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Hankel matrices for the period-doubling sequence

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Abstract

We give an explicit evaluation, in terms of products of Jacobsthal numbers, of the Hankel determinants of order a power of two for the period-doubling sequence. We also explicitly give the eigenvalues and eigenvectors of the corresponding Hankel matrices. Similar considerations give the Hankel determinants for other orders.

1. Introduction

Let $\mathbf{s} = (s_n)_{n\geq 0}$ be a sequence of real numbers. The *Hankel matrix* $M_{\mathbf{s}}(k)$ of order k associated with $\mathbf{s}$ is defined as follows:

$$M_{\mathbf{s}}(k) = \begin{bmatrix} s_0 & s_1 & \cdots & s_{k-1} \\ s_1 & s_2 & \cdots & s_k \\ \vdots & \vdots & \ddots & \vdots \\ s_{k-1} & s_k & \cdots & s_{2k-2} \end{bmatrix}. \tag{1}$$

See, for example, [17]. Note that the rows of $M_{\mathbf{s}}(k)$ are made up of successive length-k "windows" into the sequence $\mathbf{s}$.

Of particular interest are the determinants $\Delta_{\mathbf{s}}(k) = \det M_{\mathbf{s}}(k)$ of the Hankel matrices in (1), which are often quite challenging to compute explicitly. In some cases when these determinants are non-zero, they permit estimation of the irrationality measure of the associated real numbers $\sum_{n\geq 0} s_n b^{-n}$, where $b \geq 2$ is an integer; see, for example, [2,4–7,19,26]. In some sense, the

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Hankel determinants measure how "far away" the sequence $\mathbf{s}$ is from a linear recurrence with constant coefficients, since for such a sequence we have $H_{\mathbf{s}}(n) = 0$ for all sufficiently large n.

In this note we consider the Hankel determinants for a certain infinite sequence of interest, the so-called *period-doubling sequence*

$$\mathbf{d} = (d_i)_{i\geq 0} = 1011101010111011101110101011101 \cdots .$$

This sequence can be defined in various ways [8], but probably the three simplest are as follows:

- as the fixed point of the map

$$1 \to 10, \qquad 0 \to 11;$$

- as the first difference, taken modulo 2, of the Thue–Morse sequence $\mathbf{t} = 0110100110010110 \cdots$ (fixed point of the map $0 \to 01$, $1 \to 10$);
- as the sequence defined by

$$d_i = \begin{cases} 1, & \text{if } s_2(i) \not\equiv s_2(i+1) \pmod 2; \\ 0, & \text{otherwise;} \end{cases}$$

  where $s_2(i)$ is the sum of the binary digits of i when expressed in base 2.

We explicitly compute the Hankel determinants when the orders are a power of 2, and we also compute the eigenvalues and eigenvectors of the corresponding Hankel matrices. We derive recursions for Hankel determinants for all orders. Finally, we also consider the determinants for the complementary sequence

$$\overline{\mathbf{d}} = 0100010101000100010001010100010 \cdots ,$$

obtained from $\mathbf{d}$ by changing 1 to 0 and vice versa.

1.1. Previous work

By considering $\Delta_{\mathbf{d}}(n)$ modulo 2, Allouche, Peyrière, Wen, and Wen [1] proved that $\Delta_{\mathbf{d}}(n)$ is odd for all $n \geq 1$. However, they did not obtain any explicit formula for $\Delta_{\mathbf{d}}(n)$. In fact, their main focus was on the non-vanishing of the Hankel determinants for the Thue–Morse sequence on values ± 1. For this, also see Bugeaud and Han [3] and Han [14]. Recently Fu and Han [9] also studied some Hankel matrices associated with the period-doubling sequence, but they did not obtain our result.

There are only a small number of sequences defined by iterated morphisms for which the Hankel determinants are explicitly known (even for subsequences). These include the infinite Fibonacci word [18], the paperfolding sequence [9,13], the iterated differences of the Thue–Morse sequence [12], the Cantor sequence [27], and sequences related to the Thue–Morse sequence [10,11,15].

2. Hankel determinants

Here are the first few terms of the Hankel determinants for the period-doubling sequence and its complementary sequence:

k	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$\Delta_{\mathbf{d}}(k)$	1	1	−1	−3	1	1	−1	−15	1	1	−1	−3	1	1	−9	−495
$\Delta_{\overline{\mathbf{d}}}(k)$	0	−1	0	1	0	−1	0	9	0	−1	0	1	0	−1	0	225

The large values at the powers of 2 suggest something interesting is going on. Indeed, by explicit calculation we find

$$\Delta_{\mathbf{d}}(32) = -467775 \qquad \Delta_{\overline{\mathbf{d}}}(32) = 245025$$
$$\Delta_{\mathbf{d}}(64) = -448046589375 \qquad \Delta_{\overline{\mathbf{d}}}(64) = 218813450625$$
$$\Delta_{\mathbf{d}}(128) = -396822986774382287109375 \qquad \Delta_{\overline{\mathbf{d}}}(128) = 200745746250569862890625,$$

and so forth. Another obvious pattern is $\Delta_{\overline{\mathbf{d}}}(n) = 0$ for odd n.

Define $J_n = (2^n - (-1)^n)/3$, the so-called *Jacobsthal numbers* [16]. It is easy to see that

$$J_{n+1} = J_n + 2J_{n-1} \tag{2}$$
$$J_{n+1} = 2J_n + (-1)^n \tag{3}$$
$$J_{n+1} = 2^n - J_n \tag{4}$$

for $n \geq 0$. We will prove that $\Delta_{\mathbf{d}}(n)$ and $\Delta_{\overline{\mathbf{d}}}(n)$ are products of n Jacobsthal numbers, and that their factorizations are almost the same. The reason why $\Delta_{\overline{\mathbf{d}}}(n) = 0$ for odd n is that it is a product involving J_0.

In this paper we will prove

Theorem 1. *For integers $k \geq 2$ we have $\Delta_{\mathbf{d}}(2^k) = -J_{k+1} \prod_{3 \leq i \leq k} J_i^{2^{k-i}}$, and $\Delta_{\overline{\mathbf{d}}}(2^k) = J_k \prod_{3 \leq i \leq k} J_i^{2^{k-i}}$, where, as usual, the empty product evaluates to 1.*

In the proof of Theorem 1, we also obtain a complete description of the eigenvalues of $M_{\mathbf{d}}(2^k)$ and $M_{\overline{\mathbf{d}}}(2^k)$, as well as a basis for the corresponding eigenspaces.

Our second main result handles the Hankel determinants of all orders.

Theorem 2. *For all integers $n \geq 1$, the Hankel determinants $\Delta_{\mathbf{d}}(n)$ and $\Delta_{\overline{\mathbf{d}}}(n)$ are products of n Jacobsthal numbers, counted with multiplicity, and including the trivial divisors J_0, J_1, J_2 in the count. If n is even, then $J_i \Delta_{\mathbf{d}}(n) = -J_{i+1} \Delta_{\overline{\mathbf{d}}}(n)$ for some $i > 0$.*

3. 1-D and 2-D morphisms

Let Σ, Δ denote finite alphabets. A *morphism* (or *substitution*) is a map h from $\Sigma^* \to \Delta^*$ satisfying $h(xy) = h(x)h(y)$ for all strings x, y. If $\Sigma = \Delta$ we can iterate h, writing $h^1(x)$ for $h(x)$, $h^2(x)$ for $h(h(x))$, and so forth. In this paper we will need a variant of the so-called Thue–Morse morphism [24], defined as follows:

$$\rho(1) = (-1, 1) \qquad \rho(-1) = (1, -1).$$

We can also define the notion of morphisms for arrays (or matrices). A *2-D morphism* (or *2-D substitution*) can be viewed as a map from Σ to $\Delta^{r \times s}$ that is extended to matrices in the obvious way [20–22,25].

One of the most famous maps of this form is the map

$$\gamma(1) = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \qquad \gamma(-1) = \begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix},$$

which, when iterated k times, produces a *Hadamard matrix* of order 2^k. (An $n \times n$ matrix H is said to be Hadamard if all entries are ± 1 and furthermore $HH^T = nI$, where I is the identity matrix; see [23].)

We now observe that the Hankel matrix $M_{\mathbf{d}}(2^k)$ of the period-doubling sequence can be generated in a similar way, via the 2-D morphism

$$\varphi(1) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \varphi(0) = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

More precisely, $M_{\mathbf{d}}(2^k) = \varphi^k(1)$.

Similarly, $M_{\overline{\mathbf{d}}}(2^k) = \overline{\varphi}^k(0)$ for the complementary substitution $\overline{\varphi}$ which is defined as follows:

$$\overline{\varphi}(0) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad \overline{\varphi}(1) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Let $v = (a_1, a_2, \ldots, a_n)$ be a vector of length n. By diag(v) we mean the diagonal matrix

$$\begin{bmatrix} a_1 & 0 & 0 & \cdots & 0 \\ 0 & a_2 & 0 & \cdots & 0 \\ 0 & 0 & a_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_n \end{bmatrix}.$$

We now observe that the Hankel matrices of the period-doubling sequence are diagonalized by the Hadamard matrices $\gamma^k(1)$:

Theorem 3. *For $k \geq 1$ we have*

(a) $\gamma^k(1)\varphi^k(1)\gamma^k(1) = 2^k \operatorname{diag}(J_{k+1}, J_k, J_{k-1}\rho(1), J_{k-2}\rho^2(1), \ldots, J_1\rho^{k-1}(1))$ *and*
(b) $\gamma^k(1)\varphi^k(0)\gamma^k(1) = 2^{k+1} \operatorname{diag}(J_k, J_{k-1}, J_{k-2}\rho(1), \ldots, J_1\rho^{k-2}(1), J_0\rho^{k-1}(1))$.

Proof. By induction on k. The verification for $k = 1$ is left to the reader.

Now assume the results are true for k. We prove them for $k + 1$.

We start with (a). Write P_k for the vector $[J_{k+1}, J_k, J_{k-1}\rho(1), J_{k-2}\rho^2(1), \ldots, J_1\rho^{k-1}(1)]$ and Q_k for the vector $[J_k, J_{k-1}, J_{k-2}\rho(1), \ldots, J_1\rho^{k-2}(1), J_0\rho^{k-1}(1)]$. Note that from the definition of P_k and Q_k, and the fact that $J_0 = 0$, we have

$$Q_{k+1} = [P_k, \overbrace{0, 0, \ldots, 0}^{2^k}]. \tag{5}$$

Now

$$\begin{aligned}
\gamma^{k+1}(1)\varphi^{k+1}(1)\gamma^{k+1}(1) &= \begin{bmatrix} \gamma^k(1) & \gamma^k(1) \\ \gamma^k(1) & -\gamma^k(1) \end{bmatrix} \begin{bmatrix} \varphi^k(1) & \varphi^k(0) \\ \varphi^k(0) & \varphi^k(1) \end{bmatrix} \begin{bmatrix} \gamma^k(1) & \gamma^k(1) \\ \gamma^k(1) & -\gamma^k(1) \end{bmatrix} \\
&= \begin{bmatrix} \gamma^k(1)(\varphi^k(1) + \varphi^k(0)) & \gamma^k(1)(\varphi^k(1) + \varphi^k(0)) \\ \gamma^k(1)(\varphi^k(1) - \varphi^k(0)) & \gamma^k(1)(\varphi^k(0) - \varphi^k(1)) \end{bmatrix} \\
&\quad \times \begin{bmatrix} \gamma^k(1) & \gamma^k(1) \\ \gamma^k(1) & -\gamma^k(1) \end{bmatrix} \\
&= \begin{bmatrix} 2\gamma^k(1)(\varphi^k(1) + \varphi^k(0))\gamma^k(1) & \mathbf{0} \\ \mathbf{0} & 2\gamma^k(1)(\varphi^k(1) - \varphi^k(0))\gamma^k(1) \end{bmatrix} \\
&= \begin{bmatrix} 2^{k+1} \operatorname{diag}(P_k + 2Q_k) & \mathbf{0} \\ \mathbf{0} & 2^{k+1} \operatorname{diag}(P_k - 2Q_k) \end{bmatrix},
\end{aligned}$$

where by $\mathbf{0}$ we mean the appropriately-sized matrix of all 0's.

Now, from (2) and (3) we see that $[P_k + 2Q_k, P_k - 2Q_k] = P_{k+1}$, so the proof of the first claim is complete.

Now let us verify (b):

$$\begin{aligned}\gamma^{k+1}(1)\varphi^{k+1}(0)\gamma^{k+1}(1) &= \begin{bmatrix}\gamma^k(1) & \gamma^k(1)\\ \gamma^k(1) & -\gamma^k(1)\end{bmatrix}\begin{bmatrix}\varphi^k(1) & \varphi^k(1)\\ \varphi^k(1) & \varphi^k(1)\end{bmatrix}\begin{bmatrix}\gamma^k(1) & \gamma^k(1)\\ \gamma^k(1) & -\gamma^k(1)\end{bmatrix}\\ &= \begin{bmatrix}2\gamma^k(1)\varphi^k(1) & 2\gamma^k(1)\varphi^k(1)\\ \mathbf{0} & \mathbf{0}\end{bmatrix}\begin{bmatrix}\gamma^k(1) & \gamma^k(1)\\ \gamma^k(1) & -\gamma^k(1)\end{bmatrix}\\ &= \begin{bmatrix}4\gamma^k(1)\varphi^k(1)\gamma^k(1) & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{bmatrix}\\ &= 4\begin{bmatrix}2^k \operatorname{diag}(P_k) & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{bmatrix}\\ &= 2^{k+2}\operatorname{diag}(Q_{k+1}),\end{aligned}$$

where we have used (5). This completes the proof of (b). □

Corollary 4. *The eigenvalues of $M_{\mathbf{d}}(2^k)$, with their multiplicities, are as follows:*

- J_{k+1} *with multiplicity* 1.
- J_k *with multiplicity* 1.
- J_{k-i} *and* $-J_{k-i}$, *each with multiplicity* 2^{i-1}, *for* $1 \le i \le k-3$.
- 1 *and* -1, *each with multiplicity* $3 \cdot 2^{k-3}$.

Furthermore, the basis for the eigenspace of each eigenvalue can be read off from the respective columns of the Hadamard matrix $H(2^k)$.

Proof. This follows immediately from the fact that

$$M_{\mathbf{d}}(2^{k+1}) = H(2^k)M_{\mathbf{d}}(2^k)H(2^k) = 2^k \operatorname{diag}(P_k),$$

and $H(2^k) = H(2^k)^T$, and $H(2^k)H(2^k)^T = 2^k I$. □

Corollary 5. *The eigenvalues of $M_{\bar{\mathbf{d}}}(2^k)$, with their multiplicities, are almost the same:*

- J_k *with multiplicity* 1.
- $-J_k$ *with multiplicity* 1.
- J_{k-i} *and* $-J_{k-i}$, *each with multiplicity* 2^{i-1}, *for* $1 \le i \le k-3$.
- 1 *and* -1, *each with multiplicity* $3 \cdot 2^{k-3}$.

Again, the basis for the eigenspace of each eigenvalue can be read off from the respective columns of the Hadamard matrix $H(2^k)$.

Proof. This follows immediately from the fact that $M_{\bar{\mathbf{d}}}(2^k) = E_k - M_{\mathbf{d}}(2^k)$, for the matrix E_k that has all entries equal to 1, and the fact that $\gamma^k(1)E_k\gamma^k(1) = \operatorname{diag}(2^{2k}, 0, 0, \dots, 0)$. □

Finally, we get the proof of Theorem 1:

Proof. The product of the eigenvalues of $M_{\mathbf{d}}(2^k)$ is

$$-J_{k+1}\prod_{3\le i\le k} J_i^{2^{k-i}},$$

and the product of the eigenvalues of $M_{\bar{\mathbf{d}}}(2^k)$ is

$$J_k \prod_{3\le i\le k} J_i^{2^{k-i}}. \quad \square$$

4. General orders

The Hankel determinants $\Delta_{\mathbf{d}}(2^k)$ and $\Delta_{\bar{\mathbf{d}}}(2^k)$ are products of Jacobsthal numbers that correspond to eigenvalues of their associated Hankel matrices. For general n, the Hankel determinants $\Delta_{\mathbf{d}}(n)$ are also products of Jacobsthal numbers (as we will prove below), but these numbers no longer correspond to eigenvalues of the Hankel matrix $M_{\mathbf{d}}(n)$. The Hankel determinants $\Delta_{\bar{\mathbf{d}}}(n)$ are equal to zero if n is odd.

4.1. Preliminary observations

An inspection of the Hankel determinants quickly reveals recursive formulas, such as:

Proposition 6. *For $k \ge 1$ we have $\Delta_{\mathbf{d}}(3\cdot 2^k) = \Delta_{\mathbf{d}}(2^k)$ and $\Delta_{\bar{\mathbf{d}}}(3\cdot 2^k) = \Delta_{\bar{\mathbf{d}}}(2^k)$.*

Proof. We consider $\Delta_{\mathbf{d}}(3\cdot 2^k)$ first. The result is easy to check for $k = 1$. For $k \ge 2$, the corresponding Hankel matrix is easily seen to be

$$\begin{bmatrix} P & Q & P \\ Q & P & P \\ P & P & P \end{bmatrix}$$

where $P = \varphi^n(1)$, $Q = \varphi^n(0)$.

Using Gaussian elimination, we can subtract the third row from each of the first two rows, obtaining

$$\begin{bmatrix} 0 & R & 0 \\ R & 0 & 0 \\ P & P & P \end{bmatrix}.$$

Now an easy induction gives that R is an anti-diagonal matrix of all $(-1)^k$'s, so for $k \ge 2$ we have $\det R = 1$. We conclude that the determinant is indeed $\det P$.

If the same computation is carried out for $\Delta_{\bar{\mathbf{d}}}(3\cdot 2^k)$, then we arrive at

$$\begin{bmatrix} 0 & -R & 0 \\ -R & 0 & 0 \\ \overline{P} & \overline{P} & \overline{P} \end{bmatrix}$$

where $\overline{P}$ is a complementary matrix and $-R$ is an anti-diagonal matrix of all $(-1)^{k+1}$'s. We conclude that the determinant is indeed $\det \overline{P}$. $\square$

Similar computations give

Proposition 7.

(a) $\Delta_{\mathbf{d}}(5\cdot 2^k) = \Delta_{\mathbf{d}}(2^k)$ *and* $\Delta_{\bar{\mathbf{d}}}(5\cdot 2^k) = \Delta_{\bar{\mathbf{d}}}(2^k)$ *for* $k \ge 0$.
(b) $\Delta_{\mathbf{d}}(7\cdot 2^k) = \Delta_{\mathbf{d}}(2^k)$ *and* $\Delta_{\bar{\mathbf{d}}}(7\cdot 2^k) = \Delta_{\bar{\mathbf{d}}}(2^k)$ *for* $k \ge 1$.
(c) $\Delta_{\mathbf{d}}(2^k - 1) = -\prod_{3\le i\le k-1} J_i^{2^{k-i}}$ *for* $k \ge 3$.

4.2. Two recursions

We derive two recursions to compute $\Delta_{\mathbf{d}}(n)$ and $\Delta_{\overline{\mathbf{d}}}(n)$. The derivation is the same for both determinants. We restrict our attention to the first determinant, and leave it to the reader to verify the recursion for the second determinant. If the second significant digit of the binary expansion of n is one, then we apply the first recursion. If it is zero, then we apply the second recursion. Each recursion produces a power of a Jacobsthal number and reduces $\Delta_{\mathbf{d}}(n)$ to $\Delta_{\mathbf{d}}(n')$. If n has binary expansion of length k, then the binary expansion of n' is $k-1$. The recursion also produces a power of 2, which may be positive or negative, but since we know by [1] that our Hankel determinants are odd (for $\mathbf{d}$, not for $\overline{\mathbf{d}}$), we can ignore these powers.

4.2.1. Recursion one

The Hankel matrix $M_{\mathbf{d}}(n)$ is an $n \times n$ submatrix in the larger Hankel matrix $M_{\mathbf{d}}(m)$ for any $n \leq m$. We introduce some more notation. We write P_k for $M_{\mathbf{d}}(2^k)$, and Q_k for $P_k - (-1)^k D_k$, where D_k is the $2^k \times 2^k$ anti-diagonal matrix with all ones on the diagonal. Our recursion involves the matrix $M_{i,k}(j)$, which is the $j \times j$ submatrix of $J_i P_k + J_{i-1} Q_k$ consisting of the first j columns and the first j rows, where as before J_i is the ith Jacobsthal number. We denote the determinant of $M_{i,k}(j)$ by $\Delta_{i,k}(j)$. If $i = 1$, then $M_{i,k}(j)$ is equal to the period doubling Hankel matrix $M_{\mathbf{d}}(j)$, and its determinant $\Delta_{i,k}(j)$ is equal to $\Delta_{\mathbf{d}}(j)$. If $j \leq 2^{k-1}$, then the $j \times j$ blocks in P_k and Q_k coincide, $M_{i,k}(j)$ is equal to $2^{i-1} M_{\mathbf{d}}(j)$, and its determinant is equal to $2^{(i-1)j} \Delta_{\mathbf{d}}(j)$. So the only interesting values are $2^{k-1} < j \leq 2^k$, and we will only consider such j.

Lemma 8 (*Recursion One*). *If* $2^k + 2^{k-1} < j \leq 2^{k+1}$ *then*

$$\Delta_{i,k+1}(j) = \epsilon_k \cdot \frac{J_i^{2^k}}{2^{2^{k+1}-j}} \cdot \Delta_{i+1,k}(j - 2^k)$$

where

$$\epsilon_k = \begin{cases} 1, & \text{if } k > 1; \\ -1, & \text{if } k = 1. \end{cases}$$

Observe that the recursion reduces j by 2^k which is equal to the power of the Jacobsthal number that is produced by the recursion.

Proof. By definition $H_{i,k+1}(j)$ is the $j \times j$ block in the matrix $J_i P_{k+1} + J_{i-1} Q_{k+1}$, which is equal to

$$\begin{bmatrix} J_i P_k + J_{i-1} P_k & J_i Q_k + J_{i-1} P_k \\ J_i Q_k + J_{i-1} P_k & J_i P_k + J_{i-1} P_k \end{bmatrix} = \begin{bmatrix} 2^{i-1} P_k & J_i Q_k + J_{i-1} P_k \\ J_i Q_k + J_{i-1} P_k & 2^{i-1} P_k \end{bmatrix}.$$

Abbreviating this expression, we write this matrix as

$$\begin{bmatrix} A & B \\ B & A \end{bmatrix}.$$

Perform Gaussian elimination by subtracting "row" 1 of this 2×2 block matrix from row 2, and then subtract "column" 1 from column 2 to get

$$\begin{bmatrix} A & A + B \\ B - A & 0 \end{bmatrix}.$$

The lower left block $B - A$ is an anti-diagonal matrix $(-1)^{k+1} J_i D$. The upper right block $A + B$ is equal to $(J_i + 2J_{i-1})P_k + J_i Q_k = J_{i+1} P_k + J_i Q_k$, i.e., it is equal to $M_{i+1,k}(2^{k-1})$. We started out with the $j \times j$ submatrix in the entire matrix. The recursion essentially reduces it to the $(j-2^k) \times (j-2^k)$ submatrix in the upper right block $A+B$ by getting rid of the first column $\begin{bmatrix} A \\ B-A \end{bmatrix}$, as follows:

The $j \times j$ submatrix extends over the $j-2^k$ top rows of the lower block $B-A$. The first $2^{k+1}-j$ columns of this $(j-2^k)\times 2k$ submatrix are zero and the last $j-2^k$ columns form the anti-diagonal $(-1)^{k+1} J_i D_{j-2^k}$. Ignoring the sign of the determinant for the moment, the submatrix contributes a factor $J_i^{j-2^k}$ to the determinant. We can remove the final $j-2^k$ rows and the columns of the anti-diagonal matrix, after which we are left with a $2^k \times 2^k$ matrix. Let us denote it by R. It consists of the first $2^{k+1}-j$ columns of A and the first $j-2^k$ columns of $A+B$, which, as we noted above, is equal to $M_{i+1,k}(2^k)$. Another equality is $A+B = 2A + (-1)^{k+1} J_i D_k$. So our $2^k \times 2^k$ matrix R consists of a block from A and a block from $2A + (-1)^{k+1} J_i D_k$. By our conditions on j (and this is the first place in the proof where we use this), the second block has as least as many columns as the first. Perform a Gaussian elimination in which every column in the second block is divided by two and subtracted from the corresponding column in the first block. This reduces R to a matrix

$$\begin{bmatrix} 0 & N_1 \\ (-1)^k \dfrac{J_i}{2} D_{2^{k+1}-j} & N_2 \end{bmatrix}.$$

The upper right block N_1 corresponds to the $(j-2^k) \times (j-2^k)$ proper submatrix of $A+B$, which as we have seen above, is equal to $M_{i+1,k}(j-2^k)$. Here we need that $j > 2^k + 2^{k+1}$. The lower left block contributes a factor $\left(\frac{J_i}{2}\right)^{2^{k+1}-j}$. Ignoring the signs for the moment, we have reduced the matrix to $M_{i+1,k}(j-2^k)$ and have obtained a factor $\frac{J_i^{2^k}}{2^{2^{k+1}-j}}$, as required.

Now we still need to consider the sign. We found 2^k factors in total, the first $2^{k+1}-j$ were J_i and the remaining $j-2^k$ were $J_i/2$. The first came with a sign $(-1)^{k+1}$ and the latter with a sign $(-1)^k$, which together produce the sign $(-1)^j$. The determinant of an anti-diagonal D_m is equal to 1 if $m = 0$ or 1 mod 4 and -1 otherwise. We encountered both D_{j-2^k} and $D_{2^{k+1}-j}$. If $k > 1$, then this produces the sign $(-1)^j$, but if $k = 1$, it produces $(-1)^{j-1}$. Finally, we need to observe the position of these two anti-diagonal matrices as blocks in a matrix. Using that a matrix $\begin{bmatrix} 0 & S \\ T & U \end{bmatrix}$ with $s \times s$ block S and $t \times t$ block T has determinant $(-1)^{st} \det(S)\det(T)$, this produces a factor $(-1)^{(j-2^k)^2}$ for the first anti-diagonal matrix and a factor $(-1)^{(2^{k+1}-j)(j-2^k)}$ for the second. Together this produces the sign $+1$. If we consider all three factors that we found, then we see that they produce $+1$ if $k > 1$ but -1 if $k = 1$, which is ϵ_k. $\square$

For $\Delta_{\overline{\mathbf{d}}}(j)$ the computations are the same, but we need to change some signs. Whenever there is a $(-1)^{k+1}$ in the computation above, it now becomes a $(-1)^k$, and vice versa. The net result is that recursion one applies to $\overline{\mathbf{d}}$ as well.

4.2.2. Recursion two

Our second recursion deals with j that have second significant digit zero in their binary expansion.

Lemma 9 (*Recursion Two*). *If* $2^k < j \le 2^k + 2^{k-1}$ *then*

$$\Delta_{i,k+1}(j) = (-1)^j 2^{(i-1)(2^{k+1}-j)} \cdot J_i^{2j-2^{k+1}} \cdot \Delta_{1,k}(2^{k+1}-j).$$

Observe that the recursion reduces j by $2j - 2^{k+1}$, which is equal to the power of the Jacobsthal number. Also observe that recursions one and two both apply to $j = 2^k + 2^{k-1}$, and we obtain the equality

$$J_i^{2^k} \Delta_{i,k-1}(2^{k-1}) = \frac{J_i^{2^k}}{2^{k-1}} \Delta_{i+1,k-1}(2^{k-1}).$$

Proof. By the same argument as in the proof of the first recursion, we end up with the $2^k \times 2^k$ matrix R, only now the A block is at least as large as the $2A + (-1)^{k+1} J_i D_{2^k}$ block. This time, we can use the first block to reduce the second. Subtract every column of A twice from the corresponding column in the second block. We end up with the matrix

$$\begin{bmatrix} N_1 & 0 \\ N_2 & (-1)^{k+1} J_i D_{j-2^k} \end{bmatrix}.$$

The upper left block N_1 is a $(2^{k+1}-j)\times(2^{k+1}-j)$ submatrix of A, which is shorthand notation for $2^{i-1} P_k$, so N_1 is in fact equal to $2^{i-1} M_{1,k}(2^{k+1} - j)$. Remembering that we already encountered a determinant of $(-1)^{k+1} J_i D_{j-2^k}$ in the reduction, with an extra sign $(-1)^{(j-2^k)^2} = (-1)^j$, it follows that

$$\Delta_{i,k+1}(j) = (-1)^j 2^{(i-1)(2^{k+1}-j)} \Delta_{1,k}(2^{k+1} - j) \cdot \det\left((-1)^{k+1} J_i D_{j-2^k}\right)^2,$$

which reduces to the required recursion. □

Again, the computations are the same for $\overline{\mathbf{d}}$, except for the final equation. There the sign $(-1)^{k+1}$ changes to $(-1)^k$, which does not affect the outcome.

4.2.3. Applying the two recursions

The two recursions combine to reduce any $2^k < j \le 2^{k+1}$ to a $2^{k-1} \le j' \le 2^k$. Each recursion decreases the index k in $\Delta_{i,k}(j)$ by one. If we start with an odd j that has a binary expansion of length k, then after $k-2$ applications of the recursions, we end up at $\Delta_{i,2}(3)$ for some i (ignoring the additional factors that we picked up during the recursion). Then we need to apply recursion two and end at $\Delta_{1,1}(1)$, ignoring the power of two. For $\mathbf{d}$ this is equal to 1, or J_1, and for $\overline{\mathbf{d}}$ this is equal to zero, or J_0, which explains why $\Delta_{\overline{\mathbf{d}}}(j) = 0$ for odd j. It follows that if j is odd, then $\Delta_{i,k}(j)$ is a product of 2's and Jacobsthal numbers. If we start with even j then we end at $\Delta_{i,1}(2)$ after $k - 1$ applications of the recursions. Now for $\mathbf{d}$ we have that $\Delta_{i,1}(2)$ is equal to

$$\begin{vmatrix} J_i + J_{i-1} & J_{i-1} \\ J_{i-1} & J_i + J_{i-1} \end{vmatrix} = J_i(J_i + 2J_{i-1}) = J_i J_{i+1},$$

while for $\overline{\mathbf{d}}$ it is equal to

$$\begin{vmatrix} 0 & J_i \\ J_i & 0 \end{vmatrix} = -J_i^2,$$

and so the quotient of $\Delta_{\mathbf{d}}(j)$ and $\Delta_{\overline{\mathbf{d}}}(j)$ is $-J_{i+1}/J_i$ for even j. Therefore, we restrict our attention to $\mathbf{d}$, because the corresponding result for $\overline{\mathbf{d}}$ is straightforward.

If the recursion ends at $\Delta_{i,1}(2)$, then it produces two more Jacobsthal numbers. If it ends at $\Delta_{1,1}(1)$ it produces 1, or J_1. It follows that the powers of the Jacobsthal numbers in $\Delta_{\mathbf{d}}(j)$ add up to j. Of course, some powers may be trivial since $J_0 = 0$ and $J_1 = J_2 = 1$. We are now ready to prove Theorem 2, which we restate as follows.

Theorem 10. *$\Delta_{\mathbf{d}}(j)$ is a product of powers of Jacobsthal numbers $J_i^{n_i}$. The exponent n_i decreases as the index i increases, with the exception of the largest non-trivial power $J_{i+1}^{n_{i+1}}$ for which it may be true that $n_{i+1} = n_i = 1$. The sign of $\Delta_{\mathbf{d}}(j)$ depends on j* mod 4. *It is negative if and only if $j = 2$* mod 4 *or* 3 mod 4.

Proof. We start with the sign first. It is true for $j \le 4$ by direct inspection. So we may assume that $k > 1$ in the recursion and argue by induction. Recursion two reduces j to $j - 2^k$, which is equal modulo 4, without changing the sign. Recursion reduces j to $2^{k+1} - j$, so modulo 4 it interchanges 1 and 3. It also changes the sign in this case, as it should, which finishes the induction.

The recursion produces powers of Jacobsthal numbers and perhaps powers of two. But we need not compute the exponent of 2 in $\Delta_{\mathbf{d}}(j)$, since there are none [1]. Recursion one produces $J_i^{2^k}$ and increases the index i by 1. Recursion two produces $J_i^{2j-2^{k+1}}$ and resets the index i to 1. The exponent $2j - 2^{k+1}$ is at most equal to 2^k, so recursion one produces the highest power of the two. This exponent decreases (strictly) with k and it immediately follows that the n_i decrease with i. The only exception is that in the final step of the iteration, when we end with $\Delta_{i,1}(2) = J_i J_{i+1}$, we obtain two additional Jacobsthal factors. □

The following recursive formula was conjectured by Jason Bell and Kevin Hare on November 26 2015, and independently by Tewodros Amdeberhan and Victor Moll on December 6 2015:

Theorem 11. *For odd j we have*

$$\Delta_{\mathbf{d}}(2^m \cdot j) = \Delta_{\mathbf{d}}(j)^{2^m} \cdot \Delta_{\mathbf{d}}(2^m).$$

Proof. By induction. If recursion one applies to $2^m \cdot j$, then it gives

$$\Delta_{i,k+m+1}(2^m \cdot j) = \frac{J_i^{2^{k+m}}}{2^{2^{k+m+1}-j}} \cdot \Delta_{i+1,k+m}(2^m(j - 2^k)),$$

which in particular produces a Jacobsthal power $J_i^{2^{k+m}}$ and reduces $2^m \cdot j$ to $2^m(j - 2^k)$. If recursion one applies to $2^m \cdot j$, then it also does to j, and it produces the Jacobsthal power $J_i^{2^k+m}$ and reduces j to $(j - 2^k)$. Similarly, if recursion two applies, then it produces a Jacobsthal power $J_i^{2^m(2j-2^{k+1})}$ and reduces $2^m \cdot j$ to $2^m(2^{k+1} - j)$, while it produces a Jacobsthal power $J_i^{(2j-2^{k+1})}$ for j. We can ignore the powers of two, as before, and it is not hard to check that the signs are equal on both sides of the equation, so we may ignore that as well. The recursion for odd j ends at $\Delta_{i,1}(1)$, while for $2^m \cdot j$ it reaches $\Delta_{1,m}(2^m)$ at that point, and we conclude that the recursive formula holds. □

5. Hankel determinants of the shifted sequence

The Hankel matrix of the shifted sequence $s_q s_{q+1} s_{q+2} \cdots$ is given by

$$M_{\mathbf{s},q}(j) = \begin{bmatrix} s_q & s_{q+1} & \cdots & s_{q+j-1} \\ s_{q+1} & s_{q+2} & \cdots & s_{q+j} \\ \vdots & \vdots & \ddots & \vdots \\ s_{q+j-1} & s_{q+j} & \cdots & s_{q+2j-2} \end{bmatrix} \tag{6}$$

and the corresponding Hankel determinant is $\Delta_{\mathbf{s},q}(j)$. Observe that $M_{\mathbf{s},q}(j)$ occurs as a $j \times j$ submatrix in $M_{\mathbf{s}}(q + j)$. For $q > 0$ the Hankel determinants of $\mathbf{d}$ are no longer products of

Jacobsthal numbers, but the first few terms indicate some interesting patterns:

q	j															
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
0	1	1	−1	−3	1	1	−1	−15	1	1	−1	−3	1	1	−9	−495
1	0	−1	1	2	1	1	−4	11	3	−2	3	−3	−2	−7	141	354
2	1	0	−1	−1	0	1	−1	−8	1	3	0	−3	1	40	−9	−253
3	1	−1	−1	0	0	1	−3	5	−3	0	0	−3	−11	−7	−17	180
4	1	1	0	0	0	1	−1	−3	0	0	0	−3	1	1	−4	−128
5	0	−1	0	0	1	−2	3	0	0	0	0	−3	−2	−5	52	76
6	1	1	0	−1	1	3	0	0	0	0	0	−3	1	21	−4	−45
7	0	−1	−1	2	−3	0	0	0	0	0	0	−3	−8	−5	−7	26
8	1	1	−1	−3	0	0	0	0	0	0	0	−3	1	1	−1	−15
9	0	−1	1	3	0	0	0	0	0	0	3	5	3	4	−15	0
10	1	0	−1	−3	0	0	0	0	0	3	−1	−8	1	15	0	0
11	1	−1	−2	3	0	0	0	0	−3	−2	−3	11	15	0	0	0
12	1	1	−1	−3	0	0	0	−3	1	1	−1	−15	0	0	0	0
13	0	−1	1	3	0	0	3	5	3	4	−15	0	0	0	0	0
14	1	0	−1	−3	0	3	−1	−8	1	15	0	0	0	0	0	0
15	1	−1	−2	3	−3	−2	−3	11	15	0	0	0	0	0	0	0
16	1	1	−1	−3	1	1	−1	−15	0	0	0	0	0	0	0	0

One pattern that emerges from this table is that the $q = 2^k$th row starts with the first 2^{k-1} numbers of the first row, followed by 2^{k-1} zeros. This follows directly from our results.

Proposition 12. *If $j \le 2^{k-1}$ then $\Delta_{\mathbf{d},2^k}(j) = \Delta_{\mathbf{d}}(j)$, and if $2^{k-1} < j \le 2^k$ then $\Delta_{\mathbf{d},2^k}(j) = 0$.*

Proof. The Hankel matrix $M_{\mathbf{d},2^k}(2^k)$ is the lower left Q_k block of $M_{\mathbf{d}}(2^{k+1}) = \begin{bmatrix} P_k & Q_k \\ Q_k & P_k \end{bmatrix}$, which, in turn, is equal to $Q_k = \begin{bmatrix} P_{k-1} & P_{k-1} \\ P_{k-1} & P_{k-1} \end{bmatrix}$. It immediately follows that $\Delta_{\mathbf{d},2^k}(j) = \Delta_{\mathbf{d}}(j)$ if $j \le 2^{k-1}$, since this is the determinant of the $j \times j$ block in P_{k-1}, and that $\Delta_{\mathbf{d},2^k}(j) = 0$ for $2^{k-1} < j < 2^k$ since row one of the matrix is repeated in row $2^{k-1} + 1$. □

The table again indicates that something interesting is going on when j is a power of 2. A full analysis is probably not that easy. Allouche et al. [1] needed 16 recursions to resolve the Hankel table of the Thue–Morse sequence modulo 2.

6. Conclusion

We set out to study the values of Hankel determinants of the Thue–Morse sequence at powers of 2, and we ended up studying Hankel determinants of the period-doubling sequence. The values of the Hankel determinants $\Delta_{\mathbf{t}}(n)$ for the Thue–Morse sequence continue to be mysterious.

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