s}$  in a closed cell configuration and zero in an open cell configuration due to an additional reinforcement. The explanation for this is provided in the following section.



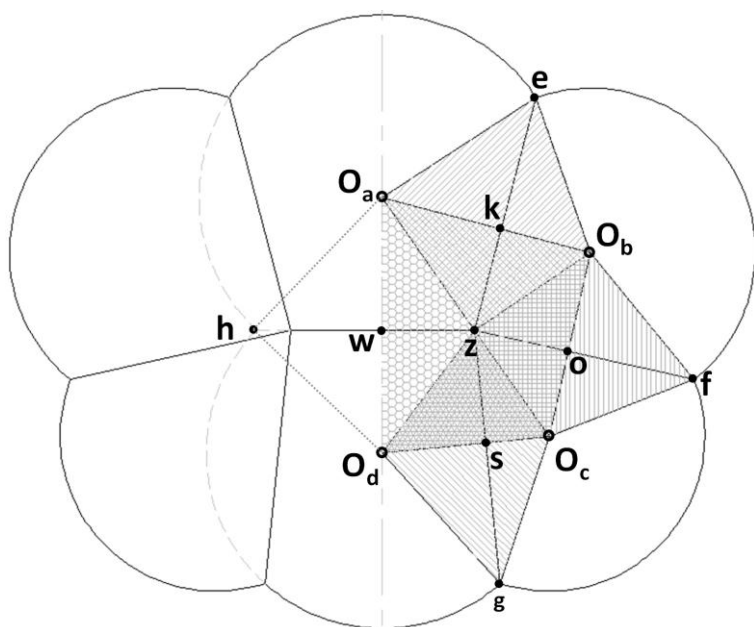
**Fig. 6.24a: Example of an open cell multi-cell multi-bubble**



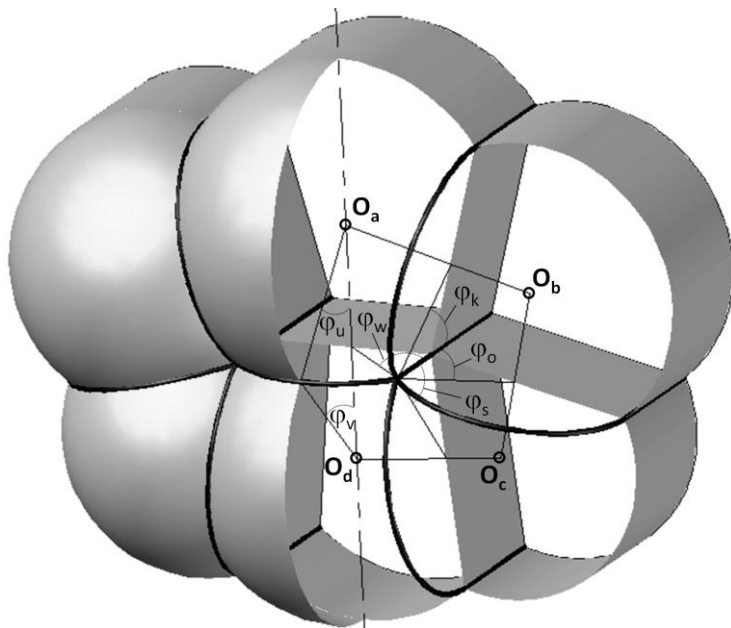
**Fig. 6.24b: Cross-section of the multi-cell multi-bubble of figure 6.24a**



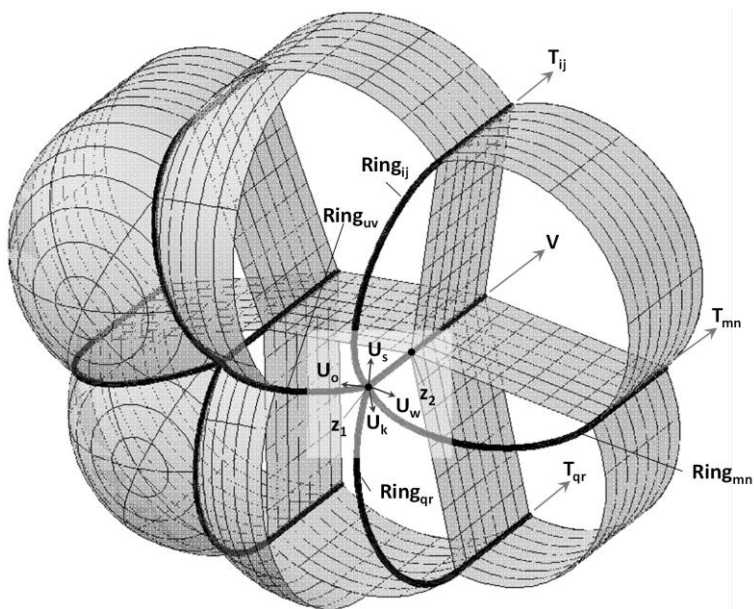
**Fig. 6.25a: Distinctive angles in the cross-section of the multi-cylinder & multi-sphere**



**Fig. 6.25b: Distinctive areas in the cross-section of the multi-cylinder & multi-sphere**



**Fig. 6.25c: Distinctive angles in the multi-sphere**



**Fig. 6.25d: Representation of the loads in the reinforcements of the multi-cell multi-bubble**

### 6.2.2. Analysis of the multi-cell multi-sphere

For the multi-sphere in figure 6.24, three reinforcement cables merge into one rod at the intersection of the three spherical membrane elements. For the multi-sphere in figure 6.25, it concerns four elements instead of three. Similarly, three or four circular walls would have one mutual line as intersection when the multi-bubbles in figures 6.24 and 6.25 would be in a closed cell configuration. First the analysis of the open cell configuration is shown and the similarities with the closed cell configuration are explained subsequently. The analysis in this section is applied to the multi-bubble displayed in figure 6.25. The methodology is exactly the same for the multi-bubble in figure 6.24 of which the results of loads and forces are presented at the end of this chapter. The loads in the reinforcement cables or circular walls are derived in the same way as in section 6.1.2.

The forces in radial direction  $F_{\phi,s_{ij}}$ ,  $F_{\phi,s_{mn}}$ ,  $F_{\phi,s_{qr}}$  and  $F_{\phi,s_{uv}}$  are defined as:

$$\begin{aligned} F_{\phi,s_{ij}} &= \frac{pR_a \cos \varphi_i + pR_b \cos \varphi_j}{2} = p \frac{|O_a O_b|}{2} \\ F_{\phi,s_{mn}} &= \frac{pR_b \cos \varphi_m + pR_c \cos \varphi_n}{2} = p \frac{|O_b O_c|}{2} \\ F_{\phi,s_{qr}} &= \frac{pR_c \cos \varphi_q + pR_d \cos \varphi_r}{2} = p \frac{|O_c O_d|}{2} \\ F_{\phi,s_{uv}} &= \frac{pR_a \cos \varphi_u + pR_d \cos \varphi_v}{2} = p \frac{|O_a O_d|}{2} \end{aligned} \quad (6.26,a-d)$$

These forces result in tensile loads in their corresponding reinforcement rings, as we have shown in eq. (6.10):

$$\begin{aligned} T_{ij} &= F_{\phi,s_{ij}} \cdot R_a \sin \varphi_i \\ T_{mn} &= F_{\phi,s_{mn}} \cdot R_b \sin \varphi_m \\ T_{qr} &= F_{\phi,s_{qr}} \cdot R_c \sin \varphi_q \\ T_{uv} &= F_{\phi,s_{uv}} \cdot R_d \sin \varphi_v \end{aligned} \quad (6.27,a-d)$$

The transversal components ( $U_s$ ,  $U_w$ ,  $U_k$  and  $U_o$ ) of these tensile loads, shown in figure 6.25d, need to be in equilibrium in order to ensure that this multi-sphere configuration is balanced. The equilibrium condition of the transversal components

is similar with the equilibrium condition of the multi-cylinder (eq. 6.4). This means that the following components need to cancel each other out (figure 6.25d):

$$\begin{aligned}
 U_k &= T_{ij} \sin \varphi_k = F_{\phi, s-ij} \cdot R_a \sin \varphi_i \sin \varphi_k \\
 U_o &= T_{mn} \sin \varphi_o = F_{\phi, s-mn} \cdot R_b \sin \varphi_m \sin \varphi_o \\
 U_s &= T_{qr} \sin \varphi_s = F_{\phi, s-qr} \cdot R_c \sin \varphi_q \sin \varphi_s \\
 U_w &= T_{uv} \sin \varphi_w = F_{\phi, s-uv} \cdot R_d \sin \varphi_v \sin \varphi_w
 \end{aligned} \tag{6.28,a-d}$$

From the geometry in figure 6.25, we notice that:

$$R_a \sin \varphi_i \sin \varphi_k = R_b \sin \varphi_m \sin \varphi_o = R_c \sin \varphi_q \sin \varphi_s = R_d \sin \varphi_v \sin \varphi_w = |z_1 z_2|$$

Where  $|z_1 z_2|$  represents the length of the communal rod. This means that the multi-sphere is in equilibrium when the forces  $F_{\phi, s-ij}$ ,  $F_{\phi, s-mn}$ ,  $F_{\phi, s-qr}$  and  $F_{\phi, s-uv}$  cancel each other out. Comparing eq. (6.26) with eq. (6.25) shows that the forces at the intersections of the multi-sphere ( $F_{\phi, s}$ ) are half the magnitude of the forces at the intersections of the multi-cylinder ( $F_{\phi, c}$ ). In the previous section was shown that the forces in eq. (6.25) cancel each other out hence this is also true for the forces in eq. (6.26). The same methodology accounts for the multi-sphere in figure 6.19 and it therefore proves that under the conditions of linear membrane theory, any multi-sphere and multi-cylinder is in equilibrium when subjected to a uniform pressure. A uniform pressure is of course trivial for an open cell multi-bubble.

The axial components of the tensile loads at the intersection of the four rings are defined as:

$$\begin{aligned}
 V_k &= T_{ij} \cos \varphi_k = F_{\phi, s-ij} \cdot R_a \sin \varphi_i \cos \varphi_k \\
 V_o &= T_{mn} \cos \varphi_o = F_{\phi, s-mn} \cdot R_b \sin \varphi_m \cos \varphi_o \\
 V_s &= T_{qr} \cos \varphi_s = F_{\phi, s-qr} \cdot R_c \sin \varphi_q \cos \varphi_s \\
 V_w &= T_{uv} \cos \varphi_w = F_{\phi, s-uv} \cdot R_d \sin \varphi_v \cos \varphi_w
 \end{aligned} \tag{6.29,a-d}$$

From the analysis of the single row multi-bubble, we know that (figures 6.25b):

$$\begin{aligned}
T_{ij} &= p \frac{|O_a O_b|}{2} \cdot R_a \sin \varphi_i = p \cdot [O_a e O_b] \\
T_{mn} &= p \frac{|O_b O_c|}{2} \cdot R_b \sin \varphi_m = p \cdot [O_b f O_c] \\
T_{qr} &= p \frac{|O_c O_d|}{2} \cdot R_c \sin \varphi_q = p \cdot [O_c g O_d] \\
T_{uv} &= p \frac{|O_d O_a|}{2} \cdot R_d \sin \varphi_v = p \cdot [O_d h O_a]
\end{aligned} \tag{6.30,a-d}$$

Where  $[xxx]$  denotes the area of the triangle governed by three points. Furthermore, interpreting figures 6.25a-c, we see that:

$$\begin{aligned}
R_a \sin \varphi_i \cos \varphi_k &= |zk| \\
R_b \sin \varphi_m \cos \varphi_o &= |zo| \\
R_c \sin \varphi_q \cos \varphi_s &= |zs| \\
R_d \sin \varphi_v \cos \varphi_w &= |zw|
\end{aligned}$$

Combining this with the relations in eq. (6.26), this gives:

$$\begin{aligned}
V_k &= p \cdot [O_a z O_b] \\
V_o &= p \cdot [O_b z O_c] \\
V_s &= p \cdot [O_c z O_d] \\
V_w &= p \cdot [O_d z O_a]
\end{aligned} \tag{6.31,a-d}$$

The total axial load induced at the intersection by the four reinforcement rings is therefore:

$$V = V_k + V_o + V_s + V_w = p \cdot [O_a O_b O_c O_d] \tag{6.32}$$

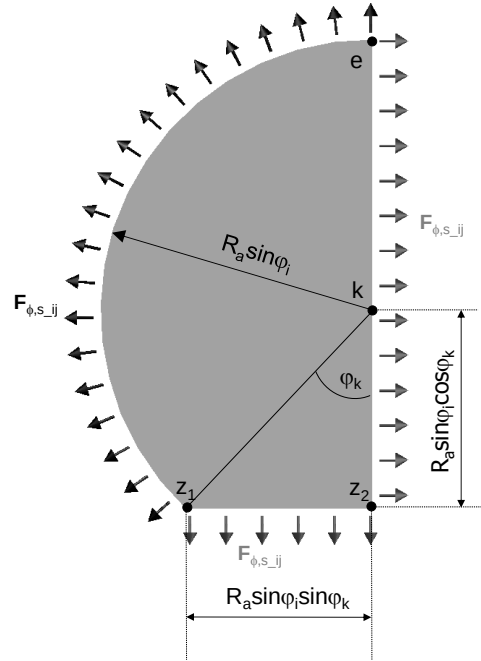
The conclusions that were drawn in chapter 2 can also be drawn for the multi-bubble. To solve for loads and forces in the multi-bubble, the solution simply depends on pressure, distances and areas. Mathematical equations confirm these conclusions but are actually not needed to solve for loads and forces. In the open cell configuration, the walls and reinforcements are uni-axially loaded.

The reinforcements in figure 6.25 can also be replaced by circular walls which would in that case experience the forces given in eq. (6.26). For the closed cell configuration, the forces  $F_{\phi,s_{ij}}$ ,  $F_{\phi,s_{mn}}$ ,  $F_{\phi,s_{qr}}$  and  $F_{\phi,s_{uv}}$  are also the axial forces working on the wall of the multi-cylinder. The reactive loads  $U_k$ ,  $U_\alpha$ ,  $U_s$ ,  $U_{wr}$ ,  $V_k$ ,  $V_\alpha$ ,  $V_s$ ,  $V_{wr}$ ,  $T_{ij}$ ,  $T_{mn}$ ,  $T_{qr}$  and  $T_{uv}$  are now distributed over the edges of the circular walls. Figure 6.26 visualizes the forces on a circular wall in case the reinforcement rings in the multi-bubble of figure 6.25 would be replaced by circular walls. The similarity between the reinforcement ring and the circular wall is obvious because the loads induced by the distributed forces in the circular wall equals the load in the ring. The loads induced by the distributed forces on this circular wall give:

$$\begin{aligned} F_{\phi,s_{ij}} \cdot |ek| &= F_{\phi,s_{ij}} \cdot R_a \sin \varphi_i = T_{ij} \\ F_{\phi,s_{ij}} \cdot |kz_2| &= F_{\phi,s_{ij}} \cdot R_a \sin \varphi_i \cos \varphi_k = V_k \\ F_{\phi,s_{ij}} \cdot |z_1z_2| &= F_{\phi,s_{ij}} \cdot R_a \sin \varphi_i \sin \varphi_k = U_k \end{aligned}$$

The procedure to analyse the forces and loads for the multi-bubble in figure 6.24 are exactly the same. For this multi-bubble, the forces are defined as (figure 6.24):

$$\begin{aligned} N_{\phi,c-a} &= pR_a & N_{\phi,z-a} &= \frac{pR_a}{2} \\ N_{\phi,c-b} &= pR_b & N_{\phi,z-b} &= \frac{pR_b}{2} \\ N_{\phi,c-c} &= pR_c & N_{\phi,z-c} &= \frac{pR_c}{2} \end{aligned}$$



**Fig. 6.26: Visualization of the forces on a circular wall that represents the closed cell version of the multi-sphere in figure 6.20**

$$\begin{aligned}
N_{\phi,s_a} &= N_{\theta,s_a} = N_{z_a} & F_{\phi,c_{ij}} &= 2F_{\phi,s_{ij}} = p \cdot |o_a o_b| \\
N_{\phi,s_b} &= N_{\theta,s_b} = N_{z_b} & F_{\phi,c_{mn}} &= 2F_{\phi,s_{mn}} = p \cdot |o_b o_c| \\
N_{\phi,s_c} &= N_{\theta,s_c} = N_{z_c} & F_{\phi,c_{qr}} &= 2F_{\phi,s_{qr}} = p \cdot |o_a o_c|
\end{aligned} \tag{6.33a-l}$$

And the loads in the reinforcements of the open cell configuration are:

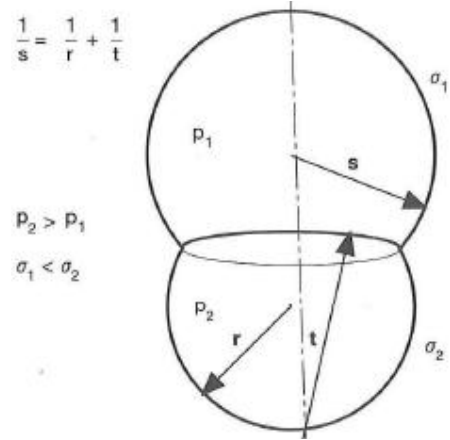
$$\begin{aligned}
T_{ij} &= p \cdot [o_a e o_b] \\
T_{mn} &= p \cdot [o_b f o_c] \\
T_{qr} &= p \cdot [o_a g o_c] \\
V &= p \cdot ([o_a z o_b] + [o_b z o_c] + [o_a z o_c]) = p \cdot [o_a o_b o_c]
\end{aligned} \tag{6.34a-d}$$

A multi-spherical pressure vessel can be analysed step by step by taking three interconnected spheres into consideration where the reference plane is the plane that goes through the centres of the three spheres. A cluster of soap bubbles as shown in figure 6.27 is the perfect example of the tremendous amount of spatial freedom the multi-sphere offers. There is however a subtle difference between a cluster of soap bubbles and a multi-spherical pressure vessel.



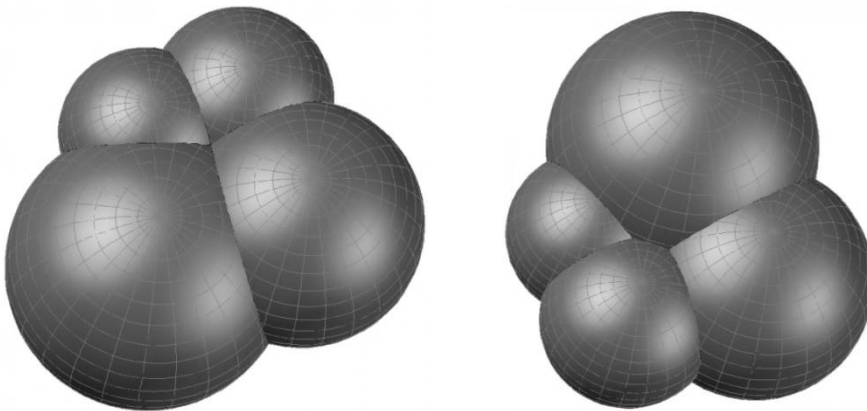
**Fig. 6.27: Soap bubbles: an example of the spatial freedom a closed-cell multi-sphere.**

The surface tension of a liquid membrane is constant which means that small soap bubbles have a higher internal pressure than larger bubbles. This means that for coalescing soap bubbles, the common wall bulges into the larger bubble as shown in figure 6.28. For multi-spherical pressure vessels on the other hand, the pressure difference in between the cells is kept the same and the common wall is therefore straight.



**Fig. 6.28: Two coalescing bubbles ( $r$  and  $s$ ) and their common surface [Beukers, 2005]**

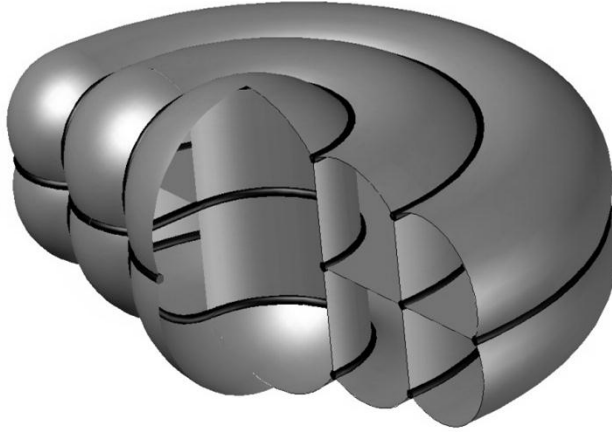
Figure 6.29a shows a multi-spherical structures that could cause some confusion because 4 spherical elements intersect at one point but the centres of the spheres are not all located in the same plane. The multi-sphere in figure 6.25 had the centres of all its spheres located in one plane which made it possible to consider the quadrangle governed by the four centres of all the spheres. The theory presented in this section is however still valid but the clue is that the multi-bubble in figure 6.29a is a multi-spherical structure with three intersecting spherical elements, similar to the multi-sphere in figure 6.24. This is clear when the multi-sphere is turned around, figure 6.29b. For this exceptional structure there are two rods that intersect each other at the location where the 4 spherical elements join each other.



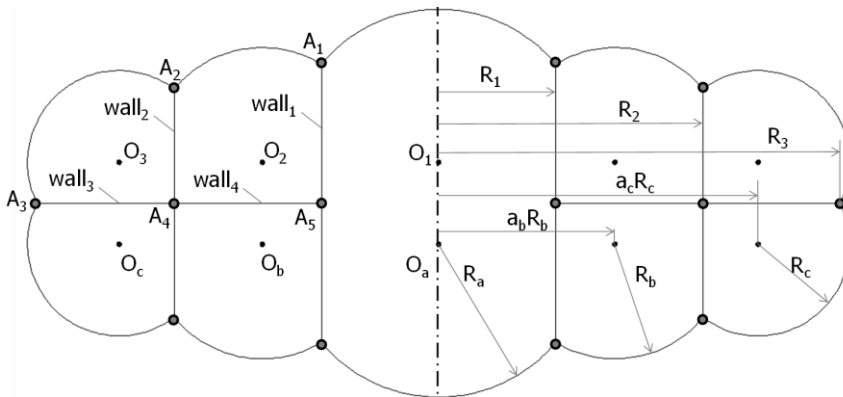
**Fig. 6.29: A multi-sphere with four and three intersecting spherical elements**

### 6.2.3 Analysis of the multi-cell multi-torus

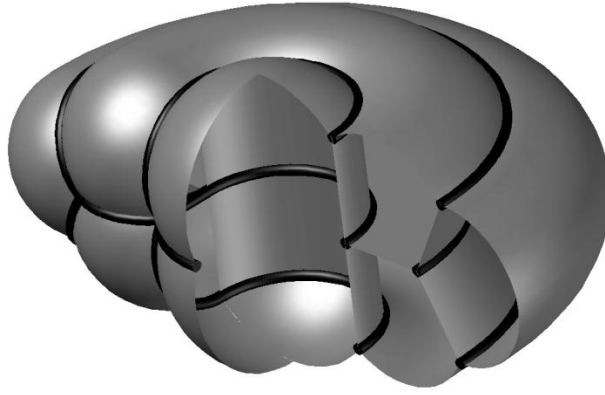
It was shown in the previous section that the solution for loads and forces in the multi-cylinder and multi-sphere simply depends on pressure and geometrical variables. This is also the case for the multi-torus, regardless of the configuration. Figures 6.30 and 6.31 show a multi-torus with respectively 4 and 3 intersecting membranes. Both figures have two axes of symmetry in order to clarify the notation. The multi-torus in both figures is coupled with a multi-sphere in order to show that the circumferential forces in the membranes and reinforcements of the multi-torus are identical to the forces in the membranes and reinforcements of the multi-sphere.



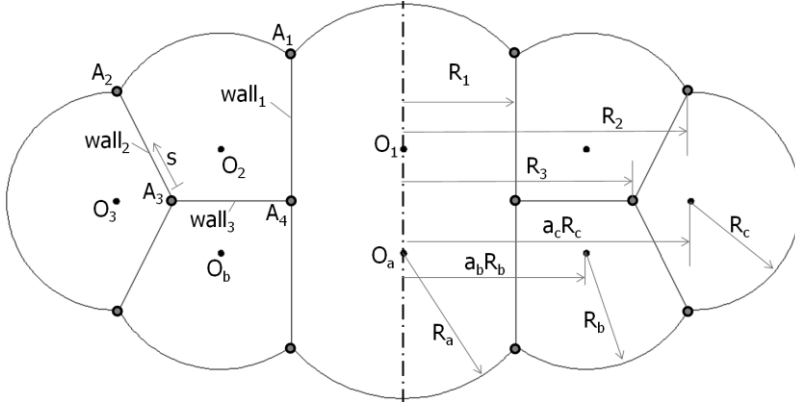
**Fig. 6.30a: 3D-view of a multi-torus with 4 intersecting membranes**



**Fig. 6.30b: Corresponding cross-section of the multi-torus shown in Fig. 6.30a**



**Fig. 6.31a: 3D-view of a multi-torus with 3 intersecting membranes**



**Fig. 6.31b: Corresponding cross-section of the multi-torus shown in Fig. 6.30a**

The forces in the membranes are defined in eqs. (6.12) and (6.13). The loads in the reinforcement rings, shown in figure 6.30b as  $A_{1r}$ ,  $A_{2r}$ ,  $A_{3r}$ ,  $A_{4r}$  and  $A_{5r}$ , are:

$$\begin{aligned}
 T_{A_1} &= p \cdot [O_1 A_1 O_2] \\
 T_{A_2} &= p \cdot [O_2 A_2 O_3] \\
 T_{A_3} &= p \cdot [O_3 A_3 O_c] \\
 T_{A_4} &= p \cdot [O_2 O_3 O_c O_b] \\
 T_{A_5} &= p \cdot [O_1 O_2 O_b O_a]
 \end{aligned} \tag{6.35a-e}$$

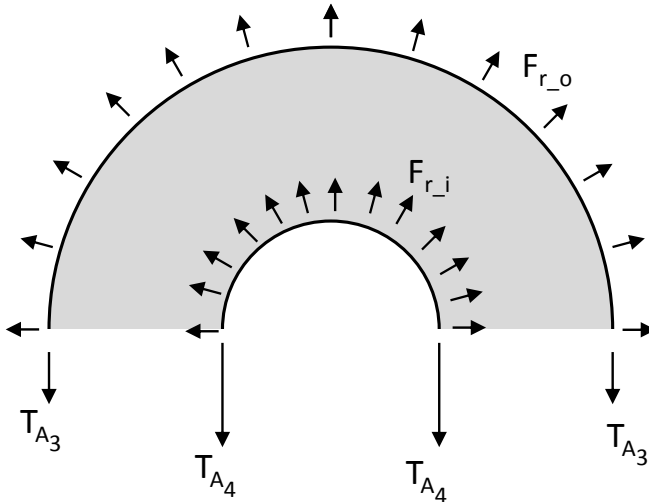
where  $[xxx]$  or  $[xxxx]$  denotes the area governed by three or four points.

The axial force in the cylindrical wall ( $wall_1$ ) is defined as the pressure working on the surface area of a flat ring characterized by the origins  $O_1$  and  $O_2$ . Similarly, the axial force in  $wall_2$  is defined as the pressure working on the surface area of a flat ring characterized by the origins  $O_2$  and  $O_3$ . The forces in  $wall_1$  and  $wall_2$  therefore become:

$$F_{y,wall_1} = \frac{p(a_b R_b)^2}{2R_1} \quad (6.36a,b)$$

$$F_{y,wall_2} = \frac{p[(a_c R_c)^2 - (a_b R_b)^2]}{2R_2}$$

The radial force in  $wall_3$  is defined as the pressure working on the surface area of a cylindrical segment characterized by the origins  $O_3$  and  $O_c$ . The radial forces vary depending on the distance from the axis of rotation. The radial force on the inside of  $wall_3$ ,  $F_{r_i}$ , is higher than the radial force on the outside,  $F_{r_o}$ , since the same load needs to be distributed over a respectively shorter and longer arc length. This is visualized in figure 6.32. The wall thickness on the inside of the horizontal wall is therefore higher than on the outside.



**Fig. 6.32: Schematic of the loads and forces in the reinforcement rings and  $wall_3$  of the multi-torus with its cross-section shown in Fig. 6.30b**

The radial force on the outside and inside of  $wall_3$  is respectively:

$$F_{r_o, wall_3} = \frac{pa_c R_c |O_3 O_c|}{R_3}, \quad F_{r_i, wall_3} = \frac{pa_c R_c |O_3 O_c|}{R_2} \quad (6.37a,b)$$

where  $/xx/$  denotes the distance between two points.

The radial forces in  $wall_4$  are found in the same manner:

$$F_{r_o, wall_4} = \frac{pa_b R_b |O_2 O_b|}{R_2}, \quad F_{r_i, wall_4} = \frac{pa_b R_b |O_2 O_b|}{R_1} \quad (6.38a,b)$$

The same methodology holds for the multi-torus in figure 6.31 with the difference that the force in  $wall_2$ ,  $F_{s_i}$  is defined as the pressure working on the surface area of a *conical* segment characterized by the origins  $O_2$  and  $O_3$ . The force in  $wall_2$  also depends on the distance from the axis of rotation. The forces and loads in the walls and reinforcement rings for the multi-torus in figure 6.31 are defined as:

$$\begin{aligned} T_{A_1} &= p \cdot [O_1 A_1 O_2] \\ T_{A_2} &= p \cdot [O_2 A_2 O_3] \\ T_{A_3} &= p \cdot [O_2 O_3 O_b] \\ T_{A_4} &= p \cdot [O_1 O_2 O_b O_a] \\ F_{y, wall_1} &= \frac{p(a_b R_b)^2}{2R_1} \\ F_{s_o, wall_2} &= \frac{p|O_2 O_3|(a_b R_b + a_c R_c)}{2R_2}, \quad F_{s_i, wall_2} = \frac{p|O_2 O_3|(a_b R_b + a_c R_c)}{2R_3} \\ F_{r_o, wall_3} &= \frac{pa_b R_b |O_2 O_b|}{R_3}, \quad F_{r_i, wall_3} = \frac{pa_b R_b |O_2 O_b|}{R_1} \end{aligned} \quad (6.39a-g)$$

Looking at the configuration in figure 6.30, it has to be noted that reinforcement ring  $A_4$  and  $A_5$  can be left out together with  $wall_3$  and  $wall_4$ . In that case, the load in ring  $A_3$  changes into:

$$T_{A_3}^* = p \cdot [O_1 O_3 A_3 O_c O_a] \quad (6.40)$$

This simplifies the design but it eliminates the possibility to connect the multi-torus to the multi-sphere.

The horizontal wall is bi-axially loaded in case the multi-torus is combined with a closed cell multi-sphere, where the external rings of the multi-sphere are replaced by a circular wall. In this case there are no reinforcement rings and there are no thickness variations in the walls of the multi-torus.

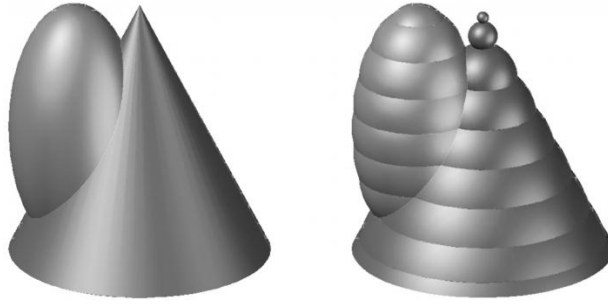
An extremely important issue for the validity of the aforementioned derivations is that the deformations must be compatible. When the stiffness of the horizontal wall is too high, the inner reinforcement ring will carry too much load and vice versa. In particular for multi-bubbles, it is important that every element undergoes the same deformation.

### **6.3 The analysis of conformable pressure vessels built up from axi-symmetric membrane elements**

The objective here is to assess the membrane forces in conformable pressure vessels of which the pressure vessel is built up from arbitrary articulated axi-symmetric shells. An example is given in figure 6.33.

The derivation of the membrane forces at the intersections of the different elements is very complex because the membrane forces are based on local reference systems. The analysis of the forces at the intersection of the ellipsoid and the cone would first require a search for intersecting coordinates. In order to do this, both bodies (the ellipsoid and the cone) need to be expressed in one global coordinate system. Cartesian coordinates would be suitable for this. The next step is to express these coordinates in the local curvilinear reference system. This is a complicated task because there is no general coordinate transformation that enables this. Lastly, the decomposition of the membrane forces tangential and perpendicular to the surface of intersection is very elaborate and complex.

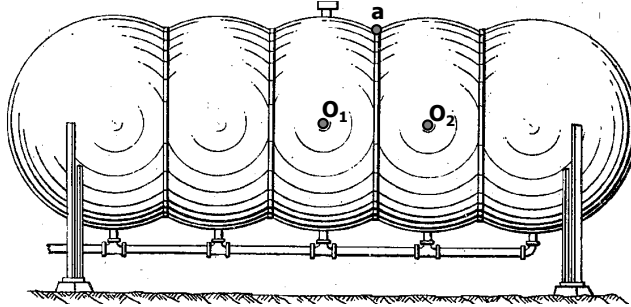
There is a different method that can be used to determine the force distribution at the surface of intersection. The way to demonstrate this is by reducing the axisymmetric shell into a multi-spherical structure.



**Fig. 6.33 a) Intersecting cone and spheroid, b) the multi-spherical equivalent of the intersecting cone and spheroid**

It was already demonstrated that multi-spherical structures are always in equilibrium. It can now be shown that any axi-symmetric shell can be reduced to a multi-spherical structure and hence, any conformable structure build up from axi-symmetric shell elements is therefore always in equilibrium.

Suppose that Jackson's reservoir in figure 6.34 is an open-cell multi-spherical cylinder. The cylinder in figure 6.34 is composed of five intersecting spherical tanks.



**Fig. 6.34: The multi-spherical tank patented by Jackson in 1944**

The forces in the spherical membranes are defined as:

$$N_{\phi,s} = N_{\theta,s} = \frac{pR}{2}$$

For the open cell configuration, the tensile load  $T$  in the ring is defined as:

$$T = F_s |o_1 a| = p \cdot [o_1 a o_2]$$

In case an infinite number of spheres are placed in an open cell configuration between  $o_1$  and  $o_2$ , i.e.  $o_2 = o_\infty$ , a cylinder would be created in which the rings are smeared out over the surface and are responsible for a change in the force distribution. In that case, the radius of the ring equals the radius of the spheres and the membrane force in the cylinder can be expressed as:

$$N_{\phi,C} = N_{\phi,S} + \frac{1}{|o_1 o_\infty|} \left( \lim_{n \rightarrow \infty} \sum_{i=1}^n T_i \right) = \frac{pR}{2} + \frac{pR \lim_{n \rightarrow \infty} \sum_{i=1}^n |o_i o_{i+1}|}{2|o_1 o_\infty|} = \frac{pR}{2} + \frac{pR}{2} = pR$$

$$N_{x,C} = N_{\theta,S} = \frac{pR}{2}$$

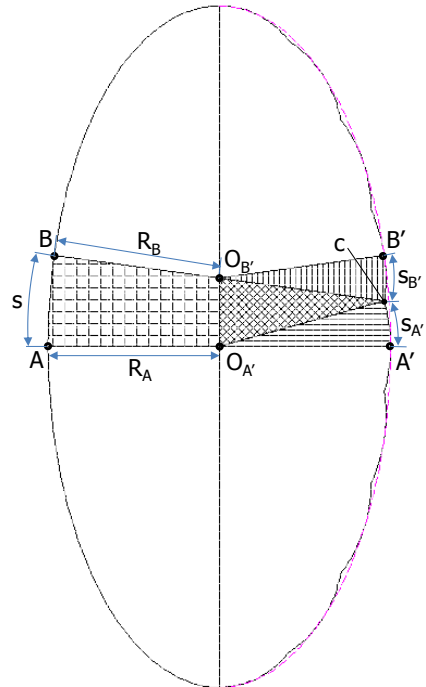
These loads correspond to the membrane forces for cylinders. All other shells can be reduced in the same manner to an open cell multi-sphere because the interconnected spheres are not required to have the same diameter. This is demonstrated in figure 6.35.

Figure 6.35 shows the cross-section of a prolate spheroid. The left side shows the original ellipse and the right side represents the multi-spherical equivalent. Considering for example point B (or B'), we notice that the hoop force at this place is  $pR_B/2$  for both the spheroid as its multi-spherical equivalent. Similarly, the average meridional force between point A and B is defined as:

$$N_\phi = \frac{p[ABO_{A'}O_{B'}]}{S}$$

Where  $[ABO_{A'}O_{B'}]$  represents the hatched area in figure 6.34. The average meridional force between A' and B' is given by:

$$N_\phi = \frac{p([AO_{A'}c] + [BO_{B'}c] + [O_{A'}cO_{B'}])}{S_{A'} + S_{B'}}$$

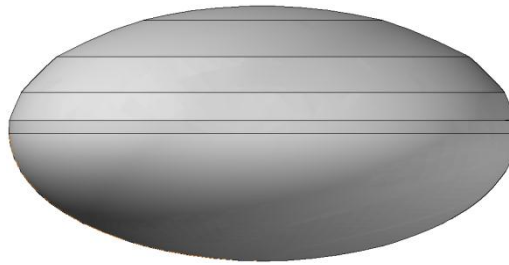


**Fig. 6.35: Cross-section of a (multi-spherical) prolate spheroid**

It is no surprise that when an infinite number of spheres are placed between OA' and OB', both sides of the cross-section become identical.

All pressurized structures can be represented by multi-spherical structures. This was already implied in the text referring to figure 3.7. Considering for example the oblate spheroid shown in figure 6.36, it can be seen that the intersections are pointed; this implies that the rings are loaded in compression and the rings therefore reduce the meridional forces, as already shown in chapter 2.

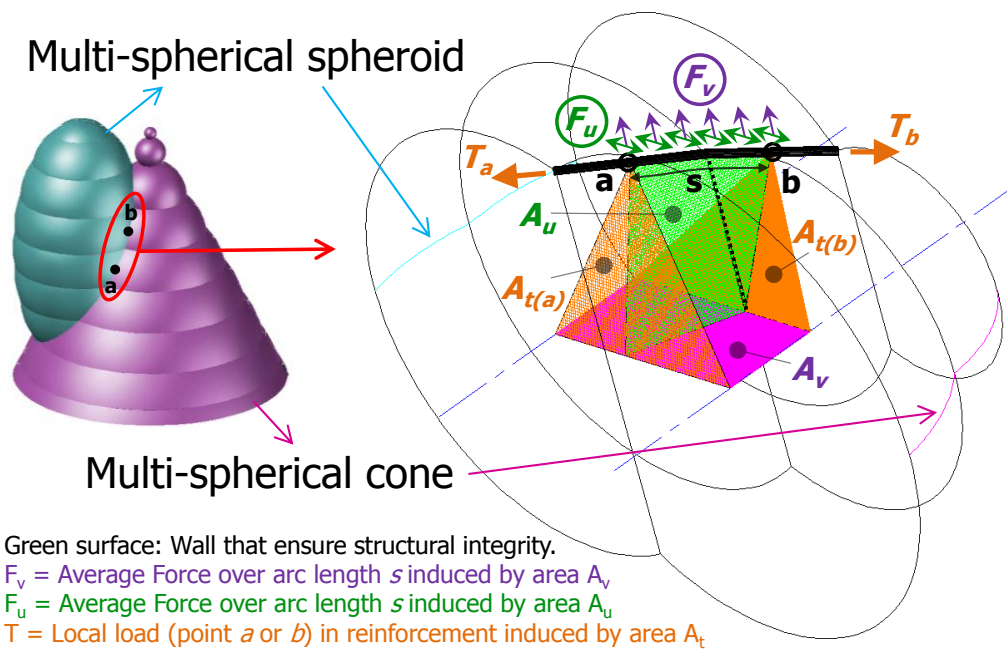
Although it is a difficult task to find analytical solutions for conformable pressure vessels built up from random axi-symmetric membrane elements, the membrane forces at the intersection can easily be assessed by correct visual assessment of surface area and arc length. The visual assessment of the membrane forces in conformable pressurized structures is illustrated by means of the examples in figures 6.37 and 6.38.



**Fig. 6.36: An oblate spheroid. The lower half shows the original smooth oblate spheroid and the upper half is its multi-spherical equivalent**

Figure 6.37 shows a detail of the intersection of the conformable structure in figure 6.33. The visual assessment is similar with respect to the visual assessment in section 6.1.4. There are two forces and one load that need to be considered. The average forces are considered between two points ( $a$  and  $b$ ) at the line of intersection between the cone and the spheroid.

The average tensile force components,  $F_u$ , that balance each other out at the intersection of the multi-bubble are found by multiplying the pressure with the ratio of the area  $A_u$  to the arclength  $s$ . The force  $F_v$  is found by multiplying the pressure with the ratio of the area  $A_v$  to the arclength  $s$ . The force  $F_v$  results from the load  $V$  (see chapter 6.2) that is distributed over the surface  $A_v$ . The load in the reinforcement at the line of intersection is found by multiplying the pressure with the triangular area  $A_t$ .



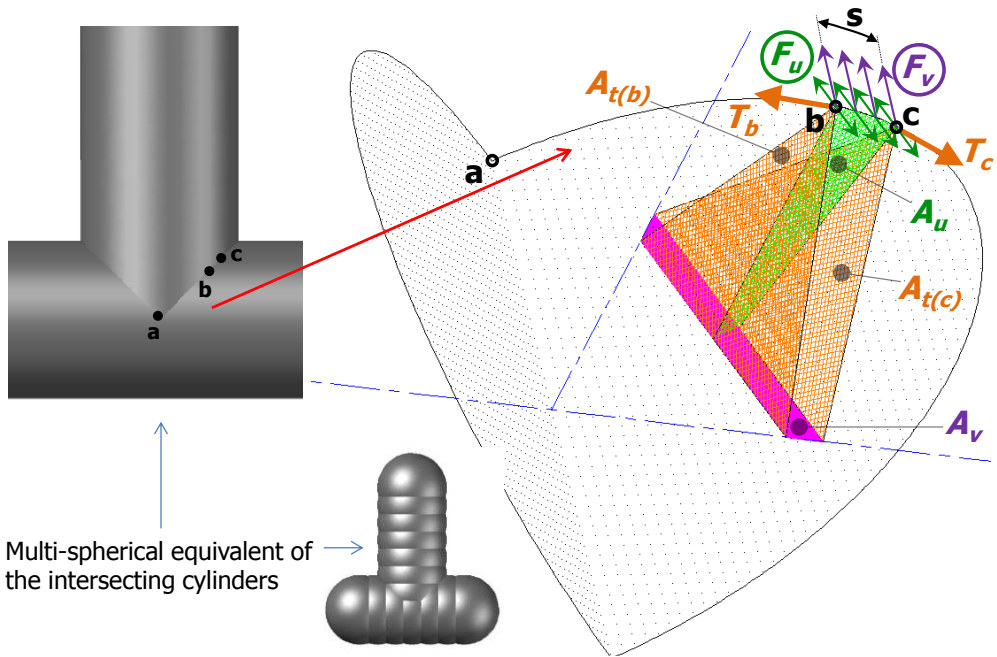
**Fig. 6.37: Visual assessment of forces and loads in arbitrary conformable pressure vessels**

All areas,  $A_u$ ,  $A_v$  and  $A_t$  are perpendicular with respect to the force or load vectors they are related to. Furthermore, the intersecting surface  $A_u$  is perpendicular with respect to  $A_t$  and  $A_v$ . The wall, surface  $A_u$ , is formed by the intersecting surface of the intersecting axis-symmetric bodies. The surface  $A_v$  is defined by the lines that connect the circular cross-sections. Generally, this is an axis of rotation but it was shown in section 2.4 that this can also be an arc. Finally, the triangular area  $A_t$  is perpendicular with respect to the line of intersection and is defined by a point at the line of intersection and the points where the perpendicular area intersects with the lines that connect the circular cross-sections.

Any configuration of intersecting axis-symmetric shells is valid. In figure 6.37, the axes are parallel with respect to each other. Figure 6.38 shows axes of revolution that are perpendicular with respect to each other.

The load  $T$  can also be distributed over the intersecting surface in a similar way as  $F_v$  results from the distributed load  $V$  in the reinforcement rod of multi-spherical pressure vessels. In contradiction with the examples shown in section 6.1.4, it is

noticeable that the forces and loads at the intersecting line and surface are not constant and vary depending on the location of the intersecting surface. This has implications on the manufacturability of these conformable structures and it is expected that in many industrial applications, instead of opting for the optimal membrane, the intersecting surface will have a constant thickness and will be over-dimensioned in order to provide structural integrity.



**Fig. 6.38: Visualization of distinctive surfaces at the intersection of the conformable pressure vessels**

## 6.4 The structural efficiency of optimal (conformable) pressurized structures

In order to investigate the structural efficiency of conformable pressurized structures, the structural efficiency of its building blocks needs to be investigated. Since all (conformable) pressure vessels can be reduced to multi-spherical pressure vessels, we only need to investigate the structural efficiency of the multi-spherical building blocks. For clarification, the analysis of the building blocks is explained via the conformable pressure vessel illustrated in figure 6.39. From the previous chapters it is known that the structural efficiency is optimal when the strain in the structural elements is everywhere and in all directions the same.

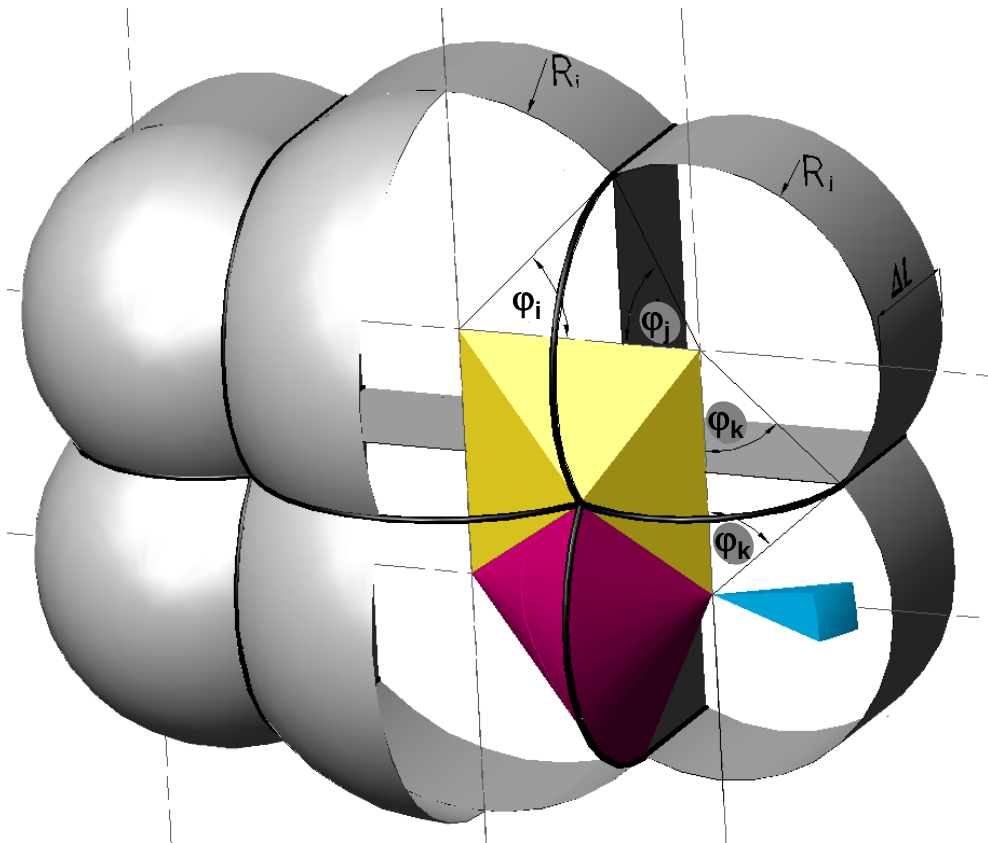


Fig. 6.39: Illustration of the multi-spherical building blocks

The exact formula for the structural efficiency depends on the failure analysis. Netting theory in this case is used for simplicity, also because weight penalties at the intersections are inevitable. This is addressed in the following chapter. Netting theory assumes that in case of the use of stiff fibers in a relative non-rigid matrix, the influence of the matrix can be discarded. Discarding the matrix means that the structure can only carry loads in the direction of the fibers. The direction of the principal stresses in the multi-bubble are perpendicular to each other and since netting theory is used for the analysis, the material needed to carry the hoop stresses is uncoupled from the material to carry the meridional stresses. The structural efficiency for the building blocks is analysed by working out the left hand side of equation 5.7. The volume is the pressurised volume accommodated in the building block. Summation of all the building blocks accounts therefore for the total volume of the conformable structure. The mass is found by multiplying the density with the volume of the structural element.

There are 3 different types of building blocks. The first building block of the multi-sphere is the spherical membrane element. The structural efficiency of the spherical element can be derived by dividing the volume element with its corresponding area element. These elements are expressed in spherical coordinates. The structural efficiency for the spherical membrane element gives:

$$\frac{pV}{m} = \frac{p \cdot \Delta V}{\rho \cdot \Delta S \cdot (t_\phi + t_\theta)} = \frac{p \cdot \iiint R^2 \sin \theta dR d\theta d\phi}{\rho \cdot \iint R^2 \sin \theta d\theta d\phi \cdot (t_\phi + t_\theta)} = \frac{pR}{3\rho(t_\phi + t_\theta)} \quad (6.41)$$

where  $\Delta V$  is the spherical volume element,  $\Delta S$  is the spherical surface area element,  $p$  is the pressure,  $\rho$  is the density and  $t$  is the required membrane thickness in meridional ( $\phi$ ) or hoop ( $\theta$ ) direction.

From linear membrane theory is known that:

$$\begin{aligned} t_\phi &= \frac{N_\phi}{\sigma} = \frac{pR}{2\sigma} \\ t_\phi &= t_\theta \end{aligned} \quad (6.42a,b)$$

where  $\bar{\sigma}$  denotes the maximum allowable stress. A simple expression for the structural efficiency is found when expressions 6.42 are substituted in equation 6.41 and simplified:

$$\frac{PV}{m} = \frac{\bar{\sigma}}{3\rho} \quad (6.43)$$

The second building block is the reinforcement ring that runs over the intersection of the two spherical membranes. The volume that this element accounts for consists of 2 partial cones. In figure 6.39, the cones have  $R_j \sin \phi_j$  as base-radius and  $R_j \cos \phi_j$  as height for one cone and  $R_i \cos \phi_i$  as height for the other cone. Considering a conical element gives:

$$\frac{pV}{m} = \frac{p \cdot \Delta V_{cone}}{\rho \cdot \Delta V_{ring}} = \frac{p \cdot \Delta V_{cone}}{\rho \cdot A_{ring} \cdot \Delta s_{ring}} \quad (6.44)$$

where  $\Delta V_{cone}$  is the volume of the cone that this ring takes account for (see figure 6.38),  $\Delta s_{ring}$  is the arc length of the ring and  $A_{ring}$  is the required cross-sectional area of the ring which depends on the load and the maximum allowable stress in the ring. From section 6.1 is known that:

$$A_{ring} = \frac{T}{\sigma} = \frac{pA_t}{\sigma} \quad (6.45)$$

Simple geometry gives:

$$\frac{\Delta V_{cone}}{A_t \cdot \Delta s_{ring}} = \frac{1}{3} \quad (6.46)$$

Substituting the expressions 6.45 and 6.46 in 6.44 leads to:

$$\frac{PV}{m} = \frac{\bar{\sigma}}{3\rho}$$

The third building block is the rod that carries the axial load  $V$  (section 6.2). The volume that this element accounts for is a pyramid of which the base is area  $A_v$  that was explained in the previous section. The third building block gives:

$$\frac{pV}{m} = \frac{\rho \cdot V_{pyr}}{\rho \cdot V_{rod}} = \frac{\rho \cdot V_{pyr}}{\rho \cdot A_{rod} \cdot L_{rod}} \quad (6.47)$$

Where  $V_{pyr}$  is the volume of the pyramid that this rod takes account for (see figure 6.39),  $L_{rod}$  is the length of the rod and therefore the height of the pyramid and  $A_{rod}$  is the required cross-sectional area of the rod.

Furthermore, we have the relations:

$$A_{rod} = \frac{V}{\sigma} = \frac{pA_v}{\sigma} \quad (6.48)$$

$$\frac{V_{pyr}}{A_v \cdot L_{rod}} = \frac{1}{3} \quad (6.49)$$

Resulting in:

$$\frac{pV}{m} = \frac{\bar{\sigma}}{3\rho}$$

This leads to the same conclusion as in chapter 5. The structural efficiency is the same for any pressurised shape when the material is in a state of uniform equal bi-axial extension. Theoretically, it makes no difference whether we consider conformable pressure vessels or any other axi-symmetric pressure vessel.

Chapter 7 will zoom in on the practical implications of building conformable pressure vessels where it will become clear that weight penalties near the intersections are inevitable.



# CHAPTER 7

## Design concepts of conformable pressure vessels

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The research described in the previous chapter is theoretical and does not provide structural solutions for the design of conformable pressure vessels. Although the detailed design of conformable pressure vessels is out of the scope of this dissertation, this chapter presents conceptual design solutions.

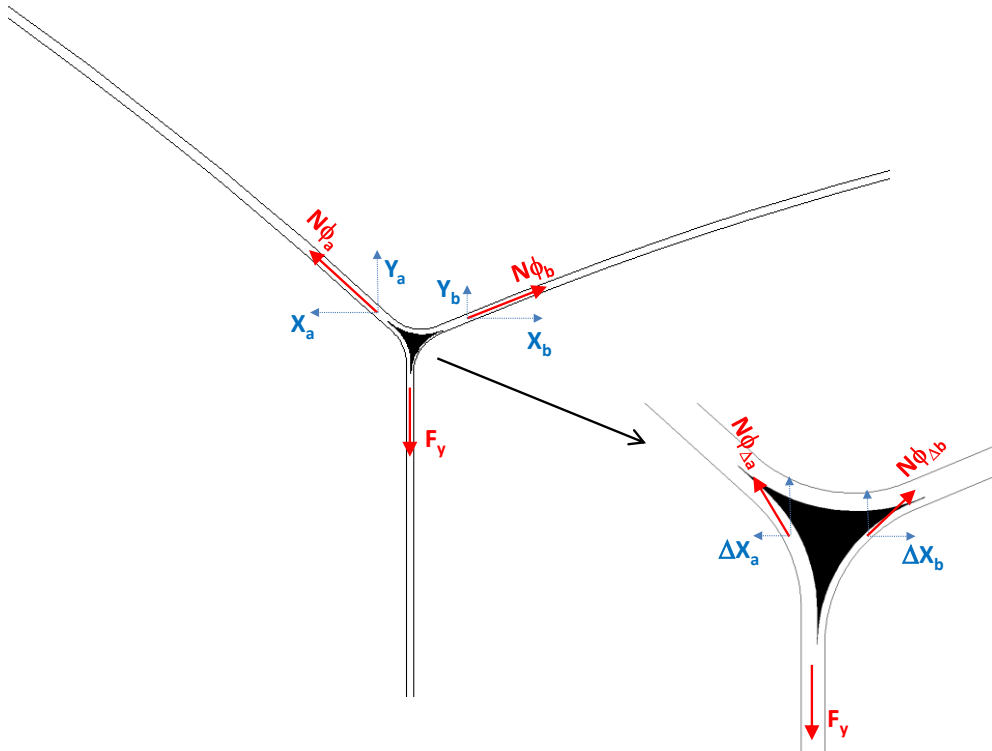
The quality of the design is in the details and this means that proper attention needs to be given to the load transfer at the intersections. First, it will be explained why the load transfer at an intersection is a point of particular interest. The subsequent section presents structural concepts taking all the structural loads into consideration. The structural design is not only dependent on the loads as functional requirements specified by the application need to be taken into consideration as well. Liquid gas and cryogenic tanks are viable applications and design concepts for these applications are therefore presented in the last section.

### 7.1 Load transfer in conformable pressure vessels

The membranes in the multi-bubble can be regarded as normal individual axis-symmetric elements when it concerns the membrane forces. The point of interest therefore lies in the intersection of the intersecting membrane elements.

Imagine a multi-bubble made of fiber reinforced materials. A structural configuration as shown in figure 7.1 is a possible arrangement showing the cross-section of 3 intersecting membranes. The assumption is made that the membrane is a composite laminate, built up from multiple plies. A part of the laminate can therefore branch off to contribute in carrying a part of the force  $F_y$  while the rest of the laminate can contribute to carry the horizontal force  $X_{by}$ , which is equal and opposite to force  $X_{\phi}$ . A detail is shown in figure 7.1, where it is seen that a part of the membrane force,  $N_{\phi}$ , is carried by a branch, denoted with  $N_{\Delta\phi}$ . When zooming

in on this branch, it is inevitable (from a practical point of view) that there are secondary forces, in this case illustrated by the horizontal force resultants  $\Delta X$ . These secondary forces are a point of concern because these forces need to be carried by the material that is encapsulated in-between these membranes. The matrix materials typically found in fibre reinforced materials would not be able to handle these loads and are prone to cracking. Caution and a good understanding of the load transfer at the intersection is required in order to make a structurally sound conformable pressure vessel.



**Fig. 7.1: Visualization of the load transfer at the intersection of a multi-bubble**

## **7.2 Structural concepts for fibre reinforced conformable pressure vessels**

The previous section showed that special attention is required for the load transfer at the intersections of conformable pressure vessels. These considerations are especially present for fibre reinforced materials. On one hand, as mentioned in chapter 6, the tailorability of fibre reinforced materials is a big advantage because it allows compatibility of deformations and therefore offers predictable behaviour and opportunities for a high structural efficiency. On the other hand, composites only have in-plane strength and stiffness. Due to the lack of strength and stiffness in the third plane, it is challenging to create a structurally sound pressure vessel at the intersection of the 3 intersecting membranes. The concerns of secondary loads are not present at the intersection of metallic multi-bubbles but metals do not possess anisotropic properties and are less advantageous with respect to structural efficiency and compatibility of deformations. This section presents several structural concepts for fibre reinforced conformable pressure vessels, taking the considerations at the intersection into account.

A distinction is made between closed-cell concepts and open-cell concepts. Closed-cell-concepts carry the load  $T$  (see chapter 6.3) via the wall or intersecting surface while open-cell-concepts carry this load by means of a reinforcement located at the line of intersection. The intersecting surface is a bi-axially loaded wall in the closed-cell configuration and a uni-axially loaded wall in the open-cell configuration. A combination of a reinforcement and a bi-axially loaded wall, each taking a part of the load  $T$  is also a possibility but the added value is disputable because it complicates the manufacturing process, hence increasing costs.

The objective of presenting several concepts is to demonstrate the practical implementation of the theory presented in chapter 6. The concepts presented in this chapter are not the only solutions and are meant as inspiration and food for thought. The concepts in this chapter have not been designed in detail and are not validated experimentally.

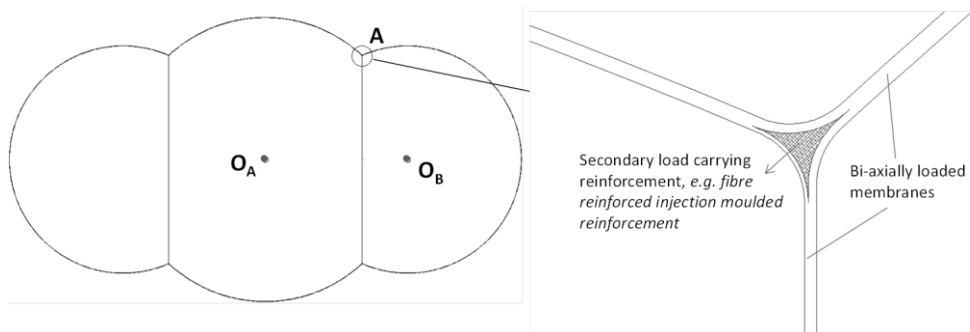
### 7.2.1 Structural concepts for closed cell multi-bubbles

The advantage of the closed cell configuration is that it minimizes the impact of local leaks and simplifies the manufacturing process if liners or gas impermeable bags are used in the multi-bubble. It is important to note that the pressure in all compartments is kept the same because otherwise the conformable tank will experience pressure levels for which it is not designed.

#### 7.2.1a: Closed-cell multi-bubble with an embedded reinforcement

##### **Structural concept**

Concept A, figure 7.2, does not differ schematically from the intersection shown in figure 7.1. The true difference here is that the membranes transfer the secondary loads via an adhesive interface to the reinforcement by shear forces. The reinforcement follows the line of intersection between the intersecting membranes.



**Fig. 7.2: Concept A: An embedded reinforcement carrying secondary loads and the load T**

The concept is simple but the design is complicated when the line of intersection is a curved path.

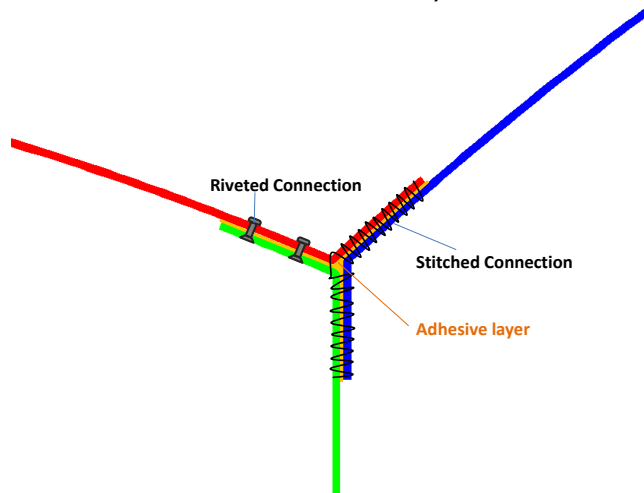
Attention should be given to the stiffness of the reinforcement. A stiff reinforcement will unload the bi-axially loaded wall. The combined stiffness of the reinforcement and the wall should therefore lead to the required deformations in order to maintain compatibility of deformations. Attention should also be given to the CTE-difference between the membranes and the reinforcement, especially for cryogenic applications. The combination of different materials causes different expansions, risking excessive residual stresses and consequently cracks in the material.

## **Manufacturing**

Concept A is created by assembling pre-manufactured reinforcements together with individual manufactured composite shells. The lay-up of the individual composite shells is such that they each take charge of their contribution to the load  $T$  and the force  $F_v$  derived in chapter 6.3. The assembled multi-bubble is subsequently overwrapped with a laminate in such a way that the complete multi-bubble has the required strength and stiffness distribution.

### **7.2.1b: Closed-cell multi-bubble with a joined intersection**

This concept is visualized in figure 7.3. Figure 7.3 shows individual pre-manufactured membranes that can be connected via out-of-plane connection methods such as riveting or stitching. Joining the elements together with the adhesive layer allows for transfer of the secondary loads.



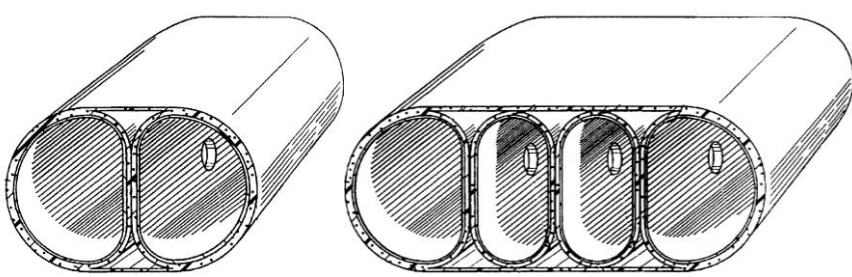
**Fig. 7.3: Concept B, individual membranes stitched or joined together**

This concept is more suitable for closed cell multi-bubbles. Intersecting walls with only uni-axial strength and stiffness (which are present in open-cell multi-bubbles) are more fragile due to the high stress concentrations caused by the out-of-plane connections.

Thin membranes with stitched connections are typically found in inflatable wings. Different joining methods such as riveting can be used when more thick-walled, hard shells need to be joined together.

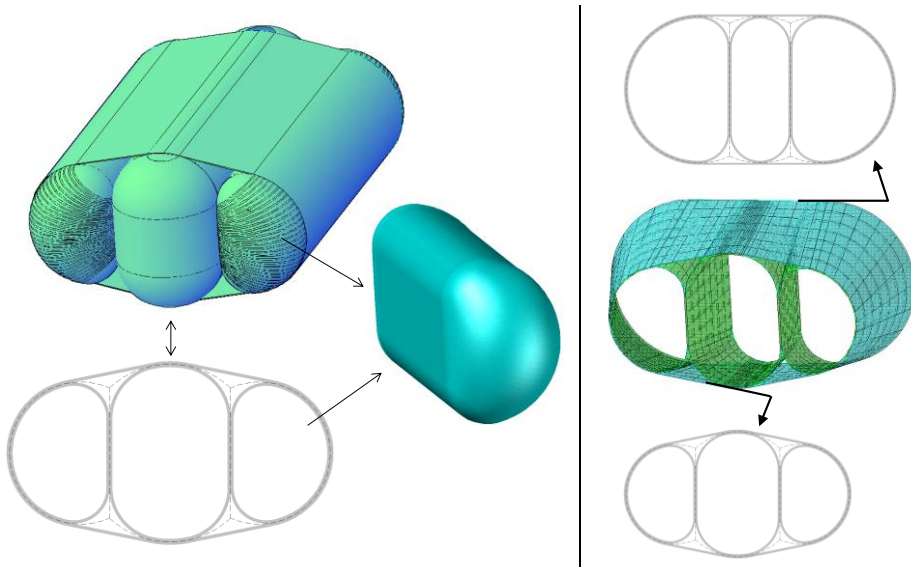
### 7.2.1c: Closed-cell multi-bubble without intersecting membranes

The intersecting membranes complicate the design of conformable pressure vessels. A design for conformable pressure vessels without intersecting membranes was invented and patented in the U.S.A. by [ATK, 2010]. Intersecting membranes are avoided by altering the design such that the membrane forces  $F_u$ , chapter 6.3, are zero at the intersections. This can be achieved by making the radius of curvature at the intersection of the individual bubbles smaller than the large radius of the two outer bubbles as shown in figure 7.4. In this way, the conformable tank can be a composition of individual filament wound cells. The individual cells comprising the tank are wound with a combination of hoop and helical composite layers. The individual cells are then joined together to form a complete tank using winding tooling that allows a final hoop overwrap to be wound over the individual cells.

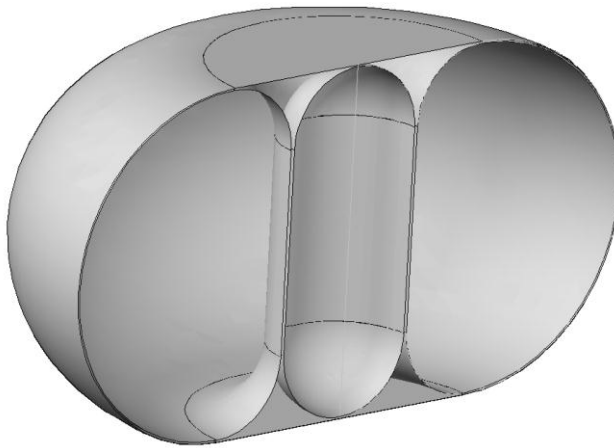


**Fig. 7.4: Patented concept from ATK, a conformable pressure vessel without intersecting membranes**

The conformable tanks of ATK, figure 7.4, are modular, multiple cells can be added in between the two outer shells. The design is now still restricted within a rectangular cuboidal shape but the same concept allows more spatial freedom, ranging from tapered cuboidal shapes to conformable toroidal vessels as is shown in figures 7.5 and 7.6 respectively. The original geometry of the multi-bubble is visualized by the dashed line in the drawings of the cross-sections.

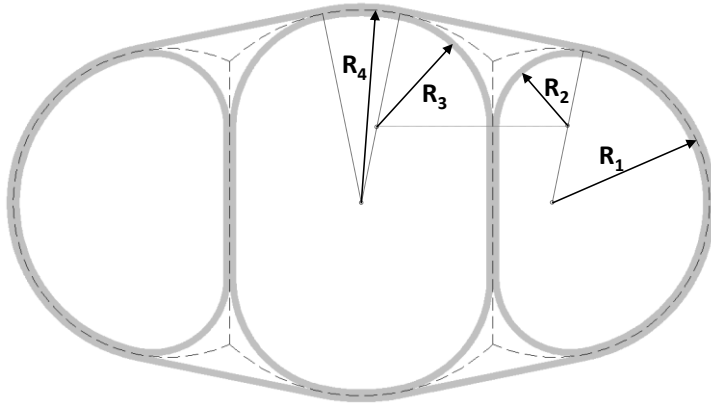


**Fig. 7.5a,b: Visualization of the spatial freedom of the concept shown in figure 7.4**



**Fig. 7.6: Example of a conformable torus without intersections**

The complications at the intersections are eliminated in this concept but the force distribution, especially in the bulkheads is complex. First, the circumferential forces in the cylindrical part of the conformable pressure vessel in figure 7.5a are explained.



**Fig. 7.7: Cross-section of the conformable pressure vessel in figure 7.5a**

The circumferential force  $N_{\phi,C}$  for the cylindrical membranes with the radius  $R_1$ ,  $R_2$ ,  $R_3$  and  $R_4$  is defined as:

$$N_{\phi,C1} = pR_1, \quad N_{\phi,C2} = pR_2, \quad N_{\phi,C3} = pR_3, \quad N_{\phi,C4} = pR_4$$

The circumferential forces in the straight wall of the outer and inner bubble are therefore:

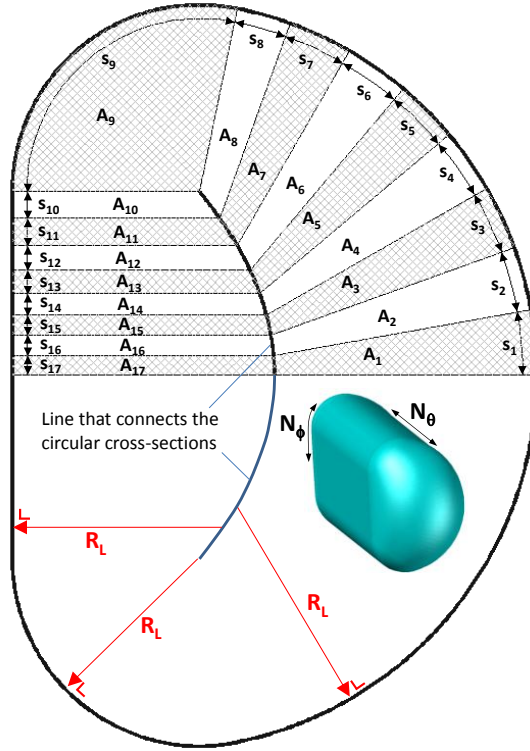
$$N_{\phi,wo} = pR_2, \quad N_{\phi,wi} = pR_3$$

Where the index 'wo' denotes 'Wall Outer bubble' and 'wi' denotes 'Wall Inner bubble'.

These force distributions are also applicable for the conformable vessel shown in figure 7.5b but it needs to be emphasized that  $R_1$  to  $R_4$  are the radii of the parallel circles ( $R_p$ ), taken perpendicular to the shell wall, similar to the cone in figure 2.10. The structural design of this conformable vessel, figure 7.5a, is therefore such that the *individual cells* carry the axial loads induced by the end caps and the circumferential force  $pR_2$  and  $pR_3$  for the outer and inner bubble respectively. The final overwrap of the assembly is uni-axially loaded and carries  $p(R_1-R_2)$  and  $p(R_4-R_3)$ . From the geometry in figure 7.7, it is clear that  $p(R_1-R_2)$  equals  $p(R_4-R_3)$ . For a standard cylinder with an axi-symmetrical end-cap, the axial load is  $pR/2$ . In this case, it would therefore be expected that the axial load of the outer bubble is

$PR_1/2$  and  $pR_2/2$ . This is not the case because the end caps are not axis-symmetrical.

Figure 7.8 visualizes the distinctive areas, lengths and radii to analyse the end-cap for this shape. This shape can be regarded as a multi-spherical pressure vessel, which is clarified by the contour lines shown in the end cap in figure 7.5a. From chapter 2 and 6 we know that we can determine the membrane forces by means of a visual assessment of the geometry.



**Fig. 7.8: Visualization of the distinctive areas, lengths and radii in the end-cap of the conformable pressure vessel shown in figure 7.5a**

The hoop forces in this end-cap, and the axial forces in the cylindrical part of this cell are:

$$N_\theta = p \frac{A_i}{s_i} \quad (7.1)$$

Dividing the end-cap into smaller areas will obviously lead to more accurate forces.

The meridional force in the end-cap is defined as:

$$N_{\phi} = p \frac{R_l}{2} \quad (7.2)$$

where  $R_l$  is the local radius and defined as the distance between the line that connects the circular cross-sections and the point that intersects the meridian perpendicularly.

It shows that the force distribution is constantly varying which means that the optimal stiffness distribution (see chapter 5) is varying as well. Obtaining the optimal membrane for the closed cell multi-bubble without intersecting membranes is a big challenge from a manufacturing point of view. However, chapter 4 has also shown that no stress-concentrations occur as long as the membrane thickness at the intersection of the end-cap with the cylindrical part is small.

The analysis of the end-cap of the inner bubble, shown in figure 7.9, also shows that the force distribution varies over the circumference of the end-cap. The end-cap of the inner bubble corresponds to the definition of a classical pressure vessel, meaning that eqs. (2.11) to (2.13) are applicable with respect to the force distribution and eq. (3.25) is applicable for the optimal stiffness distribution. The meridional radius is infinite for the cylindrical part and  $R_3$  and  $R_4$  for the curved part of the end-cap. The radius of the parallel ranges from  $R_a$  at the cylindrical part to  $R_f$  which equals  $R_4$ .

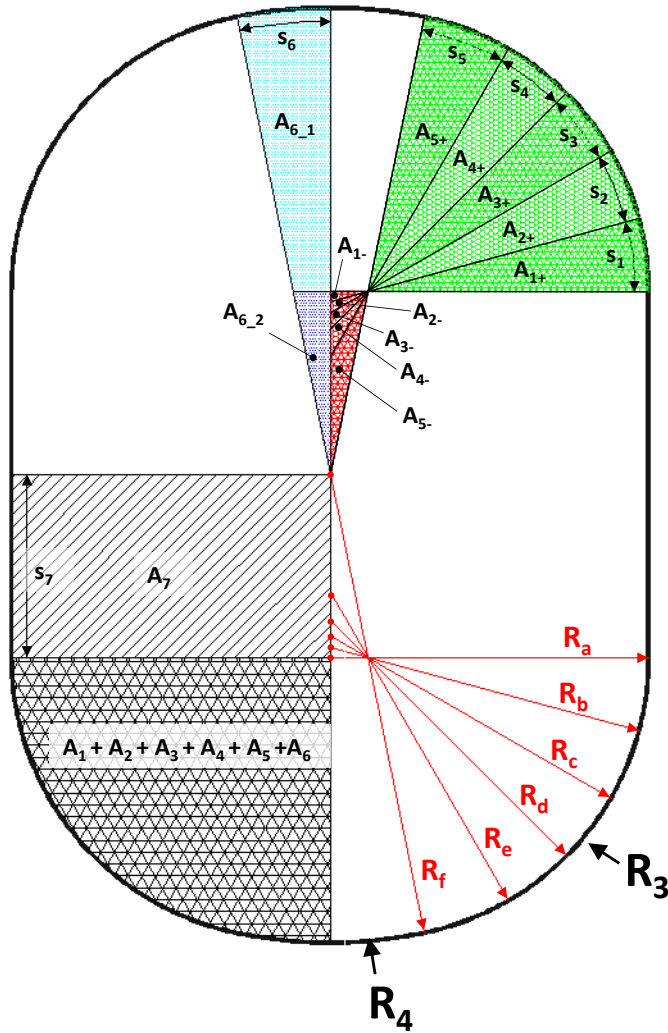
The hoop forces in the end-cap of the inner bubble are the axial forces in the cylindrical part of the inner bubble and are visualised in figure 7.9. Equation (7.1) and (7.2) are also valid but attention needs to be given to the fact that:

$$A_i = A_{i+} - A_{i-}$$

The visualisation of the 'negative areas' was explained in section 2.4. Furthermore, it is seen in figure 7.9 that the sum of all 'negative areas',  $A_{1-}$  to  $A_{5-}$ , equals  $A_{6,2}$ . This means that there is overall equilibrium with respect to the hoop forces, shown by the hatched area indicated on the lower left curved half of figure 7.9.

The meridional forces are also varying in that part of the membrane where  $R_3$  is the meridional radius. The local radius of the parallel ranges from  $R_a$  on one end to

$R_f$  on the other end. From eq. (7.2), it is therefore seen that the meridional force is also varying.



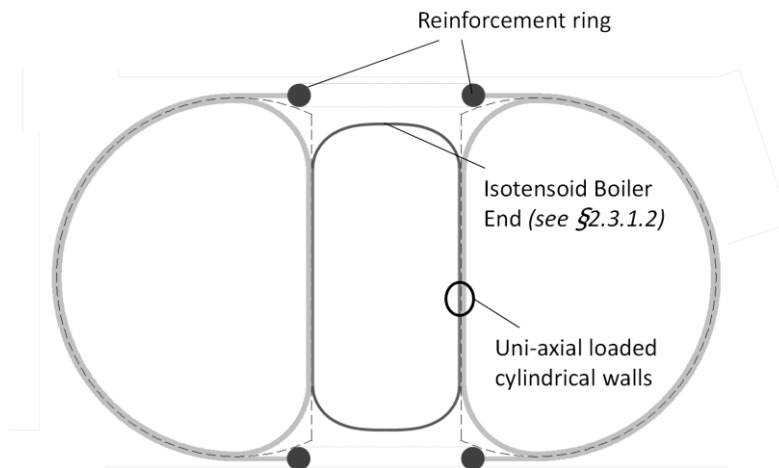
**Fig. 7.9: Visualization of the distinctive areas, lengths and radii in the end-cap of the inner bubble in the conformable pressure vessel shown in figure 7.5a**

The end-caps of the inner bubbles of the concepts shown in figure 7.4 are a lot simpler because these end caps represent simple cylinders with hemi-spherical boiler ends. The end-cap in figure 7.9 is a lot more difficult from a manufacturing point of view when the optimal pressure vessel is desired. The manufacturing of an

optimal pressure vessel with the geometry shown in figure 7.5b is extremely complex knowing that the force distribution is constantly varying at every location in the pressure vessel.

Finally, the multi-torus in figure 7.6 is easier than the conformable pressure vessel in figure 7.5. The multi-torus without intersecting membranes is a collection of regular toroidal membranes with uni-axially loaded cylindrical walls and a final overwrapped membrane that compensates the mismatch in meridional force at the location where the torus with radius  $R_1$  meets the torus with radius  $R_2$  and the torus with radius  $R_3$  meets the sphere with radius  $R_4$  in figure 7.7. There are some practical advantages of the 'flat' multi-torus shown in figure 7.6 with respect to the multi-torus that could be made by using the cross-section in figure 7.7. The overwrapped membrane of the 'flat' multi-torus in figure 7.6 is not part of the inner cell which means that this membrane can be replaced by a reinforcement ring which makes the inner cell interchangeable.

An example of this is shown in figure 7.10. The modularity that was demonstrated in figure 7.4 obviously accounts for the toroidal version of this concept as well.



**Fig. 7.10: Example of a flat multi-torus with no intersecting membranes and exchangeable inner cell**

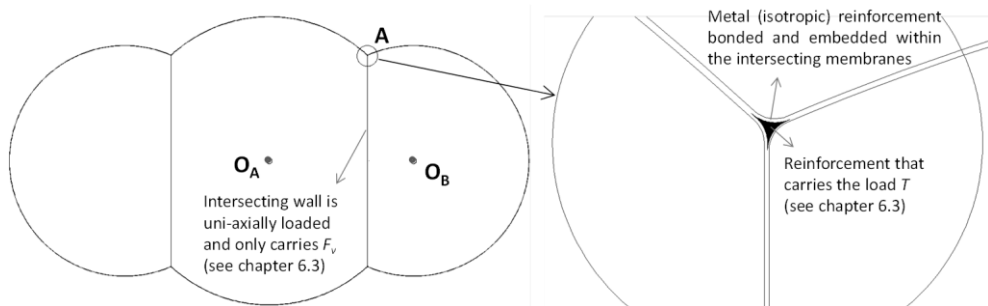
As a final concluding comment, it needs to be addressed that the closed cell multi-bubble without intersecting membranes is the only concept for which the optimal structural efficiency, derived in section 6.4 and 5.2 is theoretically possible. All other concepts inherently require extra material at the line of intersection.

### 7.2.2 Structural concepts for open-cell multi-bubbles

Open-cell multi-bubbles distinguish themselves from the closed cell by the fact that they carry the load  $T$  by a reinforcement located at the line of intersection, as shown in chapter 6.3. An open cell configuration does not necessarily mean that there are open connections at the intersecting surface in between the individual compartments. In fact, this is only the case when intersecting spheres are used in the open cell configuration. Open cell multi-bubbles will become deflated when there is a local leak because the uni-axially loaded wall does not have the ability to carry the pressurization loads in the new configuration. The open-cell multi-bubble as an application is therefore primarily interesting to be used as a conformable pressure cabin where the inner structure is configured as an open structure. Concepts are presented in case manufacturing considerations are the reason to opt for an open cell configuration.

#### 7.2.2a: Open-cell multi-bubble with an embedded reinforcement

Concept A, figure 7.11, does not differ schematically from the intersection shown in figure 7.1. The difference here is that a profile is included that has the ability to carry the load  $T$ . An example could be an embedded metal profile. The metal is isotropic and can therefore handle secondary loads as discussed in section 7.1.



**Fig. 7.11: Concept A<sub>1</sub>: An embedded reinforcement carrying secondary loads and the load  $T$**

The membranes transfer the load via the adhesive interface to the embedded reinforcement by shear forces. The cross-sectional area of the reinforcement ( $A_R$ ) is chosen such that:

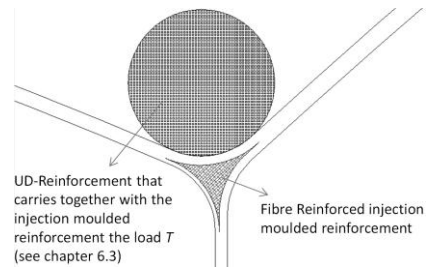
$$A_r = \frac{T}{\varepsilon E_m} \quad (7.3)$$

where  $E_m$  is the modulus of the metal reinforcement,  $\varepsilon$  is the strain in the surrounding membranes and  $T$  is the load in the reinforcement, defined as:

$$T = \frac{p[O_A A O_B]}{2} \quad (7.4)$$

The concept is simple but the design is complicated when the line of intersection is a curved path. Special attention is also required when concept A is subjected to large temperature shocks, e.g. cryogenic use, because the combination of different materials causes different expansions, risking excessive residual stresses and consequently cracks in the material.

An alternative version of concept A is shown in figure 7.12 where a fibre-reinforced injection-moulded reinforcement carries the secondary loads. This injection moulded reinforcement is easier to manufacture when the line of intersection has a complex geometry. The load  $T$  is carried by the injection-moulded reinforcement and a separate unidirectional reinforcement.

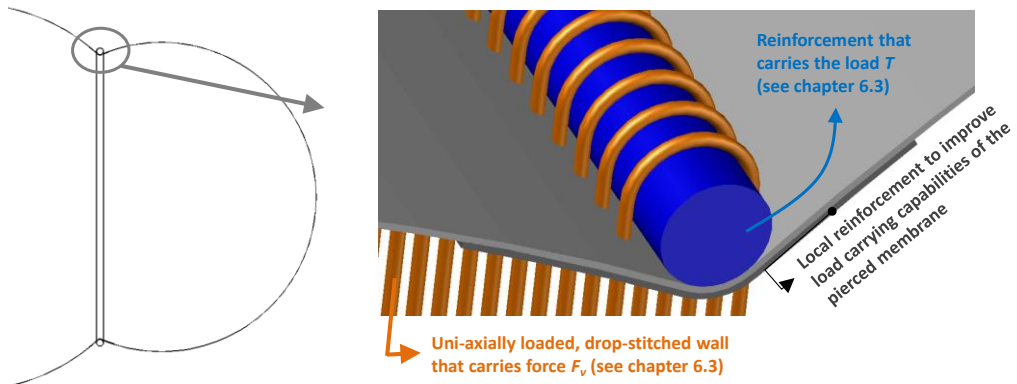


**Fig. 7.12: Concept A2: an embedded fibre reinforced injection moulded reinforcement carrying secondary loads.**

### 7.2.2b: Open-cell multi-bubble with a drop-stitched wall

The beauty of this concept is that it allows the curved membrane to be continuous. Local reinforcement is needed around the area where the membrane/shell is interrupted by the drop-stitched wall, especially when it involves drilled holes in a curved shell. A better load transfer is obtained when the fibres in the membrane are woven around the drop-stitched wall but this poses serious manufacturing challenges. The unidirectional reinforcement carries the load  $T$ . The challenge in this concept is in the sealing of the shell, unless individual liners are used. In the

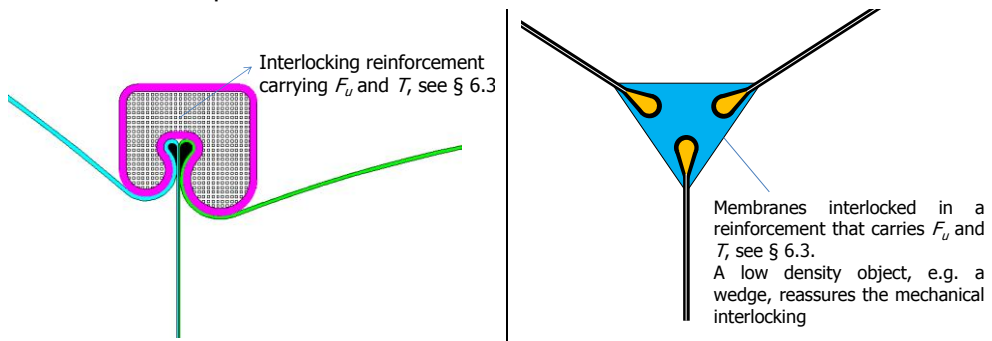
latter case, a separate liner is needed in between the closely spaced walls or a lightweight material such as a sandwich to fill up this space.



**Fig. 7.13: Open Cell Multi-bubble with a drop stitched wall**

### 7.2.2c: Open-cell multi-bubble with interlocking reinforcements

Interlocking reinforcements create a modular design where cells can be replaced by different cells. Two variants are given as example in figure 7.14 but many variations of interlocking reinforcements are possible. The first option in figure 7.14a resembles the connection between a tire and a rim. The main idea of this concept is that it carries the load  $F_{ur}$  which was explained in chapter 6.3. The concept is therefore also applicable for the closed cell multi-bubble. However, due to the inherent material usage of the interlocking connection, it makes more sense to use it for the open-cell multi-bubble.

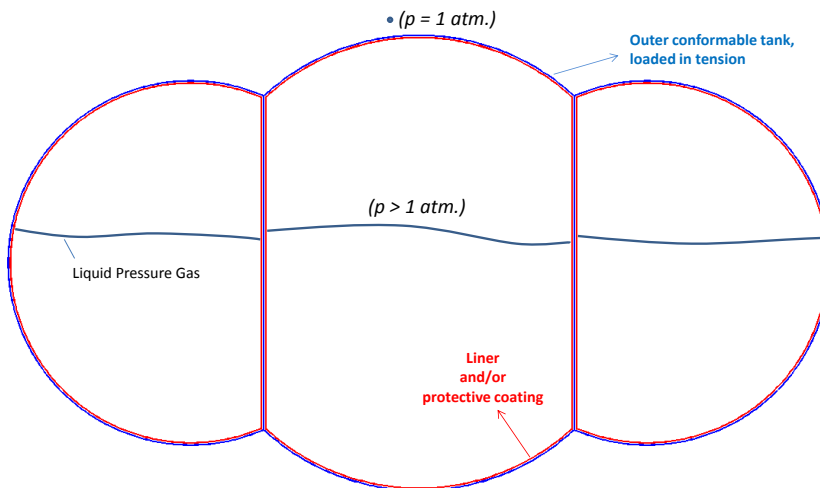


**Fig. 7.14: Two examples of an interlocking reinforcement for the open-cell multi-bubble**

### 7.3 Functional requirements of conformable tanks

Conformable pressure vessels are especially interesting for applications where liquids need to be confined to a pressurised environment. A conformable pressure vessel can use the available space for the tank more effectively since the medium is a liquid and therefore incompressible.

Take for example Liquified Petroleum Gas (LPG), this gas becomes a liquid at a temperature of 15°C and a pressure equal or greater than 8.5 bar. LPG-tanks for cars need to be able to sustain pressures of at least 30 bars. LPG-tanks are in contradiction to cryogenic tanks relatively simple because small pressures are sufficient to keep the medium in a liquid state at room temperature. This means that an ordinary multi-bubble can be used. In the case that fibre reinforced materials are used, an additional liner or coating can be used in order to seal the tank and/or protect the composite from harmful chemical effects caused by the LPG. A concept for a simple conformable liquid pressure gas tank without the need for thermal insulation is shown in figure 7.15.



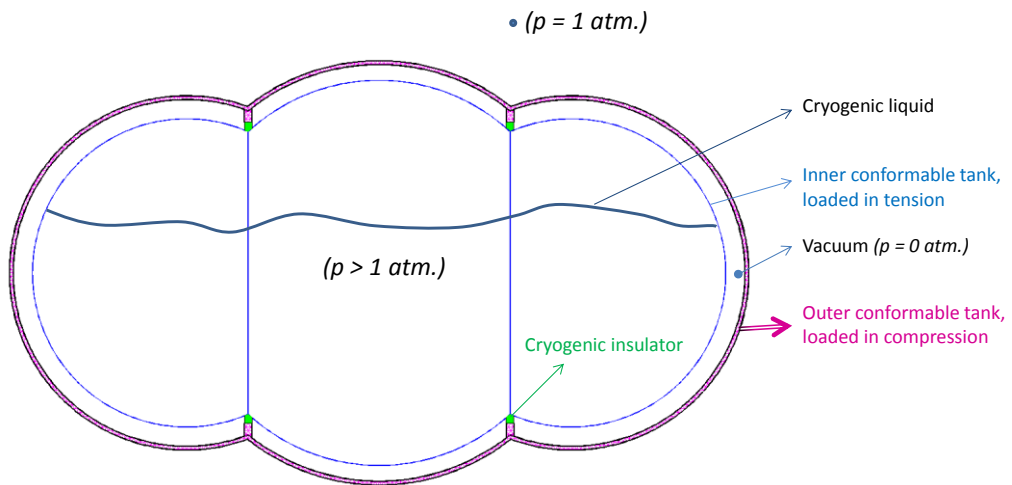
**Fig. 7.15: A conformable tank for Liquid Pressure Gas applications without thermal insulation requirements .** (*Atm. stands for atmospheric pressure*)

Cryogenic tanks require cryogenic insulation and it requires special attention when combinations of materials with different coefficients of thermal expansion (CTE), e.g. composites, are used. There are two concepts identified for conformable cryogenic tanks.

### **Cryogenic Conformable tank type 1**

This concept is a conformable tank consisting of 2 multi-bubbles. A vacuum is applied in between the tanks in order to ensure the cryogenic insulation. The inner tank is therefore loaded in tension. The inner tank can be equipped with a separate liner (see also option 2) or consist of a substrate that surveys the outgassing.

The outer tank is loaded in compression and flexural rigidity is required to prevent the outer shell from buckling. The analysis of the forces in a multi-bubble subjected to external pressure is exactly the same as the forces found in a multi-bubble subjected to internal pressure, only the sign changes, i.e. tensile forces become compressive forces. A sandwich structure can for example secure this flexural rigidity. The inner tank is suspended from the outer tank at the intersections, see figure 7.16, and at the polar caps because this is the place where the tank is filled. The outer tank relieves some of the tension in the wall of the inner conformable tank.



**Fig. 7.16: Concept 1, a cryogenic conformable tank consisting of 2 multi-bubbles with a vacuum in between.**

### **Cryogenic Conformable tank type 2**

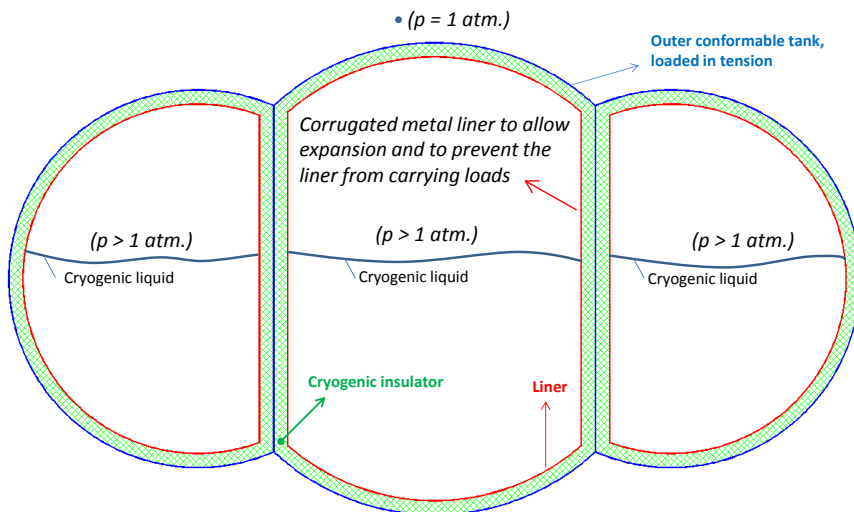
Concept 2, figure 7.17, has cryogenic insulation in between the liner and the outer load-carrying shell. The pressure subjected to the inside surface of the liner is transmitted to the outer shell through the insulation. A corrugated liner ensures

that the liner does not carry the pressurisation loads since the liner will shrink due to the cryogenic temperatures.

Multiple options for the insulation of cryogenic tanks have been investigated ranging from Multi-Layer Insulation (MLI), bulk fill cryogenic insulation materials and foams, [NASA CryoTestLab, 2005].

MLI consists of alternate layers of low-emittance radiation shields separated by low-conductivity spacing material, [Directed Technologies, Inc., 1996]. The radiation shield material is usually made up of very thin aluminium sheets or thin plastic sheets (mylar or kapton) that have been coated by a thin vapor-deposited layer of aluminium, gold, or some other high reflectivity material. Spacer materials are usually silk, nylon, or dacron netting. The insulation volume is evacuated to a near-vacuum to minimize gas convective thermal transfer.

Foam and Bulk Fill insulation material has generally a lower thermal insulation performance than MLI but it is cheaper and easier to process, especially when it involves doubly curved geometries. Examples of Bulk-Fill Cryogenic Insulation Materials are glass bubbles, perlite powder and aerogel beads. Examples of foams are polyimide foams and polyurethane foams that are contemporary choices.



**Fig. 7.17: Concept 2, a cryogenic conformable tank consisting where the liquid is separated from the multi-bubble by a liner and cryogenic insulator**

The trade-off between the two types is cost related, and mainly depends on the extra costs involved by making a second multi-bubble versus the costs involved by implementing the cryogenic insulation.

## **7.4 Concluding comments**

This chapter presented structural concepts and conceptual designs to be used for the realization of conformable pressurized tanks. The context in this chapter is based on the assumption that there are no restrictions on costs and manufacturability. Drawbacks of an optimal stiffness distribution need to be investigated with respect to manufacturability and costs.

Detailed design and prototyping of demonstrators is required to assess the feasibility of the presented concepts; this will undoubtedly give inspiration for new concepts. Acquisition for the development of a demonstrator of a cryogenic conformable tank has been initiated and granted as a continuation of this research. It is very likely that the optimal stiffness distribution is not optimal in real life for which costs are the design driver. Deviating from the optimal membrane means however that the force distribution and the deformations need to be re-examined, especially when the shell is thick-walled rather than a membrane. Qualitative insight is provided in chapter 2 to 6 to assess the effects of a non-optimal membrane. Quantitative results are obtained by reformulating the non-linear membrane theory to the specific design that is investigated. Other methods such as FEM can also be considered, especially for thick-walled applications.

It was mentioned in the introduction of chapter 6 that conformable pressure vessels with full spatial freedom have not been realised yet, mainly due to an insufficient understanding of conformable pressurised structures.

It can be concluded that the feasibility of conformable tanks with full spatial freedom is shown from a technological perspective.

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# CHAPTER 8

## Pressurization of the Blended Wing Body

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This research project fell under the CleanEra-project, a multi-disciplinary project where a team of researchers investigated new technologies to enable drastic reduction of the environmental impact of future aircraft. The focus was the Blended Wing Body (BWB) as shown in figure 8.1. The pressurisation of the cabin area of this aerodynamically superior aircraft poses challenges for aerospace engineers.



**Fig. 8.1: The Silent Aircraft BWB Design**

This chapter is a case-study and shows the involvement of the conformable pressure cabin in BWB Aircraft. First, an introduction of the BWB Aircraft is given, motivating the interest in this concept and addressing the technological challenges that have to be overcome. One of these challenges is obviously the design of the pressure cabin and section 8.3 introduces possible solutions. The feasibility of this new pressure cabin concept does not only entail structural considerations. Considerations with respect to passenger acceptance and cabin configuration also come into play. The interior configuration of the conformable pressure cabin is therefore presented in section 8.4. The concepts presented in this chapter represent *ideas* of the aircraft of the future.

## 8.1 Introduction to the Blended Wing Body Aircraft

Since the 1980s, the airline industry has gone through multiple changes with a more ruthless business model becoming prevalent. Inflating fuel prices, increasing competition, a boost in passenger numbers and heightened environmental awareness have deemed the era of conventional aircraft design stagnant. Recent preliminary studies [Hansen et al., 2008] stress that overcoming such factors is not within the scope of previous small step and iterative design procedures. These studies highlight the potential benefits in shifting towards unconventional aircraft design in order to combat the adverse effects that the airline industry now must face. Boasting qualities such as low fuel consumption and reduction in noise, the (BWB) aircraft concept is a case in point and has recently emerged at the forefront of aircraft design.

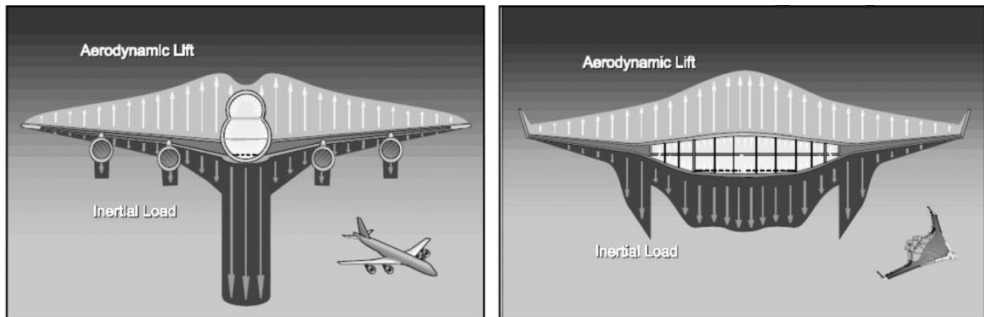
This BWB design is not a new concept. Design steps towards the configuration date back to the before the Second World War. An example is illustrated in figure 8.2, showing a BWB Bomber of the Horten Brothers that made its maiden flight in 1945.



**Fig. 8.2: Drawing of the Horten H.IX [www.warbirdsresourcegroup.org]**

Conceptual studies show that fuel savings of around 30% and a lower acoustic signature are expected from this new aircraft configuration, [Liebeck, 2004]. The absence of a separate empennage in combination with the reduced wetted area is the most significant contributor in support of these claims. Apart from the

aerodynamic advantages, also a structural advantage is expected over conventional tube-and-wing (TAW) aircraft. The structural efficiency is improved due to a lower wing loading and a large inertia relief, visualized in figure 8.3. The bending moments in the wing structure and hence the empty weight of the aircraft is reduced because the planform weight distribution is close to the aerodynamic load distribution for BWB-aircraft.



**Fig. 8.3: Inertia Distribution for Conventional Aircraft (a) and BWB (b) [Liebeck, 2004]**

While the BWB aircraft has some important advantages, the configuration of the pressure fuselage, modularity of the concept and the stability and controllability pose serious concerns to the viability of this concept. These challenges are, together with the risk involved with new concepts, the reason that BWB aircraft are not yet used in civil aviation today.

This research focused on the structural challenge to efficiently carry the pressurization loads during high-altitude cruise flight. The noncircular nature of the cabin cross section requires alternatives to the highly efficient circular shell structure of conventional aircraft. A conformable pressure cabin needs to offer a solution but it comes at the cost of increased structural complexity.

The engineering challenges with respect to the pressure fuselage are not only structurally related. There are major considerations such as passenger acceptance and practicability (operational activities and safety related issues) that need to be taken into account simultaneously.

Prior developments towards pressure cabin concepts have not provided solutions that boost the development of the BWB. Ideas and solutions with respect to structural design, passenger acceptance and the practicability of the pressure fuselage are presented in the following sections in order to enhance the feasibility of the BWB-concept.

## 8.2 Design concepts for the pressure fuselage of the BWB

This section details the concept of the centre-body pressure vessel. The centre-body pressure vessel of a BWB needs to be designed for multiple load conditions (aerodynamic loads, pressurization loads, inertia loads etc.). With respect to the BWB-centre-body, four concepts are introduced of which two concepts; the integrated skin and double shell concept were already introduced in the open literature. The concepts presented all have their pros and cons and the best concept is therefore very much dependant on the specific configuration.

### 8.2.1 The integrated skin and shell concept

The integrated concept (figure 8.4) is similar to the conventional aircraft in the sense that the aerodynamic fuselage and pressure cabin form one integrated module. The pressurization loads are carried via sandwich panels that carry the pressure-induced bending moments. It was already explained in chapter 2 that transferring the pressurization loads via bending moments is detrimental for the structural efficiency. Furthermore, the integrated skin concept is not an effective pressure fuselage because the pressurization loads are not allowed to change the aerodynamic characteristics, which makes the integrated concept stiffness dominated, increasing the weight penalty even more.

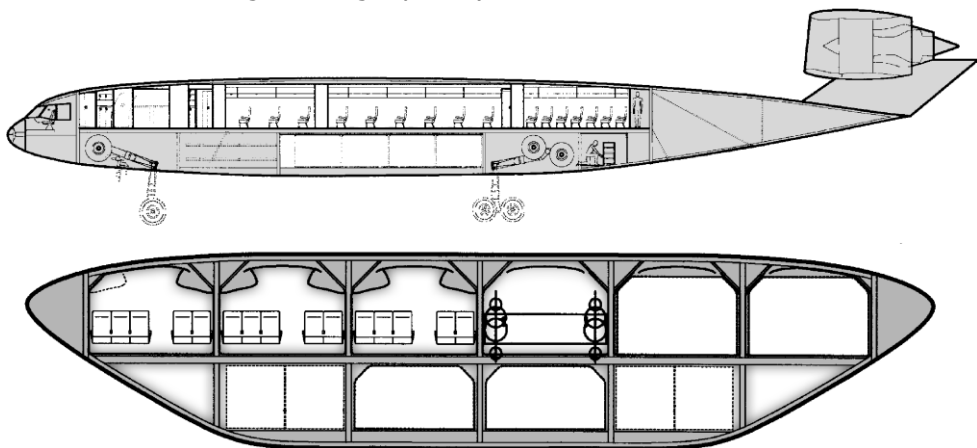
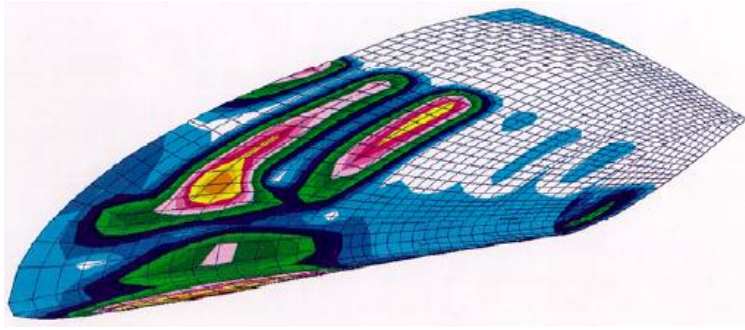


Fig. 8.4: The integrated pressure cabin (Liebeck's concept)

By making use of the finite element method, [Liebeck, 2004] has illustrated the deformation of the BWB while being pressurized, shown in figure 8.5. Variations of this concept have been introduced by having a vaulted inner facing in order to promote the in-plane loading.



**Fig. 8.5: Illustration of exaggerated skin deflection of the integrated pressure cabin [Liebeck, 2004]**

## **8.2.2 The segregated multi-bubble pressure cabin**

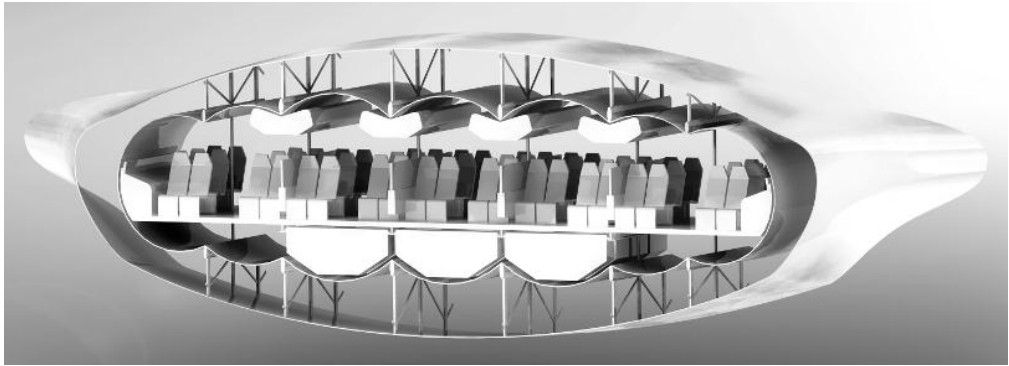
### **8.2.2.1 Concept description**

The second concept is the segregated concept, also called the “double Shell” concept. This concept consists of a multi-bubble that is designed to carry the pressurization loads. The multi-bubble is segregated from the rest of the structure that carries the aerodynamic and inertia loading. Figure 8.6 shows the format. This configuration allows for storage of electronics and other systems that do not require pressurization to be implemented in the available space between the multi-bubble and the aerodynamic shell.

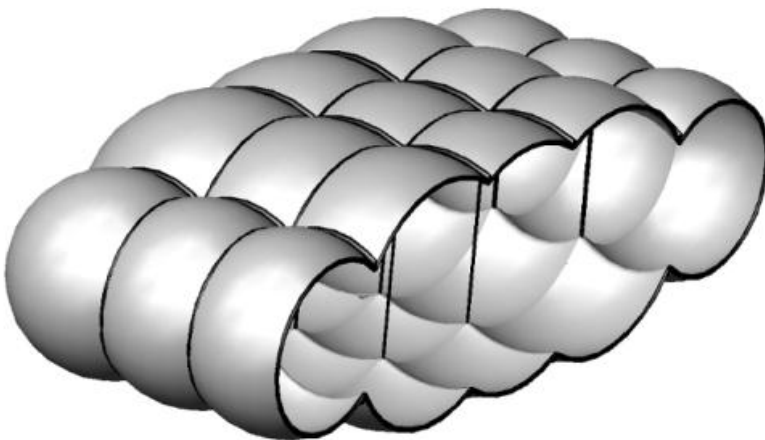
In the case of this segregated structure the pressure decreases stepwise from inside the cabin to the outside of the aerodynamic shell, with an area of static pressure present in between the multi-bubble and the aerodynamic shell.

The pressure-bearing walls are replaced by pillars and beams that are located outside the cabin in order to create an unobstructed open space. These beams are however detrimental for the structural efficiency and a way to avoid this is by making a pressure cabin that is build up from open cell multi-spheres, shown in

figure 8.7. The trade-off between the multi-sphere and the multi-cylinder with beams and pillars is a trade-off between structural efficiency vs. more usable space, interior flexibility and manufacturability. There is of course the opportunity to design a multi-bubble that closely fits the aerodynamic contour but this has severe drawbacks with respect to the interior configuration of the pressure cabin. The practicability of the interior requires a structured lay-out where the distances between pillars in the span and longitudinal direction are constant. These requirements can be fulfilled when the intersecting surfaces lie in the same plane which often leads to simple multi-bubbles, consisting of multi-spheres, multi-cylinders and multi-tori. The practicability of the interior is further discussed in section 8.3.



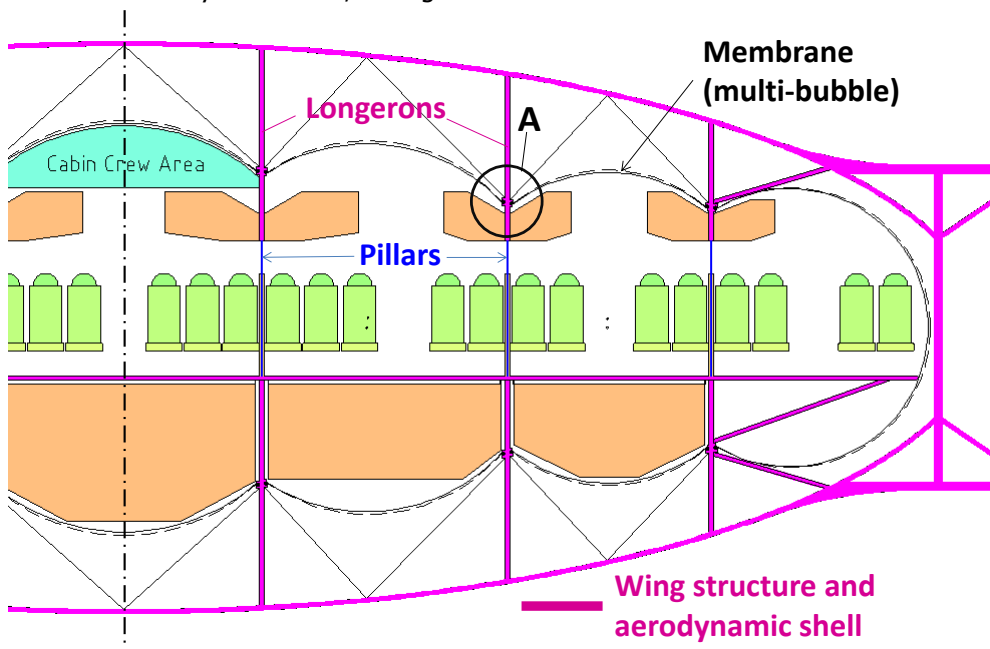
**Fig. 8.6: Design Concept of the segregated multi-cylinder pressure cabin**



**Fig. 8.7: The open-cell multi-sphere: A true in-plane loaded open structure**

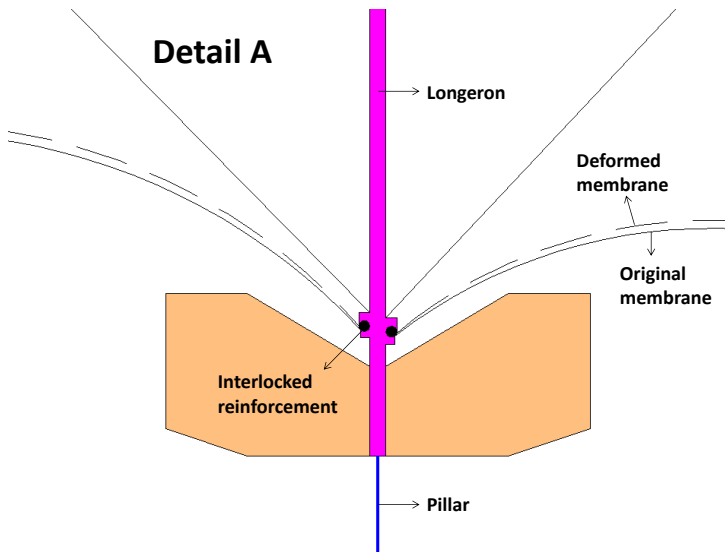
The integration of the multi-bubble with the aerodynamic shell is the biggest difficulty of the segregated concept. The expansion of the multi-bubble requires hinge supports that have the required strength and degrees of freedom or reinforcements that constrain the expansion.

A cross-section of the design and main structural members is shown in figure 8.8 of which detail A is highlighted in figure 8.9. Except for the pillars, all structural elements are continuous. The multi-cylindrical pressure cabin consists of individual membranes and reinforcements that are interlocked in the 'longitudinal beams' or 'longerons' that are part of the aerodynamic shell. The longerons are replaced by short pillars for a short section in the pressure cabin in order to generate an open configuration. The longerons are not carrying axial stresses caused by pressurization. This is done by the interlocked reinforcement that can rotate and slide freely in the axial direction. The longerons stop or are interrupted where the rear bulkhead begins. All other structural members visualized in figure 8.8 are there to create a stable platform that prevents the multi-bubble to move position within the aerodynamic shell, during aircraft manoeuvres.



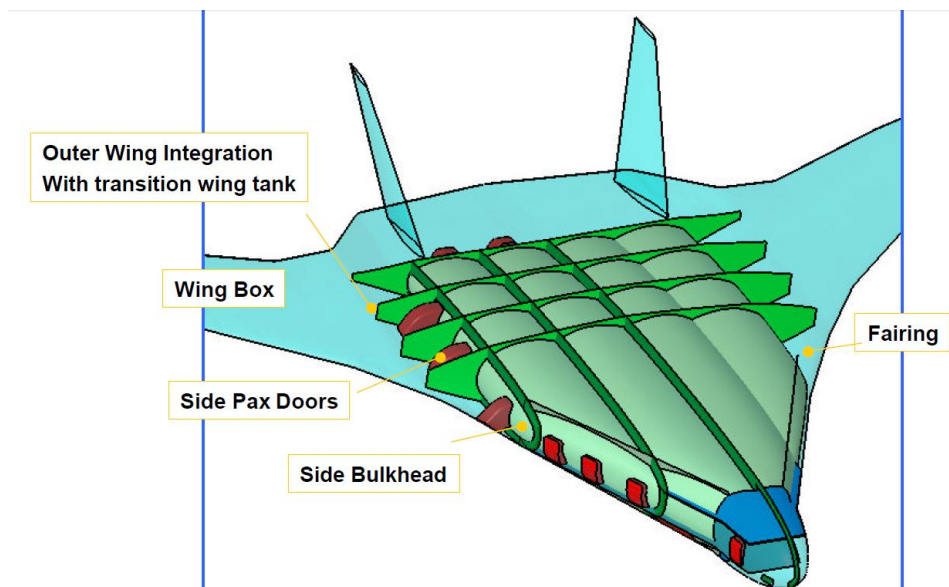
**Fig. 8.8:** Example of a cross-section of a BWB with a segregated multi-bubble pressure cabin

The example shown in figure 8.8 is especially applicable for the design concept shown in figure 8.10. Figure 8.10 shows a concept where the multi-bubble is a multi-cylinder except for the rear bulkhead. The multi-bubble in this concept is integrated with the leading edge. Figure 8.11 shows a complete segregated multi-bubble. The concept in figure 8.10 is preferred because it simplifies the design; there is no need for a double amount of doors and windows at the leading edge of the aircraft and the expansion is easier to control because the intersecting surface is located in the same plane.

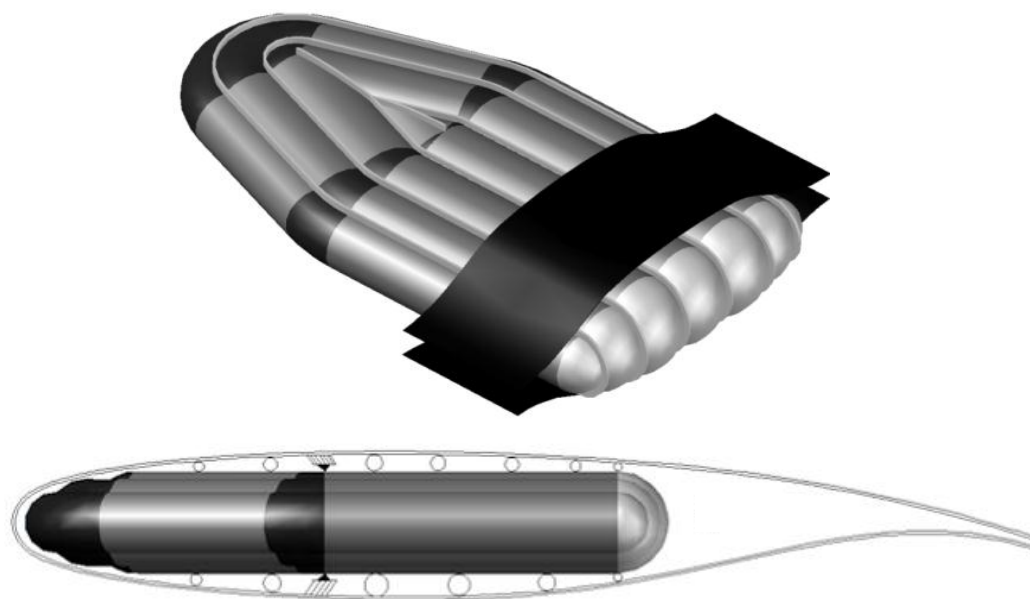


**Fig. 8.9: Detailed view of the cross-section shown in figure 8.8**

A potentially interesting solution for the concept shown in figure 8.8 is to use a multi-cylinder that has no axial expansion as explained in chapter 3.3. Although this would cause a weight penalty with respect to the weight of the membrane, it might also enable a rigid connection between the pillars and the aerodynamic shell, bypassing, the need for heavier hinge supports. This statement relies on the incorrect assumption of a rigid aerodynamic shell. It therefore needs to be investigated in detail to what extent a simplified design is feasible by tailoring the membrane deformations.



**Fig. 8.10: Design concept of the VELA (Very Efficient Large Aircraft), [Kresse, 2006]**



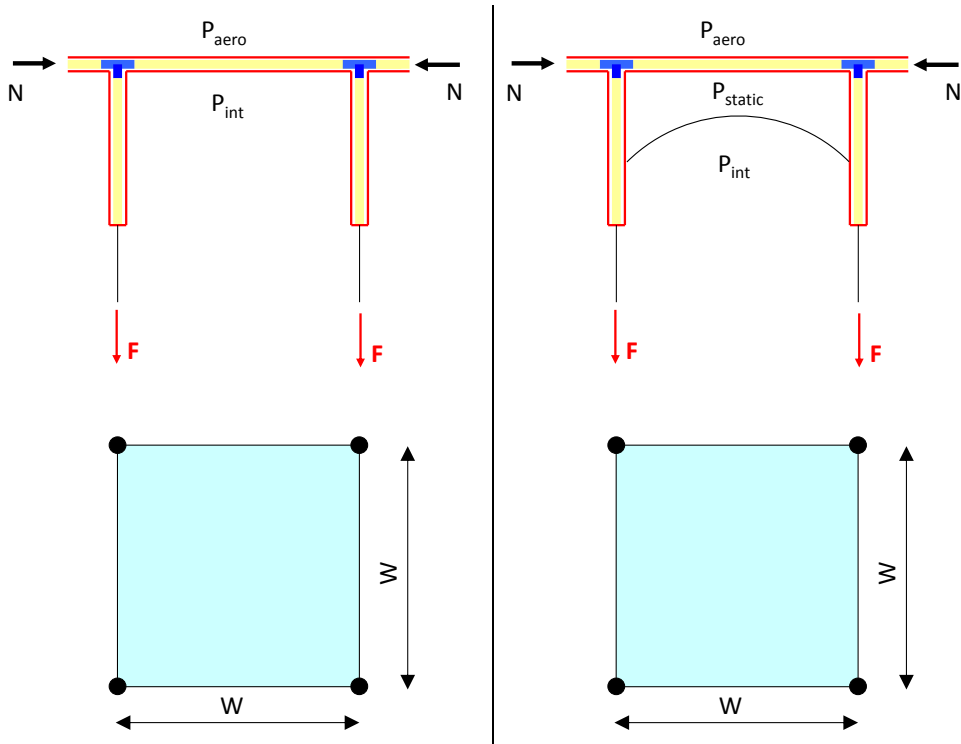
**Fig. 8.11: A fully segregated multi-bubble consisting of multi-spheres, multi-cylinders, multi-toroids and reinforcements at the intersections.**

#### **8.2.2.2 Sea-level altitude**

One additional advantage from the passenger's point of view is that the pressure inside the multi-bubble always needs to be kept the same as at sea level. The reason for this is that when a conventional aircraft makes a steep dive, it could happen that the pressure outside the fuselage is momentarily higher than inside the pressure cabin which would lead to compressive stresses inside the wall of the fuselage. The thin membrane of the multi-bubble provides no buckling resistance and this means that the pressure inside the multi-bubble needs to be in all circumstances higher than outside. A higher pressure means that more mass of air needs to be transported at cruise altitude and it means that the structure needs to be stronger which would lead to an additional weight penalty. This weight penalty is however counteracted by the fact that the pressurized volume of the multi-bubble BWB is about 1/3rd smaller than that of a conventional aircraft for the same number of passengers. This is explained by the fact that the cargo space of a conventional aircraft exceeds the required luggage space when the aircraft is exclusively used for passenger transportation. For passenger aircraft, the available floor area plays a large role in the capacity of passenger aircraft and the ratio of floor area and pressurized volume is higher for the multi-bubble than it is for a conventional aircraft. In this case, the BWB would only be used as a cargo carrier, the option of only an aerodynamic shell with pressurized cargo containers is the most promising option.

#### **8.2.2.3 Structural performance of the segregated multi-bubble**

A simplified analysis [McInally, 2011] showed that the segregated concept is lighter than the integrated concept but not by a hugely significant amount. The analysis was done by comparing a pressure cabin cell of the integrated concept with the segregated concept. The schematic of the integrated and segregated solution is given in figure 8.12. In this analysis the assumption was made that the pillar pitch width is equal to the pillar pitch depth, as illustrated in figure 8.12. The pressurization, inertia and aerodynamic loading is all carried by a sandwich panel in the integrated concept. The segregated concept segregates the loading, the pressurization loads are carried by the membrane and the inertia and aerodynamic loads are carried by the sandwich panel.



**Fig. 8.12: Schematic of the integrated and segregated solution.**

There are two reasons for this small difference in structural efficiency. The first reason comes from the higher required pressure in the pressure cabin of the segregated concept. The second reason comes from the fact that the dynamic pressure differential, ( $P_{aero} - P_{static}$ ), which causes the aircraft to become airborne also creates pressure induced bending moments in the aerodynamic shell of the BWB. These pressure differentials are small under normal flight conditions but they can become significant in extreme situations. Such extreme situations are exceptional and overall the segregated solution proves superior to the integrated concept. It allows the shell to maintain a better aerodynamic shape because the deflections of the panels under normal flight conditions are less. This in turn also reduces fatigue issues.

Liebeck addressed a potential safety issue for the separate arched pressure vessel concept: if a rupture were to occur in the thin arched skin, the cabin pressure would have to be borne by the wing skin, which must in turn be sized to carry the

pressure load. In practice, this argument is not entirely true. Considering figure 8.13, it can be seen that the inner volume of the aerodynamic shell is significantly larger (in the order of 50%) than that of the pressure vessel. This would already imply that the aerodynamic shell does not need to sustain the same pressurization loads. More importantly, it is also feasible to incorporate a system that 'opens' the aerodynamic shell in case of over-pressure due to a rupture in the membrane. This would resemble a blast valve or a large pressure regulator.

### **8.2.3 The hard shell / soft shell concept**

This concept, shown in figure 8.13 consists of a multi-bubble that carries all loads and a soft shell that provides the aerodynamic shape and transfers the dynamic pressure loads to the multi-bubble. The interior of the pressure cabin and the configuration of the aircraft systems is identical to the double shell concept. There are some significant advantages of this concept compared to the previous proposed concepts:

- The multi-bubble is from a structural topology point of view more efficiently loaded than the integrated shell concept. Even though the geometry of the aerodynamic shell is more favourable with respect to area moment of inertia, the multi-bubble's overall performance is superior because it carries the pressurization loads more efficiently.
- The geometric distortions after pressurization of a multi-bubble are smaller compared to the integrated concept. This in return results in better structural integrity, e.g. fatigue characteristics, for the hard shell/soft shell concept.

The inner shell is strength dominated, hence lighter, and only the soft outer shell has stiffness requirements in order to maintain the aerodynamic shape that is subjected to the dynamic pressure loading.

- The design of this concept is simpler than the double shell concept. The integration of the two shells is straightforward which in return will lead to weight savings.
- The hard shell, being a primary aircraft structure is protected from environmental influences by the soft shell. The requirements on impact are

therefore a lot less stringent for the hard shell / soft shell concept. This in return leads to weight savings for the hard shell.

- The higher pressure requirement of the segregated concept does not apply for the hard shell/soft shell concept. This also leads to weight savings for the hard shell.

The main disadvantage of this concept is the dead weight that the soft scale entails. The trick is to keep the volume difference between the soft shell and hard shell as small as possible. This means that a compromise needs to be found between the geometry of the aerodynamic design and the design of the multi-bubble. This compromise also accounts for the two previous presented concepts but is even more stringent for this concept. The other disadvantage of this concept is that the concept is restricted for multi-cylindrical pressure cabins, which makes this concept less flexible.

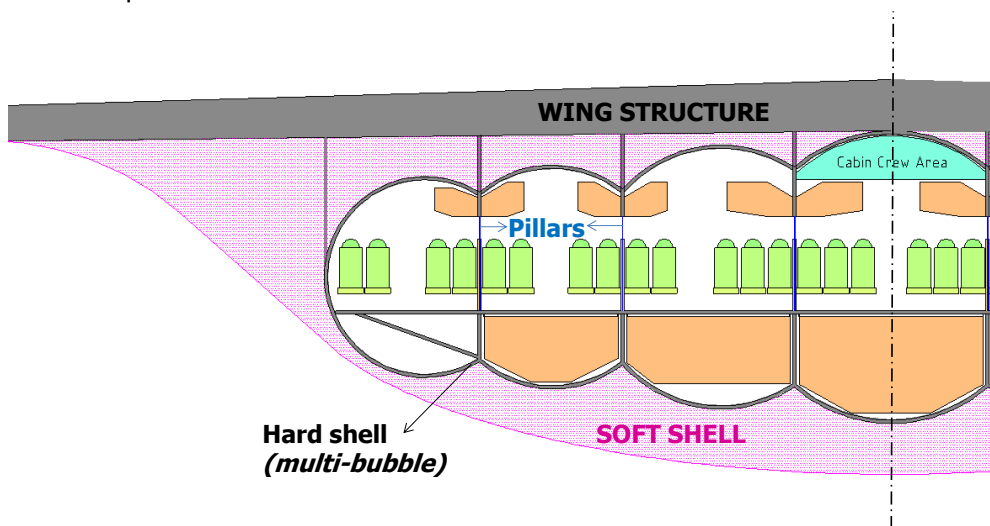


Fig. 8.13: Schematic of the hard shell/soft shell concept

## 8.2.4 The oval centre-body

### 8.2.4.1 An Inside-Out Approach to BWB Cabin Design

Although each of the previous proposed design solutions to the pressurization problem has its advantages and disadvantages, their commonality lies in the

starting point of the design process. All three cabin concepts are designed to fit within a predefined outer surface. It is assumed that this outer surface provides the best shape to fulfill all requirements on handling characteristics while maximizing aerodynamic efficiency. However, the required fuel weight is also dependent on the operative empty weight, which, in turn, is dominated by the empty weight of the aircraft. It could therefore be argued that the aerodynamic shape of the centre-body of the BWB should not be dictated by aerodynamic characteristics but that structural weight should be the prime driver for this part of the airframe design. Moreover, the local section lift coefficient at the centre-body is relatively low compared to the outboard wings if an elliptical lift distribution is assumed. Therefore, it is even more evident that it is the structural design rather than the aerodynamic design that should prevail.

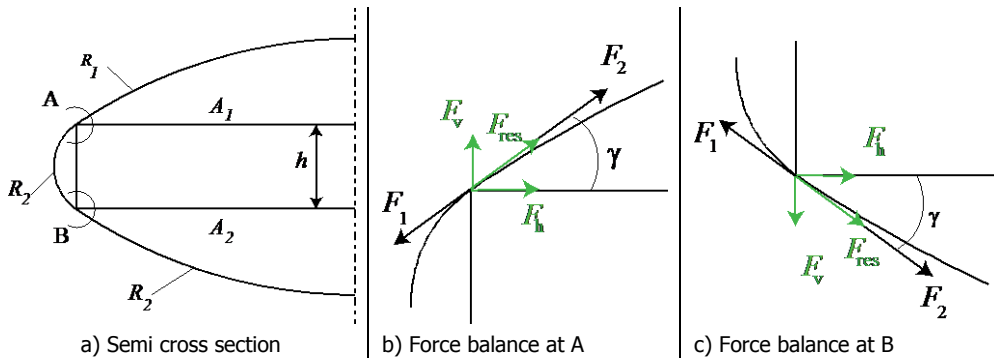
Apart from structural and aerodynamic considerations, other considerations with respect to passenger acceptance and cabin configuration come into play. These are difficult to wrap inside a simple analytical formula but play a major role in the commercial success of an(y) airplane. The vertical pillars or walls that are specified in both cabin concepts for the BWB put restrictions on cabin configuration. Positioning of chairs, galleys, toilets and other operational items are constrained by these structural members that invariably penetrate the cabin. Moreover, the spaciousness of the wide cabin, which is one of the attractive features of the BWB, is significantly hampered when structural members are positioned. From a passenger acceptance and cabin configuration point of view, it would therefore be preferred to have a spacious, naturally-illuminated cabin without vertical pillars or walls interrupting the space.

#### **8.2.4.2 Structural Concept of Oval Cross Section**

With the aforementioned passenger consideration in mind, a new cabin cross section was created that does not have the complexity of a double shell concept, the dead weight of the soft shell concept, and is still able to carry the pressurization loads via in-plane loading. Envision a centre-body cross section as four connecting arcs, two at each side that are identical, one bottom arc and one top arc. In this example the top and bottom arc can have an identical radius of curvature as shown in Figure 8.14.

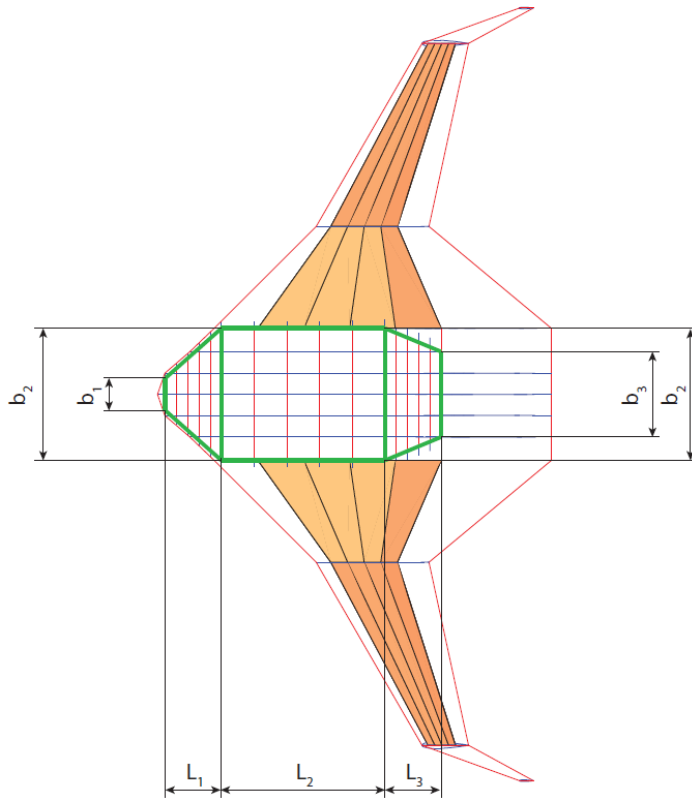
The resulting cross-section is non-cylindrical, meaning that the curvature of the side arcs are higher than those of the top and bottom arcs. Each of these three arcs can have a different radius of curvature while still allowing for a smooth transition at their connecting nodes.

The pressurization of the fuselage cross-section in figure 8.14 will only result in in-plane stresses in the fuselage skin of the oval centre-body. The in-plane stresses result in discrete forces (force  $H$  and  $V$  in figure 8.14b/c) at the connection nodes of the arcs. These discrete forces need to be redistributed with additional structural members to maintain structural integrity. Therefore, each node is connected by 2 additional panels. When pressurized, the long horizontal panels are loaded in compression while the short panels on the side are loaded in tension. In the practical application of this concept the lower horizontal member doubles as the passenger floor. In addition, the upper and lower panes also provide the carry-through structure for the wing torque box at the location of the wing.



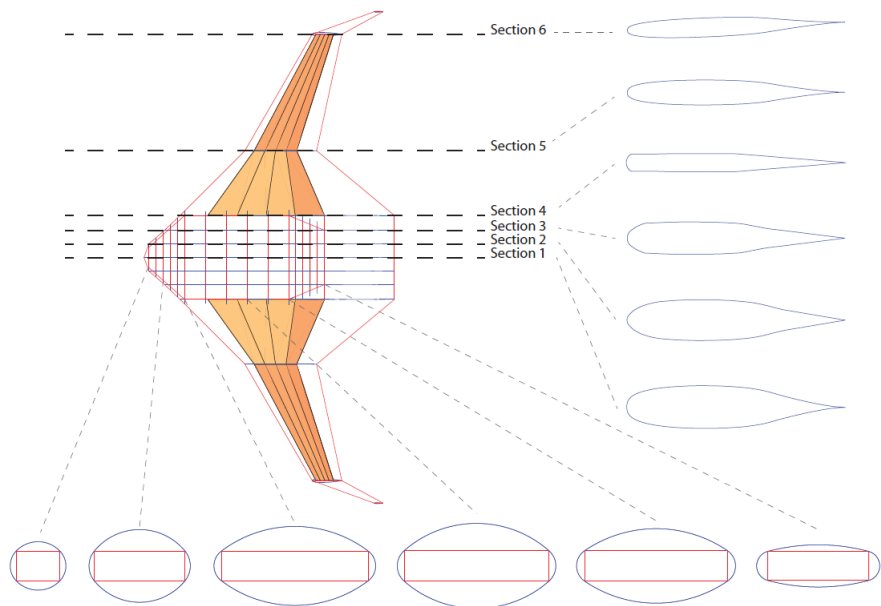
**Fig. 8.14: Cross-section of the oval centre-body and details**

The 'oval-fuselage' is defined by seven variables and a constant cabin height. Six variables are specified by the planform, shown in figure 8.15 and the seventh variable is the thickness to chord ratio of the centreline airfoil which corresponds with section 1 in figure 8.16.

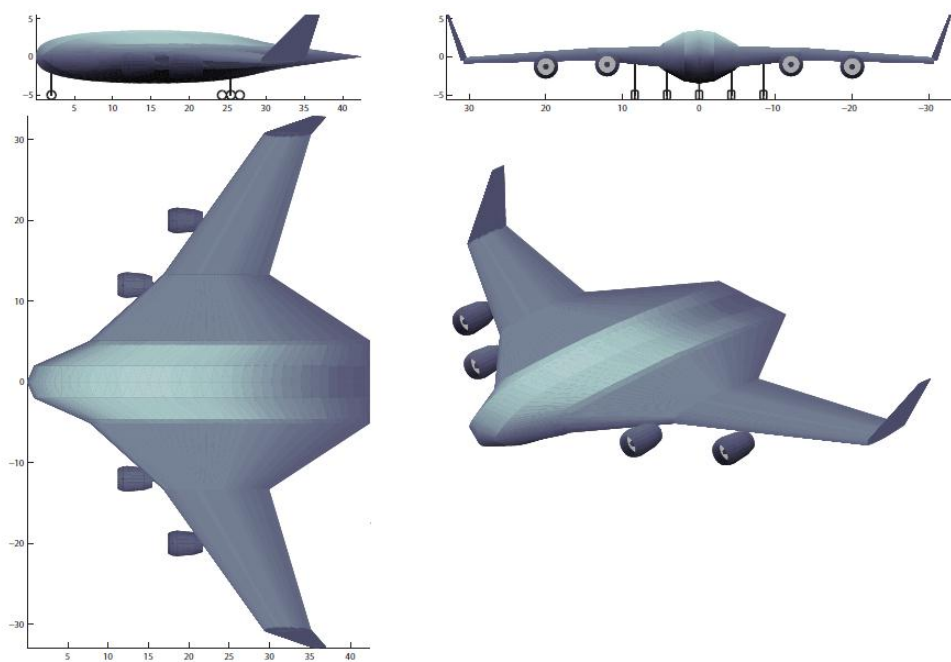


**Fig. 8.15: Planform of an 'oval-fuselage' defined by the 6 variables:  $b_1$ - $b_3$  and  $L_1$ - $L_3$**

The structural efficiency of the oval pressure vessel is lower than that of the multi-bubble because there are structural members loaded in compression. The structural efficiency of this concept reduces when the radius of the top and bottom arc is larger, hence making the oval flatter. This explains why this concept is not the best concept for all BWB designs and the success of this concept is therefore very dependent on the BWB configuration. Preliminary studies, [Hoogreef, 2012], show that the optimal configuration is found in the  $\sim 400$  passengers range with a floor area in between 250 and 300 m<sup>2</sup>. A design concept of this BWB is shown in figure 8.17.



**Fig. 8.16: Top view of the 'oval fuselage' showing the cross-sections and airfoil sections**



**Fig. 8.17: 3-view of a 400 passengers 'oval-fuselage' BWB**

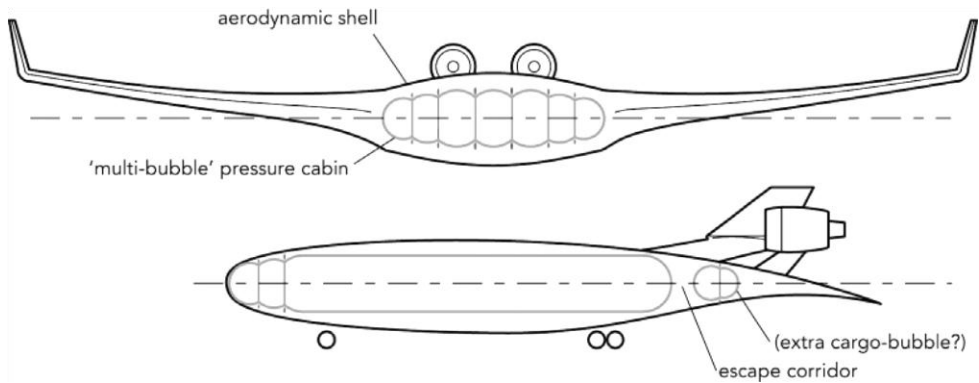
## **8.3 Interior configuration of the BWB pressure cabin**

A traditional cabin configuration cannot be blindly tagged on to a newly shaped aircraft. The topological change of the aircraft pressure cabin requires a fresh approach to the entire passenger transport experience. Although the interior configuration is a different research topic, it goes hand in hand with the exploration of new structural concepts in order to demonstrate the feasibility of the complete pressure cabin. However, previous concepts for BWB pressure cabins have not focused on the interior experience of the passenger. An explorative concept study of the configuration of the pressure cabin of a BWB was published in [van der Voet et al., 2012]. In this paper we envisioned the intended and expected use of this new aircraft and tailored the (limitations of the) technological context to human behaviour, preferences and airworthiness. The context of this research can be implemented for all structural concepts presented in section 8.2.

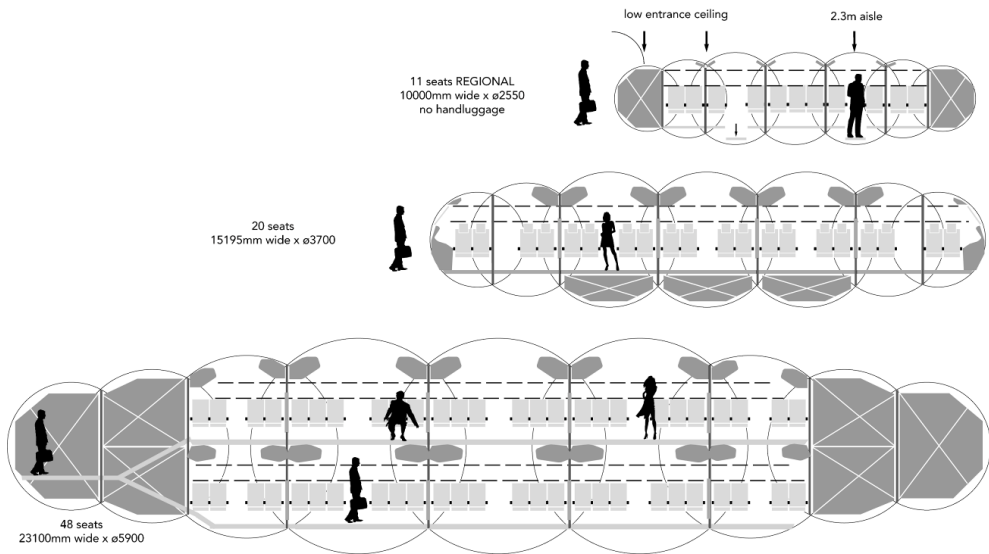
This section presents the conclusions of the explorative concept study. The solutions and considerations that are worked out for the BWB are explained by means of a worked out example for which the Boeing 777-200 served as the equivalent benchmark model.

### **8.3.1 Size and Cabin Lay-out**

The shape of the pressure cabin needs to be organized in such a way that wing-shape and cabin-shape match (figure 8.18), and is, as in any aircraft design process, an iterative process influenced by cabin requirements, performance characteristics and weight limitation. If anything is to be said about the interior use of the multi-bubble BWB aircraft, it has to be clear how the interior could be organized for passenger and cargo distribution. Depending on the height of the pressure cabin (for passengers), the cargo can either be put under the floor or in the outer sections of the cabin. Three interesting configurations, shown in figure 8.19, were worked out for the multi-bubble but also apply for other structural concepts.

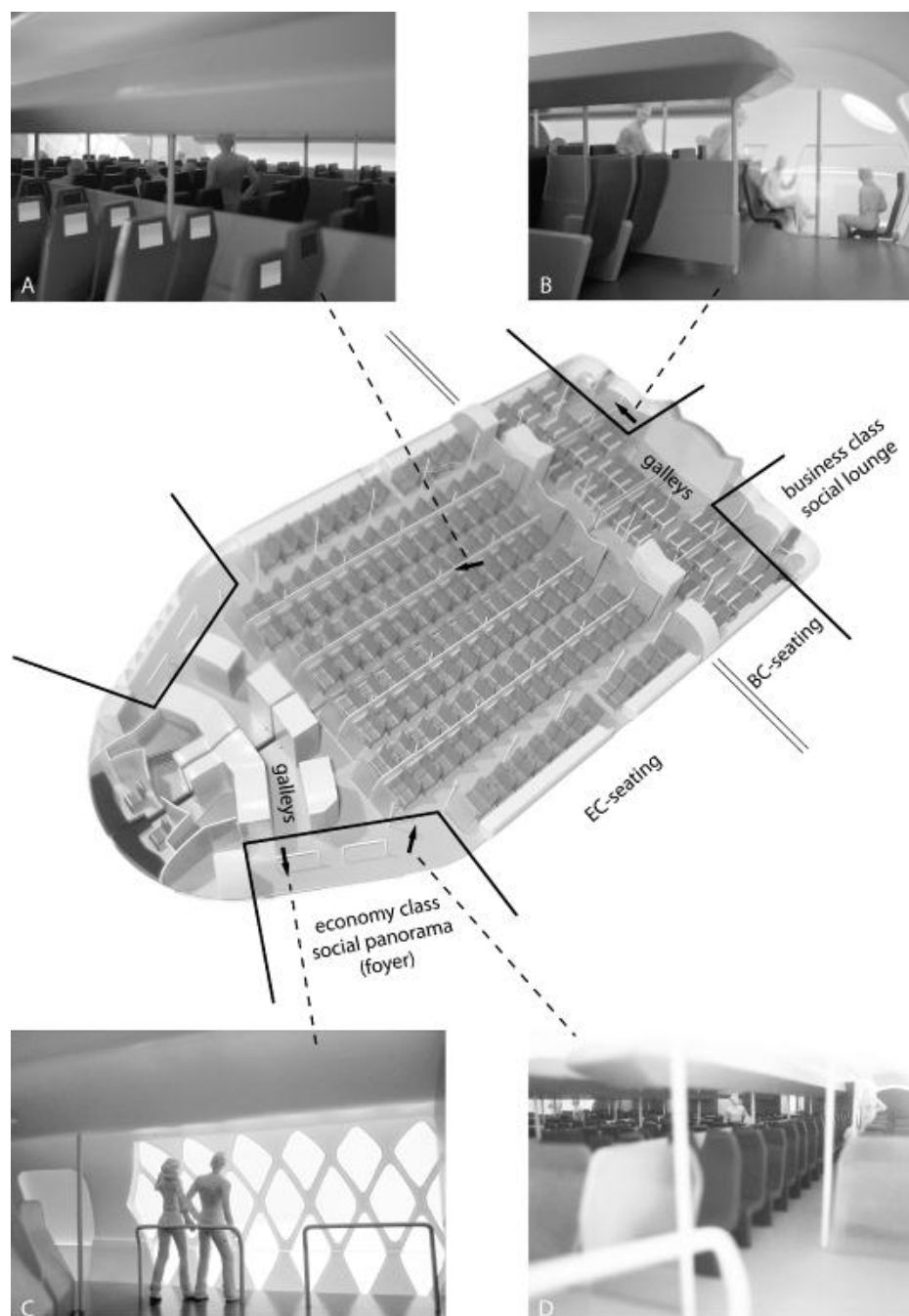


**Fig. 8.18: Schematic of a multi-bubble in the BWB**



**Fig. 8.19: Three size variants and cross section layout**

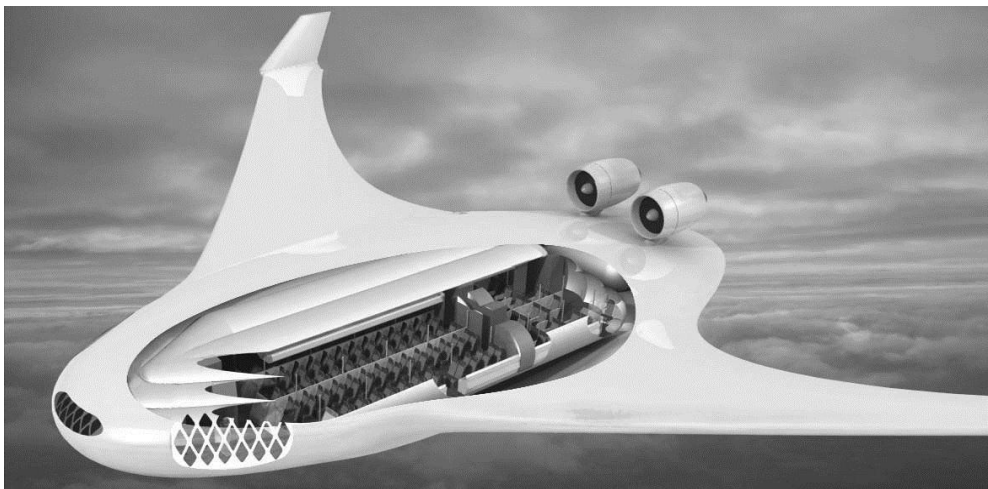
Our research has shown that for most passengers flying in a BWB will be more enjoyable than in a conventional airliner. The passengers' experience changes drastically due to the new topology of the aircraft and the introduction of shared large windows. In fact, that is its biggest selling point. The BWB gives all passengers the same view outside rather than giving just a few lucky passengers a window-view during the entire flight. An impression is given in figure 8.20.



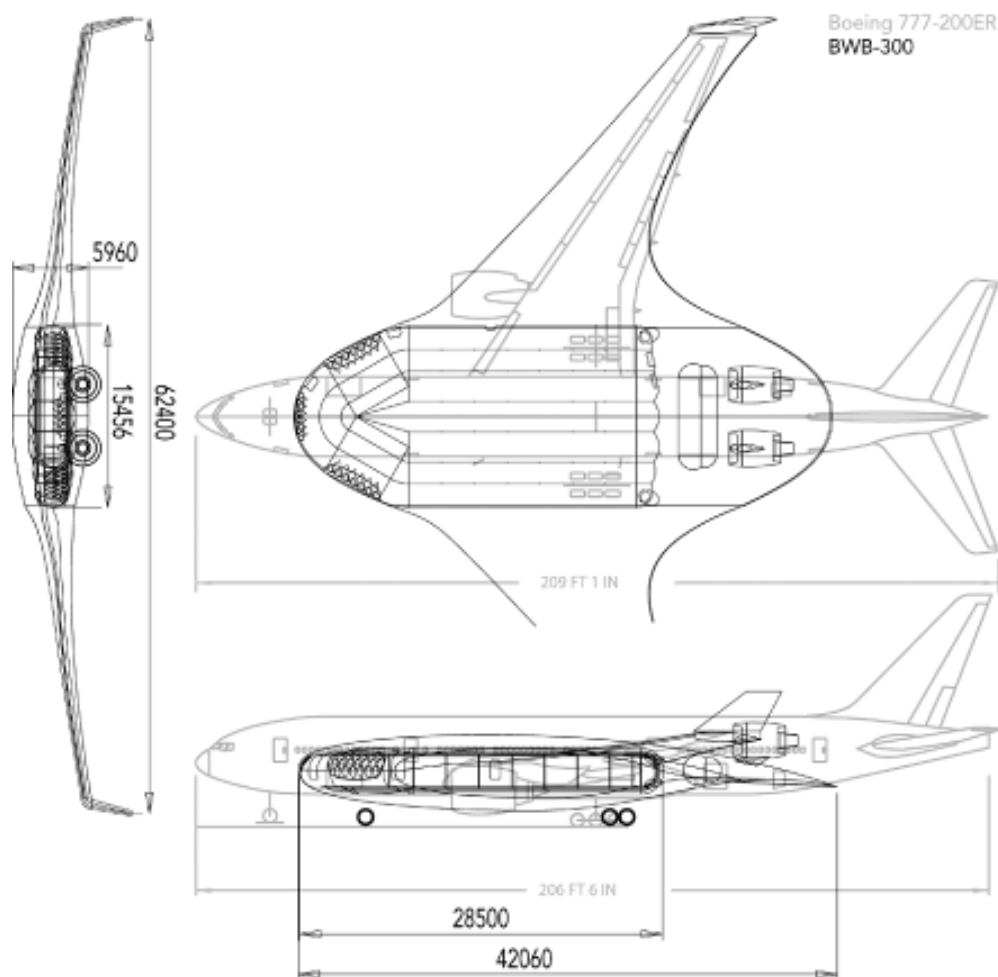
**Fig. 8.20: Interior configuration of the BWB-300**

Evacuation of BWB-aircraft will be somewhat of a challenge. Every new BWB-configuration needs to be re-assessed for this purpose. For conventional aircraft evacuation regulations can easily be implemented into guidelines. Also, the orientation possibilities are slightly worse in the BWB, as it has considerably less total window surface. Nevertheless, the BWB can be made into a comfortable and safe passenger transport in regards to orientation and evacuation. It is commonly said that a BWB is very uncomfortable for the passengers at the sides due to higher roll-rate accelerations. Even in extreme conditions passenger are subjected to less acceleration than in the tail of a conventional aircraft or even when cornering in a car. The BWB offers passengers more freedom to move around. They can walk towards the common window-areas in the front to enjoy a panoramic view, socialise and stimulate the blood-flow at the same time. Dividers in between the seat-rows are installed to improve cabin quietness and make the middle seats more comfortable. These also function as a privacy screen or wall to lean against. This makes middle seats comparable to window-seats without the private window.

Overall, we believe the introduction of the Blended Wing Body will be greeted by passengers with the same excitement as the introduction of the first jet aircraft or the Concorde. An example for which the Boeing 777 served as benchmark is shown in figures 8.20 to 8.22.



**Fig. 8.21: Cutout view of BWB-300 in 306-passenger configuration. (The exterior shell is an artist's impression.)**



**Fig. 8.22: Dimensions and comparison with the Boeing 777-200**

## **8.4 Concluding comments**

This chapter presented the BWB as a possible future aircraft concept. The solution of an efficient pressure cabin is one of the engineering challenges that have to be overcome. The introduction and development of this aircraft is not only dependent on the technological challenges but also dependent on political and economic considerations. Several concepts are presented to overcome the pressure cabin issues in case the green light for the development of this aircraft is given. It is impossible to say what concept is favourable over the other. Again, detailed studies and the development of demonstrators need to show the feasibility of each of these concepts and will also inspire engineers to modify the presented concepts. For example, it was shown that the oval fuselage is not suitable for very large BWBs. A compromise could be to have two intersecting oval fuselages with only one row in the middle where pillars interrupt the passenger cabin. The space in between these ovals outside the pressure cabin can be filled up with a soft shell. This new concept is a combination of the oval fuselage and the hard shell/soft shell concept. Endless combinations are possible and the content in this chapter is only a starting point for preliminary concept studies.

Factors such as airworthiness also have a strong influence in the concept selection. Finally, it is concluded that this thesis provides the insight to develop in-plane loaded conformable pressure cabins and implementations of this insight are presented in this chapter as future food for thought.



# CHAPTER 9

## Overall concluding comments and recommendations

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### 9.1 Concluding comments

From a structural topology point of view, efficient structures are in-plane, rather than out-of-plane, loaded structures. Pressure acts perpendicular to the shell wall and only axi-symmetric shell elements have the ability to transfer pressure via in-plane loading. The multi-bubble is a structurally efficient conformable pressurised structure that consists of any arbitrary combination of intersecting axi-symmetrical membrane structures. The multi-bubble therefore carries the pressurization loads via in-plane stresses provided there is suitable in-plane loaded reinforcement at the surface of intersection.

In general, linear membrane theory can in general be used to assess the membrane forces and deformations in the pressurised structure. Linear membrane theory is only accurate for optimal membranes and provides good results for structures for which the strains are low, say 2%. Optimal membranes are membranes for which the deformations are the same in every direction, implying that the force distribution is identical to the stiffness distribution. Structurally, optimal membrane structures have multiple benefits:

- Linear membrane theory is always valid which simplifies the analysis of forces and deformations.
- The deformed geometry is easily assessed once the strain  $\varepsilon$  is known. The deformed shape is a scale factor  $(1+\varepsilon)$  larger than the original shape
- The structural efficiency is maximized and is not related to the shape of the pressure vessel as long as the pressure is carried via in-plane loading and all the structural members are subjected under tension.
- Even more important and a logical consequence of the uniform deformation is the disposal of compatibility of deformations when

combining different membrane elements, e.g. bulkheads on cylinders and conformable pressure vessels.

All the benefits of optimal membrane structures do not apply for non-optimal membrane structures. Accurate solutions for non-optimal membranes can only be obtained by means of non-linear membrane theory. Non-linear membrane theory is difficult to implement and the simplified quasi-linear theory yields very good results for the analysis of the non-linear membrane response. The modified linear membrane theory is the energy formulation of the quasi-linear theory and enables the assessment of the solution without iterations. Together with the quasi-linear theory, the 'force method' is a practical engineering method to assess the stress analysis at the junction of two non-optimal shells of different geometry. The 'force method' allows the flexural rigidity of the shell to be taken into consideration. It is concluded that disturbances only appear in a narrow strip along the junctions of the two shells. The disturbances at the junction are minimized when the flexural stiffness of the shell at the junction is minimized.

Finally, one of the most important conclusions in this thesis is that linear membrane forces of pressurised structures can be visually assessed by taking into consideration distinctive arc lengths and surface areas perpendicular to the force vector. This visual assessment is particular useful in the analysis of conformable pressurised structures for which there are no analytical solutions.

## **9.2 Recommendations**

There is an endless amount of recommendations to make with respect to conformable pressurised structures. Aspects such as permeability, cryogenic temperatures, connections of valves, safety etc. have to be investigated with respect to the development of conformable tanks. All these aspect are investigated in the CHATT-project (*Cryogenic Hypersonic Advanced Tank Technologies*) as a continuation of this research. The CHATT project is funded by the European Union as part of the 7<sup>th</sup> framework program. The recommendations with respect to this thesis are narrowed down to issues that have been investigated in this thesis.

One of the most interesting research topics from a scientific point of view is the formulation of the nonlinear membrane response of the orthotropic multi-bubble

and the shell response of the thick-walled multi-bubble. This report presented the modified linear membrane theory, a simplified energy method to find the nonlinear membrane response. The formulation was worked out for axi-symmetric membrane structures but it is interesting to find a general formulation that is applicable for multi-bubbles. In this latter case, the minimum energy solution needs to be found for the summation of all individual membranes, subjected to displacement boundary conditions at the intersections.

The detailed design and the manufacturing of demonstrators of conformable tanks is very interesting from an engineering point of view. This will give insight into feasible fibre architecture for conformable tanks, and give answers to the need of a non-linear membrane solution for conformable tanks. Non-linear membrane solutions for specific cases are easier to establish than a general non-linear formulation. The detailed design and manufacturing of demonstrators will subsequently give insight in the assessment of the optimal conformable tank from an economic point of view.

Similar recommendations can be made for the development of conformable pressure cabins where further research will provide insight to the feasibility and potential of different concepts.

These recommendations are followed-up in the continuation of this research project.



# APPENDIX A

## In-plane response of fibre reinforced materials

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For a single phase material, it is quite easy to understand how to calculate the stiffness and strength of these materials because the properties are the same in all directions. For composites this is not the case, but it is important to know the mechanical response if we want to design composite structures. The first sections help us to understand the mechanical response of a single ply. Subsequently, we look at the mechanical response of a laminate. A laminate can be considered as a stacking of multiple plies which are mostly oriented in multiple directions because most structures are subjected to multiple load cases. In case of multiple load cases, a minimum of three fiber orientations is needed to carry all the loads (compare it with a framework).

### A.1 Fiber reinforced plies: stress – strain relationships

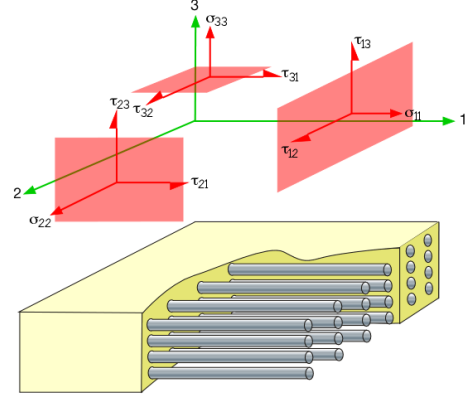
Composites are elastically anisotropic, which means that there is a linear relation between every stress component and every deformation component. This can be expressed by Hooke's law:

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{yz} \\ \tau_{xz} \\ \tau_{xy} \end{Bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & S_{14} & S_{15} & S_{16} \\ S_{21} & S_{22} & S_{23} & S_{24} & S_{25} & S_{26} \\ S_{31} & S_{32} & S_{33} & S_{34} & S_{35} & S_{36} \\ S_{41} & S_{42} & S_{43} & S_{44} & S_{45} & S_{46} \\ S_{51} & S_{52} & S_{53} & S_{54} & S_{55} & S_{56} \\ S_{61} & S_{62} & S_{63} & S_{64} & S_{65} & S_{66} \end{bmatrix} \begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \epsilon_z \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \end{Bmatrix} \quad \text{or} \quad \{\sigma_i\} = [S_{ij}] \cdot \{\epsilon_j\} \quad (\text{A.1})$$

Equation (A.1) is the notation of Hooke's law along the material axes where  $[S]$  is the stiffness matrix.

From the law of Maxwell follows  $s_{ij} = s_{ji}$ . Furthermore, from figure A.1 we see that there is symmetry along the orthogonal coordinate system, and equation (A.1) becomes:

$$(A.2) \quad \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{yz} \\ \tau_{xz} \\ \tau_{xy} \end{Bmatrix} = \begin{bmatrix} s_{11} & s_{12} & s_{13} & 0 & 0 & 0 \\ s_{12} & s_{22} & s_{23} & 0 & 0 & 0 \\ s_{13} & s_{23} & s_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & s_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & s_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & s_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \end{Bmatrix}$$



**Fig. A.1: Orthotropic fiber reinforced layer [Berthelot, 1998]**

Thin plates are in a state of generalized plane stress, the so called plane stress condition ( $\sigma_z = \tau_{xz} = \tau_{yz} = 0$ ). The stress-strain relationship for each ply in an arbitrary direction as shown in figure A.2 is:

$$(A.3) \quad \begin{Bmatrix} \sigma_{x_1} \\ \sigma_{y_1} \\ \tau_{x_1y_1} \end{Bmatrix} = \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{x_1} \\ \varepsilon_{y_1} \\ \gamma_{xy_1} \end{Bmatrix} \quad \text{or} \quad \sigma_{i\varphi} = S_{ij\varphi} \varepsilon_{j\varphi}$$

Whereby

$$(A.4) \quad S_{ij\varphi} = \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix} = [T_\sigma]^{-1} \begin{bmatrix} s_{11} & s_{12} & s_{16} \\ s_{12} & s_{22} & s_{26} \\ s_{16} & s_{26} & s_{66} \end{bmatrix} [T_\varepsilon]$$

$[T_\sigma]$  and  $[T_\varepsilon]$  are transformation matrices that allow us to calculate the elastic properties of a single ply under a different angle than the chosen coordinate system.  $[T_\sigma]$  and  $[T_\varepsilon]$  are the transformation matrices to calculate the stresses respectively the strains in any direction relative to the chosen coordinate system. In figure A.2, the material coordinate system is indicated by the XY-axes. The

elastic properties in the new  $X_1Y_1$ -coordinate system can be calculated with the transformation matrices and are given by:

$$[T_\sigma] = \begin{bmatrix} c^2 & s^2 & 2cs \\ s^2 & c^2 & -2cs \\ -cs & cs & c^2 - s^2 \end{bmatrix} \quad (\text{A.5a})$$

$$[T_\epsilon] = \begin{bmatrix} c^2 & s^2 & -cs \\ s^2 & c^2 & cs \\ 2cs & -2cs & c^2 - s^2 \end{bmatrix} \quad (\text{A.5b})$$

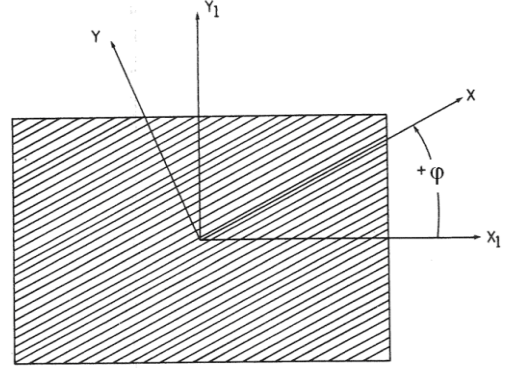
$$c = \cos \phi$$

$$s = \sin \phi$$

**Note:**

$$\epsilon_{j\phi} = [T_\epsilon] \epsilon_j \quad \text{and} \quad \epsilon_j = [T_\sigma]^T \epsilon_{j\phi}$$

Where  $[T_\sigma]^T$  is the transpose of  $[T_\sigma]$



**Fig. A.2: Different coordinate reference systems**

The stiffness matrix expressed in engineering constants reads:

$$\begin{bmatrix} S_{11} & S_{12} & S_{16} \\ S_{12} & S_{22} & S_{26} \\ S_{16} & S_{26} & S_{66} \end{bmatrix} = \begin{bmatrix} \frac{E_x}{1 - \mu_{xy}\mu_{yx}} & \frac{\mu_{xy}E_y}{1 - \mu_{xy}\mu_{yx}} & 0 \\ \frac{\mu_{xy}E_y}{1 - \mu_{xy}\mu_{yx}} & \frac{E_y}{1 - \mu_{xy}\mu_{yx}} & 0 \\ 0 & 0 & G_{xy} \end{bmatrix} \quad (\text{A.6})$$

where:

$$\frac{\mu_{xy}}{E_x} = \frac{\mu_{yx}}{E_y} \quad (\text{A.7})$$

The assumption seems to be that plies are always unidirectional, but in industry weaves are used a lot because weaves are easier to process (can be draped more easily in or on to a mould). The theory in this appendix is valid for all elastic orthotropic materials (also weaves) as long as the right mechanical properties are used.

## A.2 Strength of a single ply

The origin of fracture in composite materials is based on completely different principles than for metals. This is caused by the combination of very strong fibers in a relatively weak matrix. The fibers will carry the largest share of the load, in many cases the matrix will fail first. The strength is already influenced during the manufacturing process: during lay-up flaws such as cavities and voids can occur in the matrix that can cause stress concentrations and crack initiation. The cracks preferably do not grow perpendicular to the loading direction, since the fibers act as crack stoppers. The cracks often grow along the fibers and in between layers. The fracture strength also depends on elasticity of the resin hence on the stress distribution, influence on creep and relaxation.

Several failure criteria for composites can be found in literature. An overview is given below of the most common criteria's. For analyzing plies with failure criteria, we use the material coordinate system.

### 1) The maximum stress criterion:

This criterion assumes that no failure will occur if:

$$\bar{\sigma}_{xC} \leq \sigma_x \leq \bar{\sigma}_{xT} \quad \text{and} \quad \bar{\sigma}_{yC} \leq \sigma_y \leq \bar{\sigma}_{yT} \quad \text{and} \quad \tau_{xy} \leq \bar{\tau}_{xy} \quad (\text{A.8})$$

where  $\bar{\sigma}_{xC}$  is the maximum compressive stress in  $x$ -direction, and  $\bar{\sigma}_{xT}$  is the maximum tensile stress in  $x$ -direction. The criterion assumes that the magnitude of for example  $\sigma_y$  or  $\tau_{xy}$  has no influence on the fracture stress in  $X$ -direction, and vice versa. However  $\sigma_y$  and  $\tau_{xy}$  influence the resin, so matrix failure can play a role.

### 2) The maximum strain criterion

For the maximum strain criterion strain instead of stress is used:

$$\bar{\varepsilon}_{x\bar{C}} \leq \varepsilon_x \leq \bar{\varepsilon}_{x\bar{T}} \quad \text{and} \quad \bar{\varepsilon}_{y\bar{C}} \leq \varepsilon_y \leq \bar{\varepsilon}_{y\bar{T}} \quad \text{and} \quad \gamma_{xy} \leq \bar{\gamma}_{xy} \quad (\text{A.9})$$

The maximum strain criterion is the most used criterion in industry. The main reason for this is because it is very easy to use because the laminate strains are considered and not the plies individually. The results have a good reliability when fiber orientations are limited to 0°, 90° and ±45° degrees which are the most common orientations used in industry.

### 3) The Tsai-Hill criterion

This criterion relies on the principle of maximum allowable deformation energy

This criterion assumes that failure occurs when:

$$\frac{\frac{\sigma_x^2}{2}}{\bar{\sigma}_x} + \frac{\frac{\sigma_y^2}{2}}{\bar{\sigma}_y} - \frac{\frac{\sigma_x \sigma_y}{2}}{\bar{\sigma}_x} + \frac{\frac{\tau_{xy}^2}{2}}{\bar{\tau}_{xy}} = 1 \quad (\text{A.10})$$

The Tsai-Hill shows a reasonable correlation with experimental results, except for compression stresses, for which Tsai-Hill is often too pessimistic.

### 4) The Puck-criterion

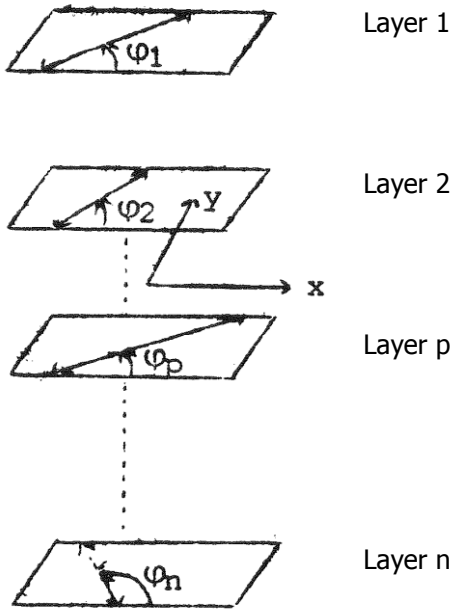
When we consider failure of unidirectional materials in a physical way, there are two failure criteria:

$$\begin{aligned} \frac{\sigma_x}{\bar{\sigma}_x} &\leq 1 && \text{Fiber cracking due to tensile stresses or micro-buckling (compression).} \\ \left( \frac{\sigma_y}{\bar{\sigma}_y} \right)^2 + \left( \frac{\tau_{xy}}{\bar{\tau}_{xy}} \right)^2 &\leq 1 && \text{Matrix cracking wherefore we can use the Von Mises criterion.} \end{aligned} \quad (\text{A.11})$$

### A.3 Laminates: Plane-stress conditions

A laminate consists of several layers of plies / weaves. The reason for this is that plies are generally very thin and cannot handle all the loads. Another reason is that there are normally multiple loading conditions and therefore, at least three fiber orientations are needed to build up a structure. In this section, the mechanical response of laminates is highlighted.

Consider an arbitrary laminate that consists of  $n$  different oriented layers of different thickness.



**Fig. A.3: Stacking of a laminate [Jong de, 1989]**

The different layers force each other, because they are glued to each other, to undergo the same  $\varepsilon_x$ ,  $\varepsilon_y$  and  $\gamma_{xy}$ .

So for the  $k^{\text{th}}$  layer, we have ( $1 \leq p \leq n$ ):

$$\sigma_{i_k} = (S_{ij})_{\varphi_k} \varepsilon_j \quad (\text{eq. A.2})$$

The average stress in the laminate follows from:

$$(\sigma_i)_{\text{lam}} = \sum_{k=1}^{k=n} \sigma_{i_k} \times \frac{t_k}{t} = \sum_{k=1}^n \left\{ (S_{ij})_{\varphi_k} \frac{t_k}{t} \right\} \varepsilon_j \quad (\text{A.12})$$

So, the in-plane stiffness of the laminate is

$$(S_{ij})_{\text{lam}} = \sum_{k=1}^n \left\{ (S_{ij})_{\varphi_k} \frac{t_k}{t} \right\} \quad (\text{A.13})$$

## A.4 Response of a laminated plate under loading

In the previous section it has been shown what the response is of a laminate in a plane stress condition. That means the plate was supposed to be so thin that all stresses and deformations in  $Z$ -direction can be neglected. The stacking sequence of the plies was unimportant and the average stiffness  $s_{ij}$  was attributed to the entire laminate, as if it was considered to be one material with the same stiffness properties.

This approach is valid if the laminate consists of a lot of plies that are stacked in such a way that the effect to the different stiffness from the plies is negligible. If this is not the case, the laminate will deform in the  $z$ -direction (bending and twisting), which cannot be neglected.

To be able to generate valid results for laminated plates, the following assumptions are made:

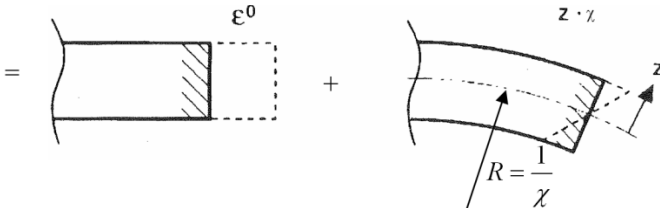
- The plate is constructed of an arbitrary number of layers of orthotropic sheets bonded together. The layers are oriented in an arbitrary direction with the  $xy$ -plane of the plate.
- The plate is thin, i.e., the thickness is much smaller than the other physical dimensions.
- In plane strains  $\varepsilon_x$ ,  $\varepsilon_y$  and  $\gamma_{xy}$  are small compared to unity.
- In order to include in-plane force effects, nonlinear terms in the equations of motion involving products of stresses and plates slopes are retained. All other nonlinear terms are neglected.
- Transverse shear strains  $\varepsilon_{xz}$  and  $\varepsilon_{yz}$  are negligible.
- Tangential displacements  $u$  and  $v$  are linear functions of the  $z$ -coordinate.
- The transverse normal strain  $\varepsilon_z$  is negligible.
- Each ply obeys Hooke's law
- The plate has constant thickness.
- There are no body forces
- Transverse shear stresses  $\sigma_{xz}$  and  $\sigma_{yz}$  vanish on the surfaces at  $z = \pm h / 2$

The response of a laminated plate under loading is formulated by the ABD-matrix. The ABD-matrix is a stiffness matrix that describes the relationship between the

loading and the deformations of a laminate. The strain deformations in a laminate are defined as:

$$\begin{aligned}\varepsilon_x &= \varepsilon_x^0 + Z\chi_x \\ \varepsilon_y &= \varepsilon_y^0 + Z\chi_y \\ \varepsilon_{xy} &= \varepsilon_{xy}^0 + Z\chi_{xy}\end{aligned}$$

(A.14a-c)



**Fig. A.4: Strain relations in a plate**

Where

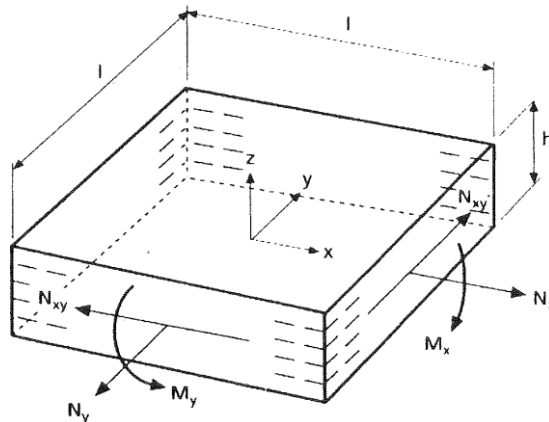
$$\begin{aligned}\varepsilon_x^0 &= \frac{\partial U^0}{\partial x} & \varepsilon_y^0 &= \frac{\partial V^0}{\partial y} & \gamma_{xy}^0 &= \frac{\partial U^0}{\partial y} + \frac{\partial V^0}{\partial x} \\ \chi_x &= -\frac{\partial^2 W}{\partial x^2} & \chi_y &= -\frac{\partial^2 W}{\partial y^2} & \chi_{xy} &= -2\frac{\partial^2 W}{\partial x \partial y}\end{aligned}$$

(A.15a-f)

The in-plane forces and moments, figure A.5, acting on a small element are:

$$\begin{aligned}N_x &= \int_{-h_b}^{h_t} \sigma_x dz & N_y &= \int_{-h_b}^{h_t} \sigma_y dz & N_{xy} &= \int_{-h_b}^{h_t} \tau_{xy} dz \\ M_x &= \int_{-h_b}^{h_t} z \sigma_x dz & M_y &= \int_{-h_b}^{h_t} z \sigma_y dz & M_{xy} &= \int_{-h_b}^{h_t} z \tau_{xy} dz\end{aligned}$$

(A.16a-f)



**Fig. A.5: In-plane loading on a laminate, [Jong de, 1989]**

By substituting eq. (A.15) and (A.3) into (eq. A.16), we can generate the constitutive equations or ABD-matrix:

$$\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \\ M_x \\ M_y \\ M_{xy} \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{16} & B_{11} & B_{12} & B_{16} \\ A_{12} & A_{22} & A_{26} & B_{12} & B_{22} & B_{26} \\ A_{16} & A_{26} & A_{66} & B_{16} & B_{26} & B_{66} \\ B_{11} & B_{12} & B_{16} & D_{11} & D_{12} & D_{16} \\ B_{12} & B_{22} & B_{26} & D_{12} & D_{22} & D_{26} \\ B_{16} & B_{26} & B_{66} & D_{16} & D_{26} & D_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \\ \chi_x \\ \chi_y \\ \chi_{xy} \end{Bmatrix} \quad (\text{A.17})$$

whereby:

$$\begin{aligned} [A] &= \int_{h_b}^{h_t} [\bar{Q}] dz \Rightarrow A_{ij} = \sum_{k=1}^K (\bar{Q}_{ij})_k (z_k - z_{k-1}) \\ [B] &= \int_{h_b}^{h_t} z [\bar{Q}] dz \Rightarrow B_{ij} = \frac{1}{2} \sum_{k=1}^K (\bar{Q}_{ij})_k (z_k^2 - z_{k-1}^2) \\ [D] &= \int_{h_b}^{h_t} z^2 [\bar{Q}] dz \Rightarrow D_{ij} = \frac{1}{3} \sum_{k=1}^K (\bar{Q}_{ij})_k (z_k^3 - z_{k-1}^3) \end{aligned} \quad (\text{A.18})$$

A laminate is generally balanced and symmetric in order to avoid undesired deformations.

In a symmetrical laminate, the ply located at a position  $+z$  is identical to the ply at  $-z$ . For a symmetrical laminate, we find that the  $[B]$  matrix is zero.

In a balanced laminate, for every unidirectional ply in the  $+\varphi$  direction (measured counterclockwise from the  $x$ -coordinate), there is an identical ply in the  $-\varphi$  direction. For these laminates  $A_{16} = A_{26} = 0$ .

The deformations and their relation with the stiffness terms in the ABD-matrix is illustrated in figure A.6.

Other terminologies typically used in composites are orthotropic material and quasi-isotropic material. A laminate is called orthotropic when the stiffness terms  $A_{16}$ ,  $A_{26}$ ,  $B_{16}$ ,  $B_{26}$ ,  $D_{16}$ , and  $D_{26}$  are zero. A quasi-isotropic laminate has the same  $A$ -matrix as an isotropic material and the stiffness terms are defined as:  $A_{11}=A_{22}$ ,  $\nu=A_{12}/A_{11}$  and  $A_{66}=A_{11}(1-\nu)/2$ .

Illustration of the coupling terms  $A_{16}$ ,  $D_{16}$ ,  $B_{16}$ ,  $B_{11}$ ,  $B_{12}$ ,  $B_{66}$ ,  $A_{12}$  and  $D_{12}$ . When the element shown in the last column is zero, there is no coupling. (The coupling terms  $A_{26}$ ,  $D_{26}$ ,  $B_{26}$ ,  $B_{22}$  can be illustrated in a similar manner by applying a force  $N_y$  and a moment  $M_y$  in the  $y$ - $z$  plane.

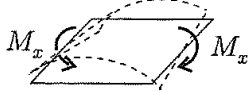
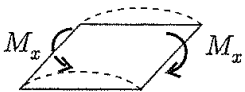
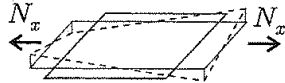
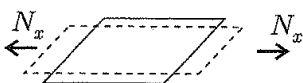
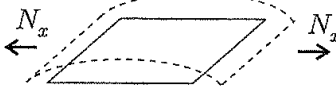
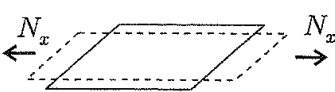
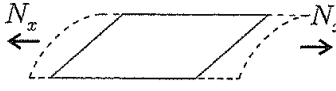
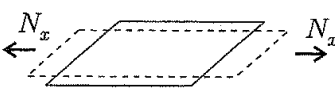
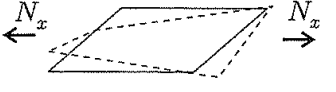
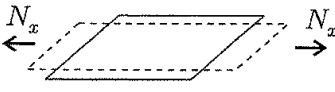
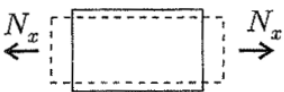
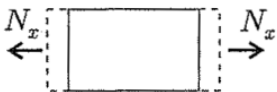
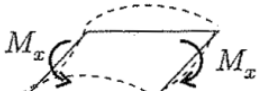
| Coupling                                                                                                    | No Coupling                                                                         | Element  |
|-------------------------------------------------------------------------------------------------------------|-------------------------------------------------------------------------------------|----------|
| Extension-shear<br>        |    | $A_{16}$ |
| Bending-twist<br>          |    | $D_{16}$ |
| Extension-twist<br>        |    | $B_{16}$ |
| In-plane-out-of-plane<br> |   | $B_{11}$ |
|                          |  | $B_{12}$ |
|                          |  | $B_{66}$ |
| Extension-extension<br>  |  | $A_{12}$ |
| Bending-bending<br>      |  | $D_{12}$ |

Fig. A.6: Significance of the laminate coupling terms [Kollar, 1999]

## A.5 Strength of a laminate

If the load acting on a component is oriented in one direction, it may seem wise to place all fibers in one direction, the loading direction. In practice, however, a multi-axial stress state will occur. To cope with this there should be a percentage of fibers in at least three directions. In most cases the laminate will be symmetrical with respect to the centerline so that the B-matrix in the coupling ABD-matrix will be zero. It will also be balanced in most cases, so that the terms  $A_{16}$  and  $A_{26}$  will be zero in the A-matrix.

Failure analysis for laminates is more complicated than for a single ply, for the following reasons:

- The stress throughout the laminate will vary strongly with ply orientation.
- Plies have a mutual interaction if one ply starts to crack
- At the free edges of a laminate shear- and tensile stresses will occur that might initiate premature delamination and failure.

It is important to make a distinction between first ply failure and overall failure of the structure. If first ply failure coincides with overall failure, you may seem to have a favorable situation in terms of optimization, but the structure will not have any residual strength and fracture will be explosive without any warning of damage by for example visual damage or stiffness reduction. These types of structures therefore need an extra low stress level, which diminishes the weight saving.

Just as for metals, we can state:

- At limit load no permanent damage must occur (matrix failure)
- At ultimate load no overall structure failure may occur (fiber failure)

The criteria from the previous paragraph can be applied, where each ply must be analyzed at a certain load. This will give information on which layer will fail first, and how it will fail. This theory is known as the continuum theory, since it handles the ply as a macroscopically homogeneous continuum.

To analyze the laminate after first ply failure, we can choose from one of the following methods:

1. Choose  $E_y$  and  $G_{xy}$  of the damaged plies zero and recalculate the distribution of the stress in all plies. Locate the new critical layers and if they have failed also, perform the procedure again.
2. Remove all failed plies out of the analysis entirely and recalculate stress distribution and residual stress.
3. Approximate the residual strength by disregarding the stiffness and strength of the resin. The analysis is done on a network of fibers, a method that is known as the netting theory.

The second method is obviously the most conservative one and is the one to use as the lower boundary for strength of the laminate.

There are a number of disadvantages associated with these methods:

- No attention is paid to the fact that layers interact with each other. Fracture of a weaker layer will be delayed because the adjacent layers support this ply. Fracture of a layer may also cause premature failure of adjacent layers.
- We also have to take edge effects into account. Through interlaminar tension and/or shear stress delamination may be caused that can be the source for permanent damage or fracture.

To calculate the strength of a laminate in terms of stresses, we use the following method:

- All plies undergo the same displacements. The strain for the laminate becomes therefore:

$$\{\varepsilon_i\} = [S_{ij}]_{lam}^{-1} \cdot \{\sigma_j\} \text{ or } \{\varepsilon_i\} = [C_{ij}]_{lam} \cdot \{\sigma_j\} \quad (\text{A.19})$$

- If the strain of the laminate, and thus the plies is known, the stresses in each ply can be calculated:

$$\{\sigma_i\}_k = [S_{ij}] \cdot [T_\sigma]^T \cdot \{\varepsilon_j\} \quad (\text{A.20})$$

For each ply, the stress will be calculated and with the Tsai-Hill criterion, each ply can be checked whether the entire laminate can handle the loads.

If the maximum strain criterion is used, the analysis becomes even more simple, and the strains in each ply only need to be calculated with the following formula:

$$\{\varepsilon_j\}_k = [T_\varepsilon] \bullet \{\varepsilon_j\} \quad (\text{A.21})$$



## APPENDIX B

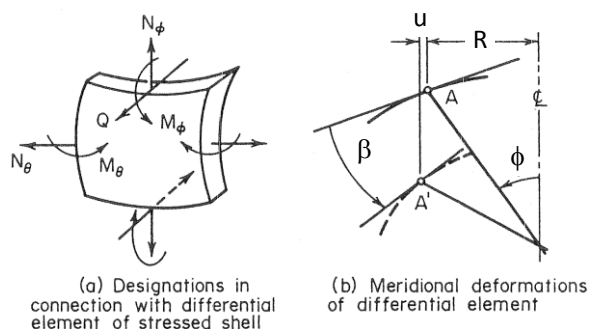
### Secondary solutions

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The secondary solutions are the loads required to bring the displaced edge of the shell into the position prescribed by boundary conditions as shown in figure B.1. Figure B.1b shows the meridional deformations of the element in figure B.1a in which point A is displaced to the position A'.

The secondary solutions are needed when the junction or the edge of a shell is analysed by using the force method which was introduced in chapter 4.

This appendix presents secondary solutions for isotropic non-shallow spherical, conical, cylindrical shells and circular plates. This appendix relies on material from [Baker, 1972].



**Fig. B.1 Unit edge loading (a) and meridional deformations (b) due to the secondary solutions [Baker, 1972]**

In a non-shallow shell, the edge disturbances die out before reaching the apex of the shell. For the cases shown, the boundaries of the shells are assumed to be free to rotate and deflect vertically and horizontally because of the action of edge loadings. Abrupt discontinuities in the shell thickness must not be present. The thickness of the shell must be uniform in the range in which the stresses are present.

Equations are here listed that can be used for closed and open shells. Open shells are shells that have an axi-symmetrical circular opening. The segments must have

a meridional length such that the disturbances due to the edge loading will die out before the opposite edge is reached. For the spherical shell this implies that the angle  $\alpha$  is equal or larger than  $20^\circ$ , see figure B.2. Edge loadings may act at the lower or upper edge of the open shell. Linear bending theory is used for the derivation of the formulas.

The circumferential deformations are obtained directly from the meridional deformations and can be obtained from the displacement  $u$  in the tables

## B.1 Spherical shells

Secondary solutions are presented in table B.1. The  $k_s$ -factor in table B.1 is a dimensionless geometric factor given by:

$$k_s = \sqrt[4]{\left(\frac{R}{t}\right)^2 3(1-\mu^2)} \quad (\text{B.1})$$

Figure B.2 represents the case in which the  $M$  and  $H$  loading acts on the upper edge of an open shell. The formulas in table B.1 can also be used,  $\phi_1 > 90^\circ$ . The actual shell in figure B.2 is imagined to be turned  $180^\circ$  as shown in figure B.3.

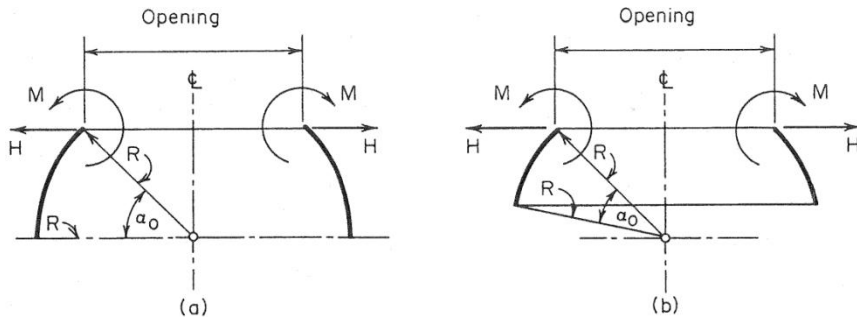
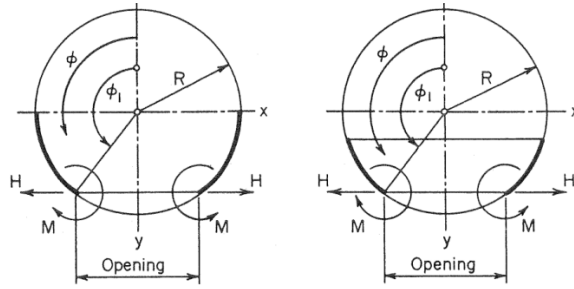
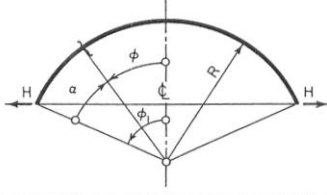
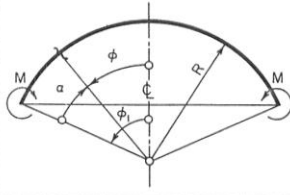


Fig. B.2 Loadings at upper edge of open spherical Shell [Baker, 1972]

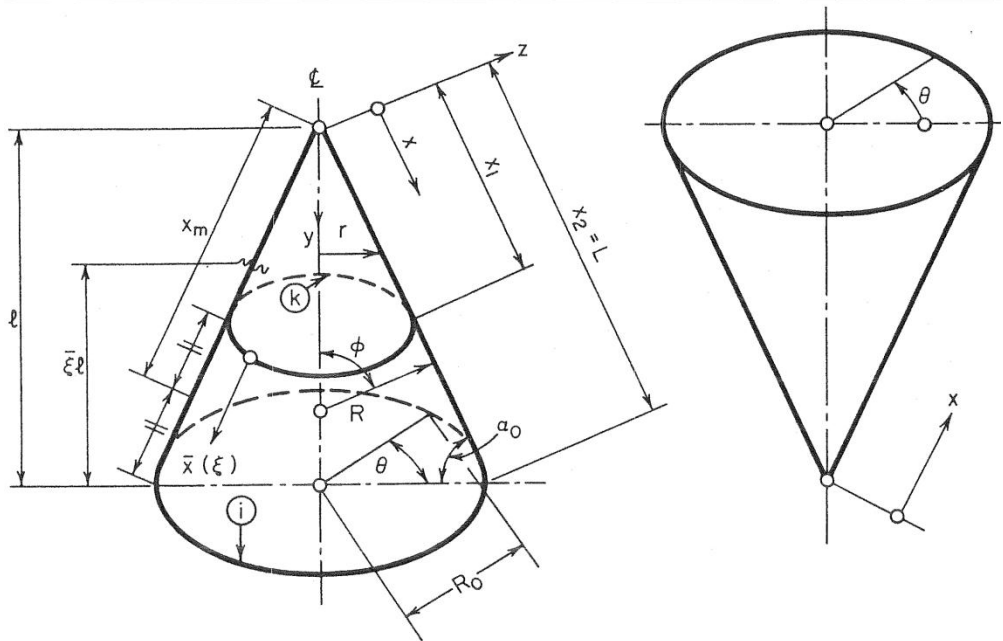


**Fig. B.3 Spherical Shell of which  $\phi_1 > 90^\circ$  [Baker, 1972]**

|                                                                                   |                                                                                                                                                               |                                                                                                                                                                                                 |  |
|-----------------------------------------------------------------------------------|---------------------------------------------------------------------------------------------------------------------------------------------------------------|-------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|--|
|  |                                                                                                                                                               |                                                                                                                |  |
| $Q_\phi$                                                                          | $\left[ -\sqrt{2} \sin \phi_1 e^{-k\alpha} \cos \left( k\alpha + \frac{\pi}{4} \right) \right] H$                                                             | $\left( +\frac{2k}{R} e^{-k\alpha} \sin k\alpha \right) M$                                                                                                                                      |  |
| $N_\phi$                                                                          | $-Q_\phi \cot \phi$                                                                                                                                           | $-Q_\phi \cot \phi$                                                                                                                                                                             |  |
| $N_\theta$                                                                        | $\left( 2k \sin \phi_1 e^{-k\alpha} \cos k\alpha \right) H$                                                                                                   | $\left[ +2\sqrt{2} \frac{k^2}{R} e^{-k\alpha} \cos \left( k\alpha + \frac{\pi}{4} \right) \right] M$                                                                                            |  |
| $M_\phi$                                                                          | $\left( \frac{R}{k} \sin \phi_1 e^{-k\alpha} \sin k\alpha \right) H$                                                                                          | $\left[ \sqrt{2} e^{-k\alpha} \sin \left( k\alpha + \frac{\pi}{4} \right) \right] M$                                                                                                            |  |
| $M_\theta$                                                                        | $\left[ -\frac{R}{k^2 \sqrt{2}} \sin \phi_1 \cot \phi e^{-k\alpha} \sin \left( k\alpha + \frac{\pi}{4} \right) \right] H + \mu M_\phi$                        | $\left( \frac{1}{k} \cot \phi e^{-k\alpha} \cos k\alpha \right) M + \mu M_\phi$                                                                                                                 |  |
| Deformations                                                                      |                                                                                                                                                               |                                                                                                                                                                                                 |  |
| $E t \beta$                                                                       | $\left[ -2\sqrt{2} k^2 \sin \phi_1 e^{-k\alpha} \sin \left( k\alpha + \frac{\pi}{4} \right) \right] H$                                                        | $\left( -\frac{4k^3}{R} e^{-k\alpha} \cos k\alpha \right) M$                                                                                                                                    |  |
| $E t u$                                                                           | $\left\{ R \sin \phi_1 e^{-k\alpha} \left[ 2k \sin \phi \cos k\alpha - \sqrt{2} \mu \cos \phi \cos \left( k\alpha + \frac{\pi}{4} \right) \right] \right\} H$ | $+ \left[ R \sin \phi (N_\theta - \mu N_\phi) \right] M + 2k e^{-k\alpha} \left[ \sqrt{2} k \sin \phi \cdot \cos \left( k\alpha + \frac{\pi}{4} \right) + \mu \cos \phi \sin k\alpha \right] M$ |  |
| For $\alpha = 0$ and $\phi = \phi_1$                                              |                                                                                                                                                               |                                                                                                                                                                                                 |  |
| $E t \beta$                                                                       | $(-2k^2 \sin \phi_1) H$                                                                                                                                       | $\left( -\frac{4k^3}{R} \right) M$                                                                                                                                                              |  |
| $E t u$                                                                           | $\left[ R \sin \phi_1 (2k \sin \phi_1 - \mu \cos \phi_1) \right] H$                                                                                           | $(+2k^2 \sin \phi_1) M$                                                                                                                                                                         |  |
| For $\phi_1 = 90^\circ$                                                           |                                                                                                                                                               |                                                                                                                                                                                                 |  |
| $E t \beta$                                                                       | $-2k^2 H$                                                                                                                                                     | $-\frac{4k^3}{R} M$                                                                                                                                                                             |  |
| $E t u$                                                                           | $2RkH$                                                                                                                                                        | $+2k^2 M$                                                                                                                                                                                       |  |

**Table B.1: Secondary solutions for the isotropic spherical shell [Baker, 1972]**

## B.2 Conical shells



**Fig. B.4 Cone nomenclature which applies to table B.2 [Baker, 1972]**

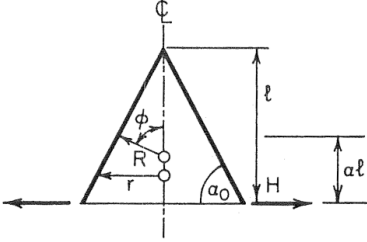
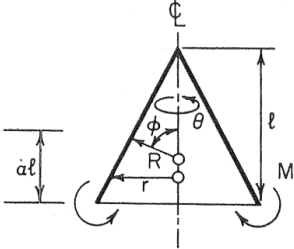
The following terms are defined for the cone:

$$k_c = \frac{1}{\sqrt{R_0 t} \sin \phi} {}^4\sqrt{3(1-\mu^2)} \quad (\text{B.2a})$$

$$R_0 = \max R \quad (\text{B.2b})$$

$$D = \frac{Et^3}{12(1-\mu^2)} \quad (\text{B.2c})$$

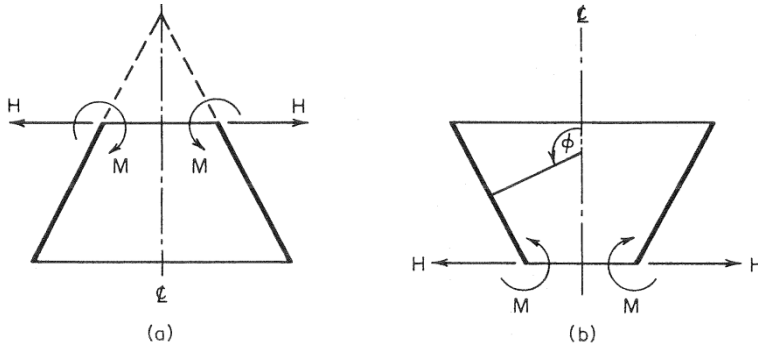
$R$  is variable and perpendicular to the meridian. The angle  $\phi$  is constant. Table B.2 presents the formulas for a closed conical shell.

| Horizontal load                                                                   |                                                                                                                                                                      | Moment loading                                                                                                                                                        |  |
|-----------------------------------------------------------------------------------|----------------------------------------------------------------------------------------------------------------------------------------------------------------------|-----------------------------------------------------------------------------------------------------------------------------------------------------------------------|--|
|  |                                                                                                                                                                      |                                                                                      |  |
| $N_\phi$                                                                          | $\left[ -\sqrt{2} \cos \phi e^{-k\alpha} \cos(k\alpha + \frac{\pi}{4}) \right] H$                                                                                    | $\left( -\frac{2k \cos \phi}{\ell} e^{-k\alpha} \sin k\alpha \right) M$                                                                                               |  |
| $N_\theta$                                                                        | $\left( -\frac{2Rk \sin^2 \phi}{\ell} e^{-k\alpha} \cos k\alpha \right) H$                                                                                           | $\left[ \frac{2\sqrt{2} Rk^2 \sin^2 \phi}{\ell^2} e^{-k\alpha} \cos(k\alpha + \frac{\pi}{4}) \right] M$                                                               |  |
| $M_\phi$                                                                          | $H \frac{\ell}{k} e^{-k\alpha} \sin k\alpha$                                                                                                                         | $\left[ -\sqrt{2} e^{-k\alpha} \sin(k\alpha + \frac{\pi}{4}) \right] M$                                                                                               |  |
| $M_\theta$                                                                        | $H \frac{\ell^2}{\sqrt{2} Rk^2} \frac{\cot \phi}{\sin \phi} e^{-k\alpha} \sin(k\alpha + \frac{\pi}{4}) + \mu M \phi$                                                 | $\left( -\frac{\ell \cot \phi e^{-k\alpha} \cos k\alpha}{Rk \sin \alpha} + \mu M \phi \right) M$                                                                      |  |
| $Q$                                                                               | $H \left[ -\sqrt{2} \sin \phi e^{-k\alpha} \cos(k\alpha + \frac{\pi}{4}) \right]$                                                                                    | $\left( -\frac{2k \sin \phi}{\ell} e^{-k\alpha} \sin k\alpha \right) M$                                                                                               |  |
| Deformations                                                                      |                                                                                                                                                                      |                                                                                                                                                                       |  |
| $u$                                                                               | $H \frac{\ell^3 e^{-k\alpha}}{2Dk^3 \sin \phi} \left[ \cos k\alpha - \mu \frac{\ell}{\sqrt{2} Rk} \frac{\cot \phi}{\sin \phi} \cos(k\alpha + \frac{\pi}{4}) \right]$ | $-\frac{\ell^2 e^{-k\alpha}}{2Dk^2 \sin \phi} \left[ \sqrt{2} \cos(k\alpha + \frac{\pi}{4}) + \mu \frac{1}{R} \frac{\cos \phi \sin k\alpha}{k \sin^2 \phi} \right] M$ |  |
| $\beta$                                                                           | $-\frac{\ell^2 e^{-k\alpha} \sin(k\alpha + \pi/4)}{\sqrt{2} Dk^2 \sin \phi} H$                                                                                       | $\frac{\ell e^{-k\alpha} \cos k\alpha}{Dk \sin \phi} M$                                                                                                               |  |
| For $\alpha = 0$                                                                  |                                                                                                                                                                      |                                                                                                                                                                       |  |
| $u$                                                                               | $\frac{\ell^3}{2Dk^3 \sin \phi} \left( 1 - \frac{\mu \ell \cot \phi}{2Rk \sin \phi} \right) H$                                                                       | $-\frac{\ell^2}{2Dk^2 \sin \phi} M$                                                                                                                                   |  |
| $\beta$                                                                           | $-\frac{\ell^2}{2Dk^2 \sin \phi} H$                                                                                                                                  | $\frac{\ell}{Dk \sin \phi} M$                                                                                                                                         |  |

Note: in this table  $\alpha$  is a coefficient which locates the section, (not to be mistaken for angle  $\alpha_0$ )

**Table B.2: Secondary solutions for the isotropic conical shell [Baker, 1972]**

Similar as with the spherical shell, loads acting on the upper edge of an open shell can be considered by taking the an angle  $\phi_1 > 90^\circ$ .



**Fig. B.5 Open conical shell loading at upper edge [Baker, 1972]**

The equations are simplified when the angle  $\alpha_0$  equals  $90^\circ$ . Table B.3 presents the simplified equations for the cylinder.

| $N_x$                  | 0                                                                                             | 0                                                                                                               |
|------------------------|-----------------------------------------------------------------------------------------------|-----------------------------------------------------------------------------------------------------------------|
| $N_\theta$             | $-H \frac{2Rk}{L} e^{-k\xi} \cos k\xi$                                                        | $M \frac{2\sqrt{2} Rk^2}{L^2} e^{-k\xi} \cos(k\xi + \pi/4)$                                                     |
| $M_x$                  | $H \frac{L}{k} e^{-k\xi} \sin k\xi$                                                           | $M [-\sqrt{2} e^{-k\xi} \sin(k\xi + \pi/4)]$                                                                    |
| $M_\theta$             | $\mu M_x$                                                                                     | $\mu M_x$                                                                                                       |
| $Q$                    | $H \sqrt{2} e^{-k\xi} \cos(k\xi + \pi/4)$                                                     | $M \frac{2k}{L} e^{-k\xi} \sin k\xi$                                                                            |
| $\beta$                | $H \frac{L^2}{\sqrt{2} k^2 D} e^{-k\xi} \sin(k\xi + \pi/4)$                                   | $M \left( -\frac{L}{Dk} e^{-k\xi} \cos k\xi \right)$                                                            |
| $U$                    | $-\frac{R}{Et} (N_\theta - \mu N_x) = H \left( \frac{L^3}{2Dk^3} e^{-k\xi} \cos k\xi \right)$ | $-\frac{R}{Et} (N_\theta - \mu N_x) = M \left[ -\frac{L^3}{\sqrt{2} Dk^2} e^{-k\xi} \cos(k\xi + \pi/4) \right]$ |
| For the case $\xi = 0$ |                                                                                               |                                                                                                                 |
| $\beta$                | $H L^2 / 2k^2 D$                                                                              | $M (-L/Dk)$                                                                                                     |
| $U$                    | $H L^3 / 2k^3 D$                                                                              | $M (-L^2 / 2Dk^2)$                                                                                              |

**Table B.3: Secondary solutions for the isotropic cylindrical shell [Baker, 1972]**

## B.3 Circular plates

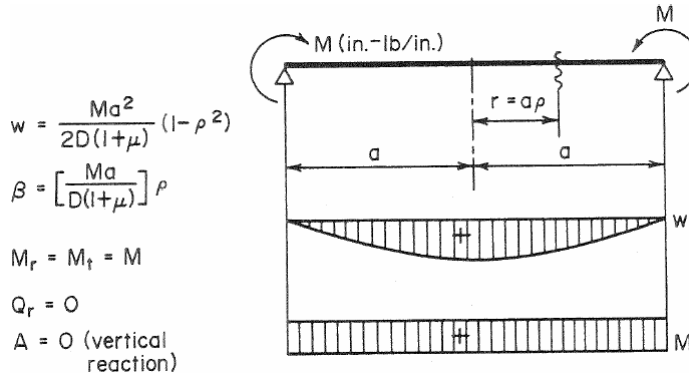


Fig. B.6: Unit edge moment solutions for circular plates [Baker, 1972]

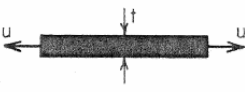
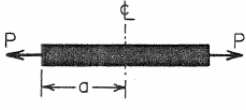
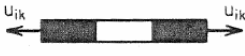
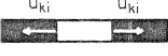
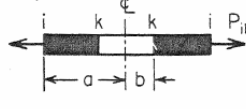
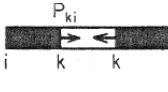
|                                         |                                                                  |                                                            |
|-----------------------------------------|------------------------------------------------------------------|------------------------------------------------------------|
| <div>Stress</div> <div>Elongation</div> |                                                                  |                                                            |
|                                         | $\frac{uEt}{a(1-\mu)}$                                           |                                                            |
| <div>Stress</div> <div>Elongation</div> |                                                                  |                                                            |
|                                         | $u_{ik} \frac{Et(1-\lambda^{-2})}{a[1-\mu+\lambda^{-2}(1+\mu)]}$ | $0$                                                        |
|                                         | $0$                                                              | $u_{ki} \frac{Et(\lambda^2-1)}{b[1-\mu+\lambda^2(1+\mu)]}$ |

Note:  $\lambda = a/b$ .

Boundary conditions: \* If  $r = a$ ,  $u = u_{ik}$ ; if  $r = b$ ,  $N_r = 0$ .

† If  $r = a$ ,  $N_r = 0$ ; if  $r = b$ ,  $u = u_{ik}$ .

Table B.4: Unit edge displacement solutions for circular plates [Baker, 1972]

|                                                                                                   |                                                                                   |                                                                                    |
|---------------------------------------------------------------------------------------------------|-----------------------------------------------------------------------------------|------------------------------------------------------------------------------------|
| <div> <div>Elongation<br/>(Reacting)</div> <div> <div>Loads<br/>(Disturbance)</div> </div> </div> |  |                                                                                    |
|                  | $\frac{Pa}{Et}(1-\mu)$                                                            |                                                                                    |
| <div> <div>Elongation</div> <div>Loads</div> </div>                                               |  |  |
|                  | $\frac{Pa}{Et(1-\lambda^{-2})} [\lambda^{-2}(1+\mu) + 1 - \mu]$                   | $\frac{2Pb}{Et(1-\lambda^{-2})}$                                                   |
|                  | $-\frac{2Pa}{Et(\lambda^2-1)}$                                                    | $-\frac{Pb}{Et(\lambda^2-1)} [\lambda^2(1+\mu) + 1 - \mu]$                         |

Note:  $\lambda = a/b$ ; vectors as indicated are applied all around the edge.

**Table B.5: Unit edge load solutions for circular plates [Baker, 1972]**





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# ABOUT THE AUTHOR



François Jeanne Joseph Marcellus Marie is a Dutchman who was born on August 7<sup>th</sup> 1979 in Maaseik, Belgium. François went to school in Belgium. In 1997, François pursued a career in engineering and did his Bachelor in Mechanical Engineering at Hogeschool Zuyd in Heerlen. Continuing his curiosity and fascination in engineering, he enrolled at the faculty of Aerospace Engineering at the Delft University of Technology in 2001, where he completed his Bachelor Degree in 2005 and his Master's Degree in 2007. He did his Master's in the chair of 'Design and Production of Composite Structures', where he graduated on the topic of structural analysis of composite plates and shells. It was in this chair where he continued as a researcher. Until 2007, François had spent much time on engineering-based extra-curricular activities and became a specialist in the design and manufacturing of prize-winning prototypes such as motorized units for wheel chairs, a gravity car, a human-powered submarine and a streamlined pronebike. Following on from this experience, he gained more in-depth knowledge in the structural design of composite structures and went on to lecture in this field. In his research position, François' major activity involved PhD research on the Design and Analysis of Conformable Pressurized Structures, but you know that already ;). The PhD research formed part of the CleanEra Project, of which François was interim project leader. In 2011, François assured the continuation of his doctoral work through the FP7-project, CHATT.

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**Academic contribution:**

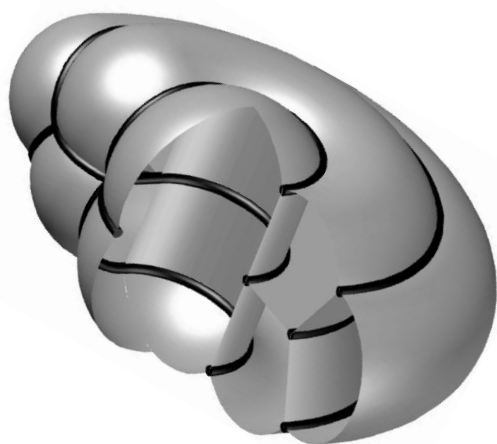
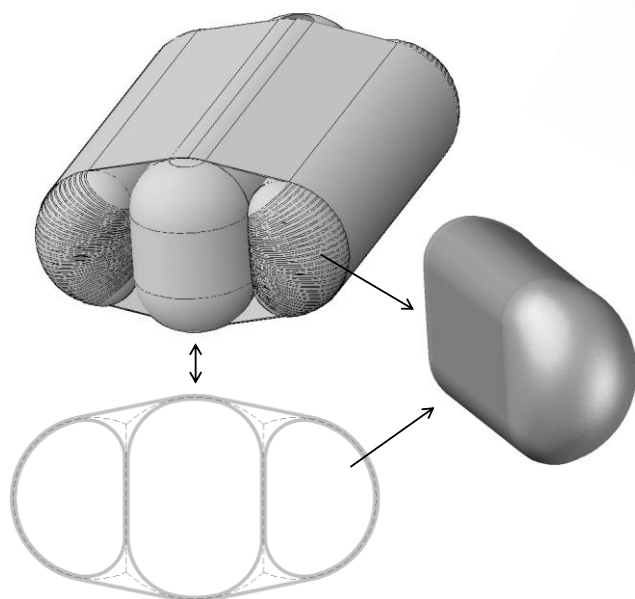
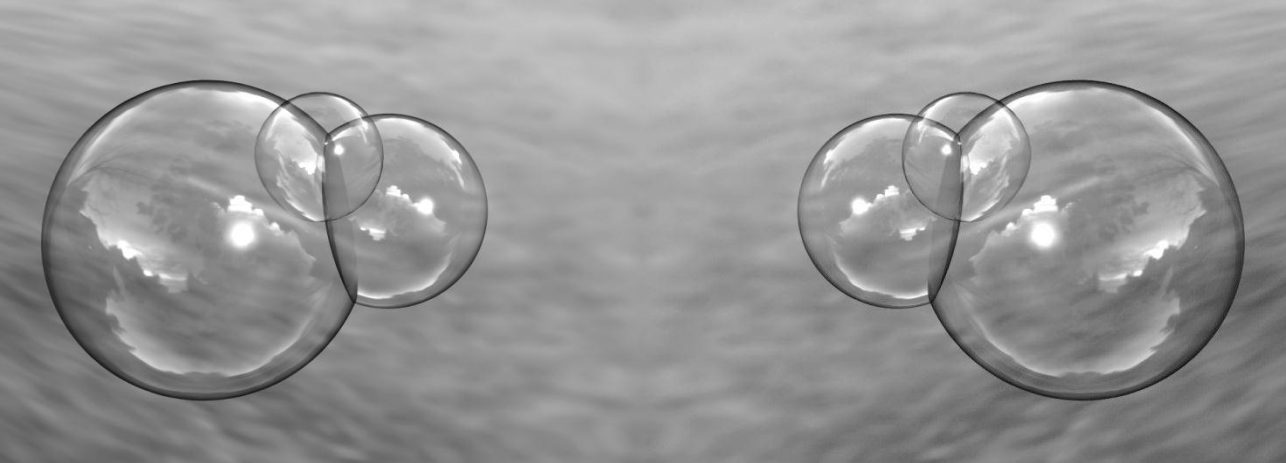
- Lecture notes and a syllabus for the course "Design of composite Aircraft Structures", Faculty of aerospace engineering, TUDelft, 2008-2012











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