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Non-tangential control of the solutions to deterministic and stochastic parabolic Cauchy problems

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"Non-tangential control of the solutions to deterministic and stochastic parabolic Cauchy problems"

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Summary

The main focus of this thesis is a stochastic parabolic Cauchy problem of the following form.

$$\begin{cases} \partial_t u(t,x) = \mathrm{div}(A(x)\nabla u(t,x)) + g(t,x)\mathrm{d}W(t), & t>0, x\in\mathbb{R}^n, \\ u(0,x) = u_0(x), & x\in\mathbb{R}^n, \end{cases} \tag{0.1}$$

with $A \in L^\infty(\mathbb{R}^n, \mathscr{M}_n(\mathbb{C}))$ satisfying uniform ellipticity estimates. In [8], a conical stochastic maximal L^p-regularity result was proved for the above problem and in [6] it was shown that for the deterministic version of the above problem ($g = 0$) we have a control of a non-tangential maximal function by the gradient of the solution. Our goal in this thesis is to develop a stochastic analogue of the determinstic result for problem (0.1).
To this end, we start by introducing tent spaces, denoted by $T^{p,2}_\beta$, and study their properties. These properties include tent spaces being Banach spaces, norm equivalences and change of aperture results. For most of these we also provide proofs.
Afterwards, we define what we mean by an elliptic operator in divergence form and also introduce the notions needed to do so. We mentioned that if $L = -\mathrm{div}(A\nabla)$ is an elliptic operator in divergence form then L has a holomorphic functional calculus in L^2. Furthermore, we show that the families $(\mathrm{e}^{-tL})_{t>0}$, $(tL\mathrm{e}^{-tL})_{t>0}$ and $(\sqrt{t}\nabla\mathrm{e}^{-tL})_{t>0}$ satisfy L^2 off-diagonal bounds.
Having given these preliminaries, we start anaylsing the relevant parts of [6]. First, we introduce the notion of maximal regularity and explain its importance for the analysis of our PDE. We define the operators $\mathcal{M}_L$, $\tilde{\mathcal{M}}_L$ and $\mathcal{R}_L$ and prove boundedness results for them.
Thereafter, we develop energy solution for (0.1) where $g = 0$ with initial data in L^2. We show that the solution is given by $\Gamma(t,0)u_0$, where Γ is a propagator. Furthermore, we show that we have the following equality

$$\Gamma(t,0)u_0 = \mathrm{e}^{-tL}u_0 + \int_0^t \mathrm{e}^{-(t-s)L}\mathrm{div}\,(A(s,\cdot) - \underline{A})\nabla\Gamma(s,\cdot)u_0\mathrm{d}s.$$

By taking $L = -\Delta$ in the above we have $u = \mathrm{e}^{-tL}u_0 + \mathcal{R}_{-\Delta}(A - I)\nabla u$. Using this with the previously mentioned boundedness results we are able to prove the non-tangential control of the solution by its gradient.
Next, we give a brief introduction to stochastic integration with respect to Brownian motion. We introduce the necessary notions from probability theory and explain how the stochastic integral is defined through the extension of an L^2 isometry. We also mention the well-known Itô isometry and Itô's formula.
Equipped with a basic understanding of stochastic integration, we analyse [8]. We give the outline of the proof of its main result, which is based on a $T^{2,2}_\beta$ estimate combined with an extrapolation result based on off-diagonal bounds. Hereafter, we explain how an application of this main result is used to get a conical stochastic maximal L^p regularity for (0.1).
Having analyzed both [8] and [6] we switch our attention to an analogue of a non-tangential maximal function estimate for (0.1). To do so, we first provide a necessary condition for the stochastic analogue of the boundedness of $\mathcal{R}_L$ for $p \geq 2$, of which the proof, again, relies on off-diagonal estimates. Afterwards, we define a Banach space $\tilde{X}^p_\beta$ through a Rademacher maximal function and prove the following result

Proposition 0.1. *Let $p \geq 2$, $\beta \geq 0$ and let $u : \mathbb{R}^{n+1}_+ \times \Omega \to \mathbb{R}^n$ be an adapted simple process. The operator $\mathcal{T}_L$ extends to a bounded operator from $L^p(\Omega; T^{p,2}_\beta)$ to $\tilde{X}^p_\beta$, where*

$$\mathcal{T}_L u(t,x,\omega) = \int_0^t \mathrm{e}^{-(t-s)L} b(u(s,\cdot,\cdot))(x,\omega)dW(s).$$

This provides a sufficient condition for our aim, if we can develop, rigorously, a solution of (0.1).
In the final chapter, we will discuss our results and the above remark regarding the sufficient condition for a stochastic analogue of a non-tangential maximal fucntion estimate with respect to the gradient of the solution.

Preface

This thesis is the result of my work for my graduation for the Master Applied Mathematics at TU Delft.

The research area of my thesis is Harmonic analysis. I got interested in analysis during the bachelor, but it was during the master level courses Functional analysis and Measure and Integration theory that I decided to specialize in analysis. After following several analysis courses, I choose Harmonic analysis to be the research area of my project, which I enjoyed a lot.

I would like to thank my supervisor Dr. D. Frey for her guidance throughout this project and for teaching me about the plethora of topics needed to finish this thesis. I, furthermore, would like to thank Prof. Dr. J.M.A.M. van Neerven for his useful comments and advice on several occasions. Last, but not least, I would like to thank my family and friends for their support, allowing me to finish this thesis.
I hope you enjoy reading this thesis.

Kind regards,

Hassan al Mahmoedi

Contents

Chapter 1

Introduction

In this thesis, we focus on a stochastic parabolic Cauchy problem of the following form.

$$\begin{cases} \partial_t u(t,x) = \operatorname{div}(A(x)\nabla u(t,x)) + g(t,x)\mathrm{d}W(t), & t>0, x\in\mathbb{R}^n,\\ u(0,x)=u_0(x), & x\in\mathbb{R}^n, \end{cases} \tag{1.1}$$

with $A \in L^\infty(\mathbb{R}^n, \mathscr{M}_n(\mathbb{C}))$ satisfying uniform ellipticity estimates

$$\begin{aligned} &\exists\Lambda>0: \quad \forall \xi,\eta\in\mathbb{C}^n, |\langle A(x)\xi,\eta\rangle| \le \Lambda|\xi||\eta| \text{ for a.e. } x\in\mathbb{R}^n,\\ &\exists\lambda>0: \quad \forall \xi\in\mathbb{C}^n, \Re e(\langle A(x)\xi,\xi\rangle) \ge \lambda|\xi|^2 \text{ for a.e. } x\in\mathbb{R}^n, \end{aligned} \tag{1.2}$$

where $g : \mathbb{R}_+ \times \mathbb{R}^n \times \Omega \to \mathbb{R}^n$ is a process with suitable measurability and integrability assumptionsa and where W is a (cylindrical) Brownian motion representing white noise.

Problem (1.1) is a stochastic partial differential equation (SPDE). To study SPDEs one utilizes stochastic calculus, which was developed in the 1950s by Itô. Via an L^2-isometry and stopping time techniques he constructed the Itô stochastic integral with respect to Brownian motion. This construction was generalized to stochastic integrals of progressively measurable processes with values in a Hilbert space H, see e.g. [10], and to H'-valued stochastic integrals with respect to an H-cylindrical Brownian motion defined by operator-valued integrands with values in the space of Hilbert-Schmidt opeartors $\mathscr{L}_2(H,H')$, see e.g Da Prato and Zabczyk [14]. Futher generalizations include stochastic integration on UMD Banach spaces, with recent works by van Neerven, Veraar and Weis [31, 29, 32].

Recently, it was shown in [8], that, under certain assumptions, the above problem satisfies a conical stochastic maximal L^p regularity with weight β. Stated somewhat informally, this means the following.

Theorem 1.1. *Let $L = -divA\nabla$ be an elliptic operator in divergence form on $\mathbb{R}^n$ with bounded measurable real-valued coefficients. Then for all $1 \le p < \infty$ and $\beta > 0$ the stochastic convolution process*

$$u(t) = \int_0^t \mathrm{e}^{-(t-s)L} g(s)\,dW(s), \quad t\ge 0,$$

satisfies the conical stochastic maximal L^p-regularity estimate

$$\mathbb{E}\|\nabla u\|^p_{T^{p,2}(\mathbb{R}_+\times\mathbb{R}^n, t^{-\beta}dt\times dx;\mathbb{R}^n)} \le C^p_{p,\beta}\mathbb{E}\|g\|^p_{T^{p,2}(\mathbb{R}_+\times\mathbb{R}^n, t^{-\beta}dt\times dx;\mathbb{R}^n)},$$

where $g \in L^p(\Omega; T^{p,2}(\mathbb{R}_+ \times \mathbb{R}^n, t^{-\beta}dt \times dx; \mathbb{R}^n))$ and where $T^{p,2}(\mathbb{R}_+ \times \mathbb{R}^n, t^{-\beta}dt \times dx; \mathbb{R}^n)$ is a weighted parabolic tent space of $\mathbb{R}^n$-valued functions on $\mathbb{R}_+ \times \mathbb{R}^n$.

The notion of (stochastic) maximal regularity playes a key role in the theory of parabolic partial differential equations. This is due to the fact that it enables one to study certain classes of 'complicated' non-linear PDEs through a fixed point problem. In the deterministic case, $L^2(\mathbb{R}^{n+1}_+)$-maximal regularity was proved by de Simon in [38]. In [39] this was extended to L^p with $p \neq 2$ for operators L generating an R-analytic semigroup. In [4] a weighted (in time) version for the L^2 boundedness was established and in [7] the corresponding result for $p \neq 2$ was proved.
For stochastic maximal regularity a classis result is due to Da Prato [13]. He proved that for $\Delta = \operatorname{div}(I\nabla)$ in (1.1), u has stochastic maximal L^2-regularity in the sense that

$$\mathbb{E}\|\nabla u\|^2_{L^2(\mathbb{R}_+;L^2(\mathbb{R}^n;\mathbb{R}^n))} \leq C^2\mathbb{E}\|g\|^2_{L^2(\mathbb{R}_+;L^2(\mathbb{R}^n;H))}.$$

This result was extended by Krylov [24, 26] to $L^p(\mathbb{R}_+;L^q(\mathbb{R}^n;H)) \to L^p(\mathbb{R}_+;L^q(\mathbb{R}^n;\mathbb{R}^n))$ for $p \geq 2$ and $2 \leq q \leq p$. Krylov also showed that under mild regularity assumption on the coefficients Δ may even be replaced by any second-order uniformly elliptic operator. Further extensions were made in [33] where, for $p > 2$, the restriction $q \leq p$ was removed and where an even greater class of operators were allowed. In all of the above results the condition $p \geq 2$ is necessary since for $1 \leq p < 2$ the corresponding result is false [25]. In [8], it is shown that u has 'conical' stochastic maximal L^p-regularity for the full range $p \in [1,\infty)$, by taking $g \in L^p(\Omega; T^{p,2}(\mathbb{R}_+ \times \mathbb{R}^n, t^{-\beta}\mathrm{d}t \times \mathrm{d}x;\mathbb{R}^n))$.

So by working on tent spaces, Theorem 1.1 shows that we can control the gradient of the solution of problem (1.1) for any $p \in [1,\infty)$. On the other hand, in [6], it was shown that for the following problem

$$\begin{cases} \partial_t u(t,x) = \operatorname{div}(A(x)\nabla u(t,x)), & t > 0, x \in \mathbb{R}^n, \\ u(0,x) = u_0(x), & x \in \mathbb{R}^n, \end{cases} \tag{1.3}$$

we have a non-tangential control of the solution by the gradient of the solution. The result reads as follows:

Proposition 1.2. *Let $1 < p < \infty, u_0 \in L^2(\mathbb{R}^n)$, and $u(t,\cdot) = \Gamma(t,0)u_0$ for all $t > 0$, where Γ is a propagator. If $\nabla u \in T^{p,2}$, then $u \in X^p$, and*

$$\|u\|_{X^p} \lesssim \|\nabla u\|_{T^{p,2}},$$

where the implicit constant is independent of u and where X^p is the subspace of functions $F \in L^2_{loc}(\mathbb{R}^{n+1}_+)$, such that

$$\|F\|_{X^p} := \left\| x \mapsto \sup_{\delta>0} \left(\fint_{\frac{\delta}{2}}^{\delta} \fint_{B(x,\sqrt{\delta})} |F(t,y)|^2 \,dy\,dt \right)^{\frac{1}{2}} \right\|_p < \infty.$$

This leads to the question wether we can establish an analogue result for problem (1.1). Investigating this question is the goal of this thesis. To do so, we analyse the papers in which the above results appear with a main focus on [6]. Thereafter, we apply the techniques from these papers to, first, establish a necessary condition for the desired result. Afterwards, we define a vector-valued version of X^p, namely $\tilde{X}^p_\beta$, through a Rademacher maximal function and prove the following result.

Proposition 1.3. *Let $p \geq 2$, $\beta \geq 0$ and let $u : \mathbb{R}^{n+1}_+ \times \Omega \to \mathbb{R}^n$ be an adapted simple process. The operator $\mathcal{T}_L$ extends to a bounded operator from $L^p(\Omega; T^{p,2}_\beta)$ to $\tilde{X}^p_\beta$, where*

$$\mathcal{T}_L u(t,x,\omega) = \int_0^t \mathrm{e}^{-(t-s)L} b(u(s,\cdot,\cdot))(x,\omega)\,dW(s).$$

This leads to the following general outline of the thesis.

First, we introduce tent spaces and elliptic operators in divergence form. We give the definitions of tent spaces and its weighted version and show that they are Banach spaces for $p \in (1, \infty)$. After that, we show how the elliptic operator in divergence form, $Lf = -\mathrm{div}(A\nabla f)$, is defined through a form method and that this operator is a maximal accretive operator. As a consequence, it generates an analytic semigroup, $(\mathrm{e}^{-tL})_{t>0}$. We also present the proof of L^2-off diagonal estimates for this semigroup and several other families of operators involving L and its semigroup. These L^2 off-diagonal estimates serve as a replacement of pointwise kernel estimates and play an important role in the proofs of the above mentioned results and in the proof of the necessary conditions that we have developed.

After these preliminaries, we give a detailed analysis of the relevant parts of [6]. We introduce several maximal regularity operators and prove boundedness results for these, wherein the L^2 off-diagonal bounds are used. Then, we focus on the L^2 setting of problem (1.3) and introduce a variation of Lions spaces. Together with a priori estimates, we then show that this Lions space is a solution space of energy solutions. In other words, we show that for $u_0 \in L^2(\mathbb{R}^n)$, the problem

$$\partial_t u = \mathrm{div}(A\nabla u), \quad u \in \dot{W}(0, \infty), \quad Tr(u) = u_0,$$

where $\dot{W}(0, \infty)$ is the aforementioned Lions space, has a global weak solution. The proof of this result also shows that the solution is obtained through propagators and provides us with some useful estimates with respect to ∇u and u_0. After proving some properties involving these propagators, we present the proof of Proposition 1.2, which also relies on Littlewood-Paley estimates.

Having given a detailed analysis of [6], we shift our focus to [8]. We first give an introduction to stochastic integration with respect to Brownian motion. We introduce the required notions to build the stochastic integral and also mention well-known results such as the the Itô isometry and Itô's formula. Afterwards, we provide a brief explanation on how Theorem 1.1 was obtained and elaborate on some of the techniques used.

Combining techniques from both [6] and [8], we then establish two new results, from which the second one provides us with a necessary condition for a stochastic analogue of Proposition 1.2. It reads as follows.

Proposition 1.4. *Let $p \geq 2$, $\beta \geq 0$ and let $u : \mathbb{R}^{n+1}_+ \times \Omega \to \mathbb{R}^n$ be an adapted simple process. Then we have*

$$\sup_{\delta>0} \mathbb{E} \left\| \left(\fint_{\frac{\delta}{2}}^{\delta} \fint_{B(\cdot,\sqrt{\delta})} |\mathcal{T}_L u|^2 \, dy \frac{dt}{t^\beta} \right)^{\frac{1}{2}} \right\|_{L^p}^p \leq C_{p,\beta,n,b} \mathbb{E} \|u\|_{T^{p,2}_\beta}^p,$$

with $C_{p,\beta,n.b}$ independent of u.

After establishing this necessary condition, we define the space $\tilde{X}^p_\beta$ through a Rademacher maximal function and show that it is a Banach space. As a last result we prove Proposition 1.3.

Lastly, we discuss the work done in this thesis and give several conluding remarks.

Chapter 2

Notation

A Banach space X of function from D_1 to D_2 will be denoted by $X(D_1; D_2)$. In most cases D_1 is either $\mathbb{R}^n$ or $\mathbb{R}^{n+1}_+$ and D_2 is $\mathbb{R}^n$ or $\mathbb{C}^n$. If no confusion can arise we will usually just write X. If the Banach space acts on a different set, then this will be added to the notation.

For the Bochner space of L^p functions from D_1 to a Banach space X, we write $L^p(D_1; X)$ or just $L^p(X)$ if no confusion can arise. Similarly, an X-valued Banach space Y will be denoted by $Y(X)$.

By $\mathscr{D}(D_1)$ or $\mathscr{D}$, we denote $\mathscr{C}_c^\infty(D_1)$, the space of compactly supported, infinitely differentiable functions on D_1, also known as the test functions. Its dual, the space of distributions on D_1, is denoted by $\mathscr{D}'(D_1)$ or $\mathscr{D}'$.
Similarly by $\mathscr{S}$, we denote the space of Schwartz function i.e. the space of functions whose whose derivatives are rapidly decreasing. Its dual, the space of tempered distributions is denoted by $\mathscr{S}'$.
We denote by $\mathscr{C}_0(L^p)$ the space of $L^p(\mathbb{R}^n)$-valued continuous functions on $[0, \infty)$ that go to 0 at infinity.

For the average value of a function f over a set B, we use the notation

$$\fint_B f(x)\mathrm{d}x := \frac{1}{|B|}\int_B f(x)\mathrm{d}x,$$

where $|B|$ is the volume of B.
By χ_B, we denote the indicator function of a set B.

Chapter 3

Tent spaces and X^p

Tent spaces were first introduced by Coifman, Meyer and Stein in [11]. Since their introduction, they have been studied by many authors and now play an important role in harmonic analysis. In this chapter we give the definition of tent spaces and study several of their properties.

For a measurable function u on $\mathbb{R}^{n+1}_+$, we define the conical square function $\mathcal{A}(u)$ by

$$\mathcal{A}(u)(x) = \left(\int_0^\infty \fint_{B(x,\sqrt{t})} |u(t,y)|^2 \mathrm{d}y \mathrm{d}t \right)^{\frac{1}{2}}, \quad x \in \mathbb{R}^n.$$

Let $p \in [1,\infty)$. The set of measurable functions u on $\mathbb{R}^{n+1}_+$ such that

$$\mathcal{A}(u) \in L^p(\mathbb{R}^n)$$

is called the (parabolic) tent space $T^{p,2}(\mathbb{R}^{n+1}_+)$. By definition we have $T^{p,2}(\mathbb{R}^{n+1}_+) \subseteq L^2_{\mathrm{loc}}(\mathbb{R}^{n+1}_+)$. The tent space norm of $u \in T^{p,2}$ is defined by

$$\|u\|_{T^{p,2}} := \|\mathcal{A}(u)\|_{L^p}.$$

Taking equivalence classes of functions that are the same almost everywhere, this indeed defines a norm: by the properties of absolute value and integral we easily get $\|cu\|_{T^{p,2}} = |c|\|u\|_{T^{p,2}}$, for some constant c, and also $\|u\|_{T^{p,2}} = 0$ implies $u = 0$ almost everywhere. For the triangle inequality, note that we can write

$$\|u\|_{T^{p,2}} = \|\mathcal{A}(u)\|_{L^p} = \left\| x \mapsto \left\| \frac{1}{|B(x,\sqrt{\cdot})|} \|u\|_{L^2(B(x,\sqrt{\cdot}))} \right\|_{L^2(\mathbb{R}_+)} \right\|_{L^p}.$$

Using three instances of Minkowski's inequality then provides us with the triangle inequality for the tent space norm. So the tent spaces are normed vector spaces. In fact they are complete normed vector spaces i.e. Banach spaces. There are various ways to prove this. We are going to show the proof from [18].
To do so, we first define the following map

$$i : T^{p,2} \to L^p \left(L^2 \left(\mathrm{d}y \frac{\mathrm{d}t}{c_n t^{\frac{n}{2}}} \right) \right),$$
$$i(u)(x,t,y) = \tilde{u}(x,t,y) = \chi_{\{(y,t):|x-y|<\sqrt{t}\}}(t,y)u(t,y),$$

where c_n is the measure of the unit ball in $\mathbb{R}^n$. By definition of the norm we have

$$\begin{aligned}\|u\|_{T^{p,2}} &= \left(\int_{\mathbb{R}^n}\left(\int_0^\infty \fint_{B(x,\sqrt{t})}|u(t,y)|^2\mathrm{d}y\mathrm{d}t\right)^{\frac{p}{2}}\mathrm{d}x\right)^{\frac{1}{p}}\\ &= \left(\int_{\mathbb{R}^n}\left(\int_0^\infty \int_{\mathbb{R}^n}\chi_{\{(y,t):|x-y|<\sqrt{t}\}}(t,y)|u(t,y)|^2\mathrm{d}y\frac{\mathrm{d}t}{c_n t^{\frac{n}{2}}}\right)^{\frac{p}{2}}\mathrm{d}x\right)^{\frac{1}{p}}\\ &= \|\tilde{u}\|_{L^p\left(L^2\left(\mathrm{d}y\frac{\mathrm{d}t}{c_n t^{\frac{n}{2}}}\right)\right)}.\end{aligned}$$

Hence, $T^{p,2}$ is isometric to $i(T^{p,2})$. This brings us to the following theorem

Theorem 3.1. *Let $1 < p < \infty$. Then the operator N, given by*

$$N(u(x,t,y)) = \chi_{\{(y,t):|x-y|<\sqrt{t}\}}(t,y)\frac{1}{c_n t^{\frac{n}{2}}}\int_{|z-t|<\sqrt{t}} u(z,t,y)\,dz,$$

defines a continuous projection from $L^p\left(L^2\left(dy\frac{dt}{c_n t^{\frac{n}{2}}}\right)\right) \to L^p\left(L^2\left(dy\frac{dt}{c_n t^{\frac{n}{2}}}\right)\right)$ whose range is $i(T^{p,2})$.

Proof. We are going to use the vector-valued Hardy-Littlewood maximal operator given by

$$M_1 f(t,y)(x) = \sup_{x\in B}\frac{1}{|B|}\int_B |f(z,t,y)|\mathrm{d}z,$$

where B denotes a Euclidean ball. By definition of N we have

$$|Nu(x,t,y)| \le \chi_{\{(y,t):|x-y|<\sqrt{t}\}}(t,y)\frac{1}{|B(y,\sqrt{t})|}\int_{B(y,\sqrt{t})}|u(z,t,y)|\mathrm{d}z \le M_1 u(t,y)(x).$$

From [36] we know that, for $1 < p < \infty$, M_1 is bounded $L^p\left(L^2\left(\mathrm{d}y\frac{\mathrm{d}t}{c_n t^{\frac{n}{2}}}\right)\right) \to L^p\left(L^2\left(\mathrm{d}y\frac{\mathrm{d}t}{c_n t^{\frac{n}{2}}}\right)\right)$. So we find

$$\|Nu\|_{L^p\left(L^2\left(\mathrm{d}y\frac{\mathrm{d}t}{c_n t^{\frac{n}{2}}}\right)\right)} \le \|M_1 u\|_{L^p\left(L^2\left(\mathrm{d}y\frac{\mathrm{d}t}{c_n t^{\frac{n}{2}}}\right)\right)} \le c_p\|u\|_{L^p\left(L^2\left(\mathrm{d}y\frac{\mathrm{d}t}{c_n t^{\frac{n}{2}}}\right)\right)}.$$

Next we need to show that N is a projection. We have

$$\begin{aligned}N(Nu)(x,t,y) &= \chi_{\{(y,t):|x-y|<\sqrt{t}\}}(t,y)\frac{1}{c_n t^{\frac{n}{2}}}\int_{|z-y|<\sqrt{t}} Nu(z,t,y)\mathrm{d}z\\ &= \chi_{\{(y,t):|x-y|<\sqrt{t}\}}(t,y)\frac{1}{c_n t^{\frac{n}{2}}}\left(\int_{|v-y|<\sqrt{t}} u(v,t,y)\mathrm{d}v\right)\left(\frac{1}{c_n t^{\frac{n}{2}}}\int_{|z-y|<\sqrt{t}}\mathrm{d}z\right)\\ &= Nu(x,t,y).\end{aligned}$$

Now let u be in the range of N and denote $h(t,y) = \frac{1}{c_n t^{\frac{n}{2}}}\int_{|z-y|<\sqrt{t}} Nu(z,t,y)\mathrm{d}z$. Then, by definition we have

$$\begin{aligned}u(x,t,y) &= \chi_{\{(y,t):|x-y|<\sqrt{t}\}}(t,y)\frac{1}{c_n t^{\frac{n}{2}}}\int_{|z-y|<\sqrt{t}} Nu(z,t,y)\mathrm{d}z\\ &= \chi_{\{(y,t):|x-y|<\sqrt{t}\}}(t,y)h(t,y)\end{aligned}$$

$$= \tilde{h}(x,t,y).$$

If $\tilde{u} \in i(T^{p,2})$ then we have

$$\begin{aligned}\tilde{u}(x,t,y) &= \chi_{\{(y,t):|x-y|<\sqrt{t}\}}(t,y)u(t,y) \\ &= \chi_{\{(y,t):|x-y|<\sqrt{t}\}}(t,y)\frac{1}{c_n t^{\frac{n}{2}}}\int_{|z-y|<\sqrt{t}} \chi_{\{(y,t):|z-y|<\sqrt{t}\}}(t,y)u(t,y)\mathrm{d}z \\ &= \chi_{\{(y,t):|x-y|<\sqrt{t}\}}(t,y)\frac{1}{c_n t^{\frac{n}{2}}}\int_{|z-y|<\sqrt{t}} \tilde{u}(z,t,y)\mathrm{d}z \\ &= N\tilde{u}(x,t,y).\end{aligned}$$

So we also have that $i(T^{p,2})$ is equal to the range of N. □

A consequence of the above theorem is the completeness result for the tent space.

Corollary 3.2. *Let $1 < p < \infty$. Then $T^{p,2}$ is a Banach space and the subspace of compactly supported functions is dense in $T^{p,2}$.*

Proof. By Theorem 3.1 we have that $i(T^{p,2})$ is a closed subspace of $L^p\left(L^2\left(\mathrm{d}y\frac{\mathrm{d}t}{c_n t^{\frac{n}{2}}}\right)\right)$ and hence is Banach. Since $T^{p,2}$ is isometric to $i(T^{p,2})$ it follows that $T^{p,2}$ itself is a Banach space as well.

To prove the density result, first note that the compactly supported functions are dense in $L^p\left(L^2\left(\mathrm{d}y\frac{\mathrm{d}t}{c_n t^{\frac{n}{2}}}\right)\right)$. Also, if f is compactly supported then there exists a compactly supported h such that $N(f) = \tilde{h}$. Now let $u \in T^{p,2}$. Then we have a sequence of compactly supported functions $(u_n)_{n\in\mathbb{N}}$ such that $u_n \to \tilde{u}$ in $L^p\left(L^2\left(\mathrm{d}y\frac{\mathrm{d}t}{c_n t^{\frac{n}{2}}}\right)\right)$. By continuity of N and calculations from the previous proof we get $\tilde{h}_n = N(u_n) \to N(\tilde{u}) = \tilde{u}$ in $L^p\left(L^2\left(\mathrm{d}y\frac{\mathrm{d}t}{c_n t^{\frac{n}{2}}}\right)\right)$. By isomtry between $T^{p,2}$ and $i(T^{p,2})$ we thus get

$$h_n \to u \quad \text{in } T^{p,2},$$

with $(h_n)_{n\in\mathbb{N}}$ a sequence of compactly supported functions. □

A useful property for the case $p = 2$ is that we have, by using Fubini and $|B(x,\sqrt{t})| = |B(y,\sqrt{t})|$,

$$\begin{aligned}\|u\|_{T^{2,2}} &= \left(\int_{\mathbb{R}^n}\int_0^\infty ⨍_{B(x,\sqrt{t})} |u(t,y)|^2\mathrm{d}y\mathrm{d}t\mathrm{d}x\right)^{\frac{1}{2}} \\ &= \left(\int_0^\infty\int_{\mathbb{R}^n} ⨍_{B(y,\sqrt{t})} |u(t,y)|^2\mathrm{d}x\mathrm{d}y\mathrm{d}t\right)^{\frac{1}{2}} \\ &= \left(\int_0^\infty\int_{\mathbb{R}^n} ⨍_{B(y,\sqrt{t})} \mathrm{d}x|u(t,y)|^2\mathrm{d}y\mathrm{d}t\right)^{\frac{1}{2}} \\ &= \left(\int_0^\infty\int_{\mathbb{R}^n} |u(t,y)|^2\mathrm{d}y\mathrm{d}t\right)^{\frac{1}{2}} \\ &= \|u\|_{L^2(L^2)},\end{aligned}$$

and hence $T^{2,2} = L^2(L^2)$.

In the above we defined our tent spaces with an integration over a ball with aperture 1 i.e. $B(x, 1\cdot\sqrt{t})$. We can also define the tent space with a different aperture: Again, we define a mapping, mapping functions on $\mathbb{R}^{n+1}_+$ to functions on $\mathbb{R}^n$, by

$$\mathcal{A}^\alpha(u)(x) = \left(\int_0^\infty \frac{1}{c_n t^{\frac{n}{2}}} \int_{B(x,\alpha\sqrt{t})} |u(t,y)|^2 \mathrm{d}y\mathrm{d}t\right)^{\frac{1}{2}}.$$

Let $p \in [1,\infty)$. The set of measurable functions u on $\mathbb{R}^{n+1}_+$ such that

$$\mathcal{A}^\alpha(u) \in L^p(\mathbb{R}^n)$$

is called the tent space $T^{p,2,\alpha}$, with norm defined by $\|u\|_{T^{p,2,\alpha}} = \|\mathcal{A}^\alpha(u)\|_{L^p}$. It turns out that this norm is equivalent with the previous one and thus defines the same sets. In [2], this equivalence was proven with sharp bounds. Since we do not necessarily need sharp bounds, we will provide a shorter proof from [18]. Let $0 < \alpha < \infty$. We define the following map

$$i_\alpha : T^{p,2,\alpha} \to L^p\left(L^2\left(\mathrm{d}y\frac{\mathrm{d}t}{c_n t^{\frac{n}{2}}}\right)\right),$$
$$i_\alpha(u)(x,t,y) = \chi_{\{(y,t):|x-y|<\alpha\sqrt{t}\}}(t,y)u(x,t,y).$$

Analogously to $\alpha = 1$ we get that $T^{p,2,\alpha}$ is isometric to $i_\alpha(T^{p,2,\alpha})$. Moreover, the operators defined by

$$N_\alpha u(x,t,y) = \chi_{\{(y,t):|x-y|<\alpha\sqrt{t}\}}(t,y)\frac{1}{c_n t^{\frac{n}{2}}}\int_{|z-y|<\alpha\sqrt{t}} u(z,t,y)\mathrm{d}z$$

are also continuous projections in $L^p\left(L^2\left(\mathrm{d}y\frac{\mathrm{d}t}{c_n t^{\frac{n}{2}}}\right)\right)$ with norms independent of α and with range $i_\alpha(T^{p,2,\alpha})$.

Proposition 3.3 (Change of angle). *Let $1 < p < \infty$ and $0 < \alpha_1, \alpha_2 < \infty$. Then $T^{p,2,\alpha_1} = T^{p,2,\alpha_2}$ with equivalent norms.*

Proof. By change of variables we may assume $\alpha_1 > \alpha_2 = 1$. For $u \in T^{p,2,\alpha_1}$ we have, using $B(x,\sqrt{t}) \subset B(x,\alpha_1\sqrt{t})$,

$$\begin{aligned}
\|u\|^p_{T^{p,2}} &= \int_{\mathbb{R}^n}\left(\int_0^\infty \frac{1}{c_n t^{\frac{n}{2}}}\int_{B(x,\sqrt{t})} |u|^2\mathrm{d}y\mathrm{d}t\right)^{\frac{p}{2}}\mathrm{d}x \\
&\le \int_{\mathbb{R}^n}\left(\int_0^\infty \frac{1}{c_n t^{\frac{n}{2}}}\int_{B(x,\sqrt{t})} \chi_{B(x,\alpha_1\sqrt{t})}|u|^2\mathrm{d}y\mathrm{d}t\right)^{\frac{p}{2}}\mathrm{d}x \\
&\le \int_{\mathbb{R}^n}\left(\int_0^\infty \frac{1}{c_n t^{\frac{n}{2}}}\int_{B(x,\alpha_1\sqrt{t})} |u|^2\mathrm{d}y\mathrm{d}t\right)^{\frac{p}{2}}\mathrm{d}x \\
&= \|u\|_{T^{p,2,\alpha_1}},
\end{aligned}$$

and hence $T^{p,2,\alpha_1} \subseteq T^{p,2}$. Now take $u \in T^{p,2}$. Then we have

$$\begin{aligned}
N_\alpha(i_{\alpha_1}(u))(x,t,y) &= \chi_{\{(y,t):|x-y|<\alpha_1\sqrt{t}\}}(t,y)\frac{1}{c_n t^{\frac{n}{2}}}\int_{|z-y|<\alpha_1\sqrt{t}} \chi_{\{(y,t):|z-y|<\alpha_1\sqrt{t}\}}(t,y)u(t,y)\mathrm{d}z \\
&= \chi_{\{(y,t):|x-y|<\alpha_1\sqrt{t}\}}(t,y)\alpha_1^n u(t,y),
\end{aligned}$$

from which we get

$$\|u\|_{T^{p,2,\alpha_1}} = \alpha_1^{-n}\|N_{\alpha_1}(i_{\alpha_1}(u))\|_{L^p\left(L^2\left(\mathrm{d}y\frac{\mathrm{d}t}{c_n t^{\frac{n}{2}}}\right)\right)} \leq C_p\alpha_1^{-n}\|u\|_{T^{p,2}}.$$

□

Remark 3.4. *The statement with sharp bounds from [2] reads as follows. Let $0 < p \leq \infty$ and $\alpha_1, \alpha_2 > 0$. There exist constants $C, C' > 0$ depending on n, p only, such that for any locally square integrable function f*

$$C \min\left\{\left(\frac{\alpha_1}{\alpha_2}\right)^{-\frac{n}{2}}, \left(\frac{\alpha_1}{\alpha_2}\right)^{-\frac{n}{p}}\right\}\|f\|_{T^{p,2,\alpha_1}} \leq \|f\|_{T^{p,2,\alpha_2}}$$
$$\leq C' \max\left\{\left(\frac{\alpha_1}{\alpha_2}\right)^{-\frac{n}{2}}, \left(\frac{\alpha_1}{\alpha_2}\right)^{-\frac{n}{p}}\right\}\|f\|_{T^{p,2,\alpha_1}}.$$

Moreover, the dependence in α_1/α_2 is best possible in the sense that this growth is attained.

We will also work with weighted (in time) tent spaces, denoted by $T^{p,2}_\beta$ for some $\beta \in \mathbb{R}$. They are defined analogously to the unweighted tent spaces, but where the time variable is integrated with respect to the measure $\frac{\mathrm{d}t}{t^\beta}$. These weighted tent spaces are also Banach spaces and have a change of aperture result. The proofs for these are similar to the unweighted case.

We will also use a variation of the non-tangential maximal function, which was introcduced by Kenig and Pipher for elliptic equations in [23].

Definition 3.5. *Let $F \in L^2_{loc}(\mathbb{R}^{n+1}_+)$. The non-tangential maximal function $\tilde{N}(F)$ is defined by*

$$\tilde{N}(F)(x) := \sup_{\delta>0}\left(⨍_{\frac{\delta}{2}}^{\delta} ⨍_{B(x,\sqrt{\delta})} |F(t,y)|^2\, dy\, dt\right)^{\frac{1}{2}}, \quad \forall x \in \mathbb{R}^n.$$

Using this non-tangential maximal function the following Banach space was defined in [23].

Definition 3.6. *Let $1 \leq p < \infty$. X^p is defined as the subspace of functions $F \in L^2_{loc}(\mathbb{R}^{n+1}_+)$ such that*

$$\|F\|_{X^p} := \|\tilde{N}(F)\|_{L^p} < \infty.$$

In contrast to the original non-tangential maximal function, defined by

$$u^* : x \mapsto \sup_{\substack{(t,y)\in(0,\infty)\times\mathbb{R}^n \\ |x-y|<\sqrt{t}}} |u(t,y)|, \tag{3.1}$$

we do not need pointwise bounds for our solutions. We can also control the modified non-tangential maximal function by the original one in the following way.

$$\sup_{\delta>0}\left(⨍_{\frac{\delta}{2}}^{\delta} ⨍_{B(x,\sqrt{\delta})} |F(t,y)|^2 \mathrm{d}y\mathrm{d}t\right)^{\frac{1}{2}} \leq \sup_{\delta>0}\left(⨍_{\frac{\delta}{2}}^{\delta} ⨍_{B(x,\sqrt{\delta})} F^*(x)\mathrm{d}y\mathrm{d}t\right)^{\frac{1}{2}}$$
$$= F^*(x).$$

Hence,

$$\|F\|_{X^p} \leq \|F^*\|_{L^p}.$$

Chapter 4

Elliptic operators

In this section we will define what a divergence form elliptic operator is and also provide a few properties that we are going to use later on. These properties and definitions are taken from [3], unless stated otherwise.
Before stating the definition of a divergence form elliptic operator we introduce the notion of a strongly continuous semigroup, an analytic semigroup and (maximal accretive) operators.

Definition 4.1. *Let X be a Banach space and $\mathcal{L}(X)$ the bounded linear operators on X. A strongly continuous semigroup on X is a map $T : \mathbb{R}_+ \to \mathcal{L}(X)$ such that*

(i) $T(0) = I$, *where I is the identity operator,*

(ii) for all $t, s \geq 0$: $T(t+s) = T(t)T(s)$,

(iii) for all $x \in X$: $\lim_{t \downarrow 0} \|T(t)x - x\|_X \to 0$.

Now let $S(t)$ be a strongly continuous semigroup on a Banach space X and denote a sector of angle ω in the complex plane by Σ_ω, i.e.,

$$\Sigma_\omega = \{z \in \mathbb{C} : |\arg z| < \omega\}.$$

Definition 4.2. *$S(t)$ is said to be analytic if there exist a $\omega \in (0, \frac{\pi}{2})$ such that the following hold:*

- *The mapping $t \to S(t)$ can be extended to Σ_ω such that the usual semigroup properties hold on Σ_ω. In other words, such that for all $s, t \in \Sigma_\omega$ we have $S(0) = I$, $S(s)S(t) = S(s+t)$ and the mapping $z \mapsto S(z)x$ is continuous for all $x \in X$.*
- *For all $z \in \Sigma_\omega \backslash \{0\}$ the mapping $z \mapsto S(z)$ is analytic in the operator norm.*

We will also need the following definitions related to operators

Definition 4.3. *Let X, Y be Banach spaces. An operator T is said to be closed if its graph, $\Gamma(T)$ is a closed set. Here, the $\Gamma(T) = \{(x, Tx) : x \in D(T)\} \subset X \oplus Y$, where $D(T)$ is the domain of T.*

Definition 4.4. *A closed operator T acting on a Hilbert space H is said to be accretive if*

$$\Re e\langle Tu, u\rangle \geq 0, \quad \textit{for all } u \in H.$$

If T has no proper accretive extension, it is called maximal accretive.

We will also introduce the notion of a bounded holomorphic funtional calculus is as was done in [16].
Let $\omega, \sigma \in \mathbb{C}$ be such that $0 \leq \omega < \sigma < \pi$. We define the following sectors in the complex plane.

$$\overline{\Sigma}_\omega := \{z \in \mathbb{C} : |\arg z| \leq \omega\}, \qquad \Sigma_\sigma^0 := \Sigma_\sigma \backslash \{0\}.$$

Note that $\overline{\Sigma}_\omega$ is a closed set and Σ_σ^0 an open set. By $H(\Sigma_\sigma^0)$ we denote the space of all holomorphic functions on Σ_σ^0. For every $\alpha, \beta > 0$,we define the following subset of $H(\Sigma_\sigma^0)$

$$H^\infty(\Sigma_\sigma^0) := \{\phi \in H(\Sigma_\sigma^0) : \|\phi\|_{L^\infty(\Sigma_\sigma^0)} < \infty\},$$
$$\Psi_{\alpha,\beta}(\Sigma_\sigma^0) := \{\phi \in H(\Sigma_\sigma^0) : \exists C : |\phi(z)| \leq C|z|^\alpha(1+|z|^{\alpha+\beta})^{-1} \text{ for every } z \in \Sigma_\sigma^0\}.$$

We also define the set $\Psi(\Sigma_\sigma^0) := \bigcup_{\alpha,\beta>0} \Psi_{\alpha,\beta}(\Sigma_\sigma^0)$.

Definition 4.5. *Let $\omega \in [0, \pi)$. A closed operator L in a Hilbert space H is said to be sectorial of angle ω if $\sigma(L) \subset \overline{\Sigma}_\omega$, where $\sigma(L)$ is the spectrum of L, and for each $\sigma > \omega$, there exists a constant $C_\sigma > 0$ such that*

$$\|(zI - L)^{-1}\| \leq C_\sigma |z|^{-1}, \quad z \notin \overline{\Sigma}_\omega.$$

Let $\omega < \theta < \sigma < \pi$ and L a sectorial operator of angle $\omega \in [0, \pi)$ in a Hilbert space H. Then for every $\phi \in \Psi(\Sigma_\sigma^0)$

$$\phi(L) := \frac{1}{2\pi \mathrm{i}} \int_{\partial \Sigma_\theta^0} \phi(\lambda)(\lambda I - L)^{-1} \mathrm{d}\lambda$$

defines a bounded operator on H. Since L is sectorial, the above integral is well-defined. Furthermore, by the extension of Cauchy's theorem, the above definition is independent of the choice of $\theta \in (\omega, \sigma)$.
Now, in addition to sectorial, assume that L is also injective. By, for example, [12] Theorem 2.3 and Theorem 3.8, we get that L has a dense domain and a dense range in H. Setting $\phi(z) := z(1+z)^{-2}$, we then get that $\phi(L)$ is injective and has dense range in H. For $f \in H^\infty(\Sigma_\sigma^0)$ we define by

$$f(L) := [\phi(L)]^{-1}(f \cdot \phi)(L)$$

a closed operator in H. If there exists a constant $c_\sigma >$ such that for all $f \in H^\infty(\Sigma_\sigma^0)$, there holds $f(L)$ is a bounded operator on H with

$$\|f(L)\| \leq c_\sigma \|f\|_{L^\infty(\Sigma_\sigma^0)},$$

we say that L has a bounded holomorphic or $H^\infty(\Sigma_\sigma^0)$ functional calculus. Having a bounded holomorphic functional calculus is equivalent to L satisfying square function estimates, i.e. for some (all) $\sigma \in (\omega, \pi)$ and some $\phi \in \Psi(\Sigma_\sigma^0)\backslash\{0\}$ there exists a $C > 0$ such that for all $u \in H$

$$C^{-1}\|u\|_H^2 \leq \int_0^\infty \|\phi(tL)u\|_H^2 \frac{\mathrm{d}t}{t} \leq C\|u\|^2.$$

For more details about functional calculi (of accretive operators) see [27].

Now we start defining what a divergence form elliptic operator is. Let $A \in L^\infty(\mathbb{R}^n, \mathscr{M}_n(\mathbb{C})))$ satisfy uniform ellipticity estimates

$$\exists \Lambda > 0 : \quad \forall \xi, \eta \in \mathbb{C}^n, |\langle A(x)\xi, \eta\rangle| \leq \Lambda|\xi||\eta|, \quad x \in \mathbb{R}^n,$$
$$\exists \lambda > 0 : \quad \forall \xi \in \mathbb{C}^n, \Re e(\langle A(x)\xi, \xi\rangle) \geq \lambda|\xi|^2, \quad x \in \mathbb{R}^n.$$

A second order divergence form operator is then formally defined as

$$Lf = -\mathrm{div}(A\nabla f).$$

We will need to justify this definition and notation. To do so, let $\mathcal{D}(L)$ be the largest subspace contained in $W^{1,2} = H^1$, the Sobolev space, such that

$$|\langle A\nabla f, \nabla g\rangle| = \left|\int_{\mathbb{R}^n} A\nabla f \cdot \nabla g \mathrm{d}x\right| \leq C\|g\|_2, \qquad \text{for all } g \in H^1,$$

where $\langle \cdot, \cdot \rangle$ is the L^2 inner product. Then we set Lf by

$$\langle Lf, g\rangle = \int_{\mathbb{R}^n} A\nabla f \cdot \overline{\nabla g} \mathrm{d}x,$$

for $f \in \mathcal{D}(L)$ and $g \in H^1$. This defines L (see [22, Chapter VI] for more information on forms). From this definition we get that L is a maximal accretive operator on L^2 and that its domain, $\mathcal{D}(L)$, is dense in H^1 (see [22, Chapter V, §3.10]).
To justify the divergence notation we first introduce the homogeneous Sobolev space in the following way: we set $\dot{H}^1(\mathbb{R}^n) = \{u \in \mathscr{D}'(\mathbb{R}^n) : \nabla u \in L^2(\mathbb{R}^n; \mathbb{C}^n)\}$ equipped with the seminorm $u \mapsto \|\nabla u\|_{L^2}$. Note that for this semi-norm, H^1 is dense in its homogeneous version $\dot{H}^1$. (Test functions are dense in $\dot{H}^1$ and are included in H^1). Hence, L extends to a bounded invertible operator from $\dot{H}^1$ into its dual space, which can be identified with $\dot{H}^{-1}(\mathbb{R}^n) = \{\mathrm{div} g : g \in L^2(\mathbb{R}^n, \mathbb{C}^n)\}$ equipped with the norm $f \mapsto \|f\|_{\dot{H}^{-1}} = \inf\{\|g\|_{L^2} : f = \mathrm{div}\, g\}$. Moreover, for all $u \in \dot{H}^1(\mathbb{R}^n)$, all $g \in L^2(\mathbb{R}^n; \mathbb{C}^n)$ and $f = \mathrm{div}\, g$, we have

$${}_{\dot{H}^{-1}}\langle f, u\rangle_{\dot{H}^1} = -{}_{L^2}\langle g, \nabla u\rangle_{L^2}.$$

Remark 4.6. *For the dual of L we have*

$$\begin{aligned}\langle Lf, g\rangle &= \langle -div(A\nabla f), g\rangle \\ &= \langle f, -div(A^*\nabla g)\rangle \\ &= \langle f, L^* g\rangle, \qquad\qquad f, g \in H^1,\end{aligned}$$

with A^* being the dual of the matrix A. This shows that the dual of L is again a divergence form elliptic operator.

Since L is a maximal accretive operator we have that $-L$ generates a bounded analytic semigroup $(\mathrm{e}^{-zL})_{z\in\Sigma_{\frac{\pi}{2}-\omega_L}}$ with $\omega_L = \inf\{\gamma \in [0, \frac{\pi}{2}) : |\arg\langle Lf, f\rangle| \leq \gamma \text{ for all } f \in \mathcal{D}(L)\}$. From [35] we also know that $(\mathrm{e}^{-tL})_{t\in\mathbb{R}_+}$ is a contraction semigroup i.e. $\|\mathrm{e}^{-tL}\|_{L^2\to L^2} \leq 1$ for all $t \in \mathbb{R}_+$.
Another consequence of L being maximal accretive, is that it also has a bounded holomorphic functional calculus on L^2 [27, Theorem 11.5].

A property involving the operator L and its semigroup that we will need later on is the following: Let $L^{\frac{1}{2}}$ be the unique maximal accretive operator such that $L^{\frac{1}{2}}L^{\frac{1}{2}} = L$. Then for all $u \in H^1$, we have, by the solution of Kato's square root problem [5],

$$\|L^{\frac{1}{2}}u\|_2 \simeq \|\nabla u\|_2 \tag{4.1}$$

and the domain of $L^{\frac{1}{2}}$ coincides with H^1. As a consequence we get

$$\sup_{t>0}\|\nabla \mathrm{e}^{-tL}u\|_{L^2} \simeq \sup_{t>0}\|L^{\frac{1}{2}}\mathrm{e}^{-tL}u\|_{L^2} = \sup_{t>0}\|\mathrm{e}^{-tL}L^{\frac{1}{2}}u\|_{L^2} \leq \|L^{\frac{1}{2}}u\|_{L^2} \simeq \|\nabla u\|_{L^2}. \tag{4.2}$$

Another property that we use on multiple occasions throughout this thesis are the off-diagonal estimates of Gaffney type.

Definition 4.7. *Let $q \in [1, 2]$ and let $\mathcal{T} = (T_t)_{t>0}$ be a family of operators acting on L^2. If for some constants $C > 0$ and $c > 0$, for all Borel sets $E, F \subseteq \mathbb{R}^n$, all $h \in L^2 \cap L^q$ with support in E and all $t > 0$ we have*

$$\|\chi_F T_t h\|_2 \leq C t^{-\frac{n}{2}(\frac{1}{q}-\frac{1}{2})} \mathrm{e}^{-\frac{c d(E,F)^2}{t}} \|h\|_q,$$

then we say that $\mathcal{T}$ satisfies $L^q - L^2$ off-diagonal estimates. Here, $d(E, F)$ denotes the distance between the sets E and F.

Whenever $q = 2$, we just say that $\mathcal{T}$ satisfies L^2 off-diagonal estimates.

Proposition 4.8. *The families $(\mathrm{e}^{-tL})_{t>0}$, $(tL\mathrm{e}^{-tL})_{t>0}$ and $(\sqrt{t}\nabla \mathrm{e}^{-tL})_{t>0}$ satisfy L^2 off-diagonal bounds.*

Proof. Let ϕ be a Lipschitz function on $\mathbb{R}^n$ with Lipschitz norm 1 and let $\rho > 0$. By the same mehod as we defined the operator L, we define $L_\rho = \mathrm{e}^{\rho\phi} L \mathrm{e}^{-\rho\phi}$. Note that for L we have

$$Lf = -\mathrm{div}(A\nabla f) = -\sum_{i,j} \partial_j a_{i,j} \partial_i f$$

and for L_ρ we have

$$L_\rho f = -\mathrm{e}^{\rho\phi} \mathrm{div}(A\nabla \mathrm{e}^{-\rho\phi} f) = -\mathrm{e}^{\rho\phi} \sum_{i,j} \partial_j a_{i,j} \partial_i (\mathrm{e}^{-\rho\phi} f),$$

where we recall that L is defined through a form and hence the a_{ij} need not to be differentiable. Using the product rule we get that L_ρ has the same principal term as L and some lower order terms, which are bounded since $\mathrm{e}^{\rho\phi}$ and $\mathrm{e}^{-\rho\phi}$ are smooth enough. In other words, if Q_ρ is the associated form, then it is bounded on H^1 and we can find a constant c depending only on dimension and the ellipticity constants of L such that

$$\Re e(Q_\rho(f)) \geq \frac{\lambda}{2} \|\nabla f\|_2^2 - c\rho^2 \|f\|_2^2, \quad f \in H^1.$$

The above shows that $L_\rho + c\rho^2$ is a maximal accretive operator on L^2 and hence generates an analytic semigroup $(\mathrm{e}^{-tL_\rho} \mathrm{e}^{-c\rho^2 t})_{t>0}$. Since c and ρ are constants we find that the semigroup $(\mathrm{e}^{-tL_\rho})_{t>0}$ exists and is analytic as well. So we have

$$\|\mathrm{e}^{-tL_\rho} f\|_2 \leq C \mathrm{e}^{c\rho^2 t} \|f\|_2, \tag{4.3}$$

for all $t > 0$ where C only depends on ellipticity constants of L and on dimension. Now let E and F be two Borel sets and $f \in L^2$, with compact support contained in E. Choose $\phi(x) = d(x, E)$. Since $L_\rho = \mathrm{e}^{\rho\phi} L \mathrm{e}^{-\rho\phi}$, viewed as multiplication of operators, we get that $\mathrm{e}^{-tL_\rho} = \mathrm{e}^{\rho\phi} \mathrm{e}^{-tL} \mathrm{e}^{-\rho\phi}$. Using this, and that the support of f is contained in E we find

$$\mathrm{e}^{-tL} = \mathrm{e}^{-\rho\phi} \mathrm{e}^{-tL_\rho} f.$$

Combining with (4.3) we thus find for all $t > 0$ and $\rho > 0$

$$\|\chi_F \mathrm{e}^{-tL} f\|_2 \leq C \mathrm{e}^{-\rho d(E,F)} \mathrm{e}^{c\rho^2 t} \|f\|_2.$$

Optimizing $-\rho d(E,F)+c\rho^2 t$ with respect to $\rho>0$ we get

$$\|\chi_F \mathrm{e}^{-tL} f\|_2 \le C\mathrm{e}^{-\frac{d(E,F)^2}{4ct}}\|f\|_2. \tag{4.4}$$

Now let $\alpha\in\mathbb{C}$ such that $|\alpha|<\frac{\pi}{2}-\omega_L$. Then $\mathrm{e}^{\mathrm{i}\alpha}L$, with coefficients $\mathrm{e}^{\mathrm{i}\alpha}A(x)$, is an operator in the same class as L. Hence we can find the same estimate as (4.4) for $\mathrm{e}^{\mathrm{i}\alpha}L$. For $z\in\sigma_{\frac{\pi}{2}-\omega_L}$ we can write $z=t\mathrm{e}^{\mathrm{i}\alpha}$ with $|\alpha|<\frac{\pi}{2}-\omega_L$ and $t\in\mathbb{R}$. In this case we have $\mathrm{e}^{-zL}=\mathrm{e}^{-t(\mathrm{e}^{\mathrm{i}\alpha}L)}$. Thus we can do the following.
Let $f,g\in L^2$, with supp $f\subseteq E$ and supp $g\subseteq F$. For $z\in\Sigma_{\omega_L}$ define

$$G(z):=\langle \mathrm{e}^{-zL}f,g\rangle=\int_{\mathbb{R}^n}\mathrm{e}^{-zL}f(x)\cdot\overline{g(x)}\mathrm{d}x.$$

Using that $(\mathrm{e}^{-zL})_{z\in\Sigma_L}$ is analytic and $\mathrm{e}^{-zL}=\mathrm{e}^{-t(\mathrm{e}^{\mathrm{i}\alpha}L)}$ we get that G is also analytic on Σ_{ω_L} and find

$$|G(z)|\le C\mathrm{e}^{-\frac{d(E,F)^2}{4ct}}\|f\|_2\|g\|_2.$$

Now fix $t>0$. We can write, by using the Cauchy integral formula,

$$tL\mathrm{e}^{-tL}=-\frac{t}{2\pi\mathrm{i}}\int_{|\zeta-t|=\eta t}\frac{\mathrm{e}^{-\zeta L}}{(\zeta-t)^2}\mathrm{d}\zeta,$$

where we choose $\eta>0$ small enough such that $\{\zeta\in\mathbb{C}:|\zeta-t|\le\eta t\}\subseteq\Sigma_L$. We thus find

$$\begin{aligned}|\langle tL\mathrm{e}^{-tL}f,g\rangle| &\le \frac{t}{2\pi}\int_{|\zeta-t|=\eta t}\frac{|G(\zeta)|}{\eta^2t^2}|\mathrm{d}\zeta|\\ &\le C\mathrm{e}^{-\frac{d(E,F)^2}{4ct}}\|f\|_2\|g\|_2\frac{1}{\eta^2 t}\frac{1}{2\pi}2\pi\eta t\\ &\le \tilde{C}\mathrm{e}^{-\frac{d(E,F)^2}{4ct}}\|f\|_2\|g\|_2.\end{aligned}$$

Hence, for all $t>0$ we find

$$\|\chi_F tL\mathrm{e}^{-tL}f\|_2\le\tilde{C}\mathrm{e}^{-\frac{d(E,F)^2}{4ct}}\|f\|_2$$

with $\tilde{C}$ only depending on dimension and ellipticity constants.
To see that $(\sqrt{t}\nabla\mathrm{e}^{-tL})_{t>0}$ also satisfies L^2 off-diagonal estimate we note that, for $f\in H^2$, we have, using ellipticity,

$$\begin{aligned}\|\sqrt{t}\nabla\mathrm{e}^{-tL}f\|_2^2 &= \langle\sqrt{t}\nabla\mathrm{e}^{-tL}f,\sqrt{t}\nabla\mathrm{e}^{-tL}f\rangle\\ &= \langle t\nabla\mathrm{e}^{-tL}f,\nabla\mathrm{e}^{-tL}f\rangle\\ &\lesssim \langle tA\nabla\mathrm{e}^{-tL}f,\nabla\mathrm{e}^{-tL}f\rangle\\ &= \langle -tL\mathrm{e}^{-tL}f,f\rangle\\ &\le \|-tL\mathrm{e}^{-tL}f\|_2\|f\|_2,\end{aligned}$$

where the implicit constant only depends on the ellipticity constants. Hence, the L^2 off-diagonal estimates of $(\sqrt{t}\nabla\mathrm{e}^{-tL})_{t>0}$ follow from the L^2 off-diagonal estimates of $(tL\mathrm{e}^{-tL})_{t>0}$. □

Remark 4.9. *Let $t>0$. We have $(\sqrt{t}\nabla\mathrm{e}^{-tL})^*=\sqrt{t}\mathrm{e}^{-tL^*}div$. Using this we find*

$$\|\sqrt{t}\nabla\mathrm{e}^{-tL}\|_{L^2\to L^2}=\sup_{f,g\in L^2}\frac{|\langle\sqrt{t}\nabla\mathrm{e}^{-tL}f,g\rangle|}{\|f\|_2\|g\|_2}=\sup_{f,g\in L^2}\frac{|\langle f,\sqrt{t}\mathrm{e}^{-tL^*}div g\rangle|}{\|f\|_2\|g\|_2}=\|\sqrt{t}\mathrm{e}^{-tL^*}div\|_{L^2\to L^2}.$$

Since L^* is a divergence form elliptic operator, as mentioned in Remark 4.6, we know that $(\sqrt{t}\nabla\mathrm{e}^{-tL^*})_{t>0}$ satisfies L^2 off-diagonal bounds as well. Switching the roles of L^* and L in the above computation thus shows us that $(\sqrt{t}\mathrm{e}^{-tL}div)_{t>0}$ also satisfies L^2 off-diagonal bounds.

Proposition 4.10. *If $T = (T_t)_{t\geq 0}$ and $S = (S_t)_{t\geq 0}$ both satisfy L^2 off-diagonal estimates, then so does $(T_tS_t)_{t\geq 0}$.*

Proof. Let $E, F \subset \mathbb{R}^n$ be Borel sets, $t > 0$ and $f \in L^2$. We want to show

$$\|\chi_F T_t(S_t\chi_E f)\|_2 \lesssim \mathrm{e}^{-\frac{cd(E,F)^2}{t}}\|\chi_E f\|_2,$$

for some constant $c > 0$. First we assume that $F \cap E = \emptyset$. We denote $d = d(E, F)$, and define

$$G_1 = \{x \in \mathbb{R}^n; d(x, E \cup F) \geq \frac{1}{3}d\},$$
$$G_2 = \{x \in \mathbb{R}^n; d(x, E) < \frac{1}{3}d\},$$
$$G_3 = \{x \in \mathbb{R}^n; d(x, F) < \frac{1}{3}d\}.$$

The above sets are mutually disjoint and we have $G_1 \cup G_2 \cup G_3 = \mathbb{R}^n$. Now we get

$$\begin{aligned}\|\chi_F T_t(S_t\chi_E f)\|_2 &= \|\chi_F T_t((\chi_{G_1} + \chi_{G_2} + \chi_{G_3})S_t\chi_E f)\|_2 \\ &\leq \|\chi_F T_t(\chi_{G_1}S_t\chi_E f)\|_2 + \|\chi_F T_t(\chi_{G_2}S_t\chi_E f)\|_2 + \|\chi_F T_t(\chi_{G_3}S_t\chi_E f)\|_2. \quad (4.5)\end{aligned}$$

Using L^2 off-diagonal estimates for T and S for the first part we get

$$\begin{aligned}\|\chi_F T_t(\chi_{G_1}S_t\chi_E f)\|_2 &\lesssim \mathrm{e}^{-\frac{c_1 d(G_1,F)^2}{t}}\|\chi_{G_1}S_t\chi_E f\|_2 \\ &\lesssim \mathrm{e}^{-\frac{c_1 d(G_1,F)^2}{t}}\mathrm{e}^{-\frac{c_2 d(G_1,E)^2}{t}}\|\chi_E f\|_2 \\ &= \mathrm{e}^{-\frac{c_1\frac{1}{3}d(E,F)^2}{t}}\mathrm{e}^{-\frac{c_2\frac{1}{3}d(E,F)^2}{t}}\|\chi_E f\|_2 \\ &= \mathrm{e}^{-\frac{\tilde{c}_1 d(E,F)^2}{t}}\|\chi_E f\|_2,\end{aligned}$$

where $\tilde{c}_1 = \frac{1}{3}c_1 + \frac{1}{3}c_2$. For the second part of (4.5) we have

$$\begin{aligned}\|\chi_F T_t(\chi_{G_2}S_t\chi_E f)\|_2 &\lesssim \mathrm{e}^{-\frac{c_1 d(G_2,F)^2}{t}}\mathrm{e}^{-\frac{c_2 d(G_2,E)^2}{t}}\|\chi_E f\|_2 \\ &\lesssim \mathrm{e}^{-\frac{c_1\frac{2}{3}d(E,F)^2}{t}}\mathrm{e}^{0}\|\chi_E f\|_2 \\ &= \mathrm{e}^{-\frac{\tilde{c}_2 d(E,F)^2}{t}}\|\chi_E f\|_2,\end{aligned}$$

where $\tilde{c}_2 = \frac{2}{3}c_1 + \frac{1}{3}c_2$. Similarly, we get for the last part of (4.5)

$$\|\chi_F T_t(\chi_{G_3}S_t\chi_E f)\|_2 \lesssim \mathrm{e}^{-\frac{\tilde{c}_3 d(E,F)^2}{t}}\|\chi_E f\|_2,$$

where $\tilde{c}_3 = \frac{1}{2}c_1 + \frac{2}{3}c_2$. Collecting the above estimate thus provides us with the desired result with $c = \tilde{c_1} + \tilde{c_2} + \tilde{c_3}$.
Now assume $F \cap E \neq \emptyset$. Then $d(E, F) = 0$ and hence $\mathrm{e}^{-\frac{d(E,F)^2}{t}} = 1$. In this case we have

$$\begin{aligned}\|\chi_F T_t(S_t\chi_E f)\|_2 &= \|\chi_F T_t(\chi_{\mathbb{R}^n}S_t\chi_E f)\|_2 \\ &\lesssim \mathrm{e}^{-\frac{c_1 d(\mathbb{R}^n,F)^2}{t}}\mathrm{e}^{-\frac{c_2 d(E,\mathbb{R}^n)^2}{t}}\|\chi_E f\|_2 \\ &= \|\chi_E f\|_2,\end{aligned}$$

which is the desired result. □

Chapter 5

Maximal regularity operators

We first introduce the notion of maximal regularity in a simple setting. Let X be a Banach space and L a closed, not neccesarily bounded, operator with domain $D(L)$ dense in X. For a measurable function $f : [0,\infty) \to X$, we consider the following problem

$$\begin{cases} u'(t) + Lu(t) &= f(t), \quad t \geq 0 \\ u(0) &= 0. \end{cases} \tag{5.1}$$

Definition 5.1. *Let $p \in (1,\infty)$. We say that L has maximal L^p-regularity if there exists $C > 0$ such that for all $f \in L^p(0,\infty;X)$, there is a unique $u \in L^p(0,\infty;D(L))$ with $u' \in L^p(0,\infty;X)$ that satisfies (5.1) for almost all $t \in (0,\infty)$ and such that*

$$\|u'\|_{L^p(0,\infty;X)} + \|Lu\|_{L^p(0,\infty;X)} \leq C\|f\|_{L^p(0,\infty;X)}.$$

If we have a bounded analytic semigroup $(T(t))_{t\geq 0}$, generated by $-L$, then the solution u of (5.1) is formally given by

$$u(t) = \int_0^t T(t-s)f(s)\mathrm{d}s, \quad t \geq 0.$$

Hence, we have

$$Lu(t) = L\int_0^t T(t-s)f(s)\mathrm{d}s = \int_0^t LT(t-s)f(s)\mathrm{d}s \quad t \geq 0,$$

Since $u'(t) = -Lu(t) + f(t)$ we thus can see if L has maximal L^p-regularity by checking if the operator $\mathcal{M}_L$, defined by

$$\mathcal{M}_L f(t) = \int_0^t LT(t-s)f(s)\mathrm{d}s, \quad t \geq 0$$

for $f \in L^p(0,\infty;X)$, is bounded in $L^p(0,\infty;X)$.

Going back to our setting of interest, we consider the divergence form elliptic operator $L = -\mathrm{div}A\nabla$ with domain $D(L) = \{u \in H^1(\mathbb{R}^n); A\nabla u \in D(\mathrm{div})\}$ and $A \in L^\infty(\mathbb{R}^n;\mathscr{M}_n(\mathbb{C}))$ satifying (1.2). As previously mentioned, $-L$ generates a bounded analytic semigroup of contractions $(\mathrm{e}^{-tL})_{t\geq 0}$ on L^2. The maximal regularity operator associated with L is then formally defined by

$$\mathcal{M}_L f(t,x) = \int_0^t L\mathrm{e}^{-(t-s)L} f(s,\cdot)(x)\mathrm{d}s. \tag{5.2}$$

The $L^2(\mathbb{R}_+ \times \mathbb{R}^n)$ boundedness of this operator was proved in [38] by de Simon using the H^∞-functional calculus. For $L^p(\mathbb{R}_+ \times \mathbb{R}^n)$ boundedness, where $p \neq 2$, it was shown in [39] that a necessary and sufficient assumption was that L generates a R-analytic semigroup. In [4], a weighted (in time) version of the L^2 boundedness was established. It was shown that, for all $\beta \in (-1,\infty)$, $\mathcal{M}_L$ extends to a bounded operator on $L^2(\mathbb{R}_+ \times \mathbb{R}^n; t^{-\beta}\mathrm{d}t\mathrm{d}x) = T^{2,2}_\beta$. For the corresponding $p \neq 2$ case, the following result, of which we will provide the proof, was established in [7].

Theorem 5.2. *Let $-T$ be a densely defined closed linear operator acting on $L^2(\mathbb{R}^n)$ and generating a bounded analytic semigriup $(\mathrm{e}^{-tT})_{t\geq 0}$ and let $\beta \in (-1,\infty)$, $p \in (\frac{2n}{n+2(1+\beta)},\infty) \cap (1,\infty)$, and $\tau = \min\{p,2\}$. If for all Borel sets $E, F \subseteq \mathbb{R}^n$, all $t > 0$ and all $f \in L^2(\mathbb{R}^n)$, $(tT\mathrm{e}^{-tT})_{t\geq 0}$ satisfies*

$$\|\chi_E tT\mathrm{e}^{-tT}\chi_F\|_2 \lesssim \left(1 + \frac{d(E,F)^2}{t}\right)^{-M} \|\chi_F f\|_2, \tag{5.3}$$

where $M > \frac{n}{2\tau}$, then $\mathcal{M}_T$ extends to a bounded operator on $T^{p,2}_\beta$.

Proof. In this proof we are going to use dyadic annuli which are defined as follows. For $x \in \mathbb{R}^n$ and $t > 0$, we introduce $C_0(x,r) = B(x,t)$, and $C_j(x,r) = B(x,2^j t)\backslash B(x,2^{j-1}t)$, for $j \geq 1$, where $B(x,t) = \{y \in \mathbb{R}^n : |x-y| < t\}$, the ball of radius t with center x.

Let $f \in \mathscr{D}$. Using Minkowski inequality we can write $\|\mathcal{M}_T f\|_{T^{p,2}_\beta} \leq \sum\limits_{k=1}^{\infty}\sum\limits_{j=0}^{\infty} I_{k,j} + \sum\limits_{j=0}^{\infty} J_j$ where

$$I_{k,j} = \left(\int_{\mathbb{R}^n}\Big(\int_0^\infty \fint_{B(x,\sqrt{t})}\Big|\int_{2^{-k-1}t}^{2^{-1}t} T\mathrm{e}^{-(t-s)T}(\chi_{C_j(x,4\sqrt{t})}f(s,\cdot))(y)\mathrm{d}s\Big|^2\mathrm{d}y\frac{\mathrm{d}t}{t^\beta}\Big)^{\frac{p}{2}}\mathrm{d}x\right)^{\frac{1}{p}},$$

$$J_j = \left(\int_{\mathbb{R}^n}\Big(\int_0^\infty \fint_{B(x,\sqrt{t})}\Big|\int_{\frac{t}{2}}^{t} T\mathrm{e}^{-(t-s)T}(\chi_{C_j(x,4\sqrt{s})}f(s,\cdot))(y)\mathrm{d}s\Big|^2\mathrm{d}y\frac{\mathrm{d}t}{t^\beta}\right)^{\frac{p}{2}}\mathrm{d}x\Big)^{\frac{1}{p}}.$$

We fix $j \geq 0$ and $k \geq 1$. Then we have for fixed $x \in \mathbb{R}^n$

$$\begin{aligned}
&\int_0^\infty \fint_{B(x,\sqrt{t})}\Big|\int_{2^{-k-1}t}^{2^{-1}t} T\mathrm{e}^{-(t-s)T}(\chi_{C_j(x,4\sqrt{t})}f(s,\cdot))(y)\mathrm{d}s\Big|^2\mathrm{d}y\frac{\mathrm{d}t}{t^\beta}\\
&\simeq \int_0^\infty \int_{B(x,\sqrt{t})}\Big|\int_{2^{-k-1}t}^{2^{-1}t}(t-s)T\mathrm{e}^{-(t-s)T}(\chi_{C_j(x,4\sqrt{t})}f(s,\cdot))(y)\frac{\mathrm{d}s}{t-s}\Big|^2\mathrm{d}y\frac{\mathrm{d}t}{t^{\beta+\frac{n}{2}}}\\
&= \int_0^\infty \int_{B(x,\sqrt{t})}\Big|\int_{2^{-k-1}t}^{2^{-1}t}(t-s)T\mathrm{e}^{-(t-s)T}(\chi_{C_j(x,4\sqrt{t})}f(s,\cdot))(y)\mathrm{d}s\Big|^2\mathrm{d}y\frac{\mathrm{d}t}{t^{\beta+\frac{n}{2}+2}}\\
&\lesssim \int_0^\infty \int_{B(x,\sqrt{t})} 2^{-k}t\int_{2^{-k-1}t}^{2^{-1}t}|(t-s)T\mathrm{e}^{-(t-s)T}(\chi_{C_j(x,4\sqrt{t})}f(s,\cdot))(y)|^2\mathrm{d}s\mathrm{d}y\frac{\mathrm{d}t}{t^{\beta+\frac{n}{2}+2}}\\
&= \int_0^\infty \int_{2^{-k-1}t}^{2^{-1}t} 2^{-k}t\int_{B(x,\sqrt{t})}|(t-s)T\mathrm{e}^{-(t-s)T}(\chi_{C_j(x,4\sqrt{t})}f(s,\cdot))(y)|^2\mathrm{d}y\frac{\mathrm{d}s\mathrm{d}t}{t^{\beta+\frac{n}{2}+2}}\\
&\lesssim \int_0^\infty \int_{2^{-k-1}t}^{2^{-1}t} 2^{-k}t\left(1+\frac{2^{j2t}}{t-s}\right)^{-2M}\|\chi_{B(x,2^{j+2}\sqrt{t})}f(s,\cdot))(y)\|_2^2\frac{\mathrm{d}s\mathrm{d}t}{t^{\beta+\frac{n}{2}+2}}\\
&\lesssim 2^{-k}2^{-4jM}\int_0^\infty\left(\int_{2^k s}^{2^{k+1}s}\frac{\mathrm{d}t}{t^{\beta+\frac{n}{2}+1}}\right)\|\chi_{B(x,2^{j+3+\frac{k}{2}}\sqrt{s})}f(s,\cdot))(y)\|_2^2\mathrm{d}s\\
&\lesssim 2^{-k(\frac{n}{2}+1+\beta)}2^{-4jM}\int_0^\infty \|\chi_{B(x,2^{j+3+\frac{k}{2}}\sqrt{s})}f(s,\cdot))(y)\|_2^2\frac{\mathrm{d}s}{s^{\beta+\frac{n}{2}}}
\end{aligned}$$

$$\simeq 2^{-k(\frac{n}{2}+1+\beta)}2^{-4jM}\int_0^\infty \fint_{B(x,2^{j+3+\frac{k}{2}}\sqrt{s})} |f(\cdot,y)(s)|^2 \mathrm{d}y\frac{\mathrm{d}s}{s^\beta}.$$

In the first equality we used that $t-s$ and t are similar in size for $s\in\cup_{k\geq 1}[2^{-k-1}t,2^{-k}t]\subseteq[0,\frac{t}{2}]$. For the first inequality we used the Cauchy-Schwarz inequality for the integral with respect to y. Afterwards we used Fubini to switch the integral. In the second inequality we used (5.3) and in the subsequent steps, again we used that $t-s$ and t are similar in size and Fubini. Using these calculations together with Remark 3.4 we thus find

$$I_{k,j}\lesssim (j+k)2^{-k(\frac{1}{2}(\frac{n}{2}+1+\beta)-\frac{n}{2\tau})}2^{-j(2M-\frac{n}{\tau})}\|f\|_{T^{p,2}_\beta},$$

where $\tau=\min\{p,2\}$. Since we have $M>\frac{n}{2\tau}$ and $\frac{n}{2}+1+\beta>\frac{2n}{2\tau}$ it follows that

$$\sum_{k=1}^\infty\sum_{j=0}^\infty \lesssim \|f\|_{T^{p,2}_\beta}.$$

Note that our restriction on β is required for $\frac{n}{2}+1+\beta>\frac{2n}{2\tau}$ to be true for $p\geq 2$.
Now we estimate J_0. We have $J_0\leq(\int_{\mathbb{R}^n}\tilde{J}_0(x)^{\frac{p}{2}}\mathrm{d}x)^{\frac{1}{p}}$ where

$$\tilde{J}_0(x)=\int_0^\infty\int_{\mathbb{R}^n}\Big|\int_{\frac{t}{2}}^t T\mathrm{e}^{-(t-s)T}g(s,\cdot)(y)\mathrm{d}s\Big|^2\frac{\mathrm{d}y\mathrm{d}t}{t^{\beta+\frac{n}{2}}},$$

with $g(s,y)=\chi_{B(x,4\sqrt{s})}(y)f(s,y)$. We have

$$\begin{aligned}\int_{\frac{t}{2}}^t T\mathrm{e}^{-(t-s)T}g(s,\cdot)(y)\mathrm{d}s &= \int_0^t T\mathrm{e}^{-(t-s)T}g(s,\cdot)(y)\mathrm{d}s-\int_0^{\frac{t}{2}} T\mathrm{e}^{-(t-s)T}g(s,\cdot)(y)\mathrm{d}s\\ &= \int_0^t T\mathrm{e}^{-(t-s)T}g(s,\cdot)(y)\mathrm{d}s-\mathrm{e}^{-\frac{t}{2}T}\int_0^{\frac{t}{2}} T\mathrm{e}^{-(\frac{t}{2}-s)T}g(s,\cdot)(y)\mathrm{d}s.\end{aligned}$$

So we can rewrite the inside integral of $\tilde{J}_0(x)$ as

$$\mathcal{M}_T g(t,\cdot)-\mathrm{e}^{-\frac{t}{2}T}\mathcal{M}_T g(\frac{t}{2},\cdot).$$

Since $\beta+\frac{n}{2}\in(-1,\infty)$ we have by [4] that $\mathcal{M}_T$ is bounded on $L^2(\mathbb{R}_+\times\mathbb{R}^n;t^{-(\beta+\frac{n}{2})}\mathrm{d}y\mathrm{d}t)$. We also have that $(\mathrm{e}^{-tT})_{t\geq 0}$ is uniformly bounded on $L^2(\mathbb{R}^n)$. We thus get, in combination with Fubini's Theorem

$$\tilde{J}_0\lesssim\int_0^\infty\|\chi_{B(x,4\sqrt{s})}f(s,\cdot)\|_2^2\frac{\mathrm{d}s}{s^{\beta+\frac{n}{2}}}.$$

As for J_j with $j\geq 1$ we have for fixed $x\in\mathbb{R}^n$

$$\begin{aligned}&\int_0^\infty\fint_{B(x,\sqrt{t})}\Big|\int_{\frac{t}{2}}^t T\mathrm{e}^{-(t-s)T}(\chi_{C_j(x,4\sqrt{s})}f(s,\cdot))(y)\mathrm{d}s\Big|^2\mathrm{d}y\frac{\mathrm{d}t}{t^\beta}\\ &\simeq\int_0^\infty\int_{B(x,\sqrt{t})}\Big|\int_{\frac{t}{2}}^t (t-s)T\mathrm{e}^{-(t-s)T}(\chi_{C_j(x,4\sqrt{s})}f(s,\cdot))(y)\frac{\mathrm{d}s}{t-s}\Big|^2\mathrm{d}y\frac{\mathrm{d}t}{t^{\beta+\frac{n}{2}}}\\ &\lesssim\int_0^\infty\int_{B(x,\sqrt{t})}\int_{\frac{t}{2}}^t |(t-s)T\mathrm{e}^{-(t-s)T}(\chi_{C_j(x,4\sqrt{s})}f(s,\cdot))(y)|^2\frac{\mathrm{d}s}{(t-s)^2}\mathrm{d}y\frac{\mathrm{d}t}{t^{\beta+\frac{n}{2}-1}}\\ &\lesssim\int_0^\infty\int_{\frac{t}{2}}^t (t-s)^{-2}\Big(1+\frac{2^{jm}t}{t-s}\Big)^{-2M}\|\chi_{B(x,2^{j+2}\sqrt{s})}f(s,\cdot)\|_2^2\frac{\mathrm{d}s}{s^{\beta+\frac{n}{2}-1}}\mathrm{d}t\end{aligned}$$

$$\lesssim 2^{-2j(2M-2)} \int_0^\infty \left(\int_s^{2s} s(t-s)^{-2} \Big(1 + \frac{2^{jm}t}{t-s}\Big)^{-2} \mathrm{d}t \right) \|\chi_{B(x,2^{j+2}\sqrt{s})} f(s,\cdot)\|_2^2 \frac{\mathrm{d}s}{s^{\beta+\frac{n}{2}}}$$
$$\lesssim 2^{-4jM} \int_0^\infty \|\chi_{B(x,2^{j+2}\sqrt{s})} f(s,\cdot)\|_2^2 \frac{\mathrm{d}s}{s^{\beta+\frac{n}{2}}}$$
$$\simeq 2^{-4jM} \int_0^\infty ⨍_{B(x,2^{j+2}\sqrt{s})} |f(\cdot,y)(s)|^2 \mathrm{d}y \frac{\mathrm{d}s}{s^\beta}$$

In the first inequality we used Cauchy Schwarz inequality. In the second we used Fubini and then (5.3). In the next step we again used Fubini and that t and s are similar in size. In the last inequality we used the change of variables $v = \frac{t}{t-s}$. Using a change of angle we thus find

$$J_j \lesssim 2^{-2jM} j 2^{j\frac{n}{\tau}} \|f\|_{T^{p,2}_\beta} = j 2^{-j(2M-\frac{n}{\tau})} \|f\|_{T^{p,2}_\beta}.$$

Now summing up all the estimates provides us with the desired result. □

Remark 5.3. *In Section 4 we have seen that the operator $L = -divA\nabla$ is a special case of a densely defined linear operator acting on $L^2(\mathbb{R}^n)$ that generates a bounded analytic semigroup $(\mathrm{e}^{-tL})_{t\geq 0}$. Furthermore, by Proposition 4.8 we also get $(tL\mathrm{e}^{-tL})_{t\geq 0}$ satisfies (5.3) for any M. So by the above theorem we get that $\mathcal{M}_L$ extends to a bounded operator on $T^{p,2}_\beta$ for $\beta \in (-1,\infty)$ and $p \in (\frac{2n}{n+2(1+\beta)}, \infty)$.*

We are also going to use a variation of $\mathcal{M}_L$, namely $\tilde{\mathcal{M}}_L$ defined by

$$\tilde{\mathcal{M}}_L f(t,\cdot) = \int_0^t \nabla \mathrm{e}^{-(t-s)L} \mathrm{div} f(s,\cdot) \mathrm{d}s. \tag{5.4}$$

Proposition 5.4. *$\tilde{\mathcal{M}}_L$ is well defined as a bounded operator from $L^1(H^2)$ to $L^\infty_{loc}(L^2)$, where $H^2 = H^2(\mathbb{R}^n;\mathbb{C}^n)$, and extends to a bounded operator on $L^2(L^2)$.*

Proof. By using (4.2), we have, for all $\tau > 0$ and $g \in H^2$,

$$\|\nabla \mathrm{e}^{-\tau L} \mathrm{div}\, g\|_{L^2} \lesssim \|\nabla \mathrm{div}\, g\|_{L^2} \lesssim \|g\|_{H^2},$$

which shows that the operator is indeed well defined.
Now for the extension to $L^2(L^2)$. By Remark 4.6 we know that the adjoint of a divergence form elliptic operator is again a divergence form elliptic operator. Let $L' = -\mathrm{div}\, A^*\nabla$. Then $(L')^* = L$. We thus know, by the solution of the Kato square root problem [5], that for $g \in L^2(\mathbb{R}^n;\mathbb{C}^n)$ we have $\|\nabla(L')^{-\frac{1}{2}} g\|_{L^2} \lesssim \|g\|_{L^2}$. Hence, $\nabla(L')^{-\frac{1}{2}} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n;\mathbb{C}^n)$ is a bounded operator. Using $(\nabla(L')^{-\frac{1}{2}})^* = -((L')^*)^{-\frac{1}{2}}\mathrm{div} = -L^{-\frac{1}{2}}\mathrm{div}$, we find that $h = L^{-\frac{1}{2}}\mathrm{div}\, g \in L^2(\mathbb{R}^n)$. Furthermore, if $g \in H^2$, then $\mathrm{div}\, g \in H^1 = D(L^{\frac{1}{2}})$ by [5]. Therefore, $Lh = L^{\frac{1}{2}}\mathrm{div}\, g \in L^2(\mathbb{R}^n)$. Hence $h \in D(L)$. Since $\nabla L^{-\frac{1}{2}}$ is L^2 bounded, [5], we get for all $g \in H^2$ and $\tau > 0$

$$\nabla \mathrm{e}^{-\tau L} \mathrm{div}\, g = \nabla L^{-\frac{1}{2}} L \mathrm{e}^{-\tau L} L^{-\frac{1}{2}} \mathrm{div}\, g \quad \text{in } L^2.$$

From this we get for all $f \in L^1(H^2)$

$$\tilde{\mathcal{M}}_L f = \nabla L^{-\frac{1}{2}} \mathcal{M}_L L^{-\frac{1}{2}} \mathrm{div}\, f.$$

Since $\mathcal{M}_L$ is bounded on $L^2(L^2)$, and $\mathscr{D} \subseteq L^1(H^2)$, we find, by density, that $\tilde{\mathcal{M}}_L$ extends to a bounded operator on $L^2(L^2)$. □

We also have that the adjoint of $\tilde{\mathcal{M}}_L$ extends boundedly to $L^2(L^2)$:

Lemma 5.5. *The adjoint of $\tilde{\mathcal{M}}_L$, initially defined for $g \in \mathscr{D}$, is given by*

$$\tilde{\mathcal{M}}_L^* g(s,x) = \int_0^\infty \nabla(\mathrm{e}^{-\sigma L})^* \, div\, g(\sigma+s,\cdot)(x)\, d\sigma, \quad (s,x) \in (0,\infty)\times\mathbb{R}^n$$

and extends to a bounded operator on $L^2(L^2)$.

Proof. Let $f, g \in \mathscr{D}$. Then

$$\begin{aligned}
\int_{\mathbb{R}} \langle \tilde{\mathcal{M}}_L f(t,\cdot), g(t,\cdot)\rangle \mathrm{d}t &= \int_{\mathbb{R}}\int_{\mathbb{R}} \chi_{(0,\infty)}(t-s)\langle \nabla \mathrm{e}^{-(t-s)L}\mathrm{div}\, f(s,\cdot), g(t,\cdot)\rangle \mathrm{d}s\mathrm{d}t \\
&\overset{\sigma=t-s}{=} \int_{\mathbb{R}}\int_{\mathbb{R}} \chi_{(0,\infty)}(\sigma)\langle \nabla \mathrm{e}^{-\sigma L}\mathrm{div}\, f(s,\cdot), g(\sigma+s,\cdot)\rangle \mathrm{d}s\mathrm{d}\sigma \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}} \chi_{(0,\infty)}(\sigma)\langle f(s,\cdot), \nabla(\mathrm{e}^{-\sigma L})^*\mathrm{div}\, g(\sigma+s,\cdot)\rangle \mathrm{d}\sigma\mathrm{d}s \\
&= \int_{\mathbb{R}} \left\langle f(s,\cdot), \int_0^\infty \nabla(\mathrm{e}^{-\sigma L})^*\mathrm{div}\, g(\sigma+s,\cdot)\mathrm{d}\sigma \right\rangle \mathrm{d}s.
\end{aligned}$$

By density of $\mathscr{D}$ we get the extension to $L^2(L^2)$. □

Proposition 5.6. *Let $\beta \in (-1,\infty)$ Then $\tilde{\mathcal{M}}_L$ extends to a bounded operator on $T^{p,2}_\beta$ for $p \in (\frac{2n}{n+2(1+\beta)},\infty)\cap(1,\infty)$.*

Proof. As in the proof of Proposition 5.4, we have the following L^2 equality

$$\nabla \mathrm{e}^{-\tau L}\mathrm{div}\, g = \nabla L^{-\frac{1}{2}} L\mathrm{e}^{-\tau L} L^{-\frac{1}{2}}\mathrm{div}\, g.$$

From this we get for all $f \in L^1(t^{-\beta}\mathrm{d}t; H^2)$

$$\tilde{\mathcal{M}}_L f = \nabla L^{-\frac{1}{2}}\mathcal{M}_L L^{-\frac{1}{2}}\mathrm{div}\, f.$$

Since $\mathcal{M}_L$ is bounded on $L^2(t^{-\beta}\mathrm{d}t; L^2)$ for $\beta \in (-1,\infty)$, see [4], and $\mathscr{D} \subseteq L^1(H^2)$, we find, by density, that $\tilde{\mathcal{M}}_L$ extends to a bounded operator on $L^2(t^{-\beta}\mathrm{d}t; L^2)$.
Now we consider the following family of operators: $(t\nabla\mathrm{e}^{-tL}\mathrm{div})_{t>0}$. We have for any $t>0$

$$t\nabla\mathrm{e}^{-tL}\mathrm{div} = \sqrt{t}\nabla\mathrm{e}^{-\frac{t}{2}L}\mathrm{e}^{-\frac{t}{2}L}\sqrt{t}\mathrm{div}.$$

By Proposition 4.8 and Remark 4.9 we know that $(\sqrt{t}\nabla\mathrm{e}^{-\frac{t}{2}L})_{t>0}$ and $(\mathrm{e}^{-\frac{t}{2}L}\sqrt{t}\mathrm{div})_{t>0}$ satisfy L^2 off-diagonal estimates. Hence, by Proposition 4.10, $(t\nabla\mathrm{e}^{-tL}\mathrm{div})_{t>0}$ satisfies L^2 off-diagonal estimates as well. Since satisfying L^2 off-diagonal estimates implies that (5.3) is satisfied for any M, we can now repeat the proof of Theorem 5.2 to obtain the desired result. □

Remark 5.7. *For $\beta = 0$, the above proposition holds for a bigger range of p: Let $q \in [1,2)$ be such that $\sup_{t>0}\|\sqrt{t}\nabla\mathrm{e}^{-tL^*}\|_{\mathscr{L}(L^s)} < \infty$ for all $s \in [2,q')$. In that case, $\tilde{\mathcal{M}}_L$ extends to a bounded operator on $T^{p,2}$ for all $p \in (p_c,\infty]$ with $p_c = \max\{\frac{nq}{n+q},\frac{2n}{n+q'}\}$. For the proof we refer to [6, Proposition 2.8].*

We will also need to following integral operator

$$\mathcal{R}_L f(t,x) := \int_0^t \mathrm{e}^{-(t-s)L}\mathrm{div} f(s,\cdot)(x)\mathrm{d}s, \tag{5.5}$$

which is defined as a bounded operator from $L^1(H^1)$, with $H^1 = H^1(\mathbb{R}^n;\mathbb{C}^n)$, to $L^\infty_{\mathrm{loc}}(L^2)$.

Proposition 5.8. *Let $p \in (0,\infty)$. The operator $\mathcal{R}_L$ extends to a bounded operator from $T^{p,2}$ to X^p.*

Proof. In this proof we are going to use dyadic annuli which are defined as follows. For $x \in \mathbb{R}^n$ and $r > 0$, we introduce $S_1(x,r) = B(x,2r)$, and $S_j(x,r) = B(x,2^{j+1}r)\setminus B(x,2^j r)$, for $j \geq 2$, where $B(x,r) = \{y \in \mathbb{R}^n : |x-y| < r\}$, the ball of radius r with center x.

We note that it is enough to show that $\|\mathcal{R}_L f\|_{X^p} \lesssim \|f\|_{T^{p,2}}$ for $f \in \mathscr{C}_c(\mathbb{R}^{n+1}_+;\mathbb{C}^n)$, since $\mathscr{C}_c(\mathbb{R}^{n+1}_+;\mathbb{C}^n)$ is contained in $L^1(H^1)$ and is dense in $T^{p,2}$ of $\mathbb{C}^n$-valued functions.
Now let $f \in \mathscr{C}_c(\mathbb{R}^{n+1}_+;\mathbb{C}^n)$. For almost all $(t,x) \in \mathbb{R}^{n+1}_+$ we have

$$\begin{aligned}
\mathcal{R}_L f &= \int_0^t \mathrm{e}^{-(t-s)L}\mathrm{div} f(s,\cdot)(x)\mathrm{d}s \\
&= \sum_{k=0}^{\infty} \int_{\frac{2^{-k}t}{2}}^{2^{-k}t} \mathrm{e}^{-(t-s)L}\mathrm{div} f(s,\cdot)(x)\mathrm{d}s \\
&= \sum_{k=0}^{\infty} \mathrm{e}^{-(1-2^{-k})tL} \int_{\frac{2^{-k}t}{2}}^{2^{-k}t} \mathrm{e}^{-(2^{-k}t-s)L}\mathrm{div} f(s,\cdot)(x)\mathrm{d}s \\
&= \sum_{k=0}^{\infty} \mathrm{e}^{-(1-2^{-k})tL} K_L f(2^{-k}t,x),
\end{aligned}$$

where $K_L f(t,x) = \int_{\frac{t}{2}}^{t} \mathrm{e}^{-(t-s)L}\mathrm{div} f(s,\cdot)(x)ds$. Fixing $x \in \mathbb{R}^n$ and $k > 0$, we get for $\delta > 0$

$$\begin{aligned}
&\left(\fint_{\frac{\delta}{2}}^{\delta} \fint_{B(x,\sqrt{\delta})} |\mathrm{e}^{-(1-2^{-k})tL} K_L f(2^{-k}t,y)|^2 \mathrm{d}y\mathrm{d}t \right)^{\frac{1}{2}} \\
&\quad \leq \sum_{j=1}^{\infty} \left(\fint_{\frac{\delta}{2}}^{\delta} \fint_{B(x,\sqrt{\delta})} |\mathrm{e}^{-(1-2^{-k})tL} (\chi_{S_j(x,\sqrt{\delta})} K_L f(2^{-k}t,\cdot))(y)|^2 \mathrm{d}y\mathrm{d}t \right)^{\frac{1}{2}} \\
&\quad \lesssim \sum_{j=1}^{\infty} 2^{(j+1)\frac{n}{2}} \left(\fint_{\frac{\delta}{2}}^{\delta} \mathrm{e}^{-c_1 \frac{4^j\delta}{(1-2^{-k})t}} \fint_{B(x,2^{j+1}\sqrt{\delta})} |K_L f(2^{-k}t,y)|^2 \mathrm{d}y\mathrm{d}t \right)^{\frac{1}{2}} \\
&\quad \lesssim \sum_{j=1}^{\infty} 2^{j\frac{n}{2}} \mathrm{e}^{-c4^j} \left(\fint_{\frac{\delta}{2}}^{\delta} \fint_{B(x,2^{j+1}\sqrt{\delta})} |K_L f(2^{-k}t,y)|^2 \mathrm{d}y\mathrm{d}t \right)^{\frac{1}{2}} \\
&\quad \leq \sum_{j=1}^{\infty} 2^{j\frac{n}{2}} \mathrm{e}^{-c4^j} \sup_{\delta'>0} \left(\fint_{\frac{2^k\delta'}{2}}^{2^k\delta'} \fint_{B(x,2^{j+1+\frac{k}{2}}\sqrt{\delta'})} |K_L f(2^{-k}t,y)|^2 \mathrm{d}y\mathrm{d}t \right)^{\frac{1}{2}} \\
&\quad = \sum_{j=1}^{\infty} 2^{j\frac{n}{2}} \mathrm{e}^{-c4^j} \sup_{\delta'>0} \left(\fint_{\frac{\delta'}{2}}^{\delta'} \fint_{B(x,2^{j+1+\frac{k}{2}}\sqrt{\delta'})} |K_L f(t,y)|^2 \mathrm{d}y\mathrm{d}t \right)^{\frac{1}{2}},
\end{aligned}$$

where in the first inequality we used Minkowski inequality to move the sum out of the integral. In the second inequality we used the L^2 off-diagonal estimates for $(\mathrm{e}^{-tL})_{t\geq 0}$. Note that for $k = 0$, we have, for any $j \geq 0$,

$$\left(\fint_{\frac{\delta}{2}}^{\delta} \fint_{B(x,\sqrt{\delta})} |K_L f(t,y)|^2 \mathrm{d}y\mathrm{d}t \right)^{\frac{1}{2}} \lesssim 2^{j\frac{n}{2}} \left(\fint_{\frac{\delta}{2}}^{\delta} \fint_{B(x,2^{j+1}\sqrt{\delta})} |K_L f(t,y)|^2 \mathrm{d}y\mathrm{d}t \right)^{\frac{1}{2}}$$

$$\leq \quad 2^{j\frac{n}{2}} \sup_{\delta'>0} \left(\fint_{\frac{\delta'}{2}}^{\delta'} \fint_{B(x,2^{j+1}\sqrt{\delta'})} |K_L f(t,y)|^2 \mathrm{d}y \mathrm{d}t \right)^{\frac{1}{2}}$$

Looking at the part in the supremum, we have for $\delta' > 0$ and $j > 0$

$$\left(\fint_{\frac{\delta'}{2}}^{\delta'} \fint_{B(x,2^{j+1+\frac{k}{2}}\sqrt{\delta'})} |K_L f(t,y)|^2 \mathrm{d}y \mathrm{d}t \right)^{\frac{1}{2}}$$
$$\leq \sum_{l=1}^{\infty} \left(\fint_{\frac{\delta'}{2}}^{\delta'} \fint_{B(x,2^{j+1+\frac{k}{2}}\sqrt{\delta'})} \Big| \int_{\frac{t}{2}}^{t} \mathrm{e}^{-(t-s)L} \mathrm{div}(\chi_{S_l(x,2^{j+1+\frac{k}{2}}\sqrt{\delta'})} f(s,\cdot))(y) \mathrm{d}s \Big|^2 \mathrm{d}y \mathrm{d}t \right)^{\frac{1}{2}}.$$

Again, we used Minkowski and the boundedness of the operator to take out the sum. Now let $l = 1$, and $t \in (\frac{\delta'}{2}, \delta')$. Then we have, using Remark 4.9

$$\begin{aligned} \Big\| \int_{\frac{t}{2}}^{t} \mathrm{e}^{-(t-s)L} \mathrm{div}(\chi_{S_1(x,2^{j+1+\frac{k}{2}}\sqrt{\delta'})} f(s,\cdot))(y) \mathrm{d}s \Big\|_2 &\leq \int_{\frac{t}{2}}^{t} \|\mathrm{e}^{-(t-s)L} \mathrm{div}(\chi_{S_1(x,2^{j+1+\frac{k}{2}}\sqrt{\delta'})} f(s,\cdot))\|_2 \mathrm{d}s \\ &\leq \int_{\frac{t}{2}}^{t} \frac{1}{\sqrt{t-s}} \|\chi_{B(x,2^{j+2+\frac{k}{2}}\sqrt{\delta'})} f(s,\cdot)\|_2 \mathrm{d}s. \end{aligned}$$

From this we find

$$\begin{aligned} &\left(\fint_{\frac{\delta'}{2}}^{\delta'} \fint_{B(x,2^{j+1+\frac{k}{2}}\sqrt{\delta'})} \Big| \int_{\frac{t}{2}}^{t} \mathrm{e}^{-(t-s)L} \mathrm{div}(\chi_{S_1(x,2^{j+1+\frac{k}{2}}\sqrt{\delta'})} f(s,\cdot))(y) \mathrm{d}s \Big|^2 \mathrm{d}y \mathrm{d}t \right)^{\frac{1}{2}} \\ &\lesssim \left(\frac{1}{\delta'} \int_{\frac{\delta'}{2}}^{\delta'} (2^{j+1+\frac{k}{2}}\sqrt{\delta'})^{-n} \int_{\mathbb{R}^n} \Big| \int_{\frac{t}{2}}^{t} \mathrm{e}^{-(t-s)L} \mathrm{div}(\chi_{S_1(x,2^{j+1+\frac{k}{2}}\sqrt{\delta'})} f(s,\cdot))(y) \mathrm{d}s \Big|^2 \mathrm{d}y \mathrm{d}t \right)^{\frac{1}{2}} \\ &= \left(\frac{1}{\delta'} \int_{\frac{\delta'}{2}}^{\delta'} (2^{j+1+\frac{k}{2}}\sqrt{\delta'})^{-n} \Big\| \int_{\frac{t}{2}}^{t} \mathrm{e}^{-(t-s)L} \mathrm{div}(\chi_{S_1(x,2^{j+1+\frac{k}{2}}\sqrt{\delta'})} f(s,\cdot))(y) \mathrm{d}s \Big\|_2^2 \mathrm{d}t \right)^{\frac{1}{2}} \\ &\leq \left(\int_{\frac{\delta'}{2}}^{\delta'} \left(\int_{\frac{t}{2}}^{t} \frac{1}{\sqrt{\delta'}} \frac{1}{\sqrt{t-s}} \|(2^{j+1+\frac{k}{2}}\sqrt{\delta'})^{-\frac{n}{2}} \chi_{B(x,2^{j+2+\frac{k}{2}}\sqrt{\delta'})} f(s,\cdot)\|_2 \mathrm{d}s \right)^2 \mathrm{d}t \right)^{\frac{1}{2}} \\ &= \|TF(t,\cdot)\|_{L^2(\frac{\delta'}{2},\delta')}, \end{aligned}$$

where T is the integral operator defined as

$$TF(t,\cdot) := \int_0^{\infty} \chi_{(\frac{t}{2},t)}(s) \frac{1}{\sqrt{\delta'}} \frac{1}{\sqrt{t-s}} F(s,\cdot) \mathrm{d}s$$

and

$$F(s,\cdot) := \|(2^{j+1+\frac{k}{2}}\sqrt{\delta'})^{-\frac{n}{2}} \chi_{B(x,2^{j+2+\frac{k}{2}}\sqrt{\delta'})} f(s,\cdot)\|_2.$$

Using that $t \in (\frac{\delta'}{2}, \delta')$ and $s \in (\frac{t}{2}, t)$, we find that

$$\begin{aligned} \sup_{s\in(\frac{\delta'}{4},\delta')} \int_{\frac{\delta'}{2}}^{\delta'} \chi_{(\frac{t}{2},t)}(s) \frac{1}{\sqrt{\delta'}} \frac{1}{\sqrt{t-s}} \mathrm{d}t &= \sup_{s\in(\frac{\delta'}{4},\delta')} \int_{\frac{\delta'}{2}}^{\delta'} \chi_{(s,2s)}(t) \frac{1}{\sqrt{\delta'}} \frac{1}{\sqrt{t-s}} \mathrm{d}t \\ &= \sup_{s\in(\frac{\delta'}{4},\delta')} \int_{\frac{\delta'}{2}\vee s}^{\delta'\wedge 2s} \frac{1}{\sqrt{\delta'}} \frac{1}{\sqrt{t-s}} \mathrm{d}t = C < \infty, \end{aligned}$$

and

$$\sup_{t\in(\frac{\delta'}{2},\delta')}\int_{\frac{t}{2}}^{t}\frac{1}{\sqrt{\delta'}}\frac{1}{\sqrt{t-s}}\mathrm{d}s<C<\infty.$$

We thus have, by Schur's lemma [17, Appendix I], $\|TF(t,\cdot)\|_{L^2(\frac{\delta'}{2},\delta')}\lesssim\|F(t,\cdot)\|_{L^2(\frac{\delta'}{2},\delta')}$, and hence

$$\left(\fint_{\frac{\delta}{2}}^{\delta}\fint_{B(x,2^{j+1+\frac{k}{2}}\sqrt{\delta'})}\Big|\int_{\frac{t}{2}}^{t}\mathrm{e}^{-(t-s)L}\mathrm{div}(\chi_{S_1(x,2^{j+1+\frac{k}{2}}\sqrt{\delta'})}f(s,\cdot))(y)\mathrm{d}s\Big|^2\mathrm{d}y\mathrm{d}t\right)^{\frac{1}{2}}$$
$$\lesssim\left(\int_{\frac{\delta'}{2}}^{\delta'}\fint_{B(x,2^{j+2+\frac{k}{2}}\sqrt{\delta'})}|f(t,y)|^2\mathrm{d}y\mathrm{d}t\right)^{\frac{1}{2}}\leq\left(\int_0^{\infty}\fint_{B(x,2^{j+3+\frac{k}{2}}\sqrt{t})}|f(t,y)|^2\mathrm{d}y\mathrm{d}t\right)^{\frac{1}{2}}$$

For $l\geq 2$ we have

$$\left(\fint_{\frac{\delta'}{2}}^{\delta'}\fint_{B(x,2^{j+1+\frac{k}{2}}\sqrt{\delta'})}\Big|\int_{\frac{t}{2}}^{t}\mathrm{e}^{-(t-s)L}\mathrm{div}(\chi_{S_l(x,2^{j+1+\frac{k}{2}}\sqrt{\delta'})}f(s,\cdot))(y)\mathrm{d}s\Big|^2\mathrm{d}y\mathrm{d}t\right)^{\frac{1}{2}}$$
$$\simeq\left(\fint_{\frac{\delta'}{2}}^{\delta'}\Big\|\int_{\frac{t}{2}}^{t}(2^{j+1+\frac{k}{2}}\sqrt{\delta'})^{-\frac{n}{2}}\chi_{B(x,2^{j+1+\frac{k}{2}}\sqrt{\delta'})}\mathrm{e}^{-(t-s)L}\mathrm{div}(\chi_{S_l(x,2^{j+1+\frac{k}{2}}\sqrt{\delta'})}f(s,\cdot))(y)\mathrm{d}s\Big\|^2\mathrm{d}t\right)^{\frac{1}{2}}$$
$$\leq\left(\int_{\frac{\delta'}{2}}^{\delta'}\left(\int_{\frac{t}{2}}^{t}\frac{1}{\sqrt{\delta'}}\Big\|(2^{j+1+\frac{k}{2}}\sqrt{\delta'})^{-\frac{n}{2}}\chi_{B(x,2^{j+1+\frac{k}{2}}\sqrt{\delta'})}\mathrm{e}^{-(t-s)L}\mathrm{div}(\chi_{S_l(x,2^{j+1+\frac{k}{2}}\sqrt{\delta'})}f(s,\cdot))\Big\|\mathrm{d}s\right)^2\mathrm{d}t\right)^{\frac{1}{2}}$$
$$\lesssim\left(\int_{\frac{\delta'}{2}}^{\delta'}\left(\int_{\frac{t}{2}}^{t}\frac{1}{\sqrt{\delta'}}\frac{1}{\sqrt{t-s}}2^{ln}\mathrm{e}^{-c\frac{4^{l+j}2^k\delta'}{t-s}}(\fint_{B(x,2^{j+l+2+\frac{k}{2}}\sqrt{\delta'})}|f(s,y)|^2\mathrm{d}y)^{\frac{1}{2}}\mathrm{d}s\right)^2\mathrm{d}t\right)^{\frac{1}{2}}$$

where in the second inequality we used the L^2-off diagonal estimates for $\mathrm{e}^{-tL}\mathrm{div}$.
Now define

$$T_2F(t,\cdot):=\int_0^{\infty}\chi_{(\frac{t}{2},t)}\frac{1}{\sqrt{\delta'}}\frac{1}{\sqrt{t-s}}\mathrm{e}^{-\frac{D\delta'}{t-s}}F(s,\cdot)\mathrm{d}s,$$

with

$$F(s,x)=\left(\fint_{B(x,2^{j+l+2+\frac{k}{2}}\sqrt{\delta'})}|f(s,y)|^2\mathrm{d}y\right)^{\frac{1}{2}}\ \text{and}\ D=c4^{l+j}2^k.$$

Using that $t\in(\frac{\delta'}{2},\delta')$ and $s\in(\frac{t}{2},t)$, we have $0\leq t-s\leq\delta'$. Also, $\mathrm{e}^{-\frac{D\delta'}{t-s}}\leq\min\{1,(\frac{t-s}{D\delta'})^N\}$ for some large $N\in\mathbb{N}$, which we will choose later. If $1\leq(\frac{t-s}{D\delta'})^N$, then we have the same integral as before and find

$$\sup_{s\in(\frac{\delta'}{4},\delta')}\int_{\frac{\delta'}{2}}^{\delta'}\chi_{(\frac{t}{2},t)}(s)\frac{1}{\sqrt{\delta'}}\frac{1}{\sqrt{t-s}}\mathrm{d}t<C\leq C\left(\frac{t-s}{D\delta'}\right)^N\leq C\left(\frac{\delta'}{D\delta'}\right)^N\lesssim D^{-N}.$$

For $(\frac{t-s}{D\delta'})^N<1$, we have

$$\sup_{s\in(\frac{\delta'}{4},\delta')}\int_{\frac{\delta'}{2}\vee s}^{\delta'\wedge 2s}\frac{1}{\sqrt{\delta'}}\frac{1}{\sqrt{t-s}}\left(\frac{t-s}{D\delta'}\right)^N\mathrm{d}t\ =\ \sup_{s\in(\frac{\delta'}{4},\delta')}\left[\frac{1}{N}\frac{1}{\sqrt{\delta'}}\sqrt{t-s}\left(\frac{t-s}{D\delta'}\right)^N\right]_{\frac{\delta'}{2}\vee s}^{\delta'\wedge 2s}$$
$$\lesssim\ D^{-N}.$$

We find similar estimates for the integration with respect to s. Thus, by Schur's lemma [17, Appendix I] we find $\|T_2F(\cdot,x)\|_{L^2(\frac{\delta'}{2},\delta')} \lesssim D^{-N}\|F(\cdot,x)\|_{L^2(\frac{\delta'}{2},\delta')}$, and hence

$$\begin{aligned}
&\left(\fint_{\frac{\delta'}{2}}^{\delta'}\fint_{B(x,2^{j+1+\frac{k}{2}}\sqrt{\delta'})}\Big|\int_{\frac{t}{2}}^{t}\mathrm{e}^{-(t-s)L}\mathrm{div}(\chi_{S_l(x,2^{j+1+\frac{k}{2}}\sqrt{\delta'})}f(s,\cdot))(y)\mathrm{d}s\Big|^2\mathrm{d}y\mathrm{d}t\right)^{\frac{1}{2}}\\
&\qquad\lesssim 2^{ln}(c4^{l+j}2^k)^{-N}\left(\fint_{\frac{\delta'}{2}}^{\delta'}\fint_{B(x,2^{j+l+2+\frac{k}{2}}\sqrt{\delta'})}|f(s,y)|^2\mathrm{d}y\mathrm{d}t\right)^{\frac{1}{2}}\\
&\qquad\leq 2^{ln}(c4^{l+j}2^k)^{-N}\left(\int_{0}^{\infty}\fint_{B(x,2^{j+l+2+\frac{k}{2}}\sqrt{t})}|f(s,y)|^2\mathrm{d}y\mathrm{d}t\right)^{\frac{1}{2}}
\end{aligned}$$

Using Remark 3.4 and summing over k,j,l we thus get

$$\begin{aligned}
\|\mathcal{R}_Lf\|_{X^p} &\lesssim \sum_{k,j,l}2^{(l+\frac{j}{2})n}\mathrm{e}^{-c4^j}(c_24^{l+j}2^k)^{-N}\left\|x\mapsto\left(\int_0^\infty\fint_{B(x,2^{j+2+l+\frac{k}{2}}\sqrt{s})}|f(s,y)|^2\mathrm{d}y\mathrm{d}s\right)^{\frac{1}{2}}\right\|_{L^p}\\
&\lesssim \sum_{k,j,l}2^{(l+\frac{j}{2})n}\mathrm{e}^{-c4^j}(c_24^{l+j}2^k)^{-N}2^{(j+l+\frac{k}{2})\tau}\|f\|_{T^{p,2}}\\
&\lesssim \|f\|_{T^{p,2}}.
\end{aligned}$$

with τ only depending on n and p and where we chose $N\geq n+\tau$. □

The general outline of the above proof will be used to prove our results in Chapters 10 and 11.

Proposition 5.9. *Let p be as in Remark 5.7. Then for $f\in T^{p,2}$ we have $\nabla\mathcal{R}_Lf\in T^{p,2}$ and $\nabla\mathcal{R}_Lf=\tilde{\mathcal{M}}_Lf$ in $T^{p,2}$.*

Proof. Let $f,g\in\mathscr{D}$. Then we have

$$\begin{aligned}
\int_{\mathbb{R}}\langle\tilde{\mathcal{M}}_Lf(t,\cdot),g(t,\cdot)\rangle\mathrm{d}t &= \int_{\mathbb{R}}\int_{\mathbb{R}}\chi_{(0,\infty)}(t-s)\langle\nabla\mathrm{e}^{-(t-s)L}\mathrm{div}\,f(s,\cdot),g(t,\cdot)\rangle\mathrm{d}s\mathrm{d}t\\
&= -\int_{\mathbb{R}}\int_{\mathbb{R}}\chi_{(0,\infty)}(t-s)\langle\mathrm{e}^{-(t-s)L}\mathrm{div}\,f(s,\cdot),\mathrm{div}\,g(t,\cdot)\rangle\mathrm{d}s\mathrm{d}t\\
&= -\int_{\mathbb{R}}\langle\mathcal{R}_Lf(s,\cdot),\mathrm{div}\,g(t,\cdot)\rangle\mathrm{d}t\\
&= \int_{\mathbb{R}}\langle\nabla\mathcal{R}_Lf(t,\cdot),g(t,\cdot)\rangle\mathrm{d}t,
\end{aligned}$$

where $\langle\cdot,\cdot\rangle$ is the L^2 inner product. So we have $\nabla\mathcal{R}_Lf=\tilde{\mathcal{M}}_Lf$ in $\mathscr{D}'$. By Remark 5.7, Proposition 5.8 and density, we get the desired result. □

The $T^{p,2}$ boundedness of the operators $\tilde{\mathcal{M}}_L$ and $\mathcal{R}_L$ will have a key role in the proof of Proposition 1.2, as will be shown in Chapter 7.

Chapter 6

Energy solutions in L^2

In this chapter we develop energy solutions for (1.3) with initial data in L^2. We first show that if the matrix A in (1.3) is dependent on both time and space then the solution can be given through propagators acting on the initial data. Afterwards, we show that if A is time independent, as in Chapter 4, then the solution can be given more explicitly using the semigroup of L and the operator $\mathcal{R}_L$ from the previous chapter.
So througout most of this chapter, the matrix A in (1.3) is in $L^\infty((0,\infty)\times\mathbb{R}^n, \mathscr{M}_n(\mathbb{C}))$ and satisfies the following uniform ellipticity estimates

$$\begin{aligned} \exists\Lambda>0: &\quad \forall\xi,\eta\in\mathbb{C}^n, |\langle A(t,x)\xi,\eta\rangle| \le \Lambda|\xi||\eta| \text{ for a.e. } t>0 \text{ and } x\in\mathbb{R}^n,\\ \exists\lambda>0: &\quad \forall\xi\in\mathbb{C}^n, \Re e(\langle A(t,x)\xi,\xi\rangle) \ge \lambda|\xi|^2 \text{ for a.e. } t>0 \text{ and } x\in\mathbb{R}^n, \end{aligned} \tag{6.1}$$

Now let us first observe that for the homogeneous Sobolev space $\dot{H}^1(\mathbb{R}^n)$, which was introduced in Chapter 4, we have $\dot{H}^1(\mathbb{R}^n)\subset\mathscr{S}'$. To see this note that for every $f\in\mathscr{S}$ there exists a $F\in L^2$ such that $f=\mathrm{div}F$ by the Hodge decomposition. So for every $u\in\dot{H}^1$ we have that $f\mapsto\langle u,f\rangle=\langle u,\mathrm{div}F\rangle=\langle-\nabla u,F\rangle$ and since we have $\nabla u, F\in L^2$, this mapping is continuous.

Next, we define a variant of the solution space used by Lions in [28] as follows

$$\dot{W}(0,\infty) := \{u\in\mathscr{D}' : u\in L^2(\dot{H}^1) \text{ and } \partial_t u\in L^2(\dot{H}^{-1})\}.$$

Lemma 6.1. *For all $u\in\dot{W}(0,\infty)$, there exists a unique $v\in\dot{W}(0,\infty)\cap\mathscr{C}_0(L^2(\mathbb{R}^n))$ and $c\in\mathbb{C}$ such that $u=v+c$. Moreover,*

$$\|v\|_{L^\infty(L^2)} \le \sqrt{2\|u\|_{L^2(\dot{H}^1)}\|\partial_t u\|_{L^2(\dot{H}^1)}}$$

Proof. Let $u\in\dot{W}(0,\infty)$ and set $w=\partial_t u+\Delta u$. Using $\Delta u=\mathrm{div}(\nabla u)$, we can find, by the properties of $\dot{W}(0,\infty)$, a $g\in L^2(L^2)$ such that $w=\mathrm{div}\,g$. By $\tau_t g$, we denote the time translation of g for a given $t\ge0$. It is defined by $\tau_t g(s,\cdot)=g(s+t,\cdot)$.
Now let $f\in L^2(\mathbb{R}^n)$ and $t\ge0$. Using that the Fourier multiplier of $\nabla\mathrm{e}^{s\Delta}$ is $\xi\mathrm{e}^{-s|\xi|^2}$, which is bounded by $\frac{1}{\sqrt{2}}$, we find, for every $\epsilon, R>0$, $\epsilon<R$,

$$\begin{aligned} |\langle\int_\epsilon^R \mathrm{e}^{s\Delta}\mathrm{div}\,(\tau_t g)(s)\mathrm{d}s, f\rangle| &\le \int_\epsilon^R |\langle\tau_t g(s),\nabla\mathrm{e}^{s\Delta}f\rangle|\mathrm{d}s\\ &\le \|\tau_t g\|_{L^2((\epsilon,R),L^2)}\|(s,x)\mapsto\nabla\mathrm{e}^{s\Delta}f(x)\|_{L^2((\epsilon,R),L^2)}\\ &\le \|\tau_t g\|_{L^2(L^2)}\|(s,x)\mapsto\nabla\mathrm{e}^{s\Delta}f(x)\|_{L^2(L^2)}\\ &\le \frac{1}{\sqrt{2}}\|\tau_t g\|_{L^2(L^2)}\|f\|_{L^2}, \end{aligned}$$

where we used Cauchy-Schwarz twice in the first inequality. So by taking weak limits in L^2, we find

$$\int_0^\infty |\langle \mathrm{e}^{s\Delta}\mathrm{div}\,(\tau_t g)(s), f\rangle|\mathrm{d}s \leq \frac{1}{\sqrt{2}}\|\tau_t g\|_{L^2(L^2)}\|f\|_{L^2}.$$

Now we set, for all $t \geq 0$

$$v(t) = -\int_t^\infty \mathrm{e}^{(s-t)\Delta}w(s)\mathrm{d}s = -\int_0^\infty \mathrm{e}^{s\Delta}\mathrm{div}\,(\tau_t g)(s)\mathrm{d}s,$$

where the integral is weakly defined by the above argument. By the same argument, we get, using Riesz representation theorem, $\|v(t)\|_{L^2} \leq \frac{1}{\sqrt{2}}\|\tau_t g\|_{L^2(L^2)}$, for all $t \geq 0$. Similarly we find

$$\|v(t) - v(t')\|_{L^2} \leq \frac{1}{\sqrt{2}}\|\tau_t g - \tau_{t'} g\|_{L^2(L^2)} \quad \forall t, t' \geq 0,$$

which shows that $v \in \mathscr{C}([0,\infty); L^2)$. Since $\lim_{t\to\infty} \|\tau_t g\|_{L^2(L^2)} = 0$, we also get $\lim_{t\to\infty} \|v(t)\|_{L^2} = 0$.
Now, we are going to prove that $u-v$ is a constant. By Lemma 5.5 we know that $\tilde{\mathcal{M}}_L^*$ is bounded on $L^2(L^2)$. So we have

$$\|\nabla v\|_{L^2(L^2)} = \|\tilde{\mathcal{M}}_{-\Delta}^* g\|_{L^2(L^2)} \lesssim \|g\|_{L^2(L^2)}.$$

from this, $\|f\|_{\dot{H}^{-1}} = \inf\{\|g\|_{L^2} : f = \mathrm{div}\, g\}$ and $\Delta v = -\mathrm{div}\,\nabla v$ we thus find

$$\|\Delta v\|_{L^2(\dot{H}^{-1})} \leq \|\nabla v\|_{L^2(L^2)} \lesssim \|g\|_{L^2(L^2)}. \tag{6.2}$$

Now let $\phi \in \mathscr{D}$. Then we have

$$\begin{aligned}
\langle \partial_t v, \phi\rangle &= -\langle v, \partial_t\phi\rangle \\
&= \int_0^\infty \left\langle \int_t^\infty \mathrm{e}^{(s-t)\Delta}w(s)\mathrm{d}s, \partial_t\phi(t)\right\rangle \mathrm{d}t \\
&= \int_0^\infty \int_0^s \left\langle \mathrm{e}^{(s-t)\Delta}w(s), \partial_t\phi(t)\right\rangle \mathrm{d}t\mathrm{d}s \\
&= \int_0^\infty \int_0^s \left\langle w(s), \mathrm{e}^{(s-t)\Delta}\partial_t\phi(t)\right\rangle \mathrm{d}t\mathrm{d}s \\
&= \int_0^\infty \left\langle w(s), \int_0^s \mathrm{e}^{(s-t)\Delta}\partial_t\phi(t)\mathrm{d}t\right\rangle \mathrm{d}s \\
&= \int_0^\infty \left\langle w(s), \int_0^s [\partial_t(\mathrm{e}^{(s-t)\Delta}\phi(t)) + \mathrm{e}^{(s-t)\Delta}\Delta\phi(t)]\mathrm{d}t\right\rangle \mathrm{d}s \\
&= \int_0^\infty \left\langle w(s), \phi(s) - \mathrm{e}^{s\Delta}\phi(0) + \int_0^s \mathrm{e}^{(s-t)\Delta}\Delta\phi(t)\mathrm{d}t\right\rangle \mathrm{d}s \\
&= \langle w, \phi\rangle - \langle v, \Delta\phi\rangle.
\end{aligned}$$

In the third equality we also used Fubini. In the last equality we did the same steps as the first few equalities but in reverse order and used that ϕ is a test function. From this and (6.2) we thus find $\partial_t v \in L^2(\dot{H}^{-1})$ and $\partial_t v + \Delta v = w$ in $L^2(\dot{H}^{-1})$.
We define the distribution $h := u - v$. We have $\partial_t h + \Delta h = w - w = 0 \in L^2(\dot{H}^{-1})$ and also $\partial_t h = \partial_t u - \partial_t v \in L^2(\dot{H}^{-1})$. So $\Delta h = -\mathrm{div}\,\nabla h \in L^2(\dot{H}^{-1})$. From this we find $\nabla h \in L^2(L^2)$, which implies $h \in L^2(\dot{H}^1)$ and hence $h \in \dot{W}(0,\infty)$. Since $\dot{H}^1 \subseteq \mathscr{S}'$, we get $h \in L^2(\mathscr{S}')$. We are now able to take the partial Fourier transform $\mathcal{F}_x$ in the space variable. Using the Fourier multiplier for Δ and that $\partial_t h + \Delta h = 0$, we thus get $\phi = \mathcal{F}_x h \in L^2(\mathscr{S}')$, which satisfies

$$\partial_t\phi - |\xi|^2\phi = 0 \text{ in } \mathscr{D}'.$$

Staying away from $\xi = 0$ and solving this first order partial equation, we find

$$\phi = \mathrm{e}^{t|\xi|^2}\alpha \quad \text{in } \mathscr{D}'((0,\infty)\times(\mathbb{R}^n\backslash\{0\})),$$

for some $\alpha \in \mathscr{D}'(\mathbb{R}^n\backslash\{0\})$. Since $\xi\phi = \mathcal{F}_x(\nabla h) \in L^2(L^2)$, we find $\xi\phi = \xi\alpha\mathrm{e}^{t|\xi|^2} \in L^2(L^2(\mathbb{R}^n\backslash\{0\}))$. But we have for any $K \subseteq \mathbb{R}^n\backslash\{0\})$ that

$$\int_0^\infty \int_K |\xi\alpha\mathrm{e}^{t|\xi|^2}|^2 \mathrm{d}\xi\mathrm{d}t = \int_K \int_0^\infty |\xi\alpha\mathrm{e}^{t|\xi|^2}|^2 \mathrm{d}t\mathrm{d}\xi = \infty,$$

where we used Fubini. We must have $\alpha = 0$ in $\mathscr{D}'(\mathbb{R}^n\backslash\{0\})$. So ϕ is supported in $(0,\infty)\times\{0\}$. This implies that there exist $\tilde{c} \in \mathscr{D}'(0,\infty)$ such that $\phi = \tilde{c}\otimes\delta_0$. We have $\partial_t\phi = |\xi|^2\phi \in L^2(\dot{H}^{-1})$. Now suppose that $\partial_t\phi \neq 0$. Then we have $\delta_0 \in \dot{H}^{-1}$, which means that $\delta_0 = \mathrm{div}F$ for some $F \in L^2$. Taking Fourier transform gives $1 = \mathcal{F}(\delta_0) = |\xi|^{-2}\hat{F}(\xi)$, which can only hold if $\hat{F}(\xi) = |\xi|^2$. Since $|\xi|^2$ is not an L^2 function, we have a contradiction. So $\partial_t\phi$ must equal 0 and hence we get $\tilde{c}$ is constant. From this we find $h = \mathcal{F}_x^{-1}\phi = \mathcal{F}_x^{-1}(\tilde{c}\otimes\delta_0) = c$ for some constant $c \in \mathbb{C}$. Hence $u = v + c$.

Uniqueness: let $v_1, v_2 \in \dot{W}(0,\infty)\cap\mathscr{C}_0(L^2)$ and $c_1, c_2 \in \mathbb{C}$ be such that $u = v_1 + c_1 = v_2 + c_2$. Define $w = v_1 - v_2 = u - c_1 - u + c_2 = c_2 - c_1$. Since $w \in \mathscr{C}_0(L^2)$, we must have $w = 0$. Hence $v_1 = v_2$ and $c_1 = c_2$.

Estimate: As shown before, we have, for all $t \geq 0$ and all $g \in L^2(L^2)$ such that $w = \mathrm{div}\, g$, that $\|v(t)\|_{L^2} \leq \frac{1}{\sqrt{2}}\|\tau_t g\|_{L^2(L^2)} = \frac{1}{\sqrt{2}}\|g\|_{L^2(L^2)}$. From this we find

$$\begin{aligned}
\sup_{t\geq 0}\|v(t)\|_{L^2} &\leq \frac{1}{\sqrt{2}}\|w\|_{L^2(\dot{H}^{-1})} \\
&\leq \frac{1}{\sqrt{2}}(\|\partial_t u\|_{L^2(\dot{H}^{-1})} + \|\Delta u\|_{L^2(\dot{H}^{-1})}) \\
&\leq \frac{1}{\sqrt{2}}(\|\partial_t u\|_{L^2(\dot{H}^{-1})} + \|\mathrm{div}\,\nabla u\|_{L^2(\dot{H}^{-1})}) \\
&\leq \frac{1}{\sqrt{2}}(\|\partial_t u\|_{L^2(\dot{H}^{-1})} + \|u\|_{L^2(\dot{H}^{1})}).
\end{aligned}$$

For $a > 0$ we define $u_a : (t,x) \mapsto a^{\frac{n}{2}}u(t,ax)$, and we apply the above inequality to u_a. We find

$$\sup_{t\geq 0}\|v(t)\|_{L^2} \leq \frac{1}{\sqrt{2}}\left(\frac{1}{a}\|\partial_t u\|_{L^2(\dot{H}^{-1})} + a\|u\|_{L^2(\dot{H}^{1})}\right).$$

Optimising in a yields

$$\sup_{t\geq 0}\|v(t)\|_{L^2} \leq \sqrt{2\|\partial_t u\|_{L^2(\dot{H}^{-1})}\|u\|_{L^2(\dot{H}^{1})}}.$$

□

Remark 6.2. *For $0 \leq a < b < \infty$, and $u, v \in \dot{W}(0,\infty)\cap\mathscr{C}([a,b];L^2)$, we have that $t \mapsto \langle u(t), v(t)\rangle \in W^{1,1}(a,b)$ and*

$$({}_{L^2}\langle u(\cdot), v(\cdot)\rangle_{L^2})' = {}_{\dot{H}^{-1}}\langle u'(\cdot), v(\cdot)\rangle_{\dot{H}^1} + {}_{\dot{H}^1}\langle u(\cdot), v'(\cdot)\rangle_{\dot{H}^{-1}} \in L^1(a,b)$$

See [1, §14].

The above lemma gives the existence of the limit $\lim_{t\to 0} u(t,\cdot)$ in $\mathscr{D}'(\mathbb{R}^n)$. This brings us to the following definition

Definition 6.3. *For each $u \in \dot{W}(0,\infty)$, we define the trace of u, $Tr(u)$, by*

$$Tr(u) = \lim_{t \to 0} u(t,\cdot) = v(0) + c.$$

Definition 6.4. *Let $0 \le a < b \le \infty, \Omega$ be an open subset of $\mathbb{R}^n$ and $Q = (a,b) \times \Omega$. A function $u \in L^2_{loc}(a,b; H^1_{loc}(\Omega))$ is called a weak solution of (1.3) on Q if*

$$\iint_Q u(t,x)\overline{\partial_t \phi(t,x)}\,dx\,dt = \iint_Q A(t,x)\nabla u(t,x) \cdot \overline{\nabla \phi(t,x)}\,dx\,dt,$$

for all $\phi \in \mathscr{C}_c^\infty(Q)$. If $Q = \mathbb{R}^{n+1}_+ := (0,\infty) \times \mathbb{R}^n$, we say that u is a global weak solution.

We get the following a priori energy estimate as a corollary of Lemma 6.1.

Corollary 6.5. *Let $u \in \mathscr{D}'$ be a global weak solution of (1.3) such that $\nabla u \in L^2(L^2)$. Then there exist $c \in \mathbb{C}$ such that $v := u - c \in \mathscr{C}_0(L^2)$ and is norm decreasing, $\nabla v = \nabla u \in L^2(L^2)$, v is also weak solution of (1.3) and*

$$\|v(0)\|_{L^2} = \|v\|_{L^\infty(L^2)} \le \sqrt{2\Lambda \|\nabla v\|_{L^2(L^2)}} \le \sqrt{\frac{\Lambda}{\lambda}}\|v(0)\|_{L^2},$$

where $v(0) = v(0,\cdot)$, and Λ, λ are the ellipticity constants from (1.2).

Proof. Since u is a weak solution of (1.3) we have $\partial_t u = \operatorname{div} g$ in $\mathscr{D}'$ with $g = A\nabla u \in L^2(L^2)$. From this we get

$$\begin{aligned} |\langle \partial_t u, \bar{\phi}\rangle| &= |\langle \operatorname{div} A\nabla u, \bar{\phi}\rangle| \\ &= |\langle A\nabla u, \overline{\nabla\phi}\rangle| \\ &\le \|A\nabla u\|_{L^2(L^2)}\|\nabla\phi\|_{L^2(L^2)} \\ &\le \Lambda\|\nabla u\|_{L^2(L^2)}\|\nabla\phi\|_{L^2(L^2)}, \end{aligned}$$

for any test function ϕ. Hence $\partial_t u \in L^2(\dot{H}^{-1})$. Also note that $\nabla u \in L^2(L^2)$ implies $u \in L^2(\dot{H}^1)$. So $u \in \dot{W}(0,\infty)$ and by lemma 6.1 we get that there exits constant $c \in \mathbb{C}$ such that $v := u - c \in \dot{W}(0,\infty) \cap \mathscr{C}_0(L^2)$, and

$$\begin{aligned} \|v\|_{L^\infty(L^2)} &\le \sqrt{2\|\partial_t u\|_{L^2(\dot{H}^{-1})}\|\nabla u\|_{L^2(L^2)}} \\ &\le \sqrt{2\|g\|_{L^2(L^2)}\|\nabla u\|_{L^2(L^2)}} \\ &\le \sqrt{2\|A\nabla u\|_{L^2(L^2)}\|\nabla u\|_{L^2(L^2)}} \\ &\le \sqrt{2\Lambda}\|\nabla u\|_{L^2(L^2)} \\ &\le \sqrt{2\Lambda}\|\nabla v\|_{L^2(L^2)}. \end{aligned}$$

Now note that constants are trivially weak solutions of (1.3), and since u is aswell, this implies that v is a weak solution too. Now let $0 < a < b$. For all $V \in L^2(a,b;\dot{H}^1(\mathbb{R}^n))$, we have

$$\begin{aligned} \int_a^b {}_{\dot{H}^{-1}}\langle \partial_s v(s,\cdot), V(s,\cdot)\rangle_{\dot{H}^1} \mathrm{d}s &= \int_a^b {}_{\dot{H}^{-1}}\langle \operatorname{div} A\nabla v(s,\cdot), V(s,\cdot)\rangle_{\dot{H}^1} \mathrm{d}s \\ &= -\int_a^b {}_{L^2}\langle A\nabla v(s,\cdot), \nabla V(s,\cdot)\rangle_{L^2} \mathrm{d}s \end{aligned}$$

$$= -\int_a^b \int_{\mathbb{R}^n} A(s,x)\nabla v(s,x) \cdot \overline{\nabla V(s,x)} \mathrm{d}x\mathrm{d}s.$$

Taking $V = v$, we have by Remark 6.2 and ellipticity

$$\begin{aligned}
\|v(a,\cdot)\|_{L^2}^2 - \|v(b,\cdot)\|_{L^2}^2 &= -\left({}_{L^2}\langle v(b,\cdot), v(b,\cdot)\rangle_{L^2} - {}_{L^2}\langle v(a,\cdot), v(a,\cdot)\rangle_{L^2}\right) \\
&= -\int_a^b \partial_t({}_{L^2}\langle v(t,\cdot), v(t,\cdot)\rangle_{L^2})\mathrm{d}t \\
&= -\int_a^b {}_{\dot{H}^{-1}}\langle \partial_t v(t,\cdot), v(t,\cdot)\rangle + {}_{\dot{H}^1}\langle v(t,\cdot), \partial_t v(t,\cdot)\rangle_{\dot{H}^{-1}}\mathrm{d}t \\
&= -\int_a^b {}_{\dot{H}^{-1}}\langle \partial_t v(t,\cdot), v(t,\cdot)\rangle_{\dot{H}^1} + {}_{\dot{H}^{-1}}\overline{\langle \partial_t v(t,\cdot), v(t,\cdot)\rangle}_{\dot{H}^1}\mathrm{d}t \\
&= -2\Re e \int_a^b {}_{\dot{H}^{-1}}\langle \partial_t v(t,\cdot), v(t,\cdot)\rangle_{\dot{H}^1}\mathrm{d}t \\
&= 2\Re e \int_a^b \int_{\mathbb{R}^n} A(t,x)\nabla v(t,x) \cdot \overline{\nabla v(t,x)}\mathrm{d}x\mathrm{d}t \\
&\geq 2\lambda \int_a^b \int_{\mathbb{R}^n} \nabla v(t,x) \cdot \overline{\nabla v(t,x)}\mathrm{d}x\mathrm{d}t \\
&= 2\lambda \|\nabla v\|_{L^2(a,b;L^2)}^2,
\end{aligned}$$

which proves the norm decreasing property. Furthermore, by letting $a \to 0$ and $b \to \infty$, we find $2\lambda\|\nabla v\|_{L^2(L^2)}^2 \leq \|v(0,\cdot)\|_{L^2}^2$, which completes the proof. □

Next, we want to prove a well-posedness result for problem (1.3): prove the existence and uniqueness for global (or local) weak solutions in some solution space X. In our case X is going to be $\dot{W}(0,\infty)$. For the proof of this result we will need the following lemma.

Lemma 6.6. *Let $A_k \in L^\infty((0,\infty); L^\infty(\mathbb{R}^n; \mathscr{M}_n(\mathbb{C})))$ for $k \in \mathbb{N}$ be such that (6.1) holds uniformly in k and*

$$A_k(t,x) \overset{k\to\infty}{\longrightarrow} A(t,x) \text{ for almost every } (t,x) \in (0,\infty) \times \mathbb{R}^n.$$

Let u_k be a global weak solution of $\partial_t u = \operatorname{div} A_k \nabla u$ for all $k \in \mathbb{N}$, and assume that

$$\sup_{k\in\mathbb{N}}(\|u_k\|_{L^\infty(L^2)} + \|\nabla u\|_{L^2(L^2)}) < \infty.$$

Then there exist a subsequence $(u_{k_j})_{j\in\mathbb{N}}$ such that $(u_{k_j})_{j\in\mathbb{N}}$ weak converges to u in $L^\infty(L^2)$ and $(\nabla u_{k_j})_{j\in\mathbb{N}}$ weak* converges in $L^2(L^2)$. The limit $u \in L^\infty(L^2)$ is then a global weak solution of (1.3) such that $\nabla u \in L^2(L^2)$.*

Proof. Let $k \in \mathbb{N}$. We know that u_k is a global weak solution of $\partial_t u = \operatorname{div} A_k \nabla u$. By assumption we have $\nabla u_k \in L^2(L^2)$. So $u_k \in L^2(\dot{H}^1)$ and $\partial_t u_k = \operatorname{div} g$ with $g = A_k \nabla u_k \in L^2(L^2)$. So $u_k \in \dot{W}(0,\infty)$. Using $u_k \in L^\infty(L^2)$, we find, by Lemma 6.1 and a modification on a set of measure 0, that $u_k \in \mathscr{C}_0(L^2)$. Thus, $(u_k(0,\cdot))_{k\in\mathbb{N}}$ is uniformly bounded in $L^2(\mathbb{R}^n)$. By assumption, we also have that $(\nabla u_k)_{k\in\mathbb{N}}$ is uniformly bounded in $L^2(L^2)$. By the sequential Banach-Alaoglu theorem and the fact that ∇ is a closed operator, we can then find a subsequence for which there exists $u \in L^\infty(L^2)$ and $u_0 \in L^2(\mathbb{R}^n)$ such that

$$\begin{aligned}
u_{k_j} &\overset{j\to\infty}{\longrightarrow} u \quad \text{weak* in } L^\infty(L^2) \\
\nabla u_{k_j} &\overset{j\to\infty}{\longrightarrow} \nabla u \quad \text{weak* in } L^2(L^2)
\end{aligned}$$

$$u_{k_j}(0,\cdot) \overset{j\to\infty}{\longrightarrow} u_0 \qquad \text{weak* in } L^2.$$

Let $\phi \in \mathscr{D}(\mathbb{R}^n)$ and $t \geq 0$. By integrated version of Remark 6.2 and using $u_k \in \mathscr{C}_0(L^2)$ for all $k \in \mathbb{N}$, we find

$$\begin{aligned} \langle u_{k_j}(t,\cdot),\phi(\cdot)\rangle - \langle u_{k_j}(0,\cdot),\phi(\cdot)\rangle &= \int_0^t {}_{\dot H^{-1}}\langle \partial_s u_{k_j}(s,\cdot),\phi(\cdot)\rangle_{\dot H^1}\mathrm{d}s \\ &= \int_0^t {}_{\dot H^{-1}}\langle \mathrm{div}\, A_{k_j}(s,\cdot)\nabla u(s,\cdot),\phi(\cdot)\rangle_{\dot H^1}\mathrm{d}s \\ &= \int_0^t -{}_{L^2}\langle A_{k_j}(s,\cdot)\nabla u(s,\cdot),\nabla\phi(\cdot)\rangle_{L^2}\mathrm{d}s. \end{aligned}$$

So we get

$$\int_{\mathbb{R}^n} u_{k_j}(t,y)\overline{\phi(y0}\mathrm{d}y = \int_{\mathbb{R}^n} u_{k_j}(0,y)\overline{\phi(y)}\mathrm{d}y - \int_0^t\int_{\mathbb{R}^n} A_{k_j}(s,y)\nabla u_{k_j}(s,y)\cdot\overline{\nabla\phi(y)}\mathrm{d}y\mathrm{d}s.$$

We know that the right hand side converges to $\int_{\mathbb{R}^n} u_0(y)\overline{\phi(y)}\mathrm{d}y - \int_0^t\int_{\mathbb{R}^n} A(s,y)\nabla u(s,y)\cdot\overline{\nabla\phi(y)}\mathrm{d}y\mathrm{d}s$, so the left hand side converges as well and the limit is equal to $\int_{\mathbb{R}^n} u(t,y)\overline{\phi(y)}\mathrm{d}y$ for almost all $t>0$. By modifying t on a set of measure 0, we can assume that the equality holds everywhere. Differentiating with respect to t and using Remark 6.2 we get

$$\begin{aligned} {}_{\dot H-1}\langle \partial_t u(t,\cdot),\phi(\cdot)\rangle_{\dot H^1} + 0 &= {}_{L^2}\langle u(t,\cdot),\phi(\cdot)\rangle_{L^2} \\ &= -{}_{L^2}\langle A(t,\cdot)\nabla u(t,\cdot),\nabla\phi(\cdot)\rangle_{L^2} \\ &= {}_{\dot H^{-1}}\langle \mathrm{div}\, A(t,\cdot)\nabla u(t,\cdot),\phi(\cdot)\rangle_{\dot H^1}. \end{aligned}$$

So we have $\partial_t u(t,\cdot) = \mathrm{div}\, A(t,\cdot)\nabla u(t,\cdot)$ in $\dot H^{-1}$ for almost all $t>0$ and therefore $\partial_t u = \mathrm{div}\, A\nabla u$ in $L^2(\dot H^{-1})$. By definition of weak solution we then find that u is a weak solution of (1.3). □

Definition 6.7. *Let $u_0 \in L^2(\mathbb{R}^n)$. We say that*

$$\partial_t u = div\, A\nabla u, \quad u \in \dot W(0,\infty), \quad Tr(u) = u_0$$

is well-posed if there exists a unique $u \in \dot W(o,\infty)$ global weak solution of (1.3) such that $Tr(u) = u_0$.

Theorem 6.8. *For all $u_0 \in L^2(\mathbb{R}^n)$, the problem*

$$\partial_t u = div\, A\nabla u, \quad u \in \dot W(0,\infty), \quad Tr(u) = u_0$$

is well-posed. Moreover, $u \in \mathscr{C}_0([0,\infty);L^2)$, $\|u(t,\cdot)\|_{L^2}$ is non increasing and

$$\|u_0\| = \|u\|_{L^\infty(L^2)} \leq \sqrt{2\Lambda}\|\nabla u\|_{L^2(L^2)} \leq \sqrt{\frac{\Lambda}{\lambda}}\|u_0\|_{L^2}.$$

Proof. We will prove this theorem in several steps, where in each step we have a slightly different A.

Step 0: Suppose A is indepent of t and let $L = -\mathrm{div}\, A\nabla$. Then, from semigroup theory, we kow that $u = \mathrm{e}^{-tL}u_0 \in \mathscr{C}_0([0,\infty);L^2(\mathbb{R}^n)) \cap \mathscr{C}^\infty(0,\infty;D(L))$ is a (strong) solution of $\partial_t u + Lu = 0$. Since $u \in D(L)$ with respect to space variable, we also get that $\nabla u \in L^2(L^2)$ and hence $u \in \dot W(0,\infty)$ as well. In the same way as in Corollary 6.5, we can then find $2\lambda\|\nabla u\|^2_{L^2(L^2)} \leq \|u_0\|^2_{L^2}$. By

$$\langle u(t,\cdot),\partial_t\phi(t,\cdot)\rangle = -\langle \partial_t u(t,\cdot),\phi\rangle\rangle = -\langle \mathrm{div}\, A\nabla u(t,\cdot),\phi\rangle\rangle = \langle A\nabla u(t,\cdot),\nabla\phi(t,\cdot)\rangle,$$

where we used the L^2 inner product, we get that u is also a global weak solution.

Step 1: Now suppose A is of the form

$$A(t,x) = \sum_{k=0}^{N} \chi_{[t_k,t_{k+1})}(t)A_k(x) + \chi_{[t_{N+1},\infty)}(t)A_{N+1}(x)$$

for some $N \in \mathbb{N}$, $(t_k)_{0\leq k\leq N+1}$ an increasing sequence in $[0,\infty)$ with $t_0 = 0$ and $(A_k)_{0\leq k\leq N+1}$ satisfying (1.2) uniformly. We set $t_{N+2} = \infty$ and $L_k = -\text{div}\, A_k\nabla$. Note that $L(t) = L_k(t)$ for $t \in [t_k, t_{k+1})$. We define

$$\Gamma_A(t,s) := \mathrm{e}^{-(t-t_j)L_j}\mathrm{e}^{-(t_j-t_{j-1})L_{j-1}} \dots \mathrm{e}^{-(t_{i+1}-s)L_i}$$

for $t \in [t_j, t_{j+1})$ and $s \in [t_i, t_{i+1})$. For $t \geq 0$ we define $u : t \mapsto \Gamma_A(t,0)u_0$. Since we have composition of analytic semigroups (of contractions), $(\mathrm{e}^{-tL_k})_{t\geq 0}$, we get that $u \in \mathscr{C}_0([0,\infty); L^2(\mathbb{R}^n))$. We now prove the desired properties of u by induction. We know that $-L_0$ generates an anlytic semigroup of contractions. So we have

$$\|(t,x) \mapsto \chi_{(0,t_1)}\Gamma_A u_0(x)\|_{L^\infty(L^2)} = \|(t,x) \mapsto \mathrm{e}^{-(t+t_1)L_0}u_0(x)\|_{L^\infty(L^2)} \leq \|u_0\|_{L^2}.$$

Now, using an argument similar to (4.2), we find

$$\begin{aligned}
\|(t,x) \mapsto \chi_{(0,t_1)}(t)\nabla u(t,x)\|_{L^2(L^2)} &= \|(t,x) \mapsto \chi_{(0,t_1)}(t)\nabla \mathrm{e}^{-t_1L_0}\mathrm{e}^{-tL_0}u_0(x)\|_{L^2(L^2)} \\
&\lesssim \|(t,x) \mapsto \chi_{(0,t_1)}(t)\nabla \mathrm{e}^{-tL_0}u_0(x)\|_{L^2(L^2)} \\
&\lesssim \|u_0\|_{L^2},
\end{aligned}$$

where the last inequality follows by step 0. From step 0, we also get that $\partial_t u(t,\cdot) \in L^2(\mathbb{R}^n)$ for all $t \in (0,t_1)$ and $\partial_t(u(t,\cdot)) = L_0u(t,\cdot) = L(t)u(t,\cdot)$ in $L^2(\mathbb{R}^n)$ for all $t \in (0,t_1)$.

Now let $k \leq N+1$ and assume:

$$\begin{gathered}
\|(t,x) \mapsto \chi_{(0.t_k)}(t)\Gamma_A(t,0)u_0(x)\|_{L^\infty(L^2)} \leq \|u_0\|_{L^2}, \\
\|(t,x) \mapsto \chi_{(0,t_k)}(t)\nabla\Gamma_A(t,0)u_0(x)\|_{L^2(L^2)} \lesssim \|u_0\|_{L^2}, \\
\text{and } \partial_t u(t,\cdot) = L(t)u(t,\cdot) \text{ in } L^2(\mathbb{R}^n) \text{ for all } t \in (0,t_k)\backslash\{t_0,\dots,t_{k-1}\}.
\end{gathered}$$

In the second assumption the implicit constant may depend on N, but we are inducting on a finite number of steps and later on in the proof we will get a constant that only depends on the ellipticity constants from (1.2). We want to show that the above also holds on $(0,t_{k+1})$. For $t \in [t_k, t_{k+1})$ we have

$$\begin{aligned}
u(t,\cdot) &= \mathrm{e}^{-(t-t_k)L_k}\mathrm{e}^{-(t_k-t_{k-1})L_{k-1}} \dots \mathrm{e}^{-t_1L_0}u_0 \\
&= \mathrm{e}^{-(t-t_k)L_k}\mathrm{e}^{-(t_k-s)L_{k-1}}\mathrm{e}^{-(s-t_{k-1})L_{k-1}}\mathrm{e}^{-(t_{k-1}-t_{k-2})L_{k-2}} \dots \mathrm{e}^{-t_1L_0}u_0 \\
&= \mathrm{e}^{-(t-t_k)L_k}\mathrm{e}^{-(t_k-s)L_{k-1}}u(s,\cdot),
\end{aligned}$$

for all $s \in (t_{k-1}, t_k)$. From this and using that $\mathrm{e}^{-(t-t_k)L_k}$ and $\mathrm{e}^{-(t_k-s)L_{k-1}}$ are contractions, we get

$$\|(t,x) \mapsto \chi_{(0,t_{k+1})}(t)u(t,x)\|_{L^\infty(L^2)} \leq \|(t,x) \mapsto \chi_{(0,t_k)}(t)u(t,x)\|_{L^\infty(L^2)} \leq \|u_0\|_{L^2}.$$

For $t \in [t_k, t_{k+1})$ we have $u(t,\cdot) = \mathrm{e}^{-(t-t_k)L_k}u(t_k,\cdot)$. By a similar argument as in the first part of the induction, we get

$$\|(t,x) \mapsto \chi_{(t_k,t_{k+1})}(t)\nabla u(t,x)\|_{L^2(L^2)} \lesssim \|u(t_k,\cdot)\|_{L^2} \leq \|u_0\|_{L^2}.$$

We also have $\partial_t u(t,\cdot) = \partial_t \mathrm{e}^{-(t-t_k)L_k} u(t_k,\cdot) = -L_k u(t,\cdot) = -Lu(t,\cdot)$ in $L^2(\mathbb{R}^n)$ for all $t \in [t_k, t_{k+1})$. By induction, we thus have proved that $u \in L^\infty(L^2) \cap L^2(\dot{H}^1)$, and that u satisfies

$$\partial_t u(t,\cdot) = -L(t)u(t,\cdot) \quad \forall t \in (0,\infty)\backslash\{t_k; k \in \mathbb{N}\}.$$

We are now going to show that u is a global weak solution of (1.3). Let $\phi \in \mathscr{D}$, and pick $M > t_{N+1}$ such that supp $\phi \subseteq (0,M) \times \mathbb{R}^n$. Since the inner product on L^2 is bounded as a operator we have that, for $j = 0,\dots,N+1$, $t \mapsto \langle u(t,\cdot), \phi(t,\cdot\rangle$ is $\mathscr{C}^1$ on (t_j, t_{j+1}) and continuous on $[t_k, t_{j+1})$. Hence

$$\begin{aligned}\int_{t_j}^{t_{j+1}} \langle u(t,\cdot), \partial_t \phi(t,\cdot)\rangle \mathrm{d}t &= \int_{t_j}^{t_{j+1}} \partial_t \langle u(t,\cdot), \phi(t,\cdot)\rangle \mathrm{d}t - \int_{t_j}^{t_{j+1}} \langle \partial_t u(t,\cdot), \phi(t,\cdot)\rangle \mathrm{d}t \\ &= \langle u(t_{j+1},\cdot), \phi(t_{j+1},\cdot)\rangle - \langle u(t_j,\cdot), \phi(t_j,\cdot)\rangle + \int_{t_j}^{t_{j+1}} \langle L_j u(t,\cdot), \phi(t,\cdot)\rangle \mathrm{d}t\end{aligned}$$

Using $\langle L_j u(t,\cdot), \phi(t,\cdot)\rangle = \langle A_j u(t,\cdot), \nabla\phi(t,\cdot)\rangle$ for all $t \in (t_j, t_{j+1})$ and that supp $\phi \subseteq (0.M) \times \mathbb{R}^n$ we get, by summing in j,

$$\int_0^\infty \int_{\mathbb{R}^n} u(t,y)\overline{\phi(t,y)} \mathrm{d}y\mathrm{d}t = \int_0^\infty \int_{\mathbb{R}^n} A(t,y)\nabla u(t,y) \cdot \overline{\nabla\phi(t,y)} \mathrm{d}y \mathrm{d}t.$$

So u is a weak solution of (1.3). By Corollary 6.5, we can find a $v \in \dot{W}(0,\infty) \cap \mathscr{C}_0(L^2(\mathbb{R}^n))$ such that v is a weak solution of (1.3) and such that $\|\nabla v\|_{L^2(L^2)} \sim \|u_0\|_{L^2}$, with constants only depending on the ellipticity constants from (1.2).

Step 2: We now consider A of the form

$$A : (t,x) \mapsto \sum_{k=0}^{\infty} \chi_{(t_k,t_{k+1})}(t) A_k(x)$$

for some increasing sequence $(t_k)_{k\in\mathbb{N}}$ with $t_0 = 0$ and $\lim_{k\to\infty} t_k = \infty$ and $(A_k)_{k\in\mathbb{N}}$ satisfying (1.2) uniformly. Define

$$\mathcal{A}_N : (t,x) \mapsto \sum_{k=0}^{N} \chi_{[t_k,t_{k+1})}(t) A_k(x) + \chi_{[t_{N+1},\infty)} A_{N+1}(x)$$

for all $N \in \mathbb{N}$. Then, for almost every $(t,x) \in (0,\infty) \times \mathbb{R}^n$ we have $\mathcal{A}_N(t,x) \overset{N\to\infty}{\longrightarrow} A(t,x)$. By the previous step we have a corresponding sequence of weak solutions, $(u_N)_{N\in\mathbb{N}}$, to $\partial_t u = \operatorname{div} \mathcal{A}_N \nabla u$. Note that we have the same initial condition for every u_N. Hence, our sequence fulfills the assumptions of Lemma 6.6. So we can find a subsequence $(u_{N_j})_{j\in\mathbb{N}}$ converging to $u \in L^\infty(L^2)$ in the weak* topology such that u is a weak solution of $\partial_t u = \operatorname{div} A\nabla u$ and $\|u\|_{L^\infty(L^2)} + \|\nabla u\|_{L^2(L^2)} \sim \|u_0\|_{L^2}$, with constants depending only on the ellipticity constants.

Step 3: Now let $A \in \mathscr{C}([0,\infty); L^\infty(\mathbb{R}^n; \mathscr{M}_n(\mathbb{C})))$. We can find an almost everywhere approximation of A by matrices of the form

$$(t,x) \mapsto \sum_{k=0}^{\infty} \chi_{[t_k,t-k+1)}(t) A_k(x), \quad \text{with } A_k = A(t_k,\cdot),$$

which satisfy (1.2) uniformly. By step 2, we obtain a family of weak solutions, $(u_j)_{j\in\mathbb{N}}$, which satisfies the assumptions of lemma 6.6. Thus we can find a weak solution u of (1.3) such that $\|u\|_{L^\infty(L^2)} + \|\nabla u\|_{L^2(L^2)} \sim \|u_0\|_{L^2}$, with constants depending only on the ellipticity constants.

step 4: We are now able to consider general $A \in L^\infty((0,\infty); L^\infty(\mathbb{R}^n; \mathscr{M}_n(\mathbb{C})))$. We approximate A by

$$\left(\tilde{A} : (t,x) \mapsto j \int_t^{t+\frac{1}{j}} A(s,x)\mathrm{d}s\right) \in \mathscr{C}([0,\infty); L^\infty(\mathbb{R}^n; \mathscr{M}_n(\mathbb{C})))$$

for $g \geq 1$, which is an almost everywhere approximation. By step 3 and Lemma 6.6, we obtain, in a similar fashion as before, the desired result.

Step 5: We still need to prove uniqueness of solutions. Let $u, v \in \dot{W}(0,\infty)$ be solutions of (1.3) with $\mathrm{Tr}(u) = \mathrm{Tr}(v)$. Now define $w := u - v \in \dot{W}(0,\infty)$. Then w is a weak solution of (1.3) such that $\mathrm{Tr}(w) = 0$. By Corollary 6.5, we can find $\tilde{w} \in \mathscr{C}_0(L^2)$ and $c \in \mathbb{C}$ such that $w = \tilde{w} + c$. We then have $0 = \mathrm{Tr}(w) = \tilde{w}(0,\cdot) + c$. Since $\tilde{w} \in \mathscr{C}_0(L^2)$, we get that $\tilde{w}(0,\cdot) = 0$. Thus, by the inequalities of Corollary 6.5, we find $\|w\|_{L^\infty(L^2)} = \|\tilde{w}\|_{L^\infty(L^2)} = 0$. □

Inspired by the above proof, we are going to find a more explicit form of our solutions. We are going to do so, first through a family of operators which are known as propagators. Afterwards, by taking $A \in L^\infty(\mathbb{R}^n, \mathscr{M}_n(\mathbb{C}))$ satisfying (1.2) we can show that these solutions can be given by using the semigroup of L and the operator $\mathcal{R}_L$.

Lemma 6.9. *There exists a family of contractions $\{\Gamma(t,s); 0 \leq s \leq t < \infty\} \subseteq \mathscr{L}(L^2)$ such that*

(1) $\Gamma(t,t) = I \quad \forall t \geq 0$.

(2) $\Gamma(t,s)\Gamma(s,r) = \Gamma(t,r) \quad \forall t \geq s \geq r$.

(3) For all $h \in L^2(\mathbb{R}^n)$. and $s \geq 0$, $t \mapsto \Gamma(t,s)h \in \mathscr{C}_0([s,\infty); L^2(\mathbb{R}^n))$.

(4) For all $u_0 \in L^2(\mathbb{R}^n)$, $(t,x) \mapsto \Gamma(t,0)u_0(x)$ is a global weak solution of (1.3).

Proof. Let $u_0 \in L^2(\mathbb{R}^n)$ and u be the solution to the problem in Theorem 6.8. We then have $u \in \mathscr{C}_0(L^2) \cap L^2(\dot{H}^1)$, with $\|u(t,\cdot)\|_{L^2} \leq \|u_0\|_{L^2}$. So we can define $\Gamma(t,0)$ as the contraction on L^2 which maps u_0 to $u(t,\cdot)$. Starting from any time $s \geq 0$ and any data $h \in L^2$, we can obtain, in a similar fashion, a unique solution $v \in \dot{W}(s,\infty) \cap \mathscr{C}_0([s,\infty); L^2)$, with $u(s,\cdot) = h$. Note that u restricted to (s,∞) is a solution aswell and hence coincides with v on (s,∞). We then define $\Gamma(t,s)$ as the contraction mapping h to $u(t,\cdot)$, when $t \geq s$. Then (1), (3) and (4) follow by construction, while (2) follows from uniqueness. □

Definition 6.10. *We call $\{\Gamma(t,s); 0 \leq s \leq t < \infty\}$ the family of propagators for (1.3).*

Lemma 6.11. *Let $f \in L^2(L^2)$ and $h \in L^2(\mathbb{R}^n)$. Let $\underline{A} \in L^\infty(\mathbb{R}^n, \mathscr{M}_n(\mathbb{C}))$ satisfy (1.2) and $L = -\mathrm{div}\, \underline{A}\nabla$. Define, for all $t > 0$*

$$u(t,\cdot) = \mathrm{e}^{-tL}h + \mathcal{R}_L f(t,\cdot),$$

where $\mathcal{R}_L$ is the bounded operator from $T^{2,2}$ to X^2 from Proposition 5.8. Then u is the unique element of $\dot{W}(0,\infty)$ such that, for all $\phi \in \mathscr{D}$,

$$\langle u, \partial_t \phi\rangle = \langle \underline{A}\nabla u, \nabla\phi\rangle + \langle f, \nabla\phi\rangle,$$

and $Tr(u) = h$.

Proof. Assume $f \in \mathscr{D}$. Define $v_0 : (t,x) \mapsto \mathrm{e}^{-tL}h(x)$ and

$$v = v_0 + \mathcal{R}_L f.$$

From semigroup theory we know that $v \in \mathscr{C}(L^2)$ and

$$\partial_t v = -Lv_0 - L\mathcal{R}_L f + \operatorname{div} f = -Lv + \operatorname{div} f.$$

By step 0 of the proof of Theorem 6.8 we know that $\nabla v_0 \in L^2(L^2)$. By Proposition 5.9 we have $\nabla \mathcal{R}_L f \in T^{2,2} = L^2(L^2)$. Hence, $\nabla v \in L^2(L^2)$, and thus

$$\begin{aligned}\langle v, \partial_t \phi\rangle &= -\langle \partial_t v, \phi\rangle \\ &= \langle Lv, \phi\rangle - \langle \operatorname{div} f, \phi\rangle \\ &= \langle \underline{A}\nabla v, \nabla\phi\rangle + \langle f, \nabla \phi\rangle,\end{aligned}$$

for all $\phi \in \mathscr{D}$. By taking $t \to 0$ we also get $\operatorname{Tr}(v) = h$.
Now, let $f \in L^2(L^2)$, and let $(f_k)_{n\in\mathbb{N}}$ be a sequence of functions in $\mathscr{D}$ converging to f in $L^2(L^2)$. Define, for all $k \in \mathbb{N}$,

$$u_k = v_0 + \mathcal{R}_L f_k, \quad \text{and} \quad u = v_0 + \mathcal{R}_L f.$$

By Proposition 5.8 we know that $\mathcal{R}_L$ is a bounded operator from $T^{2,2} = L^2(L^2)$ to X^2. So $u_k \to u$ in X^2. We also know, by Proposition 5.9, that $\nabla\mathcal{R}_L f = \tilde{\mathcal{M}}_L f$ in $L^2(L^2)$, which we know to be bounded on $L^2(L^2)$ by Proposition 5.4. So $\nabla u_k \to \nabla u$ in $L^2(L^2))$. Hence, for all $\phi \in \mathscr{D}$,

$$\langle u, \nabla\partial_t\phi\rangle = \langle \underline{A}\nabla u, \nabla\phi\rangle + \langle f, \nabla\phi\rangle.$$

Furthermore, $\operatorname{Tr}(u_k) = h$ for all $k \in \mathbb{N}$ and Tr is continuous from $\dot{W}(0,\infty)$ to L^2 by Lemma 6.1. So we also have $\operatorname{Tr}(u) = h$.
To prove uniqueness, let $\tilde{u} \in \dot{W}(0,\infty)$ be another solution of

$$\langle \tilde{u}, \nabla\phi\rangle = \langle \underline{A}\nabla\tilde{u}, \nabla\phi\rangle + \langle f, \nabla\phi\rangle,$$

for all $\phi \in \mathscr{D}$, with $\operatorname{Tr}(\tilde{u}) = h$. Define $w = u - \tilde{u} \in \dot{W}(0,\infty)$. Then, for all $\phi \in \mathscr{D}$,

$$\langle u-\tilde{u}, \partial_t\phi\rangle = \langle u, \partial_t\phi\rangle - \langle \tilde{u}, \partial_t\phi\rangle = \langle \underline{A}\nabla u, \nabla\phi\rangle - \langle \underline{A}\nabla\tilde{u}, \nabla\phi\rangle = \langle \underline{A}\nabla(u-\tilde{u}), \nabla\phi\rangle,$$

and

$$\operatorname{Tr}(w) = \operatorname{Tr}(u-\tilde{u}) = \operatorname{Tr}(u) = \operatorname{Tr}(\tilde{u}) = h - h = 0.$$

Therefore, w is a solution of

$$\partial_t w = \operatorname{div} \underline{A}\nabla w, \quad w \in \dot{W}(0,\infty), \quad \operatorname{Tr}(w) = 0,$$

and thus $u = \tilde{u}$ by Theorem 6.8. □

Corollary 6.12. *Let $A \in L^\infty(\mathbb{R}^{n+1}_+, \mathscr{M}_n(\mathbb{C}))$ and $\underline{A} \in L^\infty(\mathbb{R}^n, \mathscr{M}_n(\mathbb{C}))$ satisfy (1.2). Let $L = -\operatorname{div} \underline{A}\nabla$. For all $t > 0$ and $h \in L^2(\mathbb{R}^n)$, the following holds in $L^2(\mathbb{R}^n)$:*

$$\Gamma(t,0)h = \mathrm{e}^{-tL}h + \int_0^t \mathrm{e}^{-(t-s)L} \operatorname{div}(A(s,\cdot) - \underline{A})\nabla\Gamma(s,\cdot)h\, ds. \tag{6.3}$$

Proof. Let $h \in L^2(\mathbb{R}^n)$. Define $v_0(t,\cdot) = \mathrm{e}^{-tL}h$ and $f(t,\cdot) = (A(t,\cdot) - \underline{A})\nabla\Gamma(t,0)h$ for all $t > 0$. Then $f \in L^2(L^2)$ by proof of Theorem 6.8 (step 1) combined with Lemma 6.9. Define $u = v_0 + \mathcal{R}_L f$, and $\tilde{u}(t,\cdot) = \Gamma(t,0)h$, for all $t > 0$. Using Lemma 6.11, we get, for all $\phi \in \mathscr{D}$,

$$\langle u, \partial_t\phi\rangle = \langle \underline{A}\nabla u, \nabla\phi\rangle + \langle f, \nabla\phi\rangle = \langle \underline{A}\nabla u, \nabla\phi\rangle + \langle (A-\underline{A})\nabla\tilde{u}, \nabla\phi\rangle$$

The proof of Theorem 6.8 gives us $u \in \dot{W}(0,\infty)$ and by Lemma 6.9 we know that $\tilde{u}$ is a global weak solution of (1.3) with $\operatorname{Tr}(\tilde{u}) = h$. So we have

$$\langle u-\tilde{u}, \partial_t\phi\rangle = \langle \underline{A}\nabla u, \nabla\phi\rangle + \langle (A-\underline{A})\nabla\tilde{u}, \nabla\phi\rangle - \langle A\nabla\tilde{u}, \nabla\phi\rangle = \langle \underline{A}(u - \nabla u), \nabla\phi\rangle.$$

Therefore, $u - \tilde{u} \in \dot{W}(0,\infty)$ is a global weak solution of $\partial_t(u-\tilde{u}) = \operatorname{div} \underline{A}\nabla(u-\tilde{u})$, with $\operatorname{Tr}(u-\tilde{u}) = 0$. Hence, by theorem 6.8, we have $u = \tilde{u}$. □

Chapter 7

Controlling the maximal function

In the previous chapters we developed boundedness results for the operators $\mathcal{M}_L$ and $\mathcal{R}_L$. We also obtained energy solutions of (1.3), which are given through propagators. More explicitly, for time independent A, the solution is given as the sum of the semigroup of L acting on the initial data and the operator $\mathcal{R}_L$ acting on the gradient of the solution. Using these results, we are now able to prove the following proposition (Proposition 1.2 of the introduction).

Proposition 7.1. *Let $1 < p < \infty, u_0 \in L^2(\mathbb{R}^n)$, and $u(t,\cdot) = \Gamma(t,0)u_0$ for all $t > 0$. If $\nabla u \in T^{p,2}$, then $u \in X^p$, and*

$$\|u\|_{X^p} \lesssim \|\nabla u\|_{T^{p,2}},$$

where the implicit constant is independent of u.

Proof. Let $v(t,\cdot) = \mathrm{e}^{t\Delta}u_0$, for all $t > 0$. Using (6.3) with $L = -\Delta = \operatorname{div}\nabla$ we have that

$$\begin{aligned} \|u\|_{X^p} &= \|v + \mathcal{R}_{-\Delta}(A-I)\nabla u\|_{X^p} \\ &\lesssim \|v\|_{X^p} + \|\mathcal{R}_{-\Delta}(A-I)\nabla u\|_{X^p} \\ &\leq \|v\|_{X^p} + \|\mathcal{R}_{-\Delta}\|_{\mathcal{L}(T^{p,2},X^p)}\|A-I\|_{L^\infty}\|\nabla u\|_{T^{p,2}}, \end{aligned}$$

where we also used that $\mathcal{R}_{-\Delta}$ is a bounded operator from $T^{p,2}$ to X^p by Proposition 5.8.
Now note that

$$\|v\|_{X^p} = \|\tilde{N}(v)\|_{L^p} \leq \|\tilde{N}(v^*)\|_{L^p} = \|v^*\|_{L^p},$$

where v^* is the non-tangential maximal function defined by (3.1).
Using (6.3) again, together with the classical conical Feffermann-Stein estimate [15, Theorem 8], Proposition 5.6 and Proposition 5.9 we also have

$$\begin{aligned} \|v^*\|_{L^p} &\lesssim \|\nabla v\|_{T^{p,2}} \\ &= \|\nabla(u - \mathcal{R}_{-\Delta}(A-I)\nabla u\|_{T^{p,2}} \\ &\leq \|\nabla u\|_{T^{p,2}} + \|\tilde{\mathcal{M}}_{-\Delta}\|_{\mathcal{L}(T^{p,2},T^{p,2})}\|A-I\|_{L^\infty}\|\nabla u\|_{T^{p,2}}. \end{aligned}$$

Combining the above, we thus find the desired result. □

Corollary 7.2. *For all $u_0 \in L^2(\mathbb{R}^n)$, the problem*

$$\partial_t u = div\, A\nabla u, \quad u \in X^2 \quad Tr(u) = u_0$$

is well-posed. Moreover, the solution u is the energy solution, i.e. $u(t,\cdot) = \Gamma(t,0)u_0$ for all $t > 0$, and

$$\|u_0\|_{L^2} = \|u\|_{L^\infty(L^2)} \lesssim \|u\|_{X^2} \lesssim \|\nabla u\|_{L^2(L^2)} \leq \sqrt{\frac{1}{2\lambda}}\|u_0\|_{L^2}.$$

Proof. We will only prove existence of a solution. For the uniqueness we refer to [6, Corollary 7.2]. Let $u(t,\cdot) = \Gamma(t,0)u_0$. As in the previous chapter we have that u is the unique solution of

$$\partial_t u = \operatorname{div} A\nabla u, \quad u \in \dot{W}(0,\infty) \quad \operatorname{Tr}(u) = u_0.$$

By Proposition 7.1 and Theorem 6.8 we then have

$$\|u\|_{X^2} \lesssim \|\nabla u\|_{T^{2,2}} = \|\nabla u\|_{L^2(L^2)} \leq \sqrt{\frac{1}{2\lambda}}\|u_0\|_{L^2}.$$

□

Chapter 8

Stochastic integration

Before starting the analysis of [8], we are going to give a brief introduction of how integration with respect to Brownian motion is defined. For a detailed introduction to stochastic integration see for example [37].
We start with basic definitions and notation that are needed in order to develop the desired integral. We work on a probability space $(\Omega, \mathscr{F}, \mathbb{P})$, where Ω is a sample space, $\mathscr{F}$ a σ-algebra and $\mathbb{P}$ a probability measure.
A filtration on a probablity space $(\Omega.\mathscr{F}, \mathbb{P})$ is a collection of σ-algebras $\{\mathscr{F}_t : t \in \mathbb{R}_+\}$ such that

$$\mathscr{F}_s \subseteq \mathscr{F}_t \subseteq \mathscr{F} \quad \text{for all } 0 \leq s < t < \infty.$$

A stopping time is a random variable $\tau : \Omega \to [0, \infty]$ such that $\{\omega; \tau(\omega) \leq t\} \in \mathscr{F}_t$.

A stochastic process X is a collection of random variables $\{X_t : t \in T\}$, indexed by an index set T, defined on the same probability space. The random variables all take values in the same measurable space S and X is then called an S-valued process. In most (continuous) cases, the index set is $\mathbb{R}_+$, or a subset of it, and usually represents time.
If for each $0 \leq t < \infty$, X_t is $\mathscr{F}_t$-measurable, we call the process $X = \{X_t : t \in \mathbb{R}_+\}$ adapted to the filtration $\mathscr{F}_t$.
Now let $\mathcal{B}_{\mathbb{R}_+}$ denote the Borel σ-algebra of $\mathbb{R}_+$. A process X is called measurable if it is $\mathcal{B}_{\mathbb{R}_+} \otimes \mathscr{F}$-measurable as a function from $\mathbb{R}_+ \times \Omega$ into $\mathbb{R}^n$. Furthermore, it is called progressively measurable if X restricted to $[0, T] \times \Omega$ is $\mathcal{B}_{[0,T]} \otimes \mathscr{F}_T$-measurable for each T. Note that if a process X is progressively measurable then it is also adapted, but the reverse does not need to be true.
The process X is called continuous if it is pathwise continuous, i.e. if for almost all $\omega \in \Omega$ the path $t \mapsto X_t(\omega)$ is continuous as a function of t. If a process is continous and adapted, then it is also progressively measurable.

Now let X be a stochastic process and let $\pi = \{0 = t_0 < t_1 < \cdots < t_{m(\pi)} = t\}$ be a partition of $[0, t]$. We write down the sum of squared increments as

$$\sum_{i=0}^{m(\pi)-1} (X_{t_{i+1}} - X_{t_i})^2.$$

The above sums converge to the random variable $\langle X \rangle_t$ in probability as $\text{mesh}(\pi) = \max(t_{i+1} - t_i) \to 0$ if for each $\epsilon > 0$ there exist a $\delta > 0$ such that

$$\mathbb{P}\left\{ \left| \sum_{i=0}^{m(\pi)-1} (X_{t_{i+1}} - X_{t_i})^2 - \langle X \rangle_t \right| \geq \epsilon \right\} \leq \epsilon$$

for all partitions π with $\text{mesh}(\pi) \leq \delta$. This limit is denoted as

$$\lim_{\text{mesh}(\pi)\to 0} \sum_i (X_{t_{i+1}} - X_{t_i})^2 = \langle X \rangle_t \quad \text{in probability.}$$

If the above limit exists and furthermore we have $\langle X \rangle_0 = 0$ and for each ω the path $t \mapsto \langle X \rangle_t(\omega)$ is nondecreasing, then $\langle X \rangle = \{\langle X \rangle_t : t \in \mathbb{R}_+\}$ is called the quadratic variation process.
We can define a quadratic covariation between two processes in the following way.

Definition 8.1. *Let X and Y be two stochastic processes on the same probability space. The quadratic covariation process $\langle X, Y \rangle = \{\langle X, Y \rangle_t : t \in \mathbb{R}_+\}$ is defined by*

$$\langle X, Y \rangle = \langle \frac{1}{2}(X + Y) \rangle - \langle \frac{1}{2}(X - Y) \rangle,$$

provided the quadratic processes on the right exist.

We are interested in a particular stochastic process, namely Brownian Motion process, also known as Wiener process. Brownian Motion is the random motion of particles due to collision with the gas or liquid particles in which they reside. It is named after the botanist Robert Brown, who, in 1828, describes the irregular movement of pollen in water. In 1905, Einstein described in detail that this irregular movement was due to particle collision. The first mathematical construction of Brownian Motion was done by Norbert Wiener in 1923. The process is defined as follows.

Definition 8.2. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, $\{\mathscr{F}_t\}$ a filtration and $W = \{W_t; 0 \leq t < \infty\}$ an adapted real-valued process. If W satisfies the following properties*

(i) W is a continuous process, i.e. for almost every ω, $t \mapsto W_t(\omega)$ is continuous,

(ii) W has independent increments, i.e. for $0 \leq s < t$, $W_t - W_s$ is independent of $\mathscr{F}_s$,

(iii) for all $0 \leq s \leq t$, $W_t - W_s \sim N(0, t - s)$,

then W is called a one-dimensional Brownian motion with respect to $\{\mathscr{F}_t\}$ or $\{\mathscr{F}_t\}$-Brownian motion. If additionally W satisfies

(iv) $W_0 = 0$ almost surely

then W is a standard Brownian motion.

An $\mathbb{R}^n$-valued process $W_t = (W_t^1, \ldots, W_t^n)$ such that the coordinates $W^1, \ldots, W^n$ are independent and each W_t^i is a one-dimensional Brownian motion is called an n-dimensional Brownian motion.

We can also define Brownian motions in more general settings. One of these more generalized Brownian motions, namely cylindrical Brownian motion, will be used in the next section. The definition, as given in [30], reads as follows.

Definition 8.3. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probablity space and $\{\mathscr{F}_t\}$ a filtration. An $\{\mathscr{F}_t\}$-cylindrical Brownian motion in Hilbert space H is a bounded linear operator $W_H : L^2(\mathbb{R}_+; H) \to L^2(\Omega)$ such that*

(i) for all $f \in L^2(\mathbb{R}_+; H)$ the random variable $W_H(f)$ is centered Gaussian.

(ii) for all $t \in \mathbb{R}_+$ and $f \in L^2(\mathbb{R}_+; H)$ with support in $[0, t]$, $W_H(f)$ is $\mathscr{F}_t$-measurable.

(iii) for all $t \in \mathbb{R}_+$ and $f \in L^2(\mathbb{R}_+; H)$ with support in $[t, \infty)$, $W_H(f)$ is independent of $\mathscr{F}_t$.

(iv) for all $f_1, f_2 \in L^2(\mathbb{R}_+; H)$ we have $\mathbb{E}(W_H(f_1) \cdot W_H(f_2)) = [f_1, f_2]_{L^2(\mathbb{R}_+;H)}$.

For $h \in H$ we put

$$W_H(t)h = W_H(\chi_{(0,t]} \otimes h).$$

Most of the results we mention in this chapter will use a one-dimensional Brownian motion. At the end of the chapter we will provide the definition of a stochastic integral with respect to a cylindrical Brownian motion, but we will not need many of its properties. Also note that in the above definition, if we take $H = \mathbb{R}^n$ for any n, then W_H is just an n-dimensional Brownian motion.

The next class of stochastic processes we want to describe are martingales. Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probablity space and $\{\mathscr{F}_t\}$ a filtration. A martingale with respect to filtration $\{\mathscr{F}_t\}$ is a real-valued stochastic process $M = \{M_t : t \in \mathbb{R}_+\}$ adapted to $\{\mathscr{F}_t\}$ such that for each t, M_t is integrable and

$$\mathbb{E}(M_t|\mathscr{F}_s) = M_s \quad \text{for all } s < t.$$

An example of a martingale is the real-valued Brownian motion.
If a martingale M is square-integrable i.e. $\mathbb{E}(M_t^2) < \infty$ for all t, then M is called an L^2-martingale and we denote $M \in L^2(\Omega)$.
Now let τ be a stopping time and $X = \{X_t : t \in \mathbb{R}_+\}$ a process. The stopped process X^τ is then defined by $X_t^\tau = X_{t\wedge\tau}$. This brings us to the definition of a local martingale, which we will need in order to define integration with respect to Brownian motion.

Definition 8.4. *Let $M = \{M_t : t \in \mathbb{R}_+\}$ be a process adapted to a filtration $\{\mathscr{F}_t\}$. We say that M is a local martingale if there exists a sequence of stopping times $\tau_1 \leq \tau_2 \ldots$ such that $\mathbb{P}(\tau_k \uparrow \infty) = 1$ and for each k, the stopped process M^{τ_k} is a martingale with respect to $\{\mathscr{F}_t\}$. We say that M is local L^2-martingale if for each k, $\{\mathscr{F}_t\}$ is an L^2-martingale. For both cases, $(\tau_k)_{k\in\mathbb{N}}$ is called a localizing sequence for M.*

We now want to build the integral $\int_0^t K \mathrm{d}M_s = (K \bullet M)_t$. We do this for two different cases. For the first case we define the following two spaces.

* $\mathcal{M}_{b,c}^2 :=$ {Martingale $M : M \in L^2(\Omega)$, M continuous , $M_0 = 0$ a.s, $\mathbb{E}(\sup_{t\geq 0} M_t^2) < \infty$},
* $L^2(M) :=$ {Stoch. proc. $H : H$ progressively msr., $\mathbb{E}(\int_0^\infty H_s^2 \mathrm{d}\langle M\rangle_s < \infty, M \in \mathcal{M}_{b,c}^2$}.

Here $\langle M\rangle_s$ denotes the quadratic variation process of M and the integral is a Lebesgue-Stieltjes integral with respect to the time variable s, evaluated for fixed $\omega \in \Omega$. We equip $\mathcal{M}_{b,c}^2$ with a norm obtained through the following inner product

$$\langle M, N\rangle_{\mathcal{M}_{b,c}^2} = \mathbb{E}(\langle M, N\rangle_\infty), \;\text{ with } \langle M, N\rangle_\infty = \lim_{t\to\infty} \langle M, N\rangle_t \;\text{ in } L^1(\Omega).$$

For $L^2(M)$ we use the following norm

$$\|H\|_{L^2(M)} := \left(\mathbb{E}\left(\int_0^\infty H_s^2 \mathrm{d}\langle M\rangle_s\right)\right)^{\frac{1}{2}}.$$

Equipped with these norms, both spaces are Hilbert spaces and we have the following theorem.

Theorem 8.5. *Let $M \in \mathcal{M}^2_{b,c}$, then for all $H \in L^2(M)$ there exists a unique martingale in $\mathcal{M}^2_{b,c}$, $H \bullet M$, such that*

$$\forall N \in \mathcal{M}^2_{b,c} : \langle H \bullet M, N\rangle = H \bullet \langle M, N\rangle.$$

where $H \bullet \langle M, N\rangle = \int_0^\cdot H_s d\langle M, N\rangle_s$, the Lebesgue-Stieltjes integral.
The map $L^2(M) \to \mathcal{M}^2_{b,c}$, $H \mapsto H \bullet M$ is an isometry:

$$\|H \bullet M\|_{\mathcal{M}^2_{b,c}} = \|H\|_{L^2(M)}.$$

$H \bullet M$ is also denoted as $\int_0^\cdot H_s \mathrm{d}M_s$ and is our first case of a stochastic integral. As mentioned before, we want to build the stochastic integral with respect to Brownian motion. To be able to do that, we need to relax our assumtions slighlty since $W \notin \mathcal{M}^2_{b,c}$

Again, we define two spaces.

* $\mathcal{M}^2_{\text{loc},c} := \{M : M \text{ continuous local martingale in } L^2(\Omega), M_0 = 0 \text{ a.s}\}$.
* $L^2_{\text{loc}}(M) := \{H : H \text{ progr. msr.}, \forall t \geq 0 : \mathbb{P}(\int_0^t H_s^2 \mathrm{d}\langle M\rangle_S < \infty) = 1, M \in \mathcal{M}^2_{\text{loc},c}\}$.

We have a similar result for this spaces.

Theorem 8.6. *For all $H \in L^2_{loc}(M)$ there exists a unique martingale in $\mathcal{M}^2_{loc,c}$, $H \bullet M$, such that for all $N \in \mathcal{M}^2_{loc,c}$ we have*

$$\langle H \bullet M, N\rangle = H \bullet \langle M, N\rangle.$$

In this case we can also get the stochastic integral through a limit in probablity

Theorem 8.7. *Let $M \in \mathcal{M}^2_{loc,c}$ and $H \in L^2_{loc}(M)$. Then the stochastic integral can be realized as the following limit*

$$(H \bullet M)_t = \int_0^t H_s dM_s = \lim_{n\to\infty} \sum_{i=0}^{\pi_n - 1} H_{t_i^n}(M_{t_{i+1}^n} - M_{t_i^n}) \quad \textit{in probability,}$$

where π_n is a partition of $[0, t]$.

Since Brownian motion W is in $L^2_{\text{loc}}(M)$, we thus have defined the stochastic integral with respect to Brownian motion. Furthermore we know that $\langle W\rangle_s = s$ a.s. This brings us the a well known result, known as Itô isometry, which we will use on multiple occasions.

Theorem 8.8 (Itô isometry). *Let W be a real-valued Brownian motion and let $H \in L^2_{loc,c}(W)$. Then, for any $t \geq 0$ we have*

$$\mathbb{E}\left[\left(\int_0^t H_s dW_s\right)^2\right] = \mathbb{E}\left(\int_0^t H_s^2 ds\right).$$

We would also like to mention Itô's formula. It reads as follows.

Theorem 8.9 (Itô's formula). *Let $0 < T < \infty$ and $f \in C^{1,2}([0, T] \times D)$, where D is an open subset of $\mathbb{R}^n$. Suppose Y is a $\mathbb{R}^n$-valued continuous local martingale in $L^2(\Omega)$ such that $\overline{Y[0, T]} \subset D$ almost surely. Then*

$$\begin{aligned} f(t, Y(t)) =& f(0, Y(0)) + \int_0^t f_t(s, Y(s))ds + \sum_{i=0}^n \int_0^t f_{x_i}(s, Y(s))dY_i(s) \\ &+ \frac{1}{2} \sum_{1\leq i,j\leq n} \int_0^t f_{x_i x_j}(s, Y(s))d\langle Y_i, Y_j\rangle. \end{aligned}$$

Often, a differential notation is used to denote the Itô formula. the above formula is then expressed as follows.

$$\mathrm{d}f(t,Y(t)) = f_t(t,(Y(t))\mathrm{d}t + \sum_{i=1}^{n} f_{x_i}(t,(Y(t))\mathrm{d}Y_i(t) + \frac{1}{2}\sum_{1\le i,j\le n} f_{x_ix_j}(t,(Y(t))\mathrm{d}\langle Y_i,Y_j\rangle(t).$$

This notation is more economical than the integral notation, but it has no rigorous meaning.
A special case of the Itô formula is if we take Y to be an n-dimensional Brownian motion W. Using that $\langle W_i, W_j\rangle = s$ if $i=j$ and 0 otherwise, we find

$$f(t,W(t)) = f(0,W(0)) + \int_0^t (f_t + \frac{1}{2}\Delta f)(s,W(s))\mathrm{d}s + \int_0^t \nabla f(s,W(s))\mathrm{d}W(s).$$

We will also give a small example of the use of Itô's formula.

Example 8.10. *Let $k \ge 1$. We want to evaluate the $\int_0^t W_s^k dW_s$, where W is the standard 1-dimensional Brownian motion. To do so, we take $f(x) = (k+1)^{-1}x^{k+1}$. Then we have $f'(x) = x^k$ and $f''(x) = kx^{k-1}$. Itô's formula then gives*

$$\int_0^t W_s^k dW_s = (k+1)^{-1}W_t^{k+1} - \frac{k}{2}\int_0^t W_s^{k-1} ds,$$

where the integral on the right is a Riemann integral of the continuous function $s \mapsto W_s^{k-1}$.

Lastly, we want to define the stochastic integral with respect to a cylindrical Brownian motion. To this end, we are provided with a probablity space $(\Omega, \mathscr{F}, \mathbb{P})$, endowed with a filtration $\mathscr{F} = (\mathscr{F}_t)_{t\ge 0}$, a real Hilbert space H and $W = (W(s))_{s\ge 0}$, which denotes a $\mathscr{F}$-cylindrical Brownian motion in H.
A measurable mapping $u : \mathbb{R}_+^{n+1} \times \Omega \to \mathbb{R}^n$ is called an $\mathscr{F}$-adapted simple process if it is of the form

$$u(t,x,\omega) = \sum_{l=1}^{N} \chi_{(t_l,t_{l+1}]}(t) \sum_{m=1}^{N} \chi_{A_{ml}}(\omega)\phi_{ml}(x)$$

with $0 \le t_1 < \cdots < t_{N+1} = T < \infty$, $A_{ml} \in \mathscr{F}$, and ϕ_{ml} a simple function on $\mathbb{R}^n$ with values in H. For such processes, we define the stochastic integral with respect to an H-cylindrical Brownian motion by

$$\int_0^T u(t)\mathrm{d}W_H(t) := \sum_{l=1}^{N}\sum_{m=1}^{N} \chi_{A_{ml}}(W_H(t_{n+1})\phi_{ml} - W_H(t_n)\phi_{ml}).$$

For more information about stochastic integration with respect to a cylindrical Brownian motion see e.g [30, 31].

Chapter 9

Conical stochastic maximal regularity

In this chapter we start our analysis of [8], wherein a conical maximal L^p-regularity estimate, for $1 \leq p < \infty$, was proved for the stochastic convolution process

$$u(t) = \int_0^t \mathrm{e}^{-(t-s)L} g(s) \mathrm{d}W(s), \quad t \geq 0. \tag{9.1}$$

We will give a brief overview of the first five sections, which include the main result and its proof. Afterwards, we will describe the problem and results of the sixth section and explain how they relate to our previous work.

Let W be a cylindrical Brownian motion with values in a Hilbert space H. We consider the following stochastic heat equation in $\mathbb{R}^n$ driven by W:

$$\begin{cases} \partial_t u(t,x) = \Delta u(t,x) + g(t,x)\mathrm{d}W(t), & t \geq 0,\ x \in \mathbb{R}^n, \\ u(0,x) = 0, & x \in \mathbb{R}^n, \end{cases} \tag{9.2}$$

with process $g : \mathbb{R}_+ \times \mathbb{R}^n \times \Omega \to H$ having suitable measurabilty and integrability properties. In this case, the process $u : \mathbb{R}_+ \times \mathbb{R}^n \times \Omega \to \mathbb{R}$ given formally by the stochastic convolution (9.1) is well defined and called the mild solution of (9.2). As a consequence of a classical result due to Da Prato [13], it follows that this mild solution u has stochastic maximal L^2-regularity in the sense that

$$\mathbb{E}\|\nabla u\|^2_{L^2(\mathbb{R}_+;L^2(\mathbb{R}^n;\mathbb{R}^n))} \leq C^2 \mathbb{E}\|g\|^2_{L^2(\mathbb{R}_+;L^2(\mathbb{R}^n;H))},$$

with C independent of g and H. For $p > 2$ analogous results, extensions and generalizations were established in [24, 26, 33].
On the other hand, the corresponding results for the above in the case that $1 \leq p < 2$ was shown to be false even for $H = \mathbb{R}$ in [25]. The main result of [8] shows that the stochastic heat equation (9.2), under certain conditions, has 'conical' stochastic maximal regularity on the full range of $1 \leq p < \infty$. To state the exact result we first introduce the setting in which we work.

We are provided with a probablity space $(\Omega, \mathscr{F}, \mathbb{P})$, endowed with a filtration $\mathscr{F} = (\mathscr{F}_t)_{t\geq 0}$, a real Hilbert space H and $W = (W(s))_{s\geq 0}$, which denotes a $\mathscr{F}$-cylindrical Brownian motion in H. For an $\mathscr{F}$-adapted simple process u, as introduced at the end of previous chapter, we define the stochastic convolution operator as

$$S \diamond g(t) := \int_0^t S(t-s) g(s) \mathrm{d}W(s).$$

Whenever $S = (S(t))_{t\geq 0}$ is a strongly continuous semigroup of bounded linear operators, then the above operator is well-defined as an L^2 valued process (see e.g. [31]).
The main result of [8] then reads as follows.

Theorem 9.1 (Conical stochastic maximal L^p-regularity)**.** *Let $L = -divA\nabla$ be a divergence form elliptic operator, with $A \in L^\infty(\mathbb{R}^n; \mathscr{M}_n(\mathbb{C})))$, as introduced in Chapter 4, and let $S = (S(t))_{t\geq 0}$ be the bounded analytic contraction semigroup generated by $-L$. Then for all $1 \leq p < \infty$ and $\beta > 0$, and all adapted simple processes $g : \mathbb{R}_+ \times \mathbb{R}^n \times \Omega \to H$ one has*

$$\mathbb{E}\|\nabla S(t) \diamond g\|^p_{T^{p,2}_\beta(\mathbb{R}^n)} \leq C^p_{p,\beta}\mathbb{E}\|g\|^p_{T^{p,2}_\beta(H)},$$

with constant $C_{p,\beta}$ independent of g and H.

To prove this theorem, a $T^{2,2}_\beta$ estimate is combined with an extrapolation result based on off-diagonal bounds which gives the $T^{p,2}_\beta$ estimate.
Before explaining the $T^{2,2}_\beta$ estimate, we need to introduce the following two definitions.

Definition 9.2. *Let H_1, H_2 be Hilbert spaces. A linear operator $T : H_1 \to H_2$ is said to be a finite rank operator if the range of T is finite dimensional.*

Definition 9.3. *Let H_1, H_2 be Hilbert spaces. A linear operator $T : H_1 \to H_2$ is said to be Hilbert-Schmidt if*

$$\sup_h \sum_{j=1}^k \|Th_j\|^2 < \infty$$

where the supremum is taken over all finite orthonormal systems $h = \{h_1, \ldots, h_k\}$ in H.

We are now able to explain the $T^{2,2}_\beta$ estimate. It is a weighted analogue of a classical stochastic maximal L^2-regularity result due to Da Prato (see [14, Theorem 6.14]). Let H, E be Hilbert spaces and g an $\mathscr{F}$-adapted simple process with values in the vector space $H \otimes E$ of finite rank operators from H to E. Da Pratos result states that if $-L$ generates a bounded analytic contraction semigroup $(S(t))_{t\geq 0}$ on E, then there exists a constant $C \geq 0$, independent of g and H, such that

$$\mathbb{E}\|L^{\frac{1}{2}} S \diamond g\|^2_{L^2(\mathbb{R}_+;E)} \leq C^2\mathbb{E}\|g\|^2_{L^2(\mathbb{R}_+;\mathscr{L}_2(H,E))},$$

where $\mathscr{L}_2(H, E)$ is the space of Hilbert-Schmidt operators from H to E. The weighted analogue of this result is then formulated as follows:

Proposition 9.4. *Suppose $-L$ generates a bounded analytic contraction semigroup $S = (S(t))_{t\geq 0}$ on a Hilbert space E. Then for all $\beta \geq 0$ there exists a constant $C_\beta \geq 0$ such that for all $\mathscr{F}$-adapted simple processes $g : \mathbb{R}_+ \times \Omega \to \mathscr{L}_2(H, E)$,*

$$\mathbb{E}\|L^{\frac{1}{2}} S \diamond g\|^2_{L^2(\mathbb{R}_+,t^{-\beta};E)} \leq C^2_\beta\mathbb{E}\|g\|^2_{L^2(\mathbb{R}_+,t^{-\beta};\mathscr{L}_2(H,E))}.$$

Choosing $E = L^2(\mathbb{R}^n)$ and using the following identification, see [21, Section 9.2.a],

$$L^2(\mathbb{R}_+, t^{-\beta}\mathrm{d}t; \mathscr{L}^2(\mathbb{R}^n, L^2(\mathbb{R}^n)) = L^2(\mathbb{R}_+, t^{-\beta}\mathrm{d}t; L^2(\mathbb{R}^n)) = T^{2,2}_\beta(H), \tag{9.3}$$

we get the estimate

$$\mathbb{E}\|L^{\frac{1}{2}} S \diamond g\|^2_{T^{2,2}_\beta} \leq C^2_\beta\mathbb{E}\|g\|^2_{T^{2,2}_\beta(H)}.$$

The extrapolation result which gives the $T^{p,2}_\beta$ estimate is proved in 2 steps. The first step assumes L^2-off diagonal bounds and gives the result for $p \in [1, \infty) \cap (\frac{2n}{n+2\beta}, \infty)$. It is a stochastic analogue of Theorem 5.2 where we proved a variation of L^2 off-diagonal bounds for $p \in (\frac{2n}{n+2(1+\beta)}, \infty) \cap (1, \infty)$. It is formulated as follows:

Proposition 9.5 (Extrapolation via L^2-off diagonal bounds)**.** *Let $(T_t)_{t>0}$ be a family of bounded linear operators on L^2, let $\beta > 0$, and suppose there exists a constant $C_\beta \geq 0$, independent of g and H, such that*

$$\mathbb{E}\left\|\int_0^t T_{t-s}g(s,\cdot)\,dW(s)\right\|^2_{T^{2,2}_\beta} \leq C_\beta^2 \mathbb{E}\|g\|^2_{T^{2,2}_\beta}$$

for all $\mathscr{F}$-adapted simple process $g : \mathbb{R}_+ \times \mathbb{R}^n \times \Omega \to H$. If $(t^{\frac{1}{2}}T_t)_{t>0}$ satisfies L^2-off diagonal bounds, then, for $p \in [1,\infty) \cap (\frac{2n}{n+2\beta},\infty)$, there exists a constant $C_{p,\beta} \geq 0$, independent of g and H, such that

$$\mathbb{E}\left\|\int_0^t T_{t-s}g(s,\cdot)\,dW(s)\right\|^2_{T^{p,2}_\beta} \leq C_\beta^2 \mathbb{E}\|g\|^2_{T^{p,2}_\beta(H)}.$$

The second step gives the $T^{p,2}_\beta$ estimate for $p \in [1,\infty)$ and assuming $L^1 - L^2$-off diagonal bounds. It is formulated as follows:

Proposition 9.6 (Extrapolation via $L^1 - L^2$-off diagonal bounds)**.** *Let $(T_t)_{t>0}$ be a family of bounded linear operators on L^2, let $\beta > 0$, and suppose there exists a constant $C_\beta \geq 0$, independent of g and H, such that*

$$\mathbb{E}\left\|\int_0^t T_{t-s}g(s,\cdot)\,dW(s)\right\|^2_{T^{2,2}_\beta} \leq C_\beta^2 \mathbb{E}\|g\|^2_{T^{2,2}_\beta}$$

for all $\mathscr{F}$-adapted simple process $g : \mathbb{R}_+ \times \mathbb{R}^n \times \Omega \to H$. If $(t^{\frac{1}{2}}T_t)_{t>0}$ is a family of bounded linear operators on L^2 which satisfies $L^1 - L^2$-off diagonal bounds, then, for all $p \in [1,\infty)$, there exists a constant $C_{p,\beta} \geq 0$, independent of g and H, such that

$$\mathbb{E}\left\|\int_0^t T_{t-s}g(s,\cdot)\,dW(s)\right\|^2_{T^{p,2}_\beta} \leq C_\beta^2 \mathbb{E}\|g\|^2_{T^{p,2}_\beta(H)}.$$

With these tools, one is able to prove Theorem 9.1.

Proof of Theorem 9.1. Since $A \in L^\infty(\mathbb{R}^n;\mathscr{M}_n(\mathbb{C})))$, we have as a consequence of [9, Theorem 4 and Lemma 20], that the family of operators $(t^{\frac{1}{2}}\nabla \mathrm{e}^{-tL})_{t\geq 0}$ satisfies $L^1 - L^2$-off diagonal bounds. By (4.2) we have

$$\|L^{\frac{1}{2}}u\|_{L^2} \lesssim \|\nabla u\|_{L^2}. \tag{9.4}$$

Moreover, by the discussion in Chapter 4 we know that L is maximal accretive on L^2 and therefore the bounded analytic semigroup generated by $-L$ is a contraction semigroup on L^2. By Proposition 9.4, the mapping $g \mapsto L^{\frac{1}{2}}S \diamond g$ extends to a bounded operator from $L^2_{\mathscr{F}}(\Omega;T^{2,2}_\beta(H))$ to $L^2_{\mathscr{F}}(\Omega;T^{2,2}_\beta)$, where $L^2_{\mathscr{F}}(\Omega;T^{2,2}_\beta(H))$ denotes the closed subspace of all $\mathscr{F}$-adapted simple processes belonging to $L^2(\Omega;T^{2,2}_\beta(H))$. By using (9.4), we thus get that the mapping $g \mapsto \nabla S \diamond g$ extends to a bounded operator from $L^2_{\mathscr{F}}(\Omega;T^{2,2}_\beta(H))$ to $L^2_{\mathscr{F}}(\Omega;T^{2,2}_\beta(\mathbb{R}^n))$. The desired result now follows from Theorem 9.6.

□

As an application of the above results, the following problem is considered in Section 6 of [8]:

$$\begin{cases} \mathrm{d}u(t,x) = -Lu(t,x)\mathrm{d}t + b(\nabla u(t,x))\mathrm{d}W(t), & t \geq 0,\ x \in \mathbb{R}^n, \\ u(0,x) = u_0(x), & x \in \mathbb{R}^n, \end{cases} \tag{9.5}$$

Here, $L = -\mathrm{div}A\nabla$ is the operator introduced in Section 4 but with $A \in L^\infty(\mathbb{R}^n, \mathscr{M}_n(\mathbb{R}))$ instead of $A \in L^\infty(\mathbb{R}^n, \mathscr{M}_n(\mathbb{C}))$, W is a $\mathscr{F}$-Brownian motion relative to some filtration $\mathscr{F}$ and the function $b : \mathbb{R}^n \to \mathbb{R}$ is a globally Lipschitz continous functions with Lipschitz constant L_b, i.e

$$|b(x)| \leq L_b|x|, \quad x \in \mathbb{R}^n. \tag{9.6}$$

We take the initial value $u_0 : \mathbb{R}^n \to \mathbb{R}$ to be in the L^p realisation of the operator $L^{\frac{\beta}{2}}$, for some $0 < \beta < 1$. We denote this space by $D_p(L^{\frac{\beta}{2}})$.
Problem (9.5) is then formally rewritten to obtain a notion of a solution. This results in the following abstract initial value problem

$$\begin{cases} \mathrm{d}U(t) + LU(t)\mathrm{d}t = B(\nabla u(t,))\mathrm{d}W(t), & t \geq 0, \\ U(0) = u_0, \end{cases}$$

where

$$(B(u))(t,x) := b(u(t,x)).$$

Here, the operator B is also known as Nemytskii operator. B inherits the continuity and boundedness properties from the Lipschitz continuous function b. By property (10.2), we see that B maps $L^p_{\mathscr{F}}(\Omega; T^{p,2}_\beta(H))$ into itself.
In this formal setting we want a "mild solution" to be an adapted "process" that "satisfies" the variation of constants equation

$$U(t) = S(t)u_0 + \int_0^t S(t-s)B(\nabla U(s))\mathrm{d}W(s), \tag{9.7}$$

where S is the bounded analytic semigroup generated by $-L$. Then ∇ is formally applied to both sides of the above identity and $V = \nabla U$ is formally substituted to obtain

$$V = \nabla S(\cdot)u_0 + \nabla S \diamond B(V). \tag{9.8}$$

Definition 9.7. *Problem (9.5) is said to have conical maximal L^p-regularity with weight β if for every initial value $u_0 \in D_p(L^{\frac{\beta}{2}})$ there exists a unique element V in $L^p(\Omega; T^{p,2}_\beta(\mathbb{R}^n))$ such that (9.8) holds.*

Before stating the result regarding conical maximal L^p-regularity in the above sense, the following lemma is proved in [8].

Lemma 9.8. *There exists $\beta_0 \in (0,1]$ with following property. If $p \in (1,\infty)$ and $0 < \beta < 1$ are such that the pair $(\frac{1}{p}, \beta)$ belongs to the interior of the planar polytope with vertices $(0,0)$, $(0,\beta_0)$, $(\frac{1}{2}, 1)$, $(1,1)$, $(1,0)$, then for all $u_0 \in D_p(L^{\frac{\beta}{2}})$ the function $(t,x) \mapsto \nabla S(t)u_0(x)$ belongs to $T^{p,2}_\beta(\mathbb{R}^n)$.*

The above lemma is proved in several steps. First for $p = 2$ and $0 < \beta < 1$ in the following way.
Let $v_0 = L^{\frac{\beta}{2}}u_0$. Then we have

$$\begin{aligned} \|(t,y) \mapsto \nabla S(t)u_0(y)\|^2_{T^{p,2}_\beta(\mathbb{C}^n)} &\simeq \int_0^\infty \int_{\mathbb{R}^n} |\nabla \mathrm{e}^{-tL}u_0|^2 \mathrm{d}y \frac{\mathrm{d}t}{t^\beta} \\ &\simeq \int_0^\infty \int_{\mathbb{R}^n} |A^{\frac{1}{2}} \mathrm{e}^{-tL}u_0|^2 \mathrm{d}y \frac{ddt}{t^\beta} \end{aligned}$$

$$= \int_0^\infty \int_{\mathbb{R}^n} |(tL)^{\frac{1-\beta}{2}} \mathrm{e}^{-tL} v_0|^2 \mathrm{d}y \frac{\mathrm{d}t}{t}$$
$$\simeq \|v_0\|_2^2,$$

where in the second inequality we used (4.1) and in the last inequality we used a square function estimate of the H^∞-functional calculus of L with $\phi(tL) = (tL)^{\frac{1-\beta}{2}} \mathrm{e}^{-tL}$.
In step 2 the case $1 < p < 2$ and $0 < \beta < 1$ is proved using the first order approach from [20].
In the next step, we first claim that there exists a $\beta_0 \in (0, 1]$ such that for all $0 < \beta < \beta_0$ and $f \in L^\infty$ we have

$$\|(t, y) \mapsto t^{\frac{1-\beta}{2}} \nabla L^{-\frac{\beta}{2}} \mathrm{e}^{-tL} f(y)\|_{T_1^{\infty,2}(\mathbb{C}^n)} \lesssim \|f\|_{L^\infty},$$

where $T_\beta^{\infty,2}$ is defined as the space of all locally square integrable functions such that the Carleson measure condition

$$\int_0^{r^2} \int_B |g(t, y)|^2 \mathrm{d}y \frac{\mathrm{d}t}{t^\beta} \leq Cr^n \tag{9.9}$$

holds whenever B is a ball of radius $r > 0$, with C independent of B. Using this, together with the $p = 2$ result for all $0 < \beta < 1$ and with the fact that the spaces $T_\beta^{p,2}$ interpolate between $p = 2$ and $p = \infty$, see e.g. [11], we get that for $2 < p < \infty$ and $0 < \beta < \beta_0$ where $\beta_0 \in (0, 1]$ we have

$$\|(t, y) \mapsto t^{\frac{1-\beta}{2}} \nabla L^{-\frac{\beta}{2}} \mathrm{e}^{-tL} v_0(y)\|_{T_\beta^{p,2}(\mathbb{C}^n)} \lesssim \|v_0\|_{L^p},$$

i.e.,

$$\|(t, y) \mapsto \nabla S(t) u_0(y)\|_{T_\beta^{p,2}(\mathbb{C}^n)} \lesssim \|L^{\frac{\beta}{2}} u_0\|_{L^p}.$$

We now prove the claim. First note that up to changing the matrix $A(x)$ to $A(rx + x_0)$, which does not change the ellipticity constants, our following arguments are scale and translation invariant. So we can assume the ball B in (9.9) to be the unit ball. Let f be a bounded measurable function with compact support. Let $f_0 = f\chi_{2B}$ and $f_1 = f - f_0$. By the $p = 2$ result, we have

$$\int_0^1 \int_B |t^{\frac{1-\beta}{2}} \nabla L^{-\frac{\beta}{2}} \mathrm{e}^{-tL} f_0(y)|^2 \mathrm{d}y \frac{\mathrm{d}t}{t} \lesssim \|f_0\|_{L^2}^2 \lesssim \|f\|_{L^\infty}^2.$$

For the f_1 we will use the following representation formula.

$$\nabla L^{-\frac{\beta}{2}} \mathrm{e}^{-tL} f_1 = C \int_0^\infty \nabla s^{\frac{\beta}{2}} (sL) \mathrm{e}^{-(s+t)L} f_1 \frac{\mathrm{d}s}{s},$$

with $C > 0$ a constant independent of f_1. This equality holds in $L^2(B; \mathbb{C}^n)$, see [8] for more details. One of the ingredients to prove the above equality is the following kernel estimate of $L\mathrm{e}^{-sL}$. There are constants $c, C > 0$ and $\gamma_0 > 0$ in $(n - 2, n]$ such that

$$\forall y \notin 2B \quad \left(\int_B |\nabla_x \tilde{K}_{s+t}(x, y)|^2 \mathrm{d}x \right)^{\frac{1}{2}} \leq \frac{C}{(s + t)^{\frac{n}{4} + \frac{3}{2} + \frac{\gamma_0}{4}}} \cdot \exp\left(-\frac{c|y|^2}{s + t} \right),$$

where $\tilde{K}_s$ is the kernel of $L\mathrm{e}^{-sL}$. The proof of the above estimate is the same as the proof of the same estimate for e^{-tL}, which can be found in [9, Lemma 33]. Now define $\beta_0 = \frac{1}{2}(\gamma_0 - n + 2) \in (0, 1]$. By the representation formula and the above estimate and the fact that f_1 has support outside of $2B$, we have

$$\|\nabla L^{-\frac{\beta}{2}} \mathrm{e}^{-tL} f_1\|_{L^2(B;\mathbb{C}^n)} \simeq \left\| \int_0^\infty \nabla s^{\frac{\beta}{2}} (sL) \mathrm{e}^{-(s+t)L} f_1 \frac{\mathrm{d}s}{s} \right\|_{L^2(B;\mathbb{C}^n)}$$

$$
\begin{aligned}
&= \Big\| \int_0^\infty s^{1+\frac{\beta}{2}} \nabla L \mathrm{e}^{-(s+t)L} f_1 \frac{\mathrm{d}s}{s} \Big\|_{L^2(B;\mathbb{C}^n)} \\
&= \Big\| \int_0^\infty s^{1+\frac{\beta}{2}} \int_{|y|\geq 2} \nabla \tilde{K}_{s+t}(\cdot, y) f_1(y) \mathrm{d}y \frac{\mathrm{d}s}{s} \Big\|_{L^2(B;\mathbb{C}^n)} \\
&\leq \int_0^\infty s^{1+\frac{\beta}{2}} \int_{|y|\geq 2} \|\nabla \tilde{K}_{s+t}(\cdot, y) f_1(y)\|_{L^2(B;\mathbb{C}^n)} \mathrm{d}y \frac{\mathrm{d}s}{s} \\
&\lesssim \int_0^\infty \int_{|y|\geq 2} \frac{s^{1+\frac{\beta}{2}}}{(s+t)^{\frac{n}{4}+\frac{3}{2}+\frac{\gamma_0}{4}}} \cdot \exp\left(-\frac{c|y|^2}{s+t}\right) |f_1(y)| \mathrm{d}y \frac{\mathrm{d}s}{s} \\
&\lesssim \|f_1\|_{L^\infty} \int_{|y|\geq 2} \frac{1}{|y|^{n+\beta_0-\beta}} \mathrm{d}y \\
&\lesssim \|f_1\|_{L^\infty},
\end{aligned}
$$

for $0 < \beta < \beta_0$. Hence, $\int_0^1 \|t^{\frac{1-\beta}{2}} \nabla L^{-\frac{\beta}{2}} \mathrm{e}^{-tL} f_1\|^2_{L^2(B;\mathbb{C}^n)} \frac{\mathrm{d}t}{t} \lesssim \|f_1\|_{L^\infty}$, which proves the claim. In the final step the remaining cases are proved using Stein's complex interpolation.

One last ingredient that we will need is the Banach contraction principle. The theorem was first proved by Banach in 1922. We will provide a more recent proof due to R.S. Palais, [34].

Theorem 9.9 (Banach contraction principle)**.** *Let (X, d) be a non-empty complete metric space and $T : X \to X$ a contraction operator i.e. $d(T(x_1), T(x_2)) \leq K d(x_1, x_2)$ for all $x_1, x_2 \in X$ with $0 < K < 1$. Then T has a unique fixed point $\bar{x}$ in X i.e. $T(\bar{x}) = \bar{x}$.*

Proof. Using the triangle inequality we get

$$
d(x_1, x_2) \leq d(x_1, T(x_1)) + d(T(x_1), T(x_2)) + d(T(x_2), x_2), \quad x_1, x_2 \in X.
$$

Since $0 < K < 1$ we thus get

$$
d(x_1, x_2) \leq \frac{1}{1-K} \left(d(x_1, T(x_1)) + d(x_2, T(x_2)) \right). \tag{9.10}
$$

This shows that if x_1 and x_2 are both fixed points of T then we must have $d(x_1, x_2) = 0$. So the uniqueness is proved.
Now let T^n denote T composed with itself n times. By induction we have $d(T^n(x_1), T^n(x_2)) \leq K^n d(x_1, x_2)$. Let $x \in X$ and take $x_1 = T^n(x)$ and $x_2 = T^m(x)$ in (9.10). We get

$$
\begin{aligned}
d(T^n(x), T^m(x)) &\leq \frac{1}{1-K} (d(T^n(x), T^n(T(x))) + d(T^m(x), T^m(T(x)))) \\
&\leq \frac{K^n + K^m}{1-K} d(x, T(x)).
\end{aligned}
$$

Since $K < 1$, we get, by letting $n, m \to \infty$, that $d(T^n(x), T^m(x)) \to 0$. Hence $T^n(x)$ is a Cauchy sequence in X and by completeness of X it converges to a point $\bar{x} \in X$. Using that T is a contraction and hence a continuous mapping we get

$$
\bar{x} = \lim_{n\to\infty} T^n(x) = \lim_{n\to\infty} T(T^{n-1}(x)) = T(\lim_{n\to\infty} T^{n-1}(x)) = T(\bar{x}),
$$

which proves that $\bar{x}$ is a fixed point. □

We are now able to prove the following result.

Theorem 9.10. *Let $p \in (1,\infty)$ and $0 < \beta < 1$ be such that the pair $(\frac{1}{p},\beta)$ belongs to the interior of the planar polytope with vertices $(0,0)$, $(0,\beta_0)$, $(\frac{1}{2},1)$, $(1,1)$, $(1,0)$, where β_0 is defined in the previous lemma. Suppose that $K_{p,\beta}L_b < 1$, where $K_{p,\beta}$ is the norm of the mapping $g \mapsto \nabla S \diamond g$ from $L^p(\Omega; T^{p,2}_\beta)$ to $L^p(\Omega; T^{p,2}_\beta(\mathbb{C}^n))$ and L_b is the Lipschitz constant of b. Then Problem 9.5 has conical stochastic maximal L^p-regularity with weight β, i.e. (9.8) holds for all initial values $u_0 \in D_p(L^{\frac{\beta}{2}})$.*

Proof. We define the mapping F on $L^p_{\mathscr{F}}(\Omega; T^{p,2}_\beta(\mathbb{R}^n))$ by

$$F(v) := \nabla S(\cdot)u_0 + \nabla S \diamond B(v).$$

By Theorem 9.6 we have $\nabla S \diamond B(v)$ for all $v \in L^p_{\mathscr{F}}(\Omega; T^{p,2}_\beta(\mathbb{R}^n))$. Together with Lemma 9.8 this shows us that F maps $L^p_{\mathscr{F}}(\Omega; T^{p,2}_\beta(\mathbb{R}^n))$ into itself.
Now for v_1, v_2 we have

$$\begin{aligned}\|F(v_1) - F(v_2)\|_{L^p(\Omega;T^{p,2}_\beta(\mathbb{R}^n))} &= \|\nabla S \diamond (B(v_1) - B(v_2))\|_{L^p(\Omega;T^{p,2}_\beta(\mathbb{R}^n))} \\ &\le K_{p,\beta}\|B(v_1) - B(v_2)\|_{L^p(\Omega;T^{p,2}_\beta(\mathbb{R}^n))} \\ &\le K_{p,\beta}L_b\|v_1 - v_2\|_{L^p(\Omega;T^{p,2}_\beta(\mathbb{R}^n))}.\end{aligned}$$

Since we assumed $K_{p,\beta}L_b < 1$, the above calculation shows that F is a contraction mapping from $L^p_{\mathscr{F}}(\Omega; T^{p,2}_\beta(\mathbb{R}^n))$ into itself. By the Banach contraction principle we thus get that there exists a unique fixed point v of F, i.e. there exists a unique $v \in L^p_{\mathscr{F}}(\Omega; T^{p,2}_\beta(\mathbb{R}^n))$ such that (9.8) holds.
Note that the requirement for β to be in the described polytope is due to the use of Lemma 9.8. In that lemma, the restrictions on the polytope were obtained through the different steps of its proof. □

For the stochastic parabolic Cauchy problem we see, in the above, that we can solve for v, where v can be formally seen as the gradient of the "mild solution of (9.7)". In other words, we only have information on the gradient of the solution.
On the other hand, in Chapter 7, we showed that we can control the non-tangential maximal function of the solution of the deterministic parabolic Cauchy problem (1.3) by the gradient of the solution. To do so, we first found that the solution could be given in terms of propagators, which in turn could be described as a sum of the semigroup generated by $-L$ working on the initial solution and the operator $\mathcal{R}$ working on the gradient of the initial solution. By showing that the operator $\mathcal{R}$ is bounded from $T^{p,2}$ to X^p and combining that with conical Littlewood-Paley estimates we thus got the desired result.
We want to develop an analogue result for the stochastic problem as well. In the upcoming sections we are going to define a vector-valued version of X^p on which we prove a similar result to the deterministic problem. To do so, we use some of the techniques used for the deterministic Cauchy problem and generalize them to a stochastic setting.

Chapter 10

Necessary condition

Here, we develop two results of which the second one is a necessary condition for a stochastic analogue of 7.1. The first result will not be needed later on, but it might be interesting to mention. Ideas for the proofs of both results are inspired by the proof of Proposition 5.8. We also use techniques used to prove [8, Proposition 4.1].

We are provided with a probablity space $(\Omega, \mathscr{F}, \mathbb{P})$, endowed with a filtration $\mathscr{F} = (\mathscr{F}_t)_{t\geq 0}$, and $W = (W(s))_{s\geq 0}$, wich denotes a standard $\mathscr{F}$-Brownian motion in $\mathbb{R}^n$.
A measurable mapping $u : \mathbb{R}^{n+1}_+ \times \Omega \to \mathbb{R}^n$ is called an $\mathscr{F}$-adapted simple process if it is of the form

$$u(t, x, \omega) = \sum_{l=1}^{N} \chi_{(t_l, t_{l+1}]}(t) \sum_{m=1}^{N} \chi_{A_{ml}}(\omega)\phi_{ml}(x)$$

with $0 \leq t_1 < \cdots < t_{N+1} < \infty$, $A_{ml} \in \mathscr{F}$, and ϕ_{ml} a simple function. For such processes we define the following operator

$$\tilde{\mathcal{R}}_L u(t, x, \omega) = \int_0^t \mathrm{e}^{-(t-s)L} \mathrm{div}\ u(s, \cdot, \cdot)(x, \omega) \mathrm{d}W(s), \tag{10.1}$$

as an L^2-valued process.

Proposition 10.1. *Let $p \geq 2$, $\beta \geq 1$ and let $u : \mathbb{R}^{n+1}_+ \times \Omega \to \mathbb{R}^n$ be an adapted simple process. Then we have*

$$\sup_{\delta>0} \mathbb{E} \left\| \left(\fint_{\frac{\delta}{2}}^{\delta} \fint_{B(\cdot,\sqrt{\delta})} |\tilde{\mathcal{R}}_L u|^2 dy \frac{dt}{t^{\beta-1}} \right)^{\frac{1}{2}} \right\|_{L^p}^p \leq C_{p,\beta,n} \mathbb{E} \|u\|^p_{T^{p,2}_\beta},$$

with $C_{p,\beta,n}$ independent of u.

Proof. By Fubini we have

$$\mathbb{E} \left\| \left(\fint_{\frac{\delta}{2}}^{\delta} \fint_{B(\cdot,\sqrt{\delta})} |\tilde{\mathcal{R}}_L u|^2 \mathrm{d}y \frac{\mathrm{d}t}{t^{\beta-1}} \right)^{\frac{1}{2}} \right\|_{L^p}^p = \int_{\mathbb{R}^n} \mathbb{E} \left(\fint_{\frac{\delta}{2}}^{\delta} \fint_{B(x,\sqrt{\delta})} |\tilde{\mathcal{R}}_L u|^2 \mathrm{d}y \frac{\mathrm{d}t}{t^{\beta-1}} \right)^{\frac{p}{2}} \mathrm{d}x.$$

Using the Kahane-Khintchine inequality we find

$$\mathbb{E} \left(\fint_{\frac{\delta}{2}}^{\delta} \fint_{B(x,\sqrt{\delta})} \left| \int_0^t \mathrm{e}^{-(t-s)L} \mathrm{div}\ u(s, \cdot, \cdot)(x, \omega) \mathrm{d}W(s) \right|^2 \mathrm{d}y \frac{\mathrm{d}t}{t^{\beta-1}} \right)^{\frac{p}{2}}$$

$$= \mathbb{E}\Big(\int_0^\infty \int_{\mathbb{R}^n} |\frac{1}{\delta^{\frac{n}{4}+\frac{1}{2}}}\int_0^t \chi_{(\frac{\delta}{2},\delta)}(t)\chi_{B(x,\sqrt{\delta})}(y)\mathrm{e}^{-(t-s)L}\mathrm{div}\ u(s,\cdot,\cdot)(x,\omega)\mathrm{d}W(s)|^2\mathrm{d}y\frac{\mathrm{d}t}{t^{\beta-1}}\Big)^{\frac{p}{2}}$$

$$= \mathbb{E}\Big\|\frac{1}{\delta^{\frac{n}{4}+\frac{1}{2}}}\int_0^t \chi_{(\frac{\delta}{2},\delta)}(t)\chi_{B(x,\sqrt{\delta})}(y)\mathrm{e}^{-(t-s)L}\mathrm{div}\ u(s,\cdot,\cdot)(x,\omega)\mathrm{d}W(s)\Big\|^p_{L^2(\mathbb{R}^n\times(0,\infty);\mathrm{d}y\frac{\mathrm{d}t}{t^{\beta-1}})}$$

$$\simeq \Big(\mathbb{E}\Big\|\frac{1}{\delta^{\frac{n}{4}+\frac{1}{2}}}\int_0^t \chi_{(\frac{\delta}{2},\delta)}(t)\chi_{B(x,\sqrt{\delta})}(y)\mathrm{e}^{-(t-s)L}\mathrm{div}\ u(s,\cdot,\cdot)(x,\omega)\mathrm{d}W(s)\Big\|^2_{L^2(\mathbb{R}^n\times(0,\infty);\mathrm{d}y\frac{\mathrm{d}t}{t^{\beta-1}})}\Big)^{\frac{p}{2}}$$

$$= \Big(\mathbb{E}\fint_{\frac{\delta}{2}}^{\delta}\fint_{B(x,\sqrt{\delta})}|\int_0^t \mathrm{e}^{-(t-s)L}\mathrm{div}\ u(s,\cdot,\cdot)(x,\omega)\mathrm{d}W(s)|^2\mathrm{d}y\frac{\mathrm{d}t}{t^{\beta-1}}\Big)^{\frac{p}{2}}.$$

So we get

$$\mathbb{E}\Big\|\Big(\fint_{\frac{\delta}{2}}^{\delta}\fint_{B(\cdot,\sqrt{\delta})}|\tilde{\mathcal{R}}_L u|^2\mathrm{d}y\frac{\mathrm{d}t}{t^{\beta-1}}\Big)^{\frac{1}{2}}\Big\|^p_{L^p} \simeq \int_{\mathbb{R}^n}\Big(\mathbb{E}\fint_{\frac{\delta}{2}}^{\delta}\fint_{B(x,\sqrt{\delta})}|\tilde{\mathcal{R}}_L u|^2\mathrm{d}y\frac{\mathrm{d}t}{t^{\beta-1}}\Big)^{\frac{p}{2}}\mathrm{d}x,$$

with implicit constant depending on p.
Now define $\tilde{K}_L u(t,x,\omega) = \int_{\frac{t}{2}}^t \mathrm{e}^{-(t-s)L}\mathrm{div}\ u(s,\cdot,\cdot)(x,\omega)\mathrm{d}W(s)$. Similar to Proposition 5.8 we have

$$\tilde{\mathcal{R}}_L u(t,x,\omega) = \sum_{k=0}^{\infty}\mathrm{e}^{-(1-2^{-k})tL}\tilde{K}_L u(2^{-k}t,x,\omega) \quad \text{a.e.}$$

Fixing $x\in\mathbb{R}^n$ and $k\in\mathbb{N}\backslash\{0\}$ we get, analogous to Proposition 5.8,

$$\Big(\mathbb{E}\fint_{\frac{\delta}{2}}^{\delta}\fint_{B(x,\sqrt{\delta})}|\mathrm{e}^{-(1-2^{-k})tL}\tilde{K}_L u(2^{-k}t,y,\omega)|^2\mathrm{d}y\frac{\mathrm{d}t}{t^{\beta-1}}\Big)^{\frac{1}{2}}$$
$$\lesssim \sum_{j=1}^{\infty}2^{j\frac{n}{2}}\mathrm{e}^{-c4^j}\sup_{\delta'>0}\Big(\mathbb{E}\fint_{\frac{\delta'}{2}}^{\delta'}\fint_{B(x,2^{j+1+\frac{k}{2}}\sqrt{\delta'})}|\tilde{K}_L u(t,y,\omega)|^2\mathrm{d}y\frac{\mathrm{d}t}{t^{\beta-1}}\Big)^{\frac{1}{2}}.$$

For $k=0$ and any $j\geq 0$ we get

$$\begin{aligned}\Big(\mathbb{E}\fint_{\frac{\delta}{2}}^{\delta}\fint_{B(x,\sqrt{\delta})}|\tilde{K}_L u(t,y,\omega)|^2\mathrm{d}y\frac{\mathrm{d}t}{t^{\beta-1}}\Big)^{\frac{1}{2}} &\lesssim 2^{j\frac{n}{2}}\Big(\mathbb{E}\fint_{\frac{\delta}{2}}^{\delta}\fint_{B(x,2^{j+1}\sqrt{\delta})}|\tilde{K}_L u(t,y,\omega)|^2\mathrm{d}y\frac{\mathrm{d}t}{t^{\beta-1}}\Big)^{\frac{1}{2}}\\ &\leq 2^{j\frac{n}{2}}\sup_{\delta'>0}\Big(\mathbb{E}\fint_{\frac{\delta}{2}}^{\delta}\fint_{B(x,2^{j+1}\sqrt{\delta'})}|\tilde{K}_L u(t,y,\omega)|^2\mathrm{d}y\frac{\mathrm{d}t}{t^{\beta-1}}\Big)^{\frac{1}{2}}.\end{aligned}$$

For $\delta'>0$ and $j>0$ we have, using Minkowski's inequality,

$$\Big(\mathbb{E}\fint_{\frac{\delta'}{2}}^{\delta'}\fint_{B(x,2^{j+1+\frac{k}{2}}\sqrt{\delta'})}|\tilde{K}_L u(t,y,\omega)|^2\mathrm{d}y\frac{\mathrm{d}t}{t^{\beta-1}}\Big)^{\frac{1}{2}} \lesssim$$
$$\sum_{l=1}^{\infty}\Big(\mathbb{E}\fint_{\frac{\delta'}{2}}^{\delta'}\fint_{B(x,2^{j+1+\frac{k}{2}}\sqrt{\delta'})}\Big|\int_{\frac{t}{2}}^t \mathrm{e}^{-(t-s)L}\mathrm{div}(\chi_{S_l(x,2^{j+1+\frac{k}{2}}\sqrt{\delta'})}u(s,\cdot,\cdot))(y,\omega)\mathrm{d}W(s)\Big|^2\mathrm{d}y\frac{\mathrm{d}t}{t^{\beta-1}}\Big)^{\frac{1}{2}}.$$

Now let $l=1$ and $t\in(\frac{\delta'}{2},\delta')$. By a result due to Da Prato (see, [14, Theorem 6.14]), we have

$$\mathbb{E}\Big\|\int_{\frac{t}{2}}^t \mathrm{e}^{-(t-s)L}\mathrm{div}(v(s,\cdot,\cdot))(y,\omega)\mathrm{d}W(s)\Big\|^2_{L^2(\mathbb{R}_+,t^{-(\beta-1)}\mathrm{d}t;L^2(\mathbb{R}^n))}$$

$$
\begin{aligned}
&\lesssim \mathbb{E}\Big\| \int_{\frac{t}{2}}^{t} s^{-\frac{\beta-1}{2}} \mathrm{e}^{-(t-s)L} \mathrm{div}(v(s,\cdot,\cdot))(y,\omega) \mathrm{d}W(s)\Big\|^2_{L^2(\mathbb{R}_+;L^2(\mathbb{R}^n))} \\
&\quad + \mathbb{E}\Big\| \int_{\frac{t}{2}}^{t} (s^{-\frac{\beta-1}{2}} - t^{-\frac{\beta-1}{2}}) \mathrm{e}^{-(t-s)L} \mathrm{div}(v(s,\cdot,\cdot))(y,\omega) \mathrm{d}W(s)\Big\|^2_{L^2(\mathbb{R}_+;L^2(\mathbb{R}^n))} \\
&\lesssim \mathbb{E}\Big\| \int_{0}^{t} s^{-\frac{\beta-1}{2}} \mathrm{e}^{-(t-s)L} \mathrm{div}(v(s,\cdot,\cdot))(y,\omega) \mathrm{d}W(s)\Big\|^2_{L^2(\mathbb{R}_+;L^2(\mathbb{R}^n))} \\
&\quad + \mathbb{E}\Big\| \int_{\frac{t}{2}}^{t} (s^{-\frac{\beta-1}{2}} - t^{-\frac{\beta-1}{2}}) \mathrm{e}^{-(t-s)L} \mathrm{div}(v(s,\cdot,\cdot))(y,\omega) \mathrm{d}W(s)\Big\|^2_{L^2(\mathbb{R}_+;L^2(\mathbb{R}^n))} \\
&\lesssim \mathbb{E}\|v\|^2_{L^2(\mathbb{R}_+,t^{-(\beta-1)}\mathrm{d}t;\mathscr{L}^2(\mathbb{R}^n;L^2(\mathbb{R}^n)))} \\
&\quad + \mathbb{E}\Big\| \int_{\frac{t}{2}}^{t} (s^{-\frac{\beta-1}{2}} - t^{-\frac{\beta-1}{2}}) \mathrm{e}^{-(t-s)L} \mathrm{div}(v(s,\cdot,\cdot))(y,\omega) \mathrm{d}W(s)\Big\|^2_{L^2(\mathbb{R}_+;L^2(\mathbb{R}^n))}.
\end{aligned}
$$

We can estimate the last part as follows

$$
\begin{aligned}
&\mathbb{E}\Big\| \int_{\frac{t}{2}}^{t} (s^{-\frac{\beta-1}{2}} - t^{-\frac{\beta-1}{2}}) \mathrm{e}^{-(t-s)L} \mathrm{div}(v(s,\cdot,\cdot))(y,\omega) \mathrm{d}W(s)\Big\|^2_{L^2(\mathbb{R}_+;L^2(\mathbb{R}^n))} \\
&= \int_0^\infty \mathbb{E}\left(\int_{\frac{t}{2}}^{t} (s^{-\frac{\beta-1}{2}} - t^{-\frac{\beta-1}{2}}) \mathrm{e}^{-(t-s)L} \mathrm{div}(v(s,\cdot,\cdot))(y,\omega) \mathrm{d}W(s)\right)^2 \mathrm{d}t \\
&= \int_0^\infty \mathbb{E}\int_{\frac{t}{2}}^{t} \left((s^{-\frac{\beta-1}{2}} - t^{-\frac{\beta-1}{2}}) \mathrm{e}^{-(t-s)L} \mathrm{div}(v(s,\cdot,\cdot))(y,\omega)\right)^2 \mathrm{d}s\mathrm{d}t \\
&\lesssim \mathbb{E}\int_0^\infty \int_{\frac{t}{2}}^{t} \frac{|s^{-\frac{\beta-1}{2}} - t^{-\frac{\beta-1}{2}}|^2}{|t-s|} \|v(s,\cdot,\cdot)\|_{\mathscr{L}^2(\mathbb{R}^n;L^2(\mathbb{R}^n))} \mathrm{d}s\mathrm{d}t \\
&= \mathbb{E}\int_0^\infty \int_{\frac{t}{2}}^{t} \frac{s^{-(\beta-1)}|\left(\frac{s}{t}\right)^{\frac{\beta-1}{2}} - 1|^2}{|t-s|} \|v(s,\cdot,\cdot)\|_{\mathscr{L}^2(\mathbb{R}^n;L^2(\mathbb{R}^n))} \mathrm{d}s\mathrm{d}t \\
&= \mathbb{E}\int_0^\infty \|v(s,\cdot,\cdot)\|_{\mathscr{L}^2(\mathbb{R}^n;L^2(\mathbb{R}^n))} \int_s^{2s} \frac{|\left(\frac{s}{t}\right)^{\frac{\beta-1}{2}} - 1|^2}{|\frac{t}{s}-1|} \mathrm{d}t \frac{\mathrm{d}s}{s^\beta} \\
&\lesssim \mathbb{E}\|v\|^2_{L^2(\mathbb{R}_+,t^{-(\beta-1)}\mathrm{d}t;\mathscr{L}^2(\mathbb{R}^n;L^2(\mathbb{R}^n)))}.
\end{aligned}
$$

In the second equality we used Itô's isometry. We also used multiple instances of Fubini and that $\int_s^{2s} \frac{|\left(\frac{s}{t}\right)^{\frac{\beta-1}{2}}-1|^2}{|t-s|}\mathrm{d}t \sim s$. Set $v(x,y,t,\omega) = \chi_{(\frac{\delta'}{2},\delta')}(t)\chi_{B(x,2^{j+2+\frac{k}{2}}\sqrt{\delta'})}(y)u(y,t,\omega)$. Using the above estimates and Fubini, we thus find

$$
\begin{aligned}
&\left(\mathbb{E}\fint_{\frac{\delta'}{2}}^{\delta'} \fint_{B(x,2^{j+1+\frac{k}{2}}\sqrt{\delta'})} \Big| \int_{\frac{t}{2}}^{t} \mathrm{e}^{-(t-s)L} \mathrm{div}(\chi_{S_1(x,2^{j+1+\frac{k}{2}}\sqrt{\delta'})} u(s,\cdot,\cdot))(y,\omega) \mathrm{d}W(s)\Big|^2 \mathrm{d}y \frac{\mathrm{d}t}{t^{\beta-1}}\right)^{\frac{1}{2}} \\
&\quad \le \left(\frac{1}{\delta'\delta'^{\frac{n}{2}}} \mathbb{E}\int_0^\infty \int_{\mathbb{R}^n} \Big| \int_{\frac{t}{2}}^{t} \mathrm{e}^{-(t-s)L} \mathrm{div}(v(\cdot,\cdot,s,\cdot))(x,y,\omega) \mathrm{d}W(s)\Big|^2 \mathrm{d}y \frac{\mathrm{d}t}{t^{\beta-1}}\right)^{\frac{1}{2}} \\
&\quad = \left(\frac{1}{\delta'\delta'^{\frac{n}{2}}} \mathbb{E}\Big\| \int_{\frac{t}{2}}^{t} \mathrm{e}^{-(t-s)L} \mathrm{div}(v(\cdot,\cdot,s,\cdot))(x) \mathrm{d}W(s)\Big\|^2_{L^2(\mathbb{R}_+.t^{-(\beta-1)};L^2(\mathbb{R}^n))}\right)^{\frac{1}{2}}
\end{aligned}
$$

$$
\begin{aligned}
&\lesssim \left(\frac{1}{\delta' \delta'^{\frac{n}{2}}} \mathbb{E} \|v(x)\|^2_{L^2(\mathbb{R}_+, t^{-(\beta-1)}\mathrm{d}t; \mathscr{L}^2(\mathbb{R}^n, L^2(\mathbb{R}^n)))} \right)^{\frac{1}{2}} \\
&= \left(\mathbb{E} \int_{\frac{\delta'}{2}}^{\delta'} \fint_{B(x, 2^{j+2+\frac{k}{2}}\sqrt{\delta'})} \frac{1}{\delta'} |u(t,y,\omega)|^2 \mathrm{d}y \frac{\mathrm{d}t}{t^{\beta-1}} \right)^{\frac{1}{2}} \\
&\leq \left(\mathbb{E} \int_{\frac{\delta'}{2}}^{\delta'} \fint_{B(x, 2^{j+2+\frac{k}{2}}\sqrt{\delta'})} |u(t,y,\omega)|^2 \mathrm{d}y \frac{\mathrm{d}t}{t^{\beta}} \right)^{\frac{1}{2}} \\
&\leq \left(\mathbb{E} \int_0^{\infty} \fint_{B(x, 2^{j+3+\frac{k}{2}}\sqrt{t})} |u(t,y,\omega)|^2 \mathrm{d}y \frac{\mathrm{d}t}{t^{\beta}} \right)^{\frac{1}{2}},
\end{aligned}
$$

where we used that $\frac{1}{\delta'} \leq \frac{1}{t}$ and in the penultimate line we used the same identification as in (9.3), i.e.

$$
L^2(\mathbb{R}_+, t^{-\beta}\mathrm{d}t; \mathscr{L}^2(\mathbb{R}^n, L^2(\mathbb{R}^n)) = L^2(\mathbb{R}_+, t^{-\beta}\mathrm{d}t; L^2(\mathbb{R}^n)).
$$

For $l \geq 2$ we have, where we do not write some of the variables to keep the calculations clear,

$$
\begin{aligned}
&\mathbb{E} \fint_{\frac{\delta'}{2}}^{\delta'} \fint_{B(x, 2^{j+1+\frac{k}{2}}\sqrt{\delta'})} \Big| \int_{\frac{t}{2}}^{t} \mathrm{e}^{-(t-s)L} \mathrm{div}(\chi_{S_l(x, 2^{j+1+\frac{k}{2}}\sqrt{\delta'})} u) \mathrm{d}W(s) \Big|^2 \mathrm{d}y \frac{\mathrm{d}t}{t^{\beta-1}} \\
&= \fint_{\frac{\delta'}{2}}^{\delta'} \fint_{B(x, 2^{j+1+\frac{k}{2}}\sqrt{\delta'})} \mathbb{E} \Big| \int_{\frac{t}{2}}^{t} \mathrm{e}^{-(t-s)L} \mathrm{div}(\chi_{S_l(x, 2^{j+1+\frac{k}{2}}\sqrt{\delta'})} u) \mathrm{d}W(s) \Big|^2 \mathrm{d}y \frac{\mathrm{d}t}{t^{\beta-1}} \\
&= \fint_{\frac{\delta'}{2}}^{\delta'} \fint_{B(x, 2^{j+1+\frac{k}{2}}\sqrt{\delta'})} \mathbb{E} \int_{\frac{t}{2}}^{t} |\mathrm{e}^{-(t-s)L} \mathrm{div}(\chi_{S_l(x, 2^{j+1+\frac{k}{2}}\sqrt{\delta'})} u)|^2 \mathrm{d}s \mathrm{d}y \frac{\mathrm{d}t}{t^{\beta-1}} \\
&= \mathbb{E} \fint_{\frac{\delta'}{2}}^{\delta'} \int_{\mathbb{R}^n} \int_{\frac{t}{2}}^{t} |(2^{j+1+\frac{k}{2}}\sqrt{\delta'})^{-\frac{n}{2}} \chi_{B(x, 2^{j+1+\frac{k}{2}}\sqrt{\delta'})} \mathrm{e}^{-(t-s)L} \mathrm{div}(\chi_{S_l(x, 2^{j+1+\frac{k}{2}}\sqrt{\delta'})} u)|^2 \mathrm{d}s \mathrm{d}y \frac{\mathrm{d}t}{t^{\beta-1}} \\
&= \mathbb{E} \fint_{\frac{\delta'}{2}}^{\delta'} \int_{\frac{t}{2}}^{t} \|(2^{j+1+\frac{k}{2}}\sqrt{\delta'})^{-\frac{n}{2}} \chi_{B(x, 2^{j+1+\frac{k}{2}}\sqrt{\delta'})} \mathrm{e}^{-(t-s)L} \mathrm{div}(\chi_{S_l(x, 2^{j+1+\frac{k}{2}}\sqrt{\delta'})} u)\|_2^2 \mathrm{d}s \frac{\mathrm{d}t}{t^{\beta-1}} \\
&\lesssim \mathbb{E} \int_{\frac{\delta'}{2}}^{\delta'} \int_{\frac{t}{2}}^{t} \frac{1}{\delta'} \frac{1}{t-s} 2^{2ln} \mathrm{e}^{-2c\frac{4^{l+j}2^k\delta'}{t-s}} \fint_{B(x, 2^{j+l+2+\frac{k}{2}}\sqrt{\delta'})} |u(s,y,\omega)|^2 \mathrm{d}y \mathrm{d}s \frac{\mathrm{d}t}{t^{\beta-1}} \\
&\leq \mathbb{E} \int_{\frac{\delta'}{2}}^{\delta'} \int_{\frac{t}{2}}^{t} \frac{1}{t-s} 2^{2ln} \mathrm{e}^{-2c\frac{4^{l+j}2^k\delta'}{t-s}} \fint_{B(x, 2^{j+l+2+\frac{k}{2}}\sqrt{\delta'})} |u(s,y,\omega)|^2 \mathrm{d}y \mathrm{d}s \frac{\mathrm{d}t}{t^{\beta}}.
\end{aligned}
$$

where we used multiple instances of Fubini, L^2-off diagonal estimates for $\mathrm{e}^{-tL}\mathrm{div}$ and Itô's isometry.

Now define

$$
TF(t,\cdot,\cdot) := \int_{\frac{t}{2}}^{t} \frac{1}{t-s} \mathrm{e}^{\frac{-D\delta'}{t-s}} F(s,\cdot,\cdot)\mathrm{d}s
$$

with

$$
F(s,x,\omega) = \fint_{B(x, 2^{j+l+2+\frac{k}{2}}\sqrt{\delta'})} |u(s,y,\omega)|^2 \mathrm{d}y \quad \text{and} \quad D = c4^{l+j}2^{k+1}.
$$

We want to prove that $T : L^1((\frac{\delta'}{2}, \delta'), t^{-\beta}\mathrm{d}t) \to L^1((\frac{\delta'}{2}, \delta'), t^{-\beta}\mathrm{d}t)$ is bounded. This is equivalent to proving that $T_2 : L^1((\frac{\delta'}{2}, \delta')) \to L^1((\frac{\delta'}{2}, \delta'))$ is bounded with

$$
T_2F(t,\cdot,\cdot) := \int_{\frac{t}{2}}^{t} \frac{1}{t-s} \left(\frac{s}{t}\right)^{\beta} \mathrm{e}^{\frac{-D\delta'}{t-s}} F(s,\cdot,\cdot)\mathrm{d}s.
$$

To prove this we are going use Schur's lemma. We have $t \in (\frac{\delta'}{2}, \delta')$, $s \in (\frac{t}{2}, t)$ and $\beta \geq 1$. So we get $0 \leq t-s \leq \delta'$ and $(\frac{s}{t})^\beta \leq 1$. Also, $\mathrm{e}^{\frac{-D\delta'}{t-s}} \leq (\frac{t-s}{D\delta'})^N$ for some large N which we will choose later. With these we find

$$\begin{aligned}\sup_{s \in (\frac{\delta'}{4}, \delta')} \int_{\frac{\delta'}{2} \vee s}^{\delta' \wedge 2s} \frac{1}{t-s} \left(\frac{t-s}{D\delta'} \right)^N \mathrm{d}t &= \sup_{s \in (\frac{\delta'}{4}, \delta')} \left[\frac{1}{N} \left(\frac{t-s}{D\delta'} \right)^N \right]_{\frac{\delta'}{2} \vee s}^{\delta' \wedge 2s} \\ &\lesssim D^{-N}.\end{aligned}$$

We find similar estimates for the integral with respect to s. Thus, by Schur's lemma [17, Appendix I] we get $\|TF(\cdot, x, \omega)\|_{L^1((\frac{\delta'}{2}, \delta'), t^{-\beta}\mathrm{d}t)} \lesssim D^{-N} \|F(\cdot, x, \omega)\|_{L^1((\frac{\delta'}{2}, \delta'), t^{-\beta}\mathrm{d}t)}$, and hence

$$\begin{aligned}&\left(\mathbb{E} \fint_{\frac{\delta'}{2}}^{\delta'} \fint_{B(x, 2^{j+1+\frac{k}{2}}\sqrt{\delta'})} \Big| \int_{\frac{t}{2}}^{t} \mathrm{e}^{-(t-s)L} \mathrm{div}(\chi_{S_l(x, 2^{j+1+\frac{k}{2}}\sqrt{\delta'})} u(s, \cdot, \cdot))(y) \mathrm{d}W(s) \Big|^2 \mathrm{d}y \frac{\mathrm{d}t}{t^{\beta-1}} \right)^{\frac{1}{2}} \\ &\qquad \lesssim 2^{ln} (c 4^{l+j} 2^{k+1})^{-\frac{1}{2}N} \left(\mathbb{E} \int_{\frac{\delta'}{2}}^{\delta'} \fint_{B(x, 2^{j+l+2+\frac{k}{2}}\sqrt{\delta'})} |u(t, y, \omega)|^2 \mathrm{d}y \frac{\mathrm{d}t}{t^\beta} \right)^{\frac{1}{2}} \\ &\qquad \lesssim 2^{ln} (c 4^{l+j} 2^{k+1})^{-\frac{1}{2}N} \left(\mathbb{E} \int_{0}^{\infty} \fint_{B(x, 2^{j+l+2+\frac{k}{2}}\sqrt{t})} |u(t, y, \omega)|^2 \mathrm{d}y \frac{\mathrm{d}t}{t^\beta} \right)^{\frac{1}{2}}.\end{aligned}$$

Using the change of aperture lemma [2] and summing over k, j, l we thus get

$$\begin{aligned}&\sup_{\delta > 0} \mathbb{E} \left\| \left(\fint_{\frac{\delta}{2}}^{\delta} \fint_{B(\cdot, \sqrt{\delta})} |\tilde{\mathcal{R}}_L u|^2 \mathrm{d}y \frac{\mathrm{d}t}{t^{\beta-1}} \right)^{\frac{1}{2}} \right\|_{L^p}^p \\ &\quad \lesssim \sum_{k,j,l} 2^{(l+\frac{j}{2})n} \mathrm{e}^{-c4^j} (c_2 4^{l+j} 2^{k+1})^{-\frac{1}{2}N} \left\| \left(\mathbb{E} \int_0^\infty \fint_{B(\cdot, 2^{j+l+2+\frac{k}{2}}\sqrt{t})} |u(t, y, \omega)|^2 \mathrm{d}y \frac{\mathrm{d}t}{t^\beta} \right)^{\frac{1}{2}} \right\|_{L^p}^p \\ &\quad \leq \sum_{k,j,l} 2^{(l+\frac{j}{2})n} \mathrm{e}^{-c4^j} (c_2 4^{l+j} 2^{k+1})^{-\frac{1}{2}N} \mathbb{E} \left\| \left(\int_0^\infty \fint_{B(\cdot, 2^{j+l+2+\frac{k}{2}}\sqrt{t})} |u(t, y, \omega)|^2 \mathrm{d}y \frac{\mathrm{d}t}{t^\beta} \right)^{\frac{1}{2}} \right\|_{L^p}^p \\ &\quad \lesssim \sum_{k,j,l} 2^{(l+\frac{j}{2})n} \mathrm{e}^{-c4^j} (c_2 4^{l+j} 2^{k+1})^{-\frac{1}{2}N} 2^{(j+l+\frac{k}{2})\tau} \mathbb{E} \|u\|_{T^{p,2}_\beta}^p \\ &\quad \leq C_{p,\beta,n} \mathbb{E} \|u\|_{T^{p,2}_\beta}^p,\end{aligned}$$

with τ only depending on n and p and where we chose $N \geq 2(n+\tau)$. In the second inequality we used a combination of Fubini and Jensen's inequality to get the expectation out of the norm. (Only Fubini is needed in the case $p = 2$). The restriction $p \geq 2$ is provided by the use of Jensen's inequality since it works only with convex functions. □

We have a similar result for a slightly different operator. Let $b : \mathbb{R}^n \to \mathbb{R}$ be a globally Lipschitz continuous function that satisfies

$$|b(x)| \leq C_b |x|, \quad x \in \mathbb{R}^n \tag{10.2}$$

For adapted simple processes, u, we define the operator $\mathcal{T}$ as

$$\mathcal{T}_L u(t, x, \omega) = \int_0^t \mathrm{e}^{-(t-s)L} b(u(s, \cdot, \cdot))(x, \omega) \mathrm{d}W(s).$$

Proposition 10.2. *Let $p \geq 2$, $\beta \geq 0$ and let $u : \mathbb{R}_+^{n+1} \times \Omega \to \mathbb{R}^n$ be an adapted simple process. Then we have*

$$\sup_{\delta>0} \mathbb{E} \left\| \left(\fint_{\frac{\delta}{2}}^{\delta} \fint_{B(\cdot,\sqrt{\delta})} |\mathcal{T}_L u|^2 dy \frac{dt}{t^\beta} \right)^{\frac{1}{2}} \right\|_{L^p}^p \leq C_{p,\beta,n,b} \mathbb{E} \|u\|_{T_\beta^{p,2}}^p,$$

with $C_{p,\beta,n.b}$ independent of u.

Proof. Using Fubini and Kahane-Khintchine inequality, we have, similar to the proof of the previous proposition,

$$\mathbb{E} \left\| \left(\fint_{\frac{\delta}{2}}^{\delta} \fint_{B(\cdot,\sqrt{\delta})} |\mathcal{T}_L u|^2 \mathrm{d}y \frac{\mathrm{d}t}{t^\beta} \right)^{\frac{1}{2}} \right\|_{L^p}^p \simeq \int_{\mathbb{R}^n} \left(\mathbb{E} \fint_{\frac{\delta}{2}}^{\delta} \fint_{B(x,\sqrt{\delta})} |\mathcal{T}_L u|^2 \mathrm{d}y \frac{\mathrm{d}t}{t^\beta} \right)^{\frac{p}{2}} \mathrm{d}x,$$

with implicit constant depending on p.
Now define $\tilde{K}_L u(t, x, \omega) = \int_{\frac{t}{2}}^{t} \mathrm{e}^{-(t-s)L} b(u(s, \cdot\cdot))(x, \omega) \mathrm{d}W(s)$. Similar to Proposition 5.8 we have

$$\mathcal{T}_L u(t, x, \omega) = \sum_{k=0}^{\infty} \mathrm{e}^{-(1-2^{-k})tL} \tilde{K}_L u(2^{-k} t, x, \omega) \quad \text{a.e.}$$

Fixing $x \in \mathbb{R}^n$ and $k \in \mathbb{N}\backslash\{0\}$ we get, analogous to Proposition 5.8,

$$\left(\mathbb{E} \fint_{\frac{\delta}{2}}^{\delta} \fint_{B(x,\sqrt{\delta})} |\mathrm{e}^{-(1-2^{-k})tL} \tilde{K}_L u(2^{-k} t, y, \omega)|^2 \mathrm{d}y \frac{\mathrm{d}t}{t^\beta} \right)^{\frac{1}{2}}$$
$$\lesssim \sum_{j=1}^{\infty} 2^{j\frac{n}{2}} \mathrm{e}^{-c4^j} \sup_{\delta'>0} \left(\mathbb{E} \fint_{\frac{\delta'}{2}}^{\delta'} \fint_{B(x,2^{j+1+\frac{k}{2}}\sqrt{\delta'})} |\tilde{K}_L u(t, y, \omega)|^2 \mathrm{d}y \frac{\mathrm{d}t}{t^\beta} \right)^{\frac{1}{2}}.$$

For $k = 0$ and any $j \geq 0$ we get

$$\begin{aligned} \left(\mathbb{E} \fint_{\frac{\delta}{2}}^{\delta} \fint_{B(x,\sqrt{\delta})} |\tilde{K}_L u(t, y, \omega)|^2 \mathrm{d}y \frac{\mathrm{d}t}{t^\beta} \right)^{\frac{1}{2}} &\lesssim 2^{j\frac{n}{2}} \left(\mathbb{E} \fint_{\frac{\delta}{2}}^{\delta} \fint_{B(x,2^{j+1}\sqrt{\delta})} |\tilde{K}_L u(t, y, \omega)|^2 \mathrm{d}y \frac{\mathrm{d}t}{t^\beta} \right)^{\frac{1}{2}} \\ &\leq 2^{j\frac{n}{2}} \sup_{\delta'>0} \left(\mathbb{E} \fint_{\frac{\delta}{2}}^{\delta} \fint_{B(x,2^{j+1}\sqrt{\delta'})} |\tilde{K}_L u(t, y, \omega)|^2 \mathrm{d}y \frac{\mathrm{d}t}{t^\beta} \right)^{\frac{1}{2}}. \end{aligned}$$

For $\delta' > 0$ and $j > 0$ we have, using Minkowski's inequality,

$$\left(\mathbb{E} \fint_{\frac{\delta'}{2}}^{\delta'} \fint_{B(x,2^{j+1+\frac{k}{2}}\sqrt{\delta'})} |\tilde{K}_L u(t, y, \omega)|^2 \mathrm{d}y \frac{\mathrm{d}t}{t^\beta} \right)^{\frac{1}{2}} \leq$$
$$\sum_{l=1}^{\infty} \left(\mathbb{E} \fint_{\frac{\delta'}{2}}^{\delta'} \fint_{B(x,2^{j+1+\frac{k}{2}}\sqrt{\delta'})} \Big| \int_{\frac{t}{2}}^{t} \mathrm{e}^{-(t-s)L} (\chi_{S_l(x,2^{j+1+\frac{k}{2}}\sqrt{\delta'})} b(u(s, \cdot, \cdot)))(y, \omega) \mathrm{d}W(s) \Big|^2 \mathrm{d}y \frac{\mathrm{d}t}{t^\beta} \right)^{\frac{1}{2}}.$$

For $l = 1$, and $t \in (\frac{\delta'}{2}, \delta')$ we have, where we do not write some of the variables to keep the calculations clear,

$$\mathbb{E} \fint_{\frac{\delta'}{2}}^{\delta'} \fint_{B(x,2^{j+1+\frac{k}{2}}\sqrt{\delta'})} \Big| \int_{\frac{t}{2}}^{t} \mathrm{e}^{-(t-s)L} (\chi_{S_1(x,2^{j+1+\frac{k}{2}}\sqrt{\delta'})} b(u)) \mathrm{d}W(s) \Big|^2 \mathrm{d}y \frac{\mathrm{d}t}{t^\beta}$$

$$
\begin{aligned}
&= \fint_{\frac{\delta'}{2}}^{\delta'} \fint_{B(x,2^{j+1+\frac{k}{2}}\sqrt{\delta'})} \mathbb{E}\Big|\int_{\frac{t}{2}}^{t} \mathrm{e}^{-(t-s)L}(\chi_{S_1(x,2^{j+1+\frac{k}{2}}\sqrt{\delta'})}b(u))\mathrm{d}W(s)\Big|^2 \mathrm{d}y\frac{\mathrm{d}t}{t^\beta}\\
&= \fint_{\frac{\delta'}{2}}^{\delta'} \fint_{B(x,2^{j+1+\frac{k}{2}}\sqrt{\delta'})} \mathbb{E}\int_{\frac{t}{2}}^{t} |\mathrm{e}^{-(t-s)L}(\chi_{S_1(x,2^{j+1+\frac{k}{2}}\sqrt{\delta'})}b(u))|^2 \mathrm{d}s\mathrm{d}y\frac{\mathrm{d}t}{t^\beta}\\
&= \mathbb{E}\fint_{\frac{\delta'}{2}}^{\delta'} \int_{\mathbb{R}^n}\int_{\frac{t}{2}}^{t} |(2^{j+1+\frac{k}{2}}\sqrt{\delta'})^{-\frac{n}{2}}\chi_{B(x,2^{j+1+\frac{k}{2}}\sqrt{\delta'})}\mathrm{e}^{-(t-s)L}(\chi_{S_1(x,2^{j+1+\frac{k}{2}}\sqrt{\delta'})}b(u))|^2 \mathrm{d}s\mathrm{d}y\frac{\mathrm{d}t}{t^\beta}\\
&= \mathbb{E}\fint_{\frac{\delta'}{2}}^{\delta'} \int_{\frac{t}{2}}^{t} \|(2^{j+1+\frac{k}{2}}\sqrt{\delta'})^{-\frac{n}{2}}\chi_{B(x,2^{j+1+\frac{k}{2}}\sqrt{\delta'})}\mathrm{e}^{-(t-s)L}(\chi_{S_1(x,2^{j+1+\frac{k}{2}}\sqrt{\delta'})}b(u))\|_2^2 \mathrm{d}s\frac{\mathrm{d}t}{t^\beta}\\
&\le \mathbb{E}\fint_{\frac{\delta'}{2}}^{\delta'} \int_{\frac{t}{2}}^{t} \|(2^{j+1+\frac{k}{2}}\sqrt{\delta'})^{-\frac{n}{2}}\chi_{B(x,2^{j+2+\frac{k}{2}}\sqrt{\delta'})}b(u)\|_2^2 \mathrm{d}s\frac{\mathrm{d}t}{t^\beta}\\
&\lesssim C_b\mathbb{E}\int_{\frac{\delta'}{2}}^{\delta'} \int_{\frac{t}{2}}^{t} \frac{1}{\delta'}\fint_{B(x,2^{j+1+\frac{k}{2}}\sqrt{\delta'})} |u(s,y,\omega)|^2 \mathrm{d}y\mathrm{d}s\frac{\mathrm{d}t}{t^\beta}.
\end{aligned}
$$

Aside from multiple instances of Fubini, we used Itô's isometry, the fact that the semigroup of L is a contraction and the property of the function b. Now define

$$TF(t,\cdot,\cdot) := \int_{\frac{t}{2}}^{t} \frac{1}{\delta'}F(s,\cdot,\cdot)\mathrm{d}s$$

with

$$F(s,x,\omega) = \fint_{B(x,2^{j+l+2+\frac{k}{2}}\sqrt{\delta'})} |u(s,y,\omega)|^2\mathrm{d}y.$$

We want to prove that $T : L^1((\frac{\delta'}{2},\delta'), t^{-\beta}\mathrm{d}t) \to L^1((\frac{\delta'}{2},\delta'), t^{-\beta}\mathrm{d}t)$ is bounded. This is equivalent to proving that $T_2 : L^1((\frac{\delta'}{2},\delta')) \to L^1((\frac{\delta'}{2},\delta'))$ is bounded with

$$T_2F(t,\cdot,\cdot) := \int_{\frac{t}{2}}^{t} \frac{1}{\delta'}\left(\frac{s}{t}\right)^\beta F(s,\cdot,\cdot)\mathrm{d}s.$$

We have $t \in (\frac{\delta'}{2},\delta')$, $s \in (\frac{t}{2},t)$ and $\beta \ge 0$. So $(\frac{s}{t})^\beta \le 1$. Using these we find

$$\sup_{t\in(\frac{\delta'}{4},\delta')} \int_{\frac{t}{2}}^{t} \frac{1}{\delta'}\left(\frac{s}{t}\right)^\beta \mathrm{d}s \le \sup_{t\in(\frac{\delta'}{4},\delta')} \int_{\frac{t}{2}}^{t} \mathrm{d}s \lesssim C.$$

We find similar estimates for the integral with respect to t. Thus, by Schur's lemma [17, Appendix I] we get $\|TF(\cdot,x,\omega)\|_{L^1((\frac{\delta'}{2},\delta'),t^{-\beta}\mathrm{d}t)} \lesssim \|F(\cdot,x,\omega)\|_{L^1((\frac{\delta'}{2},\delta'),t^{-\beta}\mathrm{d}t)}$, and hence

$$
\begin{aligned}
&\left(\mathbb{E}\fint_{\frac{\delta'}{2}}^{\delta'} \fint_{B(x,2^{j+1+\frac{k}{2}}\sqrt{\delta'})} \Big|\int_{\frac{t}{2}}^{t} \mathrm{e}^{-(t-s)L}(\chi_{S_1(x,2^{j+1+\frac{k}{2}}\sqrt{\delta'})}b(u(s,\cdot,\cdot)))(y,\omega)\mathrm{d}W(s)\Big|^2 \mathrm{d}y\frac{\mathrm{d}t}{t^\beta}\right)^{\frac{1}{2}}\\
&\qquad\lesssim \left(\mathbb{E}\int_{\frac{\delta'}{2}}^{\delta'} \fint_{B(x,2^{j+2+\frac{k}{2}}\sqrt{\delta'})} |u(t,y,\omega)|^2\mathrm{d}y\frac{\mathrm{d}t}{t^\beta}\right)^{\frac{1}{2}}\\
&\qquad\lesssim \left(\mathbb{E}\int_0^\infty \fint_{B(x,2^{j+3+\frac{k}{2}}\sqrt{t})} |u(t,y,\omega)|^2\mathrm{d}y\frac{\mathrm{d}t}{t^\beta}\right)^{\frac{1}{2}}.
\end{aligned}
$$

Now we are going to consider $l \ge 2$. We have, similar the to $l = 1$ case

$$\mathbb{E}\fint_{\frac{\delta'}{2}}^{\delta'} \fint_{B(x,2^{j+1+\frac{k}{2}}\sqrt{\delta'})} \Big|\int_{\frac{t}{2}}^{t} \mathrm{e}^{-(t-s)L}(\chi_{S_l(x,2^{j+1+\frac{k}{2}}\sqrt{\delta'})}b(u(s,\cdot,\cdot)))(y,\omega)\mathrm{d}W(s)\Big|^2 \mathrm{d}y\frac{\mathrm{d}t}{t^\beta}$$

$$= \mathbb{E}⨍_{\frac{\delta'}{2}}^{\delta'} \int_{\frac{t}{2}}^{t} \|(2^{j+1+\frac{k}{2}}\sqrt{\delta'})^{-\frac{n}{2}} \chi_{B(x,2^{j+1+\frac{k}{2}}\sqrt{\delta'})} \mathrm{e}^{-(t-s)L}(\chi_{S_l(x,2^{j+1+\frac{k}{2}}\sqrt{\delta'})} b(u))\|_2^2 \mathrm{d}s \frac{\mathrm{d}t}{t^\beta}.$$

Using L^2 off-diagonal estimates for the semigroup, we then find

$$\begin{aligned}
&\mathbb{E}⨍_{\frac{\delta'}{2}}^{\delta'} \int_{\frac{t}{2}}^{t} \|(2^{j+1+\frac{k}{2}}\sqrt{\delta'})^{-\frac{n}{2}} \chi_{B(x,2^{j+1+\frac{k}{2}}\sqrt{\delta'})} \mathrm{e}^{-(t-s)L}(\chi_{S_l(x,2^{j+1+\frac{k}{2}}\sqrt{\delta'})} b(u))\|_2^2 \mathrm{d}s \frac{\mathrm{d}t}{t^\beta} \\
&\qquad \lesssim 2^{2ln} \mathbb{E} \int_{\frac{\delta'}{2}}^{\delta'} \int_{\frac{t}{2}}^{t} \frac{1}{\delta'} \mathrm{e}^{-2c\frac{4^{l+j}2^k\delta'}{t-s}} ⨍_{B(x,2^{j+l+2+\frac{k}{2}}\sqrt{\delta'})} |u(s,y,\omega)|^2 \mathrm{d}y \mathrm{d}s \frac{\mathrm{d}t}{t^\beta}.
\end{aligned}$$

Applying Schur's lemma again, with similar arguments as the previous lemma, provides us with

$$\begin{aligned}
&\left(\mathbb{E}⨍_{\frac{\delta'}{2}}^{\delta'} ⨍_{B(x,2^{j+1+\frac{k}{2}}\sqrt{\delta'})} \Big| \int_{\frac{t}{2}}^{t} \mathrm{e}^{-(t-s)L}(\chi_{S_l(x,2^{j+1+\frac{k}{2}}\sqrt{\delta'})} b(u(s,\cdot,\cdot)))(y,\omega) \mathrm{d}W(s)\Big|^2 \mathrm{d}y \frac{\mathrm{d}t}{t^\beta}\right)^{\frac{1}{2}} \\
&\qquad \lesssim 2^{ln}(c4^{l+j}2^{k+1})^{-\frac{1}{2}N} \left(\mathbb{E}\int_{\frac{\delta'}{2}}^{\delta'} ⨍_{B(x,2^{j+l+2+\frac{k}{2}}\sqrt{\delta'})} |u(t,y,\omega)|^2 \mathrm{d}y \frac{\mathrm{d}t}{t^\beta}\right)^{\frac{1}{2}} \\
&\qquad \lesssim 2^{ln}(c4^{l+j}2^{k+1})^{-\frac{1}{2}N} \left(\mathbb{E}\int_{0}^{\infty} ⨍_{B(x,2^{j+l+2+\frac{k}{2}}\sqrt{t})} |u(t,y,\omega)|^2 \mathrm{d}y \frac{\mathrm{d}t}{t^\beta}\right)^{\frac{1}{2}}.
\end{aligned}$$

The rest of the proof is the same as the proof of Proposition 10.1. □

Chapter 11

Vector-valued X^p

In this chapter we are going to define a vector-valued version of the space X^p, namely $\tilde{X}_p$, and show that the operator $\mathcal{T}_L$ is bounded from $L^p(\Omega; T^{p,2}_\beta)$ to $\tilde{X}_p$.

We are going to define $\tilde{X}^p$ through a variation of a Rademacher maximal function similar to the work done in [19, Section 7]. To do so we need Rademacher variables. A (real) Rademacher variable is a random variable $\epsilon : \tilde{\Omega} \to \{-1, 1\}$ satisfying

$$\mathbb{P}(\epsilon = -1) = \mathbb{P}(\epsilon = 1) = \frac{1}{2}.$$

A (real) Rademacher sequence, $(\epsilon_k)_{k\in\mathbb{Z}}$, is a sequence of Rademacher variables.

Definition 11.1. *Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probablity space and let $k \in \mathbb{Z}$. For a process $u : \mathbb{R}^{n+1}_+ \times \Omega \to \mathbb{R}^n$, such that $u(\omega) \in L^2_{loc}(\mathbb{R}^{n+1}_+)$, we define the functional $\tilde{N}_{k,\beta}$ by*

$$\tilde{N}_{k,\beta}(u)(x,\omega) := \left(\fint_{2^{k-1}}^{2^k} \fint_{B(x,\sqrt{2^k})} |u(t,y,\omega)|^2 \, dy \frac{dt}{t^\beta} \right)^{\frac{1}{2}}, \quad \forall x \in \mathbb{R}^n.$$

Note that we can write $\tilde{N}_{k,\beta}(u)$ as an L^2 norm in the following way

$$\begin{aligned} \fint_{2^{k-1}}^{2^k} \fint_{B(x,\sqrt{2^k})} |u(t,y,\omega)|^2 \mathrm{d}y \frac{\mathrm{d}t}{t^\beta} &\simeq \int_{\mathbb{R}} \int_{\mathbb{R}^n} \frac{\chi_{(2^{k-1},2^k)}}{2^{k-1}} \frac{\chi_{B(x,\sqrt{2^k})}}{2^{\frac{nk}{2}}} |u(t,y,\omega)|^2 \mathrm{d}y \frac{\mathrm{d}t}{t^\beta} \\ &= \Big\| \frac{\chi_{(2^{k-1},2^k)}}{2^{\frac{k-1}{2}}} \frac{\chi_{B(x,\sqrt{2^k})}}{2^{\frac{nk}{4}}} u(\cdot,\cdot,\omega) \Big\|^2_{L^2(\mathbb{R}^{n+1}_+; t^{-\beta}\mathrm{d}y \times \mathrm{d}t)} \end{aligned} \tag{11.1}$$

We are now able to define our variation of the Rademacher maximal function as follows

$$\begin{aligned} M_{\mathrm{Rad},\beta} u(x) := \sup \Bigg\{ &\mathbb{E}_{\tilde{\Omega}} \Big\| \sum_{k\in\mathbb{Z}} \epsilon_k \lambda_k \tilde{N}_{k,\beta}(u)(x,\cdot) \Big\|_{L^p(\Omega)} : \\ &\lambda = (\lambda_k)_{k\in\mathbb{Z}} \text{ finitely non-zero with } \|\lambda\|_{l^2(\mathbb{Z})} \le 1 \Bigg\}. \end{aligned}$$

Definition 11.2. *Let $1 \le p < \infty$. $\tilde{X}^p_\beta$ is defined as the space of $u : \mathbb{R}^{n+1}_+ \times \Omega \to \mathbb{R}^n$ such that*

$$\|u\|_{\tilde{X}^p_\beta} := \|M_{Rad,\beta} u\|_p < \infty.$$

It is easily verified that the above indeed defines a norm and hence $\tilde{X}^p_\beta$ is a normed vector space. Furthermore, we have the following.

Proposition 11.3. *Let $1 \leq p < \infty$. Then $\tilde{X}^p_\beta$ is a Banach space.*

Proof. We are going to use that a normed space is a Banach space if and only if an absolute convergent series is also convergent.
Suppose $(u_n)_{n\in\mathbb{N}}$ is an absolute convergent series in $\tilde{X}^p_\beta$, i.e. $\sum_{n\in\mathbb{N}} \|u_n\|_{\tilde{X}^p} < \infty$. Fixing x and using Kahane-Khintchine inequality, [19, Proposition 2.3], we have

$$\mathbb{E}_{\tilde{\Omega}} \Big\| \sum_{k\in\mathbb{Z}} \epsilon_k \lambda_k \tilde{N}_{k,\beta}(\sum_{n\in\mathbb{N}} u_n)(x,\cdot) \Big\|_{L^p(\Omega)} \simeq \Big\| \Big(\sum_{k\in\mathbb{Z}} |\lambda_k|^2 |\tilde{N}_{k,\beta}(\sum_{n\in\mathbb{N}} u_n)(x,\cdot)|^2 \Big)^{\frac{1}{2}} \Big\|_{L^p(\Omega)}$$
$$= \Big\| \Big(\sum_{k\in\mathbb{Z}} |\tilde{N}_{k,\beta}(\lambda_k \sum_{n\in\mathbb{N}} u_n)(x,\cdot)|^2 \Big)^{\frac{1}{2}} \Big\|_{L^p(\Omega)}.$$

We now define the function $T_x u(\cdot,\cdot,\omega) : \mathbb{Z} \to L^2(\mathbb{R}^{n+1}_+; t^{-\beta}\mathrm{d}y \times \mathrm{d}t)$ by

$$\left(T_x u(\cdot,\cdot,\omega)\right)(k) := \lambda_k \frac{\chi_{(2^{k-1},2^k)}}{2^{\frac{k-1}{2}}} \frac{\chi_{B(x,\sqrt{2^k})}}{2^{\frac{nk}{4}}} u(\cdot,\cdot,\omega).$$

So we have $T_x u(\cdot,\cdot,\omega) \in l^2(\mathbb{Z}; L^2(\mathbb{R}^{n+1}_+; t^{-\beta}\mathrm{d}y \times \mathrm{d}t))$. Using this we get

$$\Big\| \Big(\sum_{k\in\mathbb{Z}} |\tilde{N}_{k,\beta}(\lambda_k \sum_{n\in\mathbb{N}} u_n)(x,\cdot)|^2 \Big)^{\frac{1}{2}} \Big\|_{L^p(\Omega)} = \|T_x \sum_{n\in\mathbb{N}} u_n\|_{L^p(\Omega; l^2(\mathbb{Z}; L^2(\mathbb{R}^{n+1}_+; t^{-\beta}\mathrm{d}y\times\mathrm{d}t)))}.$$

Now we can use Minkowski's inequality to find

$$\|\sum_{n\in\mathbb{N}} u_n\|_{\tilde{X}^p} \lesssim \Big\| \sup_{\|\lambda\|_{l^2(\mathbb{Z})}\leq 1} \|T_x \sum_{n\in\mathbb{N}} u_n\|_{L^p(\Omega; l^2(\mathbb{Z}; L^2(\mathbb{R}^{n+1}_+; t^{-\beta}\mathrm{d}y\times\mathrm{d}t)))} \Big\|_p$$
$$\leq \sum_{n\in\mathbb{N}} \Big\| \sup_{\|\lambda\|_{l^2(\mathbb{Z})}\leq 1} \|T_x u_n\|_{L^p(\Omega; l^2(\mathbb{Z}; L^2(\mathbb{R}^{n+1}_+; t^{-\beta}\mathrm{d}y\times\mathrm{d}t)))} \Big\|$$
$$\simeq \sum_{n\in\mathbb{N}} \|u_n\|_{\tilde{X}^p}.$$

By assumption the last expression is finite and hence $\sum_{n\in\mathbb{N}} u_n$ converges. □

This brings us to the following proposition.

Proposition 11.4. *Let $p \geq 2$, $\beta \geq 0$ and let $u : \mathbb{R}^{n+1}_+ \times \Omega \to \mathbb{R}^n$ be an adapted simple process. The operator $\mathcal{T}_L$ extends to a bounded operator from $L^p(\Omega; T^{p,2}_\beta)$ to $\tilde{X}^p_\beta$.*

Proof. Fixing x and using Kahane-Khintchine inequality, [19, Proposition 2.3], we have

$$\mathbb{E}_{\tilde{\Omega}} \Big\| \sum_{k\in\mathbb{Z}} \epsilon_k \lambda_k \tilde{N}_{k,\beta}(\mathcal{T}_L u)(x,\cdot) \Big\|_{L^p(\Omega)} \simeq \Big\| \Big(\sum_{k\in\mathbb{Z}} |\lambda_k|^2 |\tilde{N}_{k,\beta}(\mathcal{T}_L u)(x,\cdot)|^2 \Big)^{\frac{1}{2}} \Big\|_{L^p(\Omega)}.$$

Denoting $N = (\tilde{N}_{k,\beta}(\mathcal{T}_L u)(x,\cdot))_{k\in\mathbb{Z}}$ we get

$$\Big\| \Big(\sum_{k\in\mathbb{Z}} |\lambda_k|^2 |\tilde{N}_{k,\beta}(\mathcal{T}_L u)(x,\cdot)|^2 \Big)^{\frac{1}{2}} \Big\|_{L^p(\Omega)} = \|\lambda N\|_{L^p(\Omega; l^2(\mathbb{Z}))}$$
$$= \Big(\mathbb{E} \|\lambda N\|^p_{l^2(\mathbb{Z})} \Big)^{\frac{1}{p}}$$

$$\simeq \Big(\mathbb{E}\|\lambda N\|^2_{l^2(\mathbb{Z})}\Big)^{\frac{1}{2}}$$
$$= \Big(\sum_{k\in\mathbb{Z}} |\lambda_k|^2 \mathbb{E}(|\tilde{N}_{k,\beta}(\mathcal{T}_L u)(x,\cdot)|^2)\Big)^{\frac{1}{2}}.$$

In the penultimate line we used Kahane-Khintchine inequality and in the last line we used Fubini. For each k we have

$$\mathbb{E}(|\tilde{N}_{k,\beta}(\mathcal{T}_L u)(x,\cdot)|^2) = \mathbb{E}⨍_{2^{k-1}}^{2^k} ⨍_{B(x,\sqrt{2^k})} |\mathcal{T}_L u|^2 \mathrm{d}y \frac{\mathrm{d}t}{t^\beta}.$$

Now continuing as in Proposition 10.2 we find

$$\left(\mathbb{E}⨍_{2^{k-1}}^{2^k} ⨍_{B(x,\sqrt{2^k})} |\mathcal{T}_L u|^2 \mathrm{d}y \frac{\mathrm{d}t}{t^\beta}\right)^{\frac{1}{2}} \lesssim$$
$$\sum_{m,j,l} 2^{(l+\frac{j}{2})n} \mathrm{e}^{-c4^j} (c_2 4^{l+j} 2^{m+1})^{-\frac{1}{2}M} \left(\mathbb{E}\int_0^\infty ⨍_{B(x,2^{j+l+2+\frac{m}{2}}\sqrt{t})} |u(t,y,\omega)|^2 \mathrm{d}y \frac{\mathrm{d}t}{t^\beta}\right)^{\frac{1}{2}}$$

where M is a big enough constant which we will choose later. As can be seen the above expression is independent of k. Because of this and the fact that $\sum_{k\in\mathbb{Z}} |\lambda_k|^2 \le 1$ we thus get

$$\begin{aligned}
\|\mathcal{T}_L u\|_{\tilde{X}^p} &\lesssim \sum_{m,j,l} 2^{(l+\frac{j}{2})n} \mathrm{e}^{-c4^j} (c_2 4^{l+j} 2^{m+1})^{-\frac{1}{2}M} \left\| \left(\mathbb{E}\int_0^\infty ⨍_{B(\cdot,2^{j+l+2+\frac{k}{2}}\sqrt{t})} |u(t,y,\omega)|^2 \mathrm{d}y \frac{\mathrm{d}t}{t^\beta}\right)^{\frac{1}{2}} \right\|^p_{L^p} \\
&\le \sum_{m,j,l} 2^{(l+\frac{j}{2})n} \mathrm{e}^{-c4^j} (c_2 4^{l+j} 2^{m+1})^{-\frac{1}{2}M} \mathbb{E} \left\| \left(\int_0^\infty ⨍_{B(\cdot,2^{j+l+2+\frac{k}{2}}\sqrt{t})} |u(t,y,\omega)|^2 \mathrm{d}y \frac{\mathrm{d}t}{t^\beta}\right)^{\frac{1}{2}} \right\|^p_{L^p} \\
&\lesssim \sum_{m,j,l} 2^{(l+\frac{j}{2})n} \mathrm{e}^{-c4^j} (c_2 4^{l+j} 2^{m+1})^{-\frac{1}{2}M} 2^{(j+l+\frac{m}{2})\tau} \mathbb{E}\|u\|^p_{T^{p,2}_\beta} \\
&\le C_{p,\beta,n} \mathbb{E}\|u\|^p_{T^{p,2}_\beta},
\end{aligned}$$

with τ only depending on n and p and where we chose $M \ge 2(n+\tau)$. In the second inequality we used a combination of Fubini and Jensen's inequality to get the expectation out of the norm. (Only Fubini is needed in the case $p = 2$). Again, the restriction $p \ge 2$ is provided by the use of Jensen's inequality since it works only with convex functions. In the third inequality we used the change of angle for $T^{p,2}_\beta$.

□

Chapter 12

Concluding remarks

In this final chapter, we will briefly discuss our obtained results.

As discussed in Chapter 9, the solution of (1.1) is formally given by

$$\begin{aligned} u(t,x,\omega) &= \mathrm{e}^{-tL}u_0(x,\omega) + \int_0^t \mathrm{e}^{-(t-s)L} b(u(s,\cdot,\cdot))(x,\omega)\mathrm{d}W(s) \\ &= \mathrm{e}^{-tL}u_0(x,\omega) + \mathcal{T}_L u(t,x,\omega). \end{aligned}$$

If our solution is of the above form, then we would be able to get the following estimate:

$$\|u\|_{\tilde{X}^p_\beta} \lesssim \mathbb{E}\|\nabla u\|_{T^{p,2}_\beta}.$$

We would be able to obtain this estimate by using Proposition 11.4, which stated that, for $p \geq 2$, the operator $\mathcal{T}_l$ is bounded from $T^{p,2}_\beta$ to $\tilde{X}^p_\beta$, and then reasoning as in the proof of Proposition 7.1.

A second remark we would want to make is about Proposition 11.4 itself. Based on its proof it would seem that we could have defined $\tilde{X}^p_\beta$ in a more direct manner and get a similar result. By defining $\tilde{X}^p_\beta$ as the $L^p(\mathbb{R}^n)$- norm of

$$\sup_{\delta>0} \left\| \left(\fint_{\frac{\delta}{2}}^{\delta} \fint_{B(x,\sqrt{\delta})} |u|^2 \mathrm{d}y \frac{\mathrm{d}t}{t^\beta} \right)^{\frac{1}{2}} \right\|_{L^p(\Omega)}, \qquad x \in \mathbb{R}^n,$$

and arguing as in Proposition 10.2 we would get boundedness from $T^{p,2}_\beta$ to $\tilde{X}^p_\beta$ as well. We figured this out in hindsight.

A final remark we make is with regards to our goal at the start of this thesis. We wanted a stochastic analogue of the non-tangential maximal function estimate with respect to the gradient of the solution. Ideally we wanted an estimate of the form

$$\mathbb{E}\|u\|_{X^p_\beta} \lesssim \mathbb{E}\|\nabla u\|_{T^{p,2}_\beta},$$

where X^p_β would be a weighted in time version of X^p. To get such an estimate we need to be able to get a supremum out of the expectation in our setting, which we were not able to do. This is the reason why we considered the Rademacher maximal function as a substitute of our non-tangential maximal function, but even in this case we could only get an estimate on the vector-valued version of X^p.

Bibliography

[1] W. Arendt, R. Chill, C. Seifert, H. Vogt and J. Vogt, *Form methods for evolution equations, and applications*, Lecture Notes, available at https://www.mat.tuhh.de/veranstaltungen/isem18/images/5/59/LectureNotes.pdf, 2015.

[2] P. Auscher, *Change of angle in tent spaces*, C.R. Math. Acad. Sci. Paris **349**, no. 5-6, 297-301, (2011).

[3] P. Auscher, *On necessary and sufficient conditions for L^p-estimates of Riesz transforms associated to elliptic operators on $\mathbb{R}^n$ and related estimates*, Mem. Amer. Math. Soc. **186**, no. 871, xviii+75 (2007).

[4] P. Auscher, A. Axelsson, *Remarks on maximal regularity estimates*, Herbert Amann Festschrift, Progress in Nonlinear diffirential equations and their applications, vol. **80**, pp. 45-56. Birkhäuser (2011).

[5] P. Auscher, S. Hofmann, M. Lacey, A.McIntosh and Ph.Tchamitchan, *The solution of the Kato square root problem for second order elliptic operators on $\mathbb{R}^n$*, Annals of Mathematics, Second Series, **156**, no. 2, 633-54 (2002).

[6] P. Auscher, S. Monniaux and P. Portal, *On existence and uniqueness for non-autonomous parabolic Cauchy problems with rough coefficients*, arXiv:1511.05134v2 [math.AP], (19 November 2015).

[7] P. Auscher, S. Monniaux, P. Portal, *The maximal regularity operator on tent spaces*, Communications on pure and applied analysis, Volume **11**, Number **6** (November 2012).

[8] P. Auscher, J. van Neerven, P. Portal, *Conical stochastic maximal L^p-regularity for* $1 \leq p < \infty$, Math. Ann. **359**, 863-889, (2014).

[9] P. Auscher. Ph. Tchamitchian, *Square root problem for divergence operators and related topics.* Astérisque no. 249, viii+172 (1998).

[10] A. Chojnowska-Michalik, *Stochastic differential equations in Hilbert spaces*, Banach Center Publications **5**(1), 53-74, (1979).

[11] R.R. Coifman, Y. Meyer, E.M. Stein, *Some new function spaces and their applications to harmonic analysis*, J. Funct. Anal. **62**(2), 304-335, (1985).

[12] M. Cowling, I. Doust, A. McIntosh, A. Yagi, *Banach space operators with a bounded H^∞ functional calculus*, J. Aust. Math. Soc. Ser. A, **60**(1), 51-89, (1996).

[13] G. Da Prato, *Regularity results of a convolution stochastic integral and applications to parabolic stochastic equations in a Hilbert space*, Conf. Sem. Mat. Univ. Bari **182**, 17 (1982).

[14] G. Da prato, J. Zabczyk, *Stochastic equations in infinite dimensions*, encyclopedia of mathematics and its applications, vol. 44 Cambridge University Press, Cambride (1992).

[15] C. Feffermann, M. Stein, *H^p spaces of several variables*, Acta Mathematica, **129**, 137-193, (1972).

[16] D. Frey, P.C. Kunstmann, *A $T(1)$-theorem for non-integral operators*, Math Ann. **357**, no. 1, 215-278, (2013).

[17] L. Grafakos, *Classical Fourier Analysis*, second edition, Springer, New York, NY, (2009).

[18] E. Harboure, J.L. Torrea and B.E. Viviani, *A vector-valued approach to tent spaces*, J. Anal. Math. **56**, 125-140 (1991).

[19] T.P. Hytönen, A. McIntosh, P. Portal, *Kato's square root problem in Banach spaces*, J. Funct. Anal. **254**, no. 3, 675-726, (2008).

[20] T.P. Hytönen, J.M.A.M. van Neerven, P. Portal, *Conical square function estimates in UMD Banach spaces and applications to H^∞-functional calculi*, J. Anal. Math. **106**, 317-351 (2008).

[21] T.P. Hytönen, J.M.A.M van Neerven, M.C Veraar, L. Weis, *Analysis in Banach spaces, Volume II, Probabilistic methods and Operator theory* , Preliminary draft, (20 May 2016).

[22] T. Kato, *Perturbation theory for linear operators*, Springer Verlag, New York, (1966).

[23] C. Kenig, J. Pipher, *The Neumann problem for elliptic equations with nonsmooth coefficients*, Invent. Math. **113**, no. 3, 447-509 (1993).

[24] N.V. Krylov, *A generalization of the Littlewood-Paley inequality annd some other results related to stochastic partial differential equations*, Ulam Quart. **2** 4, 16 ff (1994).

[25] N.V. Krylov, *An analytic approach to SPDEs, stochastic partial differential equations: six perspectives*, Math. Surv. Monogr. **64**, 185-242 (1999).

[26] N.V. Krylov, *On the foundation of the L_p-theory of stochastic partial differential equations*, Stochastic partial differential equations and applications-VII. Lecture Notes in Pure and Applied Mathematics, vol. 245, pp. 179-19. Chapman and Hall/CRC, Boca Raton (2006).

[27] P.C. Kunstmann, L.W. Weis. *Maximal L_p regularity for parabolic equations, Fourier multiplier theorems and H^∞-functional calculus*, Functional analytic methods for evolution equations. Lecture Notes in Mathematics, vol. 1855, pp. 65-311. Springer, Berlin (2004).

[28] J.-L. Lions, *Sur les problèmes mixtes pour certains systèmes paraboliques dans les ouverts non cylindriques*, Annales de l'institut Fourier **7**, 143-182 (1957).

[29] J.M.A.M. van Neerven, M.C. Veraar, L.W. Weis, *Conditions for stochastic integrability in UMD Banach spaces*, Banach spaces and their apllications, 124-146, Walter de Gruyter, Berlin, (2007).

[30] J.M.A.M van Neerven, M.C. Veraar, L.W. Weis, *Maximal L^p-regularity for stochastic evolution equations*, SIAM. Math. Anal. **44**(3),1372-1414 (2012).

[31] J.M.A.M. van Neerven, M.C. Veraar, L.W. Weis, *Stochastic integration in UMD Banach spaces*, Ann. Probab. **35**. 1438-1478 (2007).

[32] J.M.A.M. van Neerven, M.C. Veraar, L.W. Weis, *Stochastic integration of functions with values in a Banach space*, Studia Math. **166**, no. 2, 131-170, (2005).

[33] J.M.A.M. van Neerven, M.C. Veraar, L.W. Weis, *Stochastic maximal L^p-regularity*, Ann. Probab. **40**(2). 788-812 (2012).

[34] R.S. Palais, *A simple proof of the Banach contraction principle*, Journal of fixed point theory and applications 2(2):221-223, (December 2007).

[35] R.S. Phillips, *Dissipative operators and hyperbolic systems of partial differential equations*, Trans. Am. Math. Soc. **90**, 193-254 (1959).

[36] J.L. Rubio de Francia, F.J. Ruiz, and J.L. Torrea, *Calderón-Zygmund theory for operator-valued kernels*, Advances in Math. **62**, 7-48 (1986).

[37] T. Seppäläinen, *Basics of stochastic analysis*, Department of Mathematics, University of Wisconsin-Madison, Madison, Wisconsin 53706 (8 december 2012).

[38] L. de Simon, *Un'applicazione della teoria degli integrali singolari allo studio delle equazioni differenziali lineari astratte del primo ordine*, Rendiconti del Seminario Matematico della Universit di Padova **34**, 205-223 (1964).

[39] L. Weis, *Operator-valued Fourier multiplier theorems and maximal L_p-regularity*, Math. Ann., **319**, 735-758 (2001).