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15

Extended calculi and powers of operators

In this chapter we address two strongly interwoven topics: How to verify the boundedness of the H^∞-calculus of an operator and how to represent and estimate its fractional powers. For concrete operators such as the Laplace operator or elliptic partial differential operators, the fractional domain spaces can often be identified with certain function spaces considered in Chapter 14 and the imaginary powers of the operator are related to singular integral and pseudo-differential operators treated in Chapters 11 and 13.

Throughout this chapter, unless otherwise stated, we let A be a sectorial operator on a Banach space X. We work over the complex scalar field.

15.1 Extended calculi

In Chapter 10 we have introduced the Dunford calculus

$$f \mapsto f(A),$$

defined for functions $f \in H^1(\Sigma_\sigma)$, the space of holomorphic functions on Σ_σ that are integrable with respect to the measure $\frac{\mathrm{d}z}{z}$ (in the sense of (15.1) below). We performed a detailed study of the class of operators whose Dunford calculus, when restricted to $H^1(\Sigma_\sigma) \cap H^\infty(\Sigma_\sigma)$ extends to a functional calculus for functions in $H^\infty(\Sigma_\sigma)$.

In the present section we extend the Dunford calculus of a sectorial operator A to holomorphic functions f of polynomial growth on Σ_σ. Although the operators $f(A)$ in this calculus are generally unbounded, the mapping $f \mapsto f(A)$ still shares many properties with bounded functional calculi. This extended calculus includes all functions in $H^\infty(\Sigma_\sigma)$, and it agrees with the $H^\infty(\Sigma_\sigma)$-calculus of A when this operator has a bounded $H^\infty(\Sigma_\sigma)$-calculus. In the next section, it will enable us to define the fractional powers A^α in terms of the holomorphic functions z^α. Sectorial operators A whose imaginary powers A^{it} are bounded are of special interest in view of their close relationship with a variety of topics studied in these volumes.

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We briefly recall some notation and terminology introduced in Volume II that will be used throughout this chapter. For $0 < \sigma < \pi$ we denote by

$$\Sigma_\sigma := \{z \in \mathbb{C} \setminus \{0\} : \ |\arg(z)| < \sigma\}$$

the open sector of angle σ in the complex plane; the argument is taken in the interval $(-\pi, \pi)$. A linear operator $(A, \mathsf{D}(A))$ is *sectorial* if there exists $\sigma \in (0, \pi)$ such that the spectrum $\sigma(A)$ is contained in $\overline{\Sigma_\sigma}$ and

$$M_{\sigma,A} := \sup_{z \in \complement\overline{\Sigma_\sigma}} \|zR(z, A)\| < \infty.$$

Here, for $z \in \varrho(A)$, the resolvent set of A, $R(z, A) := (z - A)^{-1}$ denotes the resolvent of A. In this situation we say that A is *σ-sectorial with constant $M_{\sigma,A}$*. The infimum of all $\sigma \in (0, \pi)$ such that A is σ-sectorial is called the *angle of sectoriality* of A and is denoted by $\omega(A)$.

By $H^1(\Sigma_\sigma)$ we denote the Banach space of all holomorphic functions $f : \Sigma_\sigma \to \mathbb{C}$ for which

$$\|f\|_{H^1(\Sigma_\sigma)} := \sup_{|\nu|<\sigma} \|t \mapsto f(e^{i\nu}t)\|_{L^1(\mathbb{R}_+, \frac{\mathrm{d}t}{t})} < \infty. \tag{15.1}$$

Our objective in this section is to extend the Dunford calculus $f \mapsto f(A)$ to larger classes of functions. This is achieved in two steps: in Subsection 15.1.a we adjoin the constant-one function and the function $(1+z)^{-1}$. Among other things, this allows us to treat bounded rational functions as well as bounded functions such as $\exp(-z)$. This calculus provides the starting point for Subsections 15.1.b and 15.1.c, where we extend the calculus to a class of unbounded functions whose growth at the origin and at infinite is controlled by a regularising function. Among other things this, extended Dunford calculus will allow us to define fractional powers of A.

15.1.a The primary calculus

Our first aim is to extend the Dunford calculus $f \mapsto f(A)$ of a sectorial operator A to a slightly larger class of functions f for which one still obtains bounded operators, while preserving the multiplicativity of the calculus.

Definition 15.1.1. *For $0 < \sigma < \pi$ we define $E(\Sigma_\sigma)$ to be the vector space of holomorphic functions $f : \Sigma_\sigma \to \mathbb{C}$ of the form*

$$f(z) = f_0(z) + \frac{a}{1+z} + b,$$

where $f_0 \in H^1(\Sigma_\sigma) \cap H^\infty(\Sigma_\sigma)$ and $a, b \in \mathbb{C}$.

We could, more generally, allow functions $f_0 \in H^1(\Sigma_\sigma)$ here, but not much is gained by doing so because any such function belongs to $H^\infty(\Sigma_\nu)$ for all

$0 < \nu < \sigma$ (see Proposition H.1.3). This additional generality would in fact cause some inconvenience in the statement of the multiplicativity rule (Proposition 15.1.4), where one would be forced to switch to slightly smaller angles. A further advantage of the present definition is that $E(\Sigma_\sigma)$ is contained in $H^\infty(\Sigma_\sigma)$ as a linear subspace.

Lemma 15.1.2. *A bounded holomorphic function $f : \Sigma_\sigma \to \mathbb{C}$ belongs to $E(\Sigma_\sigma)$ if and only if it has* integrable limits *at 0 and ∞, by which we mean that there exist constants $c_0, c_\infty \in \mathbb{C}$ such that $f - c_0$ and $f - c_\infty$ are integrable with respect to $\frac{dz}{z}$ near 0 and ∞, respectively, in the sense that*

$$\sup_{|\nu|<\sigma} \|t \mapsto f(e^{i\nu}t) - c_0\|_{L^1((0,1),\frac{dt}{t})} < \infty$$

and

$$\sup_{|\nu|<\sigma} \|t \mapsto f(e^{i\nu}t) - c_\infty\|_{L^1((1,\infty),\frac{dt}{t})} < \infty.$$

Proof. If $f = E(\Sigma_\sigma)$ is of the form $f(z) = f_0(z) + \frac{a}{1+z} + b$ one may take $c_0 = a + b$ and $c_\infty = b$. In the converse direction, if the bounded holomorphic function $f : \Sigma_\sigma \to \mathbb{C}$ has integrable limits c_0 and c_∞ at 0 and ∞, respectively, then $f_0(z) := f(z) - \frac{c_0 - c_\infty}{1+z} - c_\infty$ belongs to $H^1(\Sigma_\sigma) \cap H^\infty(\Sigma_\sigma)$. □

The following functions belong to $E(\Sigma_\sigma)$:

$$z \mapsto \frac{z^m}{(1+z)^n} \qquad \text{for } 0 < \sigma < \pi \text{ and integers } n \geqslant m \geqslant 0;$$

$$z \mapsto \exp(-\zeta z) \qquad \text{for } 0 < \sigma < \tfrac{1}{2}\pi \text{ and } \zeta \in \Sigma_{\frac{1}{2}\pi-\sigma}.$$

For the first this follows by multiplicativity (proved in Proposition 15.1.4 below) and the fact that $z \mapsto (1+z)^{-1}$ and $z \mapsto z(1+z)^{-1} = 1-(1+z)^{-1}$ belong to $E(\Sigma_\sigma)$. For the second this follows by noting that both $\exp(-\zeta z) - (1+\zeta z)^{-1}$ and $(1 + \zeta z)^{-1} - (1 + z)^{-1}$ are in $H^1(\Sigma_\sigma) \cap H^\infty(\Sigma_\sigma)$. Another example will be encountered in the proof of Theorem 15.2.8.

Definition 15.1.3 (Primary calculus). *Let A be a sectorial operator on a Banach space X and let $\omega(A) < \sigma < \pi$. For functions $f \in E(\Sigma_\sigma)$ the bounded operator $f(A) \in \mathscr{L}(X)$ is defined by*

$$f(A) := f_0(A) + a(I + A)^{-1} + bI,$$

where

$$f(z) = f_0(z) + \frac{a}{1+z} + b$$

with $f_0 \in H^1(\Sigma_\sigma) \cap H^\infty(\Sigma_\sigma)$ and $a, b \in \mathbb{C}$, and with $f_0(A)$ defined through the Dunford calculus.

Since the constants a and b are uniquely determined by f this is well defined. For functions in $f \in H^1(\Sigma_\sigma) \cap H^\infty(\Sigma_\sigma)$ the primary calculus of a sectorial operator A agrees with the Dunford calculus. If A has a bounded $H^\infty(\Sigma_\sigma)$-calculus and $\mathsf{D}(A) \cap \mathsf{R}(A)$ is dense in X, then for functions $f \in E(\Sigma_\sigma)$ the definitions of $f(A)$ through the primary calculus agrees with that through the H^∞-calculus; this is because in the H^∞-calculus we have $\frac{1}{1+z}(A) = (I+A)^{-1}$ and $\mathbf{1}(A) = I$ by Theorem 10.2.13.

Proposition 15.1.4. *Let A be a sectorial operator on a Banach space X and let $\omega(A) < \sigma < \pi$. For all $f, g \in E(\Sigma_\sigma)$ we have $fg \in E(\Sigma_\sigma)$ and*

$$(fg)(A) = f(A)g(A).$$

Proof. Let $f, g \in E(\Sigma_\sigma)$ be represented as in Definition 15.1.1. It is clear that the product $f_0 g_0$ belongs to $H^1(\Sigma_\sigma) \cap H^\infty(\Sigma_\sigma)$ and that the product of $z \mapsto (1+z)^{-1}$ with a function in $H^1(\Sigma_\sigma) \cap H^\infty(\Sigma_\sigma)$ is in $H^1(\Sigma_\sigma) \cap H^\infty(\Sigma_\sigma)$ again. Finally,

$$\frac{1}{1+z} \cdot \frac{1}{1+z} = \frac{1}{1+z} - \frac{z}{(1+z)^2}$$

and the right-hand side is in $E(\Sigma_\sigma)$. This proves that $fg \in E(\Sigma_\sigma)$.

We have $f_0 g_0 \in H^1(\Sigma_\sigma) \cap H^\infty(\Sigma_\sigma)$, and the multiplicativity of the Dunford calculus gives

$$f_0(A) g_0(A) = (f_0 g_0)(A).$$

Also, with $\phi(z) = 1/(1+z)$ and $\zeta(z) = z/(1+z^2)$,

$$\phi(A)^2 = (I+A)^{-2} = \phi(A) - \zeta(A) = (\phi - \zeta)(A) = \phi^2(A),$$

where we used Proposition 10.2.3 to see that $\zeta(A) = A(I+A)^{-2}$ in the Dunford calculus and hence in the primary calculus. Thus it remains to check that $\phi(A) f_0(A) = (\phi f_0)(A)$. This follows by applying the resolvent identity and Cauchy's theorem to the contour integral representation of the Dunford calculus:

$$\begin{aligned}
\phi(A) f_0(A) &= \frac{1}{2\pi i} \int_\Gamma f_0(z)(I+A)^{-1} R(z,A)\,\mathrm{d}z \\
&= \frac{1}{2\pi i} \int_\Gamma \frac{f_0(z)}{1+z} [R(z,A) - R(-1,A)]\,\mathrm{d}z \\
&= \frac{1}{2\pi i} \int_\Gamma \frac{f_0(z)}{1+z} R(z,A)\,\mathrm{d}z \\
&= (\phi f_0)(A).
\end{aligned}$$

This completes the proof. □

Example 15.1.5 (Bounded rational functions). As a first application let us prove that if A is sectorial, then for all integers $m \geqslant n \geqslant 0$ we have

$$\frac{z^m}{(1+z)^n}(A) = A^m(I+A)^{-n},$$

noting that $z \mapsto \frac{z^m}{(1+z)^n}$ belongs to $E(\Sigma_\sigma)$ for all $0 < \sigma < \pi$.

By Proposition 15.1.4,

$$\begin{aligned}\frac{z^m}{(1+z)^n}(A) &= \Big(\frac{z}{1+z}(A)\Big)^m \Big(\frac{1}{1+z}(A)\Big)^{n-m} \\ &= (A(I+A)^{-1})^m (I+A)^{m-n} = A^m(I+A)^{-n},\end{aligned}$$

where we used that

$$\frac{z}{1+z}(A) = \mathbf{1}(A) - \frac{1}{1+z}(A) = I - (I+A)^{-1} = A(I+A)^{-1}.$$

Example 15.1.6 (Exponential functions). In this example we assume that A is sectorial with $\omega(A) < \frac{1}{2}\pi$. For $\omega(A) < \sigma < \frac{1}{2}\pi$ and $\zeta \in \Sigma_{\frac{1}{2}\pi-\sigma}$ define

$$\exp(-\zeta A) := \exp(-\zeta z)(A),$$

noting that $z \mapsto \exp(-\zeta z)$ belongs to $E(\Sigma_\sigma)$.

By Proposition 15.1.4,

$$\exp(-\zeta_1 A)\exp(-\zeta_2 A) = \exp(-(\zeta_1+\zeta_2)z)(A).$$

Furthermore, for all $x \in X$ and $n \geqslant 1$ we have $\exp(-\zeta A)x \in \mathsf{D}(A^n)$ and

$$(z^n \exp(-\zeta z))(A)x = A^n \exp(-\zeta A)x.$$

To see this denote the left-hand side by $g(A)$. By Proposition 15.1.4 and Example 15.1.5,

$$\begin{aligned}(I+A)^{-n}g(A) &= \frac{1}{(1+z)^n}(A)g(A) = \Big(\frac{z^n}{(1+z)^n}\exp(-\zeta z)\Big)(A) \\ &= \frac{z^n}{(1+z)^n}(A)\exp(-\zeta z)(A) = A^n(I+A)^{-n}\exp(-\zeta z)(A),\end{aligned}$$

from which the claim follows.

The preceding example connects with semigroup theory through Proposition 10.2.7 in Volume II which can be restated in the present language of primary calculus as follows.

Theorem 15.1.7. *Let A be a densely defined sectorial operator on X with angle $\omega(A) < \frac{1}{2}\pi$, and let $\omega(A) < \sigma < \frac{1}{2}\pi$. Then the bounded holomorphic C_0-semigroup $(S(z))_{z\in\Sigma_{\frac{1}{2}\pi-\sigma}}$ generated by $-A$ is given by the primary calculus through*

$$S(z) = \exp(-zA), \quad z \in \Sigma_{\frac{1}{2}\pi-\sigma},$$

where $\exp(-zA) = \exp(-z\,\cdot)(A)$ *as in the preceding example.*

15.1.b The extended Dunford calculus

Throughout this section, A is a sectorial operator on a Banach space X and we fix $\omega(A) < \sigma < \pi$. We proceed to define an extension of the primary calculus $f \mapsto f(A)$ for suitable unbounded functions f. The idea is to use a regularising function ϱ to "tame" the growth of f near the origin and at infinity. The resulting operators $f(A)$ are unbounded in general, but they nevertheless enjoy various good properties. For functions $f \in H^\infty(\Sigma_\sigma)$ and $\varrho(z) = z/(1+z)^2$ the construction proposed in the definition has already been used in Volume II (see (10.14)).

Definition 15.1.8 (Regularisers, extended Dunford calculus). *Let A be a sectorial operator on a Banach space X and let $\omega(A) < \sigma < \pi$. Let $f : \Sigma_\sigma \to \mathbb{C}$ be holomorphic. A function $\varrho \in E(\Sigma_\sigma)$ is called a* regulariser on *Σ_σ for the pair (f, A) if the following two conditions are met:*

- *$\varrho f \in E(\Sigma_\sigma)$;*
- *the operator $\varrho(A)$ defined by the primary calculus is injective.*

*We say that (f, A) is Σ_σ-*regularisable *if such a regulariser exists, and in that case we define*

$$\begin{aligned} \mathsf{D}(f(A)) &:= \big\{x \in X : (\varrho f)(A)x \in \mathsf{R}(\varrho(A))\big\}, \\ f(A)x &:= \varrho(A)^{-1}(\varrho f)(A)x, \quad x \in \mathsf{D}(f(A)). \end{aligned}$$

The mapping $f \mapsto f(A)$ is referred to as the extended calculus!Dunford *of A.*

If ϱ is a Σ_σ-regulariser for the pair (f, A), then so is $\rho\varrho$ for any $\rho \in H^1(\Sigma_\sigma) \cap H^\infty(\Sigma_\sigma)$ such that $\rho(A)$ is injective. Since $\rho\varrho \in H^1(\Sigma_\sigma) \cap H^\infty(\Sigma_\sigma)$, this shows that regularisers may be assumed to lie in $H^1(\Sigma_\sigma) \cap H^\infty(\Sigma_\sigma)$ whenever this is convenient.

In what follows we omit the prefix 'Σ_σ-' whenever the choice of the angle σ is clear from the context.

A trivial consequence of the first assertion in Proposition 15.1.4 is that if $\varrho \in E(\Sigma_\sigma)$, then for every function $f \in E(\Sigma_\sigma)$ we have $\varrho f \in E(\Sigma_\sigma)$. If $\varrho(A)$ is injective, Proposition 15.1.4 implies that for all $f \in E(\Sigma_\sigma)$ the definitions of $f(A)$ in Definitions 15.1.3 and 15.1.8 agree.

The following proposition shows that the definition of the operator $f(A)$ is independent of the regulariser.

Proposition 15.1.9 (Well-definedness). *Let A be a sectorial operator on a Banach space X and let $\omega(A) < \sigma < \pi$. Let $f : \Sigma_{\sigma\vee\tau} \to \mathbb{C}$ be holomorphic, where $\sigma, \tau \in (\omega(A), \pi)$. If $\varrho \in E(\Sigma_\sigma)$ and $\vartheta \in E(\Sigma_\tau)$ are regularisers for (f, A), then*

$$\big\{x \in X : (\varrho f)(A)x \in \mathsf{R}(\varrho(A))\big\} = \big\{x \in X : (\vartheta f)(A)x \in \mathsf{R}(\vartheta(A))\big\}$$

and, for all $x \in X$ belonging to this common set,

$$\varrho(A)^{-1}(\varrho f)(A)x = \vartheta(A)^{-1}(\vartheta f)(A)x.$$

Proof. Replacing σ and τ by $\sigma \wedge \tau$ we may assume that $\sigma = \tau$. Denote the domains defined in the statement of the lemma by $\mathsf{D}_\varrho(f(A))$ and $\mathsf{D}_\vartheta(f(A))$. If $x \in \mathsf{D}_\varrho(f(A))$, then $(\varrho f)(A)x = \varrho(A)y$ for some $y \in X$. By Proposition 15.1.4 we have $\vartheta\varrho f \in E(\Sigma_\sigma)$ and

$$\varrho(A)(\vartheta f)(A)x = (\vartheta\varrho f)(A)x = \vartheta(A)(\varrho f)(A)x = \vartheta(A)\varrho(A)y = \varrho(A)\vartheta(A)y,$$

and therefore $(\vartheta f)(A)x = \vartheta(A)y$ by the injectivity of $\varrho(A)$. This shows that $(\vartheta f)(A)x \in \mathsf{R}(\vartheta(A))$, so $x \in \mathsf{D}_\vartheta(f(A))$, and

$$\vartheta(A)^{-1}(\vartheta f)(A)x = y = \varrho(A)^{-1}(\varrho f)(A)x.$$

Interchanging the roles of ϱ and ϑ, one also sees that if $x \in \mathsf{D}_\vartheta(f(A))$, then $x \in \mathsf{D}_\varrho(f(A))$. This concludes the proof. □

The following observation is an immediate consequence of Proposition 15.1.4.

Lemma 15.1.10. *Let A be a sectorial operator on a Banach space X and let $\omega(A) < \sigma < \pi$. If $f, g : \Sigma_\sigma \to \mathbb{C}$ are holomorphic functions and $\varrho \in E(\Sigma_\sigma)$ and $\vartheta \in E(\Sigma_\sigma)$ are regularisers for (f, A) and (g, A), respectively, then $\varrho\vartheta$ is a regulariser for both (f, A) and (g, A).*

Proposition 15.1.11. *Let A be a sectorial operator on a Banach space X and let $\omega(A) < \sigma < \pi$. Let $f : \Sigma_\sigma \to \mathbb{C}$ be a holomorphic function such that the pair (f, A) is regularisable.*

(1) *the operator $f(A)$ is closed;*
(2) *if $\varrho \in E(\Sigma_\sigma)$ regularises (f, A), then $\mathsf{R}(\varrho(A)) \subseteq \mathsf{D}(f(A))$ and*

$$f(A)x = (\varrho f)(A)\varrho(A)^{-1}x, \quad x \in \mathsf{R}(\varrho(A)).$$

Proof. (1): Let $x_n \in \mathsf{D}(f(A))$ satisfy $x_n \to x$ and $f(A)x_n \to y$ in X as $n \to \infty$. Then $(\varrho f)(A)x_n \to (\varrho f)(A)x$ since $(\varrho f)(A)$ is bounded, and $\varrho(A)^{-1}[(\varrho f)(A)x_n] = f(A)x_n \to y$ by the definition of $f(A)$. The closedness of $\varrho(A)^{-1}$ implies $(\varrho f)(A)x \in \mathsf{D}(\varrho(A)^{-1}) = \mathsf{R}(\varrho(A))$ and $\varrho(A)^{-1}[(\varrho f)(A)x] = y$. By the definition of $\mathsf{D}(f(A))$, this means that $x \in \mathsf{D}(f(A))$ and $f(A)x = y$. This proves the closedness of $f(A)$.

(2): For $x \in \mathsf{R}(\varrho(A))$, say $x = \varrho(A)y$, we have

$$(\varrho f)(A)x = (\varrho f)(A)\varrho(A)y = \varrho(A)(\varrho f)(A)y \in \mathsf{R}(\varrho(A)).$$

Therefore $x \in \mathsf{D}(f(A))$ and

$$f(A)x = \varrho(A)^{-1}(\varrho f)(A)x = (\varrho f)(A)y = (\varrho f)(A)\varrho(A)^{-1}x.$$

□

Proposition 15.1.12. *Let A be a sectorial operator on a Banach space X and let $\omega(A) < \sigma < \pi$. Let $f, g : \Sigma_\sigma \to \mathbb{C}$ be holomorphic functions such that the pairs (f, A) and (g, A) are regularisable.*

(1) *for all* $a, b \in \mathbb{C}$ *the pair* $(af + bg, A)$ *is regularisable, and for all* $x \in \mathsf{D}(f(A)) \cap \mathsf{D}(g(A))$ *we have* $x \in \mathsf{D}((af + bg)(A))$ *and*

$$(af + bg)(A)x = af(A)x + bg(A)x.$$

(2) *the pair* (fg, A) *is regularisable and*

$$\mathsf{D}(f(A)g(A)) = \mathsf{D}(g(A)) \cap \mathsf{D}((fg)(A)),$$

and for all $x \in X$ *belonging to the common set we have*

$$(fg)(A)x = f(A)g(A)x.$$

In particular, $f(A)g(A)x$ *is closable. If* $g(A)$ *is bounded, then*

$$(fg)(A) = f(A)g(A)$$

with equal domains.

(3) *if* f *is zero-free and the pair* $(1/f, A)$ *is regularisable, then* $f(A)$ *is injective and*

$$\Big(\frac{1}{f}\Big)(A) = f(A)^{-1}$$

with equality of domains. In particular, if A *is injective and if we set* $\mathrm{inv}(z) := 1/z$*, then* (inv, A) *is regularisable and*

$$\mathrm{inv}(A) = A^{-1}.$$

Proof. By Lemma 15.1.10 we may select a function $\varrho \in E(\Sigma_\sigma)$ that regularises both (f, A) and (g, A) (in parts (1) and (2)), respectively both (f, A) and $(1/f, A)$ (in part (3)).

(1): It is clear that if $\varrho f, \varrho g \in E(\Sigma_\sigma)$, then $\varrho(af + bg) \in E(\Sigma_\sigma)$. The assumption $x \in \mathsf{D}(f(A)) \cap \mathsf{D}(g(A))$ implies that $(\varrho f)(A)x$ and $(\varrho g)(A)x$ belong to $\mathsf{R}(\varrho(A))$ and therefore we have $(\varrho(af + bg))(A)x \in \mathsf{R}(\varrho(A))$. Hence $x \in \mathsf{D}((af + bg)(A))$ and

$$\begin{aligned}(af + bg)(A)x &= \varrho(A)^{-1}(\varrho(af + bg))(A)x \\ &= a\varrho(A)^{-1}(\varrho f)(A)x + b\varrho(A)^{-1}(\varrho g)(A)x = af(A)x + bg(A)x.\end{aligned}$$

(2): By assumption we have $\varrho f, \varrho g \in E(\Sigma_\sigma)$. By Proposition 15.1.4 we also have $\varrho^2 fg \in E(\Sigma_\sigma)$. By multiplicativity we have $\varrho^2(A) = (\varrho(A))^2$, so $\varrho^2(A)$ is injective. It follows that the operator $(fg)(A)$ is well defined in the extended Dunford calculus.

Let $x \in \mathsf{D}(g(A)) \cap \mathsf{D}((fg)(A))$. Then, by the definition of $g(A)x$, multiplicativity, and the definition of $(fg)(A)x$,

$$\begin{aligned}(\varrho f)(A)g(A)x &= (\varrho f)(A)\varrho(A)^{-1}(\varrho g)(A)x \\ &= \varrho(A)^{-1}(\varrho f)(A)(\varrho g)(A)x\end{aligned}$$

$$
\begin{aligned}
&= \varrho(A)^{-1}(\varrho^2 fg)(A)x \\
&= \varrho(A)\varrho(A)^{-2}(\varrho^2 fg)(A)x \\
&= \varrho(A)(fg)(A)x.
\end{aligned}
$$

This shows that $(\varrho f)(A)g(A)x \in \mathsf{R}(\varrho(A))$ and therefore $g(A)x \in \mathsf{D}(f(A))$, i.e., $x \in \mathsf{D}(f(A)g(A))$, and

$$(fg)(A)x = \varrho(A)^{-1}(\varrho f)(A)g(A)x = f(A)g(A)x.$$

In the converse direction, let $x \in \mathsf{D}(f(A)g(A))$. Then $x \in \mathsf{D}(g(A))$ and $g(A)x \in \mathsf{D}(f(A))$, so $(\varrho f)(A)g(A)x \in \mathsf{R}(\varrho(A))$, say $(\varrho f)(A)g(A)x = \varrho(A)y$. Then,

$$(\varrho^2 fg)(A)x = (\varrho f)(A)(\varrho g)(A)x = \varrho(A)(\varrho f)(A)g(A)x = \varrho(A)^2 y = \varrho^2(A)y.$$

This shows that $(\varrho^2 fg)(A)x$ belongs to $\mathsf{R}(\varrho^2(A))$, so $x \in \mathsf{D}((fg)(A))$ by Proposition 15.1.9 and

$$(fg)(A)x = \varrho^2(A)^{-1}(\varrho^2 fg)(A)x = y = \varrho(A)^{-1}(\varrho f)(A)g(A)x = f(A)g(A)x.$$

By part (1) of Proposition 15.1.11 the operator $(fg)(A)$ is closed, and the above argument shows that it extends $f(A)g(A)$, so $f(A)g(A)$ is closable.

(3): Noting that $\mathsf{D}((1/f)f)(A) = \mathsf{D}(\mathbf{1}(A)) = \mathsf{D}(I) = X$, it follows from part (2) that if $x \in \mathsf{D}(f(A))$, then $x \in \mathsf{D}((1/f)(A)f(A))$ and $(1/f)(A)f(A)x = x$. Reversing the roles of f and $1/f$ we also obtain that if $x \in \mathsf{D}((1/f)(A))$, then $(1/f)(A)x \in \mathsf{D}(f(A))$ and $f(A)(1/f)(A)x = x$.

The second assertion follows by considering, e.g., the regulariser $\varrho(z) = z/(1+z)$. □

As a consequence of what has been shown in the course of the proof of part (2), and by applying (2) with f and g interchanged, we find that $f(A)$ and $g(A)$ commute in the following sense: we have

$$f(A)g(A)x = g(A)f(A)x = (fg)(A)x$$

for $x \in \mathsf{D}(f(A)) \cap \mathsf{D}(g(A)) \cap \mathsf{D}(fg(A))$.

We continue with a characterisation of the domain of $f(A)$ which, in view of later applications, we formulate in two versions. For integers $n \geqslant 1$ we write

$$r_n(z) := \frac{n}{n+z}, \qquad \zeta_n(z) := \frac{n}{n+z} - \frac{1}{1+nz}.$$

These functions belong to $E(\Sigma_\sigma)$.

Proposition 15.1.13. *Let A be a sectorial operator on a Banach space X and let $\omega(A) < \sigma < \pi$. Let $f : \Sigma_\sigma \to \mathbb{C}$ be a holomorphic function, and fix an integer $k \geqslant 1$.*

(1) *If* $\mathsf{D}(A)$ *is dense in* X *and* $r_n^k f \in E(\Sigma_\sigma)$, *then* $\mathsf{D}(A^k)$ *is densely contained in* $\mathsf{D}(f(A))$, *we have*

$$\mathsf{D}(f(A)) = \Big\{x \in X : \lim_{n\to\infty} (r_n^k f)(A)x \textit{ exists in } X\Big\},$$

and, for all $x \in \mathsf{D}(f(A))$,

$$f(A)x = \lim_{n\to\infty} (r_n^k f)(A)x.$$

(2) *If* $\mathsf{D}(A) \cap \mathsf{R}(A)$ *is dense in* X *and* $\zeta_n^k f \in E(\Sigma_\sigma)$, *then* $\mathsf{D}(A^k) \cap \mathsf{R}(A^k)$ *is densely contained in* $\mathsf{D}(f(A))$, *we have*

$$\mathsf{D}(f(A)) = \Big\{x \in X : \lim_{n\to\infty} (\zeta_n^k f)(A)x \textit{ exists in } X\Big\},$$

and, for all $x \in \mathsf{D}(f(A))$,

$$f(A)x = \lim_{n\to\infty} (\zeta_n^k f)(A)x.$$

In either case, $f(A)$ *is densely defined.*

Proof. (1): Let $\varrho(z) := r_1(z) = (1+z)^{-1}$. Then $\varrho^k = r_n^k \in E(\Sigma_\sigma)$ and $\mathsf{D}(A^k) = \mathsf{R}(\varrho^k(A))$, so the inclusion $\mathsf{R}(\varrho^k(A)) \subseteq \mathsf{D}(f(A))$ of Proposition 15.1.11 implies that $\mathsf{D}(A^k) \subseteq \mathsf{D}(f(A))$.

Let $x \in \mathsf{D}(f(A))$ and set $x_n := r_n^k(A)x$. Then $x_n \in \mathsf{D}(A^k) \subseteq \mathsf{D}(f(A))$, and by Proposition 10.1.7 we have $\lim_{n\to\infty} x_n = x$ (here we use that $\mathsf{D}(A)$ is dense) and

$$\lim_{n\to\infty} f(A)x_n = \lim_{n\to\infty} f(A)r_n^k(A)x = \lim_{n\to\infty} r_n^k(A)f(A)x = f(A)x,$$

where the middle identity follows from the second part of Proposition 15.1.11, observing that r_n^k is a regulariser for (f, A). This shows that $\mathsf{D}(A^k)$ is dense in $\mathsf{D}(f(A))$.

If $x \in \mathsf{D}(f(A))$, multiplicativity and the fact that $\varrho^k r_n^k f \in E(\Sigma_\sigma)$ imply

$$\begin{aligned} r_n^k(A)f(A)x &= \varrho^k(A)^{-1} r_n^k(A)(\varrho f)(A)x \\ &= \varrho^k(A)^{-1}(\varrho r_n^k f)(A)x = (r_n^k f)(A)x. \end{aligned}$$

Therefore $\lim_{n\to\infty}(r_n^k f)(A)x$ exists and equals $f(A)x$.

Conversely, suppose that $x \in X$ is such that $\lim_{n\to\infty}(r_n^k f)(A)x =: y$ exists. Put $z_n := r_n^k(A)(\varrho^k f)(A)x$. Then $z_n \in \mathsf{D}(A^k)$, so $z_n \in \mathsf{R}(\varrho^k(A))$. Moreover $z_n \to (\varrho^k f)(A)x$, and, by multiplicativity,

$$\begin{aligned} \varrho^k(A)^{-1} z_n &= \varrho^k(A)^{-1} r_n^k(A)(\varrho^k f)(A)x \\ &= \varrho^k(A)^{-1}(\varrho^k r_n^k f)(A)x = (r_n^k f)(A)x \to y. \end{aligned}$$

Since $\varrho^k(A)^{-1}$ is closed it follows that $(\varrho^k f)(A)x$ belongs to $\mathsf{D}(\varrho^k(A)^{-1}) = \mathsf{R}(\varrho^k(A))$, and therefore $x \in \mathsf{D}(f(A))$.

(2): This is proved in the same way as (1), replacing the use of r_n and Proposition 10.1.7 by ζ_n and Proposition 10.2.6. □

The following result improves Proposition 15.1.12(2) under an additional assumption.

Proposition 15.1.14. *Let A be a sectorial operator on a Banach space X and let $\omega(A) < \sigma < \pi$. Let $f, g : \Sigma_\sigma \to \mathbb{C}$ be holomorphic functions such that the pairs (f, A) and (g, A) are regularisable. Then $f(A)g(A)$ is closable and*

$$\overline{f(A)g(A)} = (fg)(A)$$

in each of the following two cases:

(1) $\mathsf{D}(A)$ *is dense in X, and f and g are bounded near* 0 *and have at most polynomial growth near* ∞*;*
(2) $\mathsf{D}(A) \cap \mathsf{R}(A)$ *is dense in X, and f and g have at most polynomial growth near* 0 *and* ∞*.*

Proof. The closability of $f(A)g(A)$ has already been proved in Proposition 15.1.12. We prove (1); the proof of (2) is entirely similar.

With $\varrho := r_1$ as in the previous proof, the growth assumption implies that for large enough $k \geqslant 1$ the functions $\varrho^k f$, $\varrho^k g$, and $\varrho^{2k} fg$ belong to $E(\Sigma_\sigma)$. Moreover, $\mathsf{D}(A^k) = \mathsf{R}(\varrho^k(A))$. The domain $\mathsf{D}(A^{2k})$ equals $\mathsf{R}(\varrho^{2k}(A))$, which in turn is contained in $\mathsf{D}((fg)(A))$ by Proposition 15.1.11 applied with ϱ^{2k} and fg. We also have $\mathsf{D}(A^{2k}) \subseteq \mathsf{D}(A^k) \subseteq \mathsf{D}(g(A))$, and hence $\mathsf{D}(A^{2k}) \subseteq \mathsf{D}(f(A)g(A))$ by Proposition 15.1.12. Moreover, since $\mathsf{D}(A)$ is dense in X, $\mathsf{D}(A^{2k})$ is dense in $\mathsf{D}((fg)(A))$ by Proposition 15.1.13. It follows that $\mathsf{D}(f(A)g(A))$ is dense in $\mathsf{D}((fg)(A))$. □

Theorem 15.1.15 (Composition). *Let A be a sectorial operator on a Banach space X and let $\omega(A) < \sigma < \pi$. Let $f : \Sigma_\sigma \to \mathbb{C}$ be a holomorphic function such that the pair (f, A) is regularisable, and assume that*

$$f(\Sigma_\sigma) \subseteq \Sigma_\tau$$

for some $0 < \tau < \pi$. Suppose furthermore that $f(A)$ is sectorial with $\omega(f(A)) < \tau$. If $g : \Sigma_\tau \to \mathbb{C}$ is a holomorphic function such that the pairs $(g, f(A))$ and $(g \circ f, A)$ are regularisable, then

$$g(f(A)) = (g \circ f)(A)$$

holds under either one of the following additional assumptions:

(i) $g \in E(\Sigma_\tau)$*;*
(ii) $(g, f(A))$ *admits a regulariser $\varphi \in E(\Sigma_\tau)$ such that $\varphi \circ f \in E(\Sigma_\sigma)$.*

The proof depends on the following lemma.

Lemma 15.1.16. *Let A be a sectorial operator on a Banach space X and let $\omega(A) < \sigma < \pi$. Let $f : \Sigma_\sigma \to \mathbb{C}$ be a holomorphic function such that*

$$f(\Sigma_\sigma) \subseteq \Sigma_\tau$$

for some $0 < \tau < \pi$. *If* $\varrho \in E(\Sigma_\sigma)$ *be a regulariser for* (f, A) *and* $\lambda \notin \overline{\Sigma_\tau}$, *then it is a regulariser for* $((\lambda - f)^{-1}, A)$ *as well, and*

$$\frac{1}{\lambda - f(z)}(A) = R(\lambda, f(A)).$$

Proof. By assumption, $\varrho f \in E(\Sigma_\sigma)$ and $\varrho(A)$ is injective. By Lemma 15.1.2, ϱ and ϱf have integrable limits at 0 and at ∞, say c_0, c_∞ and d_0, d_∞, respectively. Putting $f_\lambda := 1/(\lambda - f)$, we wish to show that ϱf_λ has integrable limits at 0 and at ∞; another application of Lemma 15.1.2 then implies that this function belongs to $E(\Sigma_\sigma)$, so ϱ regularises (f_λ, A). The identity in the statement of the lemma then follows from Proposition 15.1.12(3).

If $c_\infty = 0$, then $|f(\cdot) - \lambda| \geqslant \delta_1 := \mathrm{dist}(\lambda, \overline{\Sigma_\tau}) > 0$ implies that $\frac{\varrho(\cdot)}{\lambda - f(\cdot)}$ has integrable limit 0 at ∞. Suppose next that $c_\infty \neq 0$. We claim that $d_\infty / c_\infty \in \overline{\Sigma_\tau}$. Indeed, otherwise we had

$$\Big| f(\cdot) - \frac{d_\infty}{c_\infty} \Big| \geqslant \delta_2 := \mathrm{dist}(d_\infty / c_\infty, \overline{\Sigma_\tau}) > 0. \tag{15.2}$$

Since both ϱf and $\frac{d_\infty}{c_\infty}\varrho$ have integrable limit d_∞ at ∞, the identity

$$\varrho f = \varrho(f - \frac{d_\infty}{c_\infty}) + \frac{d_\infty}{c_\infty}\varrho$$

implies that $\varrho(f - \frac{d_\infty}{c_\infty})$ has integrable limit 0 at ∞. But then (15.2) would imply that ϱ has integrable limit 0 at ∞, contradicting the assumption that this integrable limit satisfies $c_\infty \neq 0$. This proves the claim.

With $\delta := \min\{\delta_1, \delta_2\}$ it now follows from

$$\begin{aligned}
&\Big|\frac{\varrho(z)}{\lambda - f(z)} - \frac{c_\infty^2}{c_\infty\lambda - d_\infty}\Big| \\
&\qquad \leqslant \Big|\frac{\varrho(z)}{\lambda - f(z)} - \frac{c_\infty \varrho(z)}{c_\infty\lambda - d_\infty}\Big| + \Big|\frac{c_\infty(\varrho(z) - c_\infty)}{c_\infty\lambda - d_\infty}\Big| \\
&\qquad = \Big|\frac{c_\infty(\varrho(z)f(z) - d_\infty) - d_\infty(\varrho(z) - c_\infty)}{(\lambda - f(z))(c_\infty\lambda - d_\infty)}\Big| + \Big|\frac{c_\infty(\varrho(z) - c_\infty)}{c_\infty\lambda - d_\infty}\Big| \\
&\qquad \leqslant \frac{1}{c_\infty \delta^2}\Big(|c_\infty||\varrho(z)f(z) - d_\infty| + |d_\infty||\varrho(z) - c_\infty|\Big) + \frac{1}{\delta}|\varrho(z) - c_\infty|
\end{aligned}$$

that $\varrho f_\lambda = \frac{\varrho(\cdot)}{\lambda - f(\cdot)}$ has integrable limit $\frac{c_\infty^2}{c_\infty\lambda - d_\infty}$ at ∞.

Replacing c_∞ and d_∞ by c_0 and d_0, in the same way one sees that ϱf_λ has integrable limit 0 at 0 if $c_0 = 0$, and integrable limit $\frac{c_0^2}{c_0\lambda - d_0}$ at 0 if $c_0 \neq 0$. □

Proof of Theorem 15.1.15. We begin with the proof of the theorem under the additional assumption made in (i), namely, that $g \in E(\Sigma_\tau)$.

Step 1 – For $g = \mathbf{1}$ the theorem is trivial since $\mathbf{1} \circ f = \mathbf{1}$ and $\mathbf{1}(f(A)) = (\mathbf{1} \circ f)(A) = I$. For $g(z) = 1/(1+z)$ we have $g(f(A)) = (I + f(A))^{-1}$ and $(g \circ f)(A) = (1 + f(z))^{-1}(A) = (I + f(A))^{-1}$, the former by Definition 15.1.3 applied to $f(A)$ and the latter by Lemma 15.1.16.

Step 2 – We now consider a general $g \in E(\Sigma_\tau)$, and write $g = g_0 + a/(1+z) + b$ with $a, b \in \mathbb{C}$ and $g_0 \in H^1(\Sigma_\tau) \cap H^\infty(\Sigma_\tau)$. Let $\varrho \in E(\Sigma_\sigma)$ be a regulariser for (f, A). By Lemma 15.1.10 we may assume that ϱ also regularises $(g \circ f, A)$, and by the observation after Definition 15.1.8 we may also assume that $\varrho \in H^1(\Sigma_\sigma) \cap H^\infty(\Sigma_\sigma)$. As the proof of Lemma 15.1.16 shows, ϱ also regularises $(\frac{1}{\lambda - f(\cdot)}, A)$ for $\lambda \notin \overline{\Sigma_\tau}$.

Fix $\omega(A) < \mu < \sigma$ and $\omega(f(A)) < \nu < \tau$. By the Dunford calculus of $f(A)$, the operator $g_0(f(A))$ is bounded and for all $x \in X$ we have

$$g_0(f(A))x = \frac{1}{2\pi i} \int_{\partial \Sigma_\nu} g_0(z) R(z, f(A))x \, \mathrm{d}z.$$

If $z \in \partial \Sigma_\nu$, then by Lemma 15.1.16 for all $x \in X$ we have

$$R(z, f(A))x = \frac{1}{z - f(\cdot)}(A)x. \tag{15.3}$$

Using (15.3) and multiplicativity of the primary calculus of A, Fubini's theorem, the Cauchy integral theorem, and keeping in mind that $\varrho \in H^1(\Sigma_\sigma) \cap H^\infty(\Sigma_\sigma)$, we obtain

$$\begin{aligned}
\varrho(A) g_0(f(A))x &= \frac{1}{2\pi i} \int_{\partial \Sigma_\nu} g_0(\lambda) \varrho(A) R(\lambda, f(A))x \, \mathrm{d}\lambda \\
&= \frac{1}{2\pi i} \int_{\partial \Sigma_\nu} g_0(\lambda) \frac{\varrho(\cdot)}{\lambda - f(\cdot)}(A)x \, \mathrm{d}\lambda \\
&= \Big(\frac{1}{2\pi i}\Big)^2 \int_{\partial \Sigma_\nu} g_0(\lambda) \Big(\int_{\partial \Sigma_\mu} \frac{\varrho(z)}{\lambda - f(z)} R(z, A)x \, \mathrm{d}z \Big) \mathrm{d}\lambda \\
&= \Big(\frac{1}{2\pi i}\Big)^2 \int_{\partial \Sigma_\mu} \varrho(z) \Big(\int_{\partial \Sigma_\nu} \frac{g_0(\lambda)}{\lambda - f(z)} \, \mathrm{d}\lambda \Big) R(z, A)x \, \mathrm{d}z \\
&= \frac{1}{2\pi i} \int_{\partial \Sigma_\mu} \varrho(z) g_0(f(z)) R(z, A)x \, \mathrm{d}z \\
&= (\varrho \cdot (g_0 \circ f))(A)x.
\end{aligned}$$

Setting $h_0(z) := a/(1+z) + b$, by Step 1 we also have $h_0(f(A)) = (h_0 \circ f)(A)$ and therefore, by Proposition 15.1.4,

$$\varrho(A) h_0(f(A))x = (\varrho \cdot (h_0 \circ f))(A)x, \quad x \in X.$$

Adding up, we obtain

$$\varrho(A) g(f(A))x = (\varrho \cdot (g \circ f))(A)x, \quad x \in X,$$

the operator $g(f(A))$ being defined by the primary functional calculus of $f(A)$. Since ϱ regularises $(g \circ f, A)$, this implies that every $x \in X$ belongs to $\mathsf{D}((g \circ f)(A))$ and

$$g(f(A))x = (g \circ f)(A)x, \quad x \in X.$$

This proves that $g(f(A)) = (g \circ f)(A)$, and both operators are bounded. This concludes the proof under the assumption made (i).

For the proof of the theorem under the assumption made in (i), let $\varphi \in E(\Sigma_\tau)$ be a regulariser for the pair $(g, f(A))$ such that $\varphi \circ f \in E(\Sigma_\tau)$, and let ρ be a regulariser for $(g \circ f, A)$.

We claim that under these circumstances, $\rho \cdot (\varphi \circ f)$ regularises $(g \circ f, A)$. To this end we must show:

- $\rho \cdot (\varphi \circ f) \cdot (g \circ f) \in E(\Sigma_\sigma)$;
- $(\rho \cdot (\varphi \circ f))(A)$ is injective.

The first assertion follows from $\rho \cdot (g \circ f) \in E(\Sigma_\sigma)$ (since ρ be a regulariser for $(g \circ f, A)$) and $\varphi \circ f \in E(\Sigma_\tau)$ (by assumption). For the second assertion we use the multiplicativity rule of Proposition 15.1.12 (noting that $(\varphi \circ f)(A) = \varphi(f(A))$ by the result of Step 2 and the fact that $\varphi \in E(\Sigma_\tau)$) to see that

$$(\rho \cdot (\varphi \circ f))(A) = \rho(A)(\varphi \circ f)(A) = \rho(A)\varphi(f(A)).$$

The right-hand side is the composition of two injective operators; this is because ρ is a regulariser for $(g \circ f, A)$ and φ is a regulariser for $(g, f(A))$. This proves the claim.

In the following computation, in (i) we use the definition of a regulariser, in (ii) we apply the result of Step 2 to $\varphi g \in E(\Sigma_\tau)$, noting that φg satisfies the conditions of the theorem since g does, (iii) follows from Proposition 15.1.12, noting that $((\varphi g) \circ f))(A) = (\varphi g)(f(A))$ is a bounded operator since $\varphi g \in E(\Sigma_\tau)$, (iv) is a simple rewriting, (v) follows from the definition of a regulariser, noting that $\rho \cdot (\varphi \circ f)$ regularises $(g \circ f, A)$, (vi) follows by another application of Proposition 15.1.12, and (vii) uses the result of Step 2 once again:

$$\begin{aligned}
\rho(A)\varphi(f(A))g(f(A)) &\overset{\text{(i)}}{=} \rho(A)(\varphi g)(f(A)) \\
&\overset{\text{(ii)}}{=} \rho(A)((\varphi g) \circ f)(A) \\
&\overset{\text{(iii)}}{=} (\rho \cdot ((\varphi g) \circ f))(A) \\
&\overset{\text{(iv)}}{=} (\rho \cdot (\varphi \circ f) \cdot (g \circ f))(A) \\
&\overset{\text{(v)}}{=} (\rho \cdot (\varphi \circ f))(A)(g \circ f)(A) \\
&\overset{\text{(vi)}}{=} \rho(A)(\varphi \circ f)(A)(g \circ f)(A) \\
&\overset{\text{(vii)}}{=} \rho(A)\varphi(f(A))(g \circ f)(A).
\end{aligned}$$

The identity $g(f(A)) = (g \circ f)(A)$ follows from this since both $\rho(A)$ and $\varphi(f(A))$ are injective. $\square$

Our next aim is to relate the extended Dunford calculus with the H^∞-calculus.

Theorem 15.1.17 (Boundedness of the extended Dunford calculus). *Let A be a sectorial operator on X with $\mathsf{D}(A) \cap \mathsf{R}(A)$ dense in X, and let $\omega(A) < \sigma < \pi$. Then for all functions $f \in H^\infty(\Sigma_\sigma)$ the pair (f, A) is regularisable and the following assertions are equivalent:*

(1) *the operator $f(A)$ defined through the extended Dunford calculus is bounded for all $f \in H^\infty(\Sigma_\sigma)$;*
(2) *the operator A has bounded $H^\infty(\Sigma_\sigma)$-calculus.*

In this situation the operators $f(A)$ defined through the extended Dunford calculus and the H^∞-calculus agree.

Proof. By Proposition 10.1.8, the density of $\mathsf{D}(A) \cap \mathsf{R}(A)$ implies that A is injective. As a consequence, for every $f \in H^\infty(\Sigma_\sigma)$ the function $\zeta(z) = z/(1+z)^2$ is a regulariser for the pair (f, A).

(1)⇒(2): By the boundedness of $f(A)$ and the closedness of $\zeta(A)^{-1}$, the identity $f(A) = \zeta(A)^{-1}(\zeta f)(A)$ (note that $\mathsf{D}(f(A)) = X$) implies that $f \mapsto f(A)$ is closed as a linear map from $H^\infty(\Sigma_\sigma)$ to $\mathscr{L}(X)$, and therefore bounded, by the closed graph theorem. Denoting its norm by M, it follows that

$$\|f(A)\| \leqslant M\|f\|_\infty$$

for all $f \in H^\infty(\Sigma_\sigma)$. In particular, this bound holds for all $f \in H^1(\Sigma_\sigma) \cap H^\infty(\Sigma_\sigma)$. For such functions the extended Dunford calculus agrees with the Dunford calculus, and therefore the estimate tells us that A has a bounded $H^\infty(\Sigma_\sigma)$-calculus.

(2)⇒(1): If $x \in \mathsf{D}(A) \cap \mathsf{R}(A)$, then $x \in \mathsf{R}(\zeta(A))$, say $x = \zeta(A)y$. For the operator $f(A)$ defined through the H^∞-calculus we have, by the multiplicativity of the H^∞-calculus,

$$f(A)x = \zeta(A)f(A)y = (\zeta f)(A)y,$$

where the operator on the right-hand side is again defined by the H^∞-calculus. We can also define the operator $(\zeta f)(A)$ through the primary calculus, and these two definitions agree (they agree for functions in $H^1(\Sigma_\sigma) \cap H^\infty(\Sigma_\sigma)$ and for the functions $(1+z)^{-1}$ and $\mathbf{1}$). It follows that

$$f(A)y = \zeta(A)^{-1}(\zeta f)(A)y.$$

Since $\mathsf{D}(A) \cap \mathsf{R}(A)$ is dense in X, this implies that $f(A) = \zeta(A)^{-1}(\zeta f)(A)$. The operator on the right-hand side equals the operator $f(A)$ defined through the extended Dunford calculus, which is therefore bounded. $\square$

We finish this section with a perturbation result that will be useful in connection with bounded imaginary powers (see the proof of Lemma 15.3.8).

Theorem 15.1.18. *Let A be a densely defined sectorial operator on a Banach space X and let $\omega(A) < \sigma < \pi$. Let $f \in H^\infty(\Sigma_\sigma)$ be given. If the operator $f(A)$, defined through the extended Dunford calculus of A is bounded, then also the operator $f(A+I)$, defined through the extended Dunford calculus of $A+I$, is bounded and we have*

$$\|f(A+I)\| \leqslant (1+M_{\sigma,A})^2(\|f(A)\| + C_\sigma \|f\|_{H^\infty(\Sigma_\sigma)}),$$

where C_σ is a constant depending only on σ and $M_{\sigma,A}$ is the sectoriality constant of A at angle σ.

Proof. Note that $\omega(A+I) \leqslant \omega(A)$ and fix $\omega(A) < \nu < \sigma$. The injectivity of $A+I$ implies that the function $\zeta(z) = z/(z+1)^2$ is a regulariser for $(f, A+I)$ for any $f \in H^\infty(\Sigma_\sigma)$. Since $\mathsf{D}(A+I) = \mathsf{D}(A)$ and $\mathsf{R}(A+I) = X$, the second part of Proposition 15.1.13 implies that $f(A+I)$ is densely defined.

By the extended Dunford calculus of $A+I$, for all $x \in \mathsf{D}(f(A+I))$ we have

$$f(A+I)x = (\zeta(A+I))^{-1} \frac{1}{2\pi i} \int_{\partial\Sigma_\nu} \zeta(z) f(z) R(z, A+I)x \,\mathrm{d}z.$$

We have $1/\zeta(z) = z + 2 + z^{-1}$, and this easily implies $(\zeta(A+I))^{-1} = (A+I) + 2I + (A+I)^{-1}$. Now,

$$\begin{aligned}
&\Big\|(2I + (A+I)^{-1}) \frac{1}{2\pi i} \int_{\partial\Sigma_\nu} \zeta(z) f(z) R(z, A+I)x \,\mathrm{d}z\Big\| \\
&\qquad \leqslant (2 + M_{\nu,A}) M_{\nu,A+I} \|x\| \Big(\frac{1}{2\pi} \int_{\partial\Sigma_\nu} \frac{1}{|z+1|^2} \,|\mathrm{d}z|\Big) \|f\|_{H^\infty(\Sigma_\sigma)}
\end{aligned}$$

and, noting that $R(z, A+I) = R(z, A) + R(z, A+I)R(z, A)$ by the resolvent identity,

$$\begin{aligned}
&\Big\|(A+I) \frac{1}{2\pi i} \int_{\partial\Sigma_\nu} \zeta(z) f(z) R(z, A+I)x \,\mathrm{d}z\Big\| \\
&\quad \leqslant \Big\|(A+I) \frac{1}{2\pi i} \int_{\partial\Sigma_\nu} \zeta(z) f(z) R(z, A)x \,\mathrm{d}z\Big\| \\
&\qquad + \Big\|\frac{1}{2\pi i} \int_{\partial\Sigma_\nu} \zeta(z) f(z) (A+I) R(z, A+I) R(z, A)x \,\mathrm{d}z\Big\| \\
&\quad \leqslant \|(A+I)\zeta(A)\| \|f(A)\| \|x\| \\
&\qquad + (1 + M_{\nu,A+I}) M_{\nu,A} \|x\| \Big(\frac{1}{2\pi} \int_{\partial\Sigma_\nu} \frac{1}{|z+1|^2} \,|\mathrm{d}z|\Big) \|f\|_{H^\infty(\Sigma_\sigma)}.
\end{aligned}$$

Since $\|A\zeta(A+I)\| = \|(A+I)A(A+I)^{-2}\| \leqslant M_{\nu,A}$ and $M_{\nu,A+I} \leqslant M_{\nu,A}$, this proves the estimate

$$\|f(A+I)x\| \leqslant (1+M_{\nu,A})^2\|f(A)\| + C_\nu(1+M_{\nu,A})^2\|f\|_{H^\infty(\Sigma_\sigma)}\|x\|$$

for $x \in \mathsf{D}(f(A+I))$, with $C_\nu = \frac{1}{\pi}\int_{\partial\Sigma_\nu} \frac{1}{|z+1|^2}\,|\mathrm{d}z|$. Since $\mathsf{D}(f(A+I))$ is dense, this estimate extends to arbitrary $x \in X$. To conclude the proof we let $\nu \uparrow \sigma$ and note that $M_{\nu,A} \to M_{\sigma,A}$ by an easy estimate based on the resolvent identity. $\square$

15.1.c Extended calculus via compensation

For functions $f \in H^\infty(\Sigma_\sigma)$ and regulariser $\varrho(z) := \zeta(z) = z/(1+z)^2$ there is different approach to the extended Dunford calculus via the Cauchy integral formula, which we outline presently.

Let A be a sectorial operator and let $f \in H^\infty(\Sigma_\sigma)$. For $\omega(A) < \tau < \sigma' < \sigma$, $\mu \in \Sigma_{\sigma'} \setminus \Sigma_\tau$, and $x \in \mathsf{D}(A) \cap \mathsf{R}(A)$ define

$$f(A)x := f(\mu)x + \frac{1}{2\pi i}\int_{\partial\Sigma_{\sigma'}} f(z)\Big(R(z,A) - \frac{1}{z-\mu}\Big)x\,\mathrm{d}z. \tag{15.4}$$

Let us check that the integrand converges absolutely. Since $x \in \mathsf{D}(A) \cap \mathsf{R}(A)$ we may pick $y \in \mathsf{D}(A)$ with $Ay = x$. Then

$$\Big\|R(z,A)x - \frac{x}{z-\mu}\Big\| = \Big\|\frac{(A-\mu)R(z,A)x}{z-\mu}\Big\| \leqslant \frac{\|R(z,A)\|}{|z-\mu|}(\|Ax\| + \mu\|x\|),$$

which is of the order $O(|z|^{-2})$ as $|z| \to \infty$ along $\partial\Sigma_\tau$. Also,

$$\Big\|R(z,A)x - \frac{x}{z-\mu}\Big\| = \Big\|R(z,A)Ay - \frac{x}{z-\mu}\Big\| \leqslant \|R(z,A)Ay\| + \frac{\|x\|}{|z-\mu|}$$

which is of the order $O(1)$ as $|z| \to 0$ along $\partial\Sigma_\tau$, noting that $\|R(z,A)Ay\| = \|R(z,A)[(A-z)+z]y\| \leqslant (1 + \|zR(z,A)\|)\|y\|$. This establishes the claim. By an application of Cauchy's theorem, $f(A)$ is independent of $\mu \in \Sigma_\tau \setminus \Sigma_{\tau'}$. Since the integrand is an integrable $\overline{\mathsf{R}(A)}$-valued function, we see that

$$f(A)x \in \overline{\mathsf{R}(A)}, \quad x \in \mathsf{D}(A) \cap \mathsf{R}(A).$$

Note that if $f \in H^1(\Sigma_\sigma) \cap H^\infty(\Sigma_\sigma)$, the above definition of $f(A)x$ agrees with (10.7).

We will now check that the definition of $f(A)x$ by (15.4) agrees with the one via Definition 15.1.1 for the regulariser $\varrho(z) = \zeta(z) = z/(1+z)^2$. Suppose that $x \in \mathsf{D}(A) \cap \mathsf{R}(A)$, say $x = \zeta(A)y$. Starting from the latter definition we have

$$f(A)x = (f\zeta)(A)y = \frac{1}{2\pi i}\int_{\partial\Sigma_\tau} \frac{zf(z)}{(1+z)^2}R(z,A)y\,\mathrm{d}z.$$

Fix $\omega(A) < \tau' < \tau$ and $\mu \in \Sigma_\tau \setminus \Sigma_{\tau'}$. To check that (15.4) agrees with Definition 15.1.1 we must show that

$$f(\mu)A(I+A)^{-2}y = \frac{1}{2\pi i}\int_{\partial\Sigma_\tau} \frac{f(z)}{z-\mu}\Big[\frac{z(z-\mu)}{(1+z)^2}R(z,A)y - \big((z-\mu)R(z,A)-I\big)x\Big]\,\mathrm{d}z.$$

By Cauchy's formula, the right-hand side integral evaluates to

$$= \Big[f(z)\Big(\frac{z(z-\mu)}{(1+z)^2}R(z,A)y - \big((z-\mu)R(z,A)-I\big)x\Big)\Big]\Big|_{z=\mu} = f(\mu)x,$$

as was to be proved. We have thus proved:

Proposition 15.1.19. *Let A be a sectorial operator on a Banach space X, let $\omega(A) < \tau' < \tau < \sigma < \pi$. Let $f \in H^\infty(\Sigma_\sigma)$. For all $\mu \in \Sigma_\tau \setminus \Sigma_{\tau'}$ and $x \in \mathsf{D}(A) \cap \mathsf{R}(A)$ the integral*

$$f(A)x := f(\mu)x + \frac{1}{2\pi i}\int_{\partial\Sigma_\tau} f(z)\Big(R(z,A) - \frac{1}{z-\mu}\Big)x\,\mathrm{d}z$$

converges absolutely and we have $f(A)x = (f\zeta)(A)y$, in agreement with the definition of $f(A)x$ through the extended Dunford calculus.

The attentive reader will have noticed that we already used this procedure in Proposition 10.2.7.

15.2 Fractional powers

In this section, we will apply the extended Dunford calculus to introduce the fractional powers A^α of a sectorial operator A. Particular instances of fractional powers such as $(-\Delta)^{1/2}$, the square root of the negative Laplacian, appear all over in Analysis. On a theoretical level, domains of fractional powers encode useful smoothness properties of the elements in their domains, and correspond to (or are closely connected with) interpolation scales between the domain $\mathsf{D}(A)$ and the underlying Banach space X. For example, if the imaginary powers A^{it}, $t \in \mathbb{R}$, are bounded operators, then for all $0 < \alpha < 1$ the fractional domain $\mathsf{D}(A^\alpha)$ equals the complex interpolation space $[X, \mathsf{D}(A)]_\alpha$ as a subspace of X, and as a Banach space up to equivalent norms. As we have seen in Chapter 4, for the negative Laplacian $A = -\Delta$ on $X = L^p(\mathbb{R}^d)$, the latter can be identified as the Bessel potential space $H^{2\alpha,p}(\mathbb{R}^d)$.

After introducing fractional powers, we establish several basic algebraic properties and prove several useful representation formulas. In the next section, we then take a closer look at the class of sectorial operators whose imaginary powers are bounded, and prove a number of non-trivial theorems connecting this property with (R-, γ-)sectoriality and boundedness of the H^∞-calculus.

15.2.a Definition and basic properties

In what follows, unless otherwise stated we let A be a sectorial operator acting in a Banach space X. When additional assumptions are needed, they will always be stated explicitly.

For $\alpha \in \mathbb{C}$ it is natural to try to define the fractional power A^α by applying the extended Dunford calculus to the function

$$f_\alpha(z) := z^\alpha := e^{\alpha \log z},$$

where we use the branch of the logarithm that is holomorphic in $\mathbb{C} \setminus (-\infty, 0]$. Let $0 < |\nu| < \sigma < \pi$. For $z = re^{i\nu}$ with $r \geqslant 0$ we have

$$|f_\alpha(z)| = |r^\alpha|\,|e^{i\nu\alpha}| \leqslant |z|^{\Re\alpha} e^{\sigma|\Im\alpha|}.$$

For all integers $m, n \in \mathbb{N}$, the function

$$\varrho_{m,n}(z) := z^m(1+z)^{-m-n}$$

belongs to $E(\Sigma_\sigma)$, and

- if $\Re\alpha > 0$, then $\varrho_{m,n} f_\alpha \in E(\Sigma_\sigma)$ for all integers $m \geqslant 0$, $n > \Re\alpha$;
- if $\Re\alpha = 0$, then $\varrho_{m,n} f_\alpha \in E(\Sigma_\sigma)$ for all integers $m, n \geqslant 1$;
- if $\Re\alpha < 0$, then $\varrho_{m,n} f_\alpha \in E(\Sigma_\sigma)$ for all integers $m > |\Re\alpha|$, $n \geqslant 0$.

The operator $\varrho_{m,n}(A) = A^m(I+A)^{-m-n}$ (cf. Example 15.1.5) is injective if $m = 0$ or A is injective (or both). This shows:

Proposition 15.2.1. *Let A be a sectorial operator on a Banach space X. The pair (f_α, A) is regularisable in each of the following two cases:*

- $\Re\alpha > 0$
- $\Re\alpha \leqslant 0$ *and A is injective.*

In the first case $\varrho_{0,n}(z) = (1+z)^{-n}$ with $n > \Re\alpha$ is a regulariser; in the second case $\varrho_{n,n}(z) = z^n(1+z)^{-2n}$ with $n > |\Re\alpha|$ is a regulariser.

In view of these considerations the extended Dunford calculus allows us to make the following definition.

Definition 15.2.2 (Fractional powers). *Let A be a sectorial operator on a Banach space X. For $\alpha \in \mathbb{C}$ we define*

$$A^\alpha := f_\alpha(A), \quad \alpha \in \mathbb{C},$$

in each of the following two cases:

- $\Re\alpha > 0$
- $\Re\alpha \leqslant 0$ *and A is injective.*

These operators are closed. Moreover, if $\Re\lambda > 0$ and $\mathsf{D}(A)$ is dense, then A^α is densely defined; if $\Re\alpha \leqslant 0$ and $\mathsf{D}(A)\cap\mathsf{R}(A)$ is dense, then A is injective and A^α is densely defined. Using the results of Section 10.1.b, These assertions follow from Proposition 15.1.11, the domain identifications $\mathsf{D}(A^n) = \mathsf{R}(\varrho_{0,n})$ and $\mathsf{D}(A^n)\cap\mathsf{R}(A^n) = \mathsf{R}(\varrho_{n,n})$, and the fact that $\mathsf{D}(A^n)$ is dense if $\mathsf{D}(A)$ dense, respectively $\mathsf{D}(A^n)\cap\mathsf{R}(A^n)$ is dense in X if $\mathsf{D}(A)\cap\mathsf{R}(A)$ is dense in X.

We begin our study of fractional powers with a consistency check.

Proposition 15.2.3. *Let A be a sectorial operator on a Banach space X. For all $n = 0, 1, 2, \dots$ and $f_n(z) = z^n$ we have*

$$f_n(A) = A^n \quad \textit{with equal domains.} \tag{15.5}$$

If in addition A is injective, this identity extends to all $n \in \mathbb{Z}$.

Proof. For $n = 0$ this reduces to the identity $\mathbf{1}(A) = I$. For $n \geqslant 1$, consider the function $\varrho_n(z) = (1+z)^{-n}$ and let $x \in \mathsf{D}(A^n) = \mathsf{R}(\varrho_n(A))$, say $x = (I+A)^{-n}y$. Then

$$f_n(A)x = \varrho_n(A)^{-1}(\varrho_n f_n)(A)x = A^n x,$$

where we used that $\varrho_n(A) = (I + A)^{-n}$ in the primary calculus, and that $(\varrho_n f_n)(A) = \frac{z^n}{(1+z)^n}(A) = A^n(I + A)^{-n}$ in the primary calculus. This proves that $A^n \subseteq f_n(A)$. In the converse direction, if $x \in \mathsf{D}(f_n(A))$, then

$$A^n(I + A)^{-n}x = (\varrho_n f_n)(A)x \in \mathsf{R}(\varrho_n(A)) = \mathsf{D}(A^n),$$

forcing $x \in \mathsf{D}(A^n)$. This completes the proof of (15.5) for $n \geqslant 1$. For $n = -1, -2, \dots$ the result follows by applying Proposition 15.1.12(3). □

From the definition of the extended Dunford calculus we immediately deduce the following result.

Proposition 15.2.4. *Let A be a sectorial operator on a Banach space X, and fix an integer $k \geqslant 1$.*

(1) *For all $x \in \mathsf{D}(A^k)$ the function $z \mapsto A^z x$ is well defined and holomorphic on $\{0 < \Re z < k\}$.*
(2) *If A is injective, then for all $x \in \mathsf{D}(A^k)\cap\mathsf{R}(A^k)$ the function $z \mapsto A^z x$ is well defined and holomorphic on $\{-k < \Re z < k\}$.*

Theorem 15.2.5. *Let A be a sectorial operator on a Banach space X, and let $\alpha, \alpha_1, \alpha_2 \in \mathbb{C}$.*

(1) *If A is injective and $\alpha \in \mathbb{C}$, then A^α is injective and*

$$A^{-\alpha} = (A^\alpha)^{-1} = (A^{-1})^\alpha \quad \textit{with equality of domains.}$$

(2) *If $\Re\alpha_1 > \Re\alpha_2 > 0$, then*

$$\mathsf{D}(A^{\alpha_1}) \subseteq \mathsf{D}(A^{\alpha_2}) \quad \textit{and} \quad \mathsf{R}(A^{\alpha_2}) \supseteq \mathsf{R}(A^{\alpha_1}),$$

(3) *If A is injective and $\Re\alpha_1 < \Re\alpha_2 < 0$, then*

$$\mathsf{D}(A^{\alpha_1}) \supseteq \mathsf{D}(A^{\alpha_2}) \quad \textit{and} \quad \mathsf{R}(A^{\alpha_1}) \subseteq \mathsf{R}(A^{\alpha_2}).$$

(4) *If $\Re\alpha_1 > 0$ and $\Re\alpha_2 > 0$, then*

$$A^{\alpha_1+\alpha_2} = A^{\alpha_1}A^{\alpha_2} \quad \textit{with equality of domains.}$$

(5) *If A is injective and $\Re\alpha_1 < 0$ and $\Re\alpha_2 < 0$, then*

$$A^{\alpha_1+\alpha_2} = A^{\alpha_1}A^{\alpha_2} \quad \textit{with equality of domains.}$$

Proof. (1): The injectivity of A^α and the identity $A^{-\alpha} = (A^\alpha)^{-1}$ follow from Proposition 15.1.12(3). The identity $A^{-\alpha} = (A^{-1})^\alpha$ follows from Theorem 15.1.15, noting that A^{-1} is sectorial with the same angle as A.

(2): We consider the regulariser $\varrho_k(z) = (1+z)^{-k}$, for which we have $\mathsf{R}(\varrho_k(A)) = \mathsf{D}(A^k)$.

Let $x \in \mathsf{D}(A^{\alpha_1})$ and fix an integer $k > \max\{\Re\alpha_2, \Re\alpha_1 - \Re\alpha_2\}$. In order to prove that $x \in \mathsf{D}(A^{\alpha_2})$ we must show that $((1+z)^{-k}z^{\alpha_2})(A)x \in \mathsf{D}(A^k)$.

Since $2k > \Re\alpha_1$, by the definition of $\mathsf{D}(A^{\alpha_1})$ we have $((1+z)^{-2k}z^{\alpha_1})(A)x \in \mathsf{D}(A^{2k})$. Using the multiplicativity of the Dunford calculus, this implies that

$$\begin{aligned} A^k(I+A)^{-2k}((1+z)^{-k}z^{\alpha_2})(A)x &= \frac{z^{k+\alpha_2}}{(1+z)^{3k}}(A)x \\ &= \frac{z^{k-(\alpha_1-\alpha_2)}}{(1+z)^k}(A)\frac{z^{\alpha_1}}{(1+z)^{2k}}(A)x \end{aligned}$$

belongs to $\mathsf{D}(A^{2k})$. It follows that $(I+A)^{-2k}((1+z)^{-k}z^{\alpha_2})(A)x \in \mathsf{D}(A^{3k})$ and $((1+z)^{-k}z^{\alpha_2})(A)x \in \mathsf{D}(A^k)$ as desired. The opposite inclusion of the ranges follows from part (4) proved below.

(3): If A is injective and $\Re\alpha_1 < \Re\alpha_2 < 0$ we can apply parts (2) and (1) with $\beta_1 = -\alpha_1$ and $\beta_2 = \alpha$, noting that $\mathsf{D}(A^{\alpha_j}) = \mathsf{D}(A^{-\beta_j}) = \mathsf{R}(A^{\beta_j})$ and $\mathsf{R}(A^{\alpha_j}) = \mathsf{R}(A^{-\beta_j}) = \mathsf{D}(A^{\beta_j})$.

(4): Let $\Re\alpha_1 > 0$ and $\Re\alpha_2 > 0$. Proposition 15.1.12 implies that $A^{\alpha_1}A^{\alpha_2}x = A^{\alpha_1+\alpha_2}x$ for all $x \in \mathsf{D}(A^{\alpha_1}A^{\alpha_2}) = \mathsf{D}(A^{\alpha_2}) \cap \mathsf{D}(A^{\alpha_1+\alpha_2})$. It remains to prove that $\mathsf{D}(A^{\alpha_1+\alpha_2}) \subseteq \mathsf{D}(A^{\alpha_2})$. But this follows from part (2).

(5): This follows from (1) and (4) by taking inverses. □

Proposition 15.2.6. *Let A be a sectorial operator on a Banach space X. Let $c \in \mathbb{C} \setminus \{0\}$ satisfy $|\arg c| < \pi - \omega(A)$. Then:*

(1) *the operator cA is sectorial with angle $\omega(cA) \leqslant \omega(A) + |\arg(c)|$, and for all $\omega(A) < \sigma < \pi - |\arg c|$ we have $M_{\sigma+|\arg c|, cA} \leqslant M_{\sigma,A}$;*
(2) *for all $\alpha \in \mathbb{C}$, and assuming A to be injective if $\Re\alpha \leqslant 0$, we have*

$$(cA)^\alpha = c^\alpha A^\alpha \quad \textit{with equality of domains.}$$

Proof. Since $(\lambda - cA)^{-1} = c^{-1}(c^{-1}\lambda - A)^{-1}$, the condition $|\arg(c)| < \pi - \omega(A)$ guarantees that cA is sectorial with $\omega(cA) \leqslant \omega(A) + |\arg c|$. Also, for $\omega(A) < \sigma < \pi - |\arg c|$ and $\lambda \in \complement\overline{\Sigma_{\sigma+|\arg c|}}$ we have $c^{-1}\lambda \in \complement\overline{\Sigma_\sigma}$ and

$$\|\lambda R(\lambda, cA)\| = \|c^{-1}\lambda R(c^{-1}\lambda, A)\| \leqslant M_{\sigma,A},$$

which gives the bound $M_{\sigma+|\arg c|,cA} \leqslant M_{\sigma,A}$.

Choose $\omega > \omega(A)$ such that $\omega + |\arg c| < \pi$. Fix $\alpha \in \mathbb{C}$. Then, for $x \in X$ and $k > |\Re\alpha|$,

$$(\rho_k f_\alpha)(cA)x = \frac{1}{2\pi i}\int_{\partial\Sigma_{\omega+|\arg c|}} \rho_k(z) z^\alpha R(z, cA)x \,\mathrm{d}z$$

with $f_\alpha(z) = z^\alpha$, and $\rho_k(z) := \varrho_{0,k}(z) = (1+z)^{-k}$ if $\Re\alpha > 0$ and $\rho_k(z) := \varrho_{k,k}(z) = z^k/(1+z)^{2k}$ if $\Re\alpha \leqslant 0$. By Cauchy's theorem we can deform the path in the above integral to $\Gamma = c\cdot\partial\Sigma_\omega$ and obtain, by a change of variables,

$$\begin{aligned}(\rho_k f_\alpha)(cA)x &= \frac{1}{2\pi i}\int_\Gamma \rho_k(z) z^\alpha c^{-1} R(c^{-1}z, A)x \,\mathrm{d}z \\ &= c^\alpha \frac{1}{2\pi i}\int_{\partial\Sigma_\omega} \rho_k(cz) z^\alpha R(z, A)x \,\mathrm{d}z = c^\alpha(\rho_n^k(c\,\cdot) f_\alpha)(A)x.\end{aligned} \tag{15.6}$$

If $x \in \mathsf{D}(f_\alpha(A))$, then $(\rho_k(c\,\cdot)f_\alpha)(A)x \in \mathsf{R}(A)$ (by the definition of $\mathsf{D}(f_\alpha(A))$, since $\rho_k(c\,\cdot)$ is a regulariser for (f_α, A)), and (15.6) implies that $(\rho_k f_\alpha)(cA)x \in \mathsf{R}(A) = \mathsf{R}(cA)$. But this implies that $x \in \mathsf{D}(f_\alpha(cA))$ (by the definition of $\mathsf{D}(f_\alpha(cA))$, since ρ_k is a regulariser for (f_α, cA)). This gives the inclusion $\mathsf{D}(f_\alpha(A)) \subseteq \mathsf{D}(f_\alpha(cA))$. The same argument in reverse direction gives the inclusion $\mathsf{D}(f_\alpha(cA)) \subseteq \mathsf{D}(f_\alpha(A))$. Moreover, for any x in this common domain,

$$\begin{aligned} f_\alpha(cA)x &= (\rho_k(cA))^{-1}(\rho_k f_\alpha)(cA)x, \\ c^\alpha f_\alpha(A)x &= c^\alpha(\rho_k(c\,\cdot)(A))^{-1}(\rho_k(c\,\cdot)f_\alpha)(A)x = (\rho_k(c\,\cdot)(A))^{-1}(\rho_k f_\alpha)(cA)x,\end{aligned}$$

the last identity being a consequence of (15.6). Since the right-hand sides are obviously equal, this gives the result. □

Theorem 15.2.7. *Let A be a sectorial operator on a Banach space X. If $0 < \alpha < \pi/\omega(A)$, then A^α is sectorial, we have*

$$\omega(A^\alpha) = \alpha\omega(A),$$

and for all $\beta \in \mathbb{C}$ we have

$$(A^\alpha)^\beta = A^{\alpha\beta} \quad \textit{with equality of domains.}$$

If A is R-sectorial and $0 < |\alpha| < \pi/\omega_R(A)$, then A^α is R-sectorial and

$$\omega_R(A^\alpha) = \alpha\omega_R(A).$$

Proof. The proof proceeds in a number of steps.

Step 1 – First consider an arbitrary $\alpha > 0$. In this step we will prove that for all $\mu \notin \overline{\Sigma_{\alpha\omega(A)}}$ we have $\mu \in \varrho(A^\alpha)$ and

$$\mu R(\mu, A^\alpha) = -|\mu|^{1/\alpha} R(-|\mu|^{1/\alpha}, A) + \psi_\tau(|\mu|^{-1/\alpha} A),$$

where $\tau = \arg \mu$ and

$$\psi_\tau(z) = \frac{e^{i\tau} z + z^\alpha}{(e^{i\tau} - z^\alpha)(1+z)}.$$

Note that $\psi_\tau \in H^1(\Sigma_\sigma)$ for all $\sigma < |\tau|/\alpha$.

A straightforward calculation shows

$$\frac{\mu}{\mu - z^\alpha} - \frac{|\mu|^{1/\alpha}}{|\mu|^{1/\alpha} + z} = \frac{\mu z + |\mu|^{1/\alpha} z^\alpha}{(\mu - z^\alpha)(|\mu|^{1/\alpha} + z)} = \psi_\tau(|\mu|^{-1/\alpha} z).$$

Hence

$$\frac{1}{\mu - z^\alpha} = \frac{1}{\mu}\Big(\frac{|\mu|^{1/\alpha}}{|\mu|^{1/\alpha} + z} + \psi_\tau(|\mu|^{-1/\alpha} z)\Big).$$

Proposition 15.1.12 implies that $(\frac{1}{\mu - (\cdot)^\alpha})(A)$ is indeed the inverse of $(\mu - (\cdot)^\alpha)(A) = \mu - A^\alpha$. Thus $\mu \in \varrho(A^\alpha)$ and

$$\begin{aligned} R(\mu, A^\alpha) &= \frac{1}{\mu}\Big(\frac{|\mu|^{1/\alpha}}{|\mu|^{1/\alpha} + z} + \psi_\tau(|\mu|^{-1/\alpha} z)\Big)(A) \\ &= \frac{1}{\mu}\Big(- |\mu|^{1/\alpha} R(-|\mu|^{1/\alpha}, A) + \psi_\tau(|\mu|^{-1/\alpha} A)\Big) \end{aligned} \tag{15.7}$$

using that if $\lambda \in \complement\overline{\Sigma_\sigma}$, then $\frac{1}{\lambda - \cdot}(A)x = R(\lambda, A)x$, and observing that $\psi_\tau(|\mu|^{-1/\alpha} A)$ is well defined and bounded by the Dunford calculus of A.

Step 2 – Now let $0 < \alpha < \pi/\omega(A)$. We will prove that the operator A^α is sectorial, with $\omega(A^\alpha) \leqslant \alpha\omega(A)$.

By Step 1, for $\tau > \alpha\omega(A)$ we have $\mu \in \varrho(A)$ if $|\arg \mu| \geqslant \tau$. Furthermore, for $\sigma \in (\omega(A), \tau/\alpha)$ have

$$\begin{aligned} \psi_\tau(|\mu|^{-1/\alpha} A) &= \frac{1}{2\pi i}\int_{\partial\Sigma_\sigma} \psi_\tau(|\mu|^{-1/\alpha} z)\frac{\mathrm{d}z}{z} \\ &= \frac{1}{2\pi i}\int_{\partial\Sigma_\sigma} \psi_\tau(z)\frac{\mathrm{d}z}{z}. \end{aligned}$$

Hence we may estimate

$$\|\psi_\tau(|\mu|^{-1/\alpha} A)\| \leqslant \frac{M_{\sigma,A}}{2\pi}\int_{\partial\Sigma_\sigma} |\psi_\tau(z)|\frac{|\mathrm{d}z|}{|z|}.$$

Therefore by (15.7) the sectoriality of A implies the sectoriality of A^α with $\omega(A^\alpha) \leqslant \alpha\omega(A)$.

Step 3 – Having proved that A^α is sectorial, the identity $(A^\alpha)^\beta = A^{\alpha\beta}$ follows from the composition rule of Theorem 15.1.15.

Since $\pi/\omega(A) > 1$ we have $0 < 1/\alpha < \pi/(\alpha\omega(A)) \leqslant \pi/\omega(A^\alpha)$. Hence we may apply the inequality of the angles of sectoriality of Step 2 to A^α to obtain $\omega(A) = \omega((A^\alpha)^{1/\alpha}) \leqslant (1/\alpha)\omega(A^\alpha)$, the equality $A = (A^\alpha)^{1/\alpha}$ being a consequence what we just proved. In combination with Step 2, this proves the equality $\omega(A^\alpha) = \alpha\omega(A)$.

Step 4 – Using Proposition 10.3.2, the final assertion is proved in the same way. □

The next theorem shows that $\alpha \mapsto \|A^\alpha x\|$ satisfies a useful log–convexity property.

Theorem 15.2.8 (Interpolation estimate). *Let A be a sectorial operator on a Banach space X. Let $\alpha, \beta, \gamma \in \mathbb{C}$ satisfy*

$$0 < \Re\alpha < \Re\gamma < \Re\beta \quad \textit{or} \quad 0 = \alpha < \Re\gamma < \Re\beta$$

and let $\theta \in (0,1)$ be such that $\Re\gamma = (1-\theta)\Re\alpha + \theta\Re\beta$. Then

$$\mathsf{D}(A^\alpha) \cap \mathsf{D}(A^\beta) \subseteq \mathsf{D}(A^\gamma),$$

and for all $x \in \mathsf{D}(A^\alpha) \cap \mathsf{D}(A^\beta)$ and $\omega(A) < \sigma < \pi$ we have

$$\|A^\gamma x\| \leqslant \frac{C}{\theta(1-\theta)} \|A^\alpha x\|^{1-\theta} \|A^\beta x\|^\theta,$$

where C is a constant depending only on $\Re\beta - \Re\alpha$, σ, and A.

Proof. Let m be the smallest integer strictly greater than $\Re\beta - \Re\alpha$. We will use the auxiliary function $\psi(z) = cz^m(1+z)^{-2m}$, where c is chosen so that $\int_0^\infty \psi(s)\,\frac{\mathrm{d}s}{s} = 1$. Then the functions

$$g(z) := \int_0^1 \psi(sz)\,\frac{\mathrm{d}s}{s} \quad \text{and} \qquad h(z) := \int_1^\infty \psi(sz)\,\frac{\mathrm{d}s}{s}$$

are well defined for all $z \in \mathbb{C}$ and satisfy

$$g(z) + h(z) = \int_0^\infty \psi(sz)\,\frac{\mathrm{d}s}{s} = \int_0^\infty \psi(s)\,\frac{\mathrm{d}s}{s} = 1.$$

We claim that $g, h \in E(\Sigma_\sigma)$. Indeed, we have

$$\begin{aligned} |g(z)| &\leqslant \int_0^1 |\psi(sz)|\,\frac{\mathrm{d}s}{s} \leqslant C_{\sigma,m}|z|^m \int_0^1 s^m\,\frac{\mathrm{d}s}{s} = \frac{|z|^m}{m}, \\ |h(z)| &\leqslant \int_1^\infty |\psi(sz)|\,\frac{\mathrm{d}s}{s} \leqslant C_{\sigma,m}|z|^{-m} \int_0^1 s^{-m}\,\frac{\mathrm{d}s}{s} = \frac{|z|^{-m}}{m}. \end{aligned}$$

It follows that g an h have integrable limits 0 at 0 and ∞ in the sense of Lemma 15.1.2, respectively. From $g = 1 - h$ and $h = 1 - g$ we see that g an h have integrable limits 1 at ∞ and 0, respectively. Therefore Lemma 15.1.2 implies the claim.

For all $t > 0$, it follows from the claim that

$$g(tA) + h(tA) = I \tag{15.8}$$

in the primary calculus of the sectorial operator tA.

Now let $x \in \mathsf{D}(A^\alpha) \cap \mathsf{D}(A^\beta)$. Then $x \in \mathsf{D}(A^\gamma)$ and (15.8) implies

$$A^\gamma x = g(tA)A^\gamma x + h(tA)A^\gamma x. \tag{15.9}$$

The functions $\widetilde{g}(z) = z^{\gamma-\beta}g(z)$ and $\widetilde{h}(z) = z^{\gamma-\alpha}h(z)$ belong to $E(\Sigma_\sigma)$; this follows from the choice of m and redoing the above computation for $\widetilde{g}$ and $\widetilde{h}$. We have

$$g(tA)A^\gamma x = \widetilde{g}(tA)(tA)^{\beta-\gamma}A^\gamma x = t^{\beta-\gamma}\widetilde{g}(tA)A^\beta x.$$

Here, the first identity can be justified by viewing $\widetilde{g}(t\,\cdot)$ as a regulariser for $(z^{\beta-\gamma}, tA)$ and noting that $A^\gamma x \in \mathsf{D}(A^{\beta-\gamma})$; the second identity follows by first applying Proposition 15.2.6 and then Theorem 15.2.5. Similarly we have

$$h(tA)A^\gamma x = t^{\alpha-\gamma}\widetilde{h}(tA)A^\alpha x.$$

From (15.9) it now follows that

$$A^\gamma x = t^{\beta-\gamma}\widetilde{g}(tA)A^\beta x + t^{\alpha-\gamma}\widetilde{h}(tA)A^\alpha x.$$

Therefore,

$$\begin{aligned}\|A^\gamma x\| &\leqslant t^{\Re\beta-\Re\gamma}\|\widetilde{g}(tA)\|\|A^\beta x\| + t^{\Re\alpha-\Re\gamma}\|\widetilde{h}(tA)\|\|A^\alpha x\| \\ &\leqslant C\big(t^{\Re\beta-\Re\gamma}\|A^\beta x\| + t^{\Re\alpha-\Re\gamma}\|\|A^\alpha x\|\big),\end{aligned}$$

where the constant C only depends on $\Re\beta - \Re\alpha$, σ, and A; we used that from the definition of the primary calculus for it follows that $\sup_{t>0}\|f(tA)\| \leqslant C < \infty$ for $f \in \{\widetilde{g}, \widetilde{h}\}$, using by (10.9) and the sectoriality of A.

Optimising the choice of $t > 0$, we arrive at the estimate

$$\|A^\gamma x\| \leqslant C\left[\Big(\frac{\theta}{1-\theta}\Big)^{1-\theta} + \Big(\frac{1-\theta}{\theta}\Big)^{\theta}\right]\|A^\alpha x\|^{1-\theta}\|A^\beta x\|^\theta.$$

Since the term in the square brackets is bounded above by $1/(\theta(1-\theta))$, this gives the second estimate. $\square$

Remark 15.2.9. It is tempting to believe that

$$g(A)x = \int_0^1 \psi(sA)x\,\frac{\mathrm{d}s}{s} \quad \text{and} \qquad h(A)x = \int_1^\infty \psi(sA)x\,\frac{\mathrm{d}s}{s},$$

but these integrals may fail to converges at 0 (the first) and ∞ (the second). Calderón's reproducing formula (Proposition 10.2.5) guarantees their convergence (as improper integrals) for elements $x \in \overline{\mathsf{D}(A) \cap \mathsf{R}(A)}$ if $z \mapsto \psi(z)$ belongs to $H^1(\Sigma_\sigma)$, and for $x \in \overline{\mathsf{D}(A) \cap \mathsf{R}(A)}$ if $z \mapsto \psi(z)\log z$ belongs to $\in H^1(\Sigma_\sigma)$. The above proof does not depend on these matters; all we needed there were bounds on the operators $g(A)$ and $h(A)$ that follow directly from the definitions of these operators through the extended Dunford calculus.

Corollary 15.2.10. *Let A be a sectorial operator on a Banach space X with $0 \in \varrho(A)$. Then for all $\Re\alpha > 0$ the operator $A^{-\alpha}$ is bounded. Moreover, for $0 < \Re\alpha < n$ we have*

$$\|A^{-\alpha}\| \leqslant \frac{CM_{\sigma,A}}{\frac{\Re\alpha}{n}(1-\frac{\Re\alpha}{n})}\|A^{-1}\|^{\Re\alpha},$$

where C is a universal constant.

Proof. Let $0 < \Re\alpha < n$. By Theorem 15.2.5 and 15.2.8, applied with $\theta = 1 - \Re\alpha/n$, for all $x \in X$ we have

$$\begin{aligned}\|A^{-\alpha}x\| = \|A^{n-\alpha}(A^{-n}x)\| &\leqslant \frac{CM_{\sigma,A}}{\frac{\Re\alpha}{n}(1-\frac{\Re\alpha}{n})}\|A^{-n}x\|^{\Re\alpha/n}\|x\|^{1-\Re\alpha/n} \\ &\leqslant \frac{CM_{\sigma,A}}{\frac{\Re\alpha}{n}(1-\frac{\Re\alpha}{n})}\|A^{-1}\|^{\Re\alpha}\|x\|,\end{aligned}$$

where C is a universal constant. It follows that $A^{-\alpha}$ is bounded and satisfies the bound in the statement of the corollary. $\square$

Proposition 15.2.11. *Let A be a sectorial operator on a Banach space X; when considering A^α for $\Re\alpha \leqslant 0$ we assume A to be injective. If A has a bounded H^∞-calculus and $0 < |\alpha| < \pi/\omega_{H^\infty}(A)$, then A^α has a bounded H^∞-calculus and $\omega_{H^\infty}(A^\alpha) = \alpha\,\omega_{H^\infty}(A)$.*

Proof. This follows directly from the identity $f(A)x = g(A^\alpha)x$ for $x \in \mathsf{D}(A) \cap \mathsf{R}(A)$ and $f \in H^\infty(\Sigma_\sigma)$, with $g \in H^\infty(\Sigma_{|\alpha|\sigma})$ given by $f(z) = g(z^\alpha)$. $\square$

If A is sectorial, then $A+\varepsilon$ is sectorial and boundedly invertible. We conclude this section a some useful result that applies in this situation.

Proposition 15.2.12. *Let A be a sectorial operator on a Banach space X. If $\mathsf{D}(A) \cap \mathsf{R}(A)$ is dense in X, then for all $\alpha > 0$ and $\varepsilon > 0$ we have $\mathsf{D}(A^\alpha) = \mathsf{D}((\varepsilon + A)^\alpha)$ with equivalent graph norms.*

Proof. The result is clear for $\alpha = 1, 2, \dots$. Next let $\alpha \in (0,1)$. The functions

$$f(z) := \frac{(\varepsilon+z)^\alpha}{\varepsilon+z^\alpha} - 1, \quad g(z) = \frac{\varepsilon+z^\alpha}{(\varepsilon+z)^\alpha} - 1$$

belong to $H^1(\Sigma_\sigma)$ for all $0 < \sigma < \pi$. For $x \in \mathsf{D}(A^k) \cap \mathsf{R}(A^k)$ with k large enough, Proposition 15.1.12 gives

$$f(A)x = (\varepsilon + A)^\alpha(\varepsilon + A^\alpha)^{-1}x - x, \quad g(A)x = (\varepsilon + A^\alpha)(\varepsilon + A)^{-\alpha})x - x.$$

Since $f(A)$ and $g(A)$ are bounded, these identities imply $\mathsf{D}(A^\alpha) = \mathsf{D}((\varepsilon+A)^\alpha)$. The equivalence of the norms follows from the open mapping theorem.

If $\beta = \alpha + n$ with $n \in \mathbb{N}$ and $\alpha \in (0, 1)$ then $\mathsf{D}((\varepsilon + A)^\beta) \subseteq \mathsf{D}((\varepsilon + A)^n)$ by Theorem 15.2.5. Thus we obtain

$$\begin{aligned}\mathsf{D}((\varepsilon + A)^\beta) &= \mathsf{D}((\varepsilon + A)^n(\varepsilon + A)^\alpha) \\ &= \{x \in \mathsf{D}((\varepsilon + A)^n) : (\varepsilon + A)^\alpha x \in \mathsf{D}((\varepsilon + A)^n)\} \\ &= \{x \in \mathsf{D}(A^n) : (\varepsilon + A)^\alpha x \in \mathsf{D}(A^n)\} \\ &= \{x \in \mathsf{D}(A^n) : A^\alpha x \in \mathsf{D}(A^n)\} \\ &= \mathsf{D}(A^n A^\alpha) = \mathsf{D}(A^\beta).\end{aligned}$$

Equivalence of norms now follows easily. □

15.2.b Representation formulas

The aim of this section is to prove various integral representations for the fractional powers of sectorial operators.

Theorem 15.2.13 (Balakrishnan). *Let A be a sectorial operator on a Banach space X and let $\omega(A) < \sigma < \pi$. For all $0 < \Re\alpha < 1$ and $x \in \mathsf{D}(A)$ we have*

$$A^\alpha x = \frac{1}{2\pi i}\int_{\partial\Sigma_\sigma} z^{\alpha-1}R(z, A)Ax\,\mathrm{d}z = \frac{\sin\pi\alpha}{\pi}\int_0^\infty t^{\alpha-1}(t + A)^{-1}Ax\,\mathrm{d}t.$$

If in addition A is densely defined and $\omega(A) < \frac{1}{2}\pi$, then for all $x \in \mathsf{D}(A)$ we have

$$A^\alpha x = \frac{1}{\Gamma(1-\alpha)}\int_0^\infty s^{-\alpha}S(t)Ax\,\mathrm{d}t,$$

where $(S(t))_{t\geqslant 0}$ is the bounded analytic C_0-semigroup generated by $-A$.

Note that $\lim_{z\downarrow 0} R(z, A)Ax = 0$ for $x \in \mathsf{D}(A)$ by Proposition 10.1.7, so the first integral is absolutely convergent. By the same reasoning the second integral is absolutely convergent. The absolute convergence of the third integral follows near $t = 0$ from the fact that $x \in \mathsf{D}(A)$, and near $t = \infty$ from the bound $\|S(t)Ax\| \leqslant Ct^{-1}\|x\|$ (see Theorem G.5.3).

Integrating by parts and using with the identity $-\alpha\Gamma(-\alpha) = \Gamma(1 - \alpha)$, the third identity in Balakrishnan's theorem may equivalently be presented as

$$A^\alpha x = \frac{1}{\Gamma(-\alpha)} \int_0^\infty t^{-\alpha-1}(S(t)x - x)\,\mathrm{d}t, \quad x \in \mathsf{D}(A).$$

The absolute convergence of this integral follows from the bound $\|S(t)x - x\| = O(t)$ as $t \downarrow 0$ for $x \in \mathsf{D}(A)$.

Proof. For all $\varepsilon > 0$ the function $z \mapsto \frac{z^\alpha}{z+\varepsilon}$ belongs to $H^1(\Sigma_\sigma) \cap H^\infty(\Sigma_\sigma)$ and therefore the operator $(\frac{z^\alpha}{z+\varepsilon})(A)$ can be defined by the Dunford calculus and is bounded. Fix $x \in \mathsf{D}(A)$. Then $x \in \mathsf{D}(A^\alpha)$, and therefore by multiplicativity of the extended Dunford calculus (Proposition 15.1.12),

$$A^\alpha x = \Big(\frac{z^\alpha}{z+\varepsilon}\Big)(A)(\varepsilon + A)x.$$

Similarly,

$$\Big(\frac{z^\alpha}{z+\varepsilon}\Big)(A)x = \Big(\frac{z^\alpha}{(z+\varepsilon)(z+1)}\Big)(A)(I + A)x.$$

Combining these identities, we compute

$$\begin{aligned}
A^\alpha x &= \varepsilon\Big(\frac{z^\alpha}{(z+\varepsilon)(z+1)}\Big)(A)(I+A)x + \Big(\frac{z^\alpha}{z+\varepsilon}\Big)(A)Ax \\
&= \frac{\varepsilon}{2\pi i}\int_{\partial\Sigma_\sigma} \frac{z^\alpha}{(z+\varepsilon)(z+1)} R(z,A)(I+A)x\,\mathrm{d}z \\
&\qquad + \frac{1}{2\pi i}\int_{\partial\Sigma_\sigma} \frac{z^\alpha}{z+\varepsilon} R(z,A)Ax\,\mathrm{d}z \\
&= (I) + (II).
\end{aligned}$$

Noting that $z \mapsto z^{\alpha-1}R(z,A)Ax$ is integrable along $\partial\Sigma_\sigma$, the term (I) tends to 0 as $\varepsilon \downarrow 0$ by dominated convergence. Also,

$$(II) = \frac{1}{2\pi i}\int_{\partial\Sigma_\sigma} \frac{z}{z+\varepsilon} z^{\alpha-1}R(z,A)Ax\,\mathrm{d}z \to \frac{1}{2\pi i}\int_{\partial\Sigma_\sigma} z^{\alpha-1}R(z,A)Ax\,\mathrm{d}z$$

as $\varepsilon \downarrow 0$ by dominated convergence. This proves the first identity.

Turning to the second identity, write $\partial\Sigma_\sigma = \Gamma_\sigma \cup \Gamma_{-\sigma}$ where $\Gamma_{\pm\sigma} = \{re^{\pm i\sigma} \in \mathbb{C} : r > 0\}$. It follows from Cauchy's theorem that

$$\begin{aligned}
A^\alpha x &= \frac{1}{2\pi i}\int_{\partial\Sigma_\sigma} z^{\alpha-1}R(z,A)Ax\,\mathrm{d}z \\
&= \frac{1}{2\pi i}\int_{\Gamma_\sigma} z^{\alpha-1}R(z,A)Ax\,\mathrm{d}z \\
&\qquad + \frac{1}{2\pi i}\int_{\Gamma_{-\sigma}} z^{\alpha-1}R(z,A)Ax\,\mathrm{d}z \\
&= -\frac{1}{2\pi i}\int_0^\infty (re^{i\sigma})^{\alpha-1}R(re^{i\sigma},A)Ax\,e^{i\sigma}\,\mathrm{d}r
\end{aligned}$$

$$
\begin{aligned}
&+ \frac{1}{2\pi i}\int_0^\infty (re^{-i\sigma})^{\alpha-1} R(re^{-i\sigma}, A)Ax\, e^{-i\sigma}\,\mathrm{d}r \\
&\to \frac{1}{2\pi i}\int_0^\infty r^{\alpha-1}(e^{-i\pi(\alpha-1)} - e^{i\pi(\alpha-1)})R(-r,A)Ax\,\mathrm{d}r \quad (\text{as } \sigma\to\pi) \\
&= \frac{\sin\alpha\pi}{\pi}\int_0^\infty r^{\alpha-1}(r+A)^{-1}Ax\,\mathrm{d}r.
\end{aligned}
$$

The minus sign in the third identity comes from the fact that Γ_σ is downwards oriented. The convergence is a consequence of the dominated convergence theorem.

To prove the third formula we use the identity just proved together with the Laplace transform representation of the resolvent (Proposition G.4.1) to get

$$
\begin{aligned}
A^\alpha x &= \frac{\sin\pi\alpha}{\pi}\int_0^\infty t^{\alpha-1}\int_0^\infty e^{-ts}S(s)Ax\,\mathrm{d}s\,\mathrm{d}t \\
&= \frac{\sin\pi\alpha}{\pi}\int_0^\infty \Big(\int_0^\infty t^{\alpha-1}e^{-ts}\,\mathrm{d}t\Big)S(s)Ax\,\mathrm{d}s \\
&= \frac{1}{\Gamma(1-\alpha)}\int_0^\infty s^{-\alpha}S(s)Ax\,\mathrm{d}s,
\end{aligned}
$$

where we used the identity $\frac{\sin\pi\alpha}{\pi} = \frac{1}{\Gamma(1-\alpha)\Gamma(\alpha)}$. □

From this theorem it is rather easy to re-deduce a special case of Theorem 15.2.8 as follows. Let $0<\alpha<1$. Let $M\geqslant 0$ be such that $\|(t+A)^{-1}\|\leqslant M/t$ for all $t>0$. By Theorem 15.2.13, for all $x\in\mathsf{D}(A)$ we have

$$
\begin{aligned}
\|A^\alpha x\| &\leqslant \Big|\frac{\sin\pi\alpha}{\pi}\Big|\int_0^\infty t^{\alpha-1}\|(t+A)^{-1}Ax\|\,\mathrm{d}t \\
&\leqslant \Big|\frac{\sin\pi\alpha}{\pi}\Big|\int_0^\rho t^{\alpha-1}\|(t+A)^{-1}A\|\|x\|\,\mathrm{d}t \\
&\qquad + \Big|\frac{\sin\pi\alpha}{\pi}\Big|\int_\rho^\infty t^{\alpha-1}\|(t+A)^{-1}\|\|Ax\|\,\mathrm{d}t \\
&\leqslant \Big|\frac{\sin\pi\alpha}{\pi\alpha}\Big|(1+M)\rho^\alpha\|x\| + \frac{\sin\pi\alpha}{\pi(1-\alpha)}\Big|M\rho^{\alpha-1}\|Ax\|
\end{aligned}
$$

with absolute convergence of all integrals. Up to this point we have assumed that $x\in\mathsf{D}(A)$. The estimate extends to general $x\in\mathsf{D}(A^\alpha)$ by approximation as in the proof of that theorem. The estimate of Theorem 15.2.8 is obtained by optimising over ρ as in the proof of the theorem.

Corollary 15.2.14. *Let A be a sectorial operator on a Banach space X and let $\omega(A)<\sigma<\pi$. Let $0<\alpha<1$ and $\lambda\in\complement\overline{\Sigma_\sigma}$.*

(1) *The operator $A^\alpha R(\lambda,A)$ is bounded and*

$$\|A^\alpha R(\lambda, A)\| \leqslant C_\alpha M_{\sigma,A}(M_{\sigma,A}+1)|\lambda|^{\alpha-1},$$

where $C_\alpha = \frac{\sin(\pi\alpha)}{\pi\alpha}\frac{1}{1-\alpha}$.

(2) *If, in addition, A is densely defined and $\omega(A) < \frac{1}{2}\pi$, and $(S(t))_{t\geqslant 0}$ denotes the bounded analytic C_0-semigroup generated by $-A$, then for all $t > 0$ the operator $A^\alpha S(t)$ is bounded and*

$$\|A^\alpha S(t)\| \leqslant C_\alpha M_A t^{-\alpha},$$

where $C_\alpha = \frac{1}{\Gamma(1-\alpha)}\int_0^\infty \tau^{-\alpha}(1+\tau)^{-1}\|x\|\,\mathrm{d}\tau$ *and* $M_A = \sup_{t>0} t\|AS(t)\|$.

From Theorem G.5.3 we recall that $\sup_{t>0} t\|AS(t)\| < \infty$.

Proof. For the first assertion, fix $\lambda \in \complement\overline{\Sigma_\sigma}$. The boundedness of $A^\alpha R(\lambda, A)$ is evident from the inclusion $\mathsf{D}(A) \subseteq \mathsf{D}(A^\alpha)$. For all $x \in \mathsf{D}(A)$, by Theorem 15.2.13 we have

$$A^\alpha R(\lambda, A)x = \frac{\sin(\pi\alpha)}{\pi}\int_0^\infty t^{\alpha-1}(t+A)^{-1}R(\lambda, A)Ax\,\mathrm{d}t.$$

We split the integral on the right into two parts and estimate them separately. First, writing $A = (A+t) - t$,

$$\begin{aligned}\Big\|\int_0^{|\lambda|} t^{\alpha-1}(t+A)^{-1}R(\lambda, A)Ax\,\mathrm{d}t\Big\| &\leqslant \int_0^{|\lambda|} t^{\alpha-1}\|[I - t(t+A)^{-1}]R(\lambda, A)x\|\,\mathrm{d}t\\ &\leqslant |\lambda|^{-1}\int_0^{|\lambda|} t^{\alpha-1}(1+M)\|\lambda R(\lambda, A)x\|\,\mathrm{d}t\\ &\leqslant \frac{M(M+1)}{\alpha}|\lambda|^{\alpha-1}\|x\|.\end{aligned}$$

Similarly, but now writing $A = (A-\lambda)+\lambda$,

$$\begin{aligned}\Big\|\int_{|\lambda|}^\infty t^{\alpha-1}(t+A)^{-1}R(\lambda, A)Ax\,\mathrm{d}t\Big\| &\leqslant (1+M)\|x\|\int_{|\lambda|}^\infty t^{\alpha-2}\|t(t+A)^{-1}\|\,\mathrm{d}t\\ &\leqslant \frac{M(M+1)}{1-\alpha}|\lambda|^{\alpha-1}\|x\|.\end{aligned}$$

Turning to the second assertion, by analyticity the operators $S(t)$ map X into $\mathsf{D}(A)$ and $\sup_{t>0} t\|AS(t)\| < \infty$. The boundedness of the operators $A^\alpha S(t)$ follows from the boundedness of $AS(t)$ and the inclusion $\mathsf{D}(A) \subseteq \mathsf{D}(A^\alpha)$. To prove the estimate, note that for all $x \in X$ we have

$$A^\alpha S(t)x = \frac{1}{\Gamma(1-\alpha)}\int_0^\infty s^{-\alpha}AS(t+s)x\,\mathrm{d}s,$$

so, for $t > 0$,

$$\begin{aligned}\|A^\alpha S(t)x\| &\leqslant \frac{C}{\Gamma(1-\alpha)}\int_0^\infty s^{-\alpha}(t+s)^{-1}\|x\|\,\mathrm{d}s\\ &= \frac{Ct^{-\alpha}}{\Gamma(1-\alpha)}\int_0^\infty \tau^{-\alpha}(1+\tau)^{-1}\|x\|\,\mathrm{d}\tau.\end{aligned}$$

□

As a corollary to Theorem 15.2.13 we have the following representation formula for the negative fractional powers of A.

Corollary 15.2.15. *Let A be an injective sectorial operator on a Banach space X and let $\omega(A) < \sigma < \pi$. For all $0 < \Re\alpha < 1$ and $x \in \mathsf{R}(A)$ we have*

$$A^{-\alpha}x = \frac{1}{2\pi i}\int_{\partial\Sigma_\sigma} z^{-\alpha}R(z,A)x\,\mathrm{d}z = \frac{\sin\pi\alpha}{\pi}\int_0^\infty t^{-\alpha}(t+A)^{-1}x\,\mathrm{d}t.$$

If, in addition, A is densely defined and $\omega(A) < \frac{1}{2}\pi$, and if $(S(t))_{t\geqslant 0}$ denotes the bounded analytic C_0-semigroup generated by $-A$, then for all $x \in \mathsf{R}(A)$ we have

$$A^{-\alpha}x = \frac{1}{\Gamma(\alpha)}\int_0^\infty t^{-\alpha}S(t)x\,\mathrm{d}t.$$

Note that if $x = Ay$ with $y \in \mathsf{D}(A)$, then $R(z,A)x = -y + zR(z,A)y$, so the first integral is absolutely convergent. In the same way it is checked that the second integral is absolutely convergent. From $\|S(t)x\| = \|AS(t)y\| = O(1/t)$ as $t \to \infty$ (by Theorem G.5.3) we see that the third integral is absolutely convergent.

Proof. Writing $x = Ay$ with $y \in \mathsf{D}(A)$ we have $A^{-\alpha}x = A^{1-\alpha}y$, and Theorem 15.2.13 gives

$$A^{-\alpha}x = A^{1-\alpha}y = \frac{1}{2\pi i}\int_{\partial\Sigma_\sigma} z^{-\alpha}R(z,A)Ay\,\mathrm{d}z = \frac{1}{2\pi i}\int_{\partial\Sigma_\sigma} z^{-\alpha}R(z,A)x\,\mathrm{d}z.$$

The second identity is proved in the same way. The third follows from the second by following the lines of the proof of Theorem 15.2.13. □

When A boundedly invertible, the identities in the corollary hold for arbitrary $x \in X$. If in addition A is densely defined, the result extends to arbitrary $\Re\alpha > 0$ as follows:

Theorem 15.2.16. *Let A be a densely defined sectorial operator on a Banach space X with $0 \in \varrho(A)$, and let $\omega(A) < \sigma < \pi$. Then for all $\Re\alpha > 0$ we have*

$$A^{-\alpha}x = \frac{1}{2\pi i}\int_{\partial(\Sigma_\sigma\setminus B_\varepsilon)} z^{-\alpha}R(z,A)x\,\mathrm{d}z, \qquad x \in X,$$

with $B_\varepsilon := \{z \in \mathbb{C} : |z| < \varepsilon\}$, where $\varepsilon > 0$ is so small that $B_\varepsilon \subseteq \varrho(A)$.

Proof. First let $x \in \mathsf{D}(A^k)$ with $k > \Re\alpha$ and set $y = \zeta(A)^{-k}x = (I + A)^{2k}A^{-k}x$, where $\zeta(z) = z/(z+1)^2$. The integral

$$T_\alpha x := \frac{1}{2\pi i}\int_{\partial(\Sigma_\sigma\setminus B_\varepsilon)} z^{-\alpha}R(z,A)x\,\mathrm{d}z$$

is absolutely convergent and defines a bounded operator T_α. We may now repeat the proof of the multiplicativity of the Dunford calculus (Theorem 10.2.2) to obtain, with $\omega(A) < \nu < \sigma$,

$$\begin{aligned}T_\alpha x = (T_\alpha \circ \zeta^k(A))y &= T_\alpha \circ \frac{1}{2\pi i}\int_{\partial(\Sigma_\nu\setminus B_{\varepsilon/2})} \zeta(z)^k R(z,A)y\,\mathrm{d}z\\ &= \frac{1}{2\pi i}\int_{\partial(\Sigma_\nu\setminus B_{\varepsilon/2})} z^{-\alpha}\zeta(z)^k R(z,A)y\,\mathrm{d}z\\ &= \frac{1}{2\pi i}\int_{\partial\Sigma_\nu} z^{-\alpha}\zeta(z)^k R(z,A)y\,\mathrm{d}z.\end{aligned}$$

In the last step, the assumption $k > |\Re\lambda|$ was used to justify the change of contour by Cauchy's theorem. By the definition of $A^{-\alpha}x$ via the extended Dunford calculus, the right hand side equals $A^{-\alpha}x$. This proves the first identity for $x \in \mathsf{D}(A^k)$. Using the second part of Proposition 15.1.13, the general case follows from it by approximation, noting that T_α is a bounded operator on X. □

Theorem 15.2.17. *Let $-A$ be the generator of a bounded C_0-semigroup $(S(t))_{t\geqslant 0}$ on X. Then A is densely defined and sectorial of angle $\omega(A) \leqslant \frac{1}{2}\pi$, for all $0 < \alpha < 1$ the operator A^α is densely defined and sectorial of angle $\omega(A^\alpha) \leqslant \frac{1}{2}\pi\alpha$, and the bounded analytic C_0-semigroup generated by $-A^\alpha$ is given by*

$$S_\alpha(t)x = \int_0^\infty f_{\alpha,t}(s)S(s)x\,\mathrm{d}s, \qquad t > 0,\ x \in X,$$

where, for $t > 0$,

$$f_{\alpha,t}(s) := \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{sz - tz^\alpha}\,\mathrm{d}z, \qquad s > 0,$$

is a non-negative function which is independent of $c > 0$ and satisfies

$$\int_0^\infty f_{\alpha,t}(s)\,\mathrm{d}s = 1.$$

Proof. By generalities from semigroup theorem (see Section G.2), the assumptions imply that A is densely defined and sectorial with $\omega(A) \leqslant \frac{1}{2}\pi$. By Proposition 15.2.7, A^α is densely defined and sectorial of angle $\frac{1}{2}\pi\alpha$ and consequently $-A^\alpha$ generates a bounded analytic C_0-semigroup by Theorem G.5.2.

By Example 15.1.6 we furthermore have $S_\alpha(t) = e_t(A^\alpha)$, where $e_t(z) = e^{-tz}$. Hence by the composition rule of Theorem 15.1.15 we have

$$S_\alpha(t) = g_{\alpha,t}(A),$$

where $g_{\alpha,t}(z) = e^{-tz^\alpha}$.

Let $\frac{1}{2}\pi < \nu < \sigma < \min\{\frac{1}{2}\pi/\alpha, \pi\}$. By the Phillips calculus (Proposition 10.7.2(2)),

$$S_\alpha(t)(A)x = \int_0^\infty f_{\alpha,t}(s)S(s)x \,\mathrm{d}s, \qquad x \in X,$$

where $f_{\alpha,t} \in L^1(\mathbb{R}_+)$ is given (with $B_\varepsilon = \{z \in \mathbb{C} : |z| = \varepsilon\}$) by

$$\begin{aligned} f_{\alpha,t}(s) &= -\frac{1}{2\pi i}\int_{\partial(\Sigma_\nu \setminus B_\varepsilon)} e^{sz-tz^\alpha}\,\mathrm{d}z \\ &= -\frac{1}{2\pi i}\int_{\partial\Sigma_\nu} e^{sz-tz^\alpha}\,\mathrm{d}z = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{sz-tz^\alpha}\,\mathrm{d}z \end{aligned}$$

for $c > 0$. The second and third identity follow from Cauchy's formula, the use of which is justified by noting that for $z = re^{i\sigma u}$ with $u \geqslant 0$ we have

$$\begin{aligned} |e^{sz-tz^\alpha}| &= \exp(sr\cos\sigma - t\Re e^{\alpha(\ln r + i\sigma)}) \\ &= \exp(sr\cos\sigma - tr^\alpha\cos(\alpha\sigma)), \end{aligned}$$

from which it follows that $z \mapsto e^{sz-tz^\alpha}$ is integrable along $\partial\Sigma_\nu$. In its stated form, Proposition 10.7.2(2) requires $g_{\alpha,t} = e^{-tz^\alpha}$ to be in $H^1(\Sigma_\sigma)$, which is not the case. The reader may check, however, that the proof still works in the present situation if we replace integration over $\partial\Sigma_\nu$ by integration over $\partial(\Sigma_\nu \setminus B_\varepsilon)$. For $\lambda > 0$ we have

$$\begin{aligned} \int_0^\infty e^{-\lambda s} f_{\alpha,t}(s)\,\mathrm{d}s &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\int_0^\infty e^{-\lambda s}e^{sz-tz^\alpha}\,\mathrm{d}s\,\mathrm{d}z \\ &= -\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{e^{-tz^\alpha}}{z-\lambda}\,\mathrm{d}z = e^{-t\lambda^\alpha}. \end{aligned} \tag{15.10}$$

Using the non-negativity of $f_{\alpha,t}$, upon passing to the limit $\lambda \downarrow 0$ gives $\int_0^\infty f_{\alpha,t}(s)\,\mathrm{d}s = 1$.

Finally, the fact that $f_{\alpha,t}$ is non-negative follows from (15.10), the fact that $\lambda \mapsto e^{-t\lambda^\alpha}$ is completely monotone and the Post–Widder real inversion theorem for the Laplace transform. We refer the reader to the Notes for further details. □

We finish with two examples.

Example 15.2.18 (Fractional derivatives). For $1 < p < \infty$, the operator $A = \mathrm{d}/\mathrm{d}t$ with domain $\mathsf{D}(A) = \{f \in W^{1,p}(0,T;X) : f(0) = 0\}$ is sectorial on $L^p(0,T;X)$ of angle $\frac{1}{2}\pi$ and for all $\Re\alpha > 0$ and $f \in L^p(0,T;X)$ we have

$$A^{-\alpha}f(x) = \frac{1}{\Gamma(\alpha)}\int_0^x (x-y)^{\alpha-1}f(y)\,\mathrm{d}y \quad \text{for almost all } x \in \mathbb{R}.$$

The operators $A^{-\alpha}$ are called the (Liouville) *fractional derivatives*. In particular,

$$A^{-1}f(x) = \int_0^x f(y)\,\mathrm{d}y$$

The operator $V := A^{-1}$ is called the *Volterra operator*. These formulas are special cases of Theorem 15.2.16 once we note that $-A$ is the generator of the C_0-semigroup on $L^p(0,T;X)$ given by

$$S(t)f(s) = \begin{cases} f(s-t), & s \in [0,T],\ s > t, \\ 0, & \text{otherwise}. \end{cases}$$

To see that the generator of this semigroup is indeed $-A$, let us denote the generator by B for the moment. It is clear that $Y := \{f \in C^1([0,T];X) : f(0) = 0\}$ is contained in $\mathsf{D}(B)$ and $Bf = -f' = -Af$ for all $f \in Y$. Since Y is also invariant under the semigroup, Y is dense in $\mathsf{D}(B)$ by Lemma G.2.4. But A is a closed operator and Y is also dense in $\mathsf{D}(A)$, and therefore $B = -A$ with equal domains.

Example 15.2.19 (Poisson semigroup). Let A be the Laplace operator on $L^p(\mathbb{R}^d;X)$, where $1 \leqslant p < \infty$ is fixed and X is a Banach space. This operator has been introduced in Section 5.5 by declaring

$$\begin{aligned} \mathsf{D}(A) &:= H^{2,p}(\mathbb{R}^d;X), \\ Af &:= \Delta f, \quad f \in \mathsf{D}(A), \end{aligned}$$

where $H^{2,p}(\mathbb{R}^d;X)$ is the Banach space of all $f \in L^p(\mathbb{R}^d;X)$ admitting a weak Laplacian Δf in $L^p(\mathbb{R}^d;X)$ (see (5.44)). As was noted in Lemma 5.5.5, $C_c^\infty(\mathbb{R}^d;X)$ is dense in $\mathsf{D}(A)$, and consequently A can be equivalently defined as the closure of the operator $f \mapsto \Delta f$ acting in $C_c^\infty(\mathbb{R}^d;X)$, where Δf is now defined in terms of the classical second order derivatives of f. For UMD spaces X and exponents $1 < p < \infty$, Proposition 5.5.4 shows that

$$H^{2,p}(\mathbb{R}^d;X) = W^{2,p}(\mathbb{R}^d;X),$$

and Theorem 5.6.11 establishes a Fourier analytic characterisation of these spaces as the Banach space of all tempered distributions $u \in \mathscr{S}'(\mathbb{R}^d;X)$ such that the tempered distribution

$$((1+4\pi^2|\cdot|^2)\widehat{u})^{\smile}$$

belongs to $L^p(\mathbb{R}^d;X)$.

Let us now return to the general situation where $1 \leqslant p < \infty$ and X is a general Banach space. From this point on, we will simply write Δ for the

Laplace operator in $L^p(\mathbb{R}^d;X)$. As was shown in Example G.5.6, $-\Delta$ is the generator of a C_0-semigroup of contractions $(H(t))_{t\geqslant 0}$ on $L^p(\mathbb{R}^d;X)$, the *heat semigroup*, given by $H(0)=I$ and

$$H(t)f := k_t * f, \quad t>0,$$

where $k_t(x)=(4\pi t)^{-d/2}e^{-|x|^2/(4t)}$ is the heat kernel. It was shown in the same example that this semigroup extends analytically to $\{z\in\mathbb{C}: \Re z>0\}$ by the formula

$$H(z)f = k_z * f, \quad \Re z>0,$$

and that this extension is uniformly bounded and strongly continuous on every sector Σ_ω with $0<\omega<\frac12\pi$. As a consequence, $-\Delta$ is a densely defined sectorial operator of angle $\omega(\Delta)=0$.

By Theorem 15.2.17, the operator $(-\Delta)^{1/2}$ is densely defined and sectorial of angle 0 and generates a bounded analytic C_0-semigroup $(P(z))_{z\in\Sigma_\omega}$ for every $0<\omega<\frac12\pi$ on $L^p(\mathbb{R}^d;X)$, the so-called *Poisson semigroup*. By Theorem 15.1.7, in the primary calculus of $(-\Delta)^{1/2}$ this semigroup is given by

$$P(z)f = \exp(-z\Delta^{1/2}), \quad z\in\Sigma_\omega,\ f\in L^p(\mathbb{R}^d;X).$$

An explicit representation is obtained from Theorem 15.2.17, from which it follows that

$$P(t)f = \int_0^\infty k_t(s)H(s)x\,\mathrm{d}s, \qquad t>0,\ f\in L^p(\mathbb{R}^d;X),$$

where, for $t>0$,

$$k_t(s) := \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{sz-tz^{1/2}}\,\mathrm{d}z, \qquad s>0.$$

is a non-negative function which is independent of $c>0$ and satisfies

$$\int_0^\infty f_{\alpha,t}(s)\,\mathrm{d}s = 1.$$

We wish to prove here that

$$P(t)f = p_t * f, \quad t\geqslant 0,$$

where

$$p_t(x) = \frac{\Gamma(\frac12(d+1))}{\pi^{\frac12(d+1)}}\frac{t}{(t^2+|x|^2)^{\frac12(d+1)}}$$

is the *Poisson kernel*. For $d=1$ it takes the simpler form

$$p_t(x) = \frac{1}{\pi}\frac{t}{t^2+x^2}.$$

By Theorem 15.2.5 we have $((-\Delta)^{1/2})^2 f = -\Delta f$ for $f \in \mathsf{D}(\Delta)$ and therefore, by the composition rule of Theorem 15.1.15,

$$\exp(t(-\Delta)^{1/2})f = \phi_t(\Delta)f, \quad f \in \mathsf{D}(\Delta),$$

with $\phi_t(z) = e^{-tz^{1/2}}$. It follows from Proposition 15.1.13 that, for $f \in \mathsf{D}(\Delta)$,

$$P(t)f = \phi_t(\Delta)f = \lim_{n\to\infty} \phi_t(\Delta)\psi_n(\Delta)f,$$

where

$$\psi_n(z) = \frac{n}{n+z}, \quad n \geqslant 1.$$

The remainder of the proof will be devoted to proving the identity

$$\phi_t(\Delta)\phi_n(\Delta)f = p_t * \psi_n(\Delta)f. \tag{15.11}$$

These functions are regularisers for $(\exp(-t\,\cdot), \Delta)$. Once this has been shown the identity

$$P(t)f = p_t * f, \quad f \in L^p(\mathbb{R}^d; X),$$

follows from Proposition 10.1.7 by passing to the limit $n \to \infty$ in (15.11).

Fixing $f \in \mathsf{D}(\Delta) \cap \mathsf{R}(\Delta)$ and $t > 0$. Below we will show that

$$e^{-tz^{1/2}} = \int_0^\infty \frac{te^{-t^2/4s}}{2\pi^{1/2}s^{3/2}} e^{-zs}\,\mathrm{d}s. \tag{15.12}$$

Assuming this identity for the moment, by Fubini's theorem and Example 15.1.6 we have

$$\begin{aligned}
\phi_t(\Delta)\psi_n(\Delta)f &= (\phi_t\psi_n)(\Delta)f \\
&= \frac{1}{2\pi i}\int_{\partial\Sigma_\sigma} e^{-tz^{1/2}}\psi_n(z)R(z,\Delta)f\,\mathrm{d}z \\
&= \frac{1}{2\pi i}\int_{\partial\Sigma_\sigma}\int_0^\infty \frac{te^{-t^2/4s}}{2\pi^{1/2}s^{3/2}} e^{-zs}\psi_n(z)R(z,\Delta)f\,\mathrm{d}s\,\mathrm{d}z \\
&= \int_0^\infty \frac{te^{-t^2/4s}}{2\pi^{1/2}s^{3/2}} \frac{1}{2\pi i}\int_{\partial\Sigma_\sigma} e^{-zs}\psi_n(z)R(z,\Delta)f\,\mathrm{d}z\,\mathrm{d}s \\
&= \int_0^\infty \frac{te^{-t^2/4s}}{2\pi^{1/2}s^{3/2}} \exp(-s\Delta)\psi_n(\Delta)f\,\mathrm{d}s.
\end{aligned} \tag{15.13}$$

On the other hand,

$$\begin{aligned}
p_t(x) &\overset{(*)}{=} \frac{1}{(4\pi)^{\frac12(d+1)}}\frac{t}{(t^2+|x|^2)^{\frac12(d+1)}}\int_0^\infty s^{-\frac12(d+1)}e^{-1/4s}\,\frac{\mathrm{d}s}{s} \\
&= \frac{t}{(4\pi)^{\frac12(d+1)}}\int_0^\infty s^{-\frac12(d+3)}e^{-(t^2+|x|^2)/4s}\,\mathrm{d}s
\end{aligned}$$

$$= \int_0^\infty \frac{te^{-t^2/4s}}{2\pi^{1/2}s^{3/2}} k_t(x)\,\mathrm{d}s,$$

where $k_s(x) = (4\pi s)^{-d/2}e^{-|x|^2/4s}$ denotes the heat kernel associated with Δ and $(*)$ follows from

$$\int_0^\infty s^{-\frac{1}{2}(d+1)}e^{-1/4s}\,\frac{\mathrm{d}s}{s} = 4^{\frac{1}{2}(d+1)}\int_0^\infty u^{\frac{1}{2}(d+1)}e^{-u}\,\frac{\mathrm{d}u}{u} = 4^{\frac{1}{2}(d+1)}\Gamma(\frac{1}{2}(d+1)).$$

Now Fubini's theorem implies

$$\begin{aligned}
&p_t * \psi_n(\Delta)f \\
&= \int_{-\infty}^\infty \int_0^\infty \frac{te^{-t^2/4s}}{(4\pi)^{1/2}s^{3/2}} k_s(\cdot - y)\frac{1}{2\pi i}\int_{\partial\Sigma_\sigma} \psi_n(z)R(z,\Delta)f(y)\,\mathrm{d}s\,\mathrm{d}z\,\mathrm{d}y \\
&= \int_0^\infty \frac{te^{-t^2/4s}}{2\pi^{1/2}s^{3/2}}\,\frac{1}{2\pi i}\int_{\partial\Sigma_\sigma} \psi_n(z)\int_{-\infty}^\infty k_s(\cdot - y)R(z,\Delta)f(y)\,\mathrm{d}y\,\mathrm{d}z\,\mathrm{d}s \\
&= \int_0^\infty \frac{te^{-t^2/4s}}{2\pi^{1/2}s^{3/2}}\,\frac{1}{2\pi i}\int_{\partial\Sigma_\sigma} \psi_n(z)\exp(-s\Delta)R(z,\Delta)f\,\mathrm{d}z\,\mathrm{d}s \\
&= \int_0^\infty \frac{te^{-t^2/4s}}{2\pi^{1/2}s^{3/2}}\exp(-s\Delta)\psi_n(\Delta)f\,\mathrm{d}s.
\end{aligned} \tag{15.14}$$

The identity (15.11) is obtained by combining (15.13) and (15.14).

It remains to prove (15.12). First, the substitution $u = c/t$ gives

$$\int_0^\infty e^{-(\frac{c}{t}-t)^2}\,\mathrm{d}t = \int_0^\infty \frac{c}{u^2}e^{-(\frac{c}{u}-u)^2}\,\mathrm{d}u.$$

Renaming the second integration variable and adding the two formulas, the substitution $s = \frac{c}{u} - u$ gives

$$\int_0^\infty \frac{c}{u^2}e^{-(\frac{c}{u}-u)^2}\,\mathrm{d}u = \frac{1}{2}\int_0^\infty \Big(1+\frac{c}{u^2}\Big)e^{-(\frac{c}{u}-u)^2}\,\mathrm{d}u = \frac{1}{2}\int_{-\infty}^\infty e^{-s^2}\,\mathrm{d}s = \frac{1}{2}\pi^{1/2}.$$

We will apply this identity with $c = \frac{1}{2}tz^{1/2}$. Completing squares and changing variables twice, we obtain

$$\begin{aligned}
e^{tz^{1/2}}\int_0^\infty \frac{te^{-t^2/4s}}{s^{3/2}}e^{-zs}\,\mathrm{d}s &= \int_0^\infty \frac{t}{s}e^{-(t/2\sqrt{s}-z^{1/2}\sqrt{s})^2}\,\frac{\mathrm{d}s}{2\sqrt{s}} \\
&= \int_0^\infty \frac{t}{u^2}e^{-(t/2u-z^{1/2}u)^2}\,\mathrm{d}u = \pi^{1/2},
\end{aligned}$$

and this is the formula (15.12) we wanted to prove.

15.3 Bounded imaginary powers

A special role is played by sectorial operators whose purely imaginary fractional powers A^{it} are bounded. As their definition requires that $\mathsf{D}(A) \cap \mathsf{R}(A)$ be dense it will be convenient to introduce the following terminology.

Definition 15.3.1 (Standard sectorial operators). *A* standard sectorial operator *is a sectorial operator A with the property that $\mathsf{D}(A) \cap \mathsf{R}(A)$ is dense in X.*

The following proposition recalls some results proved in Proposition 10.1.8.

Proposition 15.3.2. *Let A be a sectorial operator on a Banach space X. Then:*

(1) *if A is standard, then A is injective;*
(2) *A is standard if and only if it is densely defined and has dense range;*
(3) *if X is reflexive, the following assertions are equivalent:*
  (i) *A is standard sectorial;*
  (ii) *A is injective;*
  (iii) *A has dense range.*

In view of (1), the fractional powers A^α of a standard sectorial operator A are well defined for all $\alpha \in \mathbb{C}$, and all results from the previous section are applicable to A.

In applications, standardness is hardly a restrictive assumption. In most situations the Banach space will be reflexive and even UMD, and in such spaces for a sectorial operator A we have the direct sum decomposition

$$X = \mathsf{N}(A) \oplus \overline{\mathsf{R}(A)}.$$

By (2), the part of A in $\overline{\mathsf{R}(A)}$ is standard sectorial, and the part of A in $\mathsf{N}(A)$ is identically zero.

Example 15.3.3 (Standardness of the Laplacian on $L^p(\mathbb{R}^d;X)$). Let us consider the Laplace operator Δ on $L^p(\mathbb{R}^d;X)$, where $1 < p < \infty$ and X is a UMD space, with domain $\mathsf{D}(\Delta) = H^{2,p}(\mathbb{R};X)$. It is shown in Example 10.1.5 that $-\Delta$ is sectorial of angle 0 on $L^p(\mathbb{R}^d;X)$ for all $1 \leqslant p < 1$, and standard sectorial if and only if $1 < p < \infty$.

For standard sectorial operators A on a Banach space X, the next diagram summarises the main results of this section.

$$
\begin{tikzcd}
\boxed{\begin{array}{l}\gamma\text{-bounded}\\ H^\infty\text{-calculus}\end{array}} \arrow[r, Rightarrow, "(1)"] \arrow[d, Rightarrow, "(2)"'] & \boxed{\begin{array}{l}\text{bounded}\\ H^\infty\text{-calculus}\end{array}} \arrow[d, Rightarrow, "(4)"] \\
\boxed{\begin{array}{l}\gamma\text{-BIP with}\\ \omega_{\gamma-\mathrm{BIP}}(A) < \pi\end{array}} \arrow[r, Rightarrow, "(5)"] \arrow[ur, Rightarrow, "(3)"] \arrow[d, Rightarrow, "(6)"'] & \boxed{\begin{array}{l}\text{BIP with}\\ \omega_{\mathrm{BIP}}(A) < \pi\end{array}} \arrow[d, Rightarrow, "(7)"] \\
\boxed{\gamma\text{-sectorial}} \arrow[r, Rightarrow, "(8)"] & \boxed{\begin{array}{l}\text{almost}\\ \gamma\text{-sectorial}\end{array}}
\end{tikzcd}
$$

The implications (1), (2), (4), (5), and (8) are trivial. The implication (3) follows from Theorem 15.3.21, where it is also shown that equivalence holds when X has Pisier's contraction property. The implications (1)–(5) are equivalence when X is a Hilbert space. The implication (6) follows from Theorem 15.3.19, and the implication (7) is Theorem 15.3.16.

15.3.a Definition and basic properties

For $t \in \mathbb{R}$ consider the function

$$f_t(z) := z^{it} := \exp(it \log z),$$

where we use the branch of the logarithm that is holomorphic on $\mathbb{C} \setminus (-\infty, 0]$. From $|f_t(z)| = \exp(-t \arg(z))$ it follows that $f_t \in H^\infty(\Sigma_\sigma)$ for each $0 < \sigma < \pi$ and

$$\|f_t\|_{H^\infty(\Sigma_\sigma)} \leqslant \exp(\sigma |t|).$$

Thus if A is a standard sectorial operator with a bounded $H^\infty(\Sigma_\sigma)$-calculus, the operators

$$A^{it} := f_t(A)$$

are well defined as bounded operators on X. Some examples of operators with bounded imaginary powers will be discussed in Subsection 15.3.h.

When A is merely standard sectorial, we may use the extended Dunford calculus to define the operators A^{it}, $t \in \mathbb{R}$, as closed and densely defined operators in X. This suggests the following definition.

Definition 15.3.4 (BIP). *A linear operator A acting in a Banach space X is said to have* bounded imaginary powers *(briefly, A has* bounded imaginary powers*) if A is standard sectorial and the operators A^{it} are bounded for all $t \in \mathbb{R}$.*

Examples of operators with bounded imaginary powers will be given in Section 15.3.h.

Proposition 15.3.5. *If A has bounded imaginary powers, then the family $(A^{it})_{t\in\mathbb{R}}$ is a C_0-group.*

Proof. It is evident from the definition through the extended Dunford calculus that $t\mapsto A^{it}x$ is strongly measurable for all $x\in X$. We have already seen that $A^{i0}x = \mathbf{1}(A)x = x$ for all $x\in \mathsf{D}(A)\cap\mathsf{R}(A)$, so $A^{i0}=I$. The identity $A^{is}A^{it}x = A^{i(s+t)}x$ follows from Proposition 15.1.12. Proposition G.2.7 implies that $t\mapsto A^{it}x$ is continuous for all $x\in X$. □

When A has bounded imaginary powers, then by the above result and the general theory of C_0-(semi)groups, there exist constants $M\geqslant 1$ and $\omega\in\mathbb{R}$ such that

$$\|A^{it}\|\leqslant Me^{\omega|t|}.$$

This allows us to define the abscissa

$$\omega_{\rm BIP}(A) := \inf\big\{\omega\in\mathbb{R}:\ \sup_{t\in\mathbb{R}} e^{-\omega|t|}\|A^{it}\|<\infty\big\}.$$

We have the following improvement of Corollary 15.2.10 in the presence of bounded imaginary powers. The point of the estimate in part (1) is that boundedness of the imaginary powers permits us to obtain an estimate that is uniform all the way up the imaginary axis.

Proposition 15.3.6. *If A has bounded imaginary powers and $0\in\varrho(A)$, and if $\|A^{it}\|\leqslant Me^{-\omega|t|}$ for all $t\in\mathbb{R}$, then:*

(1) *the operator A^{-z} is bounded for every $\Re z\geqslant 0$, and*

$$\|A^{-z}\|\leqslant C_A M e^{\omega|\Im z|}\|A^{-1}\|^{\Re z},\quad \Re z\geqslant 0,$$

where C_A depends only on $M_A := \sup_{t>0}(1+t)\|(t+A)^{-1}\|$ and $\|A^{-1}\|$;

(2) *for all $\Re z_1\geqslant 0$ and $\Re z_2\geqslant 0$ we have $A^{-z_1}A^{-z_2} = A^{-(z_1+z_2)}$;*

(3) *for all $x\in X$ the mapping $z\mapsto A^{-z}x$ is continuous on $\{\Re z\geqslant 0\}$ and holomorphic on $\{\Re z>0\}$.*

Proof. (1) and (2): By assumption for all $t\in\mathbb{R}$ the operators A^{it} are bounded, and for $\Re z>0$ the operators A^{-z} are bounded by Corollary 15.2.10. For $\Re z_1\geqslant 0$ and $\Re z_2\geqslant 0$ the identity $A^{-z_1}A^{-z_2} = A^{-(z_1+z_2)}$ follows from Proposition 15.1.12, noting that all operators occurring in this identity are bounded.

We next prove the norm estimate. We begin by noting that

$$C_A' := \sup_{s\in[0,1]}\|A^{-s}\|<\infty$$

by Corollary 15.2.15, with a constant C_A depending only on the constant M_A (which is finite since A is boundedly invertible).

By writing $z=s+it$ with $s\in[0,1]$, it follows from the identity $A^{-z} = A^{-s}A^{-it}$ that

$$\sup_{0\leqslant \Re z\leqslant 1} \|A^{-z}\| \leqslant C_A' \sup_{t\in\mathbb{R}} \|A^{-it}\| \leqslant C_A' M e^{-\omega|t|} \leqslant C_A M e^{-\omega|t|} \|A^{-1}\|^{\Re z},$$

where $C_A = C_A'/\max\{1, \|A\|^{-1}\}$. This gives the desired bound in (1) for $0 \leqslant \Re z \leqslant 1$.

For $z = z' + n$ with $n \geqslant 1$ and $0 \leqslant \Re z' < 1$, the estimate in (1) now follows from

$$\begin{aligned}\|A^{-z}\| = \|A^{-z'-n}\| \leqslant \|A^{-z'}\|\|A^{-n}\| &\leqslant C_A M^{-\omega|t|} \|A^{-1}\|^{\Re z'} \|A^{-1}\|^n \\ &= C_A M^{-\omega|t|} \|A^{-1}\|^{\Re z}.\end{aligned}$$

(3): Fix an arbitrary integer $k \geqslant 1$ and fix an element $x \in \mathsf{D}(A^k) \cap \mathsf{R}(A^k)$. We have already seen that $z \mapsto A^{-z}x$ is holomorphic on $\{|\Re z| < k\}$; in particular $z \mapsto A^{-z}x$ is continuous on $\{0 \leqslant \Re z < k\}$. The holomorphy on $\{|\Re z| < k\}$ and continuity on $\{0 \leqslant \Re z < k\}$ of $z \mapsto A^{-z}x$ for general $x \in X$ follows by approximation $x_n \to x$ with $x_n \in \mathsf{D}(A^k) \cap \mathsf{R}(A^k)$, noting that the above norm estimate implies that the convergence $A^{-z}x_n \to A^{-z}x$ is locally uniform on $\{0 \leqslant \Re z < k\}$. □

15.3.b Identification of fractional domain spaces

An important justification for studying boundedness of imaginary powers comes from Theorem 15.3.9 below, which states that boundedness of the imaginary powers implies the coincidence of the fractional power scale and the complex interpolation scale. For the proof of this result we need some lemmas. The first extends the relation $A^\alpha A^\beta = A^{\alpha+\beta}$, which has been proved so for only for α, β satisfying $\Re\alpha \cdot \Re\beta > 0$.

Lemma 15.3.7. *If A has bounded imaginary powers, then for all $\alpha \in \mathbb{C}$ and $t \in \mathbb{R}$ we have*

$$A^\alpha A^{it} = A^{it} A^\alpha = A^{\alpha+it}$$

with equality of domains.

Proof. Since A^{it} is bounded it is clear that $\mathsf{D}(A^\alpha) = \mathsf{D}(A^{it}A^\alpha)$. From Proposition 15.1.12(2) we already know the inclusion $\mathsf{D}(A^{it}A^\alpha) \subseteq \mathsf{D}(A^{\alpha+it})$ with $A^{it}A^\alpha x = A^{\alpha+it}x$ for all $x \in \mathsf{D}(A^{it}A^\alpha)$, as well as the equality $\mathsf{D}(A^{it}A^\alpha) = \mathsf{D}(A^{\alpha+it}) \cap \mathsf{D}(A^\alpha)$. Combining these results, we obtain $A^{it}A^\alpha = A^\alpha$ with equal domains. □

The second lemma considers bounded imaginary powers for shifted operators:

Lemma 15.3.8. *If A has bounded imaginary powers, then $A + \varepsilon$ has bounded imaginary powers for all $\varepsilon > 0$. If $\|A^{it}\| \leqslant Me^{\omega|t|}$ and $\omega(A) < \sigma < \pi$, then*

$$\|(A+\varepsilon)^{it}\| \leqslant M' e^{(\omega\vee\sigma)|t|},$$

for some constant M' independent of $\varepsilon > 0$.

Proof. It is immediate from Theorem 15.1.18 applied to $\varepsilon^{-1}A$ that $A+\varepsilon$ has bounded imaginary powers. By Proposition 15.2.6 we have $(\varepsilon^{-1}A)^{it} = \varepsilon^{-it}A^{it}$ and $(A+\varepsilon)^{it} = \varepsilon^{it}(\varepsilon^{-1}A+I)^{it}$ with equal domains in both cases. Hence, by the estimates provided by Theorem 15.1.18 and Proposition 15.2.6, for any fixed $\omega(A) < \sigma < \pi$ we have

$$\begin{aligned}\|(A+\varepsilon)^{it}\| &= \|(\varepsilon^{-1}A+I)^{it}\| \\ &\leqslant (1+M_{\sigma,\varepsilon^{-1}A})^2(\|A^{it}\| + C_\sigma\|z\mapsto z^{it}\|_{H^\infty(\Sigma_\sigma)}) \\ &\leqslant (1+M_{\sigma,A})^2(Me^{\omega|t|}+C_\sigma e^{\sigma|t|}).\end{aligned}$$

□

Theorem 15.3.9 (Fractional powers through complex interpolation). *If A has bounded imaginary powers, then for all $\alpha > 0$ and $0 < \theta < 1$,*

$$\mathsf{D}(A^{\alpha\theta}) = [X, \mathsf{D}(A^\alpha)]_\theta \quad \textit{with equivalent norms.}$$

Proof. By Proposition 15.2.12 and Lemma 15.3.8 we may replace A by $A+I$ if necessary, and thereby assume without loss of generality that $0 \in \varrho(A)$. This allows us to use the results of Proposition 15.3.6.

Choose $M \geqslant 1$ and $\omega \in \mathbb{R}$ such that $\|A^{it}\| \leqslant Me^{\omega|t|}$ for all $t \in \mathbb{R}$. We begin by proving the inclusion $\mathsf{D}(A^{\alpha\theta}) \subseteq [X, \mathsf{D}(A^\alpha)]_\theta$. Fix $0 < \theta < 1$ and $x \in \mathsf{D}(A^{\alpha\theta})$, and put

$$f(z) := e^{(z-\theta)^2}A^{-\alpha z}A^{\alpha\theta}x, \quad z \in \overline{\mathbb{S}},$$

where $\mathbb{S} = \{z \in \mathbb{C} : 0 < \Re z < 1\}$ is the unit strip in the complex plane. Then f is holomorphic as an X-valued function on $\mathbb{S}$ and satisfies $f(\theta) = x$. Moreover, by Proposition 15.3.6, f is continuous and uniformly bounded on $\overline{\mathbb{S}}$. Using the notation introduced in Appendix C, to prove that $x \in [X, \mathsf{D}(A^\alpha)]_\theta$ we must check that $f \in \mathscr{H}(X, \mathsf{D}(A^\alpha))$. For this it remains to be checked that $t \mapsto f(it)$ belongs to $C_{\mathrm{b}}(\mathbb{R}; X)$ and $t \mapsto f(1+it)$ belongs to $C_{\mathrm{b}}(\mathbb{R}; \mathsf{D}(A^\alpha))$. The former follows from what has already been said, and for the latter we write $\|f(1+it)\|_{\mathsf{D}(A^\alpha)} = \|f(1+it)\| + \|A^\alpha f(1+it)\|$. Again by what has already been said, the function $t \mapsto f(1+it)$ belongs to $C_{\mathrm{b}}(\mathbb{R}; X)$. The second term can be estimated as follows:

$$\begin{aligned}\|A^\alpha f(1+it)\| &= \|e^{(1+it-\theta)^2}A^\alpha A^{-\alpha(1+it)}A^{\alpha\theta}x\| \\ &= \|e^{(1+it-\theta)^2}A^{-i\alpha t}A^{\alpha\theta}x\| \leqslant e^{(1-\theta)^2-t^2}Me^{\alpha\omega|t|}\|A^{\alpha\theta}x\|,\end{aligned}$$

and this is a bounded function of $t \in \mathbb{R}$. Here we used Lemma 15.3.7, which implies that $\mathsf{D}(A^\alpha) = \mathsf{D}(A^{-\alpha(1+it)})$ and $A^{-\alpha(1+it)}y = A^{-\alpha}A^{it}y$ for $y \in X$.

To prove the reverse inclusion $[X, \mathsf{D}(A^\alpha)]_\theta \subseteq \mathsf{D}(A^{\alpha\theta})$ we will use the results and notation of Appendix C. Fix $x \in [X, \mathsf{D}(A^\alpha)]_\theta$ and let $f \in \mathscr{H}(X, \mathsf{D}(A^\alpha))$ satisfy $f(\theta) = x$. By Corollary C.2.8 there is a sequence of functions $f_n \in \mathscr{H}_0(X, \mathsf{D}(A^\alpha); \mathsf{D}(A^\alpha))$ such that $f_n(\theta) =: x_n \to x$ in $[X, \mathsf{D}(A^\alpha)]_\theta$.

Since $\mathsf{D}(A^\alpha) \subseteq \mathsf{D}(A^{\alpha z})$ for $z \in \overline{\mathbb{S}}$ and f_n takes values in $\mathsf{D}(A^\alpha)$ we may define

$$g_n(z) := e^{(z-\theta)^2} A^{\alpha z} f_n(z), \quad z \in \overline{\mathbb{S}}.$$

With respect to the norm of X, each function g_n is bounded on $\overline{\mathbb{S}}$. By the three lines lemma,

$$\begin{aligned} \|x_n\| = \|f_n(\theta)\| &\leqslant \max\Big\{\sup_{t\in\mathbb{R}}\|f_n(it)\|, \sup_{t\in\mathbb{R}}\|f_n(1+it)\|\Big\}, \\ \|A^{\alpha\theta}x_n\| = \|g_n(\theta)\| &\leqslant \max\Big\{\sup_{t\in\mathbb{R}}\|g_n(it)\|, \sup_{t\in\mathbb{R}}\|g_n(1+it)\|\Big\}. \end{aligned}$$

Moreover, for all $t \in \mathbb{R}$,

$$\begin{aligned} \|g_n(it)\| &\leqslant e^{\theta^2-t^2}\|A^{i\alpha t}f_n(it)\| \leqslant e^{\theta^2-t^2}Me^{\alpha\omega|t|}\|f_n(it)\|, \\ \|g_n(1+it)\| &\leqslant e^{(1-\theta)^2-t^2}\|A^{i\alpha t}A^\alpha f_n(1+it)\| \\ &\leqslant e^{\theta^2-t^2}Me^{\alpha\omega|t|}\|f_n(1+it)\|_{\mathsf{D}(A^\alpha)}. \end{aligned}$$

Here we used Lemma 15.3.7, which implies that $\mathsf{D}(A^\alpha) = \mathsf{D}(A^{\alpha+i\alpha t})$ and $A^{\alpha+i\alpha t}y = A^{i\alpha t}A^\alpha y$ for $y \in \mathsf{D}(A^\alpha)$.

It follows from these estimates that $\|x_n\| \lesssim \|f_n\|_{\mathscr{H}(X,X)} \leqslant \|f_n\|_{\mathscr{H}(X,\mathsf{D}(A^\alpha))}$ and $\|A^{\alpha\theta}x_n\| \lesssim \|f_n\|_{\mathscr{H}(X,\mathsf{D}(A^\alpha))}$, and therefore $\|x_n\|_{\mathsf{D}(A^{\alpha\theta})} \lesssim \|f_n\|_{\mathscr{H}(X,\mathsf{D}(A^\alpha))}$. Replacing x_n by $x_n - x_m$ in the above argument, we find that the sequence $(x_n)_{n\geqslant 1}$ is Cauchy in $\mathsf{D}(A^{\alpha\theta})$ and therefore converges to a limit. Since $x_n \to x$ in X, this limit must be x. This proves that $x \in \mathsf{D}(A^{\alpha\theta})$ and that $\|x\|_{\mathsf{D}(A^{\alpha\theta})} \lesssim \|f\|_{\mathscr{H}(X,\mathsf{D}(A^\alpha))}$. Taking the infimum with respect to f it follows that $\|x\|_{\mathsf{D}(A^{\alpha\theta})} \lesssim \|x\|_{[X,\mathsf{D}(A^\alpha)]_\theta}$. □

This theorem self-improves in an obvious manner. Upon replacing X by $\mathsf{D}(A^\beta)$ and using that $\mathsf{D}(A^\gamma) = \mathsf{D}(A^{\gamma+it})$ we arrive at the following more general result.

Corollary 15.3.10. *If A has bounded imaginary powers, then for all $\alpha, \beta \in \mathbb{C}$ with $0 \leqslant \alpha < \beta < \infty$ we have*

$$\mathsf{D}(A^{(1-\theta)\alpha+\theta\beta}) = [\mathsf{D}(A^\alpha), \mathsf{D}(A^\beta)]_\theta$$

with equivalent norms.

Let us revisit the Laplace operator Δ on $L^p(\mathbb{R}^d; X)$, where $1 < p < \infty$ and X is a UMD space, with domain $\mathsf{D}(\Delta) = H^{2,p}(\mathbb{R}; X)$. It was already noted above that $-\Delta$ is standard sectorial of angle 0 on $L^p(\mathbb{R}^d; X)$ for all $1 \leqslant p < 1$, and by Theorem 10.2.25 it has a bounded H^∞-calculus of angle 0. As a consequence, $-\Delta$ has bounded imaginary powers. Applying Theorem 15.3.9, for all $0 < \theta < 1$ we obtain

$$\mathsf{D}((-\Delta)^\theta) = [L^p(\mathbb{R}^d; X), H^{2,p}(\mathbb{R}^d; X)]_\theta \quad \text{with equivalent norms.}$$

In Chapter 5 we have proved Seeley's theorem (Theorem 5.6.9), from which it follows that if X is a UMD space and $1 < p < \infty$, then for all $0 < \theta < 1$ we have

$$[L^p(\mathbb{R}^d; X), H^{2,p}(\mathbb{R}^d; X)]_\theta = H^{2\theta,p}(\mathbb{R}^d; X) \quad \text{with equivalent norms.}$$

Thus we obtain the following result.

Theorem 15.3.11 (Laplacian on $L^p(\mathbb{R}^d; X)$). *Consider the Laplace operator Δ on $L^p(\mathbb{R}^d; X)$, where $1 < p < \infty$ and X is a UMD space, with domain $\mathsf{D}(\Delta) = H^{2,p}(\mathbb{R}; X)$. Then for all $0 < \theta < 1$ we have*

$$\mathsf{D}((-\Delta)^\theta) = H^{2\theta,p}(\mathbb{R}^d; X) \quad \textit{with equivalent norms.}$$

15.3.c Connections with sectoriality

It is part of the definition that an operator with bounded imaginary powers is standard sectorial, but there is no obvious *a priori* relation between the abscissa $\omega_{\rm BIP}(A)$ and the angle of sectoriality $\omega(A)$. The main result of this section is the following result, which says that $\omega(A) \leqslant \omega_{\rm BIP}(A)$. Moreover, if X is a UMD space, then A is R-sectorial of angle $\omega_R(A) \leqslant \omega_{\rm BIP}(A)$.

Theorem 15.3.12 (Clément–Prüss). *Let A be an operator with bounded imaginary powers on a Banach space X, and assume that $\omega_{\rm BIP}(A) < \pi$.*

(1) *A is sectorial of angle $\omega(A) \leqslant \omega_{\rm BIP}(A)$.*
(2) *If X is a UMD space, then A is R-sectorial of angle $\omega_R(A) \leqslant \omega_{\rm BIP}(A)$.*

The key lemma is the following representation formula. It expresses the resolvent of A in terms of the imaginary powers A^{it}, and a such it provides the key insight behind the Clément–Prüss theorem.

Lemma 15.3.13 (Prüss–Sohr). *Let A be an operator with bounded imaginary powers on a Banach space X, and assume that $\omega_{\rm BIP}(A) < \pi$. Let $\lambda = re^{i\theta}$ with $r > 0$ and $|\theta| < \pi - \omega_{\rm BIP}(A)$. Then for all $x \in \mathsf{D}(A) \cap \mathsf{R}(A)$ we have*

$$(I + \lambda A)^{-1}x = \frac{1}{2}x + \frac{1}{2\pi i}\text{p.v.}\int_{-\infty}^{\infty} \frac{\pi}{\sinh(\pi t)}\lambda^{-it}A^{-it}x\,\mathrm{d}t,$$

the convergence of the principal value integral on the right-hand side being part of the assertion. Furthermore, for all $0 < s < 1$,

$$\lambda^s A^s(1 + \lambda A)^{-1}x = \frac{1}{2}\int_{\mathbb{R}} \frac{1}{\sin(\pi(s - it))}\lambda^{it}A^{it}x\,\mathrm{d}t. \tag{15.15}$$

Proof. We begin with the proof of the first identity. It proceeds in three steps.

Step 1 – First take $r = 1$ and $\theta = 0$. In this step, for all $x \in X$ we will prove that

$$\begin{aligned}\frac{1}{2}x + \frac{1}{2\pi i}\text{p.v.}\int_{-\infty}^{\infty} \frac{\pi}{\sinh(\pi s)} A^{-is}x \,\mathrm{d}s \\ = \lim_{c\downarrow 0} \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{\pi}{\sin(\pi z)} A^{-z}x \,\mathrm{d}z,\end{aligned}$$

the convergence of the principal value integral being part of the assertion. Note that the integrals occurring on right-hand side converge absolutely thanks to the estimates

$$|\sinh(\pi(c+it))| = O(e^{\pi|t|}) \quad \text{as} \quad t \to \pm\infty$$

and

$$\|A^{-c-it}x\| \leqslant Me^{\omega|t|}\|A^{-c}x\|, \quad t \in \mathbb{R},$$

for all $\omega_{\mathrm{BIP}}(A) < \omega < \pi$, with $M \geqslant 1$ a constant depending on ω.

By Cauchy's theorem we have

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{\pi}{\sin(\pi z)} A^{-z}x \,\mathrm{d}z = \frac{1}{2\pi i}\int_{\Gamma_c} \frac{\pi}{\sin(\pi z)} A^{-z}x \,\mathrm{d}z,$$

where Γ_c is the (upwards oriented) contour consisting of the union of the two half-lines $\Gamma_c^{(1)} = \{is : \ s \leqslant -c\}$ and $\Gamma_c^{(3)} = \{is : \ s \geqslant c\}$ and the semi-circle $\Gamma_c^{(2)} = \{ce^{i\vartheta} : \ \vartheta \in [-\frac{1}{2}\pi, \frac{1}{2}\pi]\}$. As $c \downarrow 0$, the contributions along the two half-lines converge to the principal value integral and the contribution along the semi-circle converges to $\frac{1}{2}x$. The latter follows by noting that $A^{-z}x \to x$ as $z \to 0$ in the closed right-half plane, by the continuity of $z \mapsto A^{-z}x$ on that set (see Proposition 15.3.6). Hence

$$\lim_{c\downarrow 0} \frac{1}{2\pi i}\int_{\Gamma_c^{(2)}} \frac{\pi}{\sin(\pi z)} \,\mathrm{d}z = \lim_{c\downarrow 0} \frac{1}{2\pi}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \frac{\pi c e^{i\varphi}}{\sin(\pi c e^{i\varphi})} \,\mathrm{d}\varphi = \frac{1}{2}$$

(since $\sin(\pi c e^{i\varphi}) = \pi c e^{i\varphi} + O(c^3)$ as $c \downarrow 0$.

Step 2 – In this step we will prove the lemma for $r = 1$ and $\theta = 0$ with $x \in \mathsf{D}(A) \cap \mathsf{R}(A)$, i.e., we show that

$$(I + A)^{-1}x = \frac{1}{2}x + \frac{1}{2\pi i}\text{p.v.}\int_{-\infty}^{\infty} \frac{\pi}{\sinh(\pi s)} A^{-is}x \,\mathrm{d}s$$

for all $x \in \mathsf{D}(A) \cap \mathsf{R}(A)$. (Note that $I + A$ is boundedly invertible as part of the definition of bounded imaginary powers, since A is assumed to be standard sectorial).

Let $y := (I + A)x$. Then

$$\begin{aligned}
&\frac{1}{2}y + \frac{1}{2\pi i}\text{p.v.}\int_{-\infty}^{\infty} \frac{\pi}{\sinh(\pi s)} A^{-is} y \,\mathrm{d}s \\
&\quad = \frac{1}{2}(I+A)x + \frac{1}{2\pi i}\text{p.v.}\int_{-\infty}^{\infty} \frac{\pi}{\sinh(\pi s)} A^{-is} x \,\mathrm{d}s \\
&\qquad + \frac{1}{2\pi i}\text{p.v.}\int_{-\infty}^{\infty} \frac{\pi}{\sinh(\pi s)} A^{1-is} x \,\mathrm{d}s \\
&\quad = \frac{1}{2}(I+A)x + \frac{1}{2\pi i}\text{p.v.}\int_{\Gamma_c} \frac{\pi}{\sin(\pi z)} A^{-z} y \,\mathrm{d}x \\
&\qquad + \frac{1}{2\pi i}\text{p.v.}\int_{\Gamma_c} \frac{\pi}{\sin(\pi z)} A^{1-z} y \,\mathrm{d}x.
\end{aligned} \tag{15.16}$$

In view of $\sin(\pi z) = -\sin(\pi(1-z))$, after a change of variable in the last integral the contributions over all four half-lines cancel and we are left with

$$\frac{1}{2}(I+A)x + \frac{1}{2\pi i}\text{p.v.}\int_{\Gamma_c^{(2)}} \frac{\pi}{\sin(\pi z)} A^{-z} x \,\mathrm{d}z - \frac{1}{2\pi i}\text{p.v.}\int_{\widetilde{\Gamma}_c^{(2)}} \frac{\pi}{\sin(\pi z)} A^{-z} x \,\mathrm{d}z,$$

where $\widetilde{\Gamma}_c^{(2)} = \{1 - ce^{i\vartheta} : \ \vartheta \in [-\frac{1}{2}\pi, \frac{1}{2}\pi]\}$. As $c \downarrow 0$, the first integral tends to $\frac{1}{2}y$ and the second to $-\frac{1}{2}Ax$. In the limit $c \downarrow 0$ the three terms on the right-hand side of (15.16) therefore add up to x. This proves the identity

$$x = \frac{1}{2}y + \frac{1}{2\pi i}\text{p.v.}\int_{-\infty}^{\infty} \frac{\pi}{\sinh(\pi s)} A^{-is} y \,\mathrm{d}s.$$

Multiplying on both sides with $(I+A)^{-1}$ gives the desired result.

Step 3 – The general case follows by applying the result of Step 2 to the operator λA, which by Proposition 15.2.6 has bounded imaginary powers and satisfies $(\lambda A)^{-is} = \lambda^{-is}A^{-is}$. This completes the proof of the first identity. Using it, and fixing $0 < s < 1$, for $x \in \mathsf{D}(A) \cap \mathsf{R}(A)$ we obtain

$$\begin{aligned}
\lambda^s A^s (I + \lambda A)^{-1} x &= \frac{1}{2}\lambda^s A^s x + \frac{1}{2\pi i}\text{p.v.}\int_{-\infty}^{\infty} \frac{\pi}{\sinh(\pi t)} \lambda^{s-it} A^{s-it} x \,\mathrm{d}t \\
&= \frac{1}{2}\lambda^s A^s x - \frac{1}{2}\int_{-\infty - is}^{\infty - is} \frac{1}{\sin(\pi(s-it))} \lambda^{it} A^{it} x \,\mathrm{d}t \\
&= \frac{1}{2}\int_{-\infty}^{\infty} \frac{1}{\sin(\pi(s-it))} \lambda^{it} A^{it} x \,\mathrm{d}t
\end{aligned}$$

by the Cauchy theorem, noting that $A^s A^{it} = A^{s+it}$ by Theorem 15.2.5 in the first step. This gives the second identity. □

Remark 15.3.14. A more direct proof of the second identity can be given as follows. Starting from the identity

$$\int_{-\infty}^{\infty} \frac{e^{2\pi i t\xi}}{\sin(\pi(s-it))} \,\mathrm{d}t = \frac{2e^{2\pi s\xi}}{1 + e^{2\pi\xi}}, \qquad 0 < s < 1, \ \xi \in \mathbb{R}, \tag{15.17}$$

the substitution $z = e^{2\pi\xi}$ gives

$$\int_{-\infty}^{\infty} \frac{z^{it}}{\sin(\pi(s-it))}\,\mathrm{d}t = \frac{2z^s}{1+z}, \qquad 0<s<1,\ z\in\mathbb{R}_+. \tag{15.18}$$

By analytic continuation this extends to all $z\in\mathbb{C}$ with $|\arg(z)|<\pi$.

Let $\lambda\in\mathbb{C}\setminus\{0\}$ as in the statement of the lemma. For $x\in\mathsf{D}(A)\cap\mathsf{R}(A)$ it follows from Proposition 15.1.19 that $A^{it}x$ is given by the Bochner integral

$$A^{it}x = \mu^{it}x + \frac{1}{2\pi i}\int_{\Gamma_\nu} z^{it}\Big(R(z,A) - \frac{1}{z-\mu}\Big)x\,\mathrm{d}z,$$

where $\omega(A) < |\arg\mu| < \nu$. Substituting this identity into the right-hand side of (15.15), a short computation involving Fubini's theorem, (15.18), and Cauchy's theorem gives the result. At the expense of some additional computations, instead of invoking Proposition 15.1.19 one may also directly use the definition for $A^{it}x$ as given in Definition 15.1.8.

Proof of Theorem 15.3.12. (1): First let $\lambda = re^{i\theta}$ with $r>0$ and $|\theta| < \pi - \omega(A)$. By subtraction we obtain the identity

$$(I+\lambda A)^{-1}x = (I+A)^{-1}x + \frac{1}{2\pi i}\mathrm{p.v.}\int_{-\infty}^{\infty}\frac{\pi}{\sinh(\pi s)}(\lambda^{-is}-1)A^{-is}x\,\mathrm{d}s$$

for $x\in\mathsf{D}(A^2)\cap\mathsf{R}(A^2)$. The crux is that the term $\lambda^{-is}-1$ is of the order $O(|s|)$ near $s=0$ and can therefore be estimated as $|\lambda^{-is}-1|\lesssim |s|\wedge 1$. Similarly, $|\sinh(s)|\lesssim (|s|\wedge 1)e^{\pi|s|}$. Therefore the principal value integral is actually absolutely convergent and bounded in x. As a consequence of this, the identity extends to arbitrary $x\in X$.

The proof is completed by observing that the integral in the right hand side of the identity

$$(I+\lambda A)^{-1}x = (I+A)^{-1}x + \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\pi}{\sinh(\pi s)}(\lambda^{-is}-1)A^{-is}x\,\mathrm{d}s$$

is absolutely convergent for any $\lambda = re^{i\theta}$ with $r>0$ and $|\theta|<\pi-\omega_{\rm BIP}(A)$. Indeed, recalling the estimates for $\lambda^{-is}-1$ and $\sinh(s)$ mentioned earlier, choosing $\omega_{\rm BIP}(A)<\omega<\pi$ so that $|\theta|<\pi-\omega$ we estimate

$$\Big\|\int_{-\infty}^{\infty}\frac{\pi}{\sinh(\pi s)}(\lambda^{-is}-1)A^{-is}x\,\mathrm{d}s\Big\| \lesssim \int_{|s|\geqslant 1}\pi e^{-\pi|s|}M_\omega e^{\omega|s|}\|x\|\,\mathrm{d}s$$

with a constant independent of x. The right-hand side defines a holomorphic extension of the function $\lambda\mapsto(I+\lambda A)^{-1}x$ to the open sector $\Sigma_{\pi-\omega_{\rm BIP}(A)}$. As a consequence the spectrum of A must be contained in the closure of $\Sigma_{\omega_{\rm BIP}(A)}$. Finally, the sectoriality estimate on the complement of this closure follows from the estimate.

(2): Fix $\omega_{\mathrm{BIP}}(A) < \omega < \nu < \pi$ and choose numbers $\lambda_n = r_n e^{i\theta_n}$ with $r_n > 0$ and $|\theta_n| < \pi - \nu$, as well as vectors $x_n \in X$; $n = 1, \dots, N$. We wish to show that there exists a constant C, independent of the choices just made, such that

$$\Big\| \sum_{n=1}^N \varepsilon_n (I + \lambda_n A)^{-1} x_n \Big\|_{L^2(\Omega;X)} \leqslant C \Big\| \sum_{n=1}^N \varepsilon_n x_n \Big\|_{L^2(\Omega;X)},$$

where $(\varepsilon_n)_{n=1}^N$ is a Rademacher sequence defined on a probability space $(\Omega, \mathbb{P})$. By a simple approximation argument, there is no loss of generality in assuming that $x_n \in \mathsf{D}(A) \cap \mathsf{R}(A)$ for all $n = 1 \dots, N$.

Since $\omega(A) \leqslant \omega_{\mathrm{BIP}}(A)$ by the Clément–Prüss theorem, Lemma 15.3.13 (with $\lambda = 1$), the representation formulas of Lemma 15.3.13 hold for $\lambda = re^{i\theta}$ with $r > 0$ and $|\theta| < \pi - \nu$, with $x \in \mathsf{D}(A^2) \cap \mathsf{R}(A^2)$, and

$$\begin{aligned}
(I + re^{i\theta} A)^{-1} x = \frac{1}{2} x &+ \frac{1}{2\pi i} \int_{-\infty}^{\infty} \psi_\theta(s) r^{-is} A^{-is} x \, \mathrm{d}s \\
&+ \frac{1}{2\pi i} \int_{-\infty}^{\infty} \eta(s) r^{-is} A^{-is} x \, \mathrm{d}s, \\
&+ \frac{1}{2\pi i} \mathrm{p.v.} \int_{-1}^{1} r^{-is} A^{-is} x \, \frac{\mathrm{d}s}{s} \\
=: \frac{1}{2} x &+ T_{r,\theta} x + S_r x + R_r x,
\end{aligned}$$

where

$$\psi_\theta(s) = \frac{\pi}{\sinh(\pi s)} (e^{\theta s} - 1), \quad \eta(s) := \frac{\pi}{\sinh(\pi s)} - \frac{\mathbf{1}_{(-1,1)}(s)}{s}.$$

Applying this to $\lambda = \lambda_n$ we obtain

$$\begin{aligned}
&\Big\| \sum_{n=1}^N \varepsilon_n (I + \lambda_n A)^{-1} x_n \Big\|_{L^2(\Omega;X)} \\
&\quad \leqslant \frac{1}{2} \Big\| \sum_{n=1}^N \varepsilon_n x_n \Big\|_{L^2(\Omega;X)} + \Big\| \sum_{n=1}^N \varepsilon_n T_{r_n,\theta_n} x_n \Big\|_{L^2(\Omega;X)} \\
&\quad\quad + \Big\| \sum_{n=1}^N \varepsilon_n S_{r_n} x_n \Big\|_{L^2(\Omega;X)} + \Big\| \sum_{n=1}^N \varepsilon_n R_{r_n} x_n \Big\|_{L^2(\Omega;X)}.
\end{aligned}$$

We will estimate the last three expressions separately.

To start with the first, we note that $|\psi_{\theta_n}(s)| \lesssim e^{(\theta_n - \pi)|s|} \leqslant e^{-\nu|s|}$. Therefore, by the Kahane contraction principle and the bound $\|A^{is}\| \leqslant M e^{\omega|s|}$,

$$\Big\| \sum_{n=1}^N \varepsilon_n T_{r_n,\theta_n} x_n \Big\|_{L^2(\Omega;X)} \leqslant \frac{1}{2\pi} \int_{-\infty}^{\infty} \Big\| A^{-is} \sum_{n=1}^N \varepsilon_n \psi_{\theta_n}(s) x_n \Big\|_{L^2(\Omega;X)} \mathrm{d}s$$

$$
\begin{aligned}
&\lesssim \frac{1}{2\pi}\int_{-\infty}^{\infty} Me^{\omega|s|}\Big\|\sum_{n=1}^N \varepsilon_n \psi_{\theta_n}(s)x_n\Big\|_{L^2(\Omega;X)}\,\mathrm{d}s \\
&\lesssim \frac{1}{2\pi}\int_{-\infty}^{\infty} Me^{(\omega-\nu)|s|}\Big\|\sum_{n=1}^N \varepsilon_n x_n\Big\|_{L^2(\Omega;X)}\,\mathrm{d}s \\
&= C_{A,\nu}\Big\|\sum_{n=1}^N \varepsilon_n x_n\Big\|_{L^2(\Omega;X)}.
\end{aligned}
$$

The second term is treated similarly, now using that $|\eta(s)| \lesssim e^{-\pi|s|}$:

$$
\begin{aligned}
\Big\|\sum_{n=1}^N \varepsilon_n S_{r_n} x_n\Big\|_{L^2(\Omega;X)} &\leqslant \frac{1}{2\pi}\int_{-\infty}^{\infty} \Big\|A^{-is}\sum_{n=1}^N \varepsilon_n \eta(s)x_n\Big\|_{L^2(\Omega;X)}\,\mathrm{d}s \\
&\lesssim \frac{1}{2\pi}\int_{-\infty}^{\infty} Me^{\omega|s|}\Big\|\sum_{n=1}^N \varepsilon_n \eta(s)x_n\Big\|_{L^2(\Omega;X)}\,\mathrm{d}s \\
&\lesssim \frac{1}{2\pi}\int_{-\infty}^{\infty} Me^{(\omega-\pi)|s|}\Big\|\sum_{n=1}^N \varepsilon_n x_n\Big\|_{L^2(\Omega;X)}\,\mathrm{d}s \\
&= C'_{A,\nu}\Big\|\sum_{n=1}^N \varepsilon_n x_n\Big\|_{L^2(\Omega;X)}.
\end{aligned}
$$

For estimating the third term we use the UMD property of X through the boundedness of the Hilbert transform on $L^2(\mathbb{R};X)$.

We begin with a preliminary observation. Let us set $U_n(s) = (r_n A)^{-is} = r_n^{-is}A^{-is}$ for brevity. Then by the Kahane contraction principle, for all $s \in \mathbb{R}$ we have

$$
\begin{aligned}
\Big\|\sum_{n=1}^N \varepsilon_n U_n(s)x_n\Big\|_{L^2(\Omega;X)} &\leqslant \Big\|\sum_{n=1}^N \varepsilon_n A^{-is}x_n\Big\|_{L^2(\Omega;X)} \\
&\leqslant Me^{\omega|s|}\Big\|\sum_{n=1}^N \varepsilon_n x_n\Big\|_{L^2(\Omega;X)}.
\end{aligned}
\tag{15.19}
$$

Fix $0 < \delta < 1$ and $t \in [-\frac{1}{2}, \frac{1}{2}]$. Then

$$
\begin{aligned}
\sum_{n=1}^N \varepsilon_n \int_{\delta<|s|<1} U_n(s)x_n\,\frac{\mathrm{d}s}{s} &= \sum_{n=1}^N \varepsilon_n U_n(t)\int_{\delta<|s|<1} U_n(s-t)x_n\,\frac{\mathrm{d}s}{s} \\
&= \sum_{n=1}^N \varepsilon_n U_n(t)\int_{|s|>\delta} \varphi_n(t-s)\,\frac{\mathrm{d}s}{s} \\
&\quad - \sum_{n=1}^N \varepsilon_n \int_1^{1+t} U_n(s)x_n\,\frac{\mathrm{d}s}{s}
\end{aligned}
$$

$$+\sum_{n=1}^{N}\varepsilon_n\int_{-1}^{-1+t}U_n(s)x_n\,\frac{\mathrm{d}s}{s},$$

where $\varphi_n(\tau)=\mathbf{1}_{(-1,1)}(\tau)U_n(-\tau)x_n$. Integrating over $t\in(-\frac{1}{2},\frac{1}{2})$, we obtain

$$\begin{aligned}\sum_{n=1}^{N}\varepsilon_n\int_{\delta<|s|<1}U_n(s)x_n\,\frac{\mathrm{d}s}{s}&=\sum_{n=1}^{N}\varepsilon_n\int_{-\frac{1}{2}}^{\frac{1}{2}}U_n(t)\int_{|s|>\delta}\varphi_n(t-s)\,\frac{\mathrm{d}s}{s}\,\mathrm{d}t\\&\quad-\sum_{n=1}^{N}\varepsilon_n\int_{-\frac{1}{2}}^{\frac{1}{2}}\int_{1}^{1+t}U_n(s)x_n\,\frac{\mathrm{d}s}{s}\,\mathrm{d}t\\&\quad+\sum_{n=1}^{N}\varepsilon_n\int_{-\frac{1}{2}}^{\frac{1}{2}}\int_{-1}^{-1+t}U_n(s)x_n\,\frac{\mathrm{d}s}{s}\,\mathrm{d}t.\end{aligned}$$

Since X is UMD and $\phi_n\in L^2(\mathbb{R};X)$, the limit

$$\lim_{\delta\downarrow 0}\int_{|s|>\delta}\varphi_n(\cdot-s)\,\frac{\mathrm{d}s}{s}=\lim_{\substack{\delta\downarrow 0\\ R\to\infty}}\int_{\delta<|s|<R}\varphi_n(\cdot-s)\,\frac{\mathrm{d}s}{s}$$

exists in $L^2(\mathbb{R};X)$ by Theorem 5.1.1 and equals $\pi H\phi_n$, where H is the Hilbert transform. As a result we obtain

$$\begin{aligned}\sum_{n=1}^{N}\varepsilon_n R_{r_n}x_n&=\sum_{n=1}^{N}\varepsilon_n\mathrm{p.v.}\int_{-1}^{1}U_n(s)x_n\,\frac{\mathrm{d}s}{s}\\&=\sum_{n=1}^{N}\varepsilon_n\lim_{\delta\downarrow 0}\int_{\delta<|s|<1}U_n(s)x_n\,\frac{\mathrm{d}s}{s}\\&=\pi\sum_{n=1}^{N}\varepsilon_n\int_{-\frac{1}{2}}^{\frac{1}{2}}U_n(t)H\varphi_n(t)\,\mathrm{d}t\\&\qquad-\sum_{n=1}^{N}\varepsilon_n\int_{-\frac{1}{2}}^{\frac{1}{2}}\int_{1}^{1+t}U_n(s)x_n\,\frac{\mathrm{d}s}{s}\,\mathrm{d}t\\&\qquad+\sum_{n=1}^{N}\varepsilon_n\int_{-\frac{1}{2}}^{\frac{1}{2}}\int_{-1}^{-1+t}U_n(s)x_n\,\frac{\mathrm{d}s}{s}\,\mathrm{d}t\\&=:(I)+(II)+(III).\end{aligned}$$

It remains to estimate the three terms on the right-hand side. For estimating (I) we use that $\|H\|_{\mathscr{L}(L^2(\mathbb{R};X))}\leqslant 2\beta^+_{2,X}\beta^-_{2,X}$ (see Theorem 5.1.13). Applying the Kahane–Khintchine inequality, this gives

$$\Big\|\sum_{n=1}^{N}\varepsilon_n\int_{-1/2}^{1/2}U_n(t)H\varphi_n(t)\,\mathrm{d}t\Big\|_{L^2(\Omega;X)}$$

$$
\begin{aligned}
&\eqsim \Big\| \sum_{n=1}^N \varepsilon_n \int_{-1/2}^{1/2} U_n(t) H\varphi_n(t)\,\mathrm{d}t \Big\|_{L^1(\Omega;X)} \\
&= \Big\| \int_{-1/2}^{1/2} \sum_{n=1}^N \varepsilon_n U_n(t) H[\mathbf{1}_{(-1,1)}(\cdot) U_n(-\,\cdot\,)x_n](t)\,\mathrm{d}t \Big\|_{L^1(\Omega;X)} \\
&\leqslant \int_{-1/2}^{1/2} \Big\| \sum_{n=1}^N \varepsilon_n U_n(t) H[\mathbf{1}_{(-1,1)}(\cdot) U_n(-\,\cdot\,)x_n](t) \Big\|_{L^1(\Omega;X)} \mathrm{d}t \\
&\leqslant M \int_{-1/2}^{1/2} \Big\| \sum_{n=1}^N \varepsilon_n H[\mathbf{1}_{(-1,1)}(\cdot) U_n(-\,\cdot\,)x_n](t) \Big\|_{L^1(\Omega;X)} \mathrm{d}t \\
&= M_{1/2}\mathbb{E} \int_{-1/2}^{1/2} \Big\| H\Big[\mathbf{1}_{(-1,1)}(\cdot) \sum_{n=1}^N \varepsilon_n U_n(-\,\cdot\,)x_n\Big](t) \Big\| \,\mathrm{d}t \\
&\leqslant M_{1/2}\mathbb{E} \Big\| H\Big[\mathbf{1}_{(-1,1)}(\cdot) \sum_{n=1}^N \varepsilon_n U_n(-\,\cdot\,)x_n\Big] \Big\|_{L^2(\mathbb{R};X)} \\
&\leqslant 2\beta^+_{2,X}\beta^-_{2,X} M_{1/2}\mathbb{E} \Big\| \sum_{n=1}^N \varepsilon_n \mathbf{1}_{(-1,1)}(\cdot) U_n(-\,\cdot\,)x_n \Big\|_{L^2(\mathbb{R};X)}
\end{aligned}
$$

and, by (15.19),

$$
\begin{aligned}
&\mathbb{E} \Big\| \sum_{n=1}^N \varepsilon_n \mathbf{1}_{(-1,1)}(\cdot) U_n(-\,\cdot\,)x_n \Big\|_{L^2(\mathbb{R};X)}. \\
&\qquad = \mathbb{E} \Big\| \sum_{n=1}^N \varepsilon_n(\cdot) U_n(-\,\cdot\,)x_n \Big\|_{L^2(-1,1;X)} \\
&\qquad \leqslant \Big\| \sum_{n=1}^N \varepsilon_n(\cdot) U_n(-\,\cdot\,)x_n \Big\|_{L^2(\Omega;L^2(-1,1;X))} \\
&\qquad = \Big\| \sum_{n=1}^N \varepsilon_n(\cdot) U_n(-\,\cdot\,)x_n \Big\|_{L^2(-1,1;L^2(\Omega;X))} \\
&\qquad \leqslant M_{1/2} \Big\| \sum_{n=1}^N \varepsilon_n x_n \Big\|_{L^2(-1,1;L^2(\Omega;X))} \\
&\qquad = M_{1/2} \Big\| \sum_{n=1}^N \varepsilon_n x_n \Big\|_{L^2(\Omega;X))}
\end{aligned}
$$

where $M_{1/2} := \sup_{|t|\leqslant 1/2} \|A^{-it}\|$.

To estimate (II) we use (15.19) again:

$$
\Big\| \sum_{n=1}^N \varepsilon_n \int_{-1/2}^{1/2} \int_1^{1+t} U_n(s)x_n \,\frac{\mathrm{d}s}{s}\,\mathrm{d}t \Big\|_{L^2(\Omega;X)}
$$

$$
\begin{aligned}
&\leqslant \int_{-1/2}^{1/2} \int_1^{1+t} \Big\| \sum_{n=1}^N \varepsilon_n U_n(s) x_n \Big\|_{L^2(\Omega;X)} \frac{\mathrm{d}s}{s} \,\mathrm{d}t \\
&\leqslant M_2 \int_{-1/2}^{1/2} \int_1^{1+t} \Big\| \sum_{n=1}^N \varepsilon_n x_n \Big\|_{L^2(\Omega;X)} \frac{\mathrm{d}s}{s} \,\mathrm{d}t \\
&\leqslant M_2 \Big\| \sum_{n=1}^N \varepsilon_n x_n \Big\|_{L^2(\Omega;X)}
\end{aligned}
$$

where $M_2 := \sup_{|t|\leqslant 2} \|A^{-it}\|$.

The estimation of (III) is entirely similar. □

15.3.d Connections with almost γ-sectoriality

We have consistently limited our treatment of the H^∞-calculus and related topics to sectorial operators. It is of some interest to consider the wider class of so-called *almost sectorial* operators, defined as follows.

Definition 15.3.15 (Almost sectorial operators). *Let $\sigma \in (0,\pi)$. A linear operator A acting in a Banach space X is called:*

(i) σ-almost sectorial *if $\sigma(A) \subseteq \overline{\Sigma_\sigma}$ and the set*

$$\big\{\lambda A R(\lambda,A)^2 : \ \lambda \in \mathbb{C} \setminus \overline{\Sigma_\sigma}\big\}$$

is uniformly bounded;

(ii) σ-almost γ-sectorial *if $\sigma(A) \subseteq \overline{\Sigma_\sigma}$ and the set*

$$\big\{\lambda A R(\lambda,A)^2 : \ \lambda \in \mathbb{C} \setminus \overline{\Sigma_\sigma}\big\}$$

is γ-bounded.

The operator A is called almost sectorial*, respectively almost* γ-sectorial *if it is σ-almost sectorial, respectively σ-almost γ-sectorial, for some $\sigma \in (0,\pi)$.*

Almost R-sectorial operators are defined similarly, replacing γ-boundedness by R-boundedness.

For an almost sectorial, respectively an almost γ-sectorial operator A, we define

$$
\begin{aligned}
\widetilde{\omega}(A) &:= \inf\big\{\sigma \in (0,\pi) : \ A \text{ is } \sigma\text{-almost sectorial}\big\}, \\
\widetilde{\omega}_\gamma(A) &:= \inf\big\{\sigma \in (0,\pi) : \ A \text{ is } \sigma\text{-almost } \gamma\text{-sectorial}\big\}.
\end{aligned}
$$

The identity

$$\lambda A R(\lambda,A)^2 = [\lambda R(\lambda,A)]^2 - \lambda R(\lambda,A)$$

shows that every (γ-)sectorial operator is almost (γ-)sectorial and

$$\widetilde{\omega}(A) \leqslant \omega(A), \quad \text{respectively} \quad \widetilde{\omega}_\gamma(A) \leqslant \omega_\gamma(A).$$

The above definitions may appear somewhat *ad hoc* at first sight, but the motivation to introduce them is as follows. The operators $\lambda R(\lambda, A)$ used in the definition of sectoriality can be represented in the primary calculus of A as

$$\lambda R(\lambda, A) = r_\lambda(A) \quad \text{with} \quad R_\lambda(z) = \frac{\lambda}{\lambda - z}.$$

Indeed, the functions r_λ belong to the class $E(\Sigma_\sigma)$ introduced in Section 15.1.a as long as $0 < \sigma < |\Re\lambda|$. They do not belong to $H^1(\Sigma_\sigma)$, however, and this fact is responsible for some of the technical issues encountered in several proofs. In contrast, the operators $\lambda AR(\lambda, A)^2$ used in the definition of almost sectoriality can be represented in the Dunford calculus of A, for we have

$$\lambda AR(\lambda, A)^2 = \widetilde{r}_\lambda(A) \quad \text{with} \quad \widetilde{r}_\lambda(z) = \frac{\lambda z}{(\lambda - z)^2}.$$

Indeed, the functions $\widetilde{r}_\lambda$ belong to $H^1(\Sigma_\sigma)$ for $0 < \sigma < |\Re\lambda|$. Further motivation will be given in the Notes at the end of the chapter.

The following result gives a version of the (second part of) Clément–Prüss theorem (Theorem 15.3.12) holds without making any assumptions on the Banach space X. The price to pay is that only almost γ-sectoriality is obtained:

Theorem 15.3.16. *Let A be an operator with bounded imaginary powers on a Banach space X. Then A is almost γ-sectorial of angle $\widetilde{\omega}_\gamma(A) \leqslant \omega_{\mathrm{BIP}}(A)$.*

Proof. Fix $\omega_{\mathrm{BIP}}(A) < \theta' < \theta < \pi$ and suppose that $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$ are nonzero and satisfy $|\arg(\lambda_k)| \geqslant \theta$. Note that $|\arg(\mu_k)| \leqslant \pi - \theta$. Set $\mu_k := -1/\lambda_k$. Then for all choices $x_1, \ldots, x_n \in X$ we have, by Lemma 15.3.13,

$$\begin{aligned}
&\mathbb{E}\Big\|\sum_{k=1}^n \gamma_k \lambda_k^{1/2} A^{1/2} R(\lambda_k, A) x_k\Big\| \\
&= \mathbb{E}\Big\|\sum_{k=1}^n \gamma_k \mu_k^{1/2} A^{1/2} (1 + \mu_k A)^{-1} x_k\Big\| \\
&\leqslant \frac{1}{2}\int_{\mathbb{R}} \frac{1}{|\sin(\pi(\frac12 - it))|} \mathbb{E}\Big\|\sum_{k=1}^n \gamma_k \mu_k^{it} A^{it} x_k\Big\| \,\mathrm{d}t \\
&\overset{(*)}{\leqslant} \frac{1}{2}\int_{\mathbb{R}} \frac{e^{(\pi-(\theta-\theta'))|t|}}{|\sin(\pi(\frac12 - it))|} \|e^{-(\pi-(\theta-\theta'))|t|} A^{it}\| \Big(\sup_{1\leqslant k\leqslant n} |\mu_k^{it}|\Big) \mathbb{E}\Big\|\sum_{k=1}^n \gamma_k x_k\Big\| \,\mathrm{d}t \\
&\overset{(**)}{\leqslant} \frac{1}{2}\int_{\mathbb{R}} \frac{e^{(\pi-(\theta-\theta'))|t|}}{|\sin(\pi(\frac12 - it))|} \,\mathrm{d}t \sup_{t\in\mathbb{R}} \|e^{-\theta'|t|} A^{it}\| \,\mathbb{E}\Big\|\sum_{k=1}^n \gamma_k x_k\Big\| \\
&= C\mathbb{E}\Big\|\sum_{k=1}^n \gamma_k x_k\Big\|,
\end{aligned}$$

where in $(*)$ we used the contraction principle and in $(**)$ the fact that for $|\arg(\mu)| \leqslant \pi - \theta$ and $t \in \mathbb{R}$ we have

$$\left\|e^{-(\pi-(\theta-\theta'))|t|}A^{it}\right\||\mu^{it}| = \left\|e^{-(\pi-(\theta-\theta'))|t|}e^{-\arg(\mu)t}A^{it}\right\| \leqslant \left\|e^{-\theta'|t|}A^{it}\right\| \leqslant C',$$

where $C' := \sup_{t\in\mathbb{R}} \|e^{-\theta'|t|}A^{it}\|$ is finite since $\omega_{\mathrm{BIP}}(A) < \theta'$, and where

$$C := \frac{C'}{2}\int_{\mathbb{R}} \frac{e^{(\pi-(\theta-\theta'))|t|}}{|\sin(\pi(\frac{1}{2}-it))|}\,\mathrm{d}t.$$

We have shown that the family

$$\left\{\lambda^{1/2}A^{1/2}R(\lambda,A) : |\arg(\lambda)| \geqslant \theta\right\}$$

is γ-bounded. Taking squares, it follows that the family

$$\left\{\lambda AR(\lambda,A)^2 : |\arg(\lambda)| \geqslant \theta\right\}$$

is γ-bounded as well. Moreover we see that $\widetilde{\omega}_\gamma(A) \leqslant \theta$. This being true for all $\omega_{\mathrm{BIP}}(A) < \theta < \pi$, it follows that $\widetilde{\omega}_\gamma(A) \leqslant \omega_{\mathrm{BIP}}(A)$. □

15.3.e Connections with γ-sectoriality

We start with a definition.

Definition 15.3.17 (γ-bounded imaginary powers). *An operator A is said to have* γ-bounded imaginary powers *(briefly, A has* γ-BIP*) if it has bounded imaginary powers and the family*

$$\{A^{it} : |t| \leqslant 1\}$$

is γ-bounded.

If A has γ-bounded imaginary powers, the group property $A^{is}A^{it} = A^{i(s+t)}$ combined with Proposition 8.1.20 (or rather, the elementary bound in the discussion preceding it) implies that set

$$\{e^{-\omega|t|}A^{it} : t \in \mathbb{R}\}$$

is γ-bounded for large enough $\omega > 0$. Thus it makes sense to define the abscissa

$$\omega_{\gamma\text{-BIP}}(A) := \inf\left\{\omega \geqslant 0 : \ \{e^{-\omega|t|}A^{it} : t \in \mathbb{R}\} \text{ is } \gamma\text{-bounded}\right\}.$$

Replacing γ-boundedness by R-boundedness, we may similarly introduce operators A with *R-BIP* along with their abscissa $\omega_{R\text{-BIP}}(A)$ Since finite cotype implies equivalence of Rademacher sums and Gaussian sums (Corollary 7.2.10), an operator A on a Banach space with finite cotype has R-bounded imaginary powers if and only A has γ-bounded imaginary powers. As the ensuing proofs will make clear, operators with γ-bounded imaginary powers can be effectively studied using the continuous square functions introduced in Section 10.4.b. It is for this reason that our results will be stated for operators with γ-bounded imaginary powers. The analogous results for operators with R-bounded imaginary powers automatically follow if the underlying Banach space has finite cotype.

Proposition 15.3.18. *If A has γ-bounded imaginary powers, then $\omega_{\mathrm{BIP}}(A) = \omega_{\gamma\text{-BIP}}(A)$.*

Proof. Let $\omega_{\mathrm{BIP}}(A) < \nu < \theta$. For each $n \in \mathbb{Z}$ the singleton $\{A^{in}\}$ is γ-bounded, with γ-bound $\gamma(\{A^{in}\} = \|A^{in}\| \leqslant Me^{\nu|n|}$, where M is a constant independent of $n \in \mathbb{Z}$. By Proposition 8.1.20 (with $p = 1$ and $q = \infty$), the set

$$\{e^{-\theta|n|}A^{in} : \ n \in \mathbb{Z}\}$$

is γ-bounded. Combined with the fact that $\{A^{is} : \ s \in [-1, 1]\}$ is γ-bounded, by Proposition 8.1.19(3) we obtain that $\omega_{\gamma\text{-BIP}}(A) < \theta$. $\square$

We have seen in Theorem 15.3.12 that bounded imaginary powers imply sectoriality with angle $\omega(A) \leqslant \omega_{\mathrm{BIP}}(A)$. The next theorem provides the analogue for γ-bounded imaginary powers.

Theorem 15.3.19. *If A has γ-bounded imaginary powers with $\omega_{\gamma\text{-BIP}} < \pi$, then A is γ-sectorial with $\omega_\gamma(A) \leqslant \omega_{\gamma\text{-BIP}}(A)$.*

Proof. The proof proceeds in three steps.

Step 1 – In Steps 2 and 3 we will prove that each of the families of operators

$$\Gamma_s := \{t^s A^s (1 + tA)^{-1} : \ t > 0\}, \qquad \text{where} \ \ 0 < s < \frac{1}{2},$$

is γ-bounded, uniformly with respect to the parameter $s \in (0, \frac{1}{2})$. In the present step we show how the theorem follows from this.

For $x \in \mathsf{D}(A) \cap \mathsf{R}(A)$ we have

$$\lim_{s \downarrow 0} t^s A^s (I + tA)^{-1} x = (I + tA)^{-1} x.$$

Hence by Fatou's lemma, for all finite sequences $x_1, \ldots, x_n \in \mathsf{D}(A) \cap \mathsf{R}(A)$ and $t_1, \ldots, t_n > 0$ we have

$$\begin{aligned}
&\mathbb{E}\Big\| \sum_{k=1}^n \gamma_k (I + t_k A)^{-1} x_k \Big\|^2 \\
&\qquad \leqslant \liminf_{s \downarrow 0} \mathbb{E}\Big\| \sum_{k=1}^n \gamma_k t_k^s A^s (I + t_k A)^{-1} x_k \Big\|^2 \leqslant C \mathbb{E}\Big\| \sum_{k=1}^n \gamma_k x_k \Big\|^2,
\end{aligned}$$

where C is any finite upper bound for the γ-bounds of the families Γ_s, $s \in (0, \frac{1}{2})$. This proves that the set $\{(I + tA)^{-1} : t > 0\}$ is γ-bounded.

Applying this reasoning to operators $e^{i\theta}$ with $0 < |\theta| < \pi - \omega_{\gamma\text{-BIP}}$ (and noting that the identity $(e^{\pm i\theta} A)^{it} = e^{\mp\theta} A^{it}$ implies that these operators still have γ-bounded imaginary powers) and using Proposition 8.5.8 to extrapolate γ-boundedness from the boundary of a sector to the full sector, it follows that A is γ-sectorial and $\omega_\gamma(A) \leqslant \omega_{\gamma\text{-BIP}}(A)$.

Step 2 – We now turn to the proof of the γ-boundedness of the families Γ_s uniformly with respect to $s \in (0, \frac{1}{2})$. We claim that it suffices to prove that for all $f \in \mathscr{S}(\mathbb{R}; X)$ we have

$$\Big\| t \mapsto \int_{\mathbb{R}} k_s(t-u) A^{i(t-u)} f(u)\,\mathrm{d}u \Big\|_{\gamma(\mathbb{R};X)} \leqslant C \|f\|_{\gamma(\mathbb{R};X)}, \tag{15.20}$$

where the constant C is independent of $0 < s < \frac{1}{2}$ and

$$k_s(t) := \frac{1}{2\sin(\pi(s - it))}, \qquad t \in \mathbb{R}.$$

Indeed, suppose that (15.20) has been proved. By Fubini's theorem and the second identity of Lemma 15.3.13, for all $\xi \in \mathbb{R}$ we have

$$\begin{aligned} &\int_{\mathbb{R}} \int_{\mathbb{R}} k_s(t-u) A^{i(t-u)} f(u)\, e^{-2\pi i t\xi}\,\mathrm{d}u\,\mathrm{d}t \\ &\quad = \int_{\mathbb{R}} k_s(t) A^{it} e^{-2\pi i t\xi}\,\mathrm{d}t \int_{\mathbb{R}} f(u) e^{-2\pi i u\xi}\,\mathrm{d}u = e^{-2\pi\xi s} A^s (1 + e^{-2\pi\xi} A)^{-1} \widehat{f}(\xi). \end{aligned}$$

Hence by (15.20) and the fact, observed in Example 9.6.5, that the Fourier transform extends to an isometry on $\gamma(\mathbb{R}; X)$, we obtain

$$\|\xi \mapsto e^{-2\pi s\xi} A^s (1 + e^{-2\pi\xi} A)^{-1} \widehat{f}(\xi)\|_{\gamma(\mathbb{R};X)} \leqslant C\|f\|_{\gamma(\mathbb{R};X)} = C\|\widehat{f}\|_{\gamma(\mathbb{R};X)}.$$

Since the Fourier transform maps $\mathscr{S}(\mathbb{R}; X)$ onto itself and this space is dense in $\gamma(\mathbb{R}; X)$, this estimate extends to all strongly measurable function $g\colon S \to X$ representing an element of $\gamma(\mathbb{R}; X)$ by density. Then converse to the γ-multiplier theorem (Proposition 9.5.6) implies that Γ_s is γ-bounded, with γ-bound at most C.

Step 3 – To complete the proof of the theorem it remains to prove the bound (15.20) with a uniform constant C independent of $s \in (0, \frac{1}{2})$. We start with the observation that by (15.18) we have

$$\widehat{k_s}(\xi) = \frac{e^{-s\xi}}{1 + e^{-\xi}},$$

which implies that $\widehat{k_s} \in L^\infty(\mathbb{R})$ uniformly in $s \in (0, \frac{1}{2})$.

Fix $s \in (0, \frac{1}{2})$. For $n \in \mathbb{Z}$, set $I_n := [2n - 1, 2n + 1)$ and define, for $\varphi \in C_{\mathrm{c}}(\mathbb{R})$,

$$T_s^{(n)}\varphi(u) := \int_{\mathbb{R}} K_s^{(n)}(u,v)\varphi(v)\,\mathrm{d}v, \qquad u \in \mathbb{R},$$

where

$$K_s^{(n)}(u,v) := \sum_{j\in\mathbb{Z}} k_s(u-v)\mathbf{1}_{I_j}(u)\mathbf{1}_{I_{j+n}}(v).$$

This sum trivially converges pointwise in (t,v), since each such point is contained in at most one rectangle $I_j \times I_{j+n}$. We wish to show that the operator $T_s^{(n)}$ thus defined extends to a bounded operator on $L^2(\mathbb{R})$, uniformly in $s \in (0, \frac{1}{2})$.

For $\varphi \in C_c(\mathbb{R})$ we have, by the disjointness of the intervals I_j, monotone convergence, and a change of variables,

$$\begin{aligned}
\|T_s^{(n)}\varphi\|_2^2 &= \int_{\mathbb{R}} \Big| \sum_{j\in\mathbb{Z}} \mathbf{1}_{I_j}(u) \int_{\mathbb{R}} k_s(u-v)\mathbf{1}_{I_{j+n}}(v)\varphi(v)\,\mathrm{d}v\Big|^2 \mathrm{d}u \\
&= \int_{\mathbb{R}} \sum_{j\in\mathbb{Z}} \mathbf{1}_{I_j}(u)\Big| \int_{\mathbb{R}} k_s(u-v)\mathbf{1}_{I_{j+n}}(v)\varphi(v)\,\mathrm{d}v\Big|^2 \mathrm{d}u \\
&\leqslant \sum_{j\in\mathbb{Z}} \int_{\mathbb{R}} \Big| \int_{\mathbb{R}} k_s(u-v)\mathbf{1}_{I_{j+n}}(v)\varphi(v)\,\mathrm{d}v\Big|^2 \mathrm{d}u \\
&= \sum_{j\in\mathbb{Z}} \|\widehat{k}_s \widehat{\mathbf{1}_{I_{j+n}}\varphi}\|_2^2 \leqslant \sum_{j\in\mathbb{Z}} \|\widehat{\mathbf{1}_{I_{j+n}}\varphi}\|_2^2 \\
&= \sum_{j\in\mathbb{Z}} \int_{I_{j+n}} |\varphi(u)|^2\,\mathrm{d}u = \|\varphi\|_2^2.
\end{aligned}$$

This shows that $T_s^{(n)}$ extends to a bounded operator on $L^2(\mathbb{R})$. Moreover, since

$$|K_s^{(n)}(u,v)| \leqslant |k_s(u-v)|\mathbf{1}_{\{|u-v|\geqslant 2(|n|-1)\}}$$

and

$$|k_s(u)| \leqslant \frac{1}{2|\sinh(\pi u)|} \lesssim e^{-\pi|u|}, \qquad |u| \geqslant 1,$$

by Young's inequality we have

$$\|T_s^{(n)}\| \leqslant \|K_s^{(n)}\|_1 \lesssim \int_{\{|u|\geqslant 2(|n|-1)\}} e^{-\pi|u|}\,\mathrm{d}u \lesssim e^{-2\pi|n|}, \qquad |n| \geqslant 2.$$

By the γ-extension theorem (Theorem 9.6.1), the operators $T_s^{(n)}$ extend to bounded operators on $\gamma(\mathbb{R};X)$ and

$$\|T_s^{(n)}\|_{\mathscr{L}(\gamma(\mathbb{R}:X))} \leqslant C_0\, e^{-2\pi|n|}.$$

for some absolute constant $C_0 \geqslant 0$.

Define $p : \mathbb{R} \to \mathbb{Z}$ by $p(t) := 2j$ when $t \in I_j$. Then $|p(t) - t| \leqslant 1$ for all $t \in \mathbb{R}$. Let $\omega_{\gamma\text{-BIP}}(A) < \theta < \pi$ and let $C_1, C_2 > 0$ be such that

$$\gamma\{A^{is} : s \in [-1,1]\} \leqslant C_1, \quad \gamma\{A^{is} : s \in \mathbb{R}\} \leqslant C_2\, e^{\theta|s|}.$$

We may of course relate these constants, but that would only complicate the estimate below a bit. Fix a Schwartz function $f \in \mathscr{S}(\mathbb{R}; X)$ and an integer $n \in \mathbb{Z}$. If (u,t) belongs to the support of $K_s^{(n)}$, then $u \in I_j$ and $v \in I_{j+n}$ for some $j \in Z$, from which it follows that $p(u) = p(v) - 2n$. Therefore we may estimate

$$\begin{aligned}
&\Big\| u \mapsto \int_{\mathbb{R}} K_s^{(n)}(u,v) A^{i(u-v)} f(v)\,\mathrm{d}v \Big\|_{\gamma(\mathbb{R};X)} \\
&= \Big\| u \mapsto \int_{\mathbb{R}} K_s^{(n)}(u,v) A^{i(u-p(u)+p(v)-v-2n)} f(u)\,\mathrm{d}u \Big\|_{\gamma(\mathbb{R};X)} \\
&\leqslant C_1 \Big\| u \mapsto \int_{\mathbb{R}} K_s^{(n)}(u,v) A^{i(p(v)-v-2n)} f(v)\,\mathrm{d}u \Big\|_{\gamma(\mathbb{R};X)} \\
&\leqslant C_0 C_1\, e^{-2\pi|n|} \big\| u \mapsto A^{i(p(u)-u-2n)} f(u) \big\|_{\gamma(\mathbb{R};X)} \\
&\leqslant C_0 C_1^2 C_2\, e^{-2(\pi-\theta)|n|} \| f \|_{\gamma(\mathbb{R};X)}
\end{aligned}$$

using the γ-multiplier theorem (Theorem 9.5.1) in the second and fourth step. Since

$$k_s(u-v) = \sum_{n \in \mathbb{Z}} K_s^{(n)}(u,v), \qquad u, v \in \mathbb{R},$$

the bound (15.20) now follows from the triangle inequality. □

15.3.f Connections with boundedness of the H^∞-calculus

It has already been observed that standard sectorial operators with a bounded H^∞-calculus have bounded imaginary powers and $\omega_{\mathrm{BIP}}(A) \leqslant \omega_{H^\infty}(A)$, the angle of the H^∞-calculus of A (see Definition 10.2.10). The following theorem gives a more precise version of this result.

Theorem 15.3.20 (Cowling–Doust–McIntosh–Yagi). *If A is a standard sectorial operator with a bounded $H^\infty(\Sigma_\sigma)$-calculus for some $\omega(A) < \sigma < \pi$, then A has bounded imaginary powers and*

$$\omega_{\mathrm{BIP}}(A) = \omega_{H^\infty}(A).$$

Moreover,

$$\|A^{it}\| \leqslant M_{\sigma,A}^\infty e^{\sigma|t|}, \quad t \in \mathbb{R},$$

where $M_{\sigma,A}^\infty$ is the boundedness constant of the $H^\infty(\Sigma_\sigma)$-calculus of A.

Proof. It remains to prove the inequality $\omega_{H^\infty}(A) \leqslant \omega_{\mathrm{BIP}}(A)$. In view of the Clément–Prüss theorem (Theorem 15.3.12), which asserts that $\omega(A) \leqslant \omega_{\mathrm{BIP}}(A)$, it suffices to prove that if $\omega(A) < \mu < \nu \leqslant \sigma$ with $\|A^{it}\| \leqslant Me^{\mu|t|}$ for all $t \in \mathbb{R}$, then A has a bounded $H^\infty(\Sigma_\nu)$-calculus.

To this end let $f \in H^\infty(\Sigma_\nu)$. We will show that

$$f(z) = \sum_{k\in\mathbb{Z}} z^{ik} f_k(z), \quad z \in \Sigma_\nu, \tag{15.21}$$

for suitable functions $f_k \in H^\infty(\Sigma_\sigma)$ satisfying

$$\sum_{k\in\mathbb{Z}} e^{\mu|k|}\|f_k\|_{H^\infty(\Sigma_\sigma)} \leqslant C\|f\|_{H^\infty(\Sigma_\nu)} \tag{15.22}$$

with constant $C \geqslant 0$ independent of f. Once this has been shown, we may set

$$f(A) := \sum_{k\in\mathbb{Z}} A^{ik} f_k(A),$$

with convergence in the norm of $\mathscr{L}(X)$; here, the operators $f_k(A)$ are defined through the $H^\infty(\Sigma_\sigma)$-calculus of A. The bound (15.22) implies that

$$\|f(A)\| \leqslant CM\|f\|_\infty. \tag{15.23}$$

To complete the proof that A admits a bounded $H^\infty(\Sigma_\nu)$-calculus, we will show that for $f \in H^1(\Sigma_\nu) \cap H^\infty(\Sigma_\nu)$ the operator $f(A)$ thus defined agrees with the Dunford calculus of A.

Step 1 – In this step we prove everything up to and including (15.23). Using the change of variables $z = e^w$ we transform sectors to horizontal strips and must show that every $g \in H^\infty(S_\nu)$ can be expressed as

$$g(w) = \sum_{k\in\mathbb{Z}} e^{ikw} g_k(w), \quad w \in S_\nu,$$

where $S_\theta = \{z \in \mathbb{C} : |\Im(z)| < \theta\}$ and the functions $g_k \in H^\infty(S_\sigma)$ satisfy

$$\sum_{k\in\mathbb{Z}} e^{\mu|k|}\|g_k\|_{H^\infty(S_\sigma)} \leqslant C\|g\|_{H^\infty(S_\nu)}.$$

Let $\phi \in C_{\mathrm{c}}(\mathbb{R})$ satisfy

(i) $0 \leqslant \phi(x) \in \mathbf{1}_{(-1,1)}(\xi)$ for all $\xi \in \mathbb{R}$,
(ii) $\sum_{k\in\mathbb{Z}} \phi(\xi - k) = 1$ for all $\xi \in \mathbb{R}$,

and set

$$g_k(w) := \int_{\mathbb{R}} \check{\phi}(w-t)g(t)e^{-ikt}\,\mathrm{d}t, \quad w \in S_\sigma.$$

By the Paley–Wiener theorem, $\check{\phi}$ is an entire function with sufficient decay to ensure the convergence of the integral for every $w \in \mathbb{C}$. Fixing $w \in S_\sigma$

and $k \in \mathbb{Z}$, and sing Cauchy's theorem to shift the path of integration, for $\epsilon \in \{-1, 1\}$ we may write

$$g_k(w) := \int_{\mathbb{R}} \check{\phi}(w - t - i\epsilon\nu) g(t + i\epsilon\nu) e^{-ik(t+i\epsilon\nu)} \, \mathrm{d}t, \quad w \in S_\sigma.$$

Taking $\epsilon = -\operatorname{sgn}(k)$ gives the bound

$$\|g_k\|_\infty \leqslant C_{\sigma,\nu} e^{-\nu|k|} \|g\|_\infty, \quad k \in \mathbb{Z},$$

with

$$C_{\sigma,\nu} = \sup_{|y|<\sigma+\nu} \int_{\mathbb{R}} |\check{\phi}(x + iy)| \, \mathrm{d}x < \infty.$$

Setting $h_k(\xi) := \phi(\xi - k)\widehat{g}(\xi)$, a simple calculation gives

$$\widecheck{h_k}(w) = \int_{\mathbb{R}} \check{\phi}(w - t) g(t) e^{-ik(w-t)} \, \mathrm{d}t = e^{ikw} g_k(w), \quad w \in S_\sigma.$$

Since $\sum_{k\in\mathbb{Z}} h_k(\xi) = \widehat{g}(\xi)$ for all $\xi \in \mathbb{R}$, the result follows by taking inverse Fourier transforms.

Step 2 – It remains to show that for $f \in H^1(\Sigma_\nu) \cap H^\infty(\Sigma_\nu)$ the operator $f(A)$ defined by (15.21) agrees with the Dunford calculus. For this it suffices to observe that for such functions f, the functions g_k constructed in Step 1 belong to $H^1(S_\sigma) \cap H^\infty(S_\sigma)$ and

$$\|g_k\|_{H^1(S_\sigma)} \leqslant C_{\sigma,\nu} e^{-\nu|k|} \|g\|_{H^1(S_\nu)}, \quad k \in \mathbb{Z},$$

with $C_{\sigma,\nu}$ as before. It follows that the sum defining $f(A)$ also converges in $H^1(\Sigma_\nu)$. The required consistency now follows by interchanging summation and integration, along with the fact that $(z \mapsto z^{ik} f_k(z))(A) = A^{ik} f_k(A)$ in the extended Dunford calculus, hence *a posteriori* also in the Dunford calculus. □

With Theorem 15.3.19 at our disposal we will now investigate the connection between the γ-boundedness of the imaginary powers A^{it} and the boundedness of the H^∞-calculus of A. In preparation of the next result, it is useful to point out that in some of these results in Chapter 10 the finite cotype assumption can be dropped if one defines discrete square functions in terms of Gaussian sums instead of using Rademacher sums. To be explicit, assuming Definition 10.4.1 to have been modified in this way, the finite cotype assumption can be dropped in the following results:

- Proposition 10.4.15(2). Indeed, the proof uses the finite cotype assumption only to pass from Gaussian sums to the Rademacher sums used in the definition of discrete square functions.
- Theorem 10.4.16(1). Indeed, the finite cotype assumption was only used to apply Proposition 10.4.15(2).

- Proposition 10.4.20. Indeed, the finite cotype assumption was only used to apply Proposition 10.4.15(2).

The next theorem establishes the connection between γ-bounded imaginary powers and boundedness of the H^∞-calculus.

Theorem 15.3.21 (Bounded H^∞-calculus $\Leftrightarrow$ γ-BIP). *Let A be standard sectorial on a Banach space X.*

(1) *If A has γ-bounded imaginary powers with $\omega_{\gamma\text{-}BIP}(A) < \pi$, then A has a bounded H^∞-calculus and*

$$\omega_{H^\infty}(A) \leqslant \omega_{\gamma\text{-}BIP}(A).$$

(2) *If A has a bounded H^∞-calculus and X has Pisier's contraction principle, then A has γ-bounded imaginary powers and*

$$\omega_{\gamma\text{-}BIP}(A) \leqslant \omega_{H^\infty}(A).$$

Since Pisier's contraction principle implies finite cotype (Corollary 7.5.13), a sectorial operator A acting in a Banach space with this property has γ-bounded imaginary powers if and only A has R-bounded imaginary powers, and in that case

$$\omega_{\gamma\text{-BIP}}(A) = \omega_{R\text{-BIP}}(A).$$

Before turning to the proof of the theorem, we isolate a lemma which is essentially contained in the proof of Theorem 10.4.16. For the reader's convenience we repeat the argument here.

Lemma 15.3.22. *Let A be standard sectorial, let $\omega(A) < \sigma < \pi$, and suppose that there is a constant $C \geqslant 0$ such that for all $\psi \in H^1(\Sigma_\sigma)$ and $x^* \in \mathsf{D}(A^*) \cap \mathsf{R}(A^*)$ we have*

$$\big\| t \mapsto \psi(tA^*)x^* \big\|_{\gamma(\mathbb{R}_+, \frac{\mathrm{d}t}{t}; X^*)} \leqslant C \|\psi\|_{H^1(\Sigma_\sigma)} \|x^*\|.$$

Then for all non-zero $\phi \in H^1(\Sigma_\sigma)$ there is a constant $c_\phi \geqslant 0$ such that for all $x \in \mathsf{D}(A) \cap \mathsf{R}(A)$ we have

$$\|x\| \leqslant 2Cc_\phi M_{\sigma,A} \big\| t \mapsto \phi(tA)x \big\|_{\gamma(\mathbb{R}_+, \frac{\mathrm{d}t}{t}; X)}.$$

Note that the assumptions on x, x^*, ϕ, and ψ imply that $t \mapsto \phi(tA)x \in \gamma(\mathbb{R}_+, \frac{\mathrm{d}t}{t}; X)$ and $t \mapsto \psi(tA^*)x^* \in \gamma(\mathbb{R}_+, \frac{\mathrm{d}t}{t}; X^*)$ by Lemma 10.4.14 (which only assumes sectoriality and can therefore be applied to both A and A^*).

Proof. Fix a non-zero $\phi \in H^1(\Sigma_\sigma)$ and fix an arbitrary $\psi \in H^1(\Sigma_\sigma)$ such that $\int_0^\infty \phi(t)\psi(t) \frac{\mathrm{d}t}{t} = 1$. For all $x \in \mathsf{D}(A) \cap \mathsf{R}(A)$ and $x^* \in \mathsf{D}(A^*) \cap \mathsf{R}(A^*)$, from the reproducing formula of Proposition 10.2.5, the trace duality inequality of Theorem 9.2.14, and our assumption we obtain

$$
\begin{aligned}
|\langle x, x^*\rangle| &= \Big| \int_0^\infty \langle \phi(tA)\psi(tA)x, x^*\rangle \frac{\mathrm{d}t}{t} \Big| \\
&\leqslant \big\| t \mapsto \phi(tA)x \big\|_{\gamma(\mathbb{R}_+, \frac{\mathrm{d}t}{t}; X)} \big\| t \mapsto \psi(tA)^* x^* \big\|_{\gamma(\mathbb{R}_+, \frac{\mathrm{d}t}{t}; X^*)} \\
&\leqslant C \|\psi\|_{H^1(\Sigma_\sigma)} \big\| t \mapsto \phi(tA)x \big\|_{\gamma(\mathbb{R}_+, \frac{\mathrm{d}t}{t}; X)} \|x^*\|,
\end{aligned}
$$

where we used that $\psi(tA)^* = \psi(tA^*)$. Taking the supremum over all $x^* \in \mathsf{D}(A^*) \cap \mathsf{R}(A^*)$ of norm $\leqslant 1$, the result now follows from Lemma 10.2.19, with $c_\phi = \inf\{\|\psi\|_{H^1(\Sigma_\sigma)} : \int_0^\infty \phi(t)\psi(t) \frac{\mathrm{d}t}{t} = 1\}$. □

Proof of Theorem 15.3.21. (1): Fix $\omega_{\gamma\text{-BIP}}(A) < \sigma < \pi$. Then the set $\{e^{-\sigma|t|}A^{it} : t \in \mathbb{R}\}$ is γ-bounded.

Step 1 – In this step we prove that for all $\vartheta > \sigma$ and $x \in X$ the function $t \mapsto e^{-\vartheta|t|}A^{it}x$ belongs to $\gamma(\mathbb{R}; X)$.

By the result of Example 9.4.12 (taking $H = \mathbb{C}$), the function $t \mapsto e^{-(\vartheta-\sigma)|t|} \otimes x$ belongs to $\gamma(\mathbb{R}; X)$ and

$$
\big\| t \mapsto e^{-(\vartheta-\sigma)|t|} \otimes x \big\|_{\gamma(\mathbb{R}; X)} = \| t \mapsto e^{-(\vartheta-\sigma)|t|} \|_{L^2(\mathbb{R})} \|x\| \eqsim \frac{1}{(\vartheta-\sigma)^{1/2}} \|x\|.
$$

Hence by the γ-multiplier theorem (Theorem 9.5.1), $t \mapsto e^{-\vartheta|t|}A^{it}x$ belongs to $\gamma_\infty(\mathbb{R}; X)$ and

$$
\big\| t \mapsto e^{-\vartheta|t|}A^{it}x \big\|_{\gamma_\infty(\mathbb{R}; X)} \lesssim \frac{1}{(\vartheta-\sigma)^{1/2}} \gamma(\{e^{-\sigma|t|}A^{it} : t \in \mathbb{R}\}). \tag{15.24}
$$

We claim that the functions $t \mapsto e^{-\vartheta|t|}A^{it}x$ actually belong to the closed subspace $\gamma(\mathbb{R}; X)$ of $\gamma_\infty(\mathbb{R}; X)$. To prove this, let B be the generator of the C_0-group $(A^{it})_{t\in\mathbb{R}}$. For all $x \in \mathsf{D}(B)$ and all $0 < a < b < \infty$ and $-\infty < a < b < 0$ the function $t \mapsto e^{-\vartheta|t|}A^{it}x$ belongs to $C^1([a, b]; X)$, and hence to $\gamma(a, b; X)$ by Proposition 9.7.1. Since $\mathsf{D}(B)$ is dense in X, the claim now follows from Corollary 9.5.2.

Step 2 – The formula

$$
a^{-\frac{1}{2}+it} = \frac{\cosh(\pi t)}{\pi} \int_0^\infty u^{-\frac{1}{2}+it}(u+a)^{-1}\,\mathrm{d}u, \quad a > 0,\ t \in \mathbb{R}, \tag{15.25}
$$

may be proved by a contour integration argument. Alternatively, it can be obtained from a standard identity for the Mellin transform of the function $(1+t)^{-1}$ and some substitutions.

Set $\theta := \pi - \vartheta$. By analytic continuation, the identity (15.25) extends to complex $a \in \mathbb{C} \setminus (-\infty, 0]$. For $z \in \Sigma_{\pi-\theta}$ we may substitute $a = e^{-i\theta}z$ to obtain, after a bit of algebra,

$$
e^{\theta t} z^{it} = \frac{\cosh(\pi t)}{\pi} e^{\frac{1}{2}i\theta} \int_0^\infty u^{-\frac{1}{2}+it} z^{1/2} (e^{i\theta}u + z)^{-1}\,\mathrm{d}u.
$$

Since $\omega(A) < \pi - \theta$ (this is because $\omega(A) \leqslant \omega_{\mathrm{BIP}}(A) = \omega_{\gamma\text{-BIP}}(A)$ by the Clément–Prüss theorem and Proposition 15.3.18, and $\omega_{\gamma\text{-BIP}}(A) < \vartheta = \pi - \theta$ by assumption), we can apply Lemma 10.2.17 (with $p = 1$) to this identity and obtain, for all $x \in X$,

$$\begin{aligned} e^{\theta t} A^{it} x &= \frac{\cosh(\pi t)}{\pi} \int_0^\infty e^{\frac{1}{2} i\theta} u^{-\frac{1}{2}+it} A^{1/2} (e^{i\theta} u + A)^{-1} x \,\mathrm{d}u \\ &= \frac{\cosh(\pi t)}{\pi} \int_{-\infty}^\infty e^{itv} e^{\frac{1}{2}v + \frac{1}{2} i\theta} A^{1/2} (e^{i\theta} e^v + A)^{-1} x \,\mathrm{d}v, \end{aligned} \tag{15.26}$$

where the second identity results from the substitution $u = e^v$.

By Step 1, the function $t \mapsto e^{-\vartheta |t|} A^{it} x$ belongs to $\gamma(\mathbb{R}; X)$. Since $\cosh(\pi t) \sim e^{\pi t}$ and $\pi - \theta = \vartheta$, this implies that the function

$$t \mapsto \frac{e^{-\theta |t|} A^{it} x}{\cosh(\pi t)}$$

belongs to $\gamma(\mathbb{R}; X)$.

By Theorem 9.6.1, the γ-extension of Fourier–Plancherel transform is an isometry from $\gamma(\mathbb{R}; X)$ onto itself. Dividing both sides of (15.26) by $\cosh(\pi t)$ and applying this isometry, it follows that the function

$$v \mapsto e^{\pi v + \frac{1}{2} i\theta} A^{1/2} (e^{i\theta} e^{2\pi v} + A)^{-1} x$$

belongs to $\gamma(\mathbb{R}; X)$ and

$$\begin{aligned} \big\| v \mapsto e^{\pi v + \frac{1}{2} i\theta} A^{1/2} (e^{i\theta} e^{2\pi v} + A)^{-1} x \big\|_{\gamma(\mathbb{R}; X)} &\eqsim \Big\| t \mapsto \frac{e^{\theta t}}{\cosh(\pi t)} A^{it} x \Big\|_{\gamma(\mathbb{R}; X)} \\ &\eqsim \big\| t \mapsto e^{-\vartheta |t|} A^{it} x \big\|_{\gamma(\mathbb{R}; X)} \\ &\lesssim_A \frac{1}{\vartheta - \sigma} \|x\|, \end{aligned}$$

using (15.24) in the last step. Substituting back $e^v = u$ and leaving out terms of modulus one since they do not affect the γ-norms,

$$\big\| u \mapsto u^{1/2} A^{1/2} (e^{i\theta} u + A)^{-1} x \big\|_{\gamma(\mathbb{R}, \frac{\mathrm{d}u}{u}; X)} \lesssim_A \frac{1}{\vartheta - \sigma} \|x\|.$$

The term in the norm on the left-hand side is of the form $\phi(u^{-1} A)$ with $\phi(z) = z^{1/2} (e^{i\theta} + z)^{-1}$. This function belongs to $H^1(\Sigma_{\vartheta'})$ for all $0 < \vartheta' < \vartheta = \pi - \theta$, and the estimate can be interpreted as giving the square function estimate

$$\big\| t \mapsto \phi(tA) x \big\|_{\gamma(\mathbb{R}_+, \frac{\mathrm{d}t}{t}; X)} \lesssim \|x\|, \quad x \in X.$$

Note that up to this point we only have used that A is sectorial and has bounded imaginary powers (the γ-sectoriality assumption will only be used towards the end of the proof). Because of this, we can apply the same reasoning to the part $A^{\odot}$ of A^* in $X^{\odot} := \overline{\mathsf{D}(A^*)}$. Indeed, this operator is sectorial and has bounded imaginary powers on $X^{\odot}$ and $(A^{\odot})^{it}x^* = (A^{it})^*x^*$ for $x^* \in X^{\odot}$; we leave the easy verification as an exercise to the reader. Together with the identity $\phi(tA)^*x^* = \phi(tA^{\odot})x^*$, which is equally easy to verify, this gives the dual square function estimate

$$\big\|t \mapsto \phi(tA)^*x^*\big\|_{\gamma(\mathbb{R}_+,\frac{\mathrm{d}t}{t};X^*)} \lesssim \|x^*\|, \quad x^* \in X^{\odot} = \overline{\mathsf{D}(A^*)}.$$

Hence by Lemma 15.3.22,

$$\|x\| \lesssim \big\|t \mapsto \phi(tA)x\big\|_{\gamma(\mathbb{R}_+,\frac{\mathrm{d}t}{t};X)}, \quad x \in \mathsf{D}(A) \cap \mathsf{R}(A),$$

with an implied constant independent of x. We may now apply Theorem 10.4.19 (noting that thanks to Theorem 15.3.19 we have $\omega_\gamma(A) \leqslant \omega_{\gamma\text{-BIP}}(A)$) to conclude that A has a bounded $H^\infty(\Sigma_{\vartheta'})$-calculus for all $\omega_{\gamma\text{-BIP}}(A) < \vartheta' < \theta$. This completes the proof.

(2): Let A have a bounded $H^\infty(\Sigma_\sigma)$-calculus for some $\omega(A) < \sigma < \pi$, and let $\vartheta > \sigma$. Recalling the bound $|z^{it}| \leqslant e^{|t||\arg(z)|}$, the R-boundedness (and hence the γ-boundedness, as the Pisier contraction property implies finite cotype) of the set $\{e^{-\theta|t|}A^{it}\}$ follows from Theorem 10.3.4(3). This shows that A has γ-bounded imaginary powers and $\omega_{\gamma\text{-BIP}}(A) \leqslant \vartheta$. □

15.3.g The Hilbert space case

The last main result of this chapter is the following characterisation of sectorial operators on Hilbert spaces with bounded imaginary powers.

Theorem 15.3.23. *For any standard sectorial operator A on a Hilbert space H the following assertions are equivalent:*

(1) *A has a bounded H^∞-calculus;*
(2) *A has bounded imaginary powers.*

In this situation we have

$$\omega_{H^\infty}(A) = \omega_{\mathrm{BIP}}(A).$$

If in addition we have $0 \in \varrho(A)$, then the above conditions are equivalent to

(3) $\mathsf{D}(A^{1/2}) = (H, \mathsf{D}(A))_{\frac{1}{2},2}$ *with equivalent norms.*

In view of the equivalence of uniform boundedness and γ-boundedness for families of Hilbert space operators, the equivalence of (1) and (2) is a special case of the results in the preceding subsection. A version of the equivalence of these conditions with (3) for general Banach spaces will be discussed in the Notes at the end of the chapter.

Proof. It remains to prove the implications (2)⇒(3)⇒(1) under the additional assumption $0 \in \varrho(A)$. As a preliminary observation we point out that this assumption implies that we have equivalences of norms

$$\|x\|_{\mathsf{D}(A^{1/2})} \eqsim \|A^{1/2}x\| \tag{15.27}$$

and

$$\|x\|_{(H,\mathsf{D}(A))_{\frac{1}{2},2}} \eqsim \big\|\lambda \mapsto \lambda^{1/2}A(\lambda+A)^{-1}x\big\|_{L^2(\mathbb{R}_+,\frac{\mathrm{d}t}{t};H)}. \tag{15.28}$$

Indeed, (15.27) follows by writing $x = A^{-1/2}A^{1/2}x$ and using Corollary 15.2.10 to get

$$\|A^{1/2}x\| \leqslant \|x\| + \|A^{1/2}x\| \leqslant (\|A^{-1/2}\|+1)\|A^{1/2}x\|.$$

The equivalence (15.28) follows from Proposition K.4.1.

(2)⇒(3): The equality $\mathsf{D}(A^{1/2}) = (H,\mathsf{D}(A))_{\frac{1}{2},2}$ is an immediate consequence of Peetre's theorem (Theorem C.4.1), which in the present situation implies that for each $\theta \in (0,1)$ we have

$$(H,\mathsf{D}(A))_{\theta,2} = [H,\mathsf{D}(A)]_\theta \quad \text{with equivalent norms,}$$

and Theorem 15.3.9, which identifies $[H,\mathsf{D}(A)]_{1/2}$ as the fractional domain space $\mathsf{D}(A^{1/2})$ up to an equivalent norm.

(3)⇒(1): On H define

$$|\!|\!| x |\!|\!| := \big\|\lambda \mapsto \lambda^{1/2}A^{1/2}(\lambda+A)^{-1}x\big\|_{L^2(\mathbb{R}_+,\frac{\mathrm{d}t}{t};H)}.$$

In view of (15.27) and (15.28) and the assumption in (3), we have the norm equivalences

$$|\!|\!| x |\!|\!| \eqsim \|A^{-1/2}x\|_{(H,\mathsf{D}(A))_{\frac{1}{2},2}} \eqsim \|A^{-1/2}x\|_{\mathsf{D}(A^{1/2})} \eqsim \|x\|.$$

Consequently, $|\!|\!| \cdot |\!|\!|$ defines an equivalent Hilbertian norm on H. Recalling that $\gamma(L^2(\mathbb{R}_+\frac{\mathrm{d}t}{t}),H) = L^2(\mathbb{R}_+,\frac{\mathrm{d}t}{t};H)$ isometrically, the implication now follows from Theorem 10.4.21. □

15.3.h Examples

It has already been noted that every standard sectorial operator A with a bounded $H^\infty(\Sigma_\sigma)$-calculus for some $0<\sigma<\pi$ has bounded imaginary powers. Here we wish to highlight two examples:

Example 15.3.24 (Laplacian). Let $1<p<\infty$ and let X be a Banach space. It was already noted in the discussion preceding Theorem 15.3.11 that if X is a UMD space, then the negative of the Laplace operator Δ on $L^p(\mathbb{R}^d;X)$ with domain $\mathsf{D}(\Delta) = H^{2,p}(\mathbb{R}^d;X)$ has bounded imaginary powers. In the converse direction, it was shown in Section 10.5 that if $-\Delta$ has bounded imaginary powers on $L^p(\mathbb{R}^d;X)$, then X is a UMD space.

Example 15.3.25 (First derivative). Let $1 < p < \infty$ and let X be a UMD space.

(1) The operator $A = \mathrm{d}/\mathrm{d}x$ on $L^p(\mathbb{R};X)$ with domain $\mathsf{D}(A) = W^{1,p}(\mathbb{R};X)$ has bounded imaginary powers with angle $\frac{1}{2}\pi$.
(2) The operator $A = \mathrm{d}/\mathrm{d}t$ on $L^p(\mathbb{R}_+;X)$ with domain $\mathsf{D}(A) = \{f \in W^{1,p}(\mathbb{R}_+;X) : f(0) = 0\}$ has bounded imaginary powers with angle $\frac{1}{2}\pi$.
(3) The operator $A = \mathrm{d}/\mathrm{d}t$ on $L^p(0,T;X)$ with domain $\mathsf{D}(A) = \{f \in W^{1,p}(0,T;X) : f(0) = 0\}$ has bounded imaginary powers with angle $\frac{1}{2}\pi$ and, more precisely, we have the estimate

$$\|A^{is}\| \lesssim_T (1+s^2)e^{\frac{1}{2}\pi|s|}, \quad s \in \mathbb{R}.$$

For the proofs of (1), (2), and the first part of (3) one may observe that in each of these three cases A is standard sectorial.

In the case (1), $-A$ generates the translation group on $L^p(\mathbb{R};X)$, and in the other two cases $-A$ is the generator of the C_0-semigroup on $L^p(I;X)$ (with $I = \mathbb{R}_+$ resp. $(0,T)$) given by

$$S(t)f(s) = \begin{cases} f(s-t), & s \in I,\ s > t, \\ 0, & \text{otherwise.} \end{cases}$$

All three semigroups are contractive and, in the scalar-valued case, positive. It follows that we can apply the Hieber–Prüss theorem (Theorem 10.7.12), which gives that each of these operators has a bounded H^∞-calculus of angle $\frac{1}{2}\pi$. It then follows from Theorem 15.3.20 that each of the operators has bounded imaginary powers.

15.4 Strip type operators

It has already been noted in Volume II that the theories of Hardy spaces over a sector and a strip large rather similar. This similarity can be lifted to the operator level by introducing the ‘strip’ version of sectorial operators. Such operator admit again a Dunford calculus, a primary calculus, and an extended calculus, and one may ask about the boundedness of their H^∞-calculus. Since this topic is somewhat peripheral to the mainstream of these volumes, we will not embark on a systematic exploration of strip type operator, but rather concentrate on the relationship between sectorial operators and strip type operators. We have already seen several examples, both in Volume II and the present volume, where the relationship between sectorial operators and bisectorial operators (the mediating function being $z \mapsto z^2$) can be exploited in the study of sectorial operators. Likewise the connection with strip type operators (the mediating function being $z \mapsto e^z$) can sometimes be exploited. At the end of this section we demonstrate this by giving a proof of the Dore–Venni theorem by using the properties of strip type operators.

15.4.a Nollau's theorem

For $\vartheta > 0$ let

$$\mathbb{S}_\vartheta := \{z \in \mathbb{C} : |\Im z| < \vartheta\}$$

be the strip of height ϑ. From Appendix H we recall the definition of the Hardy space $H^p(\mathbb{S}_\vartheta)$, $1 \leqslant p \leqslant \infty$, as the Banach space of all holomorphic functions $f : \mathbb{S}_\vartheta \to \mathbb{C}$ for which the norm

$$\|f\|_{H^p(\mathbb{S}_\vartheta)} := \sup_{|y|<\vartheta} \|t \mapsto f(t+iy)\|_{L^p(\mathbb{R})}$$

is finite.

Definition 15.4.1. *A linear operator A acting in a Banach space X is said to be of* strip type $\omega > 0$ *if $\sigma(A) \subseteq \overline{\mathbb{S}_\omega}$ and*

$$\sup_{z \notin \overline{\mathbb{S}_\omega}} (|\Im z| - \omega)\|R(z, A)\| < \infty.$$

It is said to be of standard strip type $\omega > 0$ *if it is* strip type $\omega > 0$ *and* $\mathsf{D}(A) \cap \mathsf{R}(A)$ *is dense in X.*

The operator A is said to be of *(standard) strip type* if it is of (standard) strip type ω for some $\omega > 0$. The number

$$\omega^{\mathbb{S}}(A) := \inf\{\omega > 0 : \ A \text{ is of strip type } \omega\}$$

is called the *height* of A.

Example 15.4.2. By the easy part of the Hille–Yosida theorem, if iA is the generator of a C_0-group $(U(t))_{t\in\mathbb{R}}$ satisfying $\|U(t)\| \leqslant Me^{\omega|t|}$ for all $t \in \mathbb{R}$ and certain $M \geqslant 1$ and $\omega > 0$, then A is of strip type ω.

Theorem 15.4.3 (Nollau). *If A is standard sectorial, then* $\log(A)$ *is of standard strip type with $\omega^{\mathbb{S}}(A) \leqslant \omega(A)$, and the following Poisson type formula holds:*

$$R(z, \log(A)) = -\int_0^\infty \frac{1}{(z - \log t)^2 + \pi^2}(t + A)^{-1}\,\mathrm{d}t, \quad |\Im z| > \pi.$$

Proof. We proceed in two steps.

Step 1 – First we assume in addition that A is bounded and invertible. Let $\omega(A) < \nu' < \nu < \sigma < \pi$ and fix $\lambda \in \mathbb{C}$ with $|\Im\lambda| > \pi$ and $\mu \in \Sigma_\nu \setminus \Sigma_{\nu'}$. The function $z \mapsto 1/(\lambda - \log z)$ is holomorphic and bounded on Σ_σ. Let $x \in X$. Then by Proposition 15.1.19,

$$\frac{1}{\lambda - \log}(A)x$$

$$
\begin{aligned}
&= \frac{1}{\lambda - \log\mu}x + \frac{1}{2\pi i}\int_{\partial\Sigma_\nu} \frac{1}{\lambda - \log z}\Big(R(z,A) - \frac{1}{z-\mu}\Big)x\,\mathrm{d}z \\
&= \frac{1}{\lambda - \log\mu}x - \frac{1}{2\pi i}\int_0^\infty \frac{e^{i\nu}}{\lambda - i\nu - \log t}\Big(R(te^{i\nu},A) - \frac{1}{te^{i\nu}-\mu}\Big)x\,\mathrm{d}t \\
&\qquad\qquad + \frac{1}{2\pi i}\int_0^\infty \frac{e^{-i\nu}}{\lambda + i\nu - \log t}\Big(R(e^{-i\nu},A) - \frac{1}{te^{-i\nu}-\mu}\Big)x\,\mathrm{d}t.
\end{aligned}
$$

By dominated convergence we may pass to the limit $\nu \to \pi$ and obtain

$$
\begin{aligned}
&\frac{1}{\lambda - \log}(A)x \\
&\quad = \frac{1}{\lambda - \log\mu}x - \frac{1}{2\pi i}\int_0^\infty \frac{1}{\lambda - i\pi - \log t}\Big((t+A)^{-1} - \frac{1}{t+\mu}\Big)x\,\mathrm{d}t \\
&\qquad\qquad + \frac{1}{2\pi i}\int_0^\infty \frac{1}{\lambda + i\pi - \log t}\Big((t+A)^{-1} - \frac{1}{t+\mu}\Big)x\,\mathrm{d}t \\
&\quad = \frac{1}{\lambda - \log\mu}x - \int_0^\infty \frac{1}{(\lambda - \log t)^2 + \pi^2}\Big((t+A)^{-1} - \frac{1}{t+\mu}\Big)x\,\mathrm{d}t \\
&\quad = -\int_0^\infty \frac{1}{(\lambda - \log t)^2 + \pi^2}(t+A)^{-1}x\,\mathrm{d}t,
\end{aligned}
$$

where we used that

$$
\begin{aligned}
\int_0^\infty \frac{1}{(\lambda - \log t)^2 + \pi^2}\frac{1}{t+\mu}\,\mathrm{d}t &= -\lim_{\nu\to\pi}\frac{1}{2\pi i}\int_{\partial\Sigma_\nu}\frac{1}{\lambda - \log z}\frac{1}{z-\mu}\,\mathrm{d}z \\
&= -\frac{1}{\lambda - \log\mu}.
\end{aligned}
$$

By the multiplicativity of the extended calculus, $\frac{1}{\lambda-\log}(A)$ is inverse to $\lambda - \log(A)$. This gives $\lambda \in \varrho(\log(A))$ as well as the identity for the resolvent. The resolvent estimate follows from the following estimates, where we write $z = x + iy$ and set $M := \sup_{t>0}\|t(t+A)^{-1}\|$:

$$
\begin{aligned}
\|R(z,\log(A))\| &\leqslant M\int_0^\infty \frac{1}{|(z-\log t)^2 + \pi^2|}\,\frac{\mathrm{d}t}{t} \\
&\leqslant M\int_{-\infty}^\infty \frac{1}{|(z-s)^2+\pi^2|}\,\mathrm{d}s \\
&= M\int_{-\infty}^\infty \frac{1}{(((x-s)^2 - y^2 + \pi^2)^2 + (2(x-s)y)^2)^{1/2}}\,\mathrm{d}s \\
&= M\int_{-\infty}^\infty \frac{1}{((r^2 - y^2 + \pi^2)^2 + 4r^2y^2)^{1/2}}\,\mathrm{d}r \\
&\leqslant M\int_{-\infty}^\infty \frac{1}{r^2 + y^2 - \pi^2}\,\mathrm{d}r \\
&= \frac{M\pi}{(y^2-\pi^2)^{1/2}}
\end{aligned}
$$

$$\leqslant \frac{M\pi}{|y| - \pi},$$

where we used the elementary inequalities

$$\begin{aligned}(r^2 - y^2 + \pi^2)^2 + 4r^2y^2 &= r^4 + y^4 + \pi^4 + 2r^2y^2 + 2\pi^2r^2 - 2\pi^2y^2 \\ &\geqslant r^4 + y^4 + \pi^4 + 2r^2y^2 - 2\pi^2r^2 - 2\pi^2y^2 \\ &= (r^2 + y^2 - \pi^2)^2\end{aligned}$$

and (keeping in mind that $|y| > \pi$, so $2|y| - \pi > \pi$)

$$y^2 - \pi^2 \geqslant y^2 - \pi(2|y| - \pi) = (|y| - \pi)^2.$$

This proves the theorem under the additional assumption that A is bounded and has bounded inverse.

Step 2 – To deduce the general case, for $\varepsilon > 0$ we consider the operators

$$A_\varepsilon = (A + \varepsilon)(I + \varepsilon A)^{-1}.$$

For $\lambda \geqslant 0$ we have

$$\begin{aligned}\lambda + A_\varepsilon &= \lambda(I + \varepsilon A)(I + \varepsilon A)^{-1} + (A + \varepsilon)(I + \varepsilon A)^{-1} \\ &= (\lambda + \varepsilon + (\lambda\varepsilon + 1)A)(I + \varepsilon A)^{-1},\end{aligned}$$

and therefore $\lambda + A_\varepsilon$ is invertible. For $\lambda > 0$ we estimate

$$\begin{aligned}\|(\lambda + A_\varepsilon)^{-1}\| &= \|(I + \varepsilon A)(\lambda + \varepsilon + (\lambda\varepsilon + 1)A)^{-1}\| \\ &= \frac{\varepsilon}{\lambda\varepsilon + 1}\Big\|(\frac{1}{\varepsilon} + A)(\frac{\lambda + \varepsilon}{\lambda\varepsilon + 1} + A)^{-1}\Big\| \\ &= \frac{\varepsilon}{\lambda\varepsilon + 1}\Big\|I + (\frac{1}{\varepsilon} - \frac{\lambda + \varepsilon}{\lambda\varepsilon + 1})(\frac{\lambda + \varepsilon}{\lambda\varepsilon + 1} + A)^{-1}\Big\| \\ &\leqslant \frac{1}{\lambda} + \frac{\varepsilon}{\lambda\varepsilon + 1}\Big(\frac{1}{\varepsilon} - \frac{\lambda + \varepsilon}{\lambda\varepsilon + 1}\Big)\frac{\lambda\varepsilon + 1}{\lambda + \varepsilon}M_A \\ &= \frac{1}{\lambda} + \frac{1 - \varepsilon^2}{(\lambda + \varepsilon)(\lambda\varepsilon + 1)}M_A \\ &\leqslant \frac{1 + M_A}{\lambda},\end{aligned}$$

where $M_A = \sup_{\lambda > 0} \|\lambda(\lambda + A)^{-1}$. It follows that

$$\sup_{\varepsilon > 0}\Big(\sup_{\lambda > 0} \|\lambda(\lambda + A_\varepsilon)^{-1}\|\Big) \leqslant 1 + M_A,$$

and therefore the operators A_ε are uniformly sectorial. In particular the results of Step 1 apply to A_ε, with bounded that are uniform in $\varepsilon > 0$.

Step 3 – Take $0 < \nu < \pi$ close enough to π so that $\partial\Sigma_\delta$ is contained in the resolvent set of each A_ε. Noting that $R(z, A_\varepsilon)x \to R(z, A)x$ uniformly on

$\partial\Sigma_\delta$ (this follows from uniform sectoriality and similar estimates as above), we may pass to the limit $\varepsilon \downarrow 0$ and obtain

$$\begin{aligned}
\frac{1}{\lambda-\log}(A)x &= \frac{1}{\lambda-\log\mu}x + \frac{1}{2\pi i}\int_{\partial\Sigma_\nu}\frac{1}{\lambda-\log z}\Big(R(z,A)-\frac{1}{z-\mu}\Big)x\,\mathrm{d}z \\
&= \lim_{\varepsilon\downarrow 0}\frac{1}{\lambda-\log\mu}x + \frac{1}{2\pi i}\int_{\partial\Sigma_\nu}\frac{1}{\lambda-\log z}\Big(R(z,A_\varepsilon)-\frac{1}{z-\mu}\Big)x\,\mathrm{d}z \\
&= \lim_{\varepsilon\downarrow 0} -\int_0^\infty \frac{1}{(\lambda-\log t)^2+\pi^2}(t+A_\varepsilon)^{-1}x\,\mathrm{d}t \\
&= -\int_0^\infty \frac{1}{(\lambda-\log t)^2+\pi^2}(t+A)^{-1}x\,\mathrm{d}t.
\end{aligned}$$

Next we show that $\lambda \in \varrho(\log(A))$ and $\frac{1}{\lambda-\log}(A)$ is a two-sided inverse for $\lambda-\log(A)$. For $x \in \mathsf{D}(A^2)\cap\mathsf{R}(A^2)$, say $x = \zeta^2(A)y$, by the general properties of the extended Dunford calculus we have $\frac{1}{\lambda-\log}(A)x \in \mathsf{D}(A)\cap\mathsf{R}(A) \subseteq \mathsf{D}(\log(A))$ and

$$\begin{aligned}
(\lambda-\log(A))\frac{1}{\lambda-\log}(A)x &= \zeta(A)(\lambda-\log(A))\frac{\zeta}{\lambda-\log}(A)y \\
&= (\zeta(\lambda-\log))(A)\frac{\zeta}{\lambda-\log}(A)y \\
&= \Big(\zeta(\lambda-\log)\frac{\zeta}{\lambda-\log}\Big)(A)y \\
&= \zeta^2(A)y = x
\end{aligned}$$

and similarly $\frac{1}{\lambda-\log}(A)(\lambda-\log(A))x = x$. By density and closedness, these identities extend to general $x \in X$ and $x \in \mathsf{D}(\log(A))$, respectively.

Finally, the strip type estimate for A follows from the corresponding estimate for A_ε proved above, by letting $\varepsilon \downarrow 0$ and using dominated convergence once more. □

One may set up a Dunford calculus and extended Dunford calculus for strip type operators in much the same way as we did for sectorial operators as follows. For an operator A of strip type and $f \in H^1(\mathbb{S}_\sigma)$, where $\sigma > \omega^{\mathbb{S}}(A)$, the Dunford integral

$$f(A)x := \frac{1}{2\pi i}\int_{\partial\mathbb{S}_\nu} f(z)R(z,A)x\,\mathrm{d}x,$$

defines a bounded operator $f(A)$ on X. The defining integral converges absolutely and by Cauchy's theorem it is independent of the choice of ν. Moreover,

$$\|f(A)\| \leqslant \limsup_{\nu\downarrow\omega^{\mathbb{S}}(A)}\frac{1}{2\pi}\frac{C}{\nu-\omega}\int_{|\Im z|=\nu}|f(z)|\,|\mathrm{d}z| \leqslant \frac{1}{\pi}\frac{C}{\sigma-\omega^{\mathbb{S}}(A)}\|f\|_{H^1(\mathbb{S}_\sigma)}.$$

The elementary properties of the extended Dunford calculus extend to the strip case.

15.4.b Monniaux's theorem

We have seen (Proposition 15.3.5) that if an operator B in a Banach space X has bounded imaginary powers, then the bounded operators B^{it} form a C_0-group on X. In this subsection we will show that if X is a UMD space, then conversely every C_0-group on X of growth type less than π with injective generator is of the form $U(t) = B^{it}$ for some operator B in X with bounded imaginary powers:

Theorem 15.4.4 (Monniaux). *Let $(U(t))_{t\in\mathbb{R}}$ be a C_0-group on a UMD space X satisfying $\|U(t)\| \leqslant Me^{\omega|t|}$ for all $t \in \mathbb{R}$ and some $M \geqslant 1$ and $0 \leqslant \omega < \pi$. Assume furthermore that its generator iA is injective. Then there exists an operator B in X with bounded imaginary powers, given by*

$$B^{it} = U(t), \quad t \in \mathbb{R}.$$

Moreover, we have $A = \log(B)$ with equal domains.

Intuitively, one has $B = e^A$; the identity $B^{it} = U(t)$ then corresponds to the intuition that $(e^A)^{it} = e^{itA}$.

The proof of the theorem relies on several ingredients. The first is the following lemma.

Lemma 15.4.5. *Let $(U(t))_{t\in\mathbb{R}}$ be a C_0-group on a UMD space X. If, for some $M \geqslant 1$ and $\omega \in \mathbb{R}$, we have $\|U(t)\| \leqslant Me^{\omega|t|}$ for all $t \in \mathbb{R}$, then for all $x \in X$ the principal value integral*

$$\text{p.v.} \int_{-1}^{1} U(t)x \, \frac{\mathrm{d}t}{t}$$

converges in X and has norm

$$\Big\|\text{p.v.} \int_{-1}^{1} U(t)x \, \frac{\mathrm{d}t}{t}\Big\| \leqslant 6C^2 \hbar_{2,X} \|x\|,$$

where $\hbar_{2,X} := \|H\|_{\mathscr{L}(L^2(\mathbb{R};X))}$, and $C := \sup_{|t|\leqslant 2} \|U(t)\|$.

Proof. All we need to do is stripping the Rademacher sums from the estimates in the last part of the proof of Theorem 15.3.12(2). For the reader's convenience we include the proof that results from this.

Fix $0 < \delta < 1$, $s \in [-\frac{1}{2}, \frac{1}{2}]$, and $x \in X$. Then

$$\begin{aligned}
\int_{\delta<|t|<1} U(t)x \, \frac{\mathrm{d}t}{t} &= U(s) \int_{\delta<|t|<1} U(t-s)x \, \frac{\mathrm{d}s}{s} \\
&= U(s) \int_{|t|>\delta} \varphi_x(s-t) \, \frac{\mathrm{d}t}{t} \\
&\qquad - \int_{1}^{1+s} U(t)x \, \frac{\mathrm{d}t}{t} + \int_{-1}^{-1+s} U(t)x \, \frac{\mathrm{d}t}{t},
\end{aligned}$$

where $\varphi_x(\tau) = \mathbf{1}_{(-1,1)}(\tau)U(-\tau)x$. Integrating over $[-\frac{1}{2}, \frac{1}{2}]$, we obtain

$$\begin{aligned}\int_{\delta<|t|<1} U(t)x \,\frac{\mathrm{d}t}{t} &= \int_{-\frac{1}{2}}^{\frac{1}{2}} U(s) \int_{|t|>\delta} \varphi_x(s-t)\,\frac{\mathrm{d}t}{t}\,\mathrm{d}s \\ &\quad - \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_1^{1+s} U(t)x\,\frac{\mathrm{d}t}{t}\,\mathrm{d}s + \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{-1}^{-1+s} U(t)x\,\frac{\mathrm{d}t}{t}\,\mathrm{d}s.\end{aligned}$$

Since X is UMD and $\phi_x \in L^2(\mathbb{R}; X)$, the limit

$$\lim_{\delta\downarrow 0} \int_{|t|>\delta} \varphi_x(\cdot - t)\,\frac{\mathrm{d}t}{t} = \lim_{\substack{\delta\downarrow 0\\ R\to\infty}} \int_{\delta<|t|<R} \varphi_x(\cdot - t)\,\frac{\mathrm{d}t}{t}$$

exists in $L^2(\mathbb{R}; X)$ by Theorem 5.1.1 and equals $\pi H\phi_x$, where H denotes the Hilbert transform. As a result we obtain

$$\begin{aligned}\text{p.v.}\int_{-1}^{1} U(t)x\,\frac{\mathrm{d}t}{t} &= \lim_{\delta\downarrow 0} \int_{\delta<|t|<1} U(t)x\,\frac{\mathrm{d}t}{t} \\ &= \pi \int_{-\frac{1}{2}}^{\frac{1}{2}} U(s)H\varphi_x(s)\,\mathrm{d}s \\ &\quad - \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_1^{1+s} U(t)x\,\frac{\mathrm{d}t}{t}\,\mathrm{d}s + \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{-1}^{-1+s} U(t)x\,\frac{\mathrm{d}t}{t}\,\mathrm{d}s \\ &=: I + II + III.\end{aligned}$$

With constants $C := \sup_{|t|\leqslant 2} \|U(t)\|$ and $\hbar_{2,X} := \|H\|_{\mathscr{L}(L^2(\mathbb{R};X))}$, we have

$$\|I\| \leqslant \pi C\Big(\int_{-\frac{1}{2}}^{\frac{1}{2}} \|H\varphi_x(s)\|^2\,\mathrm{d}s\Big)^{\frac{1}{2}} \leqslant \pi C\hbar_{2,X}\|\varphi_x\|_{L^2(\mathbb{R};X)} \leqslant \sqrt{2}\pi C^2\hbar_{2,X}\|x\|.$$

The other two terms are elementary with

$$\|II\| \leqslant \int_{-\frac{1}{2}}^{\frac{1}{2}} C\Big|\int_1^{1+s} \frac{\mathrm{d}t}{t}\Big|\|x\|\,\mathrm{d}s \leqslant C\log 2\|x\|,$$

and III can bounded in exactly the same way. Note that both $C \geqslant \|U(0)\| = 1$ and $\hbar_{2,X} \geqslant 1$, and $\sqrt{2}\pi + 2\log 2 < 6$. □

We will use this lemma for the second ingredient for the proof of Theorem 15.4.4, a primary calculus for strip type operators. We work under the assumptions of Theorem 15.4.4 and let $\omega < \sigma < \pi$. For functions

$$g \in L^1_\omega(\mathbb{R}) = \big\{g \in L^1_{\mathrm{loc}}(\mathbb{R}) : t \mapsto e^{\omega|t|}g(t) \in L^1(\mathbb{R})\big\}$$

we define the bounded operator $\widehat{g}(A) := \varPhi_g(A)$ by the Phillips calculus (see Section 10.7.a):

$$\widehat{g}(A)x := \int_0^\infty g(t)U(t)x\,\mathrm{d}t, \quad x \in X.$$

Obviously,

$$\|\widehat{g}(A)\|_{\mathscr{L}(X)} \leqslant M\|g\|_{L^1_\omega(\mathbb{R})},$$

where $\|g\|_{L^1_\omega(\mathbb{R})} := \|t \mapsto e^{\omega|t|}g(t)\|_{L^1(\mathbb{R})}$ and M is in Theorem 15.4.4. The following lemma enables us to enrich this calculus with certain bounded functions in $H^\infty(\mathbb{S}_\sigma)$ which have limits for $\Re z \to \pm\infty$.

Lemma 15.4.6. *The Fourier transform of the distribution*

$$h(t) := \mathrm{p.\,v.}\,\frac{1}{t}\mathbf{1}_{(-1,1)}(t) \qquad \Big(\langle h, \phi\rangle := \lim_{\varepsilon\downarrow 0}\int_{\varepsilon<|t|<1}\phi(t)\,\frac{\mathrm{d}t}{t} \quad \forall \phi \in \mathscr{S}(\mathbb{R})\Big),$$

equals

$$\widehat{h}(\xi) = \int_{-1}^1 e^{-2\pi it\xi}\,\frac{\mathrm{d}t}{t} = 2\int_0^1 \sin(2\pi t\xi)\,\frac{\mathrm{d}t}{t}$$

and its analytic continuation to $\mathbb{S}_\sigma$ *satisfies*

$$\lim_{\substack{|\Im z|<\sigma\\ \Re z\to\infty}} \widehat{h}(z) = \pm\pi.$$

Proof. The first assertion follows by elementary computation and the second from the standard improper integral

$$\int_0^\infty \sin t\,\frac{\mathrm{d}t}{t} = \frac{\pi}{2}$$

and a change of variables. □

For small $\varepsilon > 0$ let $h_\varepsilon(t) := t^{-1}\mathbf{1}_{(-1,-\varepsilon]\cup[\varepsilon,1)}(t)$. Applying the Phillips calculus, we obtain

$$\widehat{h_\varepsilon}(A)x = \int_{-1}^1 h_\varepsilon(t)U(t)x\,\mathrm{d}t$$

and therefore the principal value integral

$$\widehat{h}(A)x := \mathrm{p.v.}\int_{-\pi}^{\pi} U(t)x\,\frac{\mathrm{d}t}{t}$$

exists and satisfies

$$\|\widehat{h}(A)x\| \leqslant 6C^2\hbar_{2,X}\|x\|$$

with constants as in Lemma 15.4.5.

Now we are in a position to define our primary calculus:

Definition 15.4.7 (Primary calculus). *Let A be a strip type operator and let $\omega^{\mathbb{S}}(A) < \omega < \sigma$. For functions $f : \mathbb{S}_\sigma \to \mathbb{C}$ of the form*

$$f = \widehat{g} + a\widehat{h} + b$$

with $a, b \in \mathbb{C}$, $g \in L^1_\omega(\mathbb{R})$, and h as above, we set

$$f(A) := \widehat{g}(A) + a\widehat{h}(A) + bI.$$

The condition on f is satisfied if f' is bounded and there exists an $\alpha > 1$ such that

$$f'(z) = O(|z|^\alpha) \quad \text{as} \quad |\Re z| \to \infty. \tag{15.29}$$

The primary calculus enjoys similar properties as the one for sectorial operators; in particular it is multiplicative and consistent with the Dunford calculus. The proof is elementary but a bit tedious and it is therefore left to the reader.

For every $r \in \mathbb{R}$ the primary calculus can be applied to $A + r$ in place of A, noting that $i(A + r)$ generates the C_0-group $(e^{irt}U(t))_{t\in\mathbb{R}}$. It is immediate from the above constructions that the estimates are uniform with respect to r, i.e., for all $f : \mathbb{S}_\sigma \to \mathbb{C}$ of the above form we have

$$\sup_{r\in\mathbb{R}} \|f(A + r)\| < \infty. \tag{15.30}$$

We will now exploit the fact that, for $0 \leqslant |\omega| < \sigma < \pi$, the exponential function $z \mapsto e^z$ maps the line $\Im z = \omega$ bijectively onto the ray $\arg(z) = \omega$. Thus, it maps the strip $\mathbb{S}_\sigma$ bijectively onto the sector Σ_σ. From this, we infer that if $\lambda \in \complement\overline{\Sigma_\sigma}$, then the function

$$r_\lambda(z) := \frac{1}{\lambda - e^z} \tag{15.31}$$

is bounded and holomorphic on $\mathbb{S}_\sigma$. What is more, this function is of the form discussed above and therefore $r_\lambda(A)$ is well defined in the primary calculus (as its derivative satisfies (15.29)).

Remark 15.4.8. In hindsight, one could have introduced the primary calculus using the functions r_λ instead of h. This would restore the symmetry with the definition of the primary calculus for sectorial operator. Hoever, this would require an independent construction of the operators $r_\lambda(A) = R(\lambda, B)$ by different means.

By the algebraic properties of the functions r_λ and the multiplicativity properties of the calculus, the operators R_λ form a *pseudo-resolvent* in the the sense of the following proposition.

Lemma 15.4.9 (Pseudo-resolvents). *Let $V \subseteq \mathbb{C}$ be a non-empty connected open set and let $(R_\lambda)_{\lambda\in V}$ be a family of bounded operators on a Banach space X satisfying the resolvent identity*

$$R_\lambda - R_\mu = (\mu - \lambda)R_\lambda R_\mu, \quad \lambda, \mu \in V.$$

If R_{λ_0} is injective for some $\lambda_0 \in V$, then there exists a unique closed operator B on X such that $V \subseteq \varrho(B)$ and $R_\lambda = (\lambda - B)^{-1}$ for all $\lambda \in V$.

Proof. The resolvent identity implies that any two R_λ and R_μ commute.

If $R_\lambda x = 0$, the identity $R_\lambda - R_{\lambda_0} = (\lambda_0 - \lambda)R_{\lambda_0}R_\lambda$ implies that $R_{\lambda_0}x = 0$ and therefore $x = 0$. It follows that R_λ is injective. If $y \in \mathsf{R}(R_{\lambda_0})$, there is a unique $x \in X$ such that $y = R_{\lambda_0}x$. Then $y = R_\lambda(I - (\lambda_0 - \lambda)R_{\lambda_0})x \in \mathsf{R}(R_\lambda)$.

This shows that $\mathsf{N} := \mathsf{N}(R_\lambda) = \{0\}$ and $\mathsf{R} := \mathsf{R}(R_\lambda)$ are independent of $\lambda \in V$. Hence if $y \in \mathsf{R}$, then for all $\lambda \in V$ there is a unique $x_\lambda \in X$ such that $y = R_\lambda x_\lambda$. Then, by the resolvent identity,

$$(\mu - \lambda)R_\lambda R_\mu y = R_\lambda y - R_\mu y = R_\lambda R_\mu x_\mu - R_\mu R_\lambda x_\lambda = R_\lambda R_\mu(x_\mu - x_\lambda).$$

It follows that $(\mu - \lambda)y = (x_\mu - x_\lambda)$. This implies that $\mu y - x_\mu = \lambda y - x_\lambda$ is independent of $\lambda, \mu \in V$. Denoting this element by By, we obtain a linear operator $B : y \mapsto \lambda y - x_\lambda$ with domain $\mathsf{D}(B) = \mathsf{R}(R_{\lambda_0})$. It satisfies

$$R_\lambda(\lambda - B)y = R_\lambda x_\lambda = y,$$

so R_λ is a left inverse to $\lambda - B$. Applying this to $R_\lambda y$ instead of y we also obtain $R_\lambda(\lambda - B)R_\lambda y = R_\lambda y$, and the injectivity of R_λ therefore gives $(\lambda - B)R_\lambda y = y$, so R_λ is a right inverse to $\lambda - B$. This proves that $\lambda \in \varrho(B)$ and $(\lambda - B)^{-1} = R_\lambda$. That B is closed follows from the fact that its resolvent set contains the non-empty set V. $\square$

This construction gives us a rigorous way to construct the operator e^A as the closed operator B given by the lemma.

Proof of Theorem 15.4.4. By Lemma 15.4.9 there exists a unique closed operator B on X such that $R_\lambda = r_\lambda(A) = (\lambda - B)^{-1}$ for all $\lambda \in V := \complement\overline{\Sigma_\sigma}$. We note that

$$\lambda R(\lambda, B) = (I - \lambda^{-1}B)^{-1} = r_1(A - \log\lambda),$$

and therefore (15.30) implies that B is σ-sectorial.

Since sectorial operators on reflexive Banach spaces are densely defined (see Proposition 10.1.9), the operator B is densely defined. We claim that B in injective. If $\lambda, \mu \in \complement\overline{\Sigma_\sigma}$, then

$$(\lambda - B)^{-1}(\mu - B)^{-1}B \subseteq f_{\lambda,\mu}(B) \text{ with } f_{\lambda,\mu}(z) = \frac{e^z}{(\lambda - e^z)(\mu - e^z)}.$$

The operator $f_{\lambda,\mu}(B)$ is injective in view if the identity

$$f_{\lambda,\mu}(B) = -\frac{1}{\lambda - e^z}(B)\frac{\mu^{-1}}{\mu^{-1} - e^{-z}}(B).$$

This proves the claim. As a consequence of Proposition 15.3.2, we obtain that B is standard sectorial.

The identity $B^{it} = U(t)$ follows by writing out the definition of B^{it} in the extended calculus of B. This results in an expression involving a Dunford integral containing the resolvent of B. This resolvent can be expressed, via the definition of the primary calculus, in terms of the Phillips calculus of the C_0-group U. The details are laborious and are left to the reader. From this, and the general properties of the extended calculus, it follows that $A = \log(B)$. □

15.4.c The Dore–Venni theorem

In this section we apply Monniaux's theorem (Theorem 15.4.4) to prove the celebrated Dore–Venni theorem on the closedness of sums of closed operators. We base our proof on the following lemma. It uses the fact that if iG is the generator of a bounded C_0-group, then G is bisectorial of angle 0 (see Example 10.6.3). In what follows we use that the extended Dunford calculus can be developed also for bisectorial operators. When we cite results from Section 15.1, it is understood that we actually refer to their bisectorial counterparts. We leave it to the reader to verify that these counterparts do indeed hold.

Lemma 15.4.10. *Let $(U(t))_{t\in\mathbb{R}}$ and $(V(t))_{t\in\mathbb{R}}$ be commuting C_0-groups on a Banach space X with generators $-iA$ and $-iB$, respectively, such that*

$$\|U(t)\| \leqslant M_A e^{\omega_A|t|} \quad \text{and} \quad \|V(t)\| \leqslant M_B e^{\omega_B|t|}$$

for all $t \in \mathbb{R}$, where $M_A, M_B \geqslant 1$ and $\omega_A, \omega_B \geqslant 0$ satisfy $\omega_A + \omega_B \leqslant \pi$. Let $-iC$ denote the generator of the C_0-group $W(t) = U(t)V(t)$, $t \in \mathbb{R}$. Then for all $x \in \mathsf{D}(e^A e^B)$ we have $x \in \mathsf{D}(e^C)$ and

$$e^A e^B x = e^C x.$$

Proof. We begin by noting that

$$\|U(t)\| \leqslant M_A M_B e^{(\omega_A+\omega_B)|t|}, \quad t \in \mathbb{R}.$$

In what follows we fix $\omega_A + \omega_B < \sigma < \pi$.

Fix $0 < \vartheta < \frac{1}{2}\pi$ and consider the functions $f, g \in H^1(\Sigma_\vartheta)$ given by

$$f(z) = \begin{cases} e^{2z}(1+e^z)^{-3}, & z \in \Sigma_\vartheta^+, \\ e^{-z}(1+e^{-z})^{-3}, & z \in \Sigma_\vartheta^-, \end{cases}$$

and

$$g(z) = \begin{cases} e^{z}(1+e^{z})^{-3}, & z \in \Sigma_\vartheta^+, \\ e^{-2z}(1+e^{-z})^{-3}, & z \in \Sigma_\vartheta^-. \end{cases}$$

For $G \in \{A, B, C\}$ the operators $f(G)$ and $g(G)$ are well defined and bounded in the bisectorial Dunford calculus. Moreover, by (the bisectorial counterpart of) Proposition 15.1.12 these operators are injective and $e^z = f(z)/g(z)$ implies

$$e^G = g(G)^{-1} f(G).$$

Our aim is to prove that

$$f(A)f(B)g(C) = g(A)g(B)f(C). \tag{15.32}$$

Once we have shown this, from $f(A)g(B)^{-1} \subseteq g(B)^{-1}f(A)$ (this follows by observing that $\mathsf{D}(g(B)^{-1}) = \mathsf{R}(g(B))$ and using that $f(A)$ and $g(B)$ commute, we obtain

$$\begin{aligned} e^A e^B &= g(A)^{-1} f(A) g(B)^{-1} f(B) \\ &\subseteq g(A)^{-1} g(B)^{-1} f(A)) f(B) \\ &= (g(A)^{-1} g(B)^{-1} g(C)^{-1})(g(C) f(A)) f(B)) \\ &= (g(C)^{-1} g(A)^{-1} g(B)^{-1})(g(A) g(B) f(C)) \\ &= (g(C)^{-1} g(A)^{-1} g(B)^{-1})(g(B) g(A) f(C)) \\ &= g(C)^{-1} f(C) = e^C, \end{aligned}$$

using that $g(A)$ and $g(B)$ commute in the penultimate equality.

The proof of (15.32) relies on the properties of the Phillips calculus (see Section 10.7.a). We recall from Proposition 10.7.2 and Lemma 10.7.3 that if iG generates a bounded C_0-group $(W(t))_{t\in\mathbb{R}}$ on X, then for $0 < \vartheta < \frac{1}{2}\pi$ and $h \in H^1(\Sigma_\vartheta^{\rm bi})$ one has

$$h(G)x = \int_{-\infty}^{\infty} \phi_h(t) W(t) x \,\mathrm{d}t, \quad x \in X,$$

where $\phi_h \in L^1(\mathbb{R})$ is given by

$$\widehat{\phi_h}(\xi) = h(-2\pi\xi), \quad \xi \in \mathbb{R}.$$

Applying this to $G \in \{-A, -B, -C\}$ and $h \in \{f, g\}$, and keeping in mind that $-iA$, $-iB$, and $-iC$ generate the groups $U(t)$, $V(t)$, and $U(t)V(t)$, respectively, the identity (15.32) takes the form

$$\begin{aligned} &\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} \phi_g(r)\phi_f(t)\phi_f(s) U(t+r) V(s+r) \,\mathrm{d}r \,\mathrm{d}s \,\mathrm{d}t \\ &\qquad = \int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} \phi_f(r)\phi_g(t)\phi_g(s) U(t+r) V(s+r) \,\mathrm{d}r \,\mathrm{d}s \,\mathrm{d}t, \end{aligned}$$

or equivalently,

$$\int_{\mathbb{R}}\int_{\mathbb{R}} \phi_g(r)F(t,s)U(t)V(s)\,\mathrm{d}s\,\mathrm{d}t = \int_{\mathbb{R}}\int_{\mathbb{R}} \phi_g(r)G(t,s)U(t)V(s)\,\mathrm{d}s\,\mathrm{d}t$$

with

$$F(t,s) = \int_{\mathbb{R}} \phi_g(r)\phi_f(t-r)\phi_f(s-r)\,\mathrm{d}r, \quad G(t,s) = \int_{\mathbb{R}} \phi_f(r)\phi_g(t-r)\phi_g(s-r)\,\mathrm{d}r.$$

Taking Fourier transforms, we obtain

$$\begin{aligned}\widehat{F}(x,y) &= \int_{\mathbb{R}}\int_{\mathbb{R}} F(t,s)e^{-2\pi i(tx+sy)}\,\mathrm{d}t\,\mathrm{d}s \\ &= \int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} \phi_g(r)\phi_f(t-r)\phi_f(s-r)e^{-2\pi i(tx+sy)}\,\mathrm{d}r\,\mathrm{d}t\,\mathrm{d}s \\ &= \int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} \phi_g(r)\phi_f(t-r)\phi_f(s-r)e^{-2\pi itx}e^{-2\pi isy}e^{-2\pi ir(x+y)}\,\mathrm{d}r\,\mathrm{d}t\,\mathrm{d}s \\ &= \widehat{\phi_g}(x+y)\widehat{\phi_f}(x)\widehat{\phi_f}(y) \\ &= g(-2\pi(x+y))f(-2\pi x)f(-2\pi y)\end{aligned}$$

and similarly

$$\widehat{G}(x,y) = f(-2\pi(x+y))g(-2\pi x)g(-2\pi y).$$

It is evident from the definitions of f and g that the two right-hand sides are equal. Therefore $F = G$ be the uniqueness of Fourier transforms. This completes the proof of (15.32). □

Theorem 15.4.11 (Dore–Venni). *Suppose that A and B are resolvent commuting standard sectorial operators on a UMD Banach space X. If both A and B have bounded imaginary powers with*

$$\omega_{\mathrm{BIP}}(A) + \omega_{\mathrm{BIP}}(B) < \pi,$$

then there exists a constant $K \geqslant 0$ such that

$$\|Ax\| + \|Bx\| \leqslant K\|(A+B)x\|, \quad x \in \mathsf{D}(A) \cap \mathsf{D}(B).$$

As a consequence $A + B$, with its natural domain $\mathsf{D}(A+B) = \mathsf{D}(A) \cap \mathsf{D}(B)$, is closed.

Proof. Fix $\omega_{\mathrm{BIP}}(A) + \omega_{\mathrm{BIP}}(B) < \omega < \pi$. Since A and B resolvent commute, the operators $U_A(s) = A^{is}$ and $U_B(t) = B^{it}$ commute for all $s, t \in \mathbb{R}$ and $U(t) := A^{it}B^{-it}$ is a C_0-group satisfying $\|U(t)\| \leqslant Ke^{\omega|t|}$ for all $t \in \mathbb{R}$ and some $K \geqslant 1$. Hence by Monniaux's Theorem 15.4.4 we have $U(t) = C^{it}$, $t \in \mathbb{R}$, for some standard sectorial operator C having bounded imaginary powers. The generators of the C_0-groups equal $i\log A$, $-i\log B$, and $i\log C$. Since $I+C$ is invertible there is a constant $M\geqslant 0$ such that have

$$\|x\|\leqslant M\|x+Cx\|, \quad x\in \mathsf{D}(C).$$

By Lemma 15.4.10 applied to $i\log A$ and $-i\log B$, for all $x\in\mathsf{D}(AB^{-1})$ we have $x\in\mathsf{D}(C)$ and $AB^{-1}x=Cx$. Hence for all $x\in\mathsf{D}(A)\cap\mathsf{D}(B)$ we have $Bx\in\mathsf{D}(AB^{-1})\subseteq\mathsf{D}(C)$, and therefore

$$\|Bx\|\leqslant M\|Bx+CBx\|=M\|Bx+Ax\|.$$

The same argument with the roles of A and B interchanged gives the inequality

$$\|Ax\|\leqslant M\|Bx+CBx\|=M\|Bx+Ax\|.$$

Together, these two inequalities imply the inequality in the statement of the theorem. This implies the closedness of $A+B$ by a routine argument. □

15.5 The bisectorial H^∞-calculus revisited

The bisectorial H^∞-calculus has been introduced and studied in Section 10.6. The purpose of the present section is to study in more detail the spectral projections associated with the left and right halves of the bisector. These can be thought of as abstract Riesz projections. The main result is Theorem 15.5.2, which establishes that if A is a standard bisectorial operator with a bounded $H^\infty(\Sigma_\sigma^{\rm bi})$-calculus, then A^2 is a standard 2σ-sectorial operator and

$$\mathsf{D}((A^2)^{1/2})=\mathsf{D}(A),$$

with equivalence of norms

$$\|(A^2)^{1/2}x\|\eqsim\|Ax\|.$$

We use the notation introduced in Section 10.6. Specifically, for $0<\omega<\frac12\pi$ we define $\Sigma_\omega^+:=\Sigma_\omega$ and $\Sigma_\omega^-:=-\Sigma_\omega$, and denote by

$$\Sigma_\omega^{\rm bi}:=\Sigma_\omega^+\cup\Sigma_\omega^-$$

the *bisector* of angle ω. Recall that a linear operator A on a Banach space X is said to be *bisectorial* if there exists an $\omega\in(0,\frac12\pi)$ such that the spectrum $\sigma(A)$ is contained in $\overline{\Sigma_\omega^{\rm bi}}$ and

$$M_{\omega,A}^{\rm bi}:=\sup_{z\in\complement\overline{\Sigma_\omega^{\rm bi}}}\|zR(z,A)\|<\infty.$$

In this situation we say that A is ω-*bisectorial.* The infimum of all $\omega\in(0,\frac12\pi)$ such that A is ω-bisectorial is called the *angle of bisectoriality* of A and is denoted by $\omega^{\rm bi}(A)$.

15.5.a Spectral projections

What distinguishes the theory of bisectorial operators from its sectorial counterpart is the possibility to consider the functions $\mathbf{1}_{\Sigma^+}$ and $\mathbf{1}_{\Sigma^-}$, both of which are bounded and holomorphic as functions on the bisector $\Sigma^{\mathrm{bi}} = \Sigma^+ \cup \Sigma^-$. If a bisectorial operator A has a bounded $H^\infty(\Sigma^{\mathrm{bi}})$-calculus, the operators $\mathbf{1}_{\Sigma^+}(A)$ and $\mathbf{1}_{\Sigma^-}(A)$ are well defined as bounded operators on $\overline{\mathsf{D}(A) \cap \mathsf{R}(A)}$ and take the role of "spectral projections" associated with the sectors Σ^+ and Σ^-. (The reason for writing quotations is that one has to be a bit careful here since 0 may belong to the spectrum of A.) From the multiplicativity of the H^∞-calculus it follows that the operators $\mathbf{1}_{\Sigma^+}$ and $\mathbf{1}_{\Sigma^-}$ are indeed projections, and that they are mutually orthogonal in the sense that

$$\mathbf{1}_{\Sigma^+}(A)\mathbf{1}_{\Sigma^-}(A) = \mathbf{1}_{\Sigma^-}(A)\mathbf{1}_{\Sigma^+}(A) = 0.$$

The injectivity of A on $\overline{\mathsf{D}(A) \cap \mathsf{R}(A)}$, which follows from Proposition 10.1.8, will be seen to imply the identity

$$\mathbf{1}_{\Sigma^-}(A) + \mathbf{1}_{\Sigma^-}(A) = I.$$

The importance of the operators $\mathbf{1}_{\Sigma^+}(A)$ and $\mathbf{1}_{\Sigma^-}(A)$ stems from their analogy to the Riesz projections; in particular, their difference

$$\mathbf{1}_{\Sigma^+}(A) - \mathbf{1}_{\Sigma^-}(A) =: \operatorname{sgn}(A)$$

may be thought of as an abstract analogue of the Hilbert transform.

Proposition 15.5.1. *If A is a bisectorial operator on a Banach space X with a bounded $H^\infty(\Sigma^{\mathrm{bi}}_\sigma)$-calculus for some $\omega^{\mathrm{bi}}(A) < \sigma < \frac{1}{2}\pi$, then the operators*

$$P^+ := \mathbf{1}_{\Sigma^+_\sigma}(A), \quad P^- := \mathbf{1}_{\Sigma^-_\sigma}(A),$$

are well defined as bounded projections on $\overline{\mathsf{D}(A) \cap \mathsf{R}(A)}$. As such they are mutually orthogonal in the sense that

$$P^+P^- = P^-P^+ = 0,$$

and complementary in the sense that

$$P^+ + P^- = I.$$

Denoting the parts of A in $X^+ := \mathsf{R}(P^+)$ and $X^- := \mathsf{R}(P^-)$ by A^+ and A^- respectively, then both A^+ and $-A^-$ are sectorial on X^+ and X^- and have bounded $H^\infty(\Sigma^+_\sigma)$-calculus on these spaces. We have

$$\sigma(A^\pm) \subseteq \sigma(A) \cap \overline{\Sigma^\pm_\sigma},$$

and if $\mathsf{D}(A) \cap \mathsf{R}(A)$ is dense we also have

$$(\sigma(A) \cap \overline{\Sigma^\pm_\sigma}) \setminus \{0\} \subseteq \sigma(A^\pm) \setminus \{0\}.$$

There is some abuse of notation in writing $X^\pm$ for the range of $P^\pm$, as these operators are projections defined on $\overline{\mathsf{D}(A) \cap \mathsf{R}(A)}$ only, but we do not want to overburden the notation.

Proof. That the operators P^+ and P^- are mutually orthogonal projections on $\overline{\mathsf{D}(A) \cap \mathsf{R}(A)}$ and add up to the identity follows from the general properties of the H^∞-calculus. It is also clear that both projections commute with the resolvent of A. Thus the spaces X^+ and X^- are invariant under the resolvent of A. Denote by A^+ and A^- the parts of A to X^+ and X^-. It suffices to prove that A^+ has the asserted properties; the result for A^- then follows by applying it to the bisectorial operator $-A$.

Step 1a – In this step we prove that $\sigma(A^+) \subseteq \sigma(A) \cap \overline{\Sigma_\sigma^+}$.

Let us write $p^+(z) = \mathbf{1}_{\Sigma_\sigma^+}(z)$ (so $p^+(A)x = P^+x$) and $r_\mu(z) = (\mu - z)^{-1}$ for brevity.

The crux of the argument is the observation that if $\mu \in \complement\overline{\Sigma_\sigma^+}$, then $r_\mu p^+$ is a bounded holomorphic function on the bisector $\Sigma_\sigma^{\rm bi}$ even when $\mu \in \overline{\Sigma_\sigma^-}$. Therefore the operator $(r_\mu p^+)(A)$ is well defined by the $H^\infty(\Sigma_\sigma^{\rm bi})$-calculus of A.

We shall prove next that the restriction of this operator to X^+ is a two-sided inverse of $\mu - A^+$. This will show that inclusion $\sigma(A^+) \subseteq \overline{\Sigma_\sigma^+}$. By general spectral considerations we also have $\sigma(A^+) \subseteq \sigma(A)$, and together these inclusions prove

$$\sigma(A^+) \subseteq \sigma(A) \cap \overline{\Sigma_\sigma^+}.$$

First we prove that the restriction of $(r_\mu p^+)(A)$ to X^+ is a two-sided inverse of $\mu - A^+$ for $\mu \in \complement\overline{\Sigma_\sigma^{\rm bi}}$. Fix $x \in \overline{\mathsf{D}(A) \cap \mathsf{R}(A)}$. We have $r_\mu(A)x = R(\mu, A)x$ by the properties of the $H^\infty(\Sigma_\sigma^{\rm bi})$-calculus. The multiplicativity of this calculus then implies

$$\begin{aligned}(r_\mu p^+)(A)x &= r_\mu(A)p^+(A)x = R(\mu, A)P^+x,\\ (r_\mu p^+)(A)x &= (p^+ r_\mu)(A)x = P^+R(\mu, A)x.\end{aligned} \tag{15.33}$$

It follows that X^+ is invariant under $R(\mu, A)$. Multiplying the first identity on the left and the second on the right by $\mu - A^+$ we see $(r_\mu p^+)(A)$ is a two-sided inverse to $\mu - A^+$. It follows that $\mu \in \varrho(A^+)$ and

$$R(\mu, A^+) = (r_\mu p^+)(A)|_{X^+}. \tag{15.34}$$

We now consider the case of a general $\mu \in \complement\overline{\Sigma_\sigma}$, which will be handled by the resolvent identity. To this end fix an arbitrary $\lambda \in \complement\overline{\Sigma_\sigma^{\rm bi}}$. The scalar identity

$$\frac{1}{\mu - z} = \frac{1}{\lambda - z} + \frac{\lambda - \mu}{(\lambda - z)(\mu - z)}$$

translates, after multiplying with $p^+(z)$ and using the additivity and multiplicativity of the H^∞-calculus, into the identity

$$(r_\mu p^+)(A)x = (r_\lambda p^+)(A)x + (\lambda - \mu)(r_\lambda p^+)(A)(r_\mu p^+)(A)x, \tag{15.35}$$

still for $x \in \overline{\mathsf{D}(A) \cap \mathsf{R}(A)}$. Among other things it implies that X^+ is invariant under $(r_\mu p^+)(A)$, since we have just proved that $(r_\lambda p^+)(A) = P^+ R(\mu, A)$ maps into X^+. By (15.33) and (15.34) (applied with λ in place of μ), the right-hand side of (15.35) (hence also the left-hand side) belongs to $\mathsf{D}(A)$, and for $x \in X^+$ we obtain

$$\begin{aligned}(\mu - A)(r_\mu p^+)(A)x &= (\mu - \lambda)(r_\mu p^+)(A)x + (\lambda - A)(r_\mu p^+)(A)x \\ &= (\mu - \lambda)(r_\mu p^+)(A)x + \Big[x + (\lambda - \mu)(r_\mu p^+)(A)x\Big] \\ &= x.\end{aligned}$$

It follows that $(r_\mu p^+)(A)x \in \mathsf{D}(A^+)$ and $(r_\mu p^+)(A)$ is a right inverse of $\mu - A^+$ on X^+. Also, using (15.34) (again with λ in place of μ) and the fact that $(r_\lambda p^+)(A)$ and $(r_\mu p^+)(A)$ in (15.35) commute, for $x \in \mathsf{D}(A^+)$ we obtain

$$\begin{aligned}(r_\mu p^+)(A)(\mu - A^+)x &= (r_\mu p^+)(A)(\mu - \lambda)x + (r_\mu p^+)(A)(\lambda - A^+)]x \\ &= (\mu - \lambda)(r_\mu p^+)(A)x + [x + (\lambda - \mu)(r_\mu p^+)(A)x] = x,\end{aligned}$$

and therefore $(r_\mu p^+)(A)$ is also a right inverse.

Step 1b – We now prove that if $\mu \neq 0$ belongs to $\sigma(A) \cap \overline{\Sigma_\sigma^+}$ and $\mathsf{D}(A) \cap \mathsf{R}(A)$ is dense, then $\mu \in \sigma(A^+)$.

To this end let $\mu \in \overline{\Sigma_\sigma^+}$. Since $\mu \neq 0$, we have $\mu \in \complement\overline{\Sigma_\sigma^-}$ and therefore, by the version of Step 1a for A^-, $\mu \in \varrho(A^-)$. Now if we also had $\mu \in \varrho(A^+)$, then along the decomposition $X = \overline{\mathsf{D}(A) \cap \mathsf{R}(A)} = X^+ \oplus X^-$, the operator $R(\mu, A^+) \oplus R(\mu, A^-)$ would be a two-sided inverse for $\mu - A$ and it would follow that $\mu \in \varrho(A)$. Hence for $\mu \neq 0$ this proves the inclusion $\sigma(A) \cap \overline{\Sigma_\sigma^+} \subseteq \sigma(A^+)$.

Step 2 – We next establish the resolvent bound for A^+. The uniform boundedness of $zR(z, A^+)$ on $\complement\overline{\Sigma_\sigma}^{\mathrm{bi}}$ follows from the uniform boundedness of $zR(z, A)$ on this set by taking restrictions. In particular, for any $\frac{1}{2}\pi < \vartheta < \pi - \sigma$, $zR(z, A^+)$ is uniformly bounded on the two rays $re^{\pm i\vartheta}$, and then the sectorial version of the three lines lemma implies the uniform boundedness of $zR(z, A^+)$ on Σ_ϑ^-.

Step 3 – The prescription $f(A^+) := (p^+ f)(A)$ defines a bounded linear multiplicative mapping from $H^\infty(\Sigma_\sigma^+)$ into $\mathscr{L}(\overline{\mathsf{D}(A) \cap \mathsf{R}(A)})$ that agrees with the Dunford calculus of A^+ for functions $f \in H^1(\Sigma_\sigma^+) \cap H^\infty(\Sigma_\sigma^+)$. This immediately implies that A^+ has a bounded $H^\infty(\Sigma_\sigma^+)$-calculus. $\square$

15.5.b Sectoriality versus bisectoriality

The next theorem establishes a relationship between a bisectorial operator A and the square root of the sectorial operator A^2. In the proof we shall use the fact, which can be routinely checked by redoing the arguments of Section 15.1, that an extended Dunford calculus can be set up for bisectorial operators and that it enjoys similar properties as in the sectorial case.

Theorem 15.5.2. *If A is a standard bisectorial operator with a bounded $H^\infty(\Sigma_\sigma^{\rm bi})$-calculus on a Banach space X, then A^2 is a standard 2σ-sectorial operator and*

$$\mathsf{D}((A^2)^{1/2}) = \mathsf{D}(A).$$

For all x in this common domain we have

$$\|(A^2)^{1/2}x\| \eqsim \|Ax\|$$

with constants independent of x.

Proof. The function $a(z) := (z^2)^{1/2}$ is holomorphic on $\Sigma_\sigma^{\rm bi}$ and equals z on Σ_σ^+ and $-z$ on Σ_σ^-. Likewise, the function $\mathrm{sgn}(z) := z/(z^2)^{1/2}$ is holomorphic on $\Sigma_\sigma^{\rm bi}$ and equals 1 on Σ_σ^+ and -1 on Σ_σ^-. Thus, $\mathrm{sgn}(z)a(z) = z$ for all $z \in \Sigma_\sigma^{\rm bi}$. By the multiplicativity of the extended functional calculus (cf. Proposition 15.1.12 for the sectorial case), it follows that, if $x \in \mathsf{D}((A^2)^{1/2})$, then $x \in \mathsf{D}(A)$ and $Ax = \mathrm{sgn}(A)(A^2)^{1/2}x$. Taking norms, we find that

$$\|Ax\| \leqslant M\|(A^2)^{1/2}x\|,$$

where M is the boundedness constant of the $H^\infty(\Sigma_\sigma^{\rm bi})$-calculus of A.

In the same way, the identity $a(z) = \mathrm{sgn}(z)z$ implies that, if $x \in \mathsf{D}((A^2)^{1/2})$, then $x \in \mathsf{D}(A)$ and $(A^2)^{1/2}x = \mathrm{sgn}(A)Ax$. Taking norms gives

$$\|(A^2)^{1/2}x\| \leqslant M\|Ax\|.$$

□

It is of some interest to interpret this theorem for the Hodge–Dirac operator D on $L^2(\mathbb{R}^d) \oplus L^2(\mathbb{R}^d;\mathbb{C}^d)$ of Example 10.6.5,

$$D = \begin{pmatrix} 0 & \nabla^* \\ \nabla & 0 \end{pmatrix}.$$

where $\nabla^* = -\mathrm{div}$ is the adjoint of ∇. Its square is of the form

$$D^2 = \begin{pmatrix} -\Delta & 0 \\ 0 & * \end{pmatrix},$$

where $*$ equals (at least formally) $-\nabla\circ\mathrm{div}$. Taking $g(z) = \mathrm{sgn}(z) := z/(z^2)^{1/2}$, then (at least formally)

$$g(D) = D(D^2)^{-1/2} = \begin{pmatrix} 0 & -\mathrm{div} \\ \nabla & 0 \end{pmatrix} \cdot \begin{pmatrix} (-\Delta)^{-1/2} & 0 \\ 0 & * \end{pmatrix} = \begin{pmatrix} 0 & * \\ \nabla/(-\Delta)^{1/2} & 0 \end{pmatrix}.$$

Thus we see the Riesz transform

$$R = \nabla(-\Delta)^{1/2}$$

appear as an entry in the functional calculus of D.

15.6 Notes

Section 15.1

The idea to use regularising functions to extend the functional calculus to suitable classes of unbounded functions goes back to McIntosh [1986]. A comprehensive discussion of extended functional calculi is presented by Haase [2006]; see also Haase [2020]. Our treatment in Sections 15.1 and 15.2 is based on Haase [2006] and Kunstmann and Weis [2004]. The proof of Theorem 15.1.18 is taken from the former reference.

Section 15.2

The first unified account of the theory of fractional powers was undertaken by Komatsu in a series of papers starting with Komatsu [1966]. This paper contains the results presented here and much more. Some earlier works on the subject are due to Balakrishnan [1960], Hille and Phillips [1957], Kato [1960, 1961], Krasnosel′skiĭ and Sobolevskiĭ [1959], Phillips [1952], Watanabe [1961], and Yosida [1960]. Modern accounts include Albrecht, Duong, and McIntosh [1996], Denk, Hieber, and Prüss [2003], Dore [1999, 2001], Haase [2006], and Martínez Carracedo and Sanz Alix [2001]. Our treatment barely scratches the surface of this rich and vast subject, and we have only included the most basic results needed for the treatment of bounded imaginary powers. Our approach based on the extended Dunford calculus has the advantage of keeping the technical details at a minimum, but the price to be paid is that we must make somewhat restrictive assumptions on the operator A.

Theorem 15.2.8 is from Komatsu [1966], but the proof presented here is taken from Haase [2006]. Theorem 15.2.13, 15.2.17, and 15.2.16 are due to Balakrishnan [1960] (see also Yosida [1980]). Some authors take one of the formulas in the first and third theorem as the definition of the fractional powers. For further information on the classical approach to fractional powers via integral representations, we refer the reader to the monographs Butzer and Berens [1967] and Martínez Carracedo and Sanz Alix [2001]. A complete proof of the non-negativity of the function $f_{\alpha,t}$ in Theorem 15.2.17 can be found in Yosida [1980, Proposition IX.11.2].

Section 15.3

A detailed account of the theory of bounded imaginary powers is presented by Haase [2006]; see also Amann [1995] and Prüss and Simonett [2016].

Example 15.3.25 is from Dore and Venni [1987]. Lemma 15.3.8 is from Prüss and Sohr [1990], where a different proof based on properties of the Mellin transform is given. Alternative proofs were obtained subsequently by Monniaux [1997] and Uiterdijk [1998]. The elementary proof presented here, based on the perturbation result of Theorem 15.1.18, is taken from Haase

[2006]. Theorem 15.3.9, identifying domains of fractional powers with complex interpolation spaces in the presence of bounded imaginary powers, is due to Seeley [1971]; see Triebel [1978] for references to earlier results in this direction.

Theorem 15.3.12, connecting the angles (R-)sectoriality and bounded imaginary powers, is due to Prüss and Sohr [1990] ($\omega(A) \leqslant \omega_{\mathrm{BIP}}(A)$) and Clément and Prüss [2001] ($\omega_R(A) \leqslant \omega_{\mathrm{BIP}}(A)$, in a UMD space). The estimation of the three terms in the last part of the proof are patterned after the proof of Lemma 15.4.5, which is taken from Monniaux [1999] and extends earlier results of Zsidó [1983] and Berkson, Gillespie, and Muhly [1986a]. Lemma 15.3.13 is from Prüss and Sohr [1990]. In Remark 15.3.14, the identity (15.17) can be equivalently stated in terms of the Mellin transform; see, e.g., Titchmarsh [1986]. The two theorems about (almost) γ-sectoriality and (γ-)bounded imaginary powers, Theorem 15.3.16 ($\widetilde{\omega}_\gamma(A) \leqslant \omega_{\mathrm{BIP}}(A)$) and its proof, as well as Theorem 15.3.19 ($\omega_\gamma(A) \leqslant \omega_{\gamma\text{-}\mathrm{BIP}}(A)$), are taken from Kalton, Lorist, and Weis [2023]. Theorem 15.3.20 ($\omega_{\mathrm{BIP}}(A) = \omega_{H^\infty}(A)$) is from Cowling, Doust, McIntosh, and Yagi [1996], whose proof we follow. The result proved in this paper shows that $\omega_{H^\infty}(A) = \max\{\omega(A), \omega_{\mathrm{BIP}}(A)\}$; as was noted in the main text, this improves to $\omega_{H^\infty}(A) = \omega_{\mathrm{BIP}}(A)$ by virtue of the Clément–Prüss Theorem 15.3.12. A different proof of Theorem 15.3.20, based on the theory of Euclidean structures, is presented by Kalton, Lorist, and Weis [2023]. Theorem 15.3.21, on the equivalence of bounded H^∞-calculus and γ-bounded imaginary powers, is taken from Kalton and Weis [2016]. An alternative proof is presented by Kalton, Lorist, and Weis [2023, Theorem 4.5.6 and Corollary 4.5.7]

An example of a sectorial operator on the space c_0 without bounded imaginary powers was was given by Komatsu [1966]. Hilbert space examples were constructed subsequently by McIntosh and Yagi [1990] and Baillon and Clément [1991], where a general way to construct such examples using conditional bases was invented. Venni [1993] showed that, in any Banach space with a Schauder basis, there are densely defined sectorial operators A for which some, but not all, imaginary powers are bounded. More precisely, it can be arranged that $A^{ik\pi} = I$ if k is an even integer and $A^{ik\pi}$ is unbounded if k is an odd integer. Hieber [1996] constructed an example of a pseudo-differential operator acting in $L^p(\mathbb{R})$, $1 < p < \infty$, $p \neq 2$, that is sectorial but does not admit bounded imaginary powers. Examples of operators in $L^p(S)$, $p \neq 2$, with bounded imaginary powers but without a bounded H^∞-calculus can be found in Cowling, Doust, McIntosh, and Yagi [1996]. This reference also contains the proof of the inequality $\omega_{H^\infty}(A) \leqslant \omega_{\mathrm{BIP}}(A)$.

Theorem 15.3.23 is due to McIntosh [1986]. In this connection, it is also of interest to mention the result of Yagi [1984, Theorem B] that an invertible sectorial operator A on a Hilbert space has bounded imaginary powers if $\mathsf{D}(A^\alpha) = \mathsf{D}(A^{\star\alpha})$ for all $\alpha \in [0, \epsilon)$.

Around almost sectoriality: further results

The next result, a proof of which is given by Kalton, Lorist, and Weis [2023, Proposition 4.2.4], connects the (almost) γ-sectoriality of A to γ-boundedness of the associated semigroup.

Proposition 15.6.1. *Let A be a sectorial operator on X with $\omega(A) < \frac{1}{2}\pi$ and let $\omega(A) < \sigma < \frac{1}{2}\pi$. Then*

(i) *A is γ-sectorial with $\omega_\gamma(A) \leqslant \sigma$ if and only if the set*
$$\{e^{-zA} : \ z \in \Sigma_\nu\}$$
is γ-bounded for all $0 < \nu < \frac{1}{2}\pi - \sigma$;

(ii) *A is almost γ-sectorial with $\widetilde{\omega}_\gamma(A) \leqslant \sigma$ if and only if the set*
$$\{zAe^{-zA} : \ z \in \Sigma_\nu\}$$
is γ-bounded for all $0 < \nu < \frac{1}{2}\pi - \sigma$.

For operators A that are diagonal with respect to a Schauder basis, the notion of γ-almost sectoriality captures a natural property of the basis. In order to formulate this in the form of a proposition, we first recall that a sequence $(x_n)_{n\in\mathbb{Z}}$ in a Banach space X is called a *Schauder basis* of X if every $x \in X$ has a unique representation of the form $x = \sum_{n\in\mathbb{Z}} c_n x_n$. Associated with a Schauder basis is its sequence of coordinate projections $(P_n)_{n\in\mathbb{Z}}$ on X, defined by
$$P_n \sum_{j\in\mathbb{Z}} c_j x_j := x_n, \qquad n \in \mathbb{Z},$$
and the sequence of partial sum projections $(S_n)_{n\in\mathbb{N}}$, defined by
$$S_n \sum_{j\in\mathbb{Z}} x_j := \sum_{j=-n}^{n} x_j, \qquad n \in \mathbb{N},$$
that is, $S_n = \sum_{k=-n}^{n} P_k$. For any Schauder basis, the set of partial sum projections is uniformly bounded, and by taking differences the same is seen to be true for the set of coordinate projections.

On a Banach space X with a Schauder basis $(x_n)_{n\in\mathbb{Z}}$, we may consider the diagonal operator A defined by
$$Ax_n := 2^n x_n, \qquad x_n \in X_n, \quad n \in \mathbb{Z},$$
with its natural maximal domain. It was shown in Proposition 10.2.28 that A is sectorial of angle $\omega(A) = 0$, and that A has a bounded H^∞-calculus if and only if $(x_n)_{n\in\mathbb{Z}}$ is unconditional. The following result is due to Kalton, Lorist, and Weis [2023, Proposition 6.1.3].

Proposition 15.6.2. *For the operator A just defined, the following is true:*

(1) *A is γ-sectorial if and only if the sequence $(S_n)_{n\in\mathbb{N}}$ is γ-bounded;*
(2) *A is almost γ-sectorial if and only if the sequence $(P_n)_{n\in\mathbb{Z}}$ is γ-bounded.*

We revisit this result in the Notes of Chapter 17 in connection with the problem of finding examples of sectorial operators that are not R-sectorial.

Around the Hilbert space characterisation: the γ-interpolation method

Theorem 15.3.23 asserts that a standard sectorial operator A on a Hilbert space H has a bounded H^∞-calculus if and only if it has bounded imaginary powers. This equivalence is nothing but the specialisation to Hilbert spaces of the equivalence, for any standard sectorial operator A on a Banach space X, of having a bounded H^∞-calculus and having γ-bounded imaginary powers, as stated in Theorem 15.3.21. The aim of this section is to explain that also the third equivalence in Theorem 15.3.23 is the specialisation to Hilbert spaces of a corresponding statement for Banach spaces. Recall that this third equivalence states, under the additional assumptions that $0 \in \varrho(A)$ and $\omega(A) < \frac{1}{2}\pi$, that bounded H^∞-calculus and boundedness of imaginary powers for A are equivalent to the equality

$$\mathsf{D}(A^{1/2}) = (X, \mathsf{D}(A))_{1/2,2} \quad \text{with equivalent norms.} \tag{15.36}$$

The Banach space version of this equivalence, due to Kalton, Lorist, and Weis [2023, Corollary 5.3.9], replaces (15.36) with the condition

$$\mathsf{D}(A^{1/2}) = (X, \mathsf{D}(A))^\gamma_{1/2} \quad \text{with equivalent norms,}$$

the interpolation space on the right-hand side being obtained via the so-called *γ-interpolation method* which we briefly outline next.

A discrete version of the γ-interpolation method was already considered by Kalton, Kunstmann, and Weis [2006], where Rademacher variables were used instead of Gaussian variables. In that paper, the method was used to study perturbations of the H^∞-calculus for various differential operators. The continuous version of the Gaussian method was introduced by Suárez and Weis [2006, 2009], where Gaussian interpolation of Bochner spaces $L^p(S;X)$ and square function spaces $\gamma(S;X)$, as well as a Gaussian version of abstract Stein interpolation, was studied. An abstract framework covering the γ-interpolation method, as well as the real and complex interpolation methods, has been recently developed by Lindemulder and Lorist [2021]. The present treatment follows the memoir of Kalton, Lorist, and Weis [2023]; theorem numbers in brackets refer to this memoir. As was pointed out in this reference, the results in Kalton, Kunstmann, and Weis [2006] and Suárez and Weis [2006, 2009] were based on a draft version of the memoir.

Let (X_0, X_1) be an interpolation couple of Banach spaces and let $\theta \in (0, 1)$. We call an operator

$$T : L^2(\mathbb{R}) + L^2(\mathbb{R}, e^{-2t}\,\mathrm{d}t) \to X_0 + X_1$$

admissible, and write $T \in \mathscr{A}$, if $T \in \gamma(L^2(\mathbb{R}, e^{-2jt}\,\mathrm{d}t), X_j)$ for $j = 0, 1$. For such operators we define

$$\|T\|_{\mathscr{A}} := \max_{j=0,1} \|T_j\|_{\gamma(L^2(\mathbb{R}, e^{-2jt}\,\mathrm{d}t), X_j)},$$

where T_j denotes the operator T from $L^2(\mathbb{R}, e^{-2jt}\,\mathrm{d}t)$ into X_j. It is routine to check that $\mathscr{A}$ is complete with respect to this norm.

Denoting $e_\theta : t \mapsto e^{\theta t}$, we define $(X_0, X_1)_\theta^\gamma$ as the space of all $x \in X_0 + X_1$ for which the quantity

$$\|x\|_{(X_0,X_1)_\theta^\gamma} := \inf\big\{\|T\|_{\mathscr{A}} :\ T \in \mathscr{A},\ T(e_\theta) = x\big\}$$

is finite. This space is a quotient space of $\mathscr{A}$, and as such it is a Banach space. By [Proposition 3.3.2], the set of finite rank operators T is dense in $\mathscr{A}$; in particular, we have:

Proposition 15.6.3. *$X_0 \cap X_1$ is dense in $(X_0, X_1)_\theta^\gamma$.*

[Proposition 3.3.1] gives the following interpolation result.

Theorem 15.6.4 (γ-interpolation of operators). *Suppose that (X_0, X_1) and (Y_0, Y_1) are interpolation couples of Banach spaces. Let $S\colon X_0 + X_1 \to Y_0 + Y_1$ be a bounded operator such that $S(X_0) \subseteq Y_0$ and $S(X_1) \subseteq Y_1$. Then S maps $(X_0, X_1)_\theta^\gamma$ to $(Y_0, Y_1)_\theta^\gamma$ boundedly, with norm*

$$\|S\|_{\mathscr{L}((X_0,X_1)_\theta^\gamma,(Y_0,Y_1)_\theta^\gamma)} \leqslant \|S\|_{\mathscr{L}(X_0,Y_0)}^{1-\theta}\|S\|_{\mathscr{L}(X_1,Y_1)}^{\theta}.$$

By [Proposition 3.4.1], the norm of $(X_0, X_1)_\theta^\gamma$ can be equivalently expressed as follows.

Proposition 15.6.5. *Let $\mathscr{A}_\bullet$ be the set of all strongly measurable functions $f : \mathbb{R}_+ \to X_0 \cap X_1$ such that $t \mapsto t^j f(t)$ belongs to $\gamma(\mathbb{R}_+, \frac{\mathrm{d}t}{t}; X_j)$ for $j = 0, 1$. For $f \in \mathscr{A}_\bullet$, define*

$$\|f\|_{A_\bullet} := \max_{j=0,1}\big\|t \mapsto t^j f(t)\big\|_{\gamma(\mathbb{R}_+,\, \frac{\mathrm{d}t}{t}; X_j)}.$$

Then for all $x \in (X_0, X_1)_\theta^\gamma$ we have

$$\|x\|_{(X_0,X_1)_\theta^\gamma} = \inf\Big\{\|f\|_{\mathscr{A}_\bullet} :\ f \in \mathscr{A}_\bullet,\ \int_0^\infty t^\theta f(t)\,\mathrm{d}t = x\Big\},$$

where the integral converges in the Bochner sense in $X_0 + X_1$.

[Theorem 3.4.4] contains the following relationship of the γ-interpolation method with the real and complex methods.

Theorem 15.6.6 (Relationship with the real and complex method). *Let (X_0, X_1) be an interpolation couple of Banach spaces, and let $0 < \theta < 1$.*

(1) *If X_0 and X_1 have type $p_0, p_1 \in [1,2]$ and cotype $q_0, q_1 \in [2,\infty]$ respectively, then we have the continuous embeddings*

$$(X_0, X_1)_{\theta,p} \hookrightarrow (X_0, X_1)^{\gamma}_{\theta} \hookrightarrow (X_0, X_1)_{\theta,q}$$

where $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$ and $\frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$.

(2) *If X_0 and X_1 have type 2, then we have the continuous embedding*

$$[X_0, X_1]_{\theta} \hookrightarrow (X_0, X_1)^{\gamma}_{\theta}.$$

If X_0 and X_1 have cotype 2, then we have the continuous embedding

$$(X_0, X_1)^{\gamma}_{\theta} \hookrightarrow [X_0, X_1]_{\theta}.$$

Since a Banach space X is isomorphic to a Hilbert space if and only if X has type 2 and cotype 2 (by Kwapień's Theorem 7.3.1), we obtain the corollary (cf. the Corollary of Peetre's Theorem C.4.1 for the first equivalence):

Corollary 15.6.7. *Let (H_0, H_1) be an interpolation couple of Hilbert spaces, and let $0 < \theta < 1$. Then*

$$(H_0, H_1)_{\theta,2} = [H_0, H_1]_{\theta} = (H_0, H_1)^{\gamma}_{\theta}$$

with equivalent norms.

Section 15.4

Much of the theory developed in the first three sections of this chapter has an analogue for strip type operators. The general theory of this class of operators is developed in the book of Haase [2006], which also treats their connections with logarithms of sectorial operators. Analogues of the results of Sections 10.3 and 10.4 are presented by Kalton and Weis [2016].

Theorem 15.4.3, on the strip type property and integral representation of the logarithm $\log(A)$ of a standard sectorial A, is due to Nollau [1969]. Our proof is a variation of the presentation by Haase [2006]. The converse problem to Theorem 15.4.3, whether the exponent of a striptal operator is sectorial, is subtle; we refer to Haase [2006] for a counterexample. Theorem 15.4.4, on the identification of C_0-groups on a UMD space as bounded imaginary powers of a standard sectorial operator, can be viewed as a partial result in the positive direction. It was first proved by Monniaux [1999] with a very different argument based on the notion of analytic generator. The proof presented here is essentially that of Haase [2009]. Another proof can be found in Haase [2007].

Theorem 15.4.11 about the sum of operators, both of which have bounded imaginary powers, is due to Dore and Venni [1987] under the slightly stronger assumption on A and B that they satisfy the resolvent bounds

$$\|(t+A)^{-1}\|, \|(t+B)^{-1}\| \leqslant \frac{M}{1+t}, \quad t>0.$$

In its present form, where it is only assumed that

$$\|(t+A)^{-1}\|, \|(t+B)^{-1}\| \leqslant \frac{M}{t}, \quad t>0,$$

the result was obtained by Prüss and Sohr [1990]. The original proof of Dore and Venni [1987] is ingenious and relatively short, and has been sketched in the Notes of Chapter 5. It depends on a representation formula for $(A+B)^{-1}$ in terms of fractional powers of A and B. The refinement of these arguments by Prüss and Sohr [1990], to obtain the more general case, depends on subtle approximation arguments for operators A with bounded imaginary powers which, like the proof presented here, use the functional calculus associated with the C_0-group $(A^{it})_{t\in\mathbb{R}}$ and Mellin transform techniques.

The beautiful proof of the Dore–Venni Theorem 15.4.11 presented here is due to Haase [2007] and fits well in the mainstream of ideas developed in this volume. This paper also contains our proof of Theorem 15.4.4, which is originally due to Monniaux [1999] with a different proof based on the notion of an analytic generator. Our presentation uses some ideas of Haase [2006, Section 4.2], where a detailed presentation of the theory of strip type operators if given. With these methods, the operator $B = e^{tA}$ can also be defined using the extended Dunford calculus.

The importance of the Dore–Venni Theorem 15.4.11 is mostly historical, and the more recent sum-of-operator theorems proved in the next chapter have turned out to be more versatile in their usage. It is for this reason that we have contented ourselves with a somewhat sketchy presentation, leaving a few tedious details to the reader.

Section 15.5

The results of this section follow Duelli [2005] and Duelli and Weis [2005], where Theorem 15.5.2 ($\|(A^2)^{1/2}x\| \eqsim \|Ax\|$) is proved. By the Hieber–Prüss Theorem 10.7.10, it covers the case where iA generates a bounded C_0-group. A version of Theorem 15.5.2 (with inhomogeneous estimates) for the case that iA generates a C_0-group of exponential growth type $\omega \geqslant 0$ is due to Haase [2007].

The spectral projections $P^{\pm}$ of Proposition 15.5.1 are studied in more detail by Arendt and Zamboni [2010], Duelli [2005], and Duelli and Weis [2005]. In particular, Arendt and Zamboni [2010] show that, if A is an invertible bisectorial operator and $\delta > 0$ is so small that the closure of the ball $B(0,\delta)$ belongs to the resolvent set of A, then for $x \in \mathsf{D}(A)$, these projections are given by

$$P^{\pm}x = \frac{1}{2\pi i}\int_{\partial(\Sigma_\nu^{\pm}\setminus B(0,\delta))} R(z,A)Ax \,\frac{\mathrm{d}z}{z},$$

where $\omega^{\mathrm{bi}}(A) < \nu < \sigma$ is arbitrary. The extended Dunford calculus for bisectorial operators, in particular the analogue of Proposition 15.1.12, which was used in the proof of Theorem 15.5.2, has been studied by Duelli [2005].

The Kato square root problem

A long-standing question about fractional powers of operators, and a major motivation for the development of the theory of H^∞-calculus at large, was the *square root problem* of Kato [1961]. To present this problem, we recall that a linear operator A in a Hilbert space H with inner product $(\,|\,)$ is called

- *accretive*, if $\Re(Au|u) \geqslant 0$ for all $u \in \mathsf{D}(A)$;
- *maximal accretive*, if an extension $\widetilde{A} \supseteq A$ is accretive exactly when $\widetilde{A} = A$;
- *regularly accretive*, if A is maximally accretive and there is an associated sesquilinear form $\mathfrak{a}$ in H such that $\Re\mathfrak{a}(v, v) \geqslant 0$ for all $v \in \mathsf{D}(\mathfrak{a})$, and

$$(Au|v) = \mathfrak{a}(u, v) \text{ for all } u \in \mathsf{D}(A) \subseteq \mathsf{D}(\mathfrak{a}) \text{ and all } v \in \mathsf{D}(\mathfrak{a}).$$

For a regularly accretive operator, Kato [1961] defines its real part $\Re A$ as the unique maximal accretive operator associated with the sesquilinear form

$$\Re\mathfrak{a} : (u, v) \mapsto \frac{1}{2}[\mathfrak{a}(u, v) + \overline{\mathfrak{a}(v, u)}]$$

in the above sense. He then proceeds to show that

$$\begin{aligned} \mathsf{D}(A^\alpha) &= \mathsf{D}((A^*)^\alpha), && \text{if } A \text{ is maximal accretive and } \alpha \in [0, \tfrac{1}{2}), \\ &= \mathsf{D}((\Re A)^\alpha), && \text{if } A \text{ is regularly accretive and } \alpha \in [0, \tfrac{1}{2}), \end{aligned}$$

and that these identities can fail for $\alpha > \frac{1}{2}$, "but it is not known whether or not $\alpha = \frac{1}{2}$ can be included". Kato [1961, Remark 1] writes:

> This is perhaps not true in general. But the question is open even when A is regularly accretive. In this case it appears reasonable to suppose that both $\mathsf{D}(A^{\frac{1}{2}})$ and $\mathsf{D}((A^*)^{\frac{1}{2}})$ coincide with $\mathsf{D}((\Re A)^{\frac{1}{2}}) = \mathsf{D}(\mathfrak{a})$, where $\Re A$ is the real part of A and $\mathfrak{a}$ is the regular sesquilinear form which defines A.

As suspected by Kato [1961], a counterexample to the general case of maximal accretive operators was found shortly after by Lions [1962], but the regularly accretive case was only disproved a decade later by McIntosh [1972].

What came to be known as Kato's square root problem was subsequently redefined by McIntosh [1982], making the case that, what Kato [1961] "really had in mind", was differential operators $A = -\operatorname{div} B(x)\nabla$, where $B \in L^\infty(\mathbb{R}^d; \mathbb{C}^d)$ is such that $\Re(B(x)e|e) \geqslant \delta > 0$ for a.e. $x \in \mathbb{R}^d$ and all $e \in \mathbb{C}^d$ of norm one. For such A, the associated sesquilinear form is

$$\mathfrak{a}(u, v) = \int_{\mathbb{R}^d} (B(x)\nabla u(x)|\nabla v(x))\,\mathrm{d}x$$

with domain $\mathsf{D}(\mathfrak{a}) = \mathsf{D}(\nabla) = W^{1,2}(\mathbb{R}^d)$, and the problem thus takes the form

$$\|\sqrt{-\operatorname{div} B\nabla}u\|_{L^2(\mathbb{R}^d)} \overset{?}{\eqsim} \|\nabla u\|_{L^2(\mathbb{R}^d;\mathbb{C}^d)}. \tag{15.37}$$

McIntosh [1982] further suggested that this question was related to Calderón's problem about the L^2-boundedness of the Cauchy integral on a Lipschitz graph (discussed in the Notes of Chapter 12). As pointed out by Alan McIntosh in several oral communications with the authors of this book over the first decade of this century (the quote within the first sentence of this paragraph is also from this oral source), before his making this connection, Kato's question was generally regarded as being a soft one, several levels easier than the problem of Calderón, which everyone agreed to be hard. Nevertheless, the connection suggested by McIntosh [1982] turned out to be a fruitful one, and the combined efforts of Coifman, McIntosh, and Meyer [1982] led to a proof of both the boundedness of the Cauchy integral and, what turned out to be equivalent, McIntosh's reformulation of Kato's square root problem in dimension $d = 1$.

After this, the status of the redefined square root problem increased substantially, and important progress was made by Coifman, Deng, and Meyer [1983], Fabes, Jerison, and Kenig [1985], McIntosh [1985], Alexopoulos [1991], Journé [1991], Auscher and Tchamitchian [1998], and Auscher, Hofmann, Lewis, and Tchamitchian [2001], but it took two decades from the one-dimensional result of Coifman, McIntosh, and Meyer [1982] until a complete solution was achieved by Hofmann, Lacey, and McIntosh [2002] in dimension $d = 2$ and then by Auscher, Hofmann, Lacey, McIntosh, and Tchamitchian [2002] in all dimensions.

While heavily building on ideas and results about functional calculus of the second-order operator $A = -\operatorname{div} B\nabla$, the original solution of the square root problem was not quite a "pure" functional calculus theorem in the sense that the gradient featuring in (15.37) does not have the form $f(A)$ of objects in the functional calculus of A. This "issue" was fixed by a new approach developed by Axelsson, Keith, and McIntosh [2006] which, in contrast to the sectorial calculus of second-order operators employed by Auscher et al. [2002], was based on the bi-sectorial calculus of first-order differential operators, and promoted the relevance of bi-sectorial operators and bi-sectorial H^∞-calculus in subsequent research. Quoting the MathSciNet review of Axelsson et al. [2006] by Ian Doust, this paper provided "a remarkable consolidation of many of the ideas that have arisen in the so-called Calderón program", not only reproving the square root theorem of Auscher et al. [2002] and several other results by a unified approach, but also obtaining new geometric applications to the behaviour of the Hodge–Dirac operator on a Riemannian manifold under measurable perturbations of the Riemannian metric. In fact, the very framework of Axelsson et al. [2006] is based on a general notion of *perturbed Hodge–Dirac operators*; in the application to the Kato square root problem, these take the form

$$D_B := \begin{pmatrix} 0 & -\operatorname{div} B \\ \nabla & 0 \end{pmatrix}, \qquad \mathsf{D}(D_B) = \mathsf{D}(\nabla) \oplus \mathsf{D}(\operatorname{div} B)$$

so that, at least formally,

$$(D_B^2)^{\frac{1}{2}} = \begin{pmatrix} -\operatorname{div} B\nabla & 0 \\ 0 & -\nabla \operatorname{div} B \end{pmatrix}^{\frac{1}{2}} = \begin{pmatrix} (-\operatorname{div} B\nabla)^{\frac{1}{2}} & 0 \\ 0 & (-\nabla \operatorname{div} B)^{\frac{1}{2}} \end{pmatrix},$$
$$\mathsf{D}((D_B^2)^{\frac{1}{2}}) = \mathsf{D}((-\operatorname{div} B\nabla)^{\frac{1}{2}}) \oplus \mathsf{D}((-\nabla \operatorname{div} B)^{\frac{1}{2}}).$$

Hence, if one can establish bounded bi-sectorial H^∞-calculus of D_B, as Axelsson et al. [2006] do, the Kato conjecture (15.37) will be an immediate consequence of the estimate $\|(D_B^2)^{\frac{1}{2}} u\| \eqsim \|D_B u\|$ provided by Theorem 15.5.2.

This first-order approach of Axelsson, Keith, and McIntosh [2006] has been influential for several subsequent developments. Of particular interest for the themes of the present volumes is a version of the Kato square root theorem in $L^p(\mathbb{R}^d; X)$. This was obtained by Hytönen, McIntosh, and Portal [2008] by a Banach space extension of the methods of Axelsson et al. [2006]. In the language of the original operator $A = -\operatorname{div} B\nabla$, the result of Hytönen, McIntosh, and Portal [2008] can be stated as follows:

Theorem 15.6.8. *Let X be a UMD space, and suppose that both X and X^* have the RMF property (Definition 3.6.10). Let $B, B^{-1} \in L^\infty(\mathbb{R}^d; \mathscr{L}(\mathbb{C}^d))$, where $B^{-1}(x) := B(x)^{-1}$ is the pointwise inverse of the matrix-valued function B. Let $1 \leqslant p_- < p_+ \leqslant \infty$, and suppose that $A := -\operatorname{div} B\nabla$ is sectorial in $L^p(\mathbb{R}^d; X)$ for all $p \in J := (p_-, p_+)$. Then the following are equivalent:*

(1) *For all $p \in J$, the following four sets are R-bounded in $L^p(\mathbb{R}^d; X)$:*

$$\mathscr{A}_{a,b} := \Big\{ (t\sqrt{-\Delta})^a (I + t^2 A)^{-1} (t\sqrt{-\Delta})^b : t > 0 \Big\}, \quad a, b \in \{0, 1\}.$$

(2) *For all $p \in J$, A has a bounded H^∞-calculus in $L^p(\mathbb{R}^d; X)$, and*

$$\|\sqrt{A} u\|_{L^p(\mathbb{R}^d; X)} \eqsim \|\nabla u\|_{L^p(\mathbb{R}^d; X)^d}$$

Remark 15.6.9. Several remarks concerning Theorem 15.6.8 are in order:

(a) While (2) contains a full analogue of (15.37) in $L^p(\mathbb{R}^d; X)$, along with the bounded H^∞-calculus of independent interest, the characterising condition (1) is less satisfactory than in the scalar-valued case, as it involves non-trivial R-boundedness properties of operators on $L^p(\mathbb{R}^d; X)$. However, note that the R-boundedness of $\mathscr{A}_{0,0}$ is simply the R-sectoriality of A which, by Theorem 10.3.4(2), is a general necessary condition for the bounded H^∞-calculus contained in (2). When $X = \mathbb{C}$ and $p = 2$, verifying (1) from easy-to-check pointwise conditions on B is straightforward operator theory, and the difficult harmonic analysis enters in passing from (1) to (2). Curiously, in the Banach space valued generality, the easy part becomes unavailable but the difficult part may still be pushed through.

(b) Another shortcoming of Theorem 15.6.8 compared to the scalar-valued L^2 case is that, in order to get (2) for a given p, one needs to verify (1) for an open range of exponents $(p-\varepsilon, p+\varepsilon)$. However, it was subsequently shown by Hytönen and McIntosh [2010], and later with a different argument by Auscher and Stahlhut [2013], that conditions of type (1) self-improve from one exponent p to a small range around it. This allows one to obtain a version of Theorem 15.6.8 for a fixed p in place of the range of p as stated.

(c) The RMF property (Definition 3.6.10) and the related Rademacher maximal function (Definition 3.6.8) were first introduced by Hytönen, McIntosh, and Portal [2008] for the very needs of running the argument to prove Theorem 15.6.8, but these notions (or their extensions) have proven to be useful in other contexts, notably in the study of Banach space valued multilinear operators by Di Plinio and Ou [2018], Di Plinio, Li, Martikainen, and Vuorinen [2020b], and Amenta and Uraltsev [2020].

(d) For L^p-estimates related to Kato's square root problem in the scalar-valued case, there is an alternative approach based on a generalisation of the Calderón–Zygmund theory discussed in Chapter 11, extrapolating from the L^2-results of Auscher, Hofmann, Lacey, McIntosh, and Tchamitchian [2002]. The operators now under consideration are far less regular than those covered in Chapter 11, and the extrapolation yields their boundedness, in general, only in some subinterval $(p_-, p_+) \subseteq (1, \infty)$ determined by the details of the operator in question. Based on an extrapolation theory for "non-integral operators" developed by Blunck and Kunstmann [2003], a systematic investigation of the maximal ranges of p for various L^p estimates related to the Kato square root problem is carried out by Auscher [2007].

(e) Yet another approach to the scalar-valued L^p theory is due to Frey, McIntosh, and Portal [2018]. As in the approach of Hytönen, McIntosh, and Portal [2008] and in contrast to that of Auscher [2007], they work directly in L^p instead of extrapolating from L^2; also their "proof shows that the heart of the harmonic analysis in L^2 extends to L^p for all $p \in (1, \infty)$, while the restrictions in p come from the operator-theoretic part of the L^2 proof". A novelty in their approach is using the theory of tent spaces; on the side of the results, this allows them to dispense with the R-boundedness assumptions required by Hytönen, McIntosh, and Portal [2008].

Given the focus of these volumes on analysis in Banach spaces, we have not covered, in the discussion above, the rich literature of extensions and applications of the machinery of Auscher, Hofmann, Lacey, McIntosh, and Tchamitchian [2002] and Axelsson, Keith, and McIntosh [2006] in the L^2 theory of partial differential operators and equations; for this, we refer the reader to the numerous papers citing these pioneering contributions. The first-order approach to the Kato square root problem of Axelsson, Keith, and McIntosh [2006] has been adapted in Maas and Van Neerven [2009] to the Gaussian setting to prove a nonsymmetric version of the Meyer inequalities in Malliavin calculus. This work belongs to a circle of ideas that will be treated in Volume IV.