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RESEARCH ARTICLE

Economic Circuit Theory: Electrical Network Theory for Dynamical Economic Systems

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ABSTRACT In this paper, we develop what we refer to as economic circuit theory. Our purpose is to exploit the proven effectiveness of electrical circuit theory for the design, modeling, and analysis of complex electrical networks to economic systems; in particular, it permits us to incorporate the dynamics of price into those of the flow of physical commodities, analogous to how this is done for magnetic flux and electrical charge. The theory is agent-based, wherein agents are conceptualized as electrical components, and the dynamics are determined by matching the agents' behavioral laws with the constitutive equations of the analogous components. We take a modern graph-theoretic approach, identifying the conditions for stock-flow consistency (Kirchhoff's current law) and price clearing (Kirchhoff's voltage law). With this, we develop the theory to model representative agents (equivalent networks), single-good markets (circuits), and general competitive markets (magnetically coupled circuits). We show how to apply the theory by designing and analyzing an economic circuit model for a two-good market in detail. To prove the effectiveness of the theory, we fit and validate a circuit for the global market in crude oil to the historical data, and we examine a relatively complex circuit for a hypothesized market in hydrogen.

INDEX TERMS Economic circuit theory, graph theory, systems modeling, systems thinking.

I. INTRODUCTION

Understanding and modeling dynamics is a fundamental challenge in modeling economic systems (see, e.g., [1], [2], [3], [4]). This contrasts with the engineering disciplines, which can rely on several well-established theories and methods to model physical dynamic systems [5], [6].

Traditionally, economic theory has focused on identifying the conditions for economic equilibrium, in particular that of price equilibrium. Although recently there has been a shift of focus towards modeling disequilibrium, there is currently no generally accepted economic force law to govern the dynamics of price (see [7]). Instead, economists rely on ad hoc methods. For instance, in the Dynamic Stochastic General Equilibrium (DSGE) models common in macroeconomic theory [8], [9], [10], prices are assumed to adjust instantaneously [11]. In System Dynamics (SD) modeling common in business applications, price changes are modeled through look-up tables and other ad hoc methods [12]. Agent-Based Computational Economics (ACE) models rely on various ad-hoc behavioral heuristics for price determination [8], [11], [13], [14], [15]. Finally, in complexity economics price laws are only observed at the macro scale, see e.g. [16].

In contrast, the engineering disciplines have traditionally focused on dynamic behavior. Especially in electrical circuit theory, this has led to an effective and sophisticated modeling tradition that can rely on a rigorous theoretical basis (e.g. [5], [17]). The ubiquitous sophisticated electronic devices that are part of contemporary life testifies to the effectiveness of electrical circuit theory to model complex and highly dynamic systems.

Our vision with economic circuit theory is to exploit this effectiveness for economic modeling. In the engineering literature, there does exist an established line of research that attempts to model economic systems through an analogy with electrical networks [18], [19], [20], [21], [22]. The underlying idea is that electrical networks can be thought of as

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agent-based, with each economic agent corresponding to an electrical component. Agents exchange goods or services in the same manner as electrical components exchange electric charge, thought of as a current or flow passing through the wiring that represents the trade connections.

Our economic circuit theory builds on this line of research. However, our theory differs from this line in one crucial assumption that is essential for modeling price dynamics. This is the choice of the *magnetic flux* stored in an inductor to represent the *price* that an agent ascribes to a good. We rely on a recent publication [7], where this is first proposed and where the implied economic force law is worked out as the analog of Faraday's law. This assumption implies that the *voltage* over the inductor is interpreted as an *incentive*. Such an incentive determines the agent's price dynamics in the same manner as the voltage determines the dynamics of the flux [17].

In the aforementioned line of research, instead, price is invariably assumed to be the analog to voltage, rather than the flux. However, there is no electrical component whose constituent determines the rate of change of a voltage. Therefore, this analogy cannot be implemented in an electrical circuit to model price dynamics. Indeed, the circuits that have been proposed in these works all concern equilibrium models.

In Section II we adapt the analogy from [7] for economic circuit theory. Although many aspects of this analogy are the same as in the aforementioned line of research, we mention here several important distinctions. First and foremost, the inductor represents a location of demand rather than a source of reinvestment. Although we also use a capacitor as a storage location, it requires an incentive rather than a price to increase its inventory level. Using an inductor for the location of demand frees up the resistor, which now locates the presence of economic friction in close analogy with its physical role. Rather than a price, it requires an incentive to mediate the exchange of goods.

In Section III, we introduce the main contribution of our paper: the use of the modern topological graph-theoretic approach to electrical networks to model economic price dynamics in a unified, comprehensive, and rigorous manner for complex economic systems [17], [23], [24], [25]. Specifically, we associate the Kirchhoff loop law with a price-incentive consistency condition. It determines how, in an economic system, the incentives drive the prices of the individual agents. This contrasts with its use in the aforementioned line of research where it serves as a static balancing of prices. It complements the Kirchhoff node law serving as a stock-flow consistency requirement. Although our interpretation of the node law remains the same, their combination into the economic equivalent of Tellegen's theorem provides us with a dynamic version of Walras's law, rather than its traditional price equilibrium interpretation [22], [26].

In Section IV, we develop several concepts from electrical network analysis for their use in economics. These include the use of closed circuits for single-good markets and model-simplification techniques to aggregate agents into representative agents, including cooperative agents as series and competitive agents as parallel interconnections. In addition, we show how real electrical power sources can be used to incorporate the dynamics of exogenous sources. These include the real agents such as consumers and producers who supply economic surplus to the system.

Our development of economic circuit theory culminates in Section V, where we show that the price dynamics of an economic circuit can be modeled using a state-space representation. This relies crucially on the price-flux analog, used in conjunction with the price-incentive consistency condition. Since the flux linkage is a state variable of an electrical system, so is the price (say p) of an economic system within circuit theory. The state-space representation then specifies the time-rate of change of the price (i.e., $\dot{p} = \frac{\mathrm{d}p}{\mathrm{d}t}$). This argument would not be valid if the voltage were the analog to the price. Complementing the prices with the inventory stocks creates the state vector of a dynamical system and we detail the procedure for deriving both the state and input matrices of its state-space representation for an arbitrary economic circuit.

We then show how the state-space representation allows us to analyze the transient behavior of an economic system. We refer to this as dynamic economic scenario analysis. This is achieved in two steps. First, we establish the dynamics of the equilibrium state as a function of the exogenous sources. Then, we determine the transient behavior of the price and stock levels as they attempt to move towards this, possibly changing, equilibrium state.

In Section VI-A, we apply the theory to the design and analysis of several economic circuits. For a relatively simple system consisting of two competitive markets, we show that the design of the economic circuit proceeds in the same manner as an electrical engineer would design an electronic network: We identify the requisite elementary agents, simplify by aggregating them into representative agents, and then establishing the state-space representation of the dynamics. With that, we illustrate the dynamics by investigating the inventory and price responses in the markets due to a particular demand shock using the dynamic scenario analysis. Subsequently, we provide experimental verification of the theory with an application to the spot market for crude oil. We conclude the section with an application to a rather complex application to the design of a hypothetical hydrogen economy. This application shows the power of economic circuit theory to conceptualize, design, and build prototypes of complex economic systems even in situations where there is no data available.

II. DYNAMIC STRUCTURE

A. STOCK AND PRICE AS CONJUGATE VARIABLES

In our theory, we employ a recently published analogy [7] that enables us to incorporate price dynamics in analogous electrical networks. The economic analogs to the electrical signals resulting from this analogy are summarized in Table 1.

The critical distinction is the choice of the magnetic flux as the analog to price. In this interpretation, the inductor becomes

TABLE 1. Conjugate pairs of dynamic variables and their stock-flow relationships.

	Economic	Electrical	Relation	Units
Stock	Commodity Stock	Electric Charge	$q = \int f \mathrm{d}t$	#
	Price	Magnetic Flux	$p = \int v \mathrm{d}t$	$\frac{\$}{\#}$
Flow	Commodity Flow	Current	$f = \frac{\mathrm{d}q}{\mathrm{d}t}$	$\frac{\#}{\mathrm{yr}}$
	Incentive	Voltage	$v = \frac{\mathrm{d}p}{\mathrm{d}t}$	$\frac{\$}{\#\mathrm{yr}}$

an agent with a demand (or supply) function, that maintains a price level p. Its role is then conjugate to the role of the capacitor as an agent that maintains a commodity stock level (or scarcity) q.

The flow variable corresponding to the price as a stock variable is what [7] refers to as a want and what we will additionally refer to as an incentive or a motive (see Section III-C). In this analogy, an incentive causes a change in an agent's price level and will thereby change their flow demanded or supplied, necessitating the rest of the economic network to adapt. In the same way that a flow of commodities equals the time derivative of a commodity stock, an incentive is the time-derivative of the price. This is properly analogous to how a voltage is the time-derivative of the magnetic flux [17]. Conversely, in the same way that integrating a flow of commodities over time results in a commodity stock, integrating an incentive over time leads to a price – for example an agent's contemporaneous willingness to pay. Incorporating the effect of incentives on economic activity leads to dynamic models instead of kinematic ones.

B. BEHAVIORAL LAWS

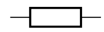
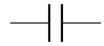
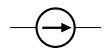
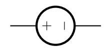
The relationships between the stock and flow variables in Table 1 are determined by the behavioral laws of elementary agents. We represent these agents by analogous electrical elements of which the constitutive relations [6] emulate the behavioral laws. The dynamics within an economic network are then emulated by the electrodynamical interactions of the analogous electrical elements.

1) MARSHALLIAN AGENTS

In Table 2, we provide a summary of the two-terminal electrical elements, their economic analogs, and behavioral laws. Two-terminal elements conduct a single current, from one terminal to the other. We refer to their economic analogs as Marshallian agents, because they are limited to processing only a single commodity flow (see Section II-B2 for a description of general Walrasian agents). The behavioral law pertaining to an agent with demand is the familiar Marshallian law of demand, relating the price p to the quantity demanded f. The storage law relates the stock level q of the storing agent to a desire to hold more (or less) of the commodity. The friction law specifies that the corresponding agent must be motivated with an incentive v in order to transport the flow f of the commodity. See [7] for an in-depth discussion of these relationships.

Insatiable and inelastic agents are active agents whose behavior is independent of other variables in the network. These agents are analogous to a voltage and a current source, respectively. The incentive or the flow supplied by an active agent may be deterministic or stochastic. When deterministic, they are specified as a function of time and, when stochastic, they follow some probability distribution.

TABLE 2. Elementary Marshallian agents as electrical two-terminal elements and their behavioral laws.

Economic	Electrical	Behavioral Law	Symbol
Agent	Component		
Demand	Inductor	$f = \varepsilon p$	
Storage	Capacitor	$v = kq$	
Friction	Resistor	$v = bf$	
Inelastic	Current Source	$f = f(t)$	
Insatiable	Voltage Source	$v = v(t)$	

2) WALRASIAN AGENTS

Agent that trade more than one type of commodity can be represented by electrical elements with four and more terminals. We refer to such agents as Walrasian agents.

In Figure 1, we show the analog to a Walrasian demander handling two separate commodities flows while maintaining a price for each. In electrical networks, this element is known as a mutual inductance. It consists of two or more coils that interact through their magnetic fields, depicted by the magnetic flux lines, and thereby couple any change in the one current (f_1) to a change in the other (f_2). The mutual inductance emulates the effect of cross-elasticity of demand where the demand for two goods are related through their relative prices. This analogy enforces our choice to model prices as magnetic flux rather than as voltages.

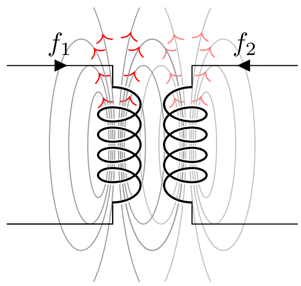


FIGURE 1. Walrasian demander for two commodities. The magnetic flux lines shared by the two Marshallian demanders quantify the price coupling.

The behavioral law of the Walrasian demander is given by a tensorial version of the Marshallian law of demand $f = \varepsilon p$ (see Table 2). For a Walrasian demander trading in G commodity types, the matrix representation of the behavioral law becomes:

$$\begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_G \end{pmatrix} = \begin{pmatrix} \varepsilon^{11} & \dots & \varepsilon^{1G} \\ \vdots & \ddots & \vdots \\ \varepsilon^{G1} & \dots & \varepsilon^{GG} \end{pmatrix} \begin{pmatrix} p_1 \\ \vdots \\ p_G \end{pmatrix}. \tag{1}$$

Here, ε^{ii} is the Marshallian price elasticity of demand and $\varepsilon^{ij}, i \neq j$ the cross-elasticity of demand. The latter is positive for substitutes, negative for complements, and zero for independent goods (see further [7] for a discussion in the context of a mechanical analog).

The other Walrasian agents can be defined in a similar manner. A Walrasian storage is equipped to store different types of commodities simultaneously, where the desirability of receiving one type is dependent on the amount of the other. Its behavioral is likewise a tensorial version of the Marshallian agent's law. Such an agent is analogous to a mutual capacitance. Walrasian frictional agents are defined similarly.

C. CAUSALITY

The dynamic structure established by the behavioral laws and the stock-flow relationships can be conveniently pictured together in what is known as a tetrahedron of state (see [6]). Figure 2 contains a tetrahedron pertinent to the economic analogy. The pairs of conjugate stock and flow variables are enclosed within dotted boxes on the horizontal and vertical, respectively. The stock-flow relationships and behavioral laws are contained within the dashed box on the left-leaning and the right-leaning diagonal, respectively. Friction connects the incentive directly with the flow over the vertical.

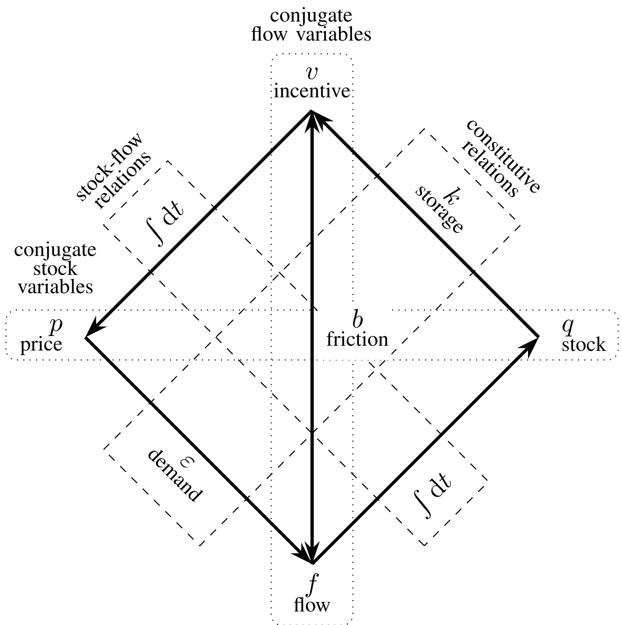


FIGURE 2. **Tetrahedron of state for economic systems.**

The causality enforced by a relationship is indicated in the tetrahedron by the direction in which the arrowhead points. For the stock-flow relationships, the causal direction coincides with that of the integration operator. Integration determines the present based on the past and, hence, is causal. To wit, the current stock is created by accumulating the preceding flows and the current price is arrived at by accruing the preceding incentives. Differentiation, on the other hand, compares the current stock to the stock a little time later. It is, therefore, forward looking and cannot be strictly causal.

For the behavioral laws, the causal direction is determined by the economic reality. The economic law of demand is a causal law, stipulating that the change in the flow is caused by a change in price (see e.g. [27]). For the storage law, one first registers the stock amount, which then becomes the cause for wanting to store more or less of the commodity in question. The friction law forms an exception, being bicausal. Either the flow is the cause of an incentive – as it would be for, e.g., a broker that offers price premiums or discounts based on a mismatch between supply and demand– or the incentive is the cause of a flow, which would for example be the case of an inventory manager that reduces the inflow of commodities based on a storage's needs.

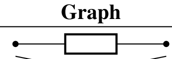
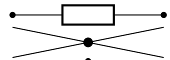
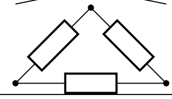
III. TOPOLOGICAL STRUCTURE OF ECONOMIC NETWORKS

Economic networks can be structured analogously to electrical networks, achieved by "wiring" agents together to facilitate trade in a predetermined manner. In network theory, a particular wiring is specified by a topological structure. This structure prescribes the transmission of currents and the balancing of voltages in the electrical network. In economic networks, we let the topology determine how agents interact with each other by prescribing the movement of commodities and the balancing of incentives among them.

A. TOPOLOGICAL CONCEPTS FOR ECONOMIC NETWORKS

In Table 3 we list the economic interpretations of the topological building blocks of a network. In the following, we consider these in detail.

TABLE 3. **Topological concepts and their economic roles.**

Topology	Economics	Graph	Symbol
Edge	Agent		$\mathcal{E}$
Node	Exchange Locus		$\mathcal{N}$
Mesh	Circular Flow		$\mathcal{M}$

Agents make up the edges of the network. We think of the commodities as flowing through the agents and incentives to be measured across agents. While in Table 3 we depict a generic electrical component, agents are represented by their specific analogous electrical component (Table 2) in a particular network.

The nodes of the network represent the loci at which the commodities are exchanged. Each node is depicted by a thick dot between any number of components. However, in practice the node comprises the entire region (the wires) between the components. Commodities flow through a node and are distributed over the incident agents, as we show in Section III-B. At each node, i.e. exchange locus, there is a particular level of desirability that results from the incident agents. The difference between the level of desirability at the two nodes at each end of an agent is equal to the agent's incentive, as we further show in Section III-C.

There is one special type of node which is referred to as a ground node in network theory. It is used as a reference desirability level, usually set to zero, to calibrate the desirability level of the remaining nodes. One can think of it as a very large reserve where the commodity is abundant such that its desirability does not change upon receipt or delivery of additional units.

In network theory, a mesh is a loop that contains no other loops. For economics, such a mesh can sustain a single circular flow of the commodity, referred to as the mesh current in network theory [23]. Agents that form a mesh can exchange commodities while having a mutual reference for the desirability level. As we show in Section III-C, meshes are necessary for agents to communicate their incentives.

With the establishment of a topological structure, we can use algebraic graph theory [25] to systematically analyze and model economic networks, rather than relying on ad-hoc methods. In itself, the use of network theory for modeling economic systems is not new (see e.g. [28], [29], [30]). However, in these attempts the agents make up the nodes of a network, while the edges represent their interactions. The disadvantage of such a choice is that the flow of the goods cannot be read-off explicitly as a network current and, consequentially, the edges cannot accommodate the behavioral laws that determine the dynamic structure. By adhering to convention in network theory, its use for modeling and design in economics becomes transparent.

B. STOCK-FLOW CONSISTENCY AND THE KIRCHHOFF NODE LAW

The analogy between electric charge and commodities implies that the Kirchhoff current law (KCL), also known as the node law, assures that the distributed commodities clear at each exchange locus (see e.g. [21], [31]). By using the KCL's formulation in algebraic graph theory [23], [25], we can efficiently ensure a stock-flow consistent distribution of commodities throughout an entire network. To do so, we first introduce the network flow vector

$$\boldsymbol{f} = \left(f_1 \ldots f_E\right)^T, \tag{2}$$

containing a flow f_j for each agent $j \in \{1, \ldots, E\}$. Then we define an incidence matrix $\boldsymbol{N}$ as follows: At a particular node we identify the edges that are incident to it. These represent the agents who exchange commodities through that particular locus. To keep track of the movements through the node, we define for each node a row of incidence descriptors. This yields a matrix with entries

$$\boldsymbol{N}_{ij} = \begin{cases} 1 & \text{if } \mathcal{E}_j \text{ provides commodities to } \mathcal{N}_i \\ -1 & \text{if } \mathcal{E}_j \text{ receives commodities from } \mathcal{N}_i \\ 0 & \text{if } \mathcal{E}_j \text{ is not incident to } \mathcal{N}_i, \end{cases} \tag{3}$$

where $i \in \{1, \ldots, N\}$ for N exchange loci.

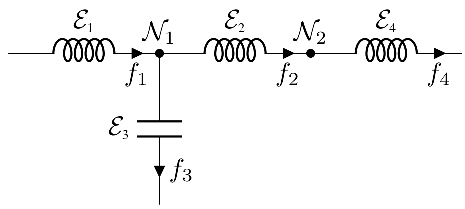


FIGURE 3. The exchange loci (nodes $\mathcal{N}_1$ and $\mathcal{N}_2$) clear the commodities among the agents incident to them. The KCL requires that $f_1 = f_2 + f_3$ and $f_2 = f_4$.

With the aid of the incidence matrix and the flow vector, the stock-flow consistency condition for an arbitrary economic network can be concisely stated as follows:

$$\boldsymbol{N}\boldsymbol{f} = \boldsymbol{0} \tag{4}$$

Here $\boldsymbol{0}$ is the N-dimensional zero flow vector.

To illustrate the procedure, consider node $\mathcal{N}_1$ from the network in Figure 3. From the choice of flow directions in the figure, we see that agent $\mathcal{E}_1$ provides commodities and therefore $\boldsymbol{N}_{11} = 1$. Agents $\mathcal{E}_2$ and $\mathcal{E}_3$ receive commodities, hence $\boldsymbol{N}_{12} = \boldsymbol{N}_{13} = -1$. Because agent $\mathcal{E}_4$ is not incident to N_1, i.e. commodities do not directly flow from the storage to this exchange locus, $\boldsymbol{N}_{14} = 0$. Collecting the entries into a row for each node in a network and arranging the rows into a matrix gives the incidence matrix for that particular network.

C. PRICE-INCENTIVE CONSISTENCY AND THE KIRCHHOFF MESH LAW

The topological structure of a network also allows us to formulate a price-incentive consistency that ensures that each price change within a network is consistent with the incentives. Its electrodynamic analog is known as the Kirchhoff voltage law (KVL), also referred to as the mesh law. This law is based on the principle of conservation of the magnetic flux linked to a mesh by the inductors [17]. For economic meshes, we link the prices of the individual demanders (or suppliers) to the mesh. A price linked in this manner represents the personal value the agent ascribes to the commodity. In [7], it is shown that the total of these personal values in a mesh is indeed a conserved quantity, similar to the total flux linkage. In this subsection, we formulate the price clearing condition in terms of a conservation law for the total value linked to the network.

A flow of value is a want, as evidenced by the *p*-*v* stock-flow relationship (see the tetrahedron of state in Figure 2). To describe the various manifestations of a want within a network, we use the analogs to some specialized terminology that has been developed in electrodynamics for this purpose (see Table 4).

At each locus, there is a level of *desirability* that the agents incident to the locus have in common. Following custom in electrodynamics, we set the desirability of the reference node (electrical ground) to zero so that no value flows there. We measure the desirability ϕ_i at node i relative to the ground level and collect these into an N-dimensional desirability vector $\boldsymbol{\phi}$.

TABLE 4. The manifestations of a want in economic networks: incentive, motive, and desirability.

Economic	Electrical	Symbol
Desirability	Electric potential	ϕ
Incentive	Voltage drop	v
Motive	Electromotive force (emf)	e

Associated to each edge, there is a unique drop v_j in desirability representing the *incentive* that the agent requires to process the commodities from one locus to the other. The flow of value associated with this drop is depicted by a curved arrow pointing in the direction of positive incentives. The curve emphasizes that the drop occurs *over* the agent (rather than *through* it as a flow). When the desirability jumps, we speak of an economic *motive*, which is generated by an active agent (Table 2). The difference in incentives and a motives is analogous to the difference in voltages and electromotive forces in electronics [17]. We adhere to the same sign convention as used in electrical network theory, with incentives directed opposite to the commodity flow and motives directed with the commodity flow (see Figure 4).

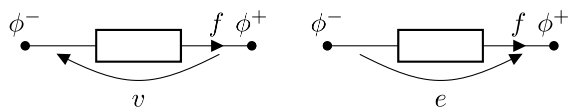


FIGURE 4. Sign convention: an incentive *v* measures the (passive) desirability drop across an agent and a motive *e* measures the desirability jump induced by an active agent.

The incidence matrix allows us to efficiently determine the network-wide incentives from the desirability levels. With $\boldsymbol{v}$ as an E-dimensional vector containing all incentives within a network, we find that

$$\boldsymbol{v} = \boldsymbol{N}^T \boldsymbol{\phi}, \tag{5}$$

where $\boldsymbol{\phi} = \left(\phi_1 \ \dots \ \phi_N\right)^T$ is a vector containing the desirability levels of all nodes in the network and $\boldsymbol{N}^T$ is the transpose of the incidence matrix (3). The use of a matrix transpose in this connection reflects the conjugacy between the flows of value and the commodities and ensures a consistent accounting of the flows and incentives.

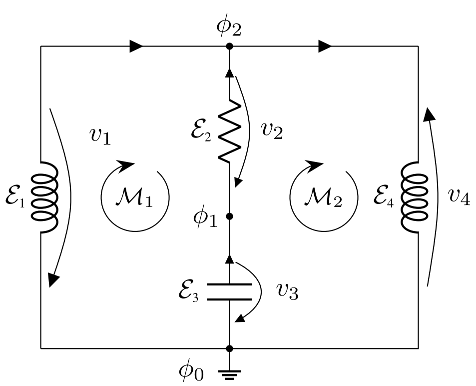


FIGURE 5. Price clearing among the agents in a circular flow. In mesh $\mathcal{M}_1$ KVL requires that $v_1 = v_2 + v_3$.

The price-incentive consistency condition can now be formulated in terms of a conservation law for economic value. To formulate the KVL for all M meshes in an economic network, we choose a direction for each circular flow and construct an $M \times E$-dimensional matrix $\boldsymbol{M}$ with entries

$$\boldsymbol{M}_{kj} = \begin{cases} 1 & \text{if } \mathcal{E}_j \text{ is in } \mathcal{M}_k \text{ and directions agree} \\ -1 & \text{if } \mathcal{E}_j \text{ is in } \mathcal{M}_k \text{ and directions disagree} \\ 0 & \text{if } \mathcal{E}_j \text{ is not part of } \mathcal{M}_k, \end{cases} \tag{6}$$

where $k \in \{1, \dots, M\}$ for M exchange loci. In algebraic graph theory, $\boldsymbol{M}$ is known as a *branch-mesh matrix* [24]. In the economic setting, we refer to it as the agent-mesh matrix. With it, the KVL becomes

$$\boldsymbol{M}\boldsymbol{v} = \boldsymbol{0}, \tag{7}$$

where $\boldsymbol{0}$ is now an $M-$dimensional zero incentive vector. For economic networks, it gives a network-wide formulation of the price-clearing condition based on the conservation of the personal economic value of the demanders and suppliers.

We illustrate this procedure with the aid of the meshes in $\mathcal{M}_1$ and $\mathcal{M}_2$ Figure 5. After constructing the incidence matrix, (5) yields the incentives of the agents: $v_1 = \phi_2 - \phi_0$, $v_2 = \phi_2 - \phi_1$, etc. In mesh $\mathcal{M}_1$, the incentive of agent $\mathcal{E}_1$ is against the direction of circular flow, while the incentives of $\mathcal{E}_2$ and $\mathcal{E}_3$ are along the flow. Hence, $\boldsymbol{M}_{11} = -1$ and $\boldsymbol{M}_{12} = \boldsymbol{M}_{13} = 1$. Agent $\mathcal{E}_4$ is not part of $\mathcal{M}_1$ and thus $\boldsymbol{M}_{14} = 0$. For the network in Figure 5, the KVL determines that $v_1 = v_2 + v_3 = -v_4$.

The use of the KVL for price-incentive consistency is new. Moreover, its formulation represents a radical departure from economic traditions concerning the role of price and, also, the existing attempts to model economic systems using circuit theory mentioned in the introduction.

In our theory, we associate a unique desirability level with a node. In economics, in contrast, it is conventional to associate a price with a locus where exchange takes place (e.g. [32]). A ground node, in particular, allows agents to dispose of the commodities at a zero desirability level, rather than at a zero price as would be understood by the traditional notion of free disposal. The use of desirability rather than price, allows us to think of the goods as flowing to where they are wanted from where they are less desired, thus establishing a cause for the change in a flow.

A price is associated with those edges that represent a demander (or a supplier). This makes the price p a property of an agent. Agents adjust their price over time following the incentives they are subject to. Subsequently, the flow supplied or demanded by such agents change according to their behavioral laws that link the flows of goods to the prices. This causal chain of dynamic responses contrasts with the typical role of price in economic models as a property of market equilibrium. In Section VI-A we show that under certain conditions, the prices of individual agents align into a common market price.

D. MARSHALLIAN MARKETS AS CLOSED CIRCUITS

A Marshallian market —i.e., a market limited to a single type of commodities— can be defined in a purely topological manner, in analogy with an electrical circuit. A circuit is a network that consists of a single closed loop, possibly containing any number of meshes [17]. For a market, this definition implies that the commodities have a trade route to arrive to an agent from any other market participant. In addition, it provides the means for the market participants to communicate their incentives among each other. Typically one adds the requirement that there is also at least one active element present that drives the circuit (e.g. a battery) and we do likewise by adding at least one active agent.

The agent-mesh matrix $\boldsymbol{M}$ can be used to systematically identify the markets within a network. The matrix's rows represent the interactions among agents within the same mesh, while its columns reflect the inter-mesh communications between agents. Consequently, markets emerge as blocks marked off by blocks of zeros in the matrix (see Figure 6). This analysis is done even more efficiently if we introduce the *markets-mesh matrix*[1]

$$\boldsymbol{T} = \boldsymbol{M}\boldsymbol{M}^T, \tag{8}$$

which is a block diagonal matrix that reveals the number of markets in a network through the number of submatrices. For the network in Figure 6, for instance, the number of submatrices is two, both when mesh $\mathcal{M}_4$ does and does not exist. In each submatrix, the diagonal entries indicate the number of agents in each mesh, while the off-diagonal entries indicate the number of agents shared between the two corresponding meshes.

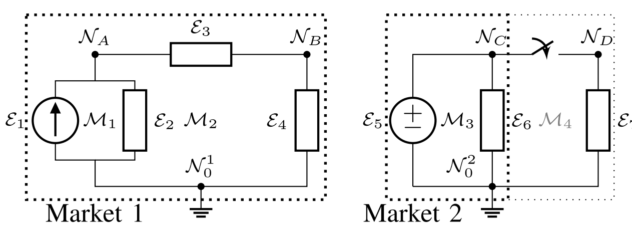


FIGURE 6. Each closed circuit represents a market.

In Figure 6 we show a network with two separate Marshallian markets. In both markets an active agent is present. One verifies indeed that in Market 1, agents $\mathcal{E}_1$ through $\mathcal{E}_4$ are able to, directly or indirectly, deliver the goods to one another through the nodes and communicate their incentives through the meshes. Agents $\mathcal{E}_5$ and $\mathcal{E}_6$ are prevented from trading with Market 1 and form separate market on their own. Agent $\mathcal{E}_7$ is not incorporated into any closed circuit, excluding it from market activities. This changes when the switch between nodes $\mathcal{N}_C$ and $\mathcal{N}_D$ closes and the agent enters the Market 2, forming a new mesh mesh $\mathcal{M}_4$ (the switch enables us to model the entrance or exit of agents). The corresponding matrices are displayed in 9. The dotted lines represent the separation of the markets and the gray entries apply when the switch between $\mathcal{N}_C$ and $\mathcal{N}_D$ is closed.

$$\boldsymbol{M} = \begin{array}{c} \\ \mathcal{M}_1 \\ \mathcal{M}_2 \\ \mathcal{M}_3 \\ \mathcal{M}_4 \end{array} \begin{array}{c} \begin{array}{ccccccc} \mathcal{E}_1 & \mathcal{E}_2 & \mathcal{E}_3 & \mathcal{E}_4 & \mathcal{E}_5 & \mathcal{E}_6 & \mathcal{E}_7 \end{array} \\ \left(\begin{array}{cccc:ccc} -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 1 & 0 & 0 & 0 \\ \hdashline 0 & 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 \end{array}\right) \end{array}, \tag{9}$$

and

$$\boldsymbol{T} = \begin{array}{c} \\ \mathcal{M}_1 \\ \mathcal{M}_2 \\ \mathcal{M}_3 \\ \mathcal{M}_4 \end{array} \begin{array}{c} \begin{array}{cccc} \mathcal{M}_1 & \mathcal{M}_2 & \mathcal{M}_3 & \mathcal{M}_4 \end{array} \\ \left(\begin{array}{cc:cc} 2 & -1 & 0 & 0 \\ -1 & 3 & 0 & 0 \\ \hdashline 0 & 0 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{array}\right) \end{array}. \tag{10}$$

E. NETWORK CONNECTIVITY AS MARKET RESILIENCE

The resilience of a market due to agents exiting can be determined by inspecting the circuit's connectivity. In network theory, the connectivity is defined to be the minimum number of edges that must be removed to disconnect two nodes [33]. In an economic network, these are the minimum number of agents that need to exit the market to disrupt the connection between two exchange loci.

Markets with a high degree of competition are naturally resilient, while markets with a high degree of cooperation are particularly vulnerable. Competitive agents trade in parallel (Section IV) and all agents have to be removed for the trade to cease. On the contrary, when agents cooperate, which corresponds to agents in series interconnection, the exit of but a single agent severs the flow of the goods between the loci. This highlights the critical role of network topology for market resilience analysis.

F. WALRASIAN MARKETS AND MAGNETIC COUPLING

Walrasian demanders couple separate Marshallian markets into a single Walrasian market by virtue of their ability to maintain prices for two distinct types of commodities. The effect of one price on another is analogous to the magnetic coupling of topologically uncoupled circuits in electrical networks [17]. This analogy underscores the necessity of modeling price as the analog of magnetic flux and not as the analog of voltage.

The price coupling between two markets is visualized in Figure 7 in the same manner as magnetic coupling is visualized. The magnetic field lines permeating the coil at a market represent the demander's price for the commodity traded on it. The density of field lines permeating both coils represent the degree to which the two prices influence each other. The behavioral law 1, then specifies how any price changes then lead to adjustments in the quantities demanded on both markets.

[1] In algebraic graph theory this matrix is not commonly used; however, it bears resemblance to the well-known Laplacian matrix $\boldsymbol{L} = \boldsymbol{N}\boldsymbol{N}^T$, where $\boldsymbol{N}$ denotes the incidence matrix. The Laplacian matrix is particularly notable for its role in determining the number of spanning trees within a graph.

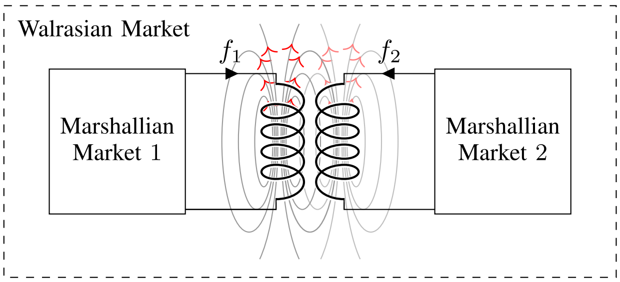


FIGURE 7. Walrasian market as two Marshallian market coupled through demand. The field lines permeating both inductors represent the price coupling between the two commodities.

G. GENERAL MARKET CLEARING AND TELLEGEN'S THEOREM

Both physical and price clearing conditions can be combined into a single general clearing condition for any economic network. In electrical network theory, the corresponding condition is known as Tellegen's theorem [25]. Using the network flow vector $\boldsymbol{f}$ (2) and the incentives vector $\boldsymbol{v}$ (5), it can be concisely formulated as:

$$\boldsymbol{f}^T \boldsymbol{v} = 0 \tag{11}$$

The product of the flows and their conjugate incentives is the rate at which the agents allocate the economic *surplus* among each other in the network (see [7]). In electrical networks, this is the electrical *power* that is being transferred between the elements in a network. Tellegen's theorem can thus be seen to assert a fundamental principle of conservation of economic surplus within an economic network, analogous to the conservation of energy in a electrical network. At a node, where the desirability level is equal among the agents, this manifests itself as physical clearing. In a mesh, where the mesh flow is shared among the agents, this manifests itself as price clearing. Together, they form a more fundamental principle of clearing of the economic surplus, consistent with the conservation law of surplus and energy (see, further, [7]).

Tellegen's theorem serves as an efficient framework for maintaining stock-flow consistency in Walrasian markets. By allowing the examination of the entire network of interconnected markets, it ensures that the sum of the products of flows and incentives around any market is zero, streamlining the process of confirming consistency across the market system without repetitive individual checks.

IV. AGGREGATE AGENTS AS CIRCUIT MODULES

Network simplification allows multiple components to be represented by a single equivalent component, reducing the complexity of a network [17]. In economic circuits, this allows us to aggregate the behavior of a group of elementary agents into a single representative behavioral law. In this section, we analyze two types of such aggregate agents.

A. ACTIVE MARSHALLIAN AGENTS

By aggregating a passive and an active agent, we obtain what we refer to as active Marshallian agents or real agents, which include consumers, producers, and traders. Active agents are analogous to real power sources in electronics, where the addition of a passive element to the ideal source is designed to capture the non-ideal behavior of the device. Active agents agents drive economic activity by allocating economic surplus [7] to the economic network.

In Figure 8, we illustrate how a consumer can be configured as an active Marshallian agent by combining the price-inelastic demand (current source) with price-elastic demand (inductor). The effective behavioral law of the consumer follows the behavioral laws of the elementary agents combined with the KCL (4):

$$f_d = u - \varepsilon_d p_d, \tag{12}$$

where u is the inelastic demand and $\varepsilon_d p_d$ is the elastic demand. This behavioral law yields the familiar downward-sloping demand line of a consumer.

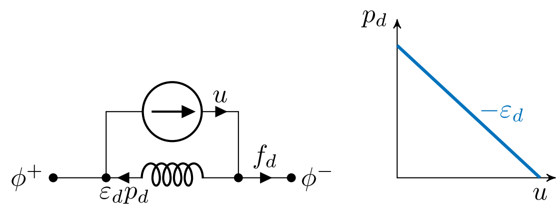


FIGURE 8. Active demander consisting of a current source and an inductor (left). The behavioral law $f_d = u - \varepsilon_d p$ results in the familiar downward sloping demand schedule of a consumer (right).

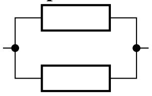
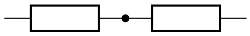
Numerous other configurations of active agents are possible. For instance, another type of consumer can be configured by taking insatiable incentives in parallel with an elastic demander and a frictional element that encapsulates the consumption itself [7]. Similarly, producers can be configured by reversing the flow direction of u, forcing the demander to act as a supplier. This leads to a flow quantity supplied of $f_s = u + \varepsilon_s p_s$ and the familiar upward sloping supply line.

B. REPRESENTATIVE AGENTS AS EQUIVALENT NETWORKS

Elementary agents of the same type can be aggregated into a single representative agent with an equivalent behavioral law. We consider two specific topologies within which the agents are aggregated: a *competitive* and a *cooperative* configuration. In a network, we configure competitive agents in parallel and cooperative agents in series (see Table 5). When in parallel —as, e.g., in a competitive market— the agents compete for the total flow available while being subjected to a common incentive and when in series —as, e.g., in a linear supply chain— the agents cooperate to establish a common flow while their incentives add up to that of a representative agent.

In Table 5 we summarize the behavioral laws for the various representative agents that can be constituted with each of the elementary Marshallian agents. In particular, it is seen that competitive demanders (or suppliers) act like a single representative demander whose price elasticity is the sum of the individual price elasticities. This is shown graphically in Figure 9 for the case of three suppliers and corresponds to the standard analysis of market supply (see e.g. [27]). The more

TABLE 5. Behavioral laws of representative agents.

Type	Competitive	Cooperative
		
Demand	$f = \left(\sum_i \varepsilon_i\right) p$	$f = \left(\sum_i \varepsilon_i^{-1}\right)^{-1} p$
Storage	$v = \left(\sum_i k_i^{-1}\right)^{-1} q$	$v = \left(\sum_i k_i\right) q$
Friction	$v = \left(\sum_i b_i^{-1}\right)^{-1} f$	$v = \left(\sum_i b_i\right) f$
Inelastic	$f = \sum_i f_i$	$f = f_1 = \cdots = f_i$
Insatiable	$v = v_1 = \ldots v_i$	$v = \sum_i v_i$

individual demanders, the more price elastic the equivalent representative demander. Conversely, when cooperating, the resultant price elasticity is the inverse of the sum of the inverse price elasticities, which is shown graphically in Figure 10. There, the equivalent supplier has a lower price elasticity, thus rendering it less and less elastic as agents are added.

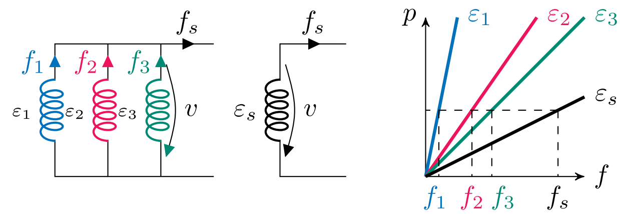


FIGURE 9. Competitive suppliers as parallel inductors (left). The three suppliers can be represented by a single equivalent inductor (center). The market supply aggregates over the individual supply schedules at the contemporary market price (right).

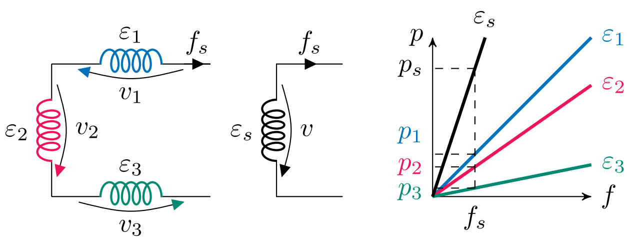


FIGURE 10. Cooperative suppliers as series inductors (left) and single equivalent supplier (center). The price p is aggregated over the supply schedules of the suppliers (right).

V. ECONOMIC NETWORKS AS DYNAMICAL SYSTEMS

The imposed analogy enables us to formulate economic networks as abstract dynamical systems and leverage the methods thereof. In this section we demonstrate how this contributes to the analysis and computation of the dynamics of economic networks.

A. STATE-SPACE REPRESENTATION

By combining the dynamic structure presented in Section II with the topological structure presented in Section III, we obtain a dynamical systems model [6]. Such a model specifies how the prices and stock levels within the economic network change in time. Here we consider behavioral laws that are linear and whose parameters do not change over time (see Table 2), leading to linear time invariant (LTI) models [34].

The assumption that the behavioral laws are LTI allows us to express the dynamics of an economic network in a state-space representation

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{u} + \boldsymbol{w}, \tag{13}$$

where the overdot forms the standard notation for a time derivative. This representation simultaneously captures the endogenous ($\boldsymbol{A}\boldsymbol{x}$), exogenous ($\boldsymbol{B}\boldsymbol{u}$), and stochastic ($\boldsymbol{w}$) influences on the dynamics of the state variables within the network, see Figure 11. As we show in Section V-C, this perspective enables us to conduct dynamic economic scenario analyses. In the following, we demonstrate how the state-space representation of an economic network is derived.

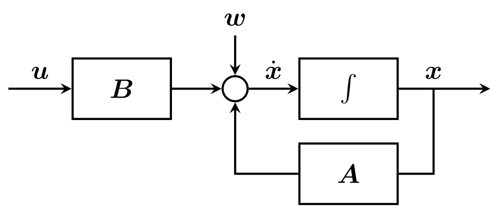


FIGURE 11. Block diagram of the state-space representation in (13).

The state of an economic network is determined by the stock levels of the storages and the price levels maintained by the demanders. We collect these into a single state vector

$$\boldsymbol{x} = \left(q_1 \; \cdots \; q_K \; p_1 \; \cdots \; p_L\right)^T, \tag{14}$$

where K is the number of storages and L the number of demanders. Of the passive agents, the frictional agents act as intermediaries in the trade activity and they do not contribute a state variable to the state vector (see Table 2). While the particular arrangement of the state variables is arbitrary, it is often convenient to arrange them into conjugate pairs (see (27)).

The inputs are determined by the active agents. Supposing that there are I active agents in a network, we have an input vector

$$\boldsymbol{u} = \left(u_1 \; \cdots \; u_I\right)^T, \tag{15}$$

of which each component is either an insatiable motive or an inelastic flow depending on the nature of the agent. Specifically, for agent j

$$u_j = \begin{cases} f_j(t) & \text{when } j \in \text{inelastic} \\ v_j(t) & \text{when } j \in \text{insatiable} \end{cases} \tag{16}$$

The particular arrangement of $\boldsymbol{u}$ is arbitrary and can be chosen for convenience. The behavior of the active agents depends on time alone. For this reason, they are properly considered to be exogenous variables, consistent with their role of inputs.

The input and state vectors determine the deterministic part of the rate of change,

$$\dot{\boldsymbol{x}} = \left(\dot{q}_1 \; \cdots \; \dot{q}_K \; \dot{p}_1 \; \cdots \; \dot{p}_L\right)^T, \tag{17}$$

of the state vector. Its components are the rates $\dot{q}_k$ at which the commodities are stocked up and the rates $\dot{p}_l$ with which the prices are adjusted. These equal the flow rate into the kth storage and the incentive over the lth demander, respectively, i.e.:

$$\dot{q}_k = f_k \qquad \text{and} \qquad \dot{p}_l = v_l$$

We use the topological structure of the network to formulate the contribution of the agents participating in the network to the rate of change $\dot{\boldsymbol{x}}$ of the state. This involves that for the stock up rate $\dot{q}_k$, we consider either one of the exchange loci, say $\mathcal{N}_n$, to which the storage is connected and clear this with the flows from the other agents into this locus consistent with the KCL. For the price movement $\dot{p}_l$ we consider the mesh $\mathcal{M}_m$ in which that demander is adjusting its price and clear it with the incentives of the remaining agents contributing to this mesh consistent with the KVL.

Next, we use the dynamic structure to express the flows and incentives, either as a function of the state $\boldsymbol{x}$ or of the time t. This is achieved by substituting the behavioral laws for the flows and incentives until we recover either a function of $\boldsymbol{x}$ or of t and arranging these obtained functions in the order in which they appear in the state vector. The procedure is summarized in (18), as shown at the bottom of the next page.[2]

The result is the state-space representation of the dynamical-systems model of the network. The procedure makes it evident that it is determined by the topological structure —as given by the matrices $\boldsymbol{N}$ and $\boldsymbol{M}$— together with dynamical structure —as given by behavioral laws f_j and v_j.

Because the behavioral laws are linear, the dependence of the vector $\dot{\boldsymbol{x}}$ on the state can be summarized by a single matrix equation $\boldsymbol{Ax}$, which captures the endogenous dynamics of the network. The matrix $\boldsymbol{A}$ is known as the state matrix and it contains the parameters of the behavioral laws of the agents within the network. The flows and incentives that depend on time are inputs, conform (16), and hence the dependence of $\dot{\boldsymbol{x}}$ on time can be summarized by the equation $\boldsymbol{Bu}$, which captures it as a dependence on the exogenous inputs. The matrix $\boldsymbol{B}$ is known as the input matrix. The sum of $\boldsymbol{Ax}$ and $\boldsymbol{Bu}$ then gives the total of the deterministic effects on $\dot{\boldsymbol{x}}$.

The stochastic agents determine the uncertainty in the dynamics of the state. We collect their contribution into a noise vector

$$\boldsymbol{w} = \left(w_1 \ \cdots \ w_{K+L}\right)^T,$$

which has the same dimension as the state vector. The influence of a stochastic agent on a storage or a price is determined by placing the components at the appropriate location in the noise vector. In the absence of any noise, a zero is entered. This arrangement allows us to express the contribution of the noise on the rate of change of the state vector as $\dot{\boldsymbol{x}} = \boldsymbol{w}$, which is added to the deterministic solution to give the state space representation in (13).

[2]Note that in this procedure each particular f_j or v_j is either a function of $\mathbf{x}$ OR a function of t.

B. EQUILIBRIUM

Equilibrium for an economic network requires that both the prices and the stocks remain stationary in time (see e.g. [34]). The requirement for equilibrium is formulated by requiring that either

$$\dot{\boldsymbol{x}} = 0 \qquad \text{or} \qquad E[\dot{\boldsymbol{x}}] = 0 \tag{19}$$

The first formulation suffices for deterministic systems, while the second formulation generalizes to stochastic systems.

The value of an equilibrium state, $\boldsymbol{x}^*$, can be found under the condition that the state matrix is invertible in the following manner:

$$\boldsymbol{x}^* = -\boldsymbol{A}^{-1}\left(\boldsymbol{Bu} + E[\boldsymbol{w}]\right) \tag{20}$$

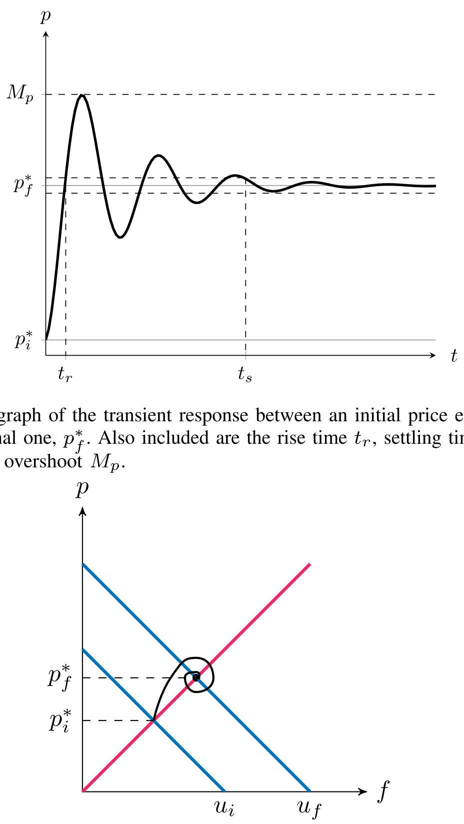
When the state matrix is not invertible, alternative approaches such as null-space analysis, numerical techniques, parameterized solutions, or the use of a psuedoinverse can be used to find equilibrium states (see e.g. [35]). The input-output perspective allows us to consider equilibria that change in time, depending on the nature of the exogenous influences ($\boldsymbol{u}$) and the mean of the stochastic influences ($E[\boldsymbol{w}]$). For instance, if the network is subjected to an exogenous sustained shock in inelastic demand, it will have two equilibrium states, an initial one before the shock and a final to which it attempts to move. In the theory of demand, this is interpreted as a shift in the demand curve (see Figure 12 in the next subsection, wherein the actual movement in time to equilibrium is analyzed). More drastically, if the network is instead subjected to a periodically changing inelastic demand, its equilibrium state changes continually over time. A changing $E[\boldsymbol{w}]$ on the other hand will result in a drift of the equilibrium state.

The existence of an equilibrium state does not guarantee that the economic network will achieve this equilibrium. When the eigenvalues of $\boldsymbol{A}$ are negative, the network is unstable and the prices and inventories will diverge from the equilibrium (see e.g. [36]). In addition, when the time-period of a varying exogenous input (such as a cyclical inelastic demand) is shorter than the time it takes the network to adjust to the changing equilibrium, the economic network finds itself chasing but not reaching equilibrium.

C. TRANSIENT ANALYSIS FOR DYNAMIC ECONOMIC SCENARIO ANALYSIS

The state-space representation of the dynamics allows us to analyze how the network transitions from one equilibrium state to another after being subjected to a shock. In control theory, this is known as transient analysis. For simple economic networks, transient responses can be determined analytically, while for more complex models they can be approximated or determined numerically [34]. For economics, the methods of transient analysis allow us to conduct *dynamic* scenario analyses, where the emphasis is on the precise movements of price levels and stock quantities within an economic network.

The plot in Figure 12 shows a typical response of a second-order system to a step input consisting of a sudden sustained shock. A corresponding second-order economic network consists of a single representative demander, storage, and frictional agent as passive agents. Such systems represent, e.g., traders who maintain both a price and an inventory (see also [7]). Under a demand shock, the price response exhibits the damped cyclical passage to the equilibrium value p_f^* from the initial equilibrium value p_i^*.



(a) Time graph of the transient response between an initial price equilibrium p_i^* to a final one, p_f^*. Also included are the rise time t_r, settling time t_s, and maximum overshoot M_p.

(b) Equilibrium states pictured as a shift in the demand curve (due to a shift from u_i to u_f) and a hypothetical transient response between these.

FIGURE 12. Transient response to a demand shock of a network consisting of a demander, storage, and trade friction.

From control theory [34], we can adapt several metrics to quantify the price movement. In Figure 12, we include the rise time t_r, the maximum overshoot M_p over the final equilibrium price, and the time t_s at which the price settles to within a certain error of the final equilibrium price. The rise time quantifies the short-run price rigidity and the overshoot shows how much the price rises above its new equilibrium before settling to it. The settling time provides a point in time separating the short run from the long run. A related concept used in econometrics is the speed-of-adjustment parameter [37]. In the absence of any friction, the system exhibits an indefinite cycle around the equilibrium state, never settling.

Figure 12b contains an illustrative picture the transient as a time-parameterized curve in the price-flow plane wherein demand and supply lines are typically graphed. The actual movement consists of a simultaneous damped oscillations along the supply and demand curves.

VI. APPLICATIONS

Economic network theory enables us to design and analyze economic systems models in the same manner as an electrical engineer would design an analyze an electronic system. In this section, we demonstrate this design process for several economic systems.

A. DESIGN AND ANALYSIS OF A TWO-COMMODITY WALRASIAN MARKET

In this subsection, we illustrate the application of the theory by analyzing a two-commodity Walrasian market shown in Figure 13. The market model serves to demonstrate the various analogies between the electrical components and agents and between the topologies and interactions introduced in this paper.

1) DESIGN OF THE NETWORK

To model the trade in two commodities we consider two Marshallian markets formed by the two circuits in Figure 13. We label the agents with a superscript $i \in \{1, 2\}$ to indicate the market in which they are active. The markets are identical with the exception of the stochastic agent $\mathcal{V}^1$ which is active in Market 1 alone.

In each market, a cooperative monopoly of, say a wholesaler $\mathcal{S}_1^i$ and a retailer $\mathcal{S}_2^i$, delivers the commodity. In addition, two storages $\mathcal{C}_1^i$ and $\mathcal{C}_2^i$ compete for the inventory stock and two brokers $\mathcal{R}_1^i$ and $\mathcal{R}_2^i$ compete to benefit from any mismatches between the flows supplied and demanded by offering premiums or discounts on the price level. Two Walrasian demanders, $\mathcal{D}_1$ and $\mathcal{D}_2$, compete for both commodities. By choosing between the commodities from both markets based on their cross-elasticity, they establish the price coupling between the two markets. The demanders are also configured in an active role as consumers by being paired with a source

$$
\begin{aligned}
\dot{q}_k &= \frac{1}{\boldsymbol{N}_{nk}}\sum_{j\neq k}\boldsymbol{N}_{nj}f_j &&= \frac{1}{\boldsymbol{N}_{nk}}\sum_{j\neq k}\boldsymbol{N}_{nj}f_j(\boldsymbol{x}) &&+\frac{1}{\boldsymbol{N}_{nk}}\sum_{j\neq k}\boldsymbol{N}_{nj}f_j(t)\\
\dot{p}_l &= \frac{1}{\boldsymbol{M}_{ml}}\sum_{j\neq l}\boldsymbol{M}_{mj}v_j &&= \frac{1}{\boldsymbol{M}_{ml}}\sum_{j\neq l}\boldsymbol{M}_{mj}v_j(\boldsymbol{x}) &&+\frac{1}{\boldsymbol{M}_{ml}}\sum_{j\neq l}\boldsymbol{M}_{mj}v_j(t)\\
\hline
\dot{\boldsymbol{x}} &= && \boldsymbol{A}\boldsymbol{x} && +\boldsymbol{B}\boldsymbol{u}
\end{aligned}
\tag{18}
$$

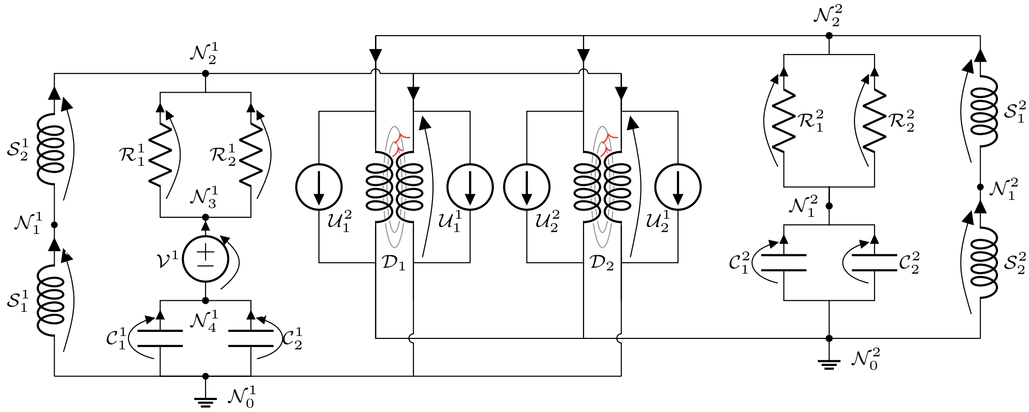


FIGURE 13. Network for a Walrasian market in two commodities. Each Marshallian submarket consists of two suppliers (inductors on the outside), two consumers (inductor-source combinations in the middle), brokers (resistors), and storages (capacitors). The Walrasian nature of the demanders is indicated by the overlapping flux lines (red arrowheads).

of inelastic demand in each market, $\mathcal{U}_1^i$ and $\mathcal{U}_2^i$, providing economic surplus. In the market for commodity 1, there is a stochastic agent $\mathcal{V}^1$ that introduces noise in the form of a stochastic insatiable motive.

2) NETWORK SIMPLIFICATION

Before deriving a state-space representation, we simplify the network by identifying any representative agents, consolidating the elementary agents based on whether they act competitively or cooperatively. From Figure 13, we immediately see that the storages $\mathcal{C}^i$, brokerages $\mathcal{R}^i$, and the inelastic agents $\mathcal{U}^i$ act competitively. We simplify each of those into a representative agent with the parameters

$$k_i = \frac{k_{\mathcal{C}_1^i} k_{\mathcal{C}_2^i}}{k_{\mathcal{C}_1^i} + k_{\mathcal{C}_2^i}}, \quad b_i = \frac{b_{\mathcal{R}_1^i} b_{\mathcal{R}_2^i}}{b_{\mathcal{R}_1^i} + b_{\mathcal{R}_2^i}}, \quad u_i = u_{\mathcal{U}_1^i} + u_{\mathcal{U}_2^i}, \tag{21}$$

respectively. We consolidate the demanders and suppliers into a single representative Walrasian agent. This requires several steps. First, we consolidate the cooperative suppliers and competitive demanders in each Marshallian market into the representative suppliers $\mathcal{S}^i$ and demanders $\mathcal{D}^i$ with respective price elasticities

$$\varepsilon_{\mathcal{S}^i} = \frac{\varepsilon_{\mathcal{S}_1^i} \varepsilon_{\mathcal{S}_2^i}}{\varepsilon_{\mathcal{S}_1^i} + \varepsilon_{\mathcal{S}_2^i}}, \quad \text{and} \quad \varepsilon_{\mathcal{D}^i} = \varepsilon_{\mathcal{D}_1^i} + \varepsilon_{\mathcal{D}_2^i}. \tag{22}$$

Then, we consolidate these into a single representative Walrasian demander/supplier $\mathcal{DS}$. To determine its elasticity tensor ε, we note that the representative supplier and demander act competitively in each Marshallian market and that the price coupling between the markets follows from the demand alone. These two observations lead to the following expressions for the principal and cross-elasticities of demand, respectively:

$$\varepsilon^{ii} = \varepsilon^{ii}_{\mathcal{S}^i} + \varepsilon^{ii}_{\mathcal{D}^i}, \qquad \varepsilon^{12} = \varepsilon^{12}_{\mathcal{D}^1} + \varepsilon^{12}_{\mathcal{D}^2} \tag{23}$$

These form the components of the elasticity tensor ε, which is the parameter of the behavioral law of the representative agent $\mathcal{DS}$. The law relates the vector of excess flows in the markets to the vector of markets' price levels.

Consolidating the elementary agents in the network of Figure 13 into representative ones leads to the simplified network given in Figure 14. Its dynamics are equivalent to those of the original design.

3) STATE-SPACE REPRESENTATION

To determine the state-space representation of the dynamics of the reduced network, we first establish its incidence and agent-mesh matrices. Following the procedure outlined in Section III, we find:

$$\boldsymbol{N} = \begin{pmatrix} -1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 1 \end{pmatrix}, \tag{24}$$

and

$$\boldsymbol{M} = \begin{pmatrix} 1 & 1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & -1 \end{pmatrix}. \tag{25}$$

Here, the rows of $\boldsymbol{N}$ in top-down order represent $\mathcal{N}_0^1, \mathcal{N}_1^1, \mathcal{N}_2^1, \mathcal{N}_3^1, \mathcal{N}_0^2, \mathcal{N}_1^2$, and $\mathcal{N}_2^2$; the rows of $\boldsymbol{M}$ represent $\mathcal{M}_1^1, \mathcal{M}_2^1, \mathcal{M}_1^2$, and $\mathcal{M}_2^2$; and the columns of both matrices represent, from left to right, $\mathcal{C}^1, \mathcal{V}^1, \mathcal{R}^1, \mathcal{U}^1, \mathcal{DS}^1, \mathcal{DS}^2, \mathcal{U}^2, \mathcal{C}^2$, and $\mathcal{R}^2$. The dotted lines indicate the separation between the markets. This is further corroborated by the markets-mesh matrix

$$\boldsymbol{T} = \begin{pmatrix} 4 & -1 & 0 & 0 \\ -1 & 2 & 0 & 0 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & -1 & 3 \end{pmatrix}, \tag{26}$$

consisting of two diagonal blocks.

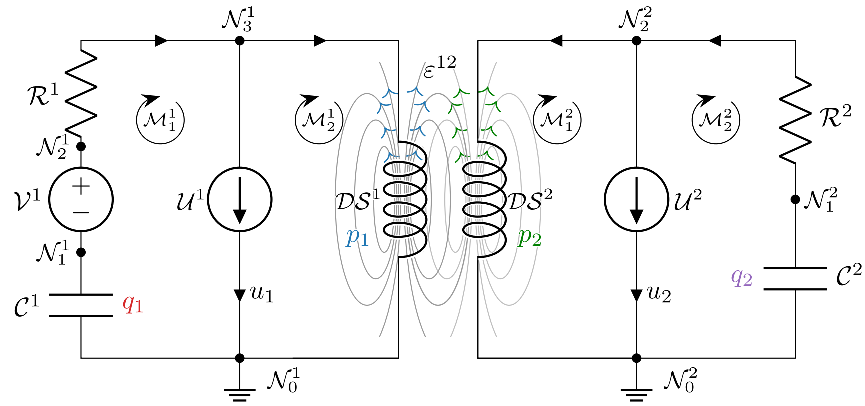


FIGURE 14. **Reduced network with representative agents.**

With the incidence and branch-mesh matrices established, our next step is to follow the procedure in Section V-A. In total, there are two price levels and two stock quantities (see Figure 14), making the state vector four-dimensional. We organize it (for the purpose of the following analysis) into two consecutive conjugate price-stock pairs, one for each market, as follows:

$$\boldsymbol{x} = \left(q_1\ p_1\ q_2\ p_2\right)^T \tag{27}$$

We have color-coded the state variables to assist with the interpretation of the simulation results in the next subsection. Figure 14 depicts the agents to which the state variables pertain.

The input vector consists of the inelastic-flow components of the consumers and the noise vector has a single component containing the motives of the stochastic agent. Following the procedure, we find the state-space model:

$$\begin{pmatrix}\dot{q}_1\\ \dot{p}_1\\ \dot{q}_2\\ \dot{p}_2\end{pmatrix} = \begin{pmatrix}0 & -\varepsilon^{11} & 0 & -\varepsilon^{12}\\ k_1 & -b_1\varepsilon^{11} & 0 & -b_1\varepsilon^{12}\\ 0 & -\varepsilon^{12} & 0 & -\varepsilon^{22}\\ 0 & -b_2\varepsilon^{12} & k_2 & -b_2\varepsilon^{22}\end{pmatrix}\begin{pmatrix}q_1\\ p_1\\ q_2\\ p_2\end{pmatrix} + \begin{pmatrix}1 & 0\\ b_1 & 0\\ 0 & 1\\ 0 & b_2\end{pmatrix}\begin{pmatrix}u_1\\ u_2\end{pmatrix} + \begin{pmatrix}0\\ w\\ 0\\ 0\end{pmatrix} \tag{28}$$

The state-space model allows us to analyze various aspects of the market's dynamics, including its causality and the conjugacy between the state variables. We consider the effects of the stochastic and inelastic agents first and then how the state matrix determines how these effects are propagated in time by the network's endogenous dynamics.

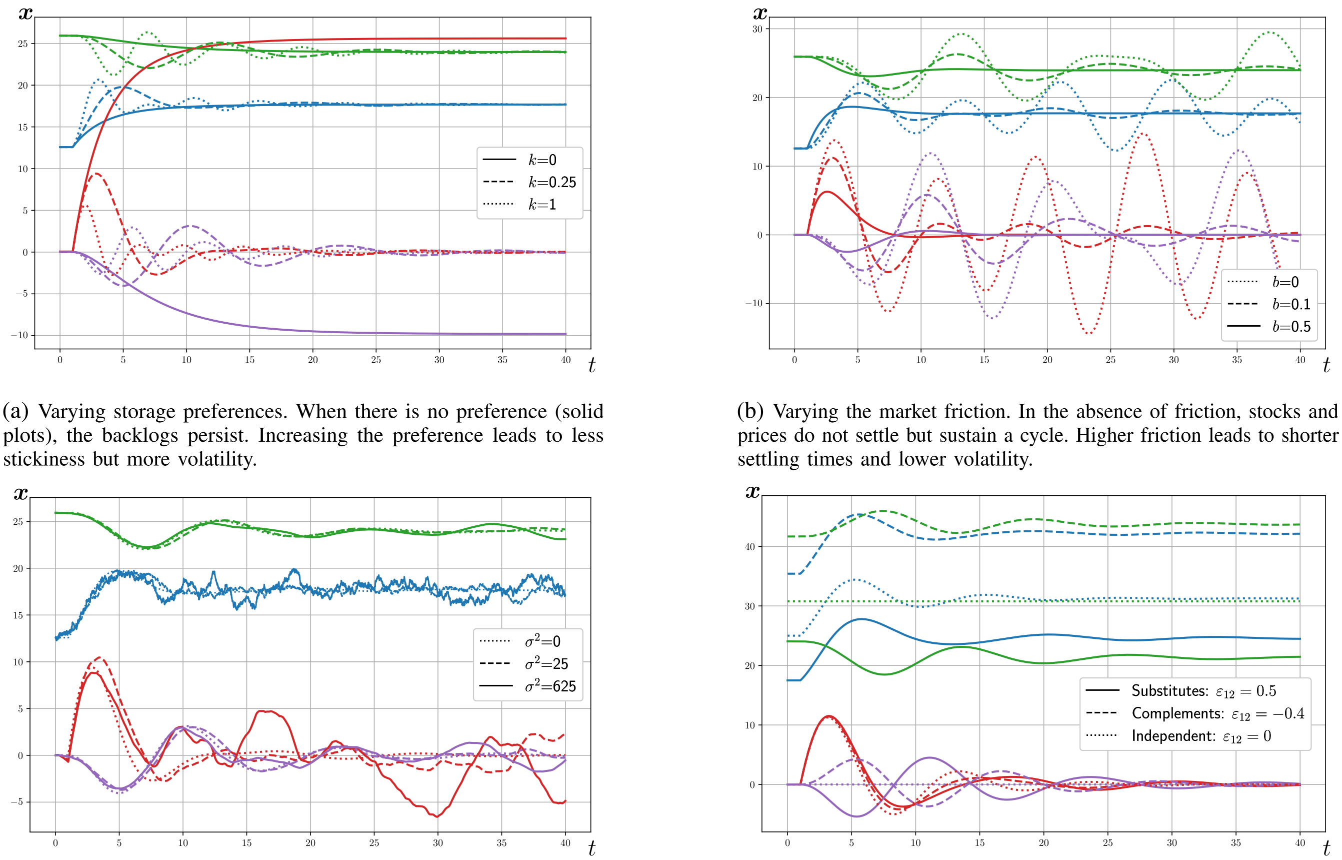
The stochastic motives are configured to directly contribute to the time rate $\dot{p}_1$ at which the price in Market 1 changes. It requires the representative demander to constantly readjust its price level in a random manner.

We next consider the input matrix that captures the exogenous effect of the inelastic agents on the dynamics. In each market, the inelastic demand directly relieves the inventory stock of the commodity it is demanding (first and third row of the input matrix). Concurrently, they adjust the market price levels, a process intermediated by the frictional representative agent who broker the flow. The latter is captured by the second and fourth row of the input matrix.

Finally, we consider the endogenous effects captured by the state matrix. The state matrix consists of four 2×2 blocks and the network consists of two coupled Marshallian second-order markets. The diagonal blocks determine the internal endogenous dynamics of the Marshallian markets and the off-diagonal blocks the coupling between them. Indeed, if we remove the price coupling by setting the cross-elasticity of demand $\varepsilon^{12} = 0$, the off-diagonal blocks contain only zeroes and the Marshallian markets become dynamically independent. For non-zero values of ε^{12}, the price in one markets affects both the flow and the price in the other. Positive values of ε^{12} lead to the quantity in the other market decreasing and the commodities are substitutes of one another. Negative values, on the other hand, characterize commodities that are complements. The zeros in the off-diagonal blocks acting on the inventory state variables show that the markets are physically disconnected.

The market friction is given by the diagonal elements of the blocks. In this position, they determine the influence of one price on itself or on the price in the other market. The internal market friction is given by the rate of discount $b_i\varepsilon^{ii}$, in units of inverse time [7], on the price that is required to adequately incentivize the intermediating broker to pass the commodities from the storage to the rest of the market. For the intermarket price coupling, the required discount rate depends on the cross-elasticity of demand instead.

The conjugacy of the state variables is borne out by the off-diagonal elements of the blocks. The price level in a market places downward pressure on the conjugate stock-up rate due to reduced demand. Consistent with a linear law of demand, the effect is proportional to the price elasticity $-\varepsilon^{ii}$. The stock level, in turn, exercises an upward pressure on the conjugate price due to the increased incentive to hold the commodity. This incentive gives a measure to the convenience of holding the commodity and it is proportional to the storage preference k_i. The stock-up rate in one market is also pressured by the price level in the other



(a) Varying storage preferences. When there is no preference (solid plots), the backlogs persist. Increasing the preference leads to less stickiness but more volatility.

(b) Varying the market friction. In the absence of friction, stocks and prices do not settle but sustain a cycle. Higher friction leads to shorter settling times and lower volatility.

(c) Influence of the variance of the disturbance on the randomness of the stocks and prices. The influence reduces as it propagates through the causal chain of the dynamics, from p_1 through q_1 to p_2 and finally q_2.

(d) Varying the cross-elasticity of demand. The price of substitute commodities move counter (solid), complements (dashed) along, and independent (dotted) do not effect each other.

FIGURE 15. Transient responses of the state variables to a step demand shock in Market 1 for various choices of the model parameters. State variables are q_1 (red), p_1 (blue), q_2 (purple), and p_2 (green). Parameters $k = k_1 = k_2$ and $b = b_1 = b_2$ for all figures and, unless indicated otherwise, $\varepsilon^{11} = 2.15$, $\varepsilon^{12} = 0.5$, $\varepsilon^{22} = 1.3$, $k = 0.25$, and $b = 0.2$.

market. This effect is proportional to the cross elasticity of demand $-\varepsilon^{12}$, consistent with the Walrasian nature of the demanders.

4) DYNAMIC SCENARIO ANALYSIS

To demonstrate the influence that the agents in the Walrasian market have on the dynamics, we subject the network to a sustained inelastic demand shock in Market 1 and analyze how it adjusts to this. Specifically, for the inputs in the state-space representation, we set:

$$u_1(t) = \begin{cases} 30\ \#_1/\text{dy}, & t \geq 1\ \text{dy} \\ 20\ \#_1/\text{dy}, & t < 1\ \text{dy} \end{cases}$$
$$\text{and} \quad u_2(t) = 20\ \#_2/\text{dy} \tag{29}$$

The shock induces both markets to move from an initial equilibrium state to a final one. In the following, we analyze the transient response of the markets under different variations of the parameter values. The results are contained in Figure 15. In addition, we have made an interactive version of the source code available.[3]

From Figure 15a we see that when $k = 0$ and there is no preference to store the commodity, Market 1 remains understocked and Market 2 remains overstocked in the long run. However, when $k > 0$ and there is a convenience to holding the stock, inventories settle at their desired level. Prices in both markets, on the other hand, remain at elevated levels in the long run, regardless of a preference to store. This in reaction to the permanent increase in quantity demanded.

Figure 15c shows the effect of the stochastic agent on the equilibrium state. The stochastic motive follows a normal distribution with zero mean and variance σ^2. We observe that the state variables have no deterministic equilibrium and only achieve an equilibrium in expectation. The intensity of the random behavior increases with increasing variance. Its effects are most pronounced on p_1 since the stochastic motive acts directly on it. It causes its conjugate storage to be affected

[3]https://colab.research.google.com/drive/1wOQ6LXt1PW74byJQyo_UBV2xnxLvlmPO?usp=sharing

and then, through the inter-market price coupling, the price and storage levels in Market 2. Each causal step in this chain attenuates the intensity of the random behavior.

From Figure 15b we see how the market friction influences the settling time. It determines the separation of the short run from the long run. When $b = 0$ and there are no trade frictions, neither stocks nor prices ever settle and, instead, remain volatile by cycling around their equilibrium values. For values of $b > 0$, the signals settle, with the settling time itself shortening with increasing trade friction, as the available the surplus is consumed more and more rapidly.

Other factors that influence the settling time are the preference to store and the price elasticity. Figure 15a shows that increasing k ensures shorter settling times, as the agents are motivated by the perceived convenience of holding the commodity. In general, settling times improve with higher price elasticities, as market participants are increasingly motivated to adjust their price. Figure 15d illustrates the effect of the cross-elasticity of demand ε^{12}. We see that independent goods have the shorter settling times and that complementary and substitutions are comparable in the long run.

As concerns the rise time, both the storage and the friction are important factors in its determination. We see that when the commodity is convenient to hold, rise times shorten and, as a result, prices appear less sticky and inventories are more responsive. A similar effect occurs with increasing friction, extending rise times and making prices stickier due to the reduction in market liquidity.

Figure 15a shows that, although higher storage preferences lead to shorter rise and settling times, this comes at the expense of increased price volatility, both in frequency and maximum overshoot. An increased convenience of storage induces the agents to turn over their inventories at a higher rate. Friction, instead, reduces volatility. Whereas rapid stock turnover rates would be attractive to benefit from anticipated changes in demand when costs are low, the added costs of trading causes the market participants to slow down their trade cycles and also reduces their desire to hold the stock too far from the desired levels.[4]

The cyclical nature of the responses allows to also identify which state variables are leading, lagging, and contrarian indicators of others. In network theory, the phase shifts between the signals is used for this purpose. From Figure 15a we see that the stock level in Market 1 leads its conjugate price by a relatively small phase shift. This is because the demand shock is met at the onset by the available inventory. The price in Market 1 then leads the price in Market 2 and finally the latter's conjugate stock level. Each step in the causal chain adding to the phase shift with the stock level in Market 1. As a result, the price in Market 2 is almost half a turn out of phase with the stock level in Market 1, with the two variables become close to being contrarian indicators of each other. The leading, lagging, and contrarian nature of the state variables is congruent to the causality embedded in the state-space model.

[4]Although this is not shown in the figure, the overshoot and the frequency disappear entirely when the friction is high enough. Instead of volatility, one observes a hyperbolic discounting of the price (see [7]).

B. SYSTEM IDENTIFICATION OF A PRICE-DYNAMICAL OIL MARKET MODEL

The use of circuit theory has the advantage that established system identification techniques for determining the parameters of dynamic models can be exploited for economic modeling. In this section, we summarize its application to a model for the spot market for crude oil. The material is taken from [38], to which the reader is referred to for the details.

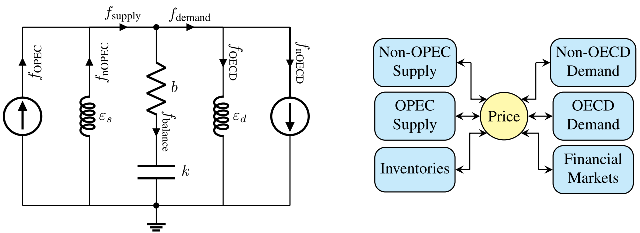


FIGURE 16. Left: Economic circuit for the oil-market model. Right: Schematic of the EIA model from [38].

Figure 16 shows a circuit representing the spot market for crude oil. The circuit is based on the bond-graph model in [38]. It contains the price drivers identified by the US Energy Information Administration (EIA) [39], also shown in the figure. These include the OPEC (current source) and non-OPEC (inductor) supply on the left, the financial markets (resistor) and inventory (capacitor) effects in the center, and the OECD (inductor) and non-OECD (current source) demand components on the right.

The circuit can be reduced to a two-state system with market price p and inventory q with the following state-space representation:

$$\begin{pmatrix} \dot{p} \\ \dot{q} \end{pmatrix} = \begin{pmatrix} -b(\varepsilon_s - \varepsilon_d) & -k \\ (\varepsilon_s - \varepsilon_d) & 0 \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix} + \begin{pmatrix} -b \\ 1 \end{pmatrix} u, \tag{30}$$

where the input $u = f_{\text{OPEC}} - f_{\text{nOECD}}$ is the net inelastic supply. The parameters (b, k, ε_s and ε_d) were estimated using grey-box identification, thus incorporating the structure of the circuit. For comparison, a third-order black-box model was trained on the same dataset.

The circuit was subjected to the historical net supply for oil during the period July 2002 to May 2003, during which a substantial supply shock occurred due to the Venezuelan oil strike. The input and the price responses are graphed in Figure 17 against the actual realized price in the period. Despite being of a lower order than the black-box model, the price response of the circuit provided a closer fit with the actual price movement (VAF=83% vs. VAF=55%). This should be contributed to the causal structure of the circuit model.

The conjugacy between price and stock variables allows us to validate the grey-box model on historical inventory data, since this dataset has not been used for identification. The results, plotted in Figure 18, show that the circuit is capable of following the dominant frequency component of the stock.

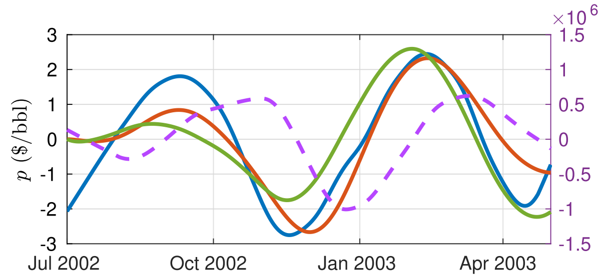


FIGURE 17. Price response as determined by the economic circuit (red) and by black-box model (green) vs. the historical price data (blue), together with the supply input (purple dashed). (From [38].)

Such a validation step would not be possible with the black-box model.

The performance of the circuit can be improved by modeling additional price drivers, thereby increasing the order of the model beyond two. For instance, the dip in measured inventory stocks during the winter of 2002-2003 appears to be due to the increase in demand during cold weather. By including such effects in the design of the circuit, its performance can be systematically improved.

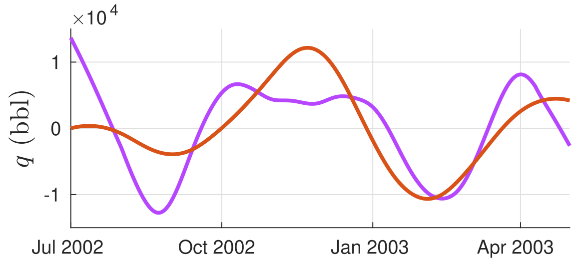


FIGURE 18. Out-of-sample prediction of inventory levels (red) vs. historical inventory data (purple). (From [38].)

C. A PRICE DYNAMIC MODEL FOR THE HYDROGEN ECONOMY

In this section, we overview an application of economic circuit theory to a more complex system: the hydrogen economy. The development of a price-dynamic hydrogen economy is considered to be an important component of the energy transition. Due to its conceptual nature, however, no historical data is available on the functioning of such an economy. In this section, we show how economic circuit theory can be used to design and analyze a price dynamic model, even in the absence of any historical data. We draw on the thesis [40], to which we refer the reader for the details.

Figure 19 contains the circuit diagram of a price-dynamic hydrogen economy. The design is based on the projected hydrogen backbone in the Netherlands, which is also shown in the figure. The model includes multiple price drivers for gaseous hydrogen: (1) its supply and demand, (2) supply from coupled markets (off-shore wind, liquid hydrogen, natural gas), (3) the current inventory (salt caverns and industry), and (4) price steering from a market operator. The state space representation of the reduced model contains four prices and three stocks for a total order of seven (see [40] for the explicit expression).

In Figure 20 we show the price response of gaseous hydrogen to a *dunkelflaute*. Also included are the price responses in several of the coupled markets. A dunkelflaute is a period of several consecutive days where no wind power is generated due to weather conditions, thus causing a downward supply shock in the wind energy. From the figure, we see that the dunkelflaute causes a very rapid price increase in gaseous hydrogen, which becomes volatile for an extended period until settling to a new equilibrium price level. The volatility contains several frequency components due to the high order of the model. The figure also shows the effect of the dunkelflaute on the prices for liquid hydrogen and natural gas, which increase due to their coupling with the gaseous hydrogen market.

The model can be used to change the design in order to improve the functioning of the hydrogen economy. Using dynamic scenario analysis, the price overshoot and settling time can be expressed as functions of the system parameters. These can then be adjusted to provide a more desirable response. For instance, one can consider adjusting the capacity of the salt cavern storage, or provide the market operator with some specific policies. The market operator can be thought of as a controller. In [40], a simple PID controller was applied and found to be both effective in improving price stability as well as intuitive in its operation.

VII. CONCLUSION AND FURTHER RESEARCH

In this paper, we propose an economic circuit theory. It is based on electrical circuit theory and, similar to its electrical counterpart, it is particularly applicable to economic systems that are out of an equilibrium state, where prices are volatile and demand is continuously changing. In addition, its basis in economic laws and its causal structure make the models predictive and interpretable. This allows for an engineering perspective on economic modeling, focusing on the design of new systems in addition to describing existing ones.

The distinguishing feature of an economic circuit model is that it combines a physical stock-flow with a price-incentive consistency condition. This combination yields a causal description of the network dynamics, with prices levels driving the flows and the conjugate stock quantities driving the incentives. This allows us to derive our main result: a state-space representation of the dynamics, i.e., a set of differential equations specifying the rate of change of the price levels and stock quantities of the market participants. The representation accounts for both endogenous and exogenous influences, the latter both deterministic and stochastic.

The economic circuit theory presented in the paper has several shortcomings and potentials for further development.

In our derivation we assume that the agents' behavioral laws are linear and time invariant. In practice, this implies that the models are valid only for relatively small variations in the price levels and stock quantities and that the parameters are stationary in the relatively short run. Nevertheless, applications to electrical systems have shown that complex dynamic

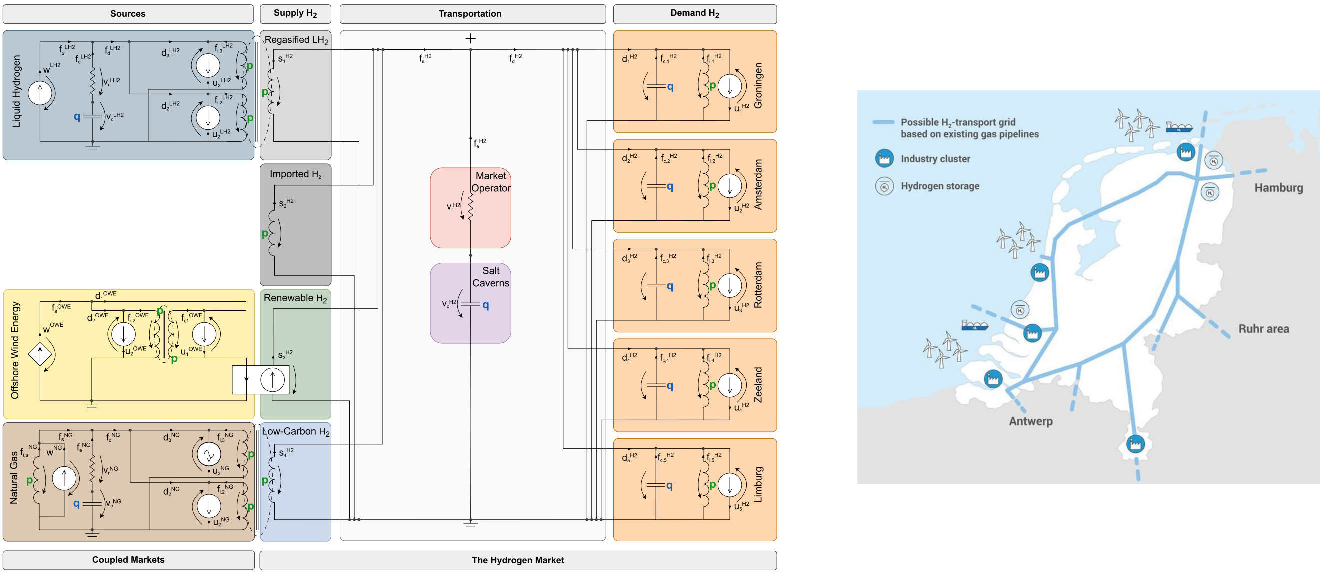


FIGURE 19. Circuit diagram of a conceptual hydrogen economy (left) and the proposed hydrogen back bone (right). (From [40].)

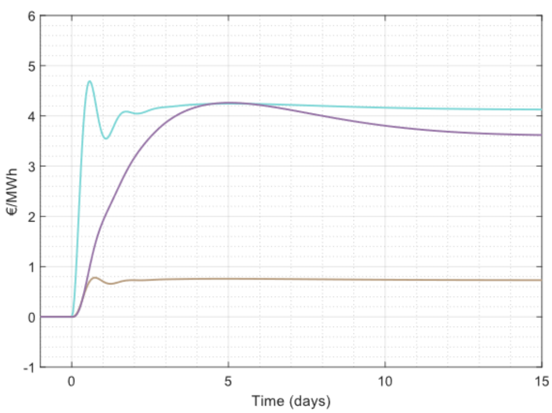


FIGURE 20. Price response of gaseous hydrogen (cyan), liquid hydrogen (purple), and natural gas (brown) to a downward shock in the supply of off-shore wind energy (a dunkelflaute). (From [40].)

behavior can be described with linear models alone, simply by increasing the order of the model. Alternatively, one can resort to nonlinear and time-varying network theories.

To test the validity of the models in real-world applications, its parameters have to be estimated. This issue is not addressed in the paper. It should be noted that the assessment burden is comparatively light, as the structure strongly constrains the class of possible models. (To wit, the example market in Section VI-A requires only six real parameters to be estimated, despite the complexity of the response.) Although efficient methods for system identification exist in the engineering literature (e.g. [41]) and identification has been applied to a comparative model in [38], more research is needed to determine what type of data sets are needed for economic systems and to determine whether these are available.

Although our theory is agent-based, its reliance on elementary agents (i.e., as electrical two-terminal components) limits its potential for agent-based modeling. Specifically, the representative agents defined in the paper consist of elementary agents of the same type. To develop agent-based systems, a more general definition of representative agents that combines several types of elementary behavior, together with the rules specifying their interaction, is required. In electrical network theory, this matter is resolved by taking a port-based approach. In a forthcoming publication, we show how it allows one to configure representative agents (known as multiports) that exhibit highly complex emergent behavior and interact through their ports. Using network analysis methods, systems can be designed in a bottom-up fashion. Using network synthesis methods, one can proceed in a top-down fashion, by prescribing the behavior and then determining an actual network that behaves accordingly.

An opportunity presented by economic circuits is that they are amenable to being controlled using methods from control theory. One thinks of a government, policy makers, or a regulating agency as separate from the system as represented by the network. With a state-space representation, an output representing some endogenous variable can be fed back and compared to a desired value. The loop is closed by the controller who provides the appropriate incentives at the input for the system to adjust its behavior. For linear systems, efficient methods exist for their control. An additional theoretical benefit is that the behavior of the system and the policy makers are not commingled, enabling the designer to circumvent the Lucas critique. We consider the exploitation of control theory a particularly fruitful avenue of investigation and we are currently actively researching it.

REFERENCES

[1] Y. Balasko, "Out-of-equilibrium price dynamics," *Econ. Theory*, vol. 33, no. 3, pp. 413–435, Sep. 2007.
[2] C. Pirrong, *Commodity Price Dynamics: A Structural Approach*. Cambridge, U.K.: Cambridge Univ. Press, 2011.

[3] C. Hommes, "The heterogeneous expectations hypothesis: Some evidence from the lab," *J. Econ. Dyn. Control*, vol. 35, no. 1, pp. 1–24, Jan. 2011.
[4] J. Lindgren, "General equilibrium with price adjustments—A dynamic programming approach," *Analytics*, vol. 1, no. 1, pp. 27–34, Jul. 2022.
[5] M. Boumans, "Visualisations for understanding complex economic systems," in *Ways of Thinking, Ways of Seeing*, C. Bissell and C. Dillon, Eds. Berlin, Germany: Springer, 2011, pp. 145–165.
[6] D. C. Karnopp, D. L. Margolis, and R. C. Rosenberg, *System Dynamics: Modeling, Simulation, and Control of Mechatronic Systems*. Hoboken, NJ, USA: Wiley, 2012.
[7] M. Mendel, "The Newtonian mechanics of demand," 2023, *arXiv:2310.17423*.
[8] Ö. Dilaver, R. C. Jump, and P. Levine, "Agent-based macroeconomics and dynamic stochastic general equilibrium models: Where do we go from here?" *J. Econ. Surv.*, vol. 32, no. 4, pp. 1134–1159, Sep. 2018.
[9] L. J. Christiano, M. S. Eichenbaum, and M. Trabandt, "On DSGE models," *J. Econ. Perspect.*, vol. 32, no. 3, pp. 113–140, 2018.
[10] T. Dessertaine, J. Moran, M. Benzaquen, and J.-P. Bouchaud, "Out-of-equilibrium dynamics and excess volatility in firm networks," *J. Econ. Dyn. Control*, vol. 138, May 2022, Art. no. 104362.
[11] B. LeBaron and L. Tesfatsion, "Modeling macroeconomies as open-ended dynamic systems of interacting agents," *Amer. Econ. Rev.*, vol. 98, no. 2, pp. 246–250, Apr. 2008.
[12] J. Sterman, *Business Dynamics, System Thinking and Modeling for a Complex World*. Boston, MA, USA: McGraw-Hill, Jan. 2000.
[13] L. Tesfatsion, "Agent-based computational economics: A constructive approach to economic theory," in *Handbook of Computational Economics*, vol. 2, 2006, ch. 16, pp. 831–880.
[14] J. D. Farmer and D. Foley, "The economy needs agent-based modelling," *Nature*, vol. 460, no. 7256, pp. 685–686, Aug. 2009.
[15] C. Gräbner, C. S. E. Bale, B. A. Furtado, B. Alvarez-Pereira, J. E. Gentile, H. Henderson, and F. Lipari, "Getting the best of both worlds? Developing complementary equation-based and agent-based models," *Comput. Econ.*, vol. 53, no. 2, pp. 763–782, Feb. 2019.
[16] J. McNerney, C. Savoie, F. Caravelli, V. M. Carvalho, and J. D. Farmer, "How production networks amplify economic growth," *Proc. Nat. Acad. Sci. USA*, vol. 119, no. 1, Jan. 2022, Art. no. e2106031118.
[17] C. Alexander and M. Sadiku, *Fundamentals of Electric Circuits*, 5th ed., Upper Saddle River, NJ, USA: Prentice-Hall, 2012.
[18] N. F. Morehouse, R. H. Strotz, and S. J. Horwitz, "An electro-analog method for investigating problems in economic dynamics: Inventory oscillations," *Econometrica*, vol. 18, no. 4, p. 313, Oct. 1950.
[19] S. Enke, "Equilibrium among spatially separated markets: Solution by electric analogue," *Econometrica*, vol. 19, no. 1, pp. 40–47, 1951.
[20] O. I. Franksen, "Mathematical programming in economics by physical analogies," *Simulation*, vol. 13, no. 2, pp. 63–87, Aug. 1969.
[21] J. W. Brewer and P. P. Craig, "Bilinear, dynamic single-ports and bond graphs of economic systems," *J. Franklin Inst.*, vol. 313, no. 4, pp. 185–196, Apr. 1982.
[22] A. Macchelli, "Towards a port-based formulation of macro-economic systems," *J. Franklin Inst.*, vol. 351, no. 12, pp. 5235–5249, Dec. 2014.
[23] G. L. Synge, "The fundamental theorem of electrical networks," *Quart. Appl. Math.*, vol. 11, no. 2, pp. 113–217, 1951.
[24] G. F. Oster and C. A. Desoer, "Tellegen's theorem and thermodynamic inequalities," *J. Theor. Biol.*, vol. 32, no. 2, pp. 219–241, Aug. 1971.
[25] F. Dörfler, J. W. Simpson-Porco, and F. Bullo, "Electrical networks and algebraic graph theory: Models, properties, and applications," *Proc. IEEE*, vol. 106, no. 5, pp. 977–1005, May 2018.
[26] J. W. Brewer, "Progress in the bond graph representations of economics and population dynamics," *J. Franklin Inst.*, vol. 328, nos. 5–6, pp. 675–696, Jan. 1991.
[27] H. R. Varian, *Intermediate Microeconomics*, 8th ed., Winnipeg, MB, Canada: The North West Company, 2010.
[28] S. P. Borgatti, A. Mehra, D. J. Brass, and G. Labianca, "Network analysis in the social sciences," *Science*, vol. 323, no. 5916, pp. 892–895, 2009.
[29] M. D. König and S. Battiston, "From graph theory to models of economic networks. a tutorial," in *Networks, Topology and Dynamics: Theory and Applications to Economics and Social Systems*. Berlin, Germany: Springer, 2009, pp. 23–63.
[30] F. Schweitzer, G. Fagiolo, D. Sornette, F. Vega-Redondo, A. Vespignani, and D. R. White, "Economic networks: The new challenges," *Science*, vol. 325, no. 5939, pp. 422–425, Jul. 2009.
[31] O. I. Franksen, "Mathematical programming in economics by physical analogies: Part II: The economic network concept," *SIMULATION*, vol. 13, no. 1, pp. 25–42, Jul. 1969.
[32] A. Marshall, *Principles of Economics*. New York, NY, USA: Macmillan, 1890.
[33] J. A. Bondy and U. S. R. Murty, *Graph Theory With Applications*. Amsterdam, The Netherlands: Elsevier, 1976.
[34] G. F. Franklin, J. D. Powell, and A. Emami-Naeini, *Feedback Control of Dynamic Systems* (What's New in Engineering Series). London, U.K.: Pearson, 2019.
[35] D. C. Lay, *Linear Algebra and Its Applications*. London, U.K.: Pearson, 2003.
[36] H. K. Khalil, *Nonlinear Systems*. Upper Saddle River, NJ, USA: Prentice-Hall, 2002.
[37] L. E. O. Svensson, "Walrasian and Marshallian stability," *J. Econ. Theory*, vol. 34, no. 2, pp. 371–379, Dec. 1984.
[38] C. Hutters, N. G. Orie, and M. B. Mendel, "Price dynamics in the oil market: A bond-graph modeling approach," *IFAC-PapersOnLine*, vol. 56, no. 2, pp. 9342–9347, 2023.
[39] EIA. (2022). *Energy & Financial Markets: What Drives Crude Oil Prices?* Accessed: Sep. 13, 2024. [Online]. Available: https://www.eia.gov/finance/markets/crudeoil/
[40] P. E. van den Berkmortel, "Modeling, design, analysis, and control of a spot market for hydrogen: An application of economic circuit theory," M.S. thesis, TU Delft, Delft, The Netherlands, 2024. [Online]. Available: https://repository.tudelft.nl/record/uuid:9b93e6b5-66b7-4f96-aafd-3909acdea478
[41] M. Verhaegen and V. Verdult, *Filtering and System Identification: A Least Squares Approach*. Cambridge, U.K.: Cambridge Univ. Press, 2007.

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