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**Towards the construction of a normal Lüroth series
(Nederlandse titel: Op weg naar de constructie van
een normale Lürothreeks)**

Verslag ten behoeve van het
Delft Institute for Applied Mathematics
als onderdeel ter verkrijging

van de graad van

**BACHELOR OF SCIENCE
in
TECHNISCHE WISKUNDE**

door

ARNOUT BOKS

**Delft, Nederland
Augustus 2009**



BSc verslag TECHNISCHE WISKUNDE

“Towards the construction of a normal Lüroth series”

(Nederlandse titel: “Op weg naar de constructie van een normale Lürothreeks”)

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Augustus, 2009

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Preface

This thesis presents the results of the research I conducted as part of my bachelor's project at TU Delft, under supervision of Cor Kraaikamp. The project was aimed at finding a normal number for the Lüroth series expansion, but actually proving the normality of a Lüroth series was not possible in the limited timespan given for this project. As a consequence I will present my work as a conjecture that a given Lüroth series is normal and give an incomplete proof. After this project we will continue trying to find the last missing proof step in order to prove the normality of the Lüroth series described in the conjecture.

I would like to thank Cor Kraaikamp for his supervision and the numerous times he proofread this thesis.

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Abstract

In this thesis we present a Lüroth series A that we conjecture to be normal. The series A is constructed by concatenating elements from a probability tree that enumerates all finite Lüroth series. We give an outline for a proof of the normality of A in four steps, of which we will prove three. Regarding the fourth step, which we cannot prove yet, we will state some remarks on its plausibility and a possible strategy for a proof. To conclude we describe the possibilities of extending the construction presented to other Generalized Lüroth Series and compare the mentioned construction to constructions for normal numbers in other expansions.

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Chapter 1

Normal numbers

A number is called *normal* (as defined by Borel [3]) in a certain representation if its digits in that representation exhibit total randomness. That is, every combination of digits occurs in that number with the ‘correct’ limiting frequency as we take more and more digits into account. For numbers written in decimal notation, this means that every group of k digits occurs with a limiting frequency of 10^{-k} .

It was already shown by Borel that almost all¹ numbers are normal in base 10. Sierpinski was the first one to give a concrete example of such a number [11], but his number is not as well-known as the constant D.G. Champernowne showed to be normal in 1933 [4]:

$$C_{10} = 0.1234567891011121314\dots,$$

which is obtained by concatenating the natural numbers (in their decimal representation) from 1 onwards. His method is easily extended to produce normal numbers for other b -adic expansions, like for the ternary expansion:

$$C_3 = 0.12101112100101102110\dots$$

Later, Erdős and Copeland proved a strong theorem [5], which asserts the normality of numbers formed by concatenating the elements of any increasing sequence of integers that is dense enough in the natural numbers. They used this theorem to prove that the number known as the Copeland-Erdős constant,

$$0.235711131719232931374143\dots,$$

formed by concatenating the sequence of prime numbers, is normal in base 10.

The concept of normality is not limited to b -adic expansions. In 1981, Adler, Keane and Smorodinsky [1] constructed a normal number with respect to the continued fraction expansion, obtained by concatenating the continued fractions of the rationals

$$\frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, \frac{1}{6}, \dots$$

A β -normal number was constructed by Ito and Shiokawa [6].

¹With “almost all” meaning all numbers except for a set of measure zero.

Chapter 2

Lüroth series

In 1883, J. Lüroth [8] introduced an expansion for real numbers $x \in (0, 1]$ given by

$$x = \frac{1}{a_1} + \frac{1}{a_1(a_1 - 1)a_2} + \cdots + \frac{1}{a_1(a_1 - 1) \cdots a_{n-1}(a_{n-1} - 1)a_n} + \cdots \quad (2.1)$$

with ‘digits’ $a_n \in \mathbb{N}, a_n \geq 2$. He proved that every irrational number has a unique infinite Lüroth expansion, and that each rational number has either a finite or a periodic expansion¹. Lüroth did not know a way to distinguish between these last two possibilities however.

The expansion studied by Lüroth is based on the operator $T_L : [0, 1] \rightarrow [0, 1]$, given by

$$\begin{aligned} T_L(x) &:= \left\lfloor \frac{1}{x} \right\rfloor \left(\left\lfloor \frac{1}{x} \right\rfloor + 1 \right) x - \left\lfloor \frac{1}{x} \right\rfloor, \quad \text{for } x \neq 0, \\ T_L(0) &:= 0. \end{aligned}$$

This operator is illustrated in Figure 2.1. The Lüroth expansion then arises when defining

$$a_1 = a_1(x) := \left\lfloor \frac{1}{x} \right\rfloor + 1,$$

so $a_1(x) = k$ if and only if $x \in I_k := \left(\frac{1}{k}, \frac{1}{k+1} \right]$, and (when $T_L^n(x) \neq 0$)

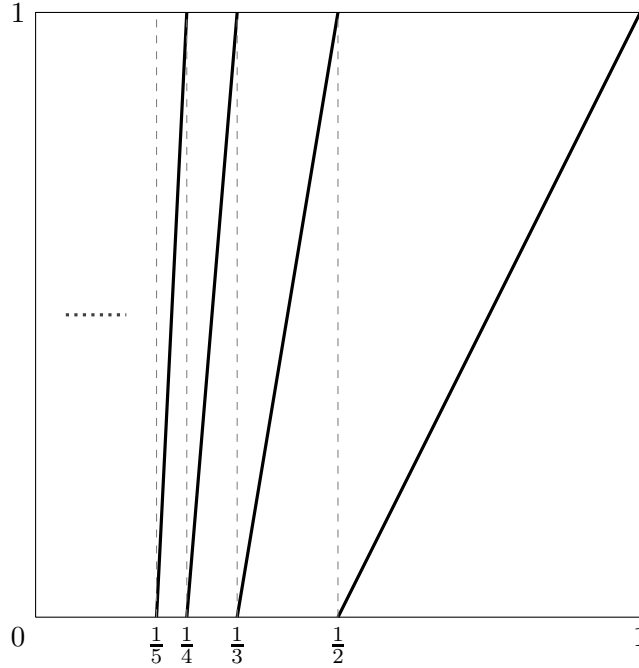
$$a_{n+1} = a_{n+1}(x) = a_1(T_L^n(x)).$$

Then, as $T_L(x) = a_1 \cdot (a_1 - 1)x - (a_1 - 1)$ and in general $T_L^n(x) = a_n \cdot (a_n - 1)T_L^{n-1}(x) - (a_n - 1)$, we find

$$\begin{aligned} x &= \frac{1}{a_1} + \frac{1}{a_1 \cdot (a_1 - 1)} T_L(x) \\ &= \frac{1}{a_1} + \frac{1}{a_1 \cdot (a_1 - 1)} \left(\frac{1}{a_2} + \frac{1}{a_2 \cdot (a_2 - 1)} T_L^2(x) \right) \\ &= \frac{1}{a_1} + \frac{1}{a_1 \cdot (a_1 - 1)} \left(\frac{1}{a_2} + \frac{1}{a_2 \cdot (a_2 - 1)} \left(\frac{1}{a_3} + \frac{1}{a_3 \cdot (a_3 - 1)} T_L^3(x) \right) \right) \\ &= \dots, \end{aligned}$$

leading to the Lüroth series expansion from 2.1. The series is guaranteed to converge to x because $a_n(x) \geq 2$ and $T_L^n(x) \in [0, 1]$.

¹Note that rational numbers with a finite Lüroth series expansion can also be written using an infinite series.

Figure 2.1: Graphical representation of the operator T_L

It was shown in [7] that T_L preserves Lebesgue measure, is ergodic, and that when we take a random $X \in (0, 1]$ the random variables $a_1(X), a_2(X), \dots$ given by the digits of X are independent and satisfy $P(a_i = k) = \frac{1}{k(k-1)}$ for $k \in \mathbb{N}, k \geq 2$. This establishes the fact that almost all Lüroth series are normal.

Later, the concept of Lüroth series was extended to a more general class of series expansions called “Generalized Lüroth Series” [2], having similar ergodic properties.

To simplify notation we will use $\langle a_1, a_2, \dots, a_n, \dots \rangle$ to denote the infinite Lüroth series

$$\frac{1}{a_1} + \frac{1}{a_1(a_1 - 1)a_2} + \dots + \frac{1}{a_1(a_1 - 1) \cdots a_{n-1}(a_{n-1} - 1)a_n} + \dots.$$

The convergents of such Lüroth series are denoted by $\langle a_1, a_2, \dots, a_n \rangle$. These are the rational numbers

$$\frac{1}{a_1} + \frac{1}{a_1(a_1 - 1)a_2} + \dots + \frac{1}{a_1(a_1 - 1) \cdots a_{n-1}(a_{n-1} - 1)a_n}.$$

By slight abuse of notation we can also denote blocks of digits by $\alpha = \langle \alpha_1, \alpha_2, \dots, \alpha_k \rangle$ and use this notation to define

$$\Delta_\alpha = \{x \in [0, 1) : \langle \alpha_1, \alpha_2, \dots, \alpha_k + 1 \rangle \leq x < \langle \alpha_1, \alpha_2, \dots, \alpha_k \rangle\},$$

the interval corresponding to all Lüroth series starting with the block α . We denote the length of such an interval by

$$\mu(\alpha) = \lambda(\Delta_\alpha),$$

where λ is the Lebesgue measure. The properties of Lüroth series mentioned above now yield

$$\mu(\alpha) = \prod_{i=1}^k \frac{1}{\alpha_i(\alpha_i - 1)},$$

and a normal Lüroth series should have the block α occurring with limiting frequency $\mu(\alpha)$.

Chapter 3

Construction of a presumably normal Lüroth series

We will now show how to construct a Lüroth series that we conjecture to be normal. We start by creating a binary tree that enumerates all finite Lüroth series, to which we will refer as the *Lüroth tree*. The root of the Lüroth tree consists of the series $\langle 2 \rangle$. It is built up recursively by attaching to each node $\langle a_1, a_2, a_3, \dots, a_n \rangle$ a child node $\langle a_1, a_2, a_3, \dots, a_n, 2 \rangle$ (addition of an extra digit 2 at the end) to the left. To the right, a child node $\langle a_1, a_2, a_3, \dots, a_n + 1 \rangle$ (incrementation by 1 of the last digit) is attached. This leads to the tree shown in Figure 3.1.

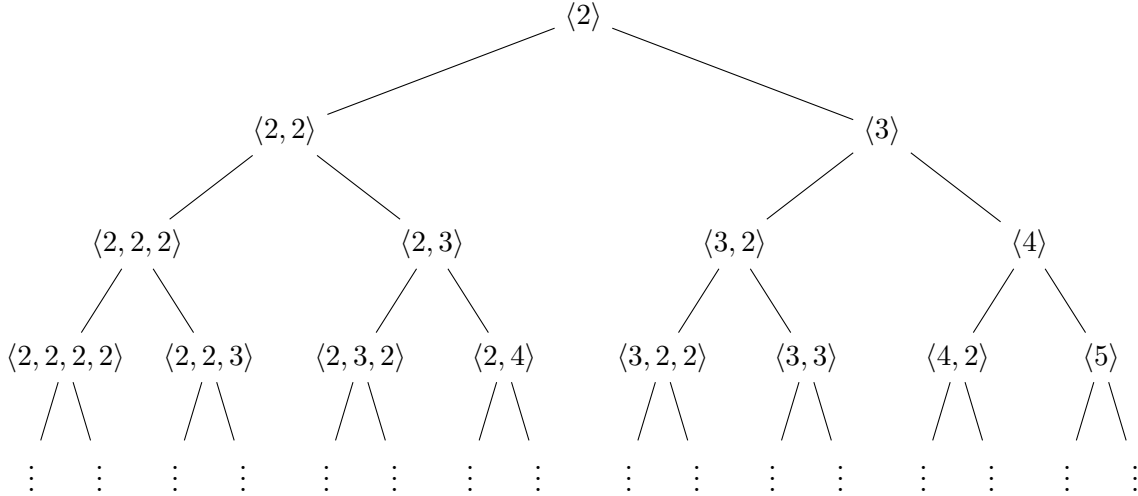


Figure 3.1: Top of the Lüroth tree

We will refer to the top level of the Lüroth tree (the root) as level 1, to the elements directly under the root as level 2, etcetera. This means the n th level will have 2^{n-1} elements.

Note that we could have swapped the left and right side of the tree, which would be more natural as this yields a binary search tree. To simplify the proof steps we will give later we have chosen not to do so. The relationship between the magnitudes of a node in the tree and its children is then formalized by the following proposition:

Proposition 1 (Ordering of nodes). *The Lüroth tree is an ‘inverted’ binary search tree, i.e. each node is smaller (in value) than each node in its left subtree, but larger than every element in its right subtree.*

Proof. Take an arbitrary node $N = \langle a_1, a_2, \dots, a_n \rangle$ from the tree. Take any node N_L from its left subtree. Then that node will be of the form $N_L = \langle a_1, a_2, \dots, a_n, a_{n+1}, \dots, a_k \rangle$, with $k \geq n+1$. Also take any node N_R in its right subtree, that will have the form $N_R = \langle a_1, a_2, \dots, a_n + m, a_{n+1}, \dots, a_k \rangle$, in which $m \in \{1, 2, 3, \dots\}$ and possibly $k = n$.

- (i) $N < N_L$: If we remember that every $a_i \geq 2$ and write down both expansions it is trivial to see that

$$\begin{aligned} N &= \frac{1}{a_1} + \dots + \frac{1}{a_1(a_1-1) \cdots a_{n-1}(a_{n-1}-1)a_n} \\ &< \frac{1}{a_1} + \dots + \frac{1}{a_1(a_1-1) \cdots a_{n-1}(a_{n-1}-1)a_n} + \dots \\ &\quad + \frac{1}{a_1(a_1-1) \cdots a_{n-1}(a_{n-1}-1)a_n(a_n-1)a_{n+1}(a_{n+1}-1) \cdots a_k} \\ &= N_L \end{aligned}$$

because of the extra positive terms in N_L that N does not have.

- (ii) $N_R < N$: In case $k = n$ we have

$$N_R = \langle a_1, a_2, \dots, a_n + m \rangle$$

and the inequality

$$\begin{aligned} N_R &= \frac{1}{a_1} + \dots + \frac{1}{a_1(a_1-1) \cdots a_{n-1}(a_{n-1}-1)} \cdot \frac{1}{a_n + m} \\ &< \frac{1}{a_1} + \dots + \frac{1}{a_1(a_1-1) \cdots a_{n-1}(a_{n-1}-1)} \cdot \frac{1}{a_n} \\ &= N \end{aligned}$$

surely holds as $\frac{1}{a_n+m} < \frac{1}{a_n}$.

Otherwise, when $k > n$, we have

$$N_R = \frac{1}{a_1} + \dots + \frac{1}{a_1(a_1-1) \cdots a_{n-1}(a_{n-1}-1)} \cdot \frac{1}{a_n + m} \cdot \left(1 + \frac{1}{a_n+m-1} \cdot \langle a_{n+1}, \dots, a_k \rangle\right).$$

Because $\langle a_{n+1}, \dots, a_k \rangle$ is a finite Lüroth series its value is less than one. This means we have that

$$N_R < \frac{1}{a_1} + \dots + \frac{1}{a_1(a_1-1) \cdots a_{n-1}(a_{n-1}-1)} \cdot \frac{1}{a_n + m} \cdot \left(1 + \frac{1}{a_n + m - 1}\right).$$

Now $\frac{1}{a_n+m} \cdot \left(1 + \frac{1}{a_n+m-1}\right) = \frac{1}{a_n+m-1} \leq \frac{1}{a_n}$ (because $m \geq 1$), and we find

$$\begin{aligned} N_R &< \frac{1}{a_1} + \dots + \frac{1}{a_1(a_1-1) \cdots a_{n-1}(a_{n-1}-1)} \cdot \frac{1}{a_n + m} \cdot \left(1 + \frac{1}{a_n + m - 1}\right) \\ &\leq \frac{1}{a_1} + \dots + \frac{1}{a_1(a_1-1) \cdots a_{n-1}(a_{n-1}-1)} \cdot \frac{1}{a_n} \\ &= N, \end{aligned}$$

which proves the inequality for this case.

□

One might be tempted to concatenate the elements in the Lüroth tree in level-order (from the root downward, from left to right), hoping this will produce a series with the desired properties. However, this is not the case. If our conjecture turns out to be true it will produce a normal number, but for a different generalized Lüroth series (see [2]) similar to the standard one. This other series also has digit set $\mathcal{D} = \{2, 3, 4, \dots\}$, but the interval $I_n = (\frac{1}{2^{n-1}}, \frac{1}{2^{n-2}}]$ corresponding to digit n will have Lebesgue measure $\lambda(I_n) = \frac{1}{2^{n-1}}$ instead of the ‘standard’ $I_n = (\frac{1}{n}, \frac{1}{n+1}]$ with $\lambda(I_n) = \frac{1}{n(n+1)}$. The proof of normality for this number is analogous to the proof that we will give later.

To overcome this problem we have to look upon the Lüroth tree as a probability tree and assign probabilities to the branches in order to achieve the correct distribution. This gives the Lüroth tree with probability distribution, shown in Figure 3.2.

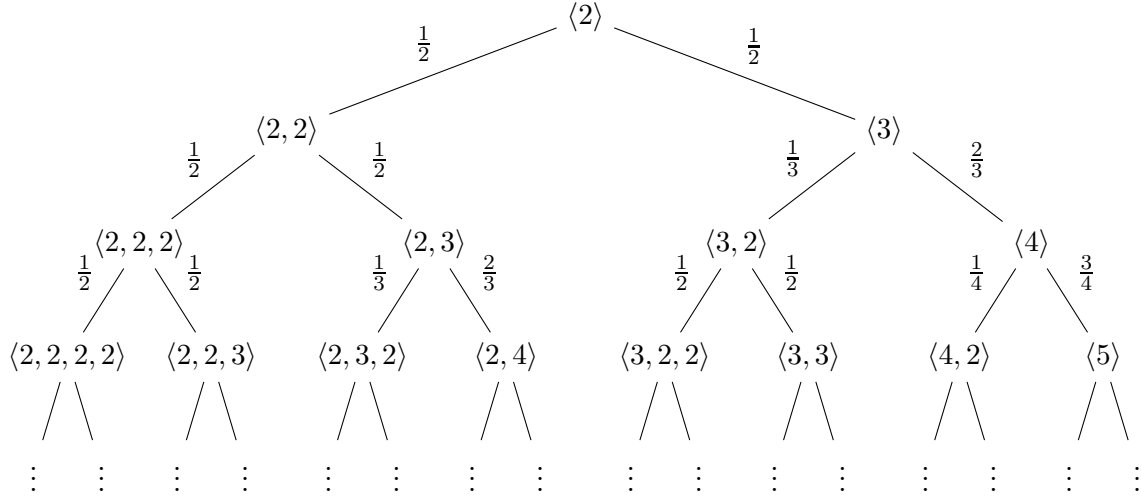


Figure 3.2: Top of the Lüroth tree with probability distribution

At each node $\langle a_1, a_2, a_3, \dots, a_n \rangle$ the probability of ‘going left’ (finishing the last digit and appending a 2) is $\frac{1}{a_n}$. The other $\frac{a_n-1}{a_n}$ of probability mass is for ‘going right’ (incrementing the last digit). Note that these probabilities only depend on the last digit.

With these probabilities, the probability of having a 4 as the first digit in a node far down in the tree is the probability of moving right twice (from the root) and then moving left. Note that this last move to the left (‘fixing’ the 4) is necessary, because otherwise the 4 could be further incremented to a 5 when we move further downward in the tree. This gives a probability of $\frac{1}{2} \cdot \frac{2}{3} \cdot \frac{1}{4} = \frac{1}{3 \cdot 4}$ as the probability that a node with depth at least 4 has a 4 as its first digit.

In general, the probability that a 2 occurring as the last digit of a node (which, as the probabilities in the tree only depend on the last digit, can either be the root or arise from a step to the left) will eventually end up to be a digit n is $\frac{1}{2} \cdot \frac{2}{3} \cdot \dots \cdot \frac{n-3}{n-2} \cdot \frac{n-2}{n-1}$ (due to $n-2$ steps to the right) multiplied by $\frac{1}{n}$ (due to the last step to the left), being $\frac{1}{n(n-1)}$ because of the telescoping product. The fact that this fraction also arises in the required limiting distribution of digits in a normal number motivates our choice for the probabilities along each edge in the tree.

We now want to construct a normal number by going through the Lüroth tree level by level, appending a representative sample from that level. Since we cannot append a node $\frac{1}{3}$ times, we will append some nodes multiple times. By doing this we can achieve that the number of times we append each node in the n th level reflects the probability of arriving at that node if we start

at the root of the tree and move downward to the n th level by following the probabilities along the edges.

To define the process that will yield our Lüroth series, we need the concept of the superfactorial (sequence M2049 in [9]) defined as

$$\text{sf}(n) = \prod_{k=1}^n k!,$$

so $\text{sf}(n)$ is the product of the first n factorials.

When constructing our Lüroth series, we start with an empty sequence and go through each level of the tree from the root downward. Going through the n th level of the tree (let's call this 'step n '), we will $\text{sf}(n)$ times pick a node from that level and append the digits from that node to the sequence we already have. So in step n we will only append digits from nodes in the n th level of the tree.

Note that at each node the probabilities for going to the left and right can be written as $\frac{1}{m}$ and $\frac{m-1}{m}$ respectively, for some $m \in \mathbb{N}, m \geq 2$. This means that when we reach that node for the i th time in that step, we will go to the left if $(i-1) \equiv 0 \pmod{m}$ and to the right if $(i-1) \not\equiv 0 \pmod{m}$, so the number of times we choose each direction complies with the probability distribution.

When in step n , we will start from the root of the tree $\text{sf}(n)$ times, follow the above procedure at each node until we reach a node from level n and append the digits of that node to our series. We will call the group of digits of one such appended node a *word*. After appending $\text{sf}(n)$ words we move on to step $n+1$.

Example 1 (Construction steps 1 through 3). The construction process described is best illustrated by an example, showing steps 1 through 3.

- **Step 1:** In this step we start with an empty Lüroth series and simply append the root. Now our series consists of just $\langle 2 \rangle$.
- **Step 2:** We will now append elements from level 2. We start at the root of the tree $\text{sf}(2) = 2$ times. The probability distribution at the root is $(\frac{1}{2}, \frac{1}{2})$, so we will go left the first time (since $(1-1) \equiv 0 \pmod{2}$). This brings us to node $\langle 2, 2 \rangle$ which is in level 2, so we will append its digits to our series to obtain $\langle 2, 2, 2 \rangle$. The second time we start at the root we go right (since $(2-1) \not\equiv 0 \pmod{2}$), arrive at $\langle 3 \rangle$ and append these digits to obtain $\langle 2, 2, 2, 3 \rangle$. This ends step 2.
- **Step 3:** We will now start at the root $\text{sf}(3) = 3! \cdot 2! \cdot 1! = 12$ times and follow the routes prescribed by the process above. Doing this we will end up at the following nodes (writing 'L' and 'R' for going left and right):

1. LL: $\langle 2, 2, 2 \rangle$
2. RL: $\langle 3, 2 \rangle$
3. LR: $\langle 2, 3 \rangle$
4. RR: $\langle 4 \rangle$
5. LL: $\langle 2, 2, 2 \rangle$
6. RR: $\langle 4 \rangle$
7. LR: $\langle 2, 3 \rangle$
8. RL: $\langle 3, 2 \rangle$
9. LL: $\langle 2, 2, 2 \rangle$
10. RR: $\langle 4 \rangle$

11. LR: $\langle 2, 3 \rangle$

12. RR: $\langle 4 \rangle$

Note that we arrived at (for example) node $\langle 4 \rangle$ (RR) exactly $\frac{1}{2} \cdot \frac{2}{3} \cdot \text{sf}(3) = 4$ times, and that the frequencies of the other nodes in level 3 also follow the probability distribution we assigned to the paths in the tree.

We will concatenate the digits from these nodes successively, so at the end of this step we obtain the Lüroth series

$$\langle 2, 2, 2, 3, 2, 2, 2, 3, 2, 2, 3, 4, 2, 2, 2, 4, 2, 3, 3, 2, 2, 2, 4, 2, 3, 4 \rangle$$

Repeating this process indefinitely we obtain an infinite Lüroth series, denoted by

$$A = \langle a_1, a_2, \dots, a_n, \dots \rangle.$$

We conjecture that A is a normal Lüroth series and show a partial proof for this statement in chapter 5.

Chapter 4

Notation

For our (partial) proof we will need some notation. We define the following symbols related to our Lüroth series A :

- W_n : the number of words we append in step n of the construction process. We have $W_n = \text{sf}(n)$.
- W_n^* : the total number of words we append in steps 1 through n of the construction process of A . This means $W_n^* = \sum_{i=1}^n W_i = \sum_{i=1}^n \text{sf}(i)$.
- d_m : the number of digits in the m th word we append.
- d_m^* : the number of digits in the first m words we append. By definition this is $d_m^* = \sum_{i=1}^m d_i$.
- D_n : the number of digits we append in the n th step of the construction process described earlier. As this step consists of appending words $W_{n-1}^* + 1$ through W_n^* we have $D_n = \sum_{m=W_{n-1}^*+1}^{W_n^*} d_m$.
- D_n^* : the number of digits our Lüroth series has after steps 1 through n of the construction process. This means that both $D_n^* = \sum_{i=1}^n D_i$ and $D_n^* = d_{W_n^*}^*$.

We will often look at the number of occurrences of a finite sequence of digits $\alpha_1, \alpha_2, \dots, \alpha_k$ in the infinite Lüroth series A or parts of it. Naturally, we only count occurrences in which these digits appear consecutively and in the same order. Two or more occurrences may overlap (i.e. share digits). For these absolute frequencies we will use the following notation:

- $f(m, \langle \alpha_1, \alpha_2, \dots, \alpha_k \rangle)$: the number of times the sequence of digits $\alpha_1, \alpha_2, \dots, \alpha_k$ occurs in the m th word that A consists of.
- $F(n, \langle \alpha_1, \alpha_2, \dots, \alpha_k \rangle)$: the number of times the sequence of digits $\alpha_1, \alpha_2, \dots, \alpha_k$ occurs in the digits that we append while performing the n th step of the construction process for A .

As above, we will denote the cumulative values (occurrences in the first n words or steps) by f^* and F^* respectively¹. In addition we will write $\mathcal{F}^*(n, \langle \alpha_1, \alpha_2, \dots, \alpha_k \rangle)$ for the number of occurrences of $\alpha_1, \alpha_2, \dots, \alpha_k$ in the first n digits of the entire series A .

¹To remember: d is for number of digits, w for number of words; uppercase letters indicate characteristics per construction step, lowercase letters per word; symbols marked with an asterisk are cumulative.

Like Champernowne [4], we will distinguish between *divided* and *undivided* occurrences of a sequence of digits. We call an occurrence undivided when it is entirely within one word. When this is not the case (so the digits $\alpha_1, \alpha_2, \dots, \alpha_k$ are split among two or more words) we call an occurrence divided. For all defined frequency symbols we will use the subscripts ‘d’ and ‘u’ when we only consider divided or undivided occurrences respectively.

Example 2. $F_u^*(3, \langle 3, 2 \rangle)$ denotes the number of times the sequence 3, 2 occurs undivided in the Lüroth series we obtain after 3 steps of the construction process. This gives us $F_u^*(3, \langle 3, 2 \rangle) = 2$, as can be seen from the Lüroth series obtained after 3 construction steps (in which word boundaries have been marked by a semicolon):

$$\langle 2; 2, 2; 3; 2, 2, 2; \underline{3, 2}; 2, 3; 4; 2, 2, 2; 4; 2, 3; \underline{3, 2}; 2, 2, 2; 4; 2, 3; 4 \rangle .$$

The number of divided occurrences of this sequence is denoted by $F_d^*(3, \langle 3, 2 \rangle)$ and we have $F_d^*(3, \langle 3, 2 \rangle) = 1$:

$$\langle 2; 2, 2; \underline{3, 2}, 2, 2; 3, 2; 2, 3; 4; 2, 2, 2; 4; 2, 3; 3, 2; 2, 2, 2; 4; 2, 3; 4 \rangle .$$

This gives us $F^*(3, \langle 3, 2 \rangle) = F_u^*(3, \langle 3, 2 \rangle) + F_d^*(3, \langle 3, 2 \rangle) = 3$ occurrences in total.

Furthermore, we adopt the convention that the natural numbers do not include 0, i.e. $\mathbb{N} = \{1, 2, 3, \dots\}$.

Chapter 5

Conjecture and partial proof

We now state our main conjecture:

Conjecture 1 (Normality of A). *The infinite Lüroth series A that we defined in Section 3 is normal. This means that for every finite Lüroth series $\alpha = \langle \alpha_1, \alpha_2, \dots, \alpha_k \rangle$ the frequency of the digits of α in the digits of A converges to the correct limiting distribution as we take more and more digits of A into account. That is*

$$\lim_{n \rightarrow \infty} \frac{\mathcal{F}^*(n, \alpha)}{n} = \mu(\alpha).$$

Remark 1 (Outline of proof). We think this conjecture can be proved in four steps, and we will provide proofs for three of these. For the fourth step, which we currently cannot prove, we will give some remarks about its plausibility. The four steps we distinguish are:

1.

$$\lim_{n \rightarrow \infty} \frac{\mathcal{F}^*(n, \alpha)}{n} = \lim_{n \rightarrow \infty} \frac{\mathcal{F}_u^*(n, \alpha)}{n}$$

(We are allowed to consider only undivided occurrences and can neglect the fact that we concatenate different words together in the construction process)

2.

$$\lim_{N \rightarrow \infty} \frac{F_u(N, \alpha)}{D_N} = \mu(\alpha)$$

(The distribution of the digits (in terms of undivided occurrences) appended in step N of the construction process converges to the correct limiting distribution as $N \rightarrow \infty$)

3.

$$\lim_{N \rightarrow \infty} \frac{F_u^*(N, \alpha)}{D_N^*} = \lim_{n \rightarrow \infty} \frac{F_u(n, \alpha)}{D_n}$$

(Because the undivided frequency of α within the digits appended in the n th construction step converges, so does the undivided frequency within the digits of the first N steps together)

4.

$$\lim_{n \rightarrow \infty} \frac{\mathcal{F}_u^*(n, \alpha)}{n} = \lim_{N \rightarrow \infty} \frac{F_u^*(N, \alpha)}{D_N^*}$$

(We may exchange the limit over the number of digits for the limit over the number of construction steps performed)

5.1 Step 1

Theorem 1 (Step 1). *When proving the normality of A , we are allowed to consider only undivided occurrences and can neglect the fact that we concatenate different words together in the construction process. That is,*

$$\lim_{n \rightarrow \infty} \frac{\mathcal{F}^*(n, \alpha)}{n} = \lim_{n \rightarrow \infty} \frac{\mathcal{F}_u^*(n, \alpha)}{n}.$$

Proof. We have

$$\lim_{n \rightarrow \infty} \frac{\mathcal{F}^*(n, \alpha)}{n} = \lim_{n \rightarrow \infty} \frac{\mathcal{F}_u^*(n, \alpha) + \mathcal{F}_d^*(n, \alpha)}{n} = \lim_{n \rightarrow \infty} \frac{\mathcal{F}_u^*(n, \alpha)}{n} + \lim_{n \rightarrow \infty} \frac{\mathcal{F}_d^*(n, \alpha)}{n},$$

provided that all limits exist. So we only have to prove that

$$\lim_{n \rightarrow \infty} \frac{\mathcal{F}_d^*(n, \alpha)}{n} = 0.$$

Note that the number of divided occurrences of α in a finite block of digits of A cannot be greater than the number of concatenations (between words) in that block, multiplied by $k - 1$. Also note that the number of concatenations is smaller (by one) than the number of words the subseries consists of. This means that

$$\lim_{n \rightarrow \infty} \frac{\mathcal{F}_d^*(n, \alpha)}{n} < \lim_{n \rightarrow \infty} \frac{(k - 1) \cdot W_{N(n)}^*}{n} = (k - 1) \cdot \lim_{n \rightarrow \infty} \frac{W_{N(n)}^*}{n},$$

in which we used $N(n)$ to denote the number of construction steps needed to get at least n digits of A . We will show that the limit at the right-hand side tends to zero as $n \rightarrow \infty$ by showing that its reciprocal goes to infinity:

$$\lim_{n \rightarrow \infty} \frac{n}{W_{N(n)}^*} = \infty.$$

Note that the term of which we take the limit denotes the average number of digits per word in the series obtained after the first $N(n)$ construction steps (when $n = D_i^*$ for some i). This means that it is sufficient to show that the average length of each word is unbounded as we execute more and more steps of the construction process.

Here we silently exchange the limit over the number of digits for a limit over the number of construction steps performed. Note that this is justified, as in the construction process the longer nodes in a level are appended to the series before the shorter ones. This means that anywhere in the middle of a construction step, the average number of digits per word is always at least as high as it is at the end of that step, justifying the change of limit.

If we look at the nodes in the $(n - 1)$ th level of the Lüroth tree, we see that in the n th level its children either keep the length of their parent (if we move right) or get one additional digit (if we move left). When performing step n of the construction process, the step to the left from each node in level $(n - 1)$ is made in at least $\frac{1}{n}$ of the cases. This means that if we denote the average number of digits in the words appended in step n by ω_n we have the inequality

$$\omega_n \geq \omega_{n-1} + \frac{1}{n}$$

from which we can derive (using that $\omega_1 = 1$) that

$$\omega_n \geq \sum_{i=1}^n \frac{1}{i}.$$

Because the harmonic series

$$\sum_{i=1}^{\infty} \frac{1}{i} = \infty$$

diverges, the average number of digits per word is unbounded and we have obtained the desired result. $\square$

5.2 Step 2

To prove step two, we need to distinguish between *fixed* and *unfixed* digits in the Lüroth tree. We will call the last digit of each treenode unfixed (because it can ‘evolve’ into another digit when we move left down the tree) and all the other digits fixed.

Note that the number of unfixed digits appended in a construction step is of the same order as the number of concatenations in that step, which (by Theorem 1) we can neglect. Thus, in this step we can also ignore the unfixed digits and only consider occurrences consisting entirely of fixed digits.

So, we will look at the frequency with which a finite sequence $\alpha = \langle \alpha_1, \alpha_2, \dots, \alpha_k \rangle$ of length k occurs undivided within the fixed digits appended in a certain step N of the construction process. As these undivided occurrences lie entirely within one word and the words appended in step N form a representative sample of level N in the Lüroth tree, we can view the mentioned frequency also as a probability: the probability that, if we take an $n \in \mathbb{N}$ ‘small enough’ compared to N and make a random walk from the root of the tree down to level N , we end up at a node that has an occurrence (consisting entirely of fixed digits) of α starting at the n th digit of that node. Note that the requirement that n is small compared to N is no problem, as we send N to infinity. In view of this, we define a random path $X_1, X_2, X_3, \dots$ from the root of the tree downwards, in which X_i is the i th node on the path. So for example $P(X_1 = \langle 2 \rangle) = 1$ (because we always start at the root) and $P(X_3 = \langle 4 \rangle) = \frac{1}{2} \cdot \frac{2}{3} = \frac{1}{3}$.

We now give a lemma which states that for any n , during our walk, we will always come across a node of length n with a trailing unfixed 2.

Lemma 1. *Fix an $n \in \mathbb{N}$. Then, for the random path $X_1, X_2, X_3, \dots$ defined above we have*

$$\sum_{i=1}^{\infty} P(X_i \text{ has length } n \text{ and ends with an unfixed } 2) = 1.$$

Proof. We prove this lemma by induction on the length n . When $n = 2$, the proof is essentially equivalent to the general inductive step, but we have included this case separately for clarity:

- (i) For $n = 1$ this is trivial as $P(X_1 = \langle 2 \rangle) = 1$ and no other node in the tree consists of just a single digit 2.
- (ii) When $n = 2$ we note that a node $\langle b_1, 2 \rangle$ for some b_1 can only occur by starting from the root of the tree (which we always do), going right zero or more times, and then going left once. We will write ‘RRL’ for the event of going right twice from the root and then going left, and use similar notation for other paths. Additionally, we will write $E_{i,n}$ for the event ‘ X_i has length n and ends with an unfixed 2’. Then, viewing the infinite sum as

conditioning over the number of times we move right, we have

$$\begin{aligned}
\sum_{i=1}^{\infty} P(E_{i,2}) &= \sum_{i=2}^{\infty} P(E_{i,2}) \\
&= P(L) + P(RL) + P(RRL) + \dots \\
&= \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{3} + \frac{1}{2} \cdot \frac{2}{3} \cdot \frac{1}{4} + \dots \\
&= \frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \dots \\
&= \sum_{i=2}^{\infty} \frac{1}{i(i-1)} \\
&= 1.
\end{aligned}$$

- (iii) Suppose that the statement of the lemma holds for a certain $n \in \mathbb{N}$. Because we know that a node with length $n+1$ cannot occur before level $n+1$ we have

$$\sum_{i=1}^{\infty} P(E_{i,n+1}) = \sum_{i=n+1}^{\infty} P(E_{i,n+1})$$

Now, $P(E_{i,n+1}|E_{j,n})$ is exactly the probability of moving right $i-j-1$ times and then moving left once, when starting from a node ending with a 2. Hence, by conditioning over the level j where we first reached a node of length n (which is only possible for $j \geq n$),

$$\begin{aligned}
\sum_{i=1}^{\infty} P(E_{i,n+1}) &= \sum_{i=n+1}^{\infty} \sum_{j=n}^{i-1} P(E_{i,n+1}|E_{j,n}) \cdot P(E_{j,n}) \\
&= \sum_{j=n}^{\infty} \left(P(E_{j,n}) \cdot \sum_{i=j+1}^{\infty} P(E_{i,n+1}|E_{j,n}) \right).
\end{aligned}$$

As we have seen earlier, provided $i \geq j+1$, $P(E_{i,n+1}|E_{j,n})$ equals $\frac{1}{(i-j+1)(i-j)}$. Hence,

$$\begin{aligned}
\sum_{i=1}^{\infty} P(E_{i,n+1}) &= \sum_{j=n}^{\infty} \left(P(E_{j,n}) \cdot \sum_{i=j+1}^{\infty} \frac{1}{(i-j+1)(i-j)} \right) \\
&= \sum_{j=n}^{\infty} \left(P(E_{j,n}) \cdot \sum_{m=1}^{\infty} \frac{1}{(m+1)m} \right) \\
&= \sum_{j=n}^{\infty} (P(E_{j,n}) \cdot 1),
\end{aligned}$$

and because $P(E_{j,n}) = 0$ for $j < n$,

$$\sum_{i=1}^{\infty} P(E_{i,n+1}) = \sum_{j=1}^{\infty} P(E_{j,n}).$$

When we now use the induction hypothesis we find the desired result:

$$\sum_{i=1}^{\infty} P(E_{i,n+1}) = 1.$$

□

Remember that the construction of the Lüroth tree has two generating operations:

- Appending a 2
- Incrementing the last digit by one

We define $L(\alpha)$ as the number of these operations we need to transform $\langle 2 \rangle$ into the series $\langle \alpha_1, \alpha_2, \alpha_3, \dots, \alpha_k, 2 \rangle$. We have

$$L(\alpha) = k + \sum_{i=1}^k (\alpha_i - 2).$$

Note that $L(\alpha)$ is also the depth of the node α in the tree.

Now we will look at the conditional probabilities of a trailing unfixed 2 eventually ‘evolving’ into the sequence α . Note that this will only occur when, from the node with the trailing 2, we first move $\alpha_1 - 2$ times to the right, then left, then $\alpha_2 - 2$ times to the right, then left again, etcetera. It does not matter what happens after we stepped to the right the last $\alpha_k - 2$ times and then to the left; now the occurrence of α is fixed and cannot change when we continue our walk. Also note that we needed $L(\alpha)$ steps for this process, so a node that has a fixed occurrence of α at position n must reside in the Lüroth tree at least $L(\alpha)$ levels lower than the node with the unfixed 2 at position n .

These observations lead to the conditional probabilities

$$\begin{aligned} & P(\alpha \text{ occurs fixed in } X_i \text{ at position } n | X_j \text{ has length } n \text{ and ends with an unfixed 2}) \\ &= \begin{cases} 0 & \text{if } i < j + L(\alpha) \\ \mu(\alpha) & \text{if } i \geq j + L(\alpha) \end{cases}, \end{aligned}$$

that can be used to prove the main Theorem of this step:

Theorem 2 (Step 2). *Fix an $n \in \mathbb{N}$ and let $\alpha = \langle \alpha_1, \alpha_2, \dots, \alpha_k \rangle$. Then, for the random path $X_1, X_2, X_3, \dots$ defined earlier we find*

$$\lim_{N \rightarrow \infty} P(\alpha \text{ occurs fixed in } X_N \text{ at position } n) = \mu(\alpha).$$

Proof. We will write $E_{j,n}$ for the event ‘ X_j has length n and ends with an unfixed 2’ like we did in Lemma 1. We note that an occurrence of α at position n must have ‘evolved’ from an unfixed 2 at that position higher up in the tree. Conditioning over the node (with index j) in which this unfixed 2 occurred we have:

$$\begin{aligned} & \lim_{N \rightarrow \infty} P(\alpha \text{ occurs fixed in } X_N \text{ at position } n) \\ &= \lim_{N \rightarrow \infty} \sum_{j=1}^N P(\alpha \text{ occurs fixed in } X_N \text{ at position } n | E_{j,n}) \cdot P(E_{j,n}) \\ &= \lim_{N \rightarrow \infty} \sum_{j=1}^{N-L(\alpha)} \mu(\alpha) \cdot P(E_{j,n}) \\ &= \mu(\alpha) \cdot \sum_{j=1}^{\infty} P(E_{j,n}). \end{aligned}$$

Now we use the result from Lemma 1 in this last expression to obtain the desired equality:

$$\lim_{N \rightarrow \infty} P(\alpha \text{ occurs fixed in } X_N \text{ at position } n) = \mu(\alpha).$$

□

5.3 Step 3

We can now easily show that, since the frequency of α within level n of the tree converges to the correct limit as $n \rightarrow \infty$, the frequency within the first N levels also converges to that limit.

Theorem 3 (Step 3). *Let $\alpha = \langle \alpha_1, \alpha_2, \dots, \alpha_k \rangle$ be a finite Lüroth series. Then*

$$\lim_{N \rightarrow \infty} \frac{F_u^*(N, \alpha)}{D_N^*} = \lim_{n \rightarrow \infty} \frac{F_u(n, \alpha)}{D_n},$$

provided the latter limit exists.

Proof. Fix an arbitrary $\varepsilon > 0$. Let $L = \lim_{n \rightarrow \infty} \frac{F_u(n, \alpha)}{D_n}$ and choose an $N_0 \in \mathbb{N}$ such that $\left| \frac{F_u(n, \alpha)}{D_n} - L \right| < \frac{1}{2}\varepsilon$ whenever $n \geq N_0$.

Now set $M = \sup_n \left| \frac{F_u(n, \alpha)}{D_n} - L \right|$. We know this supremum exists because $\frac{F_u(n, \alpha)}{D_n}$ converges. Then we have for $N > N_0$:

$$\begin{aligned} \left| \frac{F_u^*(N, \alpha)}{D_N^*} - L \right| &= \left| \left(\frac{1}{D_N^*} \sum_{n=1}^N F_u(n, \alpha) \right) - L \right| \\ &= \left| \frac{1}{D_N^*} \sum_{n=1}^N (F_u(n, \alpha) - D_n L) \right| \\ &= \left| \frac{1}{D_N^*} \sum_{n=1}^N D_n \left(\frac{F_u(n, \alpha)}{D_n} - L \right) \right| \\ &\leq \frac{1}{D_N^*} \sum_{n=1}^N \left| D_n \left(\frac{F_u(n, \alpha)}{D_n} - L \right) \right|. \end{aligned}$$

We now split this sum into two parts to obtain

$$\begin{aligned} \left| \frac{F_u^*(N, \alpha)}{D_N^*} - L \right| &\leq \frac{1}{D_N^*} \sum_{n=1}^{N_0} \left| D_n \left(\frac{F_u(n, \alpha)}{D_n} - L \right) \right| + \frac{1}{D_N^*} \sum_{n=N_0+1}^N \left| D_n \left(\frac{F_u(n, \alpha)}{D_n} - L \right) \right| \\ &< \frac{1}{D_N^*} \sum_{n=1}^{N_0} D_n M + \frac{1}{D_N^*} \sum_{n=N_0+1}^N D_n \frac{1}{2}\varepsilon \\ &= \frac{D_{N_0}^* M}{D_N^*} + \frac{(D_N^* - D_{N_0}^*)}{D_N^*} \frac{1}{2}\varepsilon. \end{aligned}$$

As $D_{N_0}^* M$ is constant and $D_N^* \rightarrow \infty$ when $N \rightarrow \infty$, we can choose an $N_1 \in \mathbb{N}$, $N_1 \geq N_0$ such that $\frac{D_{N_0}^* M}{D_N^*} < \frac{1}{2}\varepsilon$ whenever $N \geq N_1$. Then, for $N \geq N_1$:

$$\begin{aligned} \left| \frac{F_u^*(N, \alpha)}{D_N^*} - L \right| &< \frac{D_{N_0}^* M}{D_N^*} + \frac{(D_N^* - D_{N_0}^*)}{D_N^*} \frac{1}{2}\varepsilon \\ &< \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon \\ &= \varepsilon, \end{aligned}$$

which proves that also

$$\lim_{N \rightarrow \infty} \frac{F_u^*(N, \alpha)}{D_N^*} = L.$$

□

5.4 Step 4

So far we have proven that the sequence of cumulative frequencies of a certain block α (indexed by the number of digits considered) contains a convergent subsequence. This still does not guarantee that the sequence actually converges to the desired limit. To prove the latter, we have to justify the change of variables from the number of construction steps to the number of digits in the limit:

$$\lim_{n \rightarrow \infty} \frac{\mathcal{F}_u^*(n, \alpha)}{n} = \lim_{N \rightarrow \infty} \frac{F_u^*(N, \alpha)}{D_N^*}$$

First we should note that Champernowne does not include this step in his proof, but just changes variables in the limit without an explanation. His construction however has the same problem as the one described here. Adler, Keane and Smorodinsky justify the change of limit variable in their proof using the fact that their ‘construction tree’ does not grow as explosively as ours (see Section 6.2).

We have not yet succeeded in proving this last step. We will however give some remarks about its plausibility and possible strategies for proving it.

The problem with this step is that the number of digits concatenated in a construction step gets very large (compared to the number of digits in all previous steps) when moving down the Lüroth tree. This means that, even if the frequency of α tends to the desired frequency at the end of each construction step, it could still diverge arbitrarily much in the middle of a step. The crucial observation to oppose this argument probably is that the digits appended in a certain step are very similar to the digits concatenated in the previous construction step, as only the last digit of each word can be different.

Anywhere in the process of constructing the Lüroth series A we can distinguish between a number of fully completed construction steps and a certain number of ‘remainder digits’, digits appended in the current unfinished construction step. When the number of remainder digits is relatively small we can just neglect them and use the results proved in Theorem 3. When there are more remainder digits it might be possible to use the mentioned similarity with the digits in the previous construction step to prove that the frequency of α within this digits does not differ too much from the desired frequency.

We will conclude this section with a lemma that gives another criterion (that might be slightly easier to prove) for the result we wish to establish in this step:

Lemma 2. *Suppose there exists a constant C such that*

$$\limsup_{n \rightarrow \infty} \frac{\mathcal{F}_u^*(n, \alpha)}{n} < C\mu(\alpha)$$

for any block α . Then

$$\lim_{n \rightarrow \infty} \frac{\mathcal{F}_u^*(n, \alpha)}{n} = \mu(\alpha).$$

Proof. This lemma is directly adapted from [10]. □

Chapter 6

Reflection

6.1 Extension to Generalized Lüroth Series

It might be worthwhile to note that the construction we carried out can be generalized to certain other Generalized Lüroth Series expansions with infinite digit sets. One prerequisite for this is that the length of each interval corresponding to a digit is rational. Additionally, these lengths must be distinct, so that (using notation from [2]) we can form a sequence of digits

$$n_1, n_2, n_3, \dots$$

such that their interval lengths are strictly decreasing:

$$L_{n_1} > L_{n_2} > L_{n_3} > \dots$$

We can then define probabilities $P_{L,k}$ and $P_{R,k}$ that give the probability of going left and right at a node of which the last digit is n_k . The probability for going to the left is given by

$$P_{L,k} = \frac{L_k}{1 - \sum_{i=1}^{k-1} L_i},$$

and the probability of going right is its complement:

$$P_{R,k} = 1 - P_{L,k} = 1 - \frac{L_k}{1 - \sum_{i=1}^{k-1} L_i}.$$

If we now use $\langle \dots \rangle$ to denote expansions for the generalized series considered, this leads to the generalized Lüroth tree shown in Figure 6.1.

It is easily verified that in this tree the probability of reaching node $\langle n_k \rangle$ from the root and then moving left (fixing the digit n_k) is exactly L_k , so the tree has the correct underlying distribution of digits.

As the interval lengths L_k are all rational, so are the probabilities $P_{L,k}$ and $P_{R,k}$. This means that when multiplying the probabilities along a finite path downwards in the tree we also obtain a rational probability of arriving at that node. We can thus take the least common multiple of the denominators of all such probabilities at a certain level, and use that many nodes from that level to yield a representative sample from it¹. This establishes steps 2 and 3 of the proof, and only steps 1 and 4 remain to be verified.

¹Note that in our specific case we used the superfactorial for this purpose. We could have taken the least common multiple, which is smaller but more difficult to use in proofs.

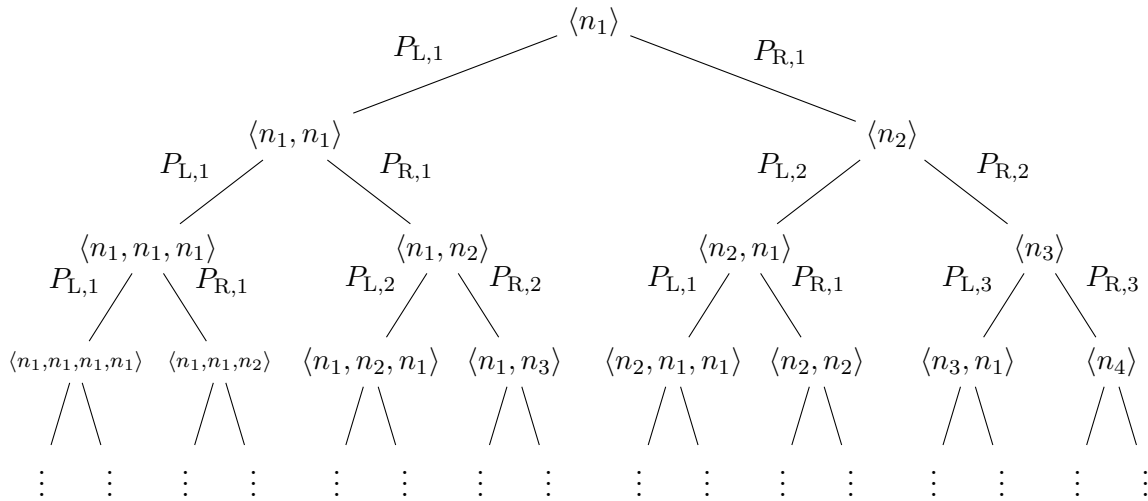


Figure 6.1: Top of the generalized Lüroth tree with probability distribution

6.2 Extension to other expansions

A little reflection shows that the method we used to construct a presumably normal Lüroth series is in a certain sense equivalent to the construction Adler, Keane and Smorodinsky used in [1], or the way Champernowne [4] constructed the normal decimal number

$$S = 0.012345678901020304 \dots 091011 \dots 99000001 \dots,$$

that he used to prove the normality of his famous constant.

All these methods have a unlimited number of steps in which they go through the interval $[0, 1)$. In each step they take a larger sample set of points from the interval than in the previous step. These sets are taken in such a way that the mesh width of the partitions they define become smaller with each step. Note that the sample set from step $n + 1$ is not necessarily a refinement of the one in step n . In every step, the elements of the sample set are represented in the expansion studied and then appended before moving on to the next step. Note that these representations must be finite. Figure 6.2 shows a comparison between the sample sets used in various constructions.

The aspect in which our construction differs from the others mentioned is that the sample sets used are not distributed uniformly in $[0, 1)$. This is caused by the fact that not all rationals have a finite Lüroth series expansion and a simple characterization of the numbers that do have one is not known yet, which makes it difficult to find a uniformly distributed set of numbers with finite expansions. The sample sets we used are constructed such that they contain only elements with finite series. To compensate for this skewed distribution we had to append each element a different number of times, so that we put more ‘mass’ on elements in larger intervals.

It might be possible to prove a theorem that, under certain conditions, asserts that constructions of this kind always yield a normal number. This will however require further research.

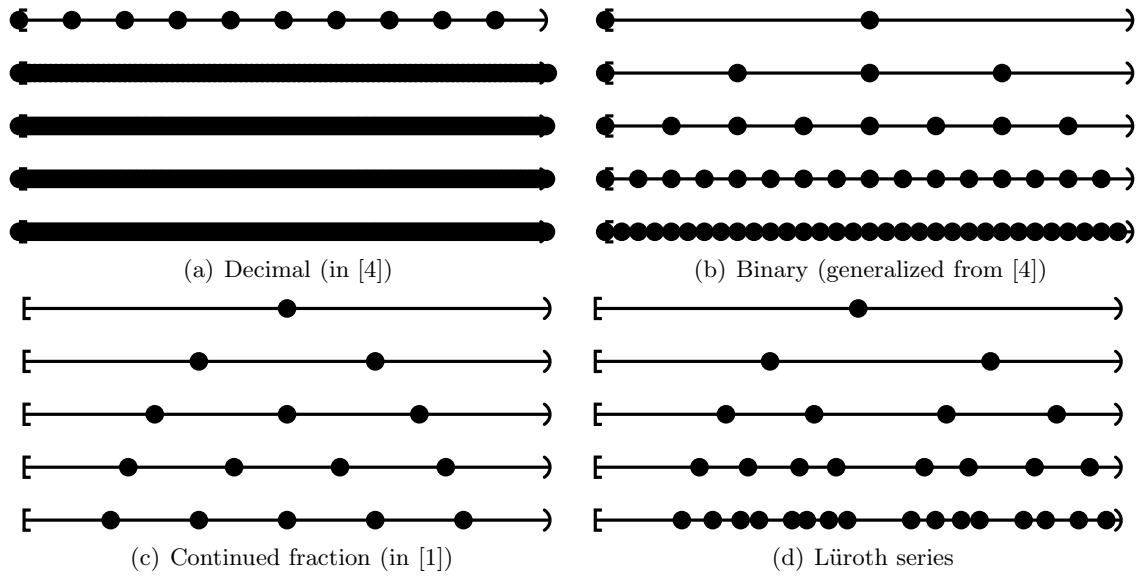


Figure 6.2: Sample sets of the interval $[0, 1]$ used in successive steps of construction processes for normal numbers in different expansions.

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