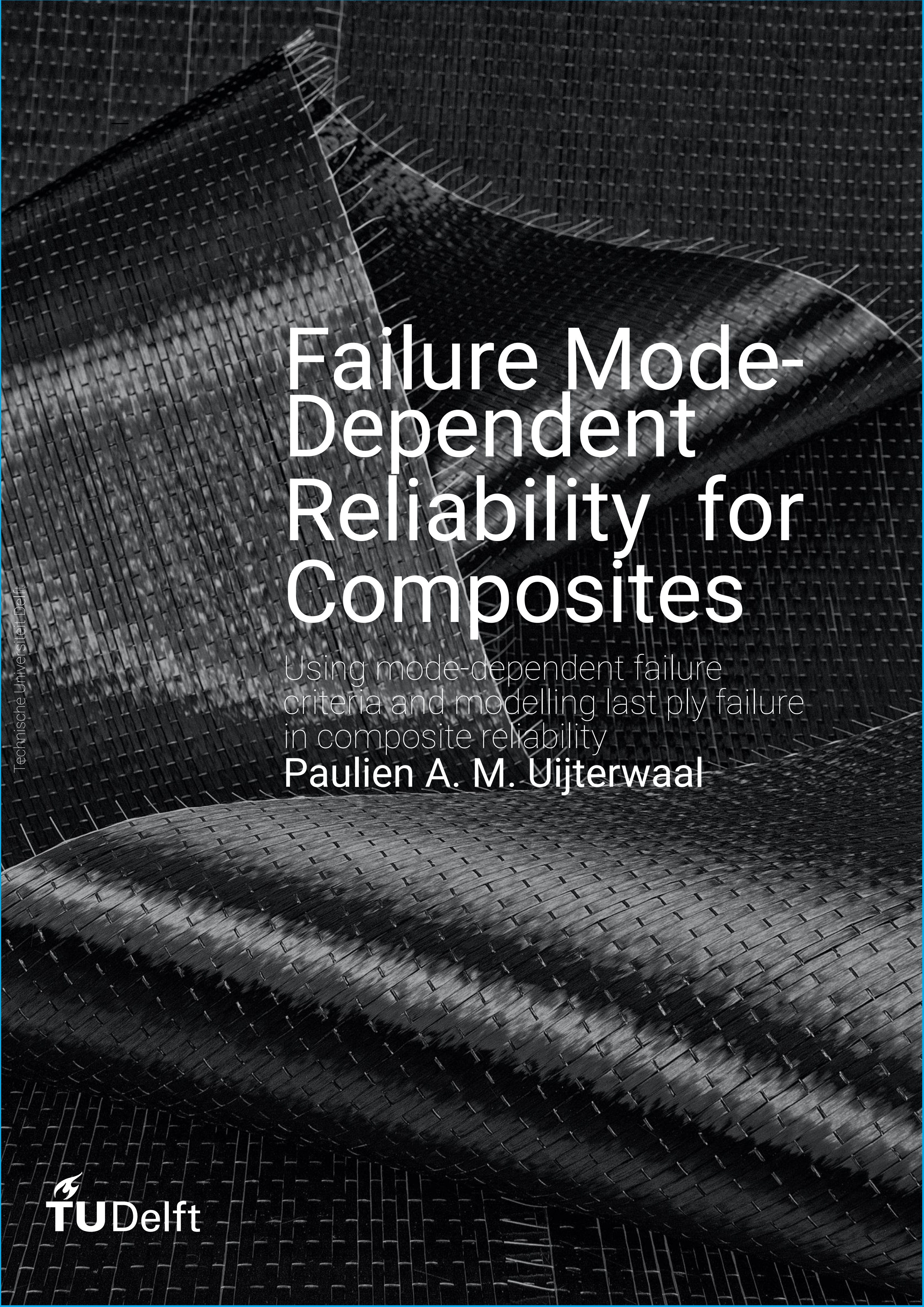

Failure Mode-Dependent Reliability for Composites

Using mode-dependent failure criteria and modelling last ply failure in composite reliability

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by

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Nomenclature

Abbreviations

CDF Cumulative Density Function

FORM First Order Reliability Method

FPF First Ply Failure

LPF Last Ply Failure

MPP Most Probable Point

PDF Probability Density Function

SORM Second Order Reliability Method

Greek symbols

β Reliability Index

λ Scale factor of a Weibull distribution

μ Mean of a normal distribution

ν_{12} Poisson's ratio

$\vec{\alpha}$ unit vector in the direction of ∇g_x

Φ Cumulative density function of the standard normal distribution

φ Probability density function of the standard normal distribution

σ Standard deviation of a normal distribution

σ_1 Principal stresses in a ply parrallel to the fibre direction

σ_2 Principal stresses in a ply perpendicular to the fibre direction

σ_{12} Principal stresses in a ply in shear

Latin Symbols

E_1 Stiffness of the ply in the direction of the fibres

E_2 Stiffness of the ply in the direction perpendicular to the fibres

F_f^c Failure index for fibre failure in compression

F_f^t Failure index for fibre failure in tension

F_m^A Failure index for matrix failure in mode A, compression

F_m^B Failure index for matrix failure in mode B, moderate transverse compression

F_m^C Failure index for matrix failure in mode C, large transverse compression

F_m^c Failure index for matrix failure in compression

F_m^t Failure index for matrix failure in tension

F_z	Cumulative density functions of the vector of random variables $\vec{z}$
f_z	Probability density functions of tthe vector of random variables $\vec{z}$
$g(x)$	Failure function
$g^*(x)$	Linearised failure function
G_{12}	Stiffness of the ply in shear
k	Shape factor of a Weibull distribution
P_f	Probability of failure
X_C	Compressive strength of the ply in the direction of the fibres
X_T	Tensile strength of the ply in the direction of the fibres
Y_C	Compressive strength of the ply in the direction perpendicular to the fibres
Y_T	Tensile strength of the ply in the direction perpendicular to the fibres
$\mathbb{R}$	Set of the reals
S	Shear strength of the ply
S_i	Failure sequence i of a given layup
$\vec{u}$	Vector of non-standardized normal variables
$\vec{x}$	Vector of standard normal variables
$\vec{z}$	Vector of random variables following the original distributions

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Preface

This report concludes my time at Delft University of Technology. I spent an amazing few years here, and I am both happy and sad to end this chapter of my life. I have learned so much at the aerospace faculty over the last seven years, both in professional aspects as in more personal aspects. In particular, at the ASM department, I found a lot of educational support and enthusiasm for the subject.

First and foremost, I would like to thank my supervisor, Dimitrios. Due to the fact that I started my literature study and thesis in spring 2020, right at the start of the corona pandemic, this undeniably affected the project. He was able to make the most of the supervision, even though we were not able to meet up in person until the day of my graduation.

Furthermore, I am grateful for my friends, in particular my neighbours, for the workday coffee breaks and the weekend board games. You have been an very important aspect for me to keep my motivation, and the semblance of a normal life in this pandemic. Finally, I would like to thank my family and boyfriend, for your unconditional support, even while we were in different countries, not knowing when we would be able to see each other again.

Paulien A. M. Uijterwaal
Delft, June 2021

Summary

The failure of composite laminates is most commonly evaluated through deterministic approaches. Based on fixed values for input parameters such as the E-modulus, a fixed value for the failure load is computed. However, the deterministic approach can't reliably predict failure in every single case. Indeed, these traditional methods predict the failure well on average, but cannot take into account the inherent probabilistic aspect of failure mechanics. It is important to be aware of how much this result can be trusted for any singular part as there is a lot of variability in composite failure.

This is where reliability methods come in. Instead of fixed material properties load, these methods use probabilistic approaches to estimate the probability of failure. This allows for a better structural assessment, and applied to aerospace engineering, could decrease the need to overdesign for the worst cases, and therefore improve the structure's efficiency.

Limitations of the FORM method in composite reliability

Several reliability methods exist, each with their strengths and weaknesses. On of the most commonly used methods is the First Order Reliability Method (FORM) together with the Rackwitz–Fiessler optimisation algorithm. This approach seems very interesting because it promises results in good agreement with Monte-Carlo results for less computational effort. However, in the case of composite reliability, it is shown that FORM is not always deliver on these promises for the following reasons:

- **Limited to normal and lognormal distributions:** A crucial step in FORM is the transformation from a general random variable to a standard normal variable. If the original variable does not follow either a normal or log-normal distribution, this transformation can't be done exactly, and will always be an approximation. This poses problems in the aerospace industry where low probability are used. However, forcing the use of lognormal or normal distributions is undesirable, as this may not represent experimental results well for many variables. The inaccuracies due to the approximation in the transformation especially poses problems for variables that are only defined on part of $\mathbb{R}$, such as Weibull distributions. In those cases, the design point can end up negative, which leads to errors. Therefore it is recommended to avoid FORM if the material parameters cannot be modelled well by normal and lognormal distributions.

- **Unreliable:** Lekou [1] noted that the algorithm does not converge for all combinations of loading. In this research it is shown that there are many reasons that the FORM method may not produce the desired result. First, the algorithm described by Madsen [2] used by Lekou can diverge when the estimate for the design point is far from the mean of the distribution used. Next, the FORM algorithm requires an optimisation step and the optimisation space has often been noted as highly non-convex with many local minima [1]. Therefore it is likely that a local minimum is found, overestimating the reliability of the composite structure. This could be solved by increasing the computational effort, but in that case there is no longer any reason to favour FORM over Monte Carlo methods.

All in all, even if FORM seems an ideal candidate on paper, in practise these problems mean that this method is not a good candidate for composite reliability. Therefore, Monte Carlo methods have been used to investigate the effects of better failure models in composite reliability.

Including better failure models in composite reliability

Current applications of reliability methods in composites are still limited by using very simple models of damage and failure[4]. By including more complex failure models, the accuracy of their predictions will improve. This can be achieved through two main aspects: changing the failure criterion to a failure mode-dependent criterion, and including damage progression.

- **Failure mode-dependence:** Almost every application of reliability to composites uses failure mode independent criteria [4]. Instead, using a failure mode-dependent criterion such as Hashin or Puck is shown to have a significant effect on the reliability prediction, with little increase in the computational

effort needed. The effects are strongest in the compressive domain, where fibre fracture is less likely to occur. The predictions of both mode-dependent failure criteria are very similar, suggesting that the difference is indeed due to the ability to predict different failure-modes, and not due to any particularities of Hashin or Puck. It is hypothesised that the model that includes mode-dependence will better fit experimental results, but further research is needed to confirm this hypothesis.

- **Last ply failure:** Reliability methods have only been applied to First Ply Failure (FPF), where it is also important to know about the Last Ply Failure (LPF). Indeed, due to the high damage tolerance of composite materials, designing for FPF only could lead to overdesign and inefficient structures. Therefore, in this research, two approaches to model LPF have been proposed. In the direct approach, the failure function uses deterministic fracture analysis to detect LPF. This approach predicts significantly lower failure probabilities, and likely underestimates the reliability of the composite structure. The second proposed method, called the Bayesian approach, is based on a probabilistic analysis of the failure of the composite laminate. It is shown that this approach is less conservative than the direct approach and are expected to better represent experimental results. However this methods is significantly more computationally expensive. Further research is needed to compare these results with experimental data.

- **Mode-dependent last ply failure** It is possible to combine both methods and predict the last ply failure reliability based on a mode-dependent failure criterion. It is shown that using this type of failure criterion has an even stronger effect on the reliability predictions for last ply failure than for first ply failure. Like for first ply failure, the differences between both mode-dependent failure criterion were small, further suggesting that it is indeed the inclusion of different failure-modes, especially related to matrix failure that has an effect on the results.

To conclude, based on Monte Carlo simulations, this research was able to show that the predictions based on more accurate failure models differ significantly from the simpler models, and it is expected that this type of model better represents experimental results.

All models are wrong, but some are useful.

-George Box

1

The need for failure mode-dependent reliability in composites

The most widely used approach to study composite failure is based on deterministic fracture mechanics. To define failure, several different failure criteria can be used. The preferred type of failure criterion is failure mode dependent, as these do not only predict that failure occurs, but also what type of failure occurs. This allows the engineer to better evaluate the impact of the failure, and whether it is critical or not. However, the deterministic approaches to composite failure are still limited as it does not take into account the material variability. Due to the small series and manual work that is common for composite materials, the variability between two products can be much higher than for products made out of classical materials such as metals.

Therefore, the classical approach will be studied in section 1.1, followed by the advantages and disadvantages of several different failure criteria in section 1.2. Then, several probability distributions will be discussed in section 1.3 to prepare for section 1.4, where the different material parameters that can be modelled as random variables can be discussed. Finally, in section 1.5, the aim of the research will be presented.

1.1. Deterministic fracture mechanics

Currently, the most widely used approach to analyse composite failure is based on deterministic fracture mechanics. In the first section, the focus will be on the basic assumptions made by this theory. In the next section we will show how failure and damage are modelled.

1.1.1. Deterministic stress analysis for composites

A composite laminate is a structure that is made out of multiple separate phases with different mechanical and chemical properties in order to create a laminate that has characteristics tailored to the use case. In this study, the focus will be on reinforced plastics composed of generally strong and stiff fibre bundles surrounded by a polymer. The fibre material is manufactured in sheets or lamina stacked on top of each other to create a laminate. In each layer, the fibres can be oriented differently, to combine into a laminate with different properties in different directions. Indeed, the material properties of the resulting structure are strongly dependent on the stacking sequence, called the layup.

One can analyse the resulting structure in several ways. The first and simplest approach is to analyse the material from a macroscale, as a homogeneous material with bulk material properties such as Strain Invariant Failure Theory (SIFT) [5]. However, this this approach should only be followed for macroscopic analysis, such as preliminary feasibility studies, as it does not properly take into account the anisotropic and heterogeneous nature of the structure [6]. As this method does not distinguish different plies in the laminate, it renders analysis of first and last ply failure impossible.

Next, one can descend at the meso scale or ply level, where the distinct behaviour of the the individual constituents of the composite (the matrix, reinforcement, and their interface) can be modelled. The most common model is Classical Laminate Theory (CLT), and is used extensively in both deterministic analysis and reliability analysis of composite structures. It requires that the structure can be assumed thin-walled, and that plane-stress conditions hold.

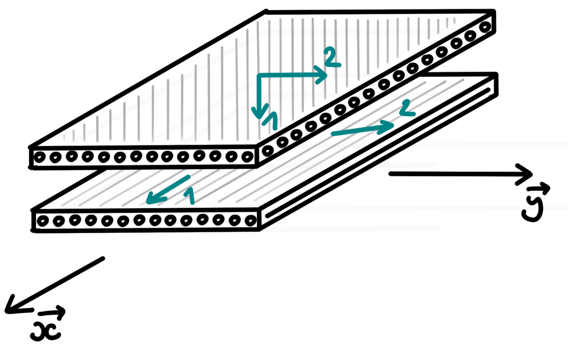


Figure 1.1: A laminate with two plies with each a local reference frame ($\vec{1}$ along the fibre direction and $\vec{2}$) perpendicular to the fibre direction, and a global reference frame ($\vec{x}, \vec{y}$)

Finally, the most detailed level analysis is at the microscopic level. In microscopic interlaminar analysis, the separate plies are not considered homogeneous, but are themselves heterogeneous materials that have microscopic internal structures comprised of fibres and matrices [7]. As one can imagine, this level of detail requires much more computational effort.

In this research, the thin-walled assumption holds, therefore CLT will be used, as it is commonly used in reliability analysis [4], well studied and balances accuracy and computational effort. In that case, one can predict the stress-strain behaviour of the homogenised composite structure based on the following assumptions [8]:

- The laminate consists of perfectly bonded layers
- The individual plies are homogenous layers, that are orthotropic and transverse isotropic, behaving similarly in both non-fibre directions.
- The structure's behaviour is linear elastic until damage occurs, there is no plastic deformation.
- The laminate deforms according to the Kirchhoff - Love assumptions, the normals to the mid-plane remain normal to the midplane even after deformation and do not change in length.
- The applied loads and moments are in-plane, e.g. a plane stress condition
- The strains $\epsilon_x, \epsilon_y, \epsilon_z$ are all much smaller than 1
- The strains $\epsilon_{xz} \approx 0$ and $\epsilon_{yz} \approx 0$

In that case, the elastic behaviour of the structure can be fully described by only 4 independent variables, the E-moduli E_1 parallel to the fibres, and E_2 perpendicular to it, the Poisson's ratio ν_{12} and the shear modulus G_{12}. Based on these variables, the behaviour of each ply of the laminate can be described. Using coordinate transformations, the behaviour of different plies can be described in a common reference frame. Then, using the fact that all the laminates will strain evenly until damage occurs, the stresses σ_1, σ_2, and σ_{12} can be computed for each of the plies.

1.1.2. Damage and last ply failure

Based on this analysis, it is possible to predict the failure of a structure. Failure of a structure can be defined as the loss of load carrying ability of the structure. In structural engineering it is often qualified as a critical state as defined by a failure criteria. Generally, this criterion is defined by an equation or set of equations dependent on the stresses as computed by the stress analysis. Once a defined threshold is exceeded, the ply is said to have failed.

Due to the multiple different materials that compose it, a composite structure can fail in many different ways. The different ways that the structure can fail are classified in failure modes. Examples of failure modes are: fibre breakage in tension, fibre debonding or delamination between plies. Not all failure criteria take all possible failure modes into account, most are limited to one or a few categories of failure. The criteria can be used once the stresses in each individual ply have been computed and are applied to each ply separately.

As soon as the threshold is exceeded for at least one ply, First Ply Failure (FPF) has occurred. As composite materials are known for their damage tolerance, it is interesting to pursue the analysis beyond the FPF to be able to design efficient structures. It is then needed to model damage on the ply. One of the simplest ways to achieve this is by decreasing the material properties of the failed ply. If the failure mode is known, this can be taken into account by decreasing the corresponding material property, for example, reducing E_2 and G_{12} for matrix failure, and reducing E_1 for fibre-dominated failure. After decreasing the material properties, the stress analysis has to be performed a second time, to verify none of the other plies fail immediately under their increased share of stress. Then the load is increased until the next ply fails, until there are no more undamaged plies remaining. This last failure is called the Last Ply Failure (LPF) [9]

So, both the FPF and LPF are very important for a failure analysis. the FPF is the point at which the failure is initiated, and the LPF is the point where the structure is lost. By having both of them one can assess the damage tolerance of the laminate under the given loading conditions.

1.2. Failure criteria for composites

Once the stresses are computed for each ply, a failure criterion can be used to determine if failure has occurred. This criterion is applied to each ply separately, and it is possible that multiple plies can fail at the same time. Three different failure criteria will be discussed, first a mode-independent criterion, Tsai-Hahn, followed by two mode-dependent criteria, Hashin and Puck.

1.2.1. Tsai-Hahn failure criterion

One of the simplest and most used type of failure criteria are quadratic failure criteria. Their main advantage is their ease of calculation as the criterion is defined by a single quadratic equation. Many different variations exist, of which Tsai-Hahn is one of the most commonly used in reliability [4]. The criterion is interactive: the stresses and failures in different direction are not decoupled and do effect one another. The Tsai-Hahn criterion states that failure occurs if and only if

$$\frac{\sigma_1^2}{X_T X_C} + \frac{\sigma_2^2}{Y_T Y_C} + \frac{\sigma_{12}^2}{S^2} - \frac{\sigma_1 \sigma_2}{\sqrt{X_T X_C Y_T Y_C}} + \left(\frac{1}{X_T} - \frac{1}{X_C}\right)\sigma_1 + \left(\frac{1}{Y_T} - \frac{1}{Y_C}\right)\sigma_2 - 1 \geq 0, \tag{1.1}$$

where σ_1, σ_2, σ_{12} are the principal stresses in the ply that have been found based on the stress analysis, X_C and X_T are the material strength in the direction of the fibres in tension and compression, Y_C and Y_T are the material strength in the perpendicular direction in tension and compression, and S is the material strength in shear [1].

The main disadvantage of quadratic failure criteria is their mode-independence. They can predict failure, but not the way in which the structure fails. This can be important as some types of failure are more critical than others. For instance, fibre debonding is generally seen as less critical, as the loss in stiffness is minimal. On the other hand, a delamination, where a crack grows between two plies parallel to the fibres, can have a much larger effect on the overall performance of the structure.

1.2.2. Hashin failure criterion

Unlike the quadratic criteria, that only have one equation to predict failure, the Hashin damage criterion has four equations that predict four different types of damage. It is based on the works of Hashin and Rotem [9]. Separating the failure modes into different equations not only allows for a better fit, but also to predict in what way the structure will fail. The Hashin damage criterion predicts four different failure modes: fibre breakage in tension F_f^t, fibre buckling in compression F_f^c, matrix cracking in tension F_m^t, and matrix crushing in compression F_m^c. For each of the failure modes, the equation is given by equations 1.2 to 1.5 [10].

$$F_f^t = \left(\frac{\sigma_1}{X_t}\right)^2 + \alpha\left(\frac{\sigma_{12}}{S_{12}}\right)^2 \geq 1 \qquad \text{and } \sigma_1 \geq 0 \tag{1.2}$$

$$F_f^c = \left(\frac{\sigma_1}{X_c}\right)^2 \geq 1 \qquad \text{and } \sigma_1 < 0 \tag{1.3}$$

$$F_m^t = \left(\frac{\sigma_2}{Y_t}\right)^2 + \left(\frac{\sigma_{12}}{S_{12}}\right)^2 \geq 1 \qquad \text{and } \sigma_2 \geq 0 \tag{1.4}$$

$$F_m^c = \left(\frac{\sigma_2}{2S_{23}}\right)^2 + \frac{\sigma_2}{Y_c}\left(\left(\frac{Y_c}{2S_{23}}\right)^2 - 1\right) + \left(\frac{\sigma_{12}}{S_{12}}\right)^2 \geq 1 \qquad \text{and } \sigma_2 < 0 \tag{1.5}$$

Where σ_1, σ_2, σ_{12} are the stresses in the laminate, X_c and X_t are the material strength in the direction of the fibres for compression and for tension, Y_c and Y_t are the material strength in the perpendicular direction to the fibres and S_{12} and S_{23} are the material strengths in shear. The coefficient α is a coefficient on $[0,1]$ to include the shear effect for fibre breakage in tension.

1.2.3. Puck failure criterion

Expanding on Hashin, Puck and Schurmann modified the criterion to improve it further. The changes include that the fibre breakage is now based on the fibre properties rather than the ply properties, and the matrix failure is subdivided into three different modes. This results in the set of five equations as given by equations 1.6 through 1.10, for fibre breakage in tension F_f^t, fibre buckling in compression F_f^c, matrix failure in compression F_m^A, matrix failure in moderate transverse compression F_m^B and matrix failure in large transverse compression F_m^C [11].

$$F_f^t = \frac{1}{\epsilon_{1t}}\left(\epsilon_1 + \frac{\nu_{12}}{E_{f_1}} m\sigma_2\right) \geq 1 \qquad \epsilon_1 + \frac{\nu_{12}}{E_{f_1}} m\sigma_2 > 0 \tag{1.6}$$

$$F_f^c = \frac{1}{\epsilon_{1t}}\left|\epsilon_1 + \frac{\nu_{12}}{E_{f_1}} m\sigma_2\right| \geq 1 - (10\gamma_{21})^2 \qquad \epsilon_1 + \frac{\nu_{12}}{E_{f_1}} m\sigma_2 < 0 \tag{1.7}$$

$$F_m^A = \sqrt{\left(\frac{\sigma_{21}^2}{S_{21}}\right)^2 + \left(1 - p_{21}^{(+)}\frac{Y_t}{S_{21}}\right)\left(\frac{\sigma_2}{Y_t}\right)^2} + p_{21}^{(+)}\frac{\sigma_2}{S_{21}} \geq 1 - \left|\frac{\sigma_1}{\sigma_{1D}}\right| \qquad \sigma_2 > 0 \tag{1.8}$$

$$F_m^B = \frac{1}{S_{21}}\left(\sqrt{\sigma_{21}{}^2 + \left(p_{21}^{(-)}\sigma_2\right)^2} + p_{21}^{(-)}\frac{\sigma_2}{S_{21}}\right) \geq 1 - \left|\frac{\sigma_1}{\sigma_{1D}}\right| \qquad 0 \leq \left|\frac{\sigma_2}{\sigma_{21}}\right| \leq \frac{R_{22}^A}{|\sigma_{21c}|} \text{ and } \sigma_2 < 0 \tag{1.9}$$

$$F_m^C = \frac{-1}{2\left(1 + p_{22}^{(-)}\right)}\left[\left(\frac{\sigma_{21}}{S_{21}}\right)^2 + \left(\frac{\sigma_2}{R_{22}^A}\right)^2\right]\frac{R_{22}^A}{\sigma_2} \geq 1 - \left|\frac{\sigma_1}{\sigma_{1D}}\right| \qquad 0 \leq \left|\frac{\sigma_{21}}{\sigma_2}\right| \leq \frac{|\sigma_{21c}|}{R_{22}^A} \text{ and } \sigma_2 < 0 \tag{1.10}$$

where σ_1, σ_2, σ_{12} are the stresses in the laminate, and Y_t is the material strength in the perpendicular direction to the fibres, S_{12} is the material strengths in shear. ϵ_1 is the strain along the fibre direction and ϵ_{1t} is the failure strain in tension along that direction. $p_{21}^{(+)}$ and $p_{21}^{(-)}$ are the slopes of the $\sigma_2 - \sigma_{21}$ fracture envelope and are mostly dependent on the type of fibre. For carbon fibre composites $p_{21}^{(+)} = 0.3$ and $p_{21}^{(-)} = 0.35$ are recommended [12]. R_A is the fracture resistance of the action plane and ν_{12} and ν_{21} are the Poisson ratios.

As one can see, this failure criterion is a lot more complex and involves a lot more variables. The effect of this is twofold, this complexity allows for a better fit than other failure criteria, and thus predicts failure more accurately. However, this also causes the need for additional testing and measurement. Furthermore, some of the variables, such as ϵ_{1t} or R_A are more difficult to measure and less commonly used, and therefore often not readily available.

1.3. Commonly used probability distributions

In the last section, it has been shown how classical laminate theory and traditional failure criteria use deterministic approaches to predict composite failure, with high accuracy. However, it can be questioned how much the results of deterministic analysis can be trusted, when there is always some inherent variation from the idealised case. Therefore, a probabilistic approach is needed. In this section, different probability distributions will be studied.

1.3.1. Normal and Lognormal distributions

The normal distribution, also known as the Gaussian distribution, is one of the most commonly used distributions. Its PDF, given in equation 1.11 is defined on $[-\infty,\infty]$, and characterized by a bell shape. It is symmetric and most of the samples will occur near the mean; the further a value is from the mean, the more unlikely it is. The distribution is fully defined by two parameters: the mean μ defining the symmetry axis, and the standard deviation σ defining the spread, or how narrow or wide the bell is [13]

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \tag{1.11}$$

The effect of these two variables can be seen in Figure 1.3. It is used to describe many different natural phenomena, from the height distribution of a population to the modelling of measurement errors in experiments.

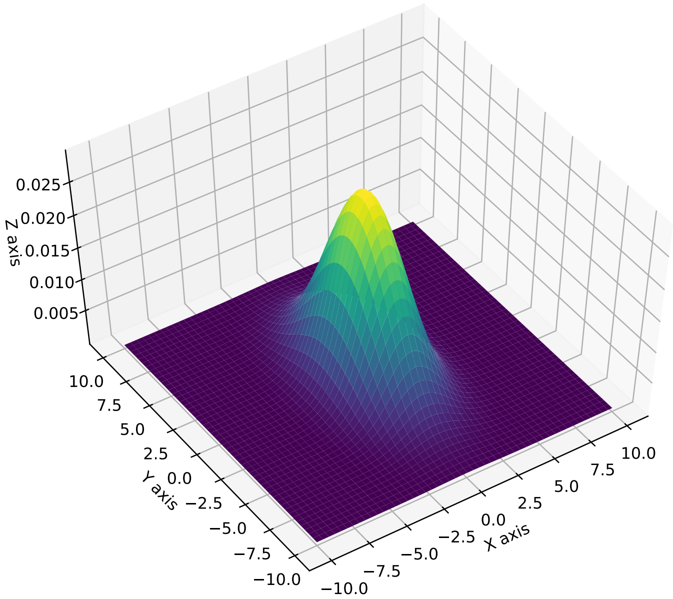


Figure 1.2: Probability density curve of a multivariate normal distribution with mean (0,0) and variance (3,10)

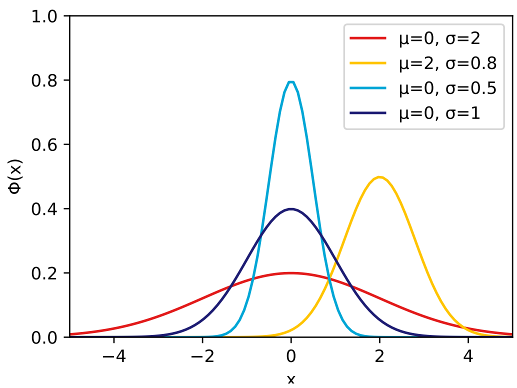


Figure 1.3: Probability density curves of normal distributions for several different parameters

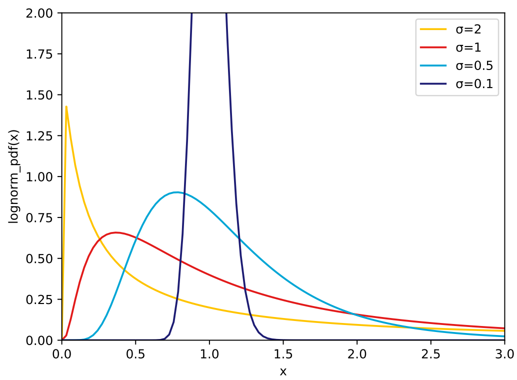


Figure 1.4: Probability density curves of lognormal distributions for several different parameters

The multivariate normal distribution (MVD) is a generalization of the normal distribution to n-dimensions. It is characterized by a vector $\vec{\mu}$ representing its centre, and a positive semi-definite $n \times n$ matrix Σ representing the covariances between the variables. If Σ is positive definite, the PDF is given by equation 1.12, and defined on $[-\infty, \infty]$. A plot of a 2-dimensional multivariate normal distribution is given by figure 1.2

$$f(\vec{x}) = (2\pi)^{-\frac{n}{2}} \det(\mathbf{\Sigma})^{-\frac{1}{2}} \, exp\left(\frac{1}{2}\left(\vec{x}-\vec{\mu}\right)^{T} \mathbf{\Sigma}^{-1} \left(\vec{x}-\vec{\mu}\right)^{2}\right) \tag{1.12}$$

A random variable X follows a log-normal distribution if the variable $Y = \ln(X)$ follows a normal distribution. The PDF of the log-normal distribution is defined on $(0, \infty)$. As shown in equation 1.13, it is defined by two parameters, μ and σ that do not represent the mean or standard deviation of X, but of Y [14]

$$f(x) = \frac{1}{x\sigma\sqrt{2\pi}} \, exp\left(-\frac{\left(ln\, x-\mu\right)^{2}}{2\sigma^{2}}\right) \tag{1.13}$$

Its probability density curve for different parameters is shown in figure 1.4. Its expected value and variance are given by equation 1.14. It is often used to model gradual changes over time, Examples of variables that follow a lognormal distribution are the length of chess games, molar mass distributions or variables used to

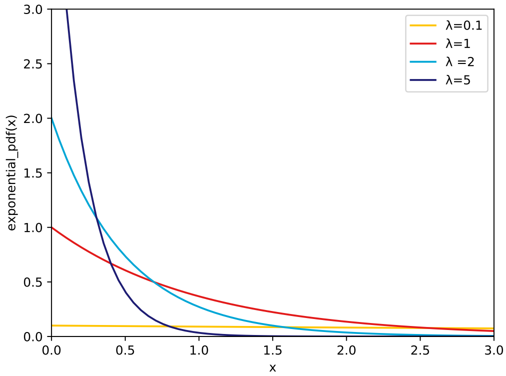


Figure 1.5: Probability density curves of exponential distributions for several different parameters

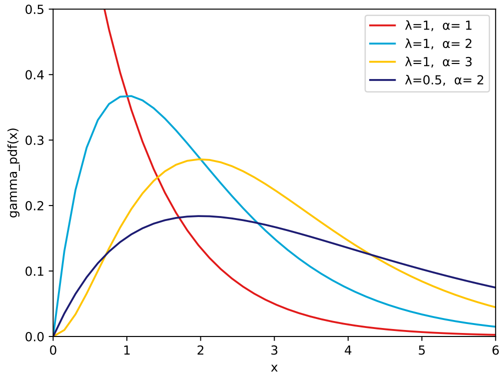


Figure 1.6: Probability density curves of gamma distributions for several different parameters

model repairs of a maintainable system.

$$E[x] = e^{\mu+\frac{1}{2}\sigma^2} \qquad Var[x] = e^{2\mu+2\sigma^2} \tag{1.14}$$

1.3.2. Exponential and Gamma distributions

A random variable has an exponential distribution if its PDF is given by equation 1.15. It is defined on $(0,\infty)$, its expected value is $1/\lambda$ and its PDF is shown in figure 1.5 for several different values of λ. It often models the distribution in time of a Poisson process. For instance, if the probability of arrival of a bus is defined by a Poisson distribution, the distribution of the interval in time between two busses follows a exponential distribution. It is memoryless, i.e. the event occurs continuously and independently at a constant rate. If one event occurs at time t, this has no effect on the likelihood of an event at time $t+\Delta t$.

$$f(x) = \lambda e^{-\lambda x} \tag{1.15}$$

Furthermore, The gamma distribution is also a continuous distribution on $(0,\infty)$ defined by two parameters $\alpha > 0$ and $\lambda > 0$ as well as $\Gamma(\alpha)$, a normalizing constant to make sure f integrates to 1.

$$f(x) = \frac{\lambda\ (\lambda x)^{\alpha-1}}{\Gamma(\alpha)} \tag{1.16}$$

The typical application of a gamma distribution is to model the results of random experiments with a exponential distribution. Indeed, if a variable follows an exponential distribution with parameter λ, the sum of n independent samples of this variable follows a gamma distribution with parameters λ and $\alpha = n$. To continue with the bus example, if the probability of the time until the next bus arrives follows an exponential distribution with parameter x, the time to wait for the next two buses follows a gamma distribution with parameters $\alpha = 2$ and $\lambda = 1/x$. It is often used in time related problems as it is only defined for $x > 0$.

1.3.3. Weibull distribution

The Weibull distribution is defined on $(0,\infty)$ by the probability density function of equation 1.17, which has been plotted for several different parameters in figure 1.7. Its parameters λ and k are both positive real numbers. Its expectation is $1/\lambda^k$ [13].

$$F(x) = \frac{k}{\lambda}\left(\frac{x}{\lambda}\right)^{k-1} e^{-(x/\lambda)^{\alpha}} \tag{1.17}$$

It can be seen as a more generalized case of the exponential distribution, where the rate of arrival of busses is not constant. It is often used to model machine failure, so that the failure rate increases with time. If $k < 1$ the failure rate decreases over time. This models the case where the recently repaired machines have a lot of chances to fail again, but once they are past a certain time after repair, are likely to remain without failure for a long time), and for $k > 1$ it increases with time, so the longer a machine has not been repaired, the more likely it is that it will fail. If $k = 1$ it is equivalent to the exponential distribution, a.k.a the failure rate is independent of time [13, 15].

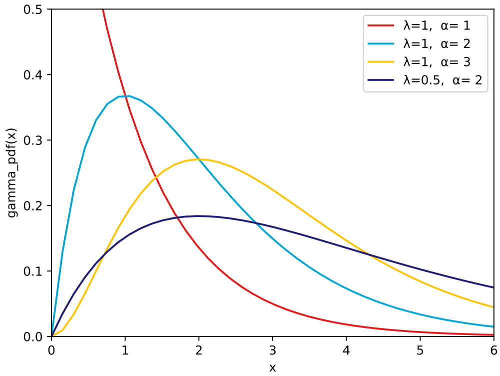


Figure 1.7: Probability density curves of Weibull distributions for several different parameters

1.4. Parameters to be modelled as random variables

In this section, different parameters that can be modelled as random variables are discussed. It is explained why they are interesting to study as random variable or not, and what type of distributions are commonly used to model them. The main variables that are studied are the material strengths, load case, geometry and stiffness. This is followed by a short overview of other variables that have been included in literature. For a select number of papers, an overview is given of the variables that are used, with their corresponding distribution.

1.4.1. Strength properties

The most common variable used in reliability studies is the material strength [4] i.e. the set of five parameters X_t, X_c, Y_t, Y_c and S. Indeed, brittle materials such as fibres used in composites often present a large variance in their strength properties. This is because this property is governed by the quality of their crystal structure, therefore the material strength for every single crystal is dependent on the proximity of flaws in their lattice [16].

The Weibull distribution is often used in material science to model the strength for brittle materials, then the parameter k is called the Weibull modulus, and quantifies how likely it is that flaws in the material are located close to each other. If the Weibull modulus is smaller than 1, the flaws are concentrated and the weakest parts fail at much lower stresses, if the value is larger than one, the failure is more likely to be due to damage growth or fatigue-like processes. However, other distributions are also used for this parameter as can be seen in table 1.1.

Besides the fact that the material strength can be very variable, it is an easy material property to measure through standardized testing, therefore this makes the distribution easy to quantify. In most cases, authors perform multiple tests, and then take the distribution that best fit their sample, which is why there is a large variety in probability distributions.

1.4.2. Stiffness properties

From section 1.1, the 4 parameters E_1, E_2, G_{12} and ν_{12} can fully describe the linear-elastic properties of a ply. E_1, G_{12} and ν_{12} are used relatively little in reliability theory. This is most likely due to the fact that these parameters are generally relatively easy to measure, and do not have such a high variability as the strength. Therefore, the benefit of modelling these parameters as a random variable is relatively small. Only G_{12} is modelled as a random variable relatively often, most likely because it is more strongly affected by the fibre-matrix interaction. However, Lekou [1] shows that when the stochastic nature of the material elastic properties is not taken into account, this does result in a serious overestimation of the reliability of the composite structure. Therefore, it can be interesting to incorporate these variables to see if they do indeed show an effect.

1.4.3. Loads

The second most common variable to be studied probabilistically is the loads [4]. Indeed, in many cases the exact loading scenario for the part can be uncertain, due to approximations made in the calculations of the

	Bacharoudis [17]	Lekou 08 [1]	Gomes [18]	Young [19]	Chen [20]	Richard [9]
X_t	log-normal	weibull	log-normal		normal	
X_c	normal	log-normal	log-normal		normal	
Y_t	normal	log-normal	log-normal			
Y_c	log-normal	log-normal	log-normal			normal
S	log-normal	weibull	log-normal			
E_1	log-normal	weibull*		normal	normal	normal
E_2	log-normal	log-normal*		normal	normal	normal
G_{12}	weibull	log-normal		normal		normal
ν_{12}	log-normal	normal*		normal		normal
ν_{21}	log-normal			normal		
loads	log-normal		log-normal	other	extrm type I	

Table 1.1: Distributions used for different input variables for a selection of variables

loads, defects or variations in stiffness of surrounding parts amongst others. As on the one hand the load can vary a lot from the design loads, and on the other hand the loading has a very large impact on the structure's reliability, this parameter that is very desirable to use as a random variable.

Unfortunately, this uncertainty on the load case also means that the load's PDF can be difficult to predict. In many cases, due to cost, it is impossible to test the structure sufficiently many times to determine a PDF. One could ask if it is indeed a good option to take the load as a variable in those cases, as the choice of the distribution and its parameters could have a large effect on the outcome. Few papers seem to explain why they choose one distribution over another for the load case when they were not able to measure it.

All in all, even if the estimate of the distribution of the load as a variable is difficult to predict in final parts, it seems better to still take it into account rather than ignoring this large source of variability.

1.4.4. Other properties

The variability in the inputs does not limit itself to the common material parameters, other variables have also been used, depending on the use case. In 2008, Lekou [1] included the thermal expansion coefficient, although the effect on the reliability was rather small. Other examples include the surface finish for a model of a composite propeller [18], or the modelling of earthquakes that require a whole new set of parameters altogether [21].

However, most of the other variables that were not mentioned in detail here are used in more niche applications. As the goal of this research is to develop a general method to include mode-dependence, an expansion into material parameters that are only relevant for certain applications is considered out of scope for this research. If the need arises, the method can be adapted as shown by Lekou [1].

1.4.5. Parameters to included in the analysis

It is no surprise that the most desired variables to use for reliability studies are also the most common, as can be seen in table 1.1. It seems obvious that strength and loading are the most interesting variables to pursue.

Strength properties are easier to measure and have a large effect on the total reliability, loads are more difficult to create a distribution for, but do have a large effect and therefore seem an interesting option. Including stiffness does not seem to improve the research, as this variable is similarly easy to measure as strength, but does not have a large impact on the outcome.

As the research is focused on a general case, more exotic choices such as the thermal expansion coefficient do not seem to create value. Thus, the set of parameters that are most interesting to use as variables are the strength properties X_t ,X_c ,Y_t ,Y_c, S and the stiffness properties E_1, E_2, ν_{12} and G_{12}, for a total of 9 variables. As this research focusses on the general case, there is no design load, on which there can be uncertainty. So, using just any design load with a distribution that has no basis in a design does not seem pertinent. Therefore, the focus will be on failure envelopes that take into account all possible combinations of two loads N_x, N_y.

The next step is to see what distribution to associate with what parameters. Although some distributions are considered a better fit than others for some variables, such as the Weibull distribution for strength, a large amount of different distributions are used for different parameters, as can be seen in 1.1. This variability is due to the fact that these authors choose their distributions not based on recommendations but based on the

		distribution	parameter 1	parameter 2	mean	skew*
X_t	[MPa]	Weibull (k,λ)	443.67	7.76	417.64	–0.733
X_c	[MPa]	log-normal (μ,σ)	5.83	0.10	340.99	0.301
Y_t	[MPa]	log-normal (μ,σ)	3.72	0.15	41.60	0.46
Y_c	[MPa]	log-normal (μ,σ)	5.13	0.13	171.13	0.39
S	[MPa]	Weibull (k,λ)	2.85	0.18	17.54	–0.39
E_1	[GPa]	Weibull (k,λ)	25.04	12.78	24.08	–0.73
E_2	[GPa]	Log-normal (μ,σ)	2.09	0.15	8.20	0.46
G_{12}	[GPa]	Log-normal (μ,σ)	0.48	0.32	1.66	1.02
ν_{12}	[-]	Normal (μ,σ)	0.305	0.06	0.305	0.0

*skew with an absolute value below 0.5 is considered low, between 0.5 and 1 is considered moderate, above 1 is considered high [13]

Table 1.2: Distributions used for different variables in Lekou [1]

best fit for an experimental dataset that they have.

One must recognize that large scale testing with lots of data points is often not possible, due to the costs involved. In the end one must pick a distribution and as long as the underlying mechanisms are not understood, this seems the best approach.A thorough investigation on which distributions to use for what variable is considered out of scope for this research. Indeed, once the method is developed, it should be easily adaptable for different parameter distributions. If there is access to experimental data, such a test for the best fit can be considered, otherwise recommendations of for instance the military handbook can be used. In this research, the distributions used will be based on those provided by Lekou [1], as given in table 1.2

1.5. The need to include failure mode dependence in composite reliability

Although the classical methods can predict the structural failure well, it is important to be aware of how much this result can be trusted for any singular part. Indeed, especially for composites, there can be a lot of variance between the material properties between parts. This is mainly due to the lack of separation between material manufacturing and forming processes. The material is created at the same time that the part is shaped, and thus can't be manufactured in fully controlled specialised factories such as for steel sheet material. But also, the material properties can be very difficult to measure, as the material behaviour in the composite structure can be different than that of the separate phases. Added to that is the existence of intralaminar failure modes, where the composite material fails due to the interaction of different phases.

Even when there could be perfect certainty about the position of every single atom and perfect knowledge of every parameter influencing the test and the test measurements, it is still up for debate, even in such a case, if a deterministic failure analysis would be able to predict failure precisely. Even a perfect, complete description of the microscopic interactions between a materials particles is not always enough to deduce its macroscopic properties [22]. It is possible that material failure is inherently a probabilistic process. Thus, to be able to reach the most optimum designs, probabilistic methods need to be included in the analysis of the laminate. This will be done trough reliability methods as described in chapter 2.

Reliability methods provide a way to take into account the inherent material variability of composite materials, and therefore make more efficient design possible. However, at the moment, the applications for reliability methods in composites have been limited to quadratic failure criteria such as Tsai-Hahn [4]. For the best failure predictions, mode dependent failure criteria are desired, as they are more accurate and also better predict effect of the failure to the structure.

Furthermore, all applications of composite reliability have been focussed of first ply failure [4]. However, as composites are especially damage tolerant, knowing about the last ply failure is equally important. Therefore there is a need to apply reliability methods to last ply failure as well. Therefore, the main research objective of this thesis is:

> How can the current reliability methods be used to predict the probability of both first and last ply failure using failure mode-dependent failure criteria?

The aim will be to combine the Hashin and Puck criterion with Monte Carlo Methods to predict both the probability of FPF and LPF. The hypothesis that this research aims to show is that the use of a mode-dependent failure criterion, rather than a quadratic failure criterion, improves the quality of the results significantly without an extreme increase in computational cost. Similarly, it is hypothesised that the reliability results for LPF differ significantly to those of FPF for some load combinations. In the ideal case, this additional information would come at an acceptable increase in computational cost.

2

Failure mode-dependence in reliability methods

There is a need to incorporate the variability of the parameters into failure mechanics. In order to fulfil this need, reliability methods are used. These methods take the input variables such as the strength, stiffness or geometry as distributions, and estimate the probability of failure.

Three different approaches to reliability methods will be discussed, and the possibilities to use them with mode-dependent failure criteria will be investigated. First analytical methods will be discussed, next semi-analytical methods will be shown and finally numerical options will be explored. This will be followed by a presentation of different approaches to apply these methods to last ply failure, as these methodologies are independent of the exact reliability methods used. Finally, the different reliability methods and approaches to LPF will all be evaluated in order to choose the the most suitable combination.

2.1. Analytical reliability methods

Analytical methods provide exact solutions to the reliability problem. Most of them predict the joint CDF using a series expansion, so they also predict the distribution of the failure variable. Analytical Methods provide fast solutions to reliability problems, in good agreement with Monte Carlo methods [4, 23]. However, they are not the most accurate methods.

2.1.1. General methodology for analytical reliability methods

Analytical methods provide exact solutions to the reliability problem. Most of them predict the joint CDF using a series expansion, so they do not only predict the probability of failure, or expected value of the probability of failure variable, but also its distribution.

The two most commonly used methods are Pearson's expanded and Edgeworth's expansion. They both start by predicting de central moments of the failure variable F. In general, the probability of failure can be given by $P(F(\vec{x}) < 1)$, with $\vec{x} = [x_1, x_2, \dots x_k]$ a vector of k random variables [23].

Using a Taylors expansion of $F(\vec{x})$ and applying the method of moments, the central moments of the PDF of F can be predicted in terms of the respective moments of the variables x_i as follows:

$$\mu_j(x_i) = \int_{-\infty}^{+\infty} \left(x_i - x_{i_m}\right)^j f(x_i)\, dx_i, \tag{2.1}$$

where $f(x_i)$ is the PDF of x_i [23]. Usually, it is recommended not to use central moments beyond fourth order, e.g. $\mu_1(x_i)$ until $\mu_4(x_i)$ since these are increasingly difficult to measure from experimental data [4]. Furthermore, as one can see, the amount of computations per central moment, as well as the number of central moments itself, scales with the number of random variables.

To improve the results, one would need higher order approximations. However, both methods depend on central moments of the input variables that are difficult and expensive to measure accurately. To measure a fourth or even a fifth moment requires a large amount of samples and is very prone to large effects from a few outliers, cancelling the advantage of the computational efficiency.

2.1.2. Including mode-dependence in analytical methods

A second path would be to improve the failure criteria used by increasing the accuracy of the predicted failure. For instance, by introducing mode-dependence. Indeed, the occurrence of a different failure mode could explain why the failure envelopes are in good agreement sometimes, but not others. However, to model material failure well, this would still require accurate measures of second and third central moments of many material properties. This goes against the low cost of the method.

Analytical methods seem best suited for initial design phases, as they provide a quick initial estimate of failure probability, but they are not the most accurate. Needing to perform many tests to characterise the material properties sufficiently well does not seem in line with the main strength of analytical methods. Thus, it does not seem advantageous to pursue the development of mode-dependent alternatives, as the costs involved with testing to gather the required inputs goes against the main strength of the methods.

2.2. Semi-analytical reliability methods

Semi Analytical methods, also called Fast Probability Integration Methods (FPIM), have been developed in order to take advantage both the speed of analytical methods and the accuracy of numerical methods. They are the most commonly used and studied in the literature [4], and all based on the same methodology. In this section, this methodology will first be explained, and the two most common variants, First Order Reliability Method (FORM) and Second Order Reliability Method (SORM) will be studied in more detail.

2.2.1. General methodology

In this section, the steps needed for the FORM and SORM methods will be detailed. To start, the inputs that are needed for the methods are: all of the variables needed to compute failure of a structure, such as strength criteria, the layup, the material properties, as well as the applied loads. For a select number of these, instead of a fixed value, a distribution is given. Most commonly, the loads and strength properties are taken as variables [4], but other variations are used, such as the thermal expansion coefficient [1]. Finally, a failure criteria is needed. Quadratic failure criteria such as Tsai-Hahn and Tsai-Wu are most common [4].

The first step is to prepare the inputs. The set of variables for which we have defined a distribution, $\vec{X}$, must be transformed to a set of non-correlated variables that all follow a standard normal distribution $\vec{U}$. Two main methods are employed: in case of non-normal correlated variables a Rosenblatt transformation is used, if the variables are non-correlated, Rackwitz–Fiessler Method is recommended [4].

As all variables follow a standard normal distribution, the joint PDF of $\vec{U}$ follows an n-dimensional standard normal distribution $\Phi(\vec{U})$. In this space, each point represents a possible state for $\vec{U}$, and the value at this point represents the possibility that this point occurs. A hypervolume can then be defined by the set of points for which the failure function $g(u) \leq 0$ corresponding to the set of all possible states of $\vec{U}$ that does not result in failure. In that case, the probability of failure can then be defined as follows:

$$P_f = 1 - P_{\bar{f}} = 1 - \int_{D|g(x)\leq 0} \Phi(\vec{U}) d\vec{u}. \tag{2.2}$$

However, this equation is very difficult to solve, because the domain is not a neat square but a blob, making the top and bottom limits interdependent for all variables. Therefore, an approximation of the integral is made.

In order to simplify the integral, the formulation of $g(u)$ must be simplified to $g^*(u)$. For FORM, this is achieved through a linearization, and for SORM, a second order approximation is made. In both cases, the approximation is made based on the most probable point of failure (MPP). As the joint PDF follows a standard normal distribution, any point at a given distance of the origin is equally likely and the further away the point is from the origin, the less likely it is to occur. From this, it follows that the MPP corresponds to the closest point to the origin on the surface $g(u) = 0$. The distance from the origin to this point is called β. Thus, the MPP can be found by using an optimization algorithm to find $u_{\vec{mpp}}$ such that

$$\text{minimise } u \quad \beta^2 = u_1^2 + u_2^2 + \cdots + u_n^2, \tag{2.3}$$

$$\text{subject to} \quad g^(\vec{u}) = 0. \tag{2.4}$$

This is done using the Rackwitz–Fiessler algorithm [1], which is further detailed in chapter 3.

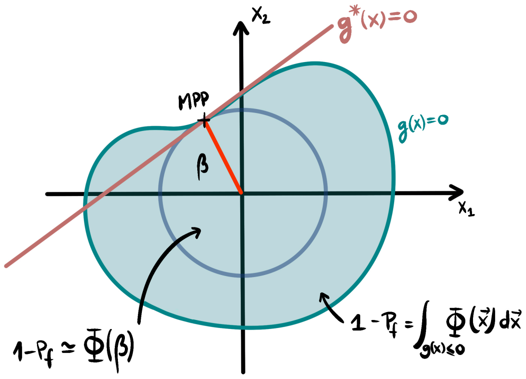


Figure 2.1: Geometrical representation of the FORM method, g*(x) is linearized at the MPP where $||\vec{\beta}||$ is minimised.

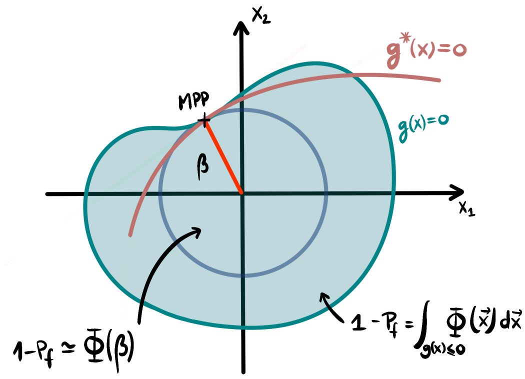


Figure 2.2: Geometrical representation of the SORM method, g*(x) is a quadratic trough the MPP where $||\vec{\beta}||$ is minimised.

2.2.2. Including failure mode-dependent criteria in FORM

The set of equations 2.3 and 2.3 need to be simplified in order to be able to be solved. In the case of FORM, the function $g(u)$ is linearized to obtain $g^*(u)$, it can then be shown that the integral of equation 2.2 is equivalent to:

$$p_f = 1 - R = 1 - \phi(\beta) = \phi(-\beta), \tag{2.5}$$

where ϕ is the distribution function of the standard one-dimensional normal distribution [4]. This can be easily found in a lookup table, rendering the integration fast and simple. FORM has been shown to have good accuracy with respect to Monte Carlo [1, 4, 24]. The method is the most studied and seems to have very much potential. Currently, the quadratic failure functions are used in almost every application, and only very few papers have used mode-dependent failure criteria [4], although this seems to be a very interesting path in order to improve the results of the analysis. There are two examples of papers that have tried to use mode dependent failure criteria are the work of Cederbaum in 1990 [24] and more recently Soliman in 2005 [25].

In Cederbaum's paper, the method to apply form with Hashin is described, formulas are derived but there work does not go any further than that. There is no discussion of the application and no evaluation of the results present in the work. Furthermore, the paper is from 1990, and in the meantime, there have been many computational advancements making a reinvestigation interesting.

In Soliman's article, the FORM method is applied with the Hashin criterion to composite cylinders subjected to axisymmetric loading [25], and compared his results to a Monte Carlo simulation. This seems to be the only application of both FORM and a mode-dependent failure criterion. He shows that the FORM results agree well with Monte Carlo results, and are less computationally expensive. Especially since this paper is from 2004, this seems promising for a similar application for infinite plate. However, as his aim was primarily to validate safety factors used, he did not compare it to a simulation using simpler failure criterion. Therefore, the question of the relative improvement by using a mode-dependent criterion is still unanswered.

It is possible to include failure-mode dependence in the FORM method by minimizing the combined failure equations. This brings along a few challenges, to start this leads to discontinuities the failure equation making the optimization step more difficult. As the failure criteria is only composed of 5 equations at most, it is simply possible to perform the optimization 5 times, and take the smallest value. As the Rackwitz–Fiessler algorithm uses unbounded optimization, it is important to make sure the optima satisfy the conditions for each failure mode.

2.2.3. Including failure mode-dependent criteria in SORM

The Second Order Expansion Method uses the second derivatives of $g(x)$ in order to improve the approximation $g^*(x)$, based on a Taylor expansion at the MPP. This can lead to three forms: parabolic, elliptic or hyperbolic depending on the sign of the curvature at the MPP. As the resulting integral can still be very difficult to evaluate, in most case a simpler shape is assumed based on only one of the curvatures. Most common are the rotational paraboloid and the non-central hypersphere based on a predetermined axis. Generally the largest positive curvature (or smallest radius of curvature) is used to stay conservative [4].

SORM leads to a more accurate prediction, but is much more computationally expensive, especially if no shape is assumed. At the moment, it seems to be much less commonly applied and studied in papers. To the best of our knowledge, no papers that have attempted to improve SORM by including mode-dependent failure criteria exist.

Similarly to the FORM methods, the SORM method should allow for mode-dependent criteria. However, this will make the optimisation step even more complex and time consuming, and the additional effort does not seem worth it at the moment.

2.3. Numerical reliability methods

The final category of reliability methods are numerical methods. This type of approach usually requires a lot of computational effort but is also often very accurate. The Extended Finite Element Method (XFEM) is a method where damage can be initiated and then propagated over time for a given model [10]. In order to use this type of method, a part geometry is needed as well as a lot of computational effort. As this research focusses on the more general case, this is considered out of scope. The focus will therefore be on the Monte Carlo Method.

2.3.1. Monte Carlo methodology

Monte Carlo methods are a class of numerical algorithms relying on repeated random sampling to approach a result. It can among others be used in statistics to approach an unknown probability depending on many different variables with known distributions. The method then consists of taking a lots of samples of the variables according to their respective distributions, and for each of them, to compute the outcome. In the case of a reliability analysis, for each of the samples one would use a failure criterion to compute if the part fails or not. Then the total probability of failure is shown in equation 2.6 [15].

$$P_f = \frac{\text{number of failed cases}}{\text{total number of cases}} \tag{2.6}$$

Monte Carlo Methods are able to represent almost any distribution. Their power lies in being able to estimate the emergent probability density function based on simple models. Indeed, there is no need for an analytical formulation of the joint PDF, which removes the need to assume some form or type of distribution, and therefore a source of error. On the other hand, they rely on a lot of computational power. Indeed, the uncertainty of the result scales with the inverse square root of the number of samples [26]. This means that the amount of simulations needed to predict the failure probability accurately can be very high. This particularly affects very large or very small probabilities, where one additional failure can change the result drastically. This makes the Monte Carlo methods particularly expensive, and thus unsuited for the needs in design.

Furthermore, the quality of the result is not only dependent on the number of simulations, but also on the quality of the (pseudo-)random number generator. Indeed, the method assumes all samples are independent, but it is possible for a bias to skew the results. For instance, if due to some limitation in the pseudo random number generator, samples that occur in the failure domain are more common, this can lead to an overestimation of the probability of failure. It is therefore very important to validate the quality of the random number generator to guarantee good results.

Despite the computational cost, and due to its high accuracy, Monte Carlo is generally used as a reference, e.g. the method is used in literature in order to compare other faster methods. As the results are of high quality, it provides a reference to compare the results. Moreover, it provides a speed threshold, as a method that provides similar results to Monte Carlo but is slower might directly be put aside. Also, Monte Carlo methods allow for a comparison that can be based on the same inputs, yielding a better evaluation of a model itself, and free of measurement errors and other differences. Finally, it is often a cheaper alternative to large scale testing, as dozens or even hundreds of tests are rarely possible outside of the computer.

2.3.2. Including failure mode dependent criteria in Monte Carlo

As the method does not include any computation of derivatives or other manipulation of the failure criterion, the process to replace the mode-independent criterion by mode-dependent criterion is very straightforward, and should not increase the computational cost significantly.

2.4. Modelling last ply failure in reliability

Reliability Methods as currently applied provide the probability of failure for first ply failure. In this section, methods to use them for last ply failure will be discussed. The first approach is called the direct approach and is discussed in section 2.4.1. The second approach is called the Bayesian approach and is discussed in sections 2.4.2 and 2.4.3.

2.4.1. Direct approach

The easiest approach would be to simply use a failure function that checks if all plies have failed in the deterministic way, and returns a Boolean variable, the last ply has failed, yes or no. As this excludes any method that is based on gradients, this approach can not be used with most optimisation algorithms. Thus this can not be used with the FORM and SORM methods, but can be used with Monte Carlo approach.

A different approach would be to keep track of the failure criteria for all the plies, and return the highest rather than the lowest, and use this as a criterion in the optimisation. This still allows to use gradients, although it will further complexify an already difficult optimisation, as new discontinuities will be added in the gradient. Furthermore, even though this does allow for variable damage, the added discontinuities that this would bring will make it even more likely that the global minimum for β will never be found. Therefore, this approach is not suited for semi-analytical methods.

Finally, when using a direct approach, for any sample, this assumes that all the plies have the same material properties. Thus if a ply has failed, it is likely that this sample had poor material properties, and the next plies are therefore more likely to fail as well. Similarly, if the sample has relatively good material properties, this will be the case for all plies, and last ply failure will be predicted as less likely. This will create an artificial correlation between the material properties of the different plies, and will under-estimate the predictions for low target probability of failure.

2.4.2. Full Bayesian approach

It is possible to approach the last ply failure from a probabilistic point of view. Suppose we have a laminate, with plies $A, B, C \ldots M, N$. This laminate's failure can be analysed as a sequence of failures resulting in the failure of the last ply. All possible sequences of failure can be broken down in a decision tree as shown in figure 2.3. For n plies, there are $n!$ possible paths on the decision tree, as many as there are permutations of the set $\{A, B, C..N\}$ [13].

For each branch on this decision tree, it is possible to compute the probability of this branch as a sequence of Bayesian events. Following the left most branch of the three, with the sequence $S_1 = (A, B, C, D)$, we can apply Bayes' Rule [13]. Then, the probability of B occurring $P(B)$ can be expressed as

$$P(B) = \frac{P(B|A) \cdot P(A)}{P(A|B)}, \tag{2.7}$$

using Bayes' formula. As, on this branch, B is defined as occurring only if A has occurred, then $P(A|B) = 1$, thus we have

$$P(B) = P(B|A) \cdot P(A). \tag{2.8}$$

A similar reasoning can be applied to compute $P(C)$ as a function of $P(B)$. Applying this recursively yields:

$$P(N) = P(N|A, B \ldots M) \times \cdots \times P(C|A, B) \times P(B|A) \times P(A), \tag{2.9}$$

Where, $P(A)$ is the probability of first ply failure for ply A, as given by any reliability method in their usual application. $P(J|A, B \ldots I)$ is the probability that ply J fails, knowing that plies A to I have all been degraded. This can be computed using any reliability method, as this is equivalent to a first ply failure analysis, where plies A to I would simply have damaged material properties from the start. So to compute $P(N)$ for this sequence S_1, n reliability analysis are needed.

As N is defined as the last ply to fail on this branch, $P(N)$ is the probability of last ply failure for this failure sequence. Of course, this is not the only possible failure sequence. The same analysis can be applied for each of the $n!$ branches of the decision tree. The probability of LPF is the sum of the probabilities given all possible failure sequences S_i, as given by

$$p_{LPF} = \sum_{i=0}^{n!} P(LPF|S_i) \tag{2.10}$$

As an example, for a simplified case with two plies, A and B, there are two possible failure sequences:

$$S_1 = (A, B) \text{ with probability of failure } P(LPF|S_1) = P(B|A) \cdot P(A)$$
$$S_2 = (B, A) \text{ with probability of failure } P(LPF|S_2) = P(A|B) \cdot P(B)$$

$$\text{then} \qquad p_{LPF} = P(LPF|S_1) + P(LPF|S_2) = P(B|A) \cdot P(A) + P(A|B) \cdot P(B)$$

In total this would require one reliability analysis for each decision point in the Bayesian tree in figure 2.3, that is to say n for the top row of the tree, $n(n-1)$ for the second row and so on until $n!$ for the bottom row. So the number of simulations needed for this analysis is

$$\text{number of simulations} = n + n(n-1) + n(n-1)(n-2) + \cdots + n! = n! \sum_{k=1}^{n-1} \frac{1}{k!} \tag{2.11}$$

This quickly grows to too many simulations to be acceptable, for only 4 plies this would require 44 simulations, and for 8 it goes to 104 554. Thus, approach has the potential to increase the computational effort over 100 000 fold. Therefore, it is not a viable solution in this form.

2.4.3. Simplified Bayesian approach

The previous approach is interesting but is not viable in the current form, as it will require too much computational power. Therefore, this approach needs to be simplified. It can be assumed that this most likely failure sequence is the biggest component of sum 2.10, and that the components of all other paths is negligible. This will overestimate the actual reliability of the laminate, but reduces the number of simulations needed drastically.

In this case, instead of needing one simulation for every decision point, the amount of simulations can be limited by choosing the most likely failure at each step. In the case with 4 plies, and the most likely failure sequence is A, B, C, D, the required simulations are coloured blue in figure 2.3. On the first layer, all four failure probabilities are required to choose the most likely one. On the next step, only three, then two, then there is only one option left. In the general case, for n plies, the amount of simulations needed is $n + (n-1) + (n-2) + \cdots + 1$. This drastically reduces the number of simulations as opposed to the full decision tree: from 44 to 10 simulations for 4 plies, from 104 554 to 36 simulations for 8 plies.

The simplified approach will overestimate the actual reliability of the laminate, but is much faster and therefore preferred. This approach can be applied using any reliability method, as the plies would simply have damaged material properties from the start. To further the analysis, it could even be possible to use the degradation factor as a variable.

2.5. Method Selection

It is clear that there is a need to include mode-dependence and damage modelling in reliability methods for composites. As described previously, mode-dependence can be included in almost any reliability method, and the Bayesian approach to last ply failure works for any reliability method.

Three types of reliability methods have been discussed, and their strengths and weaknesses have been evaluated. On one side, the strengths of analytical methods are mostly speed, rather than accuracy. The error due to the approximations of the model is already large and cannot be made smaller as they are dependent on higher order central moments. Possible improvements could be to use a better failure criterion, such as a failure-mode dependent criterion, or to predict the failure probability of LPF. However, both these paths increase the computational complexity, reducing the speed, going against the main advantage of these methods.

On the other side of the spectrum, numerical methods are more accurate, but already very computationally expensive. They are a suitable candidate to investigate the effects of mode dependence and LPF on composite reliability, however this will only increase the computational effort that is needed for this type of approach.

Between them, the semi-analytical methods are most developed at the moment. They are more accurate than analytical methods, and are more limited by the accuracy of the failure criteria, but not so computationally expensive that it will become cumbersome. Thus, they show great potential to be improved by introducing mode-dependent failure criteria. Furthermore, due to their balance between speed and accuracy, they seem to be the best start to also investigate a way to apply reliability methods to last ply failure.

In practise, there are a few problems with the semi-analytical FORM and SORM methods that do not make it the ideal candidate. Lekou [1], noted that their implementation did not converge for every combination of loads, but she provides no explanation for the cause. More convergence problems have been observed by Sundararajan [3] and the author. In the next chapter, the sources of these convergence problems will be studied, in order to assess the implications they have for composite reliability. In the end, it will be shown that the drawbacks of FORM are such that Monte Carlo methods will be preferred.

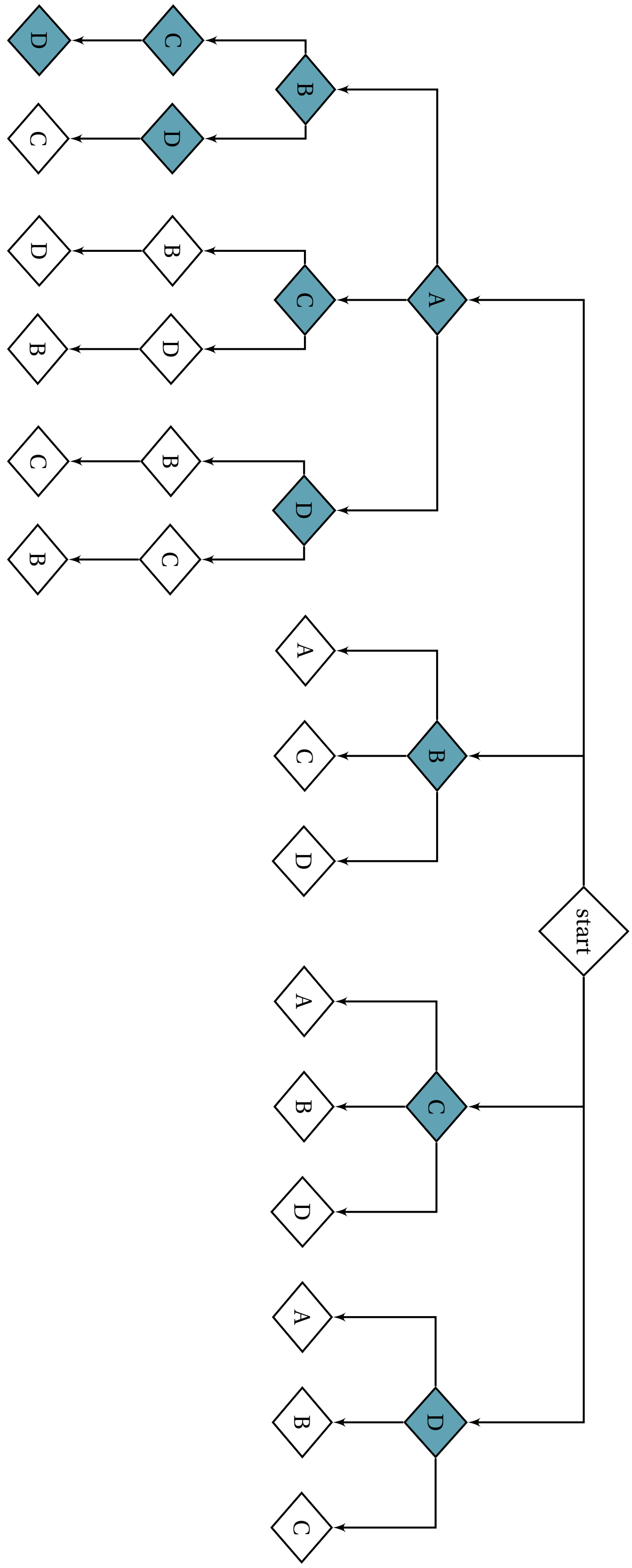


Figure 2.3: Part of the Bayesian decision tree showing all possible sequences of ply failure for a laminate with four plies for a total of 24 possible paths and 720 simulations. In blue all the simulations needed for the simplified Bayesian approach, including the simulations needed to determine the most likely ply to fail at each level, where the failure sequence is A, B, C, D.

3

Rackwitz-Fiessler

Semi-analytical methods seem very good candidates for composite reliability. Especially the FORM method described in section 2.2.2, as it promises both great accuracy and low computational effort. However, when implementing this approach, several problems arose. These are mainly related to the Rackwitz–Fiessler algorithm, and the transformation from the original variables $\vec{z}$ to the standard normal variables $\vec{x}$ in order to find the reliability index β.

In this section, different approaches to the Rackwitz-Fiessler algorithm will be discussed, in order to show their differences, and the problems encountered when applying them to composite failure will be detailed.

3.1. Hashofer-Lind algorithm

The Rackwitz–Fiessler algorithm is based on the Hashofer-Lind algorithm [2]. This algorithm finds the Most Probable Point needed for the form algorithm, given that the input variables $\vec{u}$ all follow a normal distribution $\mathcal{N}(\mu,\sigma)$ [2]. The algorithm logic is given by algorithm 1. First, the vector of random normal variables $\vec{u} = u_1, u_2 \dots u_n$ is transformed to a vector of standard normal variables $x_i = (u_i - \mu_i)/\sigma_i$ on line 3. To limit the amount of subscripts, this will be expressed as

$$\vec{x} = (\vec{u} - \vec{\mu})/\vec{\sigma}, \tag{3.1}$$

$$g_x = g\left((\vec{u} - \vec{\mu})/\vec{\sigma}\right). \tag{3.2}$$

Then, the failure function g is transformed to g_x using equation 3.2. The optimum $\vec{x}_m^*$ can then be computed iteratively, using

$$\vec{x}_{m+1} = (\vec{x}_m^T \vec{\alpha_m})\vec{\alpha_m} + \frac{g(\vec{x_m})}{||\nabla g_x(\vec{x_m})||}\vec{\alpha_m} \qquad \text{with } \vec{\alpha} = -\frac{\nabla g_x(\vec{x})}{||\nabla g_x(\vec{x})||} \tag{3.3}$$

where the left hand side is the projection of x_m along the direction of the unit vector $\vec{\alpha_m}$ and the right hand side is a correction to make sure that the new optimum fulfils $g(\vec{z}) = 0$ [2], as can be seen in figure 3.1. The gradient $\nabla g_x(\vec{x})$ can either be found analytically, or using [2]

$$\frac{\delta g_x(\vec{x})}{dx_i} = \sigma_i \frac{dg(\vec{x}\vec{\sigma} + \vec{\mu})}{\delta u_i}. \tag{3.4}$$

3.2. Rackwitz-Fiessler as described by Madsen et al

In their book “Methods of structural safety” [2] , Madsen et al. provide the most detailed explanation of the Rackwitz Fiessler algorithm that was found. The methodology as described in this book will be summarised, and some of the problems that were encountered when applying this to composite reliability will be discussed.

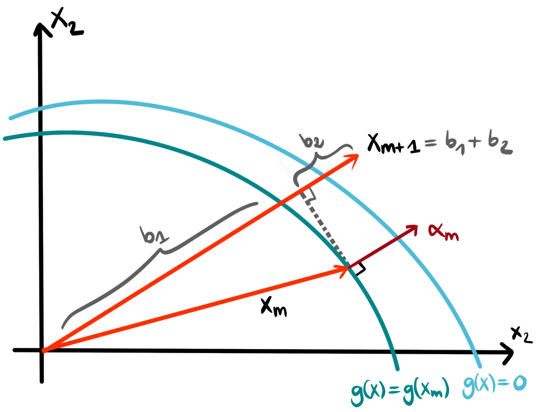


Figure 3.1: Geometrical representation of the Rackwitz-Fiessler method.

3.2.1. Methodology according to Madsen

As the Hashofer-Lind algorithm only holds for normal variables, the first step for the Rackwitz–Fiessler algorithm is to transform the original random variable $\vec{z}$ to a variable with a normal distribution $\vec{u}$. This is achieved trough the following formula applied at point $\vec{z} = \vec{z}_0$ [2]:

$$\vec{\sigma} = \frac{\varphi\left(\Phi^{-1}\left(\vec{F}_z(\vec{z}_0)\right)\right)}{\vec{f}_z(\vec{z}_0)}, \tag{3.5}$$

$$\vec{\mu} = \vec{z}_0 - \vec{\sigma}\Phi^{-1}\left(\vec{F}_z(\vec{z}_0)\right), \tag{3.6}$$

where $\vec{\sigma}$ is the vector of standard deviations of the transformed variable, $\vec{\mu}$ is its mean, $\vec{F}_z$ is the vector of CDFs of the original distribution, $\vec{f}_z$ is the vector of PDFs of the original distribution, φ applies the PDF of the standard normal distribution to each element of the vector, Φ^{-1} does the same for the inverse CDF of the standard normal distributions. Then, the transformed variable $\vec{u}$ following $\mathcal{N}(\vec{\mu}, \vec{\sigma})$ has the same value for its PDF and CDF at the point $\vec{u} = \vec{z}_0$. Thus,

$$\vec{x} = (\vec{z} - \vec{\mu})/\vec{\sigma} \tag{3.7}$$

can be approximated by a vector of standard normal distributions [2].

Algorithm 1 Hashofer-Lind [2]

initial state $\vec{z} \leftarrow \bar{z}$
while β has **not** converged **do**
  $\vec{x} \leftarrow \left(\vec{z} - \vec{\mu}\right)/\vec{\sigma}$
  Evaluate $\nabla\vec{g}_x$ and $\vec{\alpha}$ at point $\vec{x}$
  while x_m has not converged **do**
    $\vec{x}_{m+1} = (\vec{x}_m^T\vec{\alpha})\vec{\alpha} + \frac{g(\vec{x}_m)}{||\nabla g_x(x_m)||}$
  $\beta \leftarrow ||x_m^*||$
  $\vec{z} \leftarrow \vec{\mu} - \vec{\alpha}\beta\vec{\sigma}$
output β

Algorithm 2 Rackwitz–Fiessler from Madsen [2]

1: initial state $\vec{z} \leftarrow \bar{z}$
2: **while** β has **not** converged **do**
3:   $\vec{\sigma} \leftarrow f(\vec{z})$ based on 3.5
4:   $\vec{\mu} \leftarrow f(\vec{z})$ based on 3.6
5:   $\vec{x} \leftarrow \left(\vec{z} - \vec{\mu}\right)/\vec{\sigma}$
6:   Evaluate $\nabla\vec{g}_x$ and $\vec{\alpha}$ at point $\vec{x}$
7:   **while** x_m has not converged **do**
8:     $\vec{x}_{m+1} = (\vec{x}_m^T\vec{\alpha})\vec{\alpha} + \frac{g(\vec{x}_m)}{||\nabla g_x(x_m)||}$
9:   $\beta \leftarrow ||x_m^*||$
10:   $\vec{z} \leftarrow \vec{\mu} - \vec{\alpha}\beta\vec{\sigma}$
11: **output** β

Madsen then goes trough a few examples of the Rackwitz–Fiessler algorithm in chapter 5.2, that is described in algorithm 2. After choosing the average $\bar{z}$ as the initial estimate, the variable transformation is applied on lines 3 and 4. Then all of the conditions are ready for the Hashofer-Lind algorithm, that is applied on lines 5 through 10. The gradient ∇g_x is computed, along with $\vec{\alpha}$ to find the optimum $||x_m^*||$. Based on this, a new point z can be found, until β has converged. The only difference between the Hashofer-Lind algorithm and the Rackwitz–Fiessler algorithm is that $\vec{\mu}$ and $\vec{\sigma}$ need to be computed separately, and Rackwitz–Fiessler allows for variables that follow other distributions than the normal distribution.

3.2.2. Problems encountered when implementing

There are a few problems that occurred when trying to apply the Rackwitz–Fiessler algorithm as described by Madsen to composite failure. Here, these will be detailed along with their probable causes and possible solutions if these are known.

No guaranteed convergence: the examples provided in the book converge in two or three iterations, and Madsen does indeed compute a new mean and standard deviation at every iteration. However, for more complex cases that require more iterations, problems arise. It has been observed that, if the estimate for any variable z is either much smaller or much larger than the average, $f_z(z)$ is likely to be small. As σ is inversely proportional to $f_z(z)$ based on equation 3.5, σ gets larger. This means that the tails of the approximated distribution of x become very wide, and no longer represent the original distribution properly.

In the next iteration, this is likely to result in an optimum that is even further from the mean, with an even smaller $f_z(z)$, amplifying this further and further. In those cases, it has been observed that the estimate of x either tends to $-\infty$ for stiffness related variables, or $+\infty$ for lead related variables. In the end, even if it works on the simple examples provided in the book, this shows that this version of the algorithm does not always converge in more complex cases.

Convergence criterion: in his book, Madsen shows convergence by looking at the values for beta [2]. However, cases have been observed where $\beta = ||\vec{x}||$ did not change more than 0.1% between two iterations, but $||\vec{x}||$ still changed, and in actuality, β had not converged yet. Therefore, we recommend to use $\vec{z}$ or $\vec{x}$ to check for convergence.

Global vs local optimum: furthermore, for composites the optimisation space is not convex [1, 4], this means that Rackwitz–Fiessler cannot guarantee to find the global optimum, only a local optimum. The local optimum that is found is dependent on the starting point. Lekou mentions having adapted the Rackwitz–Fiessler algorithm to allow finding a global optimum, but does not detail how. In section 3.4 it is shown how, even for an algorithm that is designed to be able to get out of local optima, it can still be difficult to find the global optimum.

In this case, the approach to find a global optimum was based on multistart, e.g. running the same optimisation with different randomly selected starting points. In that case, it is not possible to guarantee that the global optimum will be found, the more starting points used, the more likely this is.

Algorithm 3 Rackwitz–Fiessler from Sundararajan [3]

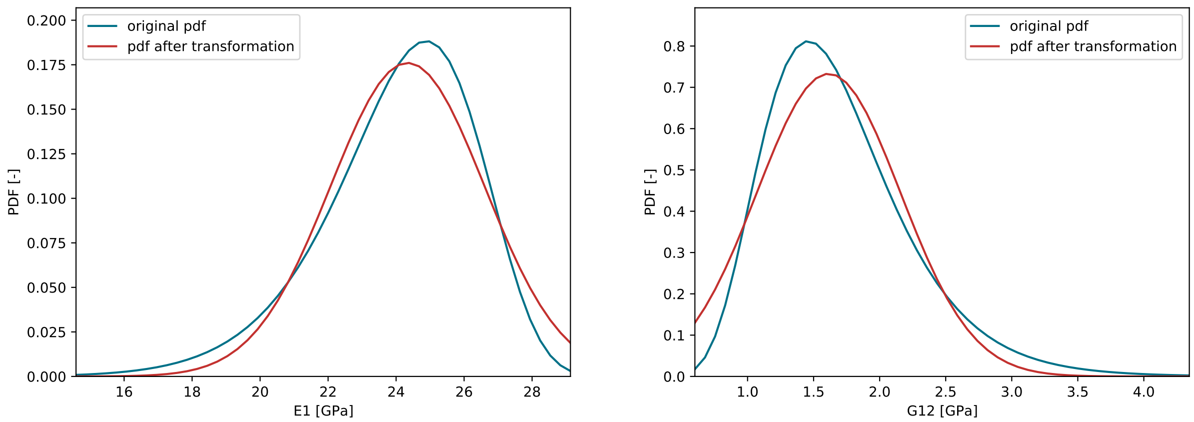
initial state $\vec{z} \leftarrow \bar{z}$
$\vec{\sigma} \leftarrow f(\vec{z})$ based on 3.5
$\vec{\mu} \leftarrow f(\vec{z})$ based on 3.6
$\vec{x} \leftarrow (\vec{z} - \vec{\mu}) / \vec{\sigma}$
while $\vec{x}$ has **not** converged **do**
    Evaluate $\nabla \vec{g}_x^m$ and $\vec{\alpha_m}$ at point $\vec{x}$
    while x_m has not converged **do**
        $\vec{x}_{m+1} = (\vec{x}_m^T \vec{\alpha})\vec{\alpha} + \frac{g(\vec{x_m})}{||\nabla g_x(x_m)||}$
    $\beta \leftarrow ||\vec{x}_m^*||$
    $\vec{x} \leftarrow -\beta\vec{\alpha}$
output β

3.3. Rackwitz-Fiessler as described by Sundararajan

The other source that has been found describing the Rackwitz-Fiessler algorithm is Sundararajan [3]. His description diverges from Madsen on a few key points, making it more likely to converge.

3.3.1. Methodology according to Sundararajan

In the third chapter of his "probabilistic structural mechanics handbook", Sundararajan provides a different version of the Rackwitz–Fiessler algorithm, described in algorithm 3. The main difference with the version of



(a) Comparison of the original pdf to the approximated pdf for E_1 following a Weibull distribution ($\sigma = 25.04$, $\eta = 12.78$) and a skew of -0.733

(b) Comparison of the original pdf to the approximated pdf for G_{12} following a lognormal distribution ($\mu = 0.48, \sigma = 032$), with a skew of 1.02

Figure 3.2: Comparison of the original PDF to the approximated PDF between the 0.1^{th} and 99.9^{th} percentiles.

Madsen is that the variable transformation is only performed once at the start, eliminating the main cause for divergence. This means that the implementation of this version of the algorithm does actuality converge most of the time. Therefore, if one is to continue research into Rackwitz–Fiessler, the algorithm as presented by Sundararajan in strongly recommend.

3.3.2. Problems with variable transformation

In the algorithm as described by Sundararajan [3], differs to the one described by Madsen on a few points only. Mainly that the variable transformation step is only performed once. This means that some of the problems that occurred in the description of Madsen are solved. However there are still two other problems related to the variable transformation in the algorithm as described by Sundararajan.

Invalidity of the transformation for skewed distributions: In figures 3.2a and 3.2b one can see that there is a difference between the original distribution and the approximated distribution. The approximation given by Madsen in equations 3.5 and 3.6 is more inaccurate the more skewed the original distribution is. In those cases, the mean and median are often further apart, meaning that the value of sigma very high, as $f_z(\vec{z})$ is small, and $F_z(\vec{z})$ is either close to 0 for variables with negative skew or close to 1 for variables with positive skew. Even when this effect is not amplified as this is only computed once, this can still destroy the validity of the transformation [3, 27]. This is especially problematic as the design point is likely in the tails of the distribution, as one can see in figure 3.2a.

Invalidity of the transformation for distributions defined on part of $\mathbb{R}$: Most material properties are only defined on $[0, \infty)$, therefore distributions that are only defined on part of the reals $\mathbb{R}$ are common. However, it is possible that the estimate for the standard normal distributed variable x is negative, and that σ is sufficiently large such that $|x\sigma| > |\mu|$. From equation 3.7, this would mean that $z = \mu + x\sigma$ is negative, which is impossible many material parameters. This is rendered more likely due to the errors in the tails for skewed distributions as described above. For instance, for G_{12} the probability of $z < 0$ goes from 0 for the original distribution, to 0.15% for the transformed distribution, which is large enough that it is possible for the design point to end up in this region.

Besides the face that negative material properties are non-sensical in many cases, this leads to errors when using distributions that are only defined on $[0, \infty]$ such as Weibull, Exponential or Gamma distributions, where a negative z renders the computation of the failure probability impossible.

For lognormal variables, these problems can be solved by using a different variable transformation, as lognormal variables can be transformed to normal variables exactly, even when there is skew [13]. This could be the reason that most applications of the FORM method often use normal and lognormal distributions, as can be seen in Table 1.1, as there would be no problems due to approximations.

In the case of different distributions, the solution proposed by Sundararajan is also to use different equations than equations 3.6 and 3.5 to transform the original variable to a normal variable. In order to better model the behaviour of the tails of the distribution the first alternative transformation is based on the mean rather than the median. This transformation is mostly recommended in cases where the original distribution is highly skewed, and is given by [3]:

$$\begin{cases} \vec{\mu} & = F_z^{-1}(0.5) = \text{median of } \vec{z}, \\ \vec{\sigma} & = \frac{z_0 - \mu}{\Phi^{-1}(F_z(z_0))}. \end{cases} \tag{3.8}$$

However, Sundararajan states that the transformation in equation 3.8 does not guarantee convergence if the median is sufficiently large. Furthermore, this transformation does not exclude the possibility that $-x\sigma > \mu$ and the resulting estimate for z is negative.

To avoid this, an alternative approach is to put a boundary on $\vec{x}$ when it reaches negative values. That it to say, equations 3.6 and 3.5 are used, unless the resulting $\vec{x}$ reaches a threshold, generally $\vec{x} \leq \vec{0}$. Then, an alternative transformation is used as is given by equation 3.9.

$$\begin{cases} \vec{\mu} & = 0, \\ \vec{\sigma} & = \frac{z_0}{\Phi^{-1}(F_z(z_0))}. \end{cases} \tag{3.9}$$

This case makes it impossible for $z = (\mu + x\sigma)$. This works fine for strength or load related variables. However, the author has observed that for stiffness related variables, the problem is generally that the design point is in the right hand tail rather than the left hand. In that case, this lower limit does not solve the problem. An upper limit could be used, but it is less obvious as to how to decide what value to use. Further research is needed into how to determine this upper limit, to ensure it does not affect the estimate too much.

However, no exact transformation from a generic distribution to a normal distribution exists. These alternative transformations can allow the method to produce results in a little more cases, but no general solution to the problem exists.

3.4. Alternatives to Rackwitz Fiessler

There is no reason that the optimisation step of the FORM method has to be performed by using the Rackwitz–Fiessler algorithm. Even if the Rackwitz fiessler is one of the fastest algorithms for this particular problem [1, 2, 4], other optimisation methods could also be used. The optimisation that is needed was described in equation 2.3 as follows.

$$\begin{aligned} \text{minimise } \vec{u} \quad & \beta^2 = u_1^2 + u_2^2 + \cdots + u_n^2 \\ \text{subject to} \quad & g^*(\vec{u}) = 0 \end{aligned}$$

As discussed before, the main challenge is that the the problem is not convex: there are many local optima [1] which makes it impossible to guarantee that a global optimum is found. The selected optimisation method should allow for non-convex spaces and should allow for an equality constraint.

In this research, attempts were made to use different algorithms that are included in the the scipy.optimize package that allow for an equality constraint. Use was made of the basinhopping[1] algorithm to attempt to find a global optimum rather than a local optimum. Even if the algorithm was able to find the correct minimum some of the time, most of the time it did not, even with a high number of starting points, as can be seen in figure

Indeed, as can be seen in table 3.3, even when using 200 different staring points and 500 basinhopping iterations, this is not a guarantee that the global minimum can be found. In most of the bottom left quadrant, the optimisation gets stuck in local minima, leading to an overestimation of the failure probability, whereas in the top left quadrant there is a point where the safe zone is reduced. With the basinhopping algorithm it is sometimes possible that a minimum that does not satisfy the constraints is accepted[1], this is likely such a case.

However, even if many starting points are needed and the global minimum is not found this method can still be useful in some cases. To start, it is much more likely to converge than Rackwitz-Fiessler, and the

[1] `https://docs.scipy.org/doc/scipy/reference/generated/scipy.optimize.basinhopping.html`

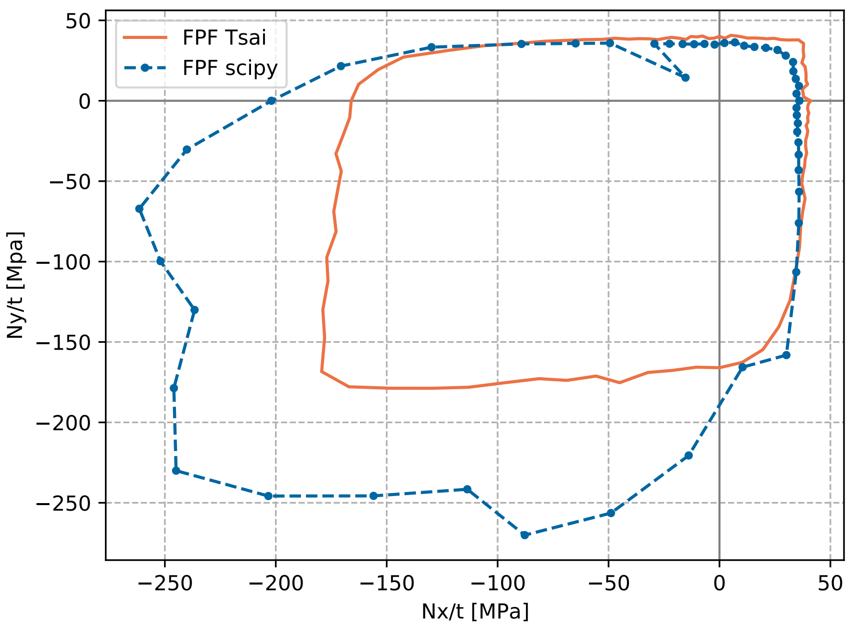


Figure 3.3: Comparison between the reliability using the FORM method when using scipy basinhopping for the optimisation to Monte Carlo results for a $[0/90]_S$ for a p_f of 10^{-4}

convergence speed of this approach is independent of the target probability of failure. Therefore, it could be interesting for preliminary design stages, especially when very small probabilities are needed (say 10^{-6}) as in those cases Monte Carlo methods quickly become very computationally expensive. This modified approach to FORM could provide an initial estimate quickly in these cases.

Unfortunately, even when using an alternative optimisation method, this does not eliminate the variable transformation that is the source of a lot of the errors plaguing the FORM method. It is still possible for the design point to be negative, and the inaccuracies of the variable transformation are still present. It is very likely that the results using this approach can only be used as a rough estimate. Therefore, no further research into alternative optimisation methods is recommended at this moment.

3.5. Discussion

FORM is a reliability method that promises accurate results for low computational effort. Unfortunately, it is shown that this method is not adapted to the needs of composite reliability.

Two different descriptions of the Rackwitz–Fiessler algorithm have been found. The method as described by Madsen [2], used by Lekou [1], is not guaranteed to converge due to the repeated variable transformation. It is strongly recommended to use the algorithm as described by Sundararajan [3].

When using any probability distribution other than normal and lognormal, the variable transformation cannot be exact, it will always be an approximation. In the case that the original variable is only defined on part of $\mathbb{R}$, it is possible that the design point according to the Rackwitz-Fiessler algorithm lies outside of the domain of definition, resulting in a point that is impossible in the original variable space.

Even for distributions that are defined on $\mathbb{R}$, the transformation is not sufficiently accurate in the tails of the distribution, this especially holds for highly skewed distributions. This means that the method does not provide the theoretically correct reliability index for $\beta \geq 2-3$ or a failure probability of about 0.02 to 0.001. These values are too large for most aerospace related applications. Alternatives variable transformations can be used, some of them are provided by Sundararajan. However, none of them can guarantee convergence for any variable distribution, they only circumvent the problem in some cases.

Furthermore, the optimisation space is highly non-convex, and the Rackwitz–Fiessler algorithm can only guarantee a local minimum. It is possible to modify the algorithm with multistart in order to find the global minimum or MPP, but this increases the computational effort significantly. Other optimisation methods than Rackwitz-Fiessler can also be used, however the ones that were tried were also unable to reliably find the global minimum. Even if a suitable alternative was found, this would still not circumvent the problems due to the variable transformation.

Thus, even if FORM sounds ideal on paper, in practise the convergence problems, limitation to normal and lognormal variables, as well as the increased computational effort due to the non-convexity of the optimisation space, make this method unsuitable for composite reliability applications. Therefore, in the next chapter, Monte Carlo methods will be used to study the effect of failure mode dependence and last ply failure on composite reliability.

4

Results and analysis for mode-dependence

In this chapter the results of the reliability analysis using the Tsai-Hahn failure criterion will be compared to the predictions that use the mode-dependent failure criteria Hashin and Puck. The difference between the predictions will be analysed in order to asses the effects of using mode-dependent criteria the FPF reliability of the laminate. First, the implementation and validation of the analysis will be discussed. Then the results will be presented for each of the three failure criteria. Finally, the implications of these results will be summarised.

4.1. Implementation of the Monte Carlo reliability analysis

In this section, the details of the implementation of the Monte Carlo reliability analysis including failure mode-dependence will be discussed. First, a short summary of the most important points for the stress analysis will be discussed, followed by the implementation of the Monte Carlo analysis and its validation.

4.1.1. Stress analysis

In order to preform the reliability analysis, a function computing the failure criterion based on the input variables and the layup has been programmed. For this analysis, it is assumed that there are only normal loads in the x and y directions of the plate, denoted N_x and N_Y. These are then normalised width respect to the total thickness of the laminate t. This means that D-matrix is not used since there is no out-of plane loading [28].

Two different layups will be analysed. First the cross-ply with a $[0,90,90,0]$ layup, and second a quasi-isotropic laminate $[0,\pm45,90]_S$. Since both theses layups are symmetric and balanced, the B-matrix is 0, and only the A-matrix is needed in this analysis [28]. These two laminates have been selected as cross-plies and quasi-isotropic laminates are among the best-studied and most used layups [28], providing a good reference for the analysis. The material used is based on the experiments and fits of Lekou [1]. The distributions used are given by table 1.1, and the PDF's of the material properties can be found in appendix A.

4.1.2. Implementation of the Monte Carlo Method

In choosing the total number of simulations per case, there is a trade off between accuracy and speed. The number of simulations must be high enough such that one failure more or less does not affect the failure probability, but not too high as to use computational effort unnecessarily.

As the target probability of failure is 10^{-4}, the number of simulations need to be at least a few multiples of 1000 in order to be able to detect such a small probability of failure. Furthermore, the simulations are highly independent, so they can be parallelised easily. The computer that was used to run these simulations has 6 cores, thus by choosing a multiple of 6 for the total number of simulation, maximum efficiency is reached. The number of simulations needed for convergence is dependent on the failure criterion used. For Tsai-Hahn convergence occurs from about 1 to $2\cdot10^5$ simulations, while for Puck, a little more is needed. Rounding up to a convenient multiple of 6 for maximum efficiency, 60 000 simulations are performed for each step.

In total 100 loading ratios N_x/N_y will be studied for each laminate, this represents 100 directions in the x,y-plane. The load will be increased stepwise until a target failure probability of 10^{-4} is reached. In order to save time in simulations, a few different step sizes are used of $[120,40,10,5,1]$MPa. Once the probability of failure is smaller than the target, the code goes back to the previous multiple and continues the search using the smaller step size, until the step size of 1 is reached.

4.1.3. Verification

The model is verified by comparing the results to those of Lekou [1], to ensure that no mistakes have been made when coding. As Lekou only models FPF, only this part of the code can be validated in this fashion. For both the $[0,90]_S$ and $[0,\pm45,90]_S$ laminate, as shown in figure 4.1 and 4.2 results show very good agreement with Lekou, suggesting that these results are valid. As Lekou describes, one can see in figures 4.1 and 4.2

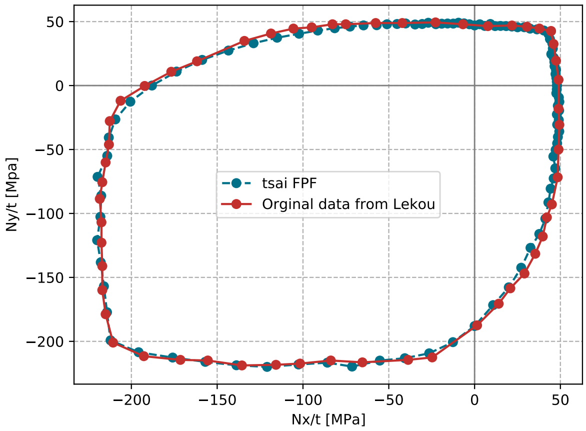


Figure 4.1: Comparison between failure locii for a $[0,90]_S$ Gl/P laminate ($P_f = 10^{-4}$) as done by Lekou [1] and the author

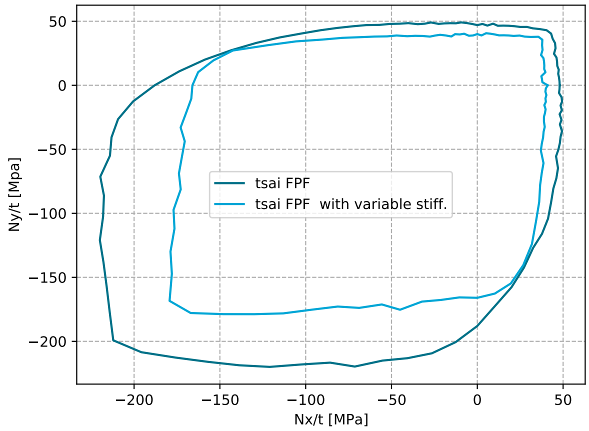


Figure 4.2: Comparison between failure locii for a $[0,90]_S$ Gl/P laminate ($P_f = 10^{-4}$) both including and excluding the stiffness properties as random variables

that the stiffness properties do have a large effect on the reliability, and that the variability of these properties cannot be neglected. If not taken into account, these result in an overestimation of the failure probability. Therefore, these simulations use all nine variables previously discussed as random variables, the strength properties X_t, X_c, Y_t, Y_c and S as well as the stiffness properties $E_1, E_2, G_{12}, \nu_{12}$. For the distributions used for each variable, refer to table 1.2.

4.2. Relative speed per failure criterion

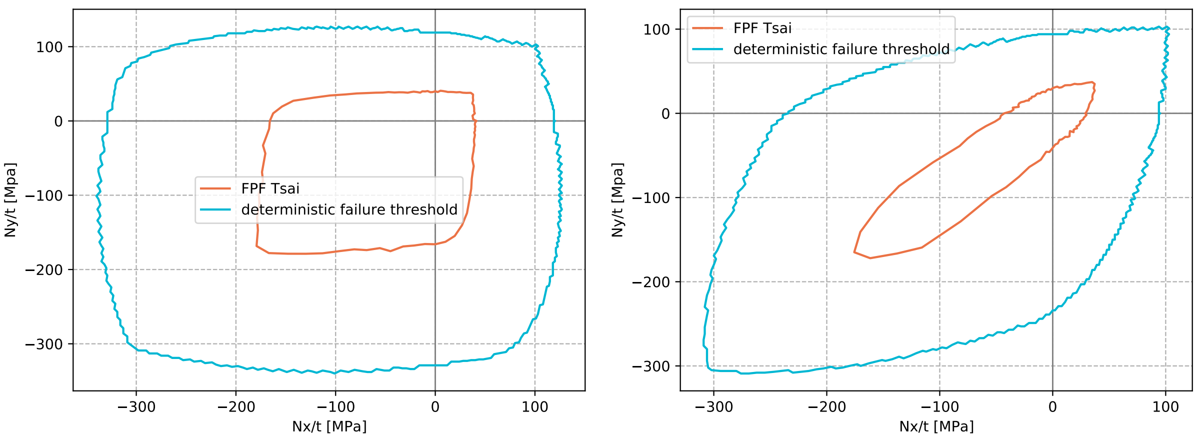
In this section the relative speed of each method is compared. The comparison is done based on the average for each loading combination and can involve more than one Monte Carlo simulation if more than one step is needed. As the code is written in Python and simulated on a desktop, these values should not be seen as an absolute reference. It is likely that the code can be significantly sped up by using C or FORTRAN in case speed is important. However this does allow for relative comparisons. As one can see the effect on computational time of using a mode-dependent failure criterion is not very large, a difference of 5 - 10% is acceptable.

	cross-ply	quasi-isotropic	increase wrt Tsai-Hahn
Tsai-Hahn	626 s	734 s	
Hashin	685 s	783 s	+ 5-10%
Puck	694 s	815 s	+ 10%

Table 4.1: Average time per loading combination simulations for each failure criterion

4.3. Using the Tsai-Hahn failure criterion

Tsai-Hahn is a quadratic mode-independent failure criterion, it is therefore unable to distinguish between different failure modes. The reliability analysis using the Tsai-Hahn failure criterion is used as a reference to compare the failure mode-dependent results to. Its formulation is given in section 1.2.



(a) For a $[0,90]_S$ laminate

(b) For a $[0,\pm45,90]_S$ laminate

Figure 4.3: Comparison between the failure locii predicted by deterministic failure methods to failure locii predicted by reliability analysis for a $P_f = 10^{-4}$ using Tsai-Hahn

4.3.1. Comparison between deterministic and probabilistic failure envelope

The first comparison to be made is between the deterministic approach and the reliability method. As can be seen in figures 4.3a and 4.3b, In both cases, the shape of the failure envelopes are relatively similar for the deterministic and probabilistic failure envelopes, however the deterministic failure threshold shows much higher failure loads. The difference between both figures is well over a typical aerospace safety factor of 1.2. By only using the deterministic analysis and a safety factor, it is possible for the design to either underperform in use, or overperform and lead to unnecessary weight increase. On the other hand, the reliability results allow for a much more concrete risk assessment. The laminate can be designed for both a target load and risk, which can very depending on the application. Thus, reliability analysis does indeed provide a better picture of the structures performance in use. Therefore, it allows to design with risk in mind.

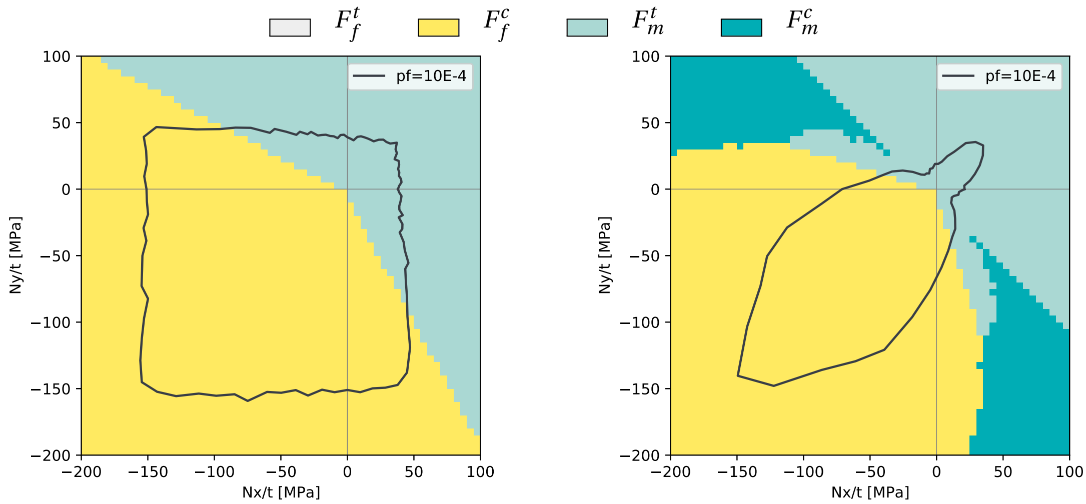
4.3.2. For the cross-ply

Even if composite fibres generally perform better in tension than compression, the structural load carrying capacity is much larger in the bottom left quadrant than in the bottom right quadrant, as can be seen in figure 4.6a. This difference was already present in the original deterministic failure envelope, but seems amplified. This is likely because the strength in tension X_t is defined trough a negatively skewed Weibull distribution, yielding in a lot of early failures, whereas the strength in compression Y_t is defined by a lognormal distribution with positive skew, where the values concentrate closer to the mean on the left side of the mean, as can be seen in figures A.5 and A.7.

As can be seen in figure 4.2, the model that includes variable stiffness properties is much more angular, it can almost be defined fully by four thresholds, whereas the transitions when excluding the stiffness properties are much smoother. In the top right quadrant, the difference is not significant. However, in the compression domain, the difference is largest, up to about 15−20% in the −135° direction, further confirming that stiffness properties have a big influence on the reliability.

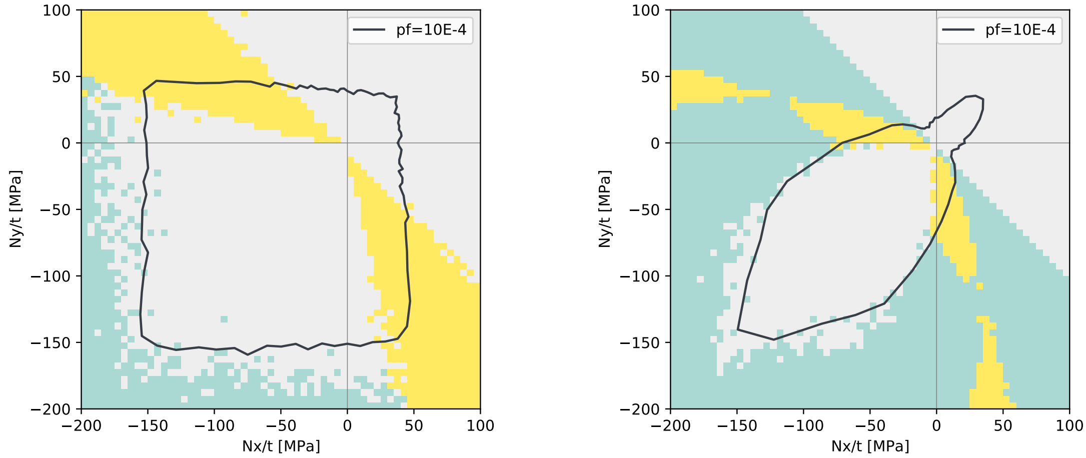
4.3.3. for the quasi-isotropic laminate

The shape of the safe zone for the quasi-isotropic laminate is much more slender and narrow as can be seen in figure 4.7c, so in most directions, the safe zone to first ply failure is much smaller. This can be explained by the fact that there are more plies in more directions, so there is a higher likelihood that some of the load is redistributed to a layer that is in tension, and as previously discussed, failures in tension are more likely at lower loads. However, the values in the top right corner and bottom left corner, along the +45 degree axis match those of the cross ply.



(a) Mode that is most likely to be highest for a $[0/90]_S$ laminate

(b) Mode that is most likely to be highest for a $[0/\pm45/90]_S$ laminate



(c) Mode that is second most likely to be highest for a $[0/90]_S$ laminate

(d) Mode that is second most likely to be highest for a $[0/\pm45/90]_S$ laminate

Figure 4.4: Failure mode whose criterion is most likely and second most likely failure to be highest the first failure of Gl/P laminate based on Hashin, and safe zone for $p_f = 10^-4$

4.4. Using the Hashin failure criterion

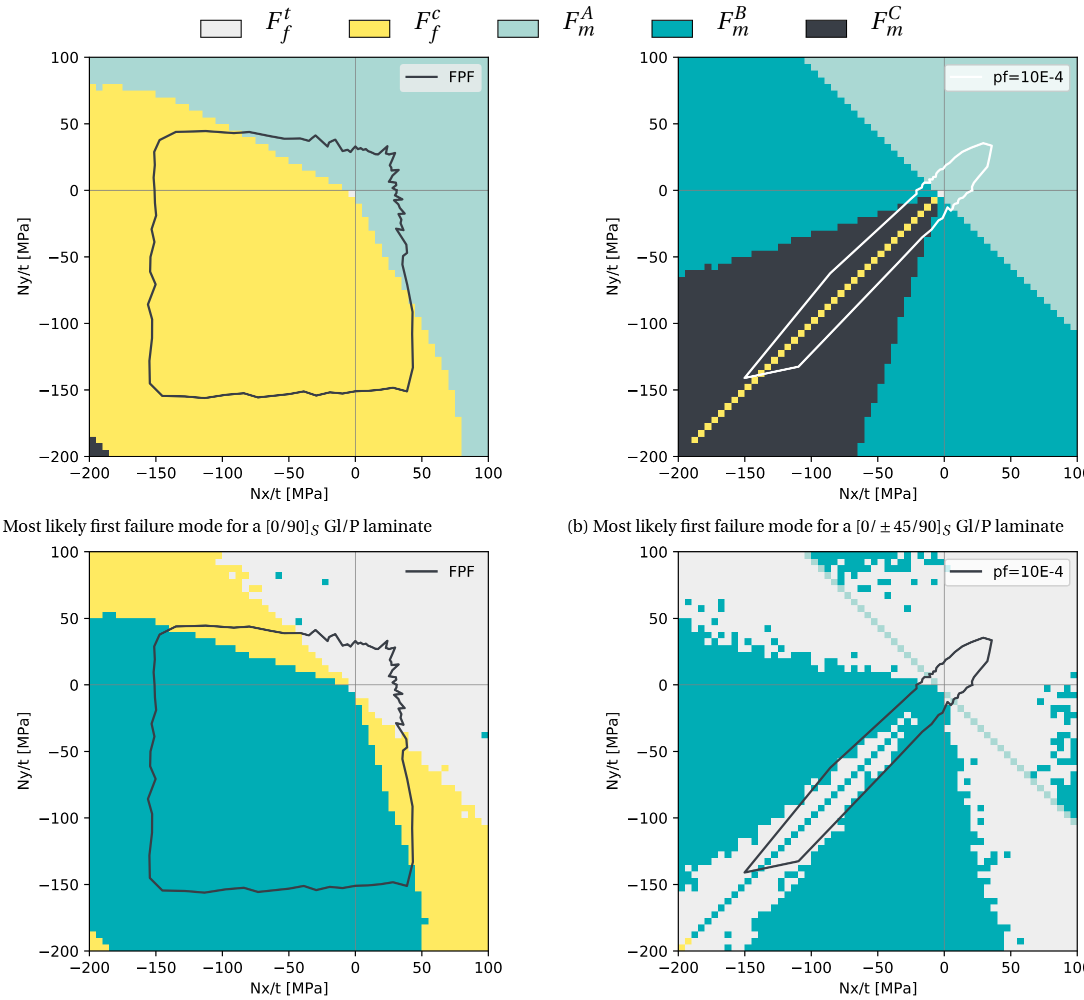
Where Tsai-Hahn can not differentiate between failure modes, the Hashin criterion distinguishes four different failure modes: F_f^t fibre breakage in tension, F_f^c fibre buckling in compression, F_m^t matrix cracking in tension and F_m^c matrix crushing in compression. The formulas for each failure mode are given in section 1.2. The Hashin failure criterion is the simplest of the two mode-dependent criteria used. Figure 4.4 shows the most likely and second most likely failure mode for each combination depending on the loading.

4.4.1. For the cross-ply

As can be seen in figure 4.6a, the resulting figure using Hashin has approximately the same shape as the one for Tsai-Hahn. In the right top quadrant, both approaches agree very well, the difference between them is no more than 5%. In the bottom left hand quarter, the difference is the strongest, in the $\vec{x}$ direction the load is 180 MPa for the Tsai-Hahn criterion, for 150 MPa based on Hashin, a difference of 16%. This can be explained by the fact that the Hashin failure criterion predicts matrix failures in this region as shown in figure 4.4c, as these are not accurately modelled by Tsai-Hahn.

4.4.2. For the quasi-isotropic laminate

As can be seen in figure 4.6b, the result for Hashin is less slender and much more wide. Again the curve is less smooth with more sudden switches in direction indicating a switch in failure mode that is not modelled well by Tsai-Hahn. The shape of the safe zone lines up quite well with the the transitions zones in figure 4.4d, suggesting a clear effect of the mode-dependent criterion.



(a) Most likely first failure mode for a $[0/90]_S$ Gl/P laminate

(b) Most likely first failure mode for a $[0/\pm 45/90]_S$ Gl/P laminate

(c) Second most likely first failure mode for a $[0/90]_S$ Gl/P laminate

(d) Second most likely first failure mode for a $[0/\pm 45/90]_S$ Gl/P laminate

Figure 4.5: Most likely and second most likely failure mode for the first failure of Gl/P laminate based on Puck

4.5. Using the Puck failure criterion

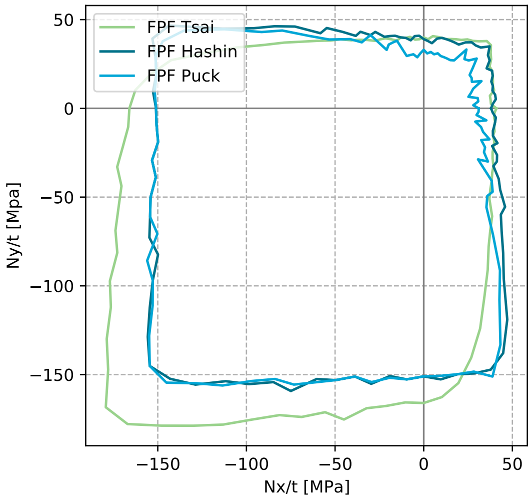
The Puck criterion is the mode complex of the mode-dependent failure criteria studied, distinguishing five different failure modes: F_f^t fibre breakage in tension, F_f^c fibre buckling in compression,F_m^A matrix failure in compression, F_m^B matrix failure in moderate transverse compression and F_m^C matrix failure in large transverse compression. The formulas for each failure mode are given in section 1.2, and the most likely failure mode dependent on the load in given by figure 4.5.

4.5.1. For the cross-ply

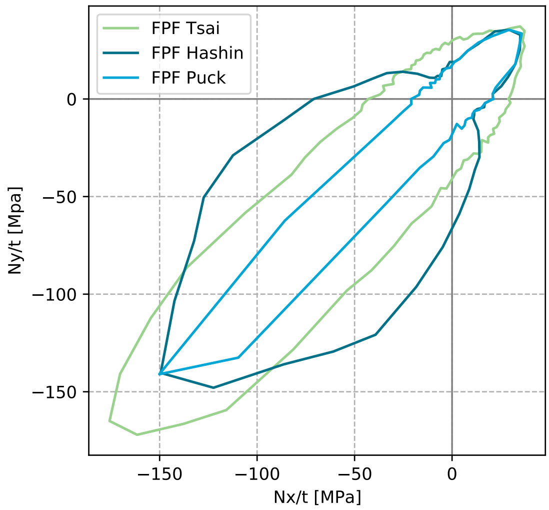
Using Puck, the results are nearly the same as for Hashin. The deviations between them are small enough that they can be explained by the Monte Carlo method. This suggests that in this case, using Puck rather than Hashin has no effect on the reliability analysis. Indeed, they both model nearly the same failure modes, even if they use different equations to model them. This suggests that it is indeed the addition of more failure modes that changes the reliability prediction, rather than any particularities of Hashin or Puck.

4.5.2. For the quasi-isotropic laminate

In this case, there is a difference between Puck and Hashin. Both model agree very well in the top right quadrant, but in the other quadrants the safe zone based on Puck is even more slender than the mode-independent safe zone. Again this suggests that matrix failure modes are likely involved in this region.



(a) Failure locii for a $[0/90]_S$ laminate using Tsai-Hahn, Hashin and Puck



(b) Failure locii for a $[0/\pm45/90]_S$ laminate using Tsai-Hahn, Hashin and Puck

Figure 4.6: First ply failure locii for a $[0/\pm45/90]_S$ and a $[0,90]_S$ Gl/P laminate ($P_f = 10^{-4}$) using Tsai-Hahn, Hashin and Puck

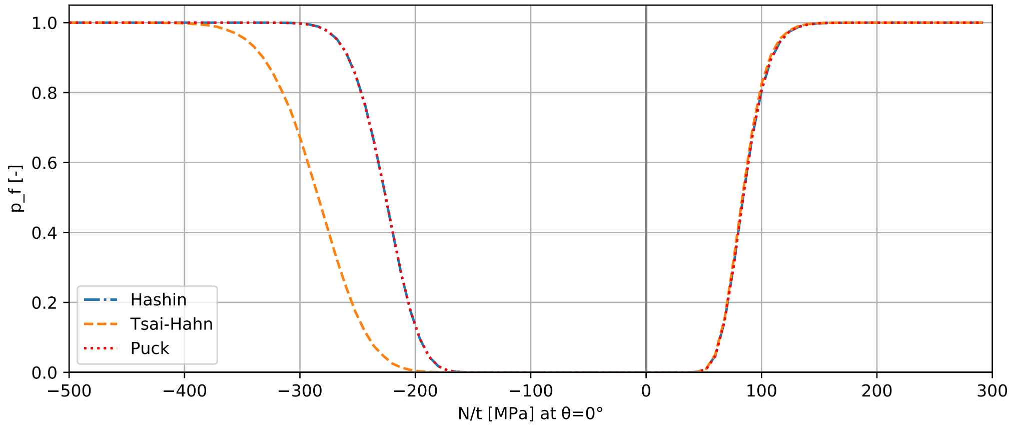
4.6. Discussion

Reliability analyses have been performed for three different failure criterion: Tsai-Hahn, Hashin and Puck. It is shown that using mode-dependent failure criterion have a significant effect on the reliability prediction for FPF of the laminate, for marginal additional computational cost. The effect is especially strong outside the top right quadrant, where more matrix failure modes are more common.

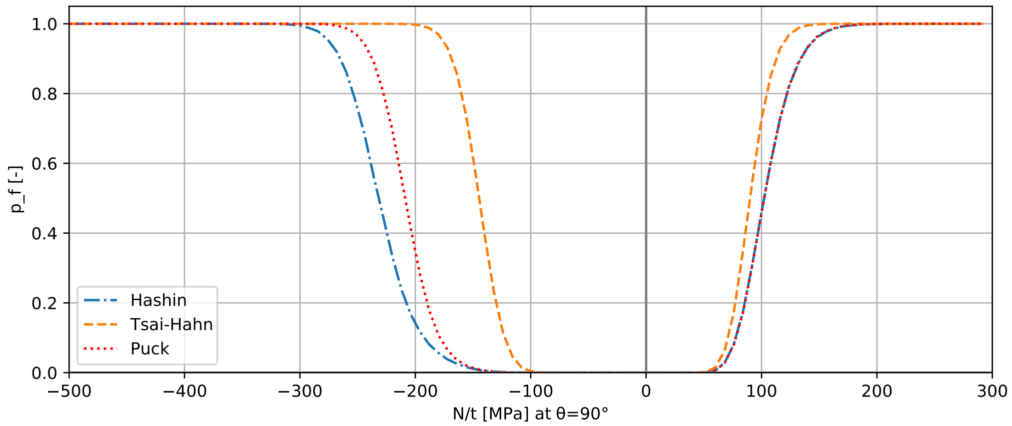
As shown in the right hand side of figures 4.7 a, b and c, and the top right quadrant of figure 4.6 a and b, in tension the three failure criteria agree well with each other. The difference between the loaf for any given probability of failure is within 5-10 percent.

However, as soon as more complex failure modes arrive, the predictions based on Tsai-Hahn differ more significantly. all three cases presented in figures 4.7, in compression, the difference between the mode-independent criterion Tsai-Hahn, and the mode dependent criteria Hashin and Puck becomes much stronger. For instance, when a compression load is oriented at 45°, this results in a drastic difference between the failure probabilities predicted, as shown in 4.7c. For a load of -300 MPa, the predicted probability of failure is only of 2.8% for Tsai-Hahn, where it is 83% and 90% for Hashin and Puck respectively. This can be explained as matrix failures are likely in this case, F_m^t for Hashin as shown in figure 4.5c and F_M^c for puck as shown in figure 4.5a.

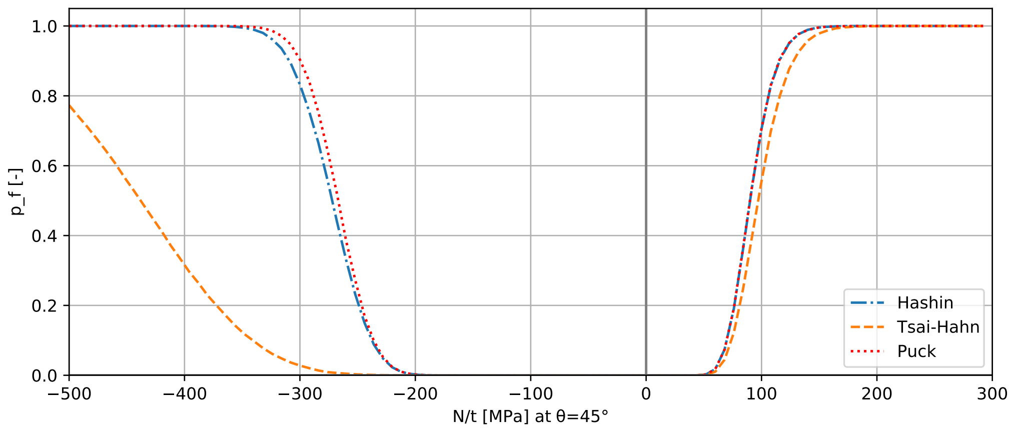
The type of criterion used is seems less critical, as for both laminates the Hashin and Puck criteria perform very similarly. Since the difference in computational effort is small and all the data is available, Puck is preferred as it includes more failure modes than Hashin, and is therefore expected to be a better fit in some cases. However, the additional material properties that are needed for Puck are more difficult to obtain, so in case that is not possible, Hashin recommended. These results still need to be validated using experimental data in order to show that using mode-dependent failure criteria do indeed improve the reliability predictions.



(a) probability of failure dependent on load in $\theta = 0$ using Tsai-Hahn, Hashin and Puck for the cross ply



(b) probability of failure dependent on load in $\theta = 90$ using Tsai-Hahn, Hashin and Puck for the cross ply



(c) probability of failure dependent on load in $\theta = 45$ using Tsai-Hahn, Hashin and Puck for the cross ply

Figure 4.7: probability of failure dependent on load in different directions using Tsai-Hahn, Hashin and Puck for the cross ply

5

Results and analysis for last ply failure

In this chapter, the results of both approaches to last ply failure in reliability will be presented, and the differences between both approaches will be analysed. As the use of mode-dependent failure criteria has an effect on the results as discussed in the previous chapter, both approaches will be used in combination with each of the three failure criteria studied.

First, the implementation and the analysis will be discussed. Then the results will be presented for each of the two laminates and each of the three failure criteria: Tsai-Hahn, Hashin and Puck. Finally, the implications of these results will be summarised.

5.1. Implementation

In this section, the details of the implementation of the Monte Carlo reliability analysis including failure-mode dependence will be discussed. Most of the assumptions discussed in section 4.1 still hold, and the variables and distribution are still those as given by tables 1.2. Here, the focus will be on the particularities related to Last Ply Failure, first the damage model that is used will be discussed, then the implementation of the Bayesian approach to LPF will be shown. Finally, the convergence of both the direct and Bayesian approach to LPF will be analysed. Similarly to the previous model, this model has also been verified by checking for convergence for each combination of failure criterion and approach to LPF.

5.1.1. Modelling damage

In order to predict LPF, a damage model needs to be defined. Most composite damage models work through degrading the properties of the failed ply, most commonly the stiffness properties [28]. These properties are multiplied by a knockdown factor, between 0 and 1. This causes the load to redistribute to the plies that have not failed yet, as these are stiffer. When combined with mode-dependent failure criterion, more complex damage models can be used, where the damage model depends on the most likely failure mode. It would even be possible to model the knockdown factor as a random variable in the reliability analysis.

In this research the focus is mostly on how to implement LPF in composite reliability. A very simple approach will be used, consisting simply of removing the ply from the laminate entirely, equivalent to a knockdown factor of 0. The effect of this is that the damage tolerance is underestimated, as the model will overestimates the stresses on the remaining plies. If there is difference between the reliability for FPF and LPF even in this extreme case, this will show that the damage tolerance of the laminate is significant. Further research is needed to determine the best approach to damage modelling in composite reliability.

5.2. Relative speed per method

In this section the relative speed of each method is compared. The comparison is done based on the average for each loading combination and can involve more than one Monte Carlo simulation if more than one step us needed. As the code is written in Python and simulated on a desktop, these values should not be seen as an absolute reference, but this does allow for relative comparisons.

The speeds are presented in table 5.1 for the quasi-isotropic laminate. For the direct approach, it is not expected that it will be a little slower, but that the difference is acceptable. However for the Bayesian approach, the number of simulations needed increases with a factor of about 6 for the quasi-isotropic laminate, so a bigger difference is expected there. Also, as for the FPF case the difference between the different failure modes was not very significantly, this is expected for both approaches to LPF as well.

As one can see in table 5.1, the direct approach increases the time needed by a factor between about 1.2 - 1.5, which is acceptable, especially since this is not expected to increase much the more plies are in the laminate. For the Bayesian approach, the difference is less large as expected, a factor of 4.5 to 5 instead of 6.5. This is likely due to the fact that each time damage is detected, the ply is eliminated, rendering the next simulations simpler. The difference in time is still acceptable for a handful of plies, but it is important to be aware that this difference will increase further with the number of plies as described in section 2.4.3

Overall, as for the FPF case, the effect on the speed of the failure mode used is not very significant. So, also in the case of LPF, using more complex failure criteria such as Hashin or Puck improves the model without much computational cost. finally It is likely that the code can be significantly sped up by using *C* or FORTRAN in case speed is important.

	FPF	Direct	increase wrt FPF	Bayes	increase wrt FPF
Tsai-Hahn	734 s	1101 s	+ 50%	3623 s	+ 395%
Hashin	783 s	991 s	+ 30%	3670 s	+ 370%
Puck	815 s	936 s	+ 15%	3662 s	+ 350%

Table 5.1: Average time per loading orientation to find the for each failure criterion for the quasi-isotropic laminate

5.3. Direct approach

The direct approach to LPF was discussed in section 2.4.1. The failure function used for the Monte Carlo analysis simply used a deterministic approach to see if LPF has occurred or not. In this section, the results from this method will be presented and the results will be compared to the FPF predictions, for all three failure criteria.

5.3.1. For the cross-ply

For any failure criterion, a clear symmetry along the +45°direction can be observed. This is because the laminate is rotationally symmetric: the layup remains the same when it its turned by a quarter turn. Thus, when rotating the load a quarter turn from the $\vec{x}$-direction to the $\vec{y}$-direction, it should come as no surprise that the material behaviour hasn't changed. Furthermore, just like in the FPF case, the Hashin and Puck failure criterion agree really well with each other, as shown in figures 5.2a.

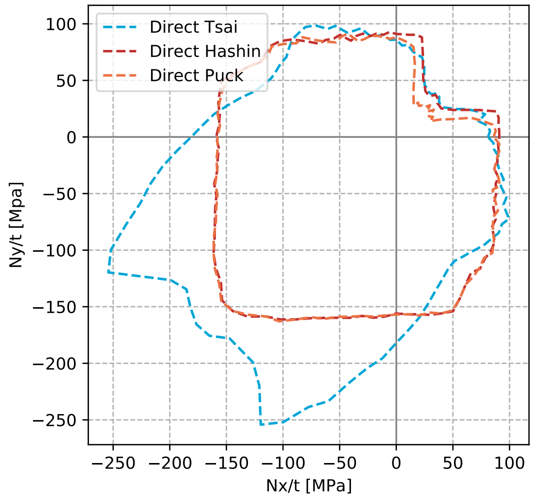
In figure 5.1, one can see that there are two possible scenarios. Either all the plies fail as soon as the first ply fails, as happens in the +45° direction, or there there is still room for the load to increase, as for instance in the 0° and +90° directions. This difference between the safe zones for FPF and the direct approach are called regions of damage tolerance. It was already noted that the safe zone was much bigger in the bottom left quadrant for FPF. However, it seems that the damage tolerance is concentrated more in the positive $\vec{x}$ and $\vec{y}$ directions. This suggests that as opposed to the first ply failure, as discussed in 4.4, after the first ply has failed, the next are not due to fibre failure in compression. This is confirmed by 5.5c, where the safe zone lines out the transition between fibre failure and matrix failure.

Overall, even though there is a clear effect of including LPF, the difference is not as strong as one might expect for a material known for damage tolerance. In the best case scenario, the difference between the loads for first and last ply failures is a factor 2. However, in most directions it is hardly noticeable, and would be encompassed by a safety factor. However, in this mode, all material properties are knocked back to zero as soon as damage is initiated, so the lack of damage tolerance could be due to this harsh model.

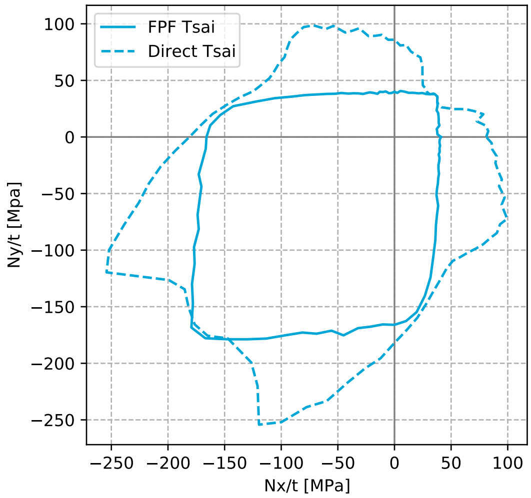
5.3.2. For the quasi-isotropic laminate

In the case of the quasi isotropic laminate, the difference between the first and last ply failure is much stronger than for the cross-ply. As can be seen in figure 5.2, the FPP and LPF safe zones only overlap in the +45° direction. The safe zone does not increase equally in all directions, but seems to increase a constant amount in direction of the top-left to bottom-right diagonal.

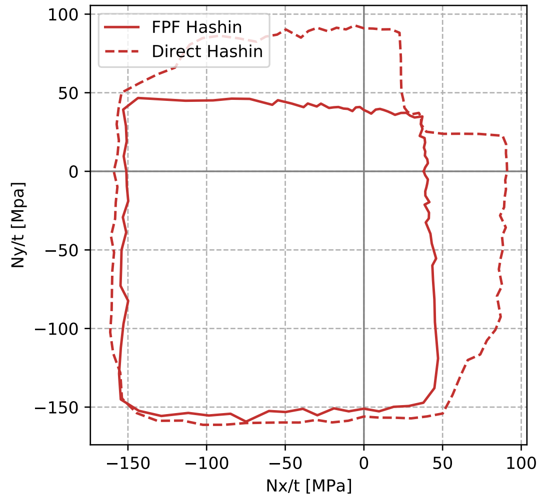
In the quasi-isotropic laminate will be more damage tolerant, as for the quasi-isotropic laminate, there are simply more plies that are liklely oriented in a good direction to withstand the load after a first ply failure.



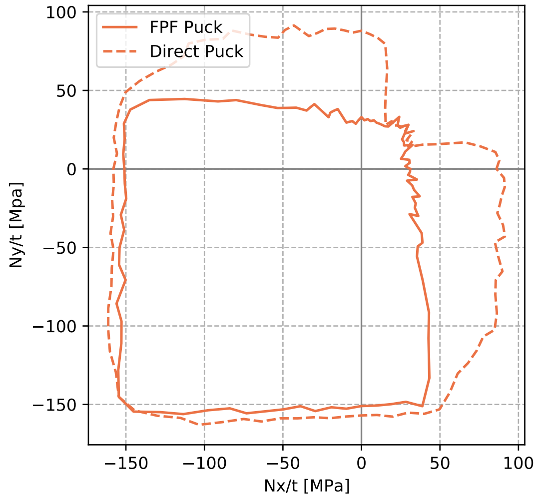
(a) Comparison of FPF failure locii using all three criteria



(b) Comparison of FPF failure locii using Tsai-Hahn



(c) Comparison of FPF failure locii using Hashin



(d) Comparison of FPF failure locii using Puck

Figure 5.1: Comparison failure locii for $[0,90]_S$ Gl/P laminate ($P_f = 10^{-4}$) using the direct approach to LPF

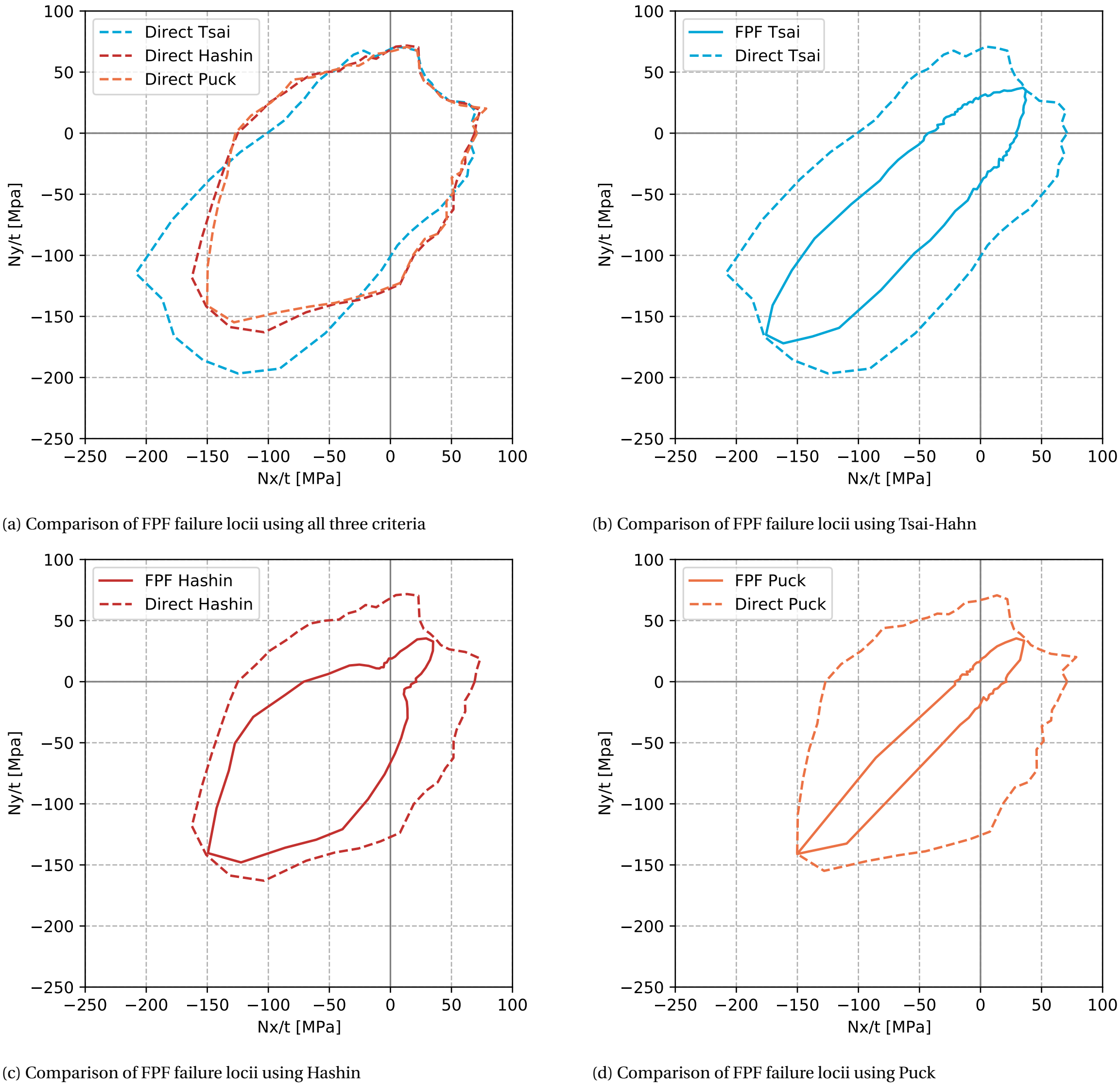
It is interesting to note that, whereas the failure envelopes of Hashin and Puck did not agree for the quasi-isotropic laminate, and the predicted failure modes do not agree very well between Puck and Hashin as shown in 5.6 and 5.5, the failure envelopes do agree really well for the LPF failure envelope as seen in figure 5.2a.

5.3.3. Discussion

Even with the harsh approach to failure, where a ply is completely eliminated as soon as damage occurs, there are still clearly regions of damage tolerance, but also regions where all plies fail at once. Being able to differentiate if the first failure of a ply will cause the failure of the entire laminate trough a domino effect, or if there is some room for damage tolerance, would be very interesting information for a designer.

Next, the failure criterion used also affects the LPF predictions. The difference is perhaps even stronger in LPF as matrix failures will become more likely the more layers have failed. An interesting note is that, even if the predicted failure modes do not agree in most cases, as shown in figures 5.6 and 5.5, the two mode-dependent criteria agree very well with each other.

However, it is hypothesised that the direct approach underestimates the damage tolerance of the laminate. As the entire laminate is based on one sample, this creates an artificial correlation between the different plies. Indeed, in the sample where one ply has poor material properties, all plies will have these same poor properties, rendering LPF more likely. There are manufacturing defects that would affect the entire laminate



(a) Comparison of FPF failure locii using all three criteria

(b) Comparison of FPF failure locii using Tsai-Hahn

(c) Comparison of FPF failure locii using Hashin

(d) Comparison of FPF failure locii using Puck

Figure 5.2: Comparison failure locii for $[0, \pm 45, 90]_S$ Gl/P laminate ($P_f = 10^{-4}$) using the direct approach to LPF

at once (for instance curing the laminate at the wrong temperature or pressure). However, many defects are more local to one layer (misalignment of a ply, weaving defects, local variation in fibre properties...). Therefore, it is expected that this is not a good representation of reality.

This could be countered by using different samples for each layer. Even if generating samples is relatively cheap, this would make the construction of the A-matrix more expensive as one would need a different stiffness matrix Q for every ply rather than for every different angle. This would negate the advantage of this model over the Bayesian model, that is to say that it is much faster, especially for laminates with many layers.

Furthermore, the direct approach only sees the last ply failure as a random event. However there is no reason that all the failures in between are not probabilistic in nature too. There is a certain level of uncertainty and variance on each ply failure, not only on the last one.

5.4. Bayesian approach

As discussed in section 2.4.3, the Bayesian approach models each ply failure as a separate random event, and then computes the probability of last ply failure based on those. In this section, first the implementation will be shortly discussed. Then the results for the Bayesian approach will be shown, for all three failure criteria, and will be compared to the predictions based on the direct approach.

5.4.1. Implementation of the Bayesian approach

For the direct approach to LPF, the failure function for first ply failure can simply be replaced with a function that returns whether or not the last ply has failed. However, the implementation of the Bayesian approach is not as straightforward. Therefore, in this section a simplified version of the implementation is presented in algorithm 4.

As the final probability of failure p_f according to this method is a product, p_f gets initialised to 1. Then, the algorithm is repeated as many times as there are plies in the laminate. For each repetition, the Monte Carlo analysis is performed for all plies. The ply with the highest probability of failure is selected, and removed. Once all plies have been removed, the final probability of failure is returned. Of course, if a different damage model is used, instead of removing the plies, their material properties need to be depreciated. However, this does not affect the methodology significantly.

Algorithm 4 Bayesian approach to LPF

$p_f \leftarrow 1$
number of plies $\leftarrow$ `count`(laminate)
while number of plies > 0 **do**
    for every ply k **do**
        $p_{f_k} \leftarrow$ `MonteCarlo`(laminate)
    $p_{f_{max}} \leftarrow$ largest p_{f_k}
    $p_f \leftarrow p_f \times p_{f_{max}}$
    `remove` ply with $p_{f_k} = p_{f_{max}}$
output p_f

5.4.2. For the cross-ply

All failure criteria are presenting distinct regions where the damage tolerance is much higher than was predicted by the direct approach. The safe zone has "lobes" that are almost, but not exactly, aligned with the $\vec{x} = 0$ and $\vec{y} = 0$ axis. In this zone, the structure is at its most efficient and last ply failure is governed by fibre failure, which typically has much higher failure strengths. The reason that these lobes are not aligned with the axis exactly is because first ply failure is more likely in tension. For instance, for the lobe that is oriented in the $-\vec{x}$ direction (to the left), this means that FPF of a 90 degree ply is more likely above the $N_y/t = 0$ axis than below. Ultimately this means that LPF is more likely when there is tension in the $\vec{y}$ direction. This orients the lobes away from the $\vec{x} = 0$ and $\vec{y} = 0$ axis.

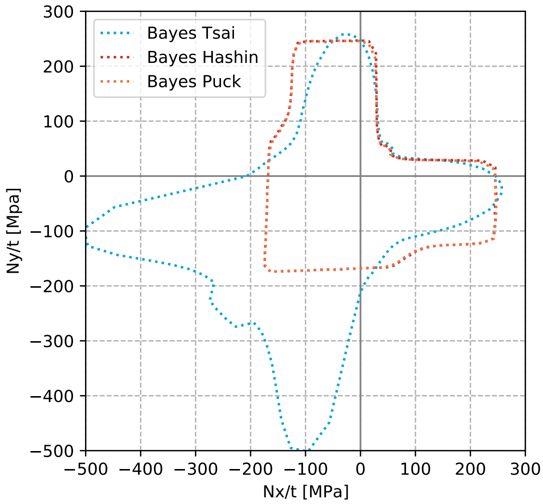
Similarly to the direct approach, the Bayesian approach does predict a very rapid failure in the bottom left quadrant for the mode-dependent criteria, but not for the Tsai-Hahn criterion. This is again related to matrix failures not being modelled well by the Tsai-Hahn criterion.

5.4.3. For the quasi-isotropic laminate

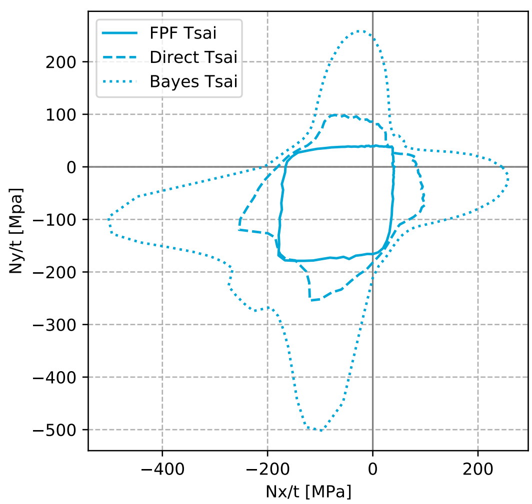
As expected, the Bayesian model does also predict zones where there is increased damage tolerance. Again, these "lobes" are oriented in the direction of the $\vec{x} = 0$ and $\vec{y} = 0$ axis, and shifted towards the negative directions for the same reasons as for the cross ply. Similarly, the Tsai-Hahn criterion predicts more damage tolerance in the bottom left quadrant than the two other criteria. In this case, this also aligns neatly with the shift in failure criteria for hashin as shown in figure 5.4c. This does not line out as neatly for the Puck criterion, even if the failure envelopes line out very neatly.

5.5. Effect of mode-dependence

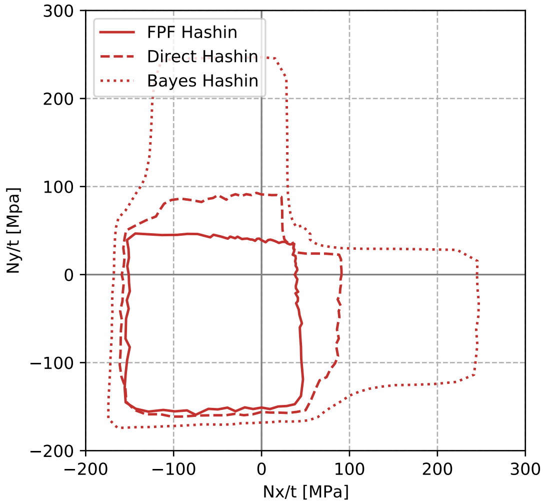
For both layups, it was already noted that the safe zone was much bigger in the bottom left quadrant for FPF and the direct approach. This also holds for the Bayesian approach to for LPF, where the damage tolerance is much higher in the bottom left quadrant. This likely has the same cause: the distribution of the strength X_t has negative skew, making early failures likely whereas the distribution of the strength Y_t has positive skew, making the variance on the left side much smaller. This effect is not present for the Puck and Hashin failure envelopes, as the failures are not governed by fibre compression in this region, as shown figure 5.5 and 5.6. In these figures, it is very notable how well both criteria agree on the safe zone, even if they do not predict the



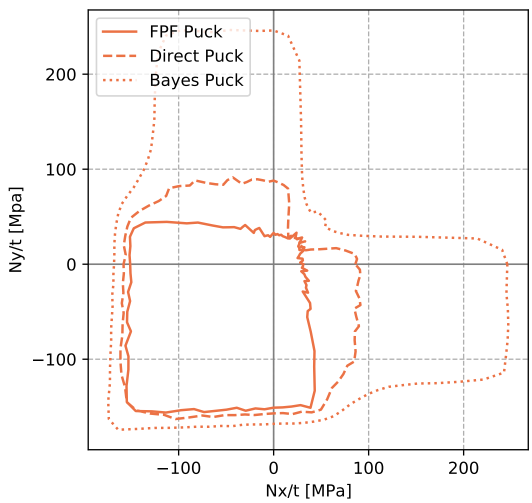
(a) Comparison of FPF failure locii using all three criteria



(b) Comparison of FPF failure locii using Tsai-Hahn



(c) Comparison of FPF failure locii using Hashin



(d) Comparison of FPF failure locii using Puck

Figure 5.3: Comparison failure locii for $[0, \pm 45, 90]_S$ Gl/P laminate ($P_f = 10^{-4}$) using the direct approach to LPF

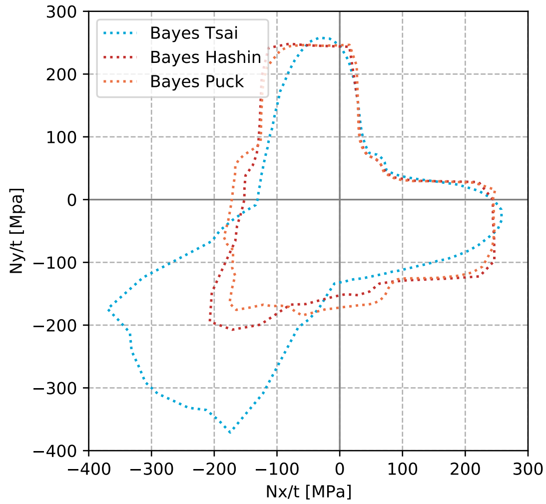
same highest failure criteria in the same regions.

Overall, like for FPF, the use of a mode-dependent failure criterion does not affect the computational effort significantly, but does affect the reliability predictions. Therefore, the use of such a mode-dependent criterion is strongly recommended. Between the Puck and Hashin criteria, the predictions agree very well. Since the inputs for Hashin are generally much more readily available, this is the recommend criterion, unless the inputs for Puck are available.

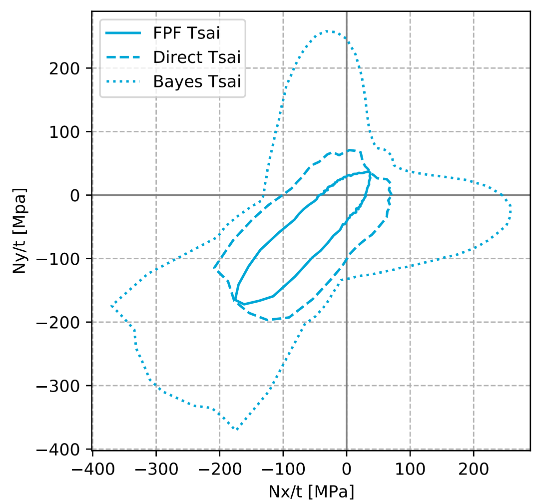
5.6. Discussion

Two different approaches to last ply reliability analysis have been used: the direct approach and the Bayesian approach. Modelling the reliability form an LPF perspective provides useful information to the designer, mainly if it is expected that the different plies fail rapidly one after another or if there is damage tolerance.

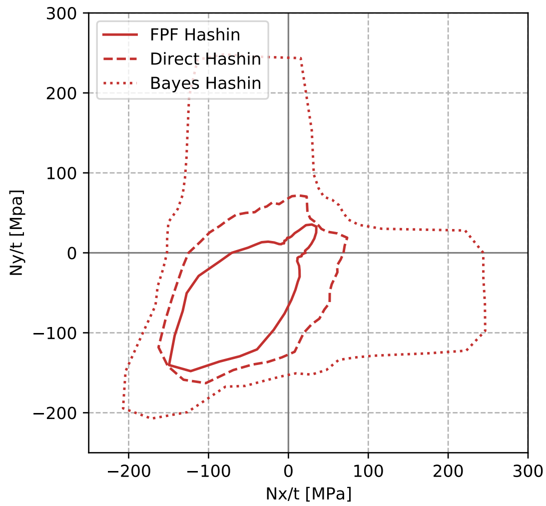
Two approaches have been discussed, the direct and the Bayesian approach. The direct approach is faster, but is expected to a lesser fit of experimental results, Indeed, the direct approach uses the same material properties for the first ply to fail and the next one. This creates a correlation between the material properties of the different plies on a single laminate, underestimating the reliability of the laminate, as compared to the Bayesian approach as can be seen in figures 5.3 and 5.4. This could be countered by using different samples for the different plies, but this would increase the computational effort needed, going against the



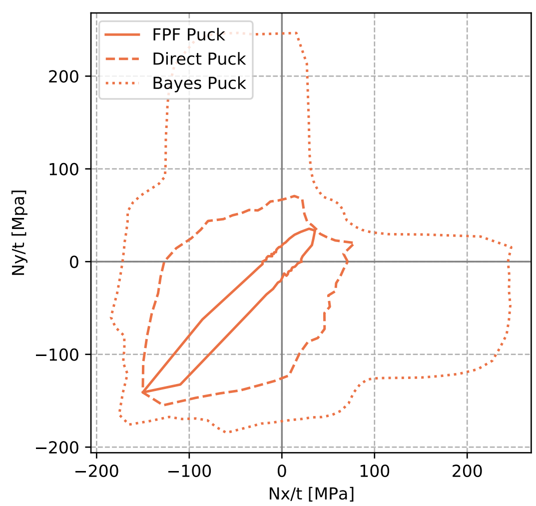
(a) Comparison of FPF failure locii using all three criteria



(b) Comparison of FPF failure locii using Tsai-Hahn



(c) Comparison of FPF failure locii using Hashin



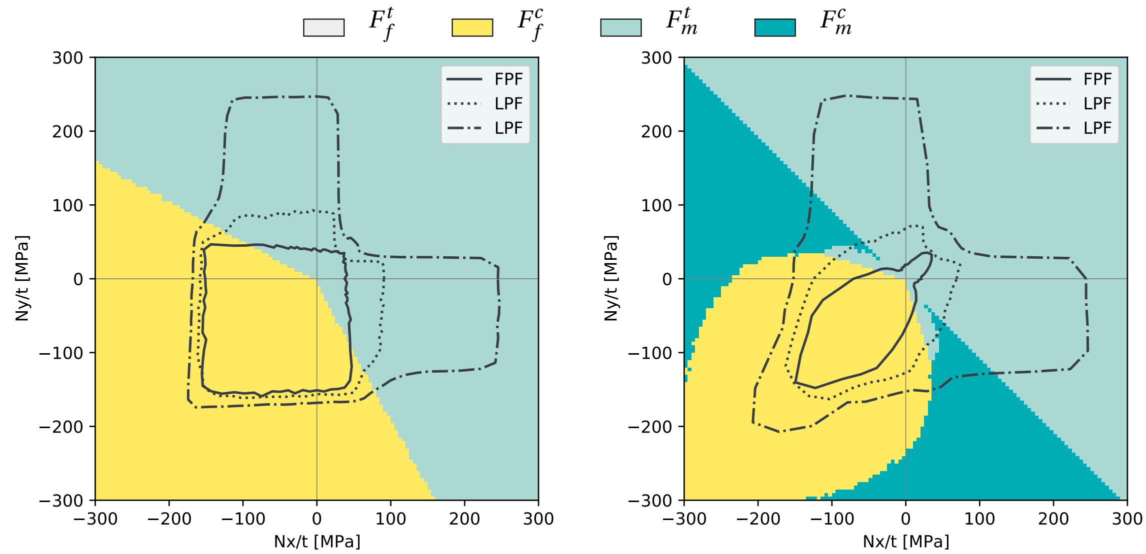
(d) Comparison of FPF failure locii using Puck

Figure 5.4: Comparison failure locii for $[0,90]_S$ Gl/P laminate ($P_f = 10^{-4}$) using the direct approach to LPF

main advantage of this method.

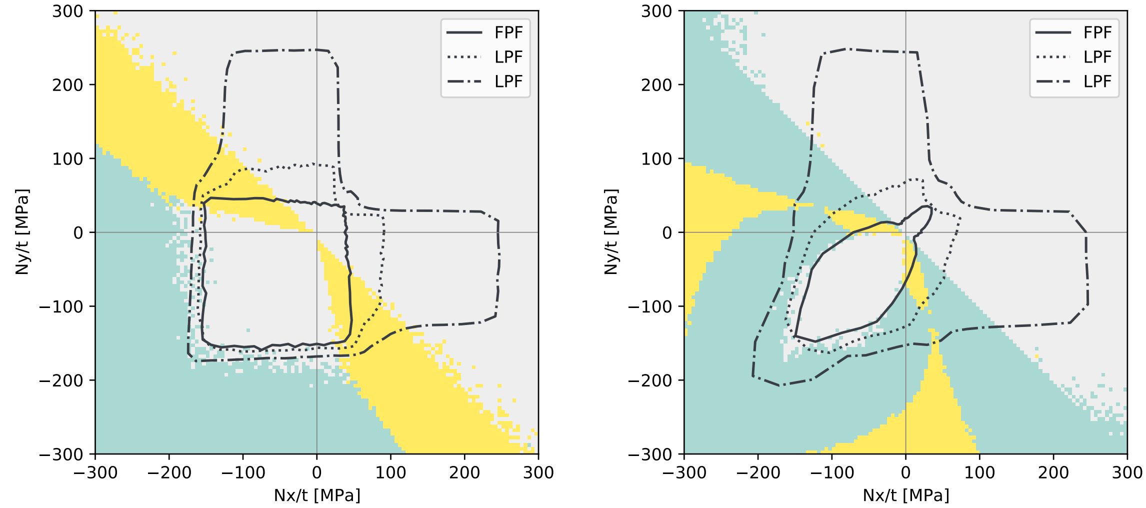
The direct approach only models the last ply failure as a random event, but any ply failure can be modelled as a random event. This is the approach used by the Bayesian method. By taking into account the probabilistic nature of every ply failure separately, the Bayesian approach is expected to be a more accurate representation of reality. Overall, the Bayesian approach is less conservative and predicts more damage tolerance. However, the Bayesian approach is much more computationally expensive as opposed to the direct approach, and the difference in computational effort grows the more plies are in the laminate. Therefore, the direct approach can be a better option in cases where there are many layers or the computational effort is especially important. All in all, it is strongly recommend to use a type of reliability analysis for last ply failure to learn more about the damage tolerance of the design.

The effect of the type of failure criterion is much more important in a LPF reliability analysis than it was in the FPF analysis. Indeed, as more complex failure modes become more likely once a few plies have failed, the mode-independent failure criterion is no longer a good, fit. Similarly to the first ply failure model, the difference between the Hashin and Puck is very small. This further shows that the difference is due to the fact that these models include failure mode-dependence, rather that only being due to the specifics of each failure criterion. So, including failure-mode dependence is very important in reliability analysis for last ply failure as well, as this affects the predictions significantly. The exact mode-dependent failure criterion used



(a) Safe zone and highest mode for a $[0/90]_S$ laminate

(b) Safe zone and second highest mode for for a $[0/\pm45/90]_S$ laminate



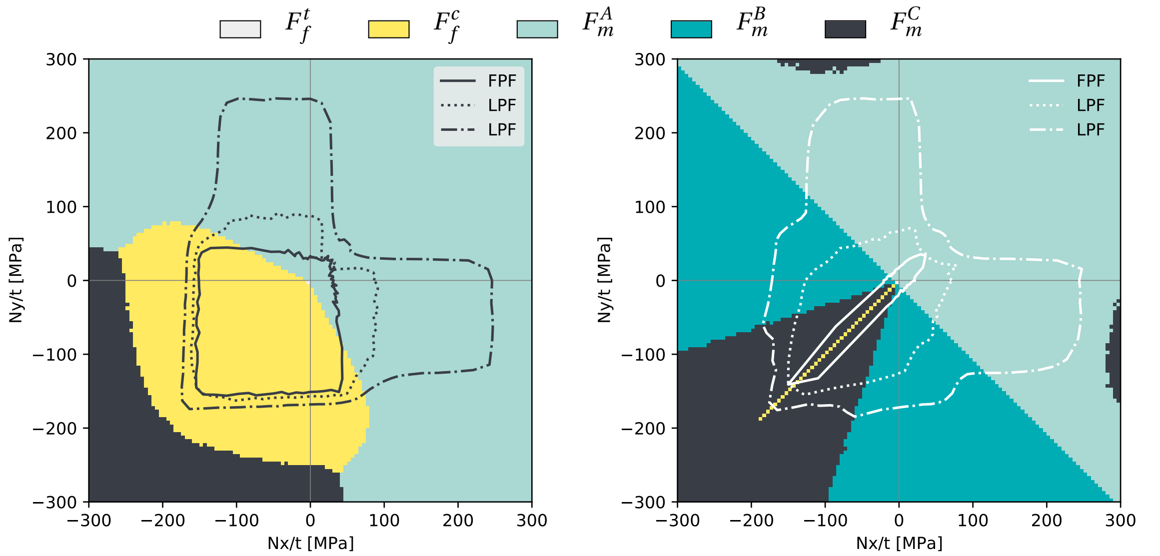
(c) Safe zone and highest mode for for a $[0/90]_S$ laminate

(d) Safe zone and second highest mode for for a $[0/\pm45/90]_S$ laminate

Figure 5.5: Comparison failure locii for a $[0/\pm45/90]_S$ and a $[0,90]_S$ Gl/P laminate ($P_f = 10^{-4}$) using Hashin with both approaches to modelling LPF, and zones where the criteria for different failure modes are most likely to be highest

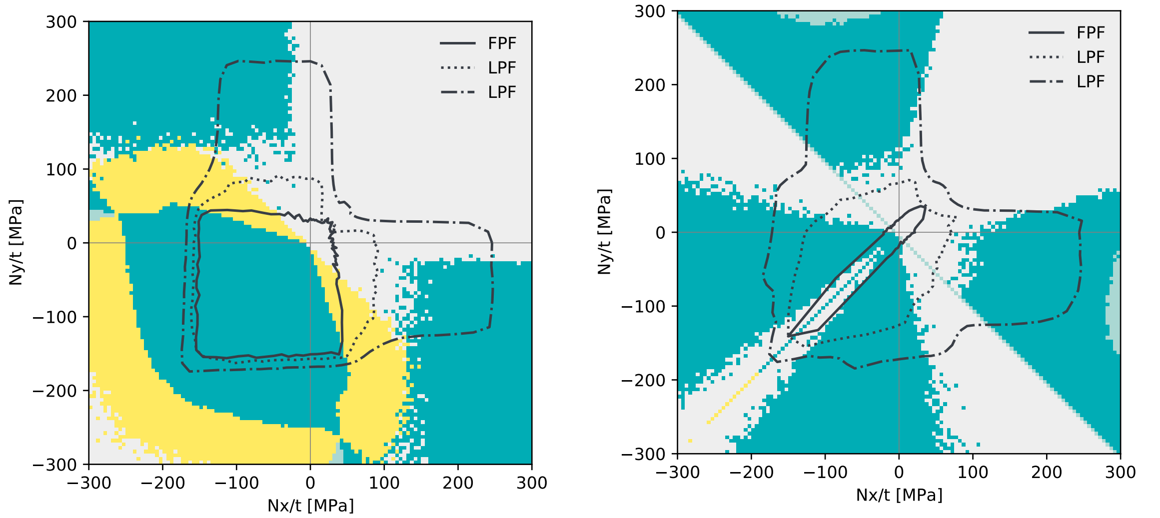
is less critical.

Finally, in this analysis, damage tolerance is underestimated due to the way failure is modelled. In this case, once there is damage initiation, it is assumed that the ply will have lost all its load-carrying ability. Of course, in practise this is rarely the case, and a ply with some damage will still be able to carry a part of the load. More research is needed into the effects of different knock-down factors and damage models and their effects on the reliability predictions.



(a) Safe zone and most likely first failure mode for a $[0/90]_S$ laminate

(b) Safe zone and most likely first failure mode for a $[0/\pm45/90]_S$ laminate



(c) Safe zone and second most likely first failure mode for a $[0/90]_S$ laminate

(d) Safe zone and second most likely first failure mode for a $[0/\pm45/90]_S$ laminate

Figure 5.6: Comparison failure locii for a $[0/\pm45/90]_S$ and a $[0,90]_S$ laminate ($P_f = 10^{-4}$) using Puck with both approaches to modelling LPF, and zones where the criteria for different failure modes are most likely to be highest

6

Conclusions and Recommendations

When designing a composite structure, it is important to focus only on the failure load, but also on how much this result can be trusted. Due to the high variance in, amongst others, material properties in composite samples, deterministic approaches can't predict a failure load that can be trusted for every single sample. Reliability methods provide a way to incorporate these probabilistic aspects of composite failure. Therefore, in this research, its was investigated how can the current reliability methods be used to predict the probability of both first and last ply failure using failure mode-dependent failure criteria.

6.1. Reliability methods for composites

There are several approaches to reliability. One of the most commonly used methods is First Order Reliability Method (FORM) [4] as it promises results on good agreement with Monte Carlo results for a lower computational cost. However it is shown that FORM is not well suited for composite reliability applications. Indeed, in order to be able to model the small probabilities needed for aerospace applications ($p_f \leq 0.01$), this method is no longer accurate, unless only normal and lognormal variables are used. Furthermore, the method has convergence problems and needs to be repeated many times to ensure good quality results. This is why it is not recommended to use FORM for composite reliability applications, and why Monte Carlo methods were used in this research.

6.2. The effects of improved failure methods on composite reliability

The goal of this research was to improve reliability predictions for composite laminates by incorporating better failure models. Two main aspects have been selected where the failure models could be improved. First the type of failure criterion was changed from quadratic (Tsai-Hahn) to mode-dependent, allowing to predict different failure modes. This was shown to have a significant effect on the predicted reliability, especially when the laminate was loaded in compression. Furthermore, it was shown that although the effect of including mode-dependence was important, the exact failure criterion used less so, since Puck and Hashin both produced very similar results.

Second, two different approaches were set up and compared to model Last Ply Failure, in order to better assess the damage tolerance of the laminate. The first model that is proposed in this research is called the direct approach, the failure function uses deterministic fracture analysis to detect LPF. This approach predicts significantly higher failure probabilities, but likely underestimates the reliability of the composite structure. Further research is needed to compare theses results with experimental data. The second approach that was proposed in this research is called the Bayesian approach, as it is based on a probabilistic analysis of the failure of the composite laminate. It is shown that this approach is less conservative than the direct approach. The results make good intuitive sense, and are expected to better represent experimental results. Further testing is needed to confirm this hypothesis. The results of both approaches need to be verified by experimental results to accurately assess their predictive quality.

6.3. Recommendations for further research

Further improvements to the model would include better modelling of damage. In this research, once a ply has failed, the ply is considered having lost all load-carrying ability. However, it would be interesting to see the effects of different knock-down factors on the reliability of the laminate. This could be especially interesting if the knock down factor varies depending on the type of failure that is predicted. This could help better reflect the different severity of the different failure modes, and allow for designs that are both reliable and are likely to fail in a favourable manner.

Another path for further investigation would be to see if the Bayesian model can be further simplified by making use of the symmetry of most layups designs. Indeed, at the moment, each ply is considered individually, but it is likely that the failures of symmetric layers are correlated, and this could be exploited to speed up the method. Finally, it is recommended that further research is put into the validation of the predictions for the mode-dependent model and the LPF model with experimental results.

A
PDF of the random variables used

In this appendix the probability density functions of the random variables used are shown in order to be able to better visualise their effects on these distributions on the reliability of the laminate. The distributions used for each of the nine variables are based on [1]. They are plotted between the 0.1^{th} and 99.9^{th} percentiles, i.e. between a CDF of 0.001 to a CDF of 0.999. The expected value of each of the distributions is shown trough a vertical line.

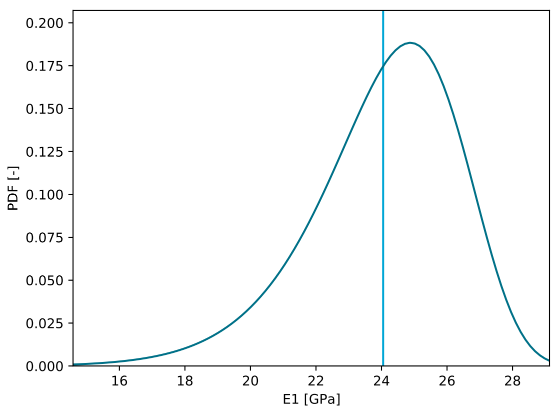


Figure A.1: PDF of the Weibull ($k = 25.0, \lambda = 12.78$) distribution of E_1

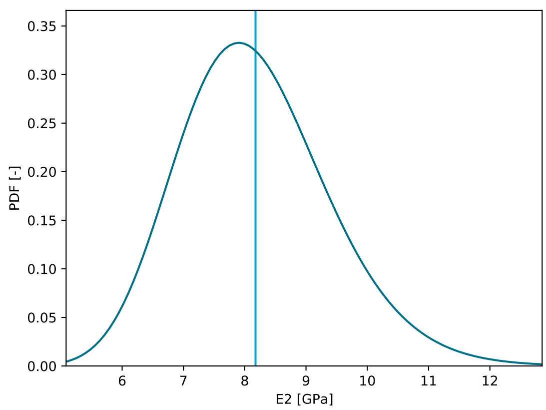


Figure A.2: PDF of the log-normal ($\mu = 2.09, \sigma = 0.15$) distribution of E_2

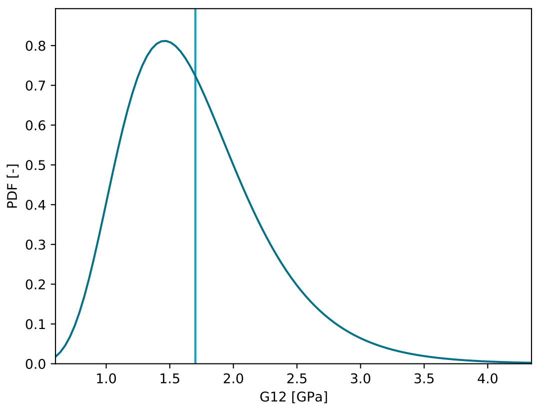


Figure A.3: PDF of the log-normal ($\mu = 0.48, \sigma = 0.32$) distribution of G_{12}

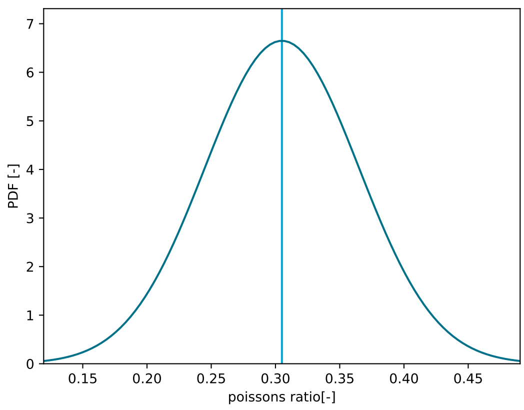


Figure A.4: PDF of the normal ($\mu = 0.305, \sigma = 0.06$) distribution of ν_{12}

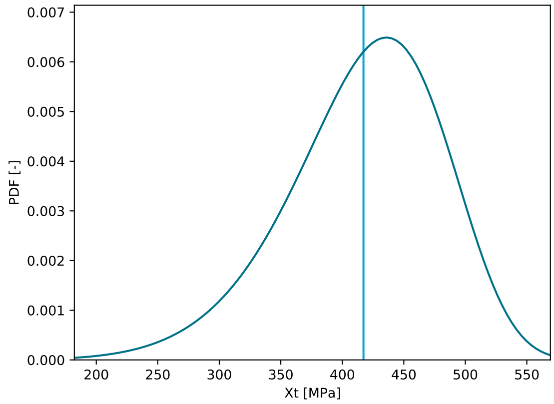


Figure A.5: PDF of the Weibull ($k = 443.67, \lambda = 7.76$) distribution of X_t

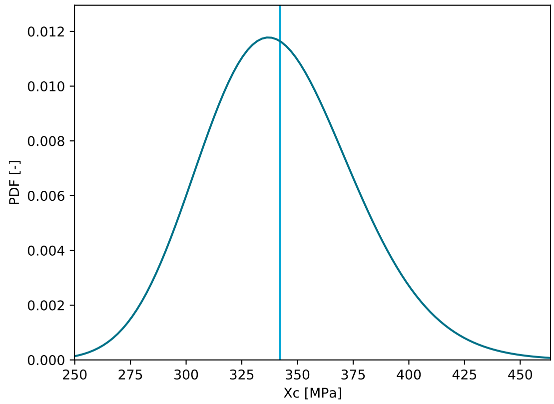


Figure A.6: PDF of the log-normal ($\mu = 5.83, \sigma = 0.1$) distribution of X_c

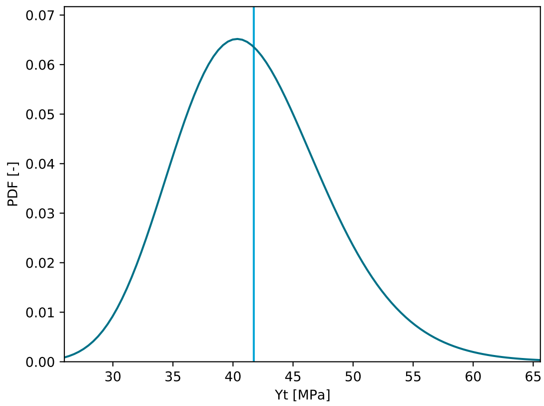


Figure A.7: PDF of the log-normal ($\mu = 3.72, \sigma = 0.15$) distribution of Y_t

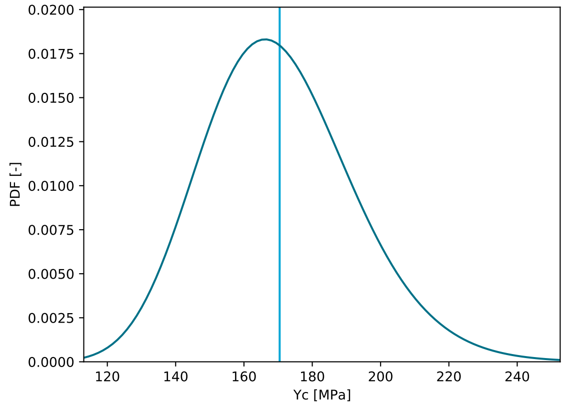


Figure A.8: PDF of the log-normal ($\mu = 5.13, \sigma = 0.13$) distribution of Y_c

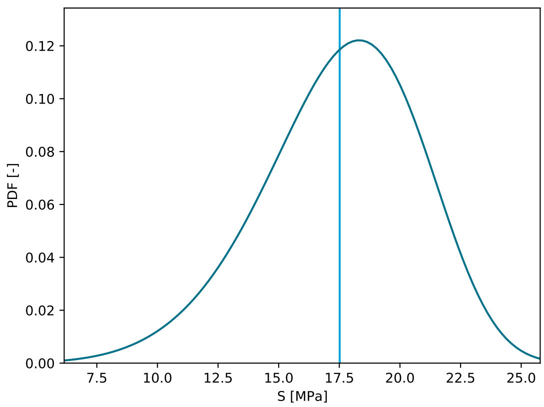


Figure A.9: PDF of the log-normal ($\mu = 2.85, \sigma = 0.18$) distribution of S

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