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Nonparametric inference for Poisson-Laguerre tessellations

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Abstract

In this paper, we consider statistical inference for Poisson-Laguerre tessellations in $\mathbb{R}^d$. The object of interest is a distribution function F which describes the distribution of the arrival times of the generator points. The function F uniquely determines the intensity measure of the underlying Poisson process. Two nonparametric estimators for F are introduced, which depend only on the points of the Poisson process that generate non-empty cells and the actual cells corresponding to these points. The proposed estimators are proven to be strongly consistent as the observation window expands unboundedly to the whole space. We also consider a stereological setting, where one is interested in estimating the distribution function associated with the Poisson process of a higher-dimensional Poisson-Laguerre tessellation, given that a corresponding sectional Poisson-Laguerre tessellation is observed.

KEYWORDS

consistency, Laguerre tessellation, Poisson point process, stochastic geometry, stereology

1 | INTRODUCTION

Tessellations have proven to be useful in a wide range of fields. For example, a Poisson-Voronoi tessellation may serve as a model for a wireless network, see Baccelli and Błaszczyszyn (2009). In cosmology, Voronoi tessellations can be used to describe the distribution of galaxies, as shown in van de Weygaert (1994). There are several generalizations of the Voronoi tessellation, such as the Laguerre tessellation, which offers more flexibility compared to the Voronoi model. In the field of materials science, Laguerre tessellations have been fitted to the so-called microstructure of a material. For instance, Laguerre tessellations were found to be accurate models for foams in Lautensack (2008), Liebscher (2015), for sintered alumina in Falco et al. (2017) and for composites in Wu et al. (2010). A major challenge in this field is that, in practice, often only 2D microscopic images of cross sections of the 3D microstructure can be obtained. By studying a 3D object via a 2D slice, there is evidently a loss of information. Inverse problems of this type, which involve the estimation of higher-dimensional information from lower-dimensional observations, belong to the field of stereology.

In this paper, we focus on statistical inference for a particular class of random tessellations known as Poisson-Laguerre tessellations. We do this both for the case where one directly observes a tessellation as well as for the case where the observed tessellation is obtained by intersecting a higher-dimensional tessellation with a hyperplane. The latter type of tessellation is often referred to as a sectional tessellation. A Laguerre tessellation in $\mathbb{R}^d$ is defined via a set of weighted points $\eta = \{(x_1, h_1), (x_2, h_2), \dots\}$, called generators. Here, x_i is a point in $\mathbb{R}^d$ and $h_i > 0$ its weight. Each generator corresponds to a set, which is either a polytope or the empty set. This set is usually called a cell, and we may also say that a generator generates this cell. The non-empty cells form a tessellation, meaning that these cells have disjoint interiors and the union of these cells equals $\mathbb{R}^d$. We refer to the subset $\eta^* \subset \eta$ of points that generate non-empty cells as the extreme points of η. We may write:

$$\eta^* := \left\{(x, h) \in \eta \,:\, C((x, h), \eta) \neq \emptyset\right\}, \tag{1}$$

where $C((x, h), \eta)$ denotes the cell associated with (x, h). A Poisson-Laguerre tessellation, which is a random tessellation, is obtained by taking η to be a Poisson (point) process on $\mathbb{R}^d \times (0, \infty)$. The intensity measure of η is assumed to be of the form $\nu_d \times \mathbb{F}$. Here, ν_d is Lebesgue measure on $\mathbb{R}^d$ and $\mathbb{F}$ is a non-zero locally finite measure concentrated on $(0, \infty)$. An example of a realization of a Poisson-Laguerre tessellation, and the corresponding realization of extreme points, is shown in Figure 1. Random Laguerre tessellations generated by an independently marked Poisson process were first studied in Lautensack (2007) and Lautensack and Zuyev (2008). We mostly follow the description of Poisson-Laguerre tessellations as given in Gusakova and Wolde-Lübke (2025). Additionally, we will also rely on the result from Gusakova and Wolde-Lübke (2025), which states that the sectional Poisson-Laguerre tessellation is again a Poisson-Laguerre tessellation. The so-called β-Voronoi tessellation as introduced in Gusakova et al. (2022) may be seen as a parametric model for a Poisson-Laguerre tessellation. In Gusakova et al. (2024) it was shown that the sectional Poisson-Voronoi tessellation is in fact a β-Voronoi tessellation.

Because tessellations are usually not directly observed in nature, typically the first step towards statistical inference for tessellations is a reconstruction step. Such a reconstruction method is used to obtain a tessellation from an image, for details see Section 9.10.1 in Chiu et al. (2013) and references therein. Therefore, when applying the methodology in this paper to real data, it needs to be combined with such a reconstruction method. It is important to point out that the reconstruction methods used in Lautensack (2008), Liebscher (2015), and Seitl et al. (2021) reconstruct a Laguerre tessellation along with the extreme points simultaneously. Effectively, statistical inference for a Poisson-Laguerre tessellation is then reduced to statistical inference for the point process η^* as in Equation (1). This appears to be the most common approach towards statistical inference for random Laguerre tessellations, and this is also the

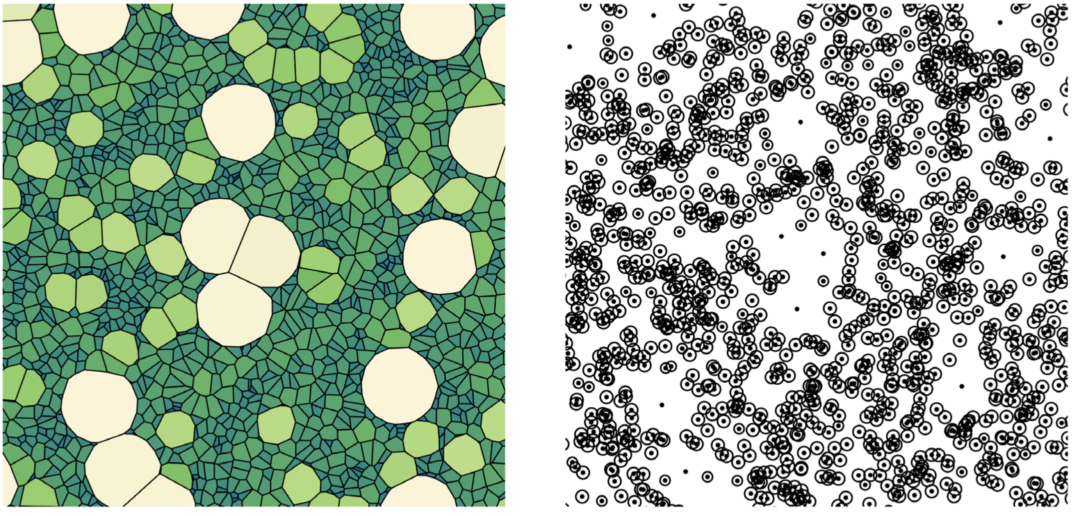

FIGURE 1 Left: A realization of a planar Poisson-Laguerre tessellation. Cells are colored according to their area. Right: The corresponding realization of extreme points. Around each point, there is a circle with a radius proportional to the weight of the point.

approach we take. For instance, in Seitl et al. (2022) a methodology is proposed for statistical inference for Laguerre tessellations, where parametric models are considered for the underlying point process. In Stoyan et al. (2021), a Laguerre tessellation, along with the corresponding extreme points, is fitted to real data. Furthermore, a statistical analysis is performed on this point process of extreme points.

Recall that the intensity measure of the underlying Poisson process η is assumed to be of the form $\nu_d \times \mathbb{F}$. For $z \geq 0$ we define $F(z) := \mathbb{F}((0, z])$, the distribution function of $\mathbb{F}$. Note that this distribution function is the only parameter in this model to be estimated. In this paper, we define nonparametric estimators for F. These estimators for F depend on both the observed Laguerre cells in a bounded observation window as well as the points of η^* in the same window. Here, it is important to realize that η^* is not necessarily a Poisson process, as it is a dependent thinning of η. The proposed estimators are proven to be consistent as the observation window expands unboundedly to the whole of $\mathbb{R}^d$. Additionally, we consider the stereological setting where the observed Poisson-Laguerre tessellation in $\mathbb{R}^{d-1}$ is obtained by intersecting a Poisson-Laguerre tessellation in $\mathbb{R}^d$ with a hyperplane. Based on this observed sectional tessellation, we introduce an estimator for the distribution function corresponding to the Poisson process of the higher-dimensional tessellation.

This paper is organized as follows. In Section 2, we introduce necessary notation and definitions. Then, the main mathematical object of interest, the Poisson Laguerre tessellation, is discussed in Section 3. In Section 4, we introduce our first estimator for F, which is based on a thinning of the extreme points. To the best of our knowledge, no estimators have been proposed in the context of Poisson-Laguerre tessellations as of yet. A second estimator for F is introduced in Section 5, which depends on all observed extreme points, as well as the volumes of the corresponding Laguerre cells. In Section 6, we consider statistical inference for Poisson-Laguerre tessellations in a stereological setting. In Section 7, we perform a simulation study for the proposed estimators to empirically verify their behavior. Finally, we conclude this paper with a discussion in Section 8.

2 | PRELIMINARIES

In this section, we introduce notation and various definitions that we need throughout this paper. Let ν_d denote Lebesgue measure on $\mathbb{R}^d$, and σ_{d-1} Lebesgue measure on the sphere $\mathbb{S}^{d-1} = \{x \in \mathbb{R}^d : \|x\| = 1\}$, also known as the spherical measure. Given $x \in \mathbb{R}^d$ and $r > 0$, we write $B(x,r) = \{y \in \mathbb{R}^d : \|x-y\| < r\}$ and $\overline{B}(x,r) = \{y \in \mathbb{R}^d : \|x-y\| \le r\}$ for the open and closed ball respectively, with radius r centered at x. We introduce the following constant:

$$\kappa_d := \nu_d\left(\overline{B}(0,1)\right) = \frac{2\pi^{\frac{d}{2}}}{\Gamma\left(1+\frac{d}{2}\right)}.$$

We may also use the following fact: $\sigma_{d-1}\left(\mathbb{S}^{d-1}\right) = d\kappa_d$. Let $A, B \subset \mathbb{R}^d$, then the sum of sets is defined as: $A + B = \{a + b : a \in A, b \in B\}$. If $x \in \mathbb{R}^d$, we also write: $A + x = \{a + x : a \in A\}$. Let $\mathcal{F}_+$ denote the space of all (not necessarily bounded) distribution functions on $(0, \infty)$.

We now introduce several definitions related to point processes. While these definitions are valid for point processes in much more general spaces, in this paper, we only consider point processes on $\mathbb{R}^d \times (0,\infty)$. For more background on the theory of point processes, we refer to Daley and Vere-Jones (2008) and Last and Penrose (2017). Suppose $\mathbb{X} = \mathbb{R}^d \times (0,\infty)$, and let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space. A measure μ on $\mathbb{X}$ is locally finite if $\mu(B) < \infty$ for all bounded $B \in \mathcal{B}(\mathbb{X})$. Here, $\mathcal{B}(\mathbb{X})$ denotes the Borel σ-algebra of $\mathbb{X}$. Let $\mathbf{N}(\mathbb{X})$ denote the space of locally finite counting measures (integer-valued measures) on $\mathbb{X}$. We equip $\mathbf{N}(\mathbb{X})$ with the usual σ-algebra $\mathcal{N}(\mathbb{X})$, which is the smallest σ-algebra on $\mathbf{N}(\mathbb{X})$ such that the mappings $\mu \mapsto \mu(B)$ are measurable for all $B \in \mathcal{B}(\mathbb{X})$. A point process on $\mathbb{X}$ is a random element η of $(\mathbf{N}(\mathbb{X}), \mathcal{N}(\mathbb{X}))$, that is a measurable mapping $\eta : \Omega \to \mathbf{N}(\mathbb{X})$. The intensity measure of a point process η on $\mathbb{X}$ is the measure Λ defined by $\Lambda(B) := \mathbb{E}(\eta(B))$, $B \in \mathcal{B}(\mathbb{X})$.

Definition 1. Suppose Λ is a σ-finite measure on $\mathbb{X}$. A Poisson process with intensity measure Λ is a point process η on $\mathbb{X}$ with the following two properties:

1. For every $B \in \mathcal{B}(\mathbb{X})$, the random variable $\eta(B)$ is Poisson distributed with mean $\Lambda(B)$.
2. For every $m \in \mathbb{N}$ and pairwise disjoint sets $B_1, \dots, B_m \in \mathcal{B}(\mathbb{X})$, the random variables $\eta(B_1), \dots, \eta(B_m)$ are independent.

Let δ denote the Dirac measure, hence for $x \in \mathbb{X}$ and $B \in \mathcal{B}(\mathbb{X})$: $\delta_x(B) = \mathbb{1}\{x \in B\}$. A counting measure μ on $\mathbb{X}$ is called simple if $\mu(\{x\}) \le 1$ for all $x \in \mathbb{X}$. As such, a simple counting measure has no multiplicities. Similarly, a point process η on $\mathbb{X}$ is called simple if $\mathbb{P}(\eta(\{x\}) \le 1, \forall x \in \mathbb{X}) = 1$. Let $\mathbf{N}_s(\mathbb{X})$ be the subset of $\mathbf{N}(\mathbb{X})$ containing all simple measures. Define: $\mathcal{N}_s(\mathbb{X}) := \{A \cap \mathbf{N}_s(\mathbb{X}) : A \in \mathcal{N}(\mathbb{X})\}$. Then, a simple point process on $\mathbb{X}$ may be seen as a random element η of $(\mathbf{N}_s(\mathbb{X}), \mathcal{N}_s(\mathbb{X}))$. If a point process is simple, it is common to identify the point process with its support and view the point process as a random set of discrete points in $\mathbb{X}$. We may, e.g., write $x \in \eta$ instead of $x \in \mathrm{supp}(\eta)$. It is common practice to switch between the interpretations of a simple point process as a random counting measure or as a random set of points, depending on which interpretation is more convenient. We will also do this throughout this paper. Enumerating the points of a simple point process in a measurable way, we may write:

$$\eta = \{x_1, x_2, \dots\}, \text{ and } \eta = \sum_{i \in \mathbb{N}} \delta_{x_i}.$$

For $v \in \mathbb{R}^d$ let S_v denote the shift operator. Suppose $\eta = \{(x_1, h_1), (x_2, h_2), \ldots\}$ is a point process with $x_i \in \mathbb{R}^d$ and $h_i > 0$. Then, we define $S_v\eta := \{(x_1 - v, h_1), (x_2 - v, h_2), \ldots\}$. Additionally, for a deterministic set $B \subset \mathbb{R}^d \times (0, \infty)$ we define $S_v B := \{(x + v, h) : (x, h) \in B\}$. Note that in the random counting measure interpretation of a point process, the definition is as follows: $S_v\eta(B) := \eta(S_v B)$, for $B \in \mathcal{B}(\mathbb{R}^d \times (0, \infty))$. This is indeed consistent with the previous definition since $S_v\eta(B) = \sum_i \delta_{(x_i, h_i)}(S_v B) = \sum_i \delta_{(x_i - v, h_i)}(B)$. We call η stationary if $S_v\eta$ and η are equal in distribution for all $v \in \mathbb{R}^d$. Throughout this paper, $(W_n)_{n\geq 1}$ is a fixed convex averaging sequence. That is, each $W_n \subset \mathbb{R}^d$ is convex and compact, and the sequence is increasing: $W_n \subset W_{n+1}$. Finally, the sequence $(W_n)_{n\geq 1}$ expands unboundedly: $\sup\{r \geq 0 : B(x, r) \subset W_n \text{ for some } x \in W_n\} \to \infty$ as $n \to \infty$.

3 | POISSON-LAGUERRE TESSELLATIONS

In this section, we describe the main mathematical object of interest in this paper, the Poisson-Laguerre tessellation. This random tessellation is a generalization of the well-known Poisson-Voronoi tessellation, and was first studied in Lautensack (2007) and Lautensack and Zuyev (2008). We will mostly follow the description of the Poisson-Laguerre tessellation as given in Gusakova and Wolde-Lübke (2025), which is subtly different. Let us start with the definition of a tessellation:

Definition 2. A tessellation of $\mathbb{R}^d$ is a countable collection $T = \{C_i : i \in \mathbb{N}\}$, of sets $C_i \subset \mathbb{R}^d$ (the cells of the tessellation) such that:

- $\text{int}(C_i) \cap \text{int}(C_j) = \emptyset$, if $i \neq j$.
- $\cup_{i\in\mathbb{N}} C_i = \mathbb{R}^d$.
- T is locally finite: $\#\{i \in \mathbb{N} : C_i \cap B \neq \emptyset\} < \infty$ for all bounded $B \in \mathcal{B}(\mathbb{R}^d)$.
- Each C_i is a compact and convex set with interior points.

Now, we will introduce the Laguerre diagram. Let $\varphi = \{(x_i, h_i)\}_{i\in\mathbb{N}}$, with $x_i \in \mathbb{R}^d$ and $h_i > 0$. Assume moreover that $x_i \neq x_j$ for $i \neq j$. The Laguerre cell associated with $(x, h) \in \varphi$ is defined as:

$$C((x, h), \varphi) = \left\{y \in \mathbb{R}^d : \|y - x\|^2 + h \leq \|y - x'\|^2 + h' \text{ for all } (x', h') \in \varphi\right\}. \tag{2}$$

The Laguerre diagram generated by φ is the set of non-empty Laguerre cells, and is denoted by $L(\varphi)$:

$$L(\varphi) := \left\{C((x, h), \varphi) : (x, h) \in \varphi \text{ and } C((x, h), \varphi) \neq \emptyset\right\}.$$

A Laguerre diagram is not necessarily a tessellation; conditions on φ are needed to ensure that $L(\varphi)$ is locally finite and that all cells are bounded. As we will discuss in a moment, the random Laguerre diagrams we consider are in fact tessellations. A Laguerre diagram has an interesting interpretation as a crystallization process. From the definition of a Laguerre cell, it follows that:

$$x \in C((x_i, h_i), \varphi) \iff \exists t \geq h_i : x \in \overline{B}\left(x_i, \sqrt{t - h_i}\right) \text{ and } x \notin \bigcup_{j\neq i} B\left(x_j, \sqrt{(t - h_j)_+}\right),$$

with $(x)_+ = \max\{x, 0\}$. Hence, we may consider the ball $B_i(t) := \overline{B}(x_i, \sqrt{(t - h_i)_+})$ which starts growing at time $t = h_i$. The ball initially grows fast, and then its growth slows down. If B_i is the

FIGURE 2 Visualization of the crystallization process. From left to right, the crystallization process is shown at times $t = 60$, $t = 80$, $t = 120$, and $t = 280$.

first ball to hit a given point $x \in \mathbb{R}^d$, then $x \in C((x_i, h_i), \varphi)$. It is possible that x_i lies in another cell $C((x_j, h_j), \varphi)$, $i \neq j$ and yet $C((x_i, h_i), \varphi)$ may be non-empty. It is also possible that a pair (x_i, h_i) does not generate a cell, essentially because its ball starts growing too late. A visualization of the crystallization process is given in Figure 2. In the literature, one can also find other parameterizations of Laguerre diagrams. For instance, the parameterization in Lautensack (2007) can be obtained as follows. Let $\psi = \{(x_i, r_i)\}_{i \in \mathbb{N}} \subset \mathbb{R}^d \times (0, \infty)$, set $h_i = -r_i^2$ for all $i \in \mathbb{N}$, and then consider the Laguerre diagram generated by $\varphi = \{(x_i, h_i)\}_{i \in \mathbb{N}}$. The pair (x_i, r_i) is then often associated with the sphere centered at x_i with radius r_i. Note that this choice leads to negative weights in our choice of parameterization. We would like to note that the weights are allowed to be negative; we consider positive weights for mathematical convenience.

Throughout this paper we assume that η is a Poisson process on $\mathbb{R}^d \times (0, \infty)$ with intensity measure $\nu_d \times \mathbb{F}$. Here, $\mathbb{F}$ is a locally finite measure concentrated on $(0, \infty)$. Because the measure $\nu_d \times \mathbb{F}$ has no atoms, η is a simple point process. From proposition 3.6. in Gusakova and Wolde-Lübke (2025) it follows that $L(\eta)$, the Laguerre diagram generated by the Poisson process η, is with probability one a tessellation. We refer to $L(\eta)$ as the Poisson-Laguerre tessellation generated by η. We do note that in the aforementioned paper, it is additionally assumed that $\mathbb{F}$ is absolutely continuous with respect to Lebesgue measure. However, this assumption is not needed for $L(\eta)$ to be a tessellation with probability one, as this is a straightforward modification of the proofs given in Gusakova and Wolde-Lübke (2025).

For $z \geq 0$ we define:

$$F(z) := \mathbb{F}((0, z]). \tag{3}$$

Thereby, this monotone function F is the only parameter in this model to be estimated. Note that F is not necessarily bounded; it is bounded if and only if $\mathbb{F}$ is a finite measure. If $\mathbb{F}$ is a finite measure one may define the constant $\lambda = \lim_{z \to \infty} F(z)$ and the probability measure $Q(\cdot) = \mathbb{F}(\cdot)/\lambda$. The intensity measure of η is then given by $\lambda \nu_d \times Q$, and η may be seen as an independently marked homogeneous Poisson process on $\mathbb{R}^d$ with mark space $(0, \infty)$. Its intensity is given by λ, and Q represents its mark distribution. In view of the crystallization process interpretation of a Laguerre diagram, we may also say that Q, and thereby F, describes the distribution of the arrival times of the generator points.

Remark 1. While we consider a Poisson process η on $\mathbb{R}^d \times (0, \infty)$, in the context of Poisson-Laguerre tessellations, also Poisson processes on $\mathbb{R}^d \times E$ have been considered for other choices of $E \subset \mathbb{R}$. We refer to Gusakova and Wolde-Lübke (2025) for an overview of various classes of Poisson-Laguerre tessellations which involve different choices of E. For instance, the choice $E = \mathbb{R}$ represents another important class of Poisson-Laguerre tessellations that requires additional investigations. For the purposes of this paper, we could also have taken $E = (a, \infty)$ for any choice of $a \in \mathbb{R}$, and have obtained analogous results. For the sake of convenience, we have chosen to take $a = 0$. However, the particular value of a does not matter in practice. Note from the definition of the Laguerre cell that adding a fixed constant c to all weights does not affect the cell. Analogously, replacing F by $\tilde{F}$, where $\tilde{F}(z) = F(z - c)$, does not affect the distribution of the resulting Poisson-Laguerre tessellation, as it shifts the distribution of the arrival times by a constant. We would also like to point out that all estimators for F in this paper only shift by c as a result of adding a constant c to all observed weights. Hence, as a consequence, when $E = (a, \infty)$, one may always add a sufficiently large constant c to all observed weights to ensure positivity of these weights. In practice, it does not matter if this a is unknown. After all, adding the true constant $c = -a$ or using $c > -h_1$, with h_1 being the smallest observed weight, will yield estimates of F which are equal up to a shift.

In the introduction of this paper, we explained that we are interested in estimators for F which depend on the observed Laguerre cells and the extreme points of η, which we denote by η^*, as defined in Equation (1). To be precise, the estimators we propose for F depend on the points of η^* in the observation window W_n, as well as the Laguerre cells corresponding to these points of η^* in W_n. Recall, $(W_n)_{n\geq 1}$ is some fixed convex averaging sequence. The reader may, e.g., keep $W_n = [-n, n]^d$ in mind as an explicit example. Note that the point process η^* may be seen as a (dependent) thinning of η, and is not necessarily a Poisson process. We conclude this section with a simulation example, with the purpose of providing an intuitive understanding of Poisson-Laguerre tessellations.

Example 1. In Figure 1, a realization is shown of a planar Poisson-Laguerre tessellation along with its realization of extreme points. The side length of the square observation window is equal to 40. For this example, we have taken $\mathbb{F}$ to be a discrete probability measure on $\{1, 8, 10\}$. Specifically, $\mathbb{F}$ is defined as: $\mathbb{F}(\{1\}) = 0.01$, $\mathbb{F}(\{8\}) = 0.04$ and $\mathbb{F}(\{10\}) = 0.95$. Hence, η may be seen as an independently marked homogeneous Poisson process, with points in $\mathbb{R}^2$ and marks in $\{1, 8, 10\}$. The homogeneous Poisson process has intensity 1, and the marks are distributed according to $\mathbb{F}$. Let us briefly discuss the image in Figure 1 in view of the crystallization process interpretation. Given the choice of $\mathbb{F}$, we expect a small number of balls corresponding to points with weight $h = 1$; these balls start growing early, and result in large cells. A larger number of points with weight $h = 8$ have balls associated with them, which start growing later, yielding cells that are a bit smaller. Finally, a very large number of points with weight $h = 10$ will generate even smaller cells.

4 | INFERENCE VIA A DEPENDENT THINNING

4.1 | Definition of an estimator

In this section, we define our first estimator for F. This estimator only depends on points (x, h) of η^* with $x \in W_n$ and for which x is located in its own Laguerre cell. The estimator is easy to compute, and the techniques used in this section will be important when we define an estimator for F based on all points of η^* in $W_n \times (0, \infty)$. Recall from the previous section that η is a Poisson process on $\mathbb{R}^d \times (0, \infty)$, $d \geq 2$, with intensity measure $\nu_d \times \mathbb{F}$. We may also write: $\eta = \{(x_1, h_1), (x_2, h_2), \ldots\}$, with $x_i \in \mathbb{R}^d$, $h_i > 0$. We start as follows, let $y \in \mathbb{R}^d$, and consider the following thinning of η:

$$\eta^y := \{(x, h) \in \eta : x + y \in C((x, h), \eta)\}. \tag{4}$$

In Equation (2), we defined $C((x, h), \eta)$, which denotes the Laguerre cell associated with the weighted point $(x, h) \in \eta$. Evidently, for every $y \in \mathbb{R}^d$, η^y only contains a subset of points of η^*. Hence, we have: $\eta^y \subset \eta^* \subset \eta$. In particular, for $y = 0$ we obtain the set of points of η^* which are contained within their own Laguerre cell. In the following lemma, we compute the intensity measure of η^y.

Lemma 1. *Let $B \in \mathcal{B}(\mathbb{R}^d)$, $y \in \mathbb{R}^d$ and $z \geq 0$, the intensity measure Λ^y of η^y satisfies:*

$$\Lambda^y(B \times (0, z]) = \nu_d(B) \int_0^z \exp\left(-\kappa_d \int_0^{\|y\|^2 + h} \left(\|y\|^2 + h - t \right)^{\frac{d}{2}} \mathrm{d}F(t) \right) \mathrm{d}F(h).$$

This intensity measure can be computed via the Mecke equation, which may, e.g., be found in Theorem 4.1 in Last and Penrose (2017). The statement is as follows:

Theorem 1 (Mecke equation). *Let Λ be a σ-finite measure on a measurable space $(\mathbb{X}, \mathcal{X})$ and let η be a point process on $\mathbb{X}$. Then η is a Poisson process with intensity measure Λ if and only if:*

$$\mathbb{E}\left(\sum_{x \in \eta} f(x, \eta) \right) = \int \mathbb{E}(f(x, \eta + \delta_x)) \Lambda(\mathrm{d}x),$$

for all non-negative measurable functions $f : \mathbb{X} \times \mathbf{N}(\mathbb{X}) \to [0, \infty]$.

Proof of Lemma 1. By definition, the intensity measure of η^y is given by:

$$\Lambda^y(B \times (0, z]) := \mathbb{E}\left(\eta^y(B \times (0, z])\right) = \mathbb{E}\left(\sum_{(x,h) \in \eta} \mathbb{1}_B(x) \mathbb{1}_{(0,z]}(h) \mathbb{1}\{x + y \in C((x, h), \eta)\} \right). \tag{5}$$

We rewrite the final indicator function in Equation (5) into a more convenient form. By the definition of a Laguerre cell, we obtain:

$$x + y \in C((x, h), \eta) \Leftrightarrow \|y\|^2 + h - h' \leq \|x + y - x'\|^2, \text{ for all } (x', h') \in \eta \Leftrightarrow \eta\left(A_{x,h,y}\right) = 0,$$

where we define the set $A_{x,h,y}$ as:

$$A_{x,h,y} = \left\{ (x', h') \in \mathbb{R}^d \times (0, \infty) : \|y\|^2 + h - h' > \|x + y - x'\|^2 \right\}.$$

Since η is a Poisson process, the random variable $\eta\left(A_{x,h,y}\right)$ is Poisson distributed with parameter $\mathbb{E}(\eta(A_{x,h,y}))$. As a consequence, the probability that $\eta\left(A_{x,h,y}\right) = 0$ is given by:

$$\begin{aligned}\mathbb{P}\left(\eta\left(A_{x,h,y}\right)=0\right) &= \exp\left(-\mathbb{E}\left(\eta\left(A_{x,h,y}\right)\right)\right)\\ &= \exp\left(-\int_{\mathbb{R}^d}\int_0^{\|y\|^2+h} \mathbb{1}\left\{\|x+y-x'\| < \sqrt{\|y\|^2+h-t}\right\}\mathrm{d}F(t)\mathrm{d}x'\right)\\ &= \exp\left(-\int_0^{\|y\|^2+h}\int_{\mathbb{R}^d} \mathbb{1}\left\{\|x'\| < \sqrt{\|y\|^2+h-t}\right\}\mathrm{d}x'\mathrm{d}F(t)\right)\\ &= \exp\left(-\kappa_d\int_0^{\|y\|^2+h}\left(\|y\|^2+h-t\right)^{\frac{d}{2}}\mathrm{d}F(t)\right). \end{aligned} \tag{6}$$

Note that (6) does not depend on x. Using (6) and the Mecke equation, the expectation in Equation (5) can be computed as follows:

$$\begin{aligned}&\mathbb{E}\left(\sum_{(x,h)\in\eta}\mathbb{1}_B(x)\mathbb{1}_{(0,z]}(h)\mathbb{1}\left\{\eta\left(A_{x,h,y}\right)=0\right\}\right)=\\ &= \int_0^\infty\int_{\mathbb{R}^d}\mathbb{1}_B(x)\mathbb{1}_{(0,z]}(h)\mathbb{P}\left(\eta\left(A_{x,h,y}\right)=0\right)\mathrm{d}x\mathrm{d}F(h) \qquad (7)\\ &= \int_0^\infty\int_{\mathbb{R}^d}\mathbb{1}_B(x)\mathbb{1}_{(0,z]}(h)\exp\left(-\kappa_d\int_0^{\|y\|^2+h}\left(\|y\|^2+h-t\right)^{\frac{d}{2}}\mathrm{d}F(t)\right)\mathrm{d}x\mathrm{d}F(h)\\ &= \nu_d(B)\int_0^z\exp\left(-\kappa_d\int_0^{\|y\|^2+h}\left(\|y\|^2+h-t\right)^{\frac{d}{2}}\mathrm{d}F(t)\right)\mathrm{d}F(h).\end{aligned}$$

In Equation (7), we used the fact that $(x,h)\notin A_{x,h,y}$ such that $\eta\left(A_{x,h,y}\right)=(\eta+\delta_{(x,h)})\left(A_{x,h,y}\right)$. ■

Recall that $\mathcal{F}_+$ denotes the space of all (not necessarily bounded) distribution functions on $(0,\infty)$. Given the statement of Lemma 1 we focus on the case $y=0$ and define for $F\in\mathcal{F}_+$ the function $G_F : [0,\infty)\to[0,\infty)$ via:

$$G_F(z) := \int_0^z\exp\left(-\kappa_d\int_0^h (h-t)^{\frac{d}{2}}\mathrm{d}F(t)\right)\mathrm{d}F(h). \tag{8}$$

For functions G_F with $F\in\mathcal{F}_+$ as in Equation (8) we obtain the following important identifiability result:

Theorem 2. *Let $F_1,F_2\in\mathcal{F}_+$, $R>0$. If $G_{F_1}(z)=G_{F_2}(z)$ for all $z\in[0,R)$ then $F_1(z)=F_2(z)$ for all $z\in[0,R)$. In particular, if $G_{F_1}=G_{F_2}$ then $F_1=F_2$.*

The key ingredient for the proof of this theorem is a variant of the Grönwall inequality. This inequality is, in particular, known for its applications in integral- and differential equations. We refer to Pachpatte (1998) for more variants of this inequality and their applications.

Theorem 3 (Theorem 1.3.3. in Pachpatte (1998)). *Suppose u, α and β are measurable non-negative functions on $[0,\infty)$. Assume that α is non-decreasing. Assume for all $z\geq 0$:*

$u, \alpha, \beta \in L^1([0, z])$. *If for all* $z \geq 0$ *the following holds*:

$$u(z) \leq \alpha(z) + \beta(z) \int_0^z u(s) \mathrm{d}s.$$

Then, for all $z \geq 0$:

$$u(z) \leq \alpha(z) \left(1 + \beta(z) \int_0^z \exp\left(\int_s^z \beta(r) \mathrm{d}r \right) \mathrm{d}s \right).$$

Note that if u, α and β satisfy the conditions in Theorem 3 and β is non-decreasing, then:

$$u(z) \leq \alpha(z)(1 + \beta(z) z \exp(\beta(z) z)). \tag{9}$$

We need to point out that in Pachpatte (1998) this theorem also includes the assumption that u, α and β are continuous. However, as noted on p. 14 in the same reference, this assumption is not needed.

Proof of Theorem 2. Let $z \geq 0$. For $i \in \{1, 2\}$ note that the (Lebesgue-Stieltjes) measures associated with G_{F_i} and F_i are mutually absolutely continuous. The corresponding Radon-Nikodym derivative is given by:

$$\frac{\mathrm{d}G_{F_i}}{\mathrm{d}F_i}(z) = \exp\left(-\kappa_d \int_0^z (z - t)^{\frac{d}{2}} \mathrm{d}F_i(t) \right).$$

Hence, we may also write:

$$F_i(z) = \int_0^z \frac{\mathrm{d}F_i}{\mathrm{d}G_{F_i}}(h) \mathrm{d}G_{F_i}(h) = \int_0^z \exp\left(\kappa_d \int_0^h (h - t)^{\frac{d}{2}} \mathrm{d}F_i(t) \right) \mathrm{d}G_{F_i}(h).$$

Via integration by parts, we may write:

$$\int_0^z (z - t)^{\frac{d}{2}} \mathrm{d}F_i(t) = 0 \cdot F_i(z) - z^{\frac{d}{2}} F_i(0) - \int_0^z F_i(t) \mathrm{d}\left((z - t)^{\frac{d}{2}} \right)(t) = \frac{d}{2} \int_0^z F_i(t)(z - t)^{\frac{d}{2} - 1} \mathrm{d}t. \tag{10}$$

Moreover, the expression in Equation (10) is a non-decreasing function of z. We now derive a general upper bound for $|F_1(z) - F_2(z)|$:

$$\begin{aligned}
&|F_1(z) - F_2(z)| \\
&= \left| \int_0^z \exp\left(\kappa_d \int_0^h (h - t)^{\frac{d}{2}} \mathrm{d}F_1(t) \right) \mathrm{d}G_{F_1}(h) - \int_0^z \exp\left(\kappa_d \int_0^h (h - t)^{\frac{d}{2}} \mathrm{d}F_2(t) \right) \mathrm{d}G_{F_2}(h) \right| \\
&\leq \left| \int_0^z \exp\left(\kappa_d \int_0^h (h - t)^{\frac{d}{2}} \mathrm{d}F_1(t) \right) \mathrm{d}G_{F_1}(h) - \int_0^z \exp\left(\kappa_d \int_0^h (h - t)^{\frac{d}{2}} \mathrm{d}F_2(t) \right) \mathrm{d}G_{F_1}(h) \right| + \\
&+ \left| \int_0^z \exp\left(\kappa_d \int_0^h (h - t)^{\frac{d}{2}} \mathrm{d}F_2(t) \right) \mathrm{d}(G_{F_1} - G_{F_2})(h) \right|.
\end{aligned} \tag{11}$$

Let us now consider the first term of (11). For $h \geq 0$ define:

$$C(h) := \max\left\{\exp\left(\kappa_d \int_0^h (h-t)^{\frac{d}{2}} \mathrm{d}F_1(t)\right), \exp\left(\kappa_d \int_0^h (h-t)^{\frac{d}{2}} \mathrm{d}F_2(t)\right)\right\}.$$

Note that C is increasing. Since $|e^x - e^y| \leq \max\{e^x, e^y\}|x-y|$ for $x, y \geq 0$ the first term in Equation (11) is bounded by:

$$\begin{aligned}
&\int_0^z \left|\exp\left(\kappa_d \int_0^h (h-t)^{\frac{d}{2}} \mathrm{d}F_1(t)\right) - \exp\left(\kappa_d \int_0^h (h-t)^{\frac{d}{2}} \mathrm{d}F_2(t)\right)\right| \mathrm{d}G_{F_1}(h) \\
&\leq \int_0^z C(h)\kappa_d \left|\int_0^h (h-t)^{\frac{d}{2}} \mathrm{d}F_1(t) - \int_0^h (h-t)^{\frac{d}{2}} \mathrm{d}F_2(t)\right| \mathrm{d}G_{F_1}(h) \\
&= \int_0^z C(h)\kappa_d \left|\frac{d}{2}\int_0^h (F_1(t) - F_2(t))(h-t)^{\frac{d}{2}-1} \mathrm{d}t\right| \mathrm{d}G_{F_1}(h) \\
&\leq \frac{d\kappa_d}{2} C(z) \int_0^z \int_0^h |F_1(t) - F_2(t)|(h-t)^{\frac{d}{2}-1} \mathrm{d}t \mathrm{d}G_{F_1}(h) \\
&\leq \frac{d\kappa_d}{2} C(z) z^{\frac{d}{2}-1} \int_0^z \int_0^z |F_1(t) - F_2(t)| \mathrm{d}t \mathrm{d}G_{F_1}(h) \\
&= \frac{d\kappa_d}{2} C(z) z^{\frac{d}{2}-1} G_{F_1}(z) \int_0^z |F_1(t) - F_2(t)| \mathrm{d}t.
\end{aligned}$$

Via integration by parts, the second term of (11) is bounded by:

$$\begin{aligned}
&\left|\exp\left(\kappa_d \int_0^z (z-t)^{\frac{d}{2}} \mathrm{d}F_2(t)\right)\left(G_{F_1}(z) - G_{F_2}(z)\right)\right| + \\
&\quad + \left|\int_0^z \left(G_{F_1}(h) - G_{F_2}(h)\right) \mathrm{d}\left(\exp\left(\kappa_d \int_0^h (h-t)^{\frac{d}{2}} \mathrm{d}F_2(t)\right)\right)(h)\right| \\
&\leq \left|G_{F_1}(z) - G_{F_2}(z)\right| \exp\left(\kappa_d \int_0^z (z-t)^{\frac{d}{2}} \mathrm{d}F_2(t)\right) + \\
&\quad + \sup_{h \in [0,z]} \left|G_{F_1}(h) - G_{F_2}(h)\right| \int_0^z \mathrm{d}\left(\exp\left(\kappa_d \int_0^h (h-t)^{\frac{d}{2}} \mathrm{d}F_2(t)\right)\right)(h) \\
&\leq \sup_{h \in [0,z]} \left|G_{F_1}(h) - G_{F_2}(h)\right| 2 \exp\left(\kappa_d \int_0^z (z-t)^{\frac{d}{2}} \mathrm{d}F_2(t)\right).
\end{aligned}$$

Combining all results, we obtain:

$$\begin{aligned}
|F_1(z) - F_2(z)| \leq{}& \frac{d\kappa_d}{2} C(z) z^{\frac{d}{2}-1} G_{F_1}(z) \int_0^z |F_1(t) - F_2(t)| \mathrm{d}t + \\
&+ \sup_{h \in [0,z]} \left|G_{F_1}(h) - G_{F_2}(h)\right| 2 \exp\left(\kappa_d \int_0^z (z-t)^{\frac{d}{2}} \mathrm{d}F_2(t)\right).
\end{aligned}$$

Applying Theorem 3 and (9) with $u(z) = |F_1(z) - F_2(z)|$ yields:

$$|F_1(z) - F_2(z)| \leq K(z) \sup_{h \in [0,z]} \left|G_{F_1}(h) - G_{F_2}(h)\right|. \tag{12}$$

Here, $K(z)$ is given by:

$$K(z) := \left(1 + \frac{d\kappa_d}{2} C(z) z^{\frac{d}{2}} G_{F_1}(z) \exp\left(\frac{d\kappa_d}{2} C(z) z^{\frac{d}{2}} G_{F_1}(z)\right)\right) 2 \exp\left(\kappa_d \int_0^z (z-t)^{\frac{d}{2}} \mathrm{d}F_2(t)\right).$$

The statement of the theorem immediately follows from Equation (12). ∎

Suppose we wish to estimate G_F, and we observe the extreme points of η within the bounded observation window W_n, as well as their Laguerre cells. We define the following unbiased estimator for G_F:

$$\begin{aligned}\hat{G}_n(z) &:= \frac{1}{\nu_d(W_n)} \sum_{(x,h)\in\eta} \mathbb{1}_{W_n}(x) \mathbb{1}_{(0,z]}(h) \mathbb{1}\{x \in C((x,h),\eta)\} \\ &= \frac{1}{\nu_d(W_n)} \sum_{(x,h)\in\eta^0} \mathbb{1}_{W_n}(x) \mathbb{1}_{(0,z]}(h). \end{aligned} \tag{13}$$

In Equation (13), η^0 represents the point process η^y as in Equation (4) with $y = 0$. Hence, G_F is a function which we can estimate and which uniquely determines F; this motivates the following definition:

Definition 3 (First inverse estimator of F). Define $\hat{F}_n^0$ to be the unique function $\hat{F}_n^0 \in \mathcal{F}_+$ which satisfies: $G_{\hat{F}_n^0}(z) = \hat{G}_n(z)$ for all $z \geq 0$, with $\hat{G}_n$ as in Equation (13).

Let us now discuss why $\hat{F}_n^0$ is well-defined. Clearly, if there exists a function $\hat{F}_n^0 \in \mathcal{F}_+$ which satisfies $G_{\hat{F}_n^0}(z) = \hat{G}_n(z)$ for all $z \geq 0$ then it is unique by Theorem 2. Suppose $(x_1,h_1),(x_2,h_2),\ldots,(x_k,h_k)$ is the sorted realization of the points of η^0 with $x_1,\ldots,x_k \in W_n$ and $h_1 \leq h_2 \leq \cdots \leq h_k$. We may write:

$$\hat{G}_n(z) = \frac{1}{\nu_d(W_n)} \sum_{i=1}^{k} \mathbb{1}\{h_i \leq z\}.$$

Set $h_0 = 0$ such that $\hat{F}_n^0(h_0) = 0$. Clearly, $\hat{G}_n$ is piecewise constant, with jump locations at $h_1,\ldots,h_k$. Recall from the proof of Theorem 2 that the Lebesgue-Stieltjes measures associated with $\hat{F}_n^0$ and $G_{\hat{F}_n^0}$ are mutually absolutely continuous. As a consequence, if $\hat{F}_n^0$ exists, it is necessarily also piecewise constant with the same jump locations as $G_{\hat{F}_n^0} = \hat{G}_n$. Therefore, if we can uniquely specify the value of $\hat{F}_n^0$ at $h_1,\ldots,h_k$, existence and uniqueness of $\hat{F}_n^0$ is established. Let $i \in \{1,\ldots,k\}$ then, for the $\hat{F}_n^0$ we are looking for:

$$\begin{aligned}\hat{G}_n(h_i) &= \hat{G}_n(h_{i-1}) + \int_{h_{i-1}}^{h_i} \exp\left(-\kappa_d \int_0^h (h-t)^{\frac{d}{2}} \mathrm{d}\hat{F}_n^0(t)\right) \mathrm{d}\hat{F}_n^0(h) \\ &= \hat{G}_n(h_{i-1}) + \exp\left(-\kappa_d \sum_{j=1}^{i} (h_i - h_j)^{\frac{d}{2}} \left(\hat{F}_n^0(h_j) - \hat{F}_n^0(h_{j-1})\right)\right)\left(\hat{F}_n^0(h_i) - \hat{F}_n^0(h_{i-1})\right) \\ &= \hat{G}_n(h_{i-1}) + \exp\left(-\kappa_d \sum_{j=1}^{i-1} (h_i - h_j)^{\frac{d}{2}} \left(\hat{F}_n^0(h_j) - \hat{F}_n^0(h_{j-1})\right)\right)\left(\hat{F}_n^0(h_i) - \hat{F}_n^0(h_{i-1})\right).\end{aligned}$$

Since $\hat{F}_n^0(h_0) = 0$, $\hat{F}_n^0$ is recursively defined via:

$$\hat{F}_n^0(h_i) = \hat{F}_n^0(h_{i-1}) + \left(\hat{G}_n(h_i) - \hat{G}_n(h_{i-1})\right) \exp\left(\kappa_d \sum_{j=1}^{i-1} (h_i - h_j)^{\frac{d}{2}} \left(\hat{F}_n^0(h_j) - \hat{F}_n^0(h_{j-1})\right)\right). \qquad (14)$$

Note that the RHS of (14) only depends on the values $\hat{F}_n^0(h_j)$ with $j < i$. So indeed, (14) completely defines $\hat{F}_n^0$. Moreover, this expression is also a convenient formula for computing $\hat{F}_n^0$ in practice.

4.2 | Consistency

In this section, we show that $\hat{F}_n^0$, as in Definition 3, is a strongly consistent estimator for F. The first step is to show that the estimator $\hat{G}_n$ as in Equation (13) for G_F is strongly consistent. For empirical estimators such as $\hat{G}_n$, their consistency follows from a spatial ergodic theorem. From Proposition 13.4.I. in Daley and Vere-Jones (2008), and the ergodicity of the Poisson process under consideration, we obtain:

Theorem 4 (Spatial ergodic theorem). *Let η be a Poisson process on $\mathbb{X} = \mathbb{R}^d \times (0, \infty)$ with intensity measure $\nu_d \times \mathbb{F}$. Here, $\mathbb{F}$ is a locally finite measure concentrated on $(0, \infty)$. Let $g(\psi, h)$ be a measurable non-negative function on $\mathbf{N}(\mathbb{X}) \times (0, \infty)$. Then, for any convex averaging sequence $(W_n)_{n\geq 1}$:*

$$\lim_{n\to\infty} \frac{1}{\nu_d(W_n)} \sum_{(x,h)\in\eta} \mathbb{1}_{W_n}(x) g(S_x\eta, h) \overset{\text{a.s.}}{=} \int_0^\infty \mathbb{E}\left(g\left(\eta + \delta_{(0,h)}, h\right)\right)\mathbb{F}(\mathrm{d}h).$$

We do note that Proposition 13.4.I in Daley and Vere-Jones (2008) is phrased in the context that $\mathbb{F}$ is a finite measure. However, like Theorem 12.2.IV in the same reference (another spatial ergodic theorem), which is stated under the assumption that $\mathbb{F}$ is locally finite, the result remains valid if $\mathbb{F}$ is locally finite. Besides the spatial ergodic theorem, we also need the following useful lemma for estimators of monotone functions:

Lemma 2. *Let $(F_n)_{n\geq 1}$ be a random sequence of monotone functions on $\mathbb{R}$, and let F be a deterministic monotone function on $\mathbb{R}$. If for all $z \in \mathbb{R}$: $\mathbb{P}(\lim_{n\to\infty} F_n(z) = F(z)) = 1$, then: $\mathbb{P}(\lim_{n\to\infty} F_n(z) = F(z),\ \forall z \in \mathbb{R}) = 1$.*

The proof of Lemma 2 is given in Appendix A. We obtain the following result:

Corollary 1. *With probability one: $\lim_{n\to\infty} \hat{G}_n(z) = G_F(z)$ for all $z \geq 0$.*

Proof. Let $z \geq 0$, by Lemma 2 it is sufficient to show that $\lim_{n\to\infty} \hat{G}_n(z) = G_F(z)$ almost surely. Using the same notation as in the proof of Lemma 1, note that $\eta(A_{x,h,0}) = S_x\eta(A_{0,h,0})$ for all $(x, h) \in \eta$ almost surely. As a consequence, $\hat{G}_n(z)$ may be written as follows:

$$\hat{G}_n(z) = \frac{1}{\nu_d(W_n)} \sum_{(x,h)\in\eta} \mathbb{1}_{W_n}(x)\mathbb{1}_{(0,z]}(h)\mathbb{1}\{S_x\eta(A_{0,h,0}) = 0\}.$$

Following the computation in the proof of Lemma 1, it is readily verified that applying the spatial ergodic theorem with $g(\psi, h) = \mathbb{1}_{(0,z]}(h)\mathbb{1}\{\psi(A_{0,h,0}) = 0\}$ yields the result with the desired limit. ■

Finally, we need the following continuity result:

Lemma 3. *Let $(F_n)_{n\geq 1}$ be a sequence of functions in $\mathcal{F}_+$ and let $F \in \mathcal{F}_+$. Let $R > 0$. If $\lim_{n\to\infty} F_n(z) = F(z)$ for all $z \in [0, R)$, then $\lim_{n\to\infty} G_{F_n}(z) = G_F(z)$ for all $z \in [0, R)$. In particular, if $\lim_{n\to\infty} F_n(z) = F(z)$ for all $z \geq 0$, then $\lim_{n\to\infty} G_{F_n}(z) = G_F(z)$ for all $z \geq 0$.*

The proof of Lemma 3 is given in Appendix A. Combining the previous results with Theorem 2, we prove the following consistency result.

Theorem 5 (Consistency of $\hat{F}_n^0$). *With probability one, $\lim_{n\to\infty} \hat{F}_n^0(z) = F(z)$ for all $z \geq 0$.*

Proof. Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space supporting a Poisson process η, with intensity measure $\nu_d \times \mathbb{F}$. By Corollary 1 there exists a set $\Omega_0 \in \mathcal{A}$ with $\mathbb{P}(\Omega_0) = 1$ such that for all $\omega \in \Omega_0$ and $z \geq 0$ we have $\lim_{n\to\infty} \hat{G}_n(z; \omega) = G_F(z)$. Let $z \geq 0$, we show that $\lim_{n\to\infty} \hat{F}_n^0(z; \omega) = F(z)$.

Pick $M > 0$ such that $F(z) < M$. For $n \in \mathbb{N}$ and $h \geq 0$, define: $\overline{F}_n(h) = \min\{\hat{F}_n^0(h; \omega), M\}$. Then, $(\overline{F}_n)_{n\geq 1}$ is a uniformly bounded sequence of monotone functions. Let $(n_l)_{l\geq 1} \subset (n)_{n\geq 1}$ be an arbitrary subsequence. By Helly's selection principle there exists a further subsequence $(n_k)_{k\geq 1} \subset (n_l)_{l\geq 1}$ such that $\overline{F}_{n_k}$ converges pointwise to some monotone function $\overline{F}$ as $k \to \infty$. This implies that $\lim_{k\to\infty} \hat{F}_{n_k}^0(h; \omega) = \lim_{k\to\infty} \overline{F}_{n_k}(h) = \overline{F}(h)$ for all $h \in [0, R)$ with $R := \sup\{h \geq 0 : \overline{F}(h) < M\}$. By Lemma 3 we obtain:

$$\lim_{k\to\infty} \hat{G}_{n_k}(h; \omega) := \lim_{k\to\infty} G_{\hat{F}_{n_k}^0(\cdot;\omega)}(h) = G_{\overline{F}}(h) \text{ for all } h \in [0, R).$$

Because the whole sequence $\hat{G}_n(h; \omega)$ converges to $G_F(h)$ as $n \to \infty$, for $h \geq 0$, we obtain $G_F(h) = G_{\overline{F}}(h)$ for all $h \in [0, R)$. Theorem 2 now yields $F(h) = \overline{F}(h)$ for all $h \in [0, R)$, and since $z \in [0, R)$ we have in particular $F(z) = \overline{F}(z)$. As a consequence: $\lim_{k\to\infty} \hat{F}_{n_k}^0(z; \omega) = F(z)$. Because the initial subsequence was chosen arbitrarily, the whole sequence converges: $\lim_{n\to\infty} \hat{F}_n^0(z; \omega) = F(z)$. ■

In Figure 3, a single realization of $\hat{G}_n$ and $\hat{F}_n^0$ are shown. We present additional simulation results in Section 7.

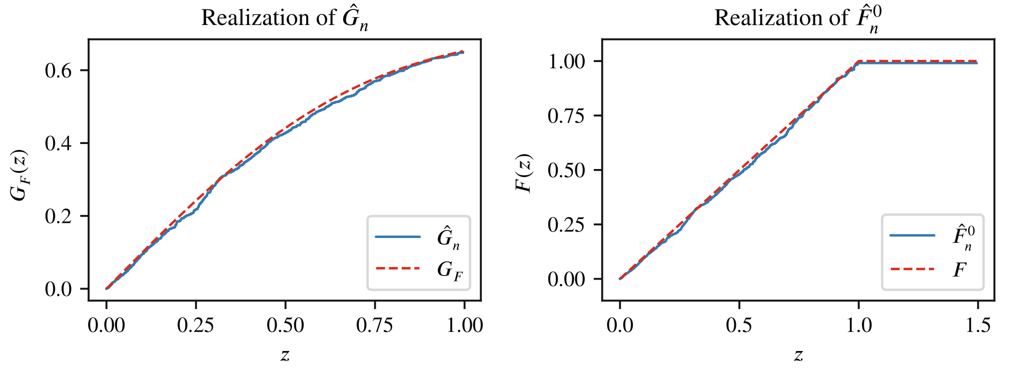

FIGURE 3 Left: A realization of $\hat{G}_n$. Right: The corresponding realization of $\hat{F}_n^0$. The actual underlying F is equal to the CDF of a uniform distribution on $(0, 1)$.

5 | INFERENCE VIA THE VOLUME-BIASED WEIGHT DISTRIBUTION

5.1 | Definition of an estimator

In this section, we define a second estimator for F, which depends on all points of η^* in $W_n \times (0, \infty)$ as well as the volumes of the Laguerre cells corresponding to these points. As such, this estimator depends on more data compared to the estimator in the previous section. First, we present a result for Poisson-Laguerre tessellations in $\mathbb{R}^d$; the estimator itself is defined specifically for the planar case ($d = 2$). Suppose for now that $\mathbb{F}$ is a finite measure, such that η may be interpreted as an independently marked Poisson process. Because $\mathbb{F}$ then determines the distribution of the weights (h-coordinates) of the points of η, a natural question is to ask how the distribution of the weights of the points of η^* is related to $\mathbb{F}$. As it turns out, it is more tractable to study a biased or weighted version of this distribution. We introduce the so-called volume-biased weight distribution in the following definition, which is also well-defined if $\mathbb{F}$ is not a finite measure:

Definition 4 (volume-biased weight distribution). Let η be a Poisson process on $\mathbb{R}^d \times (0, \infty)$, $d \geq 2$, with intensity measure $\nu_d \times \mathbb{F}$. Here, $\mathbb{F}$ is a locally finite measure concentrated on $(0, \infty)$. Let $A \in \mathcal{B}(\mathbb{R})$, define the following probability measure:

$$\mathbb{F}^V(A) := \mathbb{E}\left(\sum_{(x,h)\in\eta} \mathbb{1}_{[0,1]^d}(x) \mathbb{1}_A(h) \nu_d(C((x,h),\eta)) \right). \tag{15}$$

Consider the Poisson-Laguerre tessellation generated by η, then the interpretation of $\mathbb{F}^V$ is as follows. $\mathbb{F}^V$ describes the distribution of the random weight associated with a randomly chosen Laguerre cell, the probability of picking any given cell being proportional to its volume. Because there is an infinite number of Laguerre cells in the tessellation, care needs to be taken in making this statement precise. This can be done via Palm calculus for marked point processes, see for instance chapter 3 in Schneider and Weil (2008). Note that the sum in Equation (15) effectively only sums over points $(x, h) \in \eta$ with a Laguerre cell $C((x, h), \eta)$ of positive volume. Hence, it can also be seen as a sum over points of η^*. From its definition, it is not immediately obvious that $\mathbb{F}^V$ is a well-defined probability measure. Specifically, it is not immediately evident that $\mathbb{F}^V(\mathbb{R}) = 1$. We address this in the proof of the following theorem, where we derive the CDF (Cumulative Distribution Function) associated with $\mathbb{F}^V$.

Theorem 6. *Let η be a Poisson process as in Definition 4, and let $z \geq 0$. Define $F(z) := \mathbb{F}((0, z])$ and $F^V(z) := \mathbb{F}^V((0, z])$, the distribution functions corresponding to $\mathbb{F}$ and $\mathbb{F}^V$, respectively. The measure $\mathbb{F}^V$ is a probability measure and F^V is given by:*

$$F^V(z) = 1 - \exp\left(-\kappa_d \int_0^z (z-t)^{\frac{d}{2}} \mathrm{d}F(t)\right) + \\ + \frac{d\kappa_d}{2} \int_z^\infty \exp\left(-\kappa_d \int_0^u (u-t)^{\frac{d}{2}} \mathrm{d}F(t)\right) \int_0^z (u-h)^{\frac{d}{2}-1} \mathrm{d}F(h) \mathrm{d}u.$$

Proof. By the translation invariance of Lebesgue measure and Fubini's theorem, we may write:

$$\begin{aligned}F^V(z) &= \mathbb{E}\left(\sum_{(x,h)\in\eta} \mathbb{1}_{[0,1]^d}(x)\mathbb{1}_{(0,z]}(h)\nu_d(C((x,h),\eta)-x)\right)\\ &= \mathbb{E}\left(\sum_{(x,h)\in\eta} \mathbb{1}_{[0,1]^d}(x)\mathbb{1}_{(0,z]}(h)\int_{\mathbb{R}^d}\mathbb{1}\{y\in C((x,h),\eta)-x\}\mathrm{d}y\right)\\ &= \int_{\mathbb{R}^d}\mathbb{E}\left(\sum_{(x,h)\in\eta} \mathbb{1}_{[0,1]^d}(x)\mathbb{1}_{(0,z]}(h)\mathbb{1}\{x+y\in C((x,h),\eta)\}\right)\mathrm{d}y\\ &= \int_{\mathbb{R}^d}\mathbb{E}\left(\eta^y([0,1]^d\times(0,z])\right)\mathrm{d}y. \end{aligned} \tag{16}$$

With η^y as in Equation (4). In Lemma 1, we computed the expectation in Equation (16). Plugging in this expression, and passing to polar coordinates by substituting $y = r\theta$, with $r \geq 0$ and $\theta \in \mathbb{S}^{d-1}$, we obtain:

$$F^V(z) = \int_{\mathbb{R}^d}\int_0^z \exp\left(-\kappa_d\int_0^{\|y\|^2+h}\left(\|y\|^2+h-t\right)^{\frac{d}{2}}\mathrm{d}F(t)\right)\mathrm{d}F(h)\mathrm{d}y$$

$$= d\kappa_d\int_0^z\int_0^\infty \exp\left(-\kappa_d\int_0^{r^2+h}\left(r^2+h-t\right)^{\frac{d}{2}}\mathrm{d}F(t)\right)r^{d-1}\mathrm{d}r\mathrm{d}F(h) \tag{17}$$

$$= \frac{d\kappa_d}{2}\int_0^z\int_h^\infty \exp\left(-\kappa_d\int_0^u(u-t)^{\frac{d}{2}}\mathrm{d}F(t)\right)(u-h)^{\frac{d}{2}-1}\mathrm{d}u\mathrm{d}F(h) \tag{18}$$

$$= \frac{d\kappa_d}{2}\int_0^\infty \exp\left(-\kappa_d\int_0^u(u-t)^{\frac{d}{2}}\mathrm{d}F(t)\right)\int_0^{\min\{u,z\}}(u-h)^{\frac{d}{2}-1}\mathrm{d}F(h)\mathrm{d}u. \tag{19}$$

In Equations (17) and (19), we apply Fubini's theorem, and in Equation (18), we substitute $u = r^2 + h$. We can now write $F^V(z)$ as a sum of two integrals:

$$\begin{aligned}F^V(z) = &\frac{d\kappa_d}{2}\int_0^z \exp\left(-\kappa_d\int_0^u(u-t)^{\frac{d}{2}}\mathrm{d}F(t)\right)\int_0^u(u-h)^{\frac{d}{2}-1}\mathrm{d}F(h)\mathrm{d}u+\\ &+\frac{d\kappa_d}{2}\int_z^\infty \exp\left(-\kappa_d\int_0^u(u-t)^{\frac{d}{2}}\mathrm{d}F(t)\right)\int_0^z(u-h)^{\frac{d}{2}-1}\mathrm{d}F(h)\mathrm{d}u.\end{aligned} \tag{20}$$

The first integral of (20) can be calculated explicitly since the integrand has an explicit primitive. The first term of (20) is given by:

$$\left[-\exp\left(-\kappa_d\int_0^u(u-t)^{\frac{d}{2}}\mathrm{d}F(t)\right)\right]_0^z = 1-\exp\left(-\kappa_d\int_0^z(z-t)^{\frac{d}{2}}\mathrm{d}F(t)\right).$$

Plugging this back into (20) yields the expression for F^V as stated in the theorem. Finally, via (19) we can show that $\lim_{z\to\infty}F^V(z) = 1$. After all, the integrand in Equation (19) (considering the integral w.r.t. u) can be bounded from above using the inequality $\min\{u,z\} \le u$. Via the dominated convergence theorem, it follows that:

$$\begin{aligned}\lim_{z\to\infty} F^V(z) &= \frac{d\kappa_d}{2}\int_0^\infty \exp\left(-\kappa_d\int_0^u (u-t)^{\frac{d}{2}}\mathrm{d}F(t)\right)\lim_{z\to\infty}\int_0^{\min\{u,z\}}(u-h)^{\frac{d}{2}-1}\mathrm{d}F(h)\mathrm{d}u\\ &= \int_0^\infty \exp\left(-\kappa_d\int_0^u (u-t)^{\frac{d}{2}}\mathrm{d}F(t)\right)\frac{d\kappa_d}{2}\int_0^u (u-h)^{\frac{d}{2}-1}\mathrm{d}F(h)\mathrm{d}u\\ &= \left[-\exp\left(-\kappa_d\int_0^u (u-t)^{\frac{d}{2}}\mathrm{d}F(t)\right)\right]_0^\infty = 1.\end{aligned}$$

■

The Stieltjes integrals in the expression for F^V may be written as Lebesgue integrals using integration by parts. For instance:

$$\int_0^z (z-t)^{\frac{d}{2}}\mathrm{d}F(t) = 0\cdot F(z) - z^{\frac{d}{2}}F(0) - \int_0^z F(t)\mathrm{d}\left((z-t)^{\frac{d}{2}}\right)(t) = \frac{d}{2}\int_0^z F(t)(z-t)^{\frac{d}{2}-1}\mathrm{d}t. \tag{21}$$

As announced at the beginning of this section, we will now focus on the case $d=2$, which is important for practical applications. In that case, Theorem 6 and (21) yield the following expression for F^V.

Corollary 2. *Let $z \ge 0$, if $d=2$ the CDF F^V is given by*:

$$F^V(z) = 1-\exp\left(-\pi\int_0^z F(t)\mathrm{d}t\right) + \pi F(z)\int_z^\infty \exp\left(-\pi\int_0^u F(t)\mathrm{d}t\right)\mathrm{d}u. \tag{22}$$

Let us now introduce some convenient notation that will be used throughout this section. For $z \ge 0$, $F \in \mathcal{F}_+$ and $m \ge 0$ we define:

$$\begin{aligned} V(z;F,m) &:= 1-\exp\left(-\pi\int_0^z F(t)\mathrm{d}t\right) + \pi F(z)\left(m - \int_0^z \exp\left(-\pi\int_0^u F(t)\mathrm{d}t\right)\mathrm{d}u\right).\\ m_F &:= \int_0^\infty \exp\left(-\pi\int_0^u F(t)\mathrm{d}t\right)\mathrm{d}u.\end{aligned}$$

Note that if $m = m_F$, then $V(\cdot;F,m) = F^V$, with F^V as in Equation (22). In other words, $V(\cdot;F,m)$ is then the volume-biased weight distribution induced by F. We obtain the following identifiability result:

Theorem 7. *Let $F_1, F_2 \in \mathcal{F}_+$, let $R > 0$. If $m_{F_1} = m_{F_2}$ and $V(z;F_1,m_{F_1}) = V(z;F_2,m_{F_2})$ for all $z \in [0,R)$, then $F_1(z) = F_2(z)$ for all $z \in [0,R)$. Consequently, if $m_{F_1} = m_{F_2}$ and $V(\cdot;F_1,m_{F_1}) = V(\cdot;F_2,m_{F_2})$, then $F_1 = F_2$.*

The proof of Theorem 7 as well as the proofs of most of the remaining lemmas in this section are postponed to Appendix A. The techniques used for proving these results are similar to the techniques used in Section 4. We now define the following natural estimator for the distribution function F^V:

$$\tilde{F}_n^V(z) := \frac{1}{\nu_d(W_n)}\sum_{(x,h)\in\eta}\mathbb{1}_{W_n}(x)\mathbb{1}_{(0,z]}(h)\nu_d(C((x,h),\eta)). \tag{23}$$

Alternatively, the following estimator for F^V may be defined:

$$\hat{F}_n^V(z) := \frac{\sum_{(x,h)\in\eta} \mathbb{1}_{W_n}(x)\mathbb{1}_{(0,z]}(h)\nu_d(C((x,h),\eta))}{\sum_{(x,h)\in\eta} \mathbb{1}_{W_n}(x)\nu_d(C((x,h),\eta))}. \quad (24)$$

Remark 2. Note that the estimators $\hat{F}_n^V$ and $\tilde{F}_n^V$ for F^V do not incorporate edge effects. For instance, a Laguerre cell may be partially observed through the observation window W_n, such that computation of the estimators requires information outside of the window. In practice, one could artificially shrink the observation window such that the estimators can be computed based on this smaller window.

Similarly to $\hat{F}_n^0$, we can define an inverse estimator for F using an estimator for F^V. We choose to use $\hat{F}_n^V$ for this purpose, since it satisfies $\lim_{z\to\infty} \hat{F}_n^V(z) = 1$, in general this is not the case for $\tilde{F}_n^V$. In view of Theorem 7, we need to keep in mind that the constant m_F is unknown. We can resolve this by first using $\hat{F}_n^0$ to estimate m_F. That is, we define:

$$\hat{m}_n := m_{\hat{F}_n^0} = \int_0^\infty \exp\left(-\pi \int_0^u \hat{F}_n^0(t)\mathrm{d}t\right)\mathrm{d}u. \quad (25)$$

Finally, we define our second estimator for F as follows:

Definition 5 (Second inverse estimator of F). Define $\hat{F}_n$ to be the unique function $\hat{F}_n \in \mathcal{F}_+$ which satisfies for all $z \geq 0$:

$$\hat{F}_n^V(z) = 1 - \exp\left(-\pi \int_0^z \hat{F}_n(t)\mathrm{d}t\right) + \pi\hat{F}_n(z)\left(\hat{m}_n - \int_0^z \exp\left(-\pi \int_0^u \hat{F}_n(t)\mathrm{d}t\right)\mathrm{d}u\right), \quad (26)$$

with $\hat{F}_n^V$ as in Equation (24) and $\hat{m}_n$ as in Equation (25). That is, $\hat{F}_n$ is the unique function $\hat{F}_n \in \mathcal{F}_+$ which satisfies for all $z \geq 0$: $V(z; \hat{F}_n, \hat{m}_n) = \hat{F}_n^V(z)$.

We again discuss why $\hat{F}_n$ is well-defined. If there exists a function $\hat{F}_n \in \mathcal{F}_+$ which satisfies $V(z; \hat{F}_n, \hat{m}_n) = \hat{F}_n^V(z)$ for all $z \geq 0$ then it is unique by Theorem 7. From Equation (26), we see that $\hat{F}_n$ cannot be the zero function. Moreover, we see that $\hat{F}_n$ should satisfy the following:

$$m_{\hat{F}_n} = \lim_{z\to\infty} \int_0^z \exp\left(-\pi \int_0^u \hat{F}_n(t)\mathrm{d}t\right)\mathrm{d}u = \hat{m}_n - \lim_{z\to\infty} \frac{\hat{F}_n^V(z) - 1 + \exp\left(-\pi\int_0^z \hat{F}_n(t)\mathrm{d}t\right)}{\pi\hat{F}_n(z)} = \hat{m}_n.$$

The final equality follows from the fact that $\lim_{z\to\infty} \hat{F}_n^V(z) = 1$ and $\lim_{z\to\infty} \hat{F}_n(z) > 0$, since $\hat{F}_n$ is non-zero. Therefore, $\hat{F}_n$ necessarily satisfies:

$$\hat{F}_n^V(z) = 1 - \exp\left(-\pi \int_0^z \hat{F}_n(t)\mathrm{d}t\right) + \pi\hat{F}_n(z)\int_z^\infty \exp\left(-\pi \int_0^u \hat{F}_n(t)\mathrm{d}t\right)\mathrm{d}u. \quad (27)$$

Recall from Equation (22) that this means that $\hat{F}_n^V$ is the volume-biased weight distribution induced by $\hat{F}_n$. Suppose $(x_1, h_1), (x_2, h_2), \ldots, (x_m, h_k)$ is the sorted realization of the points of η^* with $x_1, \ldots, x_k \in W_n$ and $h_1 \leq h_2 \leq \cdots \leq h_k$. Clearly, $\hat{F}_n^V$ is piecewise constant, with jump locations at $h_1, \ldots, h_k$. In the proof of Theorem 7 we observe that the Lebesgue-Stieltjes measures associated with $\hat{F}_n$ and $V(\cdot; \hat{F}_n, \hat{m}_n) = V(\cdot; \hat{F}_n, m_{\hat{F}_n}) = \hat{F}_n^V$ are mutually absolutely continuous. As a consequence, $\hat{F}_n$ is necessarily also piecewise constant with the same jump locations as $\hat{F}_n^V$.

Therefore, we simply need to specify the value of $\hat{F}_n$ at $h_1, \ldots, h_k$. Taking $z = h_1$ in Equation (26), and using the fact that $\int_0^{h_1} \hat{F}_n(t)\mathrm{d}t = 0$ we can solve for $\hat{F}_n(h_1)$:

$$\hat{F}_n(h_1) = \frac{\hat{F}_n^V(h_1)}{\pi(\hat{m}_n - h_1)}. \tag{28}$$

In Appendix B an explicit formula for $\hat{m}_n$ is given, which also shows that $\hat{m}_n > h_1$. Let $i \in \{2, \ldots, k\}$ then, via Equation (18) from the proof of Theorem 6, it follows that for the $\hat{F}_n$ we are looking for:

$$\begin{aligned}\hat{F}_n^V(h_i) &= \hat{F}_n^V(h_{i-1}) + \pi \int_{h_{i-1}}^{h_i} \int_h^\infty \exp\left(-\pi \int_0^u \hat{F}_n(t)\mathrm{d}t\right) \mathrm{d}u \mathrm{d}\hat{F}_n(h) \\ &= \hat{F}_n^V(h_{i-1}) + \pi \int_{h_i}^\infty \exp\left(-\pi \int_0^u \hat{F}_n(t)\mathrm{d}t\right) \mathrm{d}u \big(\hat{F}_n(h_i) - \hat{F}_n(h_{i-1})\big).\end{aligned}$$

Hence,

$$\pi \int_{h_i}^\infty \exp\left(-\pi \int_0^u \hat{F}_n(t)\mathrm{d}t\right) \mathrm{d}u = \frac{\hat{F}_n^V(h_i) - \hat{F}_n^V(h_{i-1})}{\hat{F}_n(h_i) - \hat{F}_n(h_{i-1})}. \tag{29}$$

Equation (27) may be used to obtain an expression for $\hat{F}_n^V(h_i)$, plugging (29) into this expression and solving for $\hat{F}_n(h_i)$ yields:

$$\begin{aligned}\hat{F}_n(h_i) &= \hat{F}_n(h_{i-1}) \left(\frac{\hat{F}_n^V(h_i) - 1 + \exp\left(-\pi \int_0^{h_i} \hat{F}_n(t)\mathrm{d}t\right)}{\hat{F}_n^V(h_{i-1}) - 1 + \exp\left(-\pi \int_0^{h_i} \hat{F}_n(t)\mathrm{d}t\right)} \right) \\ &= \hat{F}_n(h_{i-1}) \left(\frac{\hat{F}_n^V(h_i) - 1 + \exp\left(-\pi \sum_{j=1}^{i-1} (h_i - h_j)\big(\hat{F}_n(h_j) - \hat{F}_n(h_{j-1})\big)\right)}{\hat{F}_n^V(h_{i-1}) - 1 + \exp\left(-\pi \sum_{j=1}^{i-1} (h_i - h_j)\big(\hat{F}_n(h_j) - \hat{F}_n(h_{j-1})\big)\right)} \right).\end{aligned} \tag{30}$$

Note that the RHS of (30) only depends on the values $\hat{F}_n(h_j)$ with $j < i$. Hence, (28) along with (30) completely defines $\hat{F}_n$. From Equation (30), it is evident that $\hat{F}_n \in \mathcal{F}_+$, and this expression may be used to compute $\hat{F}_n$ in practice.

5.2 | Consistency

In this section, we show that $\hat{F}_n$, as in Definition 5, is a strongly consistent estimator for F. We start with a Lemma which implies that $\hat{m}_n$ is a strongly consistent estimator for m_F.

Lemma 4. *Let* $(F_n)_{n\geq 1}$ *be a sequence in* $\mathcal{F}_+$*, and let* $F \in \mathcal{F}_+$ *be non-zero. If* $\lim_{n\to\infty} F_n(z) = F(z)$ *for all* $z \geq 0$*, then* $\lim_{n\to\infty} m_{F_n} = m_F$.

Next, we show that $\tilde{F}_n^V$ and $\hat{F}_n^V$ are strongly consistent and uniformly strongly consistent estimators of F^V, respectively.

Lemma 5. *With probability one,* $\lim_{n\to\infty} \tilde{F}_n^V(z) = F^V(z)$ *for all* $z \geq 0$. *Additionally, with probability one we have* $\lim_{n\to\infty} \|\hat{F}_n^V - F^V\|_\infty = 0$. *Here,* $\tilde{F}_n^V$ *and* $\hat{F}_n^V$ *are given by* (23) *and* (24), *respectively.*

Proof. We first show that with probability one, $\lim_{n\to\infty} \tilde{F}_n^V(z) = F^V(z)$ for all $z \geq 0$. Let $z \geq 0$, by Lemma 2 it is sufficient to show that $\lim_{n\to\infty} \tilde{F}_n^V(z) = F^V(z)$ almost surely. Again, we apply the spatial ergodic theorem (Theorem 4). This can be done since for all $(x, h) \in \eta$ we have: $C((x, h), \eta) - x = C((0, h), S_x\eta)$. Hence, by the translation invariance of Lebesgue measure, $\tilde{F}_n^V(z)$ may be written as:

$$\tilde{F}_n^V(z) = \frac{1}{\nu_d(W_n)} \sum_{(x,h)\in\eta} \mathbb{1}_{W_n}(x)\mathbb{1}_{(0,z]}(h)\nu_d(C((0, h), S_x\eta)).$$

So indeed, the spatial ergodic theorem yields $\lim_{n\to\infty} \tilde{F}_n^V(z) = F^V(z)$ almost surely. Similarly, we may argue that $\lim_{n\to\infty} \tilde{F}_n^V(\infty) = 1$ almost surely. Since $\hat{F}_n^V(z) = \tilde{F}_n^V(z)/\tilde{F}_n^V(\infty)$, we obtain via the continuous mapping theorem that $\lim_{n\to\infty} \hat{F}_n^V(z) = F^V(z)$ almost surely. The uniform strong consistency follows from repeating the steps in the proof of the Glivenko-Cantelli theorem. ■

We need one more lemma before we prove the consistency result for $\hat{F}_n$.

Lemma 6. *Let* $(F_n)_{n\geq 1}$ *be a sequence in* $\mathcal{F}_+$, *and let* $F \in \mathcal{F}_+$. *Let* $(m_n)_{n\geq 1}$ *be a sequence in* $(0, \infty)$ *and let* $m > 0$. *If* $\lim_{n\to\infty} F_n(z) = F(z)$ *for all* $z \geq 0$ *and* $\lim_{n\to\infty} m_n = m$, *then* $\lim_{n\to\infty} V(z; F_n, m_n) = V(z; F, m)$ *for all* $z \geq 0$.

Theorem 8 (Consistency of $\hat{F}_n$). *With probability one,* $\lim_{n\to\infty} \hat{F}_n(z) = F(z)$ *for all* $z \geq 0$.

Proof. Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space supporting a Poisson process η, with intensity measure $\nu_2 \times \mathbb{F}$. By Lemmas 4 and 5, there exists a set $\Omega_0 \in \mathcal{A}$ with $\mathbb{P}(\Omega_0) = 1$ such that for all $\omega \in \Omega_0$ and $z \geq 0$ we have $\lim_{n\to\infty} \hat{F}_n^V(z; \omega) = V(z; F, m_F)$ and $\lim_{n\to\infty} \hat{m}_n(\omega) = m_F$. Let $z \geq 0$, we show that $\lim_{n\to\infty} \hat{F}_n(z; \omega) = F(z)$.

Pick $M > 0$ such that $F(z) < M$. For $n \in \mathbb{N}$ and $h \geq 0$, define: $\overline{F}_n(h) = \min\{\hat{F}_n(h; \omega), M\}$. Then, $(\overline{F}_n)_{n\geq 1}$ is a uniformly bounded sequence of monotone functions. Let $(n_l)_{l\geq 1} \subset (n)_{n\geq 1}$ be an arbitrary subsequence. By Helly's selection principle there exists a further subsequence $(n_k)_{k\geq 1} \subset (n_l)_{l\geq 1}$ such that $\overline{F}_{n_k}$ converges pointwise to some monotone function $\overline{F}$ as $k \to \infty$. This implies that $\lim_{k\to\infty} \hat{F}_{n_k}(h; \omega) = \lim_{k\to\infty} \overline{F}_{n_k}(h) = \overline{F}(h)$ for all $h \in [0, R)$ with $R := \sup\{h \geq 0 : \overline{F}(h) < M\}$. By Lemma 6, we obtain along this subsequence:

$$\lim_{k\to\infty} \hat{F}_{n_k}^V(h) := \lim_{k\to\infty} V(h; \hat{F}_{n_k}(\cdot, \omega), \hat{m}_{n_k}(\omega)) = V(h; \overline{F}, m_F) \text{ for all } h \in [0, R).$$

Because the whole sequence $\hat{F}_n^V(h; \omega)$ converges to $V(h; F, m_F)$ as $n \to \infty$, for $h \geq 0$, we obtain $V(h; F, m_F) = V(h; \overline{F}, m_F)$ for all $h \in [0, R)$. Theorem 7 now yields $F(h) = \overline{F}(h)$ for all $h \in [0, R)$, since $z \in [0, R)$, we have in particular $F(z) = \overline{F}(z)$. As a consequence: $\lim_{k\to\infty} \hat{F}_{n_k}(z; \omega) = F(z)$. Because the initial subsequence was chosen arbitrarily, the whole sequence converges: $\lim_{n\to\infty} \hat{F}_n(z; \omega) = F(z)$. ■

Remark 3 (Density estimation). For many practical applications, it is reasonable to assume $\mathbb{F}$ is absolutely continuous with respect to Lebesgue measure and has a density f. One could then define estimators for f via kernel smoothing. Let k be a symmetric kernel, $\tau > 0$ and $z \in \mathbb{R}$, then we may define:

$$\hat{f}_n^0(z) = \int_0^\infty \frac{1}{\tau} k\left(\frac{z-t}{\tau}\right) \mathrm{d}\hat{F}_n^0(t) \quad \text{and} \quad \hat{f}_n(z) = \int_0^\infty \frac{1}{\tau} k\left(\frac{z-t}{\tau}\right) \mathrm{d}\hat{F}_n(t). \tag{31}$$

In the classical context of kernel density estimation, it is well-known which choices of the bandwidth parameter τ lead to consistent and/ or optimal rates of convergence, see for instance chapter 24 in van der Vaart (1998). We also refer to Groeneboom and Jongbloed (2014) for various examples of density estimators obtained via smoothing of estimators of distribution functions. In our setting, it is not yet clear which choices of τ lead to consistent estimators, because much of the behavior of the estimators $\hat{F}_n$ and $\hat{F}_n^0$ is unknown. In practice, one can still apply the density estimators in Equation (31) by manually choosing a value for τ, being careful to take a value which does not lead to under- or oversmoothing.

6 STEREOLOGY

In this section, we study a special type of Poisson-Laguerre tessellations, namely sectional Poisson-Laguerre tessellations. By this, we mean that we intersect a Poisson-Laguerre tessellation with a hyperplane, and we consider the resulting tessellation in this hyperplane. In Theorem 4.1, in Gusakova and Wolde-Lübke (2025) it was shown that intersecting a Poisson-Laguerre tessellation in $\mathbb{R}^d$ with a hyperplane, yields a tessellation in this hyperplane which is again a Poisson-Laguerre tessellation. Because our parameterization is subtly different from the setting in Gusakova and Wolde-Lübke (2025), we derive the intensity measure of the Poisson process corresponding to the sectional Poisson-Laguerre tessellation, for which we also use a different argument.

Suppose we observe the extreme points and the corresponding cells of a Poisson-Laguerre tessellation $L(\eta)$ in $\mathbb{R}^{d-1}$, through the observation window W_n. The underlying Poisson process η has intensity measure $\nu_{d-1} \times \mathbb{F}$, where $\mathbb{F}$ is a locally finite measure concentrated on $(0, \infty)$. Hence, we may use any of the estimators in the previous two sections to estimate $F(z) = \mathbb{F}((0, z])$. Throughout this section, we assume that $\overline{F}_n$ is a piecewise constant, strongly consistent estimator for F. Now, this Poisson-Laguerre tessellation in $\mathbb{R}^{d-1}$ is the sectional tessellation corresponding to a Poisson-Laguerre tessellation $L(\Psi)$ in $\mathbb{R}^d$. The Poisson process Ψ of this higher-dimensional tessellation has intensity measure $\nu_d \times \mathbb{H}$, where $\mathbb{H}$ is a locally finite measure concentrated on $(0, \infty)$. For $z \geq 0$ define: $H(z) := \mathbb{H}((0, z])$. In this section, we show how F is related to H, and how a consistent estimator for F can be used to obtain a (locally) consistent estimator for H. Thereby, we have a solution to the stereological problem. First, we need the following lemma for obtaining an expression for F in this stereological setting:

Lemma 7. *Let $\varphi \subset \mathbb{R}^d \times (0, \infty)$ be an at most countable set. Let $\theta \in \mathbb{S}^{d-1}$ and $s \in \mathbb{R}$. Define the hyperplane $T := \{y \in \mathbb{R}^d : \langle \theta, y \rangle = s\}$. For $(x, h) \in \varphi$, with $x \in \mathbb{R}^d$ and $h > 0$ let*:

$$x' := x - (\langle \theta, x \rangle - s)\theta$$
$$h' := h + \|x' - x\|^2 = h + (\langle \theta, x \rangle - s)^2.$$

Note that $x' \in T$ and define $\varphi' := \{(x', h') : (x, h) \in \varphi\}$. Then, for all $(x, h) \in \varphi$: $C((x, h), \varphi) \cap T = C'((x', h'), \varphi')$ with:

$$C'((x', h'), \varphi') = \{y \in T : \|y - x'\|^2 + h' \le \|y - \overline{x}\|^2 + \overline{h} \text{ for all } (\overline{x}, \overline{h}) \in \varphi'\}.$$

Proof. Let $y \in T$ and $(x, h) \in \varphi$, then a direct computation yields:

$$\|x' - y\|^2 = \|x - y\|^2 - 2(\langle \theta, x \rangle - s)\langle x - y, \theta \rangle + (\langle \theta, x \rangle - s)^2 = \|x - y\|^2 - h' + h.$$

Since $\|x' - y\|^2 + h' = \|x - y\|^2 + h$, the claim follows. ∎

This lemma describes the set of weighted points that generates a sectional Laguerre diagram. We now apply this to the Poisson-Laguerre tessellation generated by the Poisson process Ψ. Because a Poisson-Laguerre tessellation is stationary and isotropic, the choice of hyperplane does not affect the distribution of the sectional tessellation. For $x \in \mathbb{R}^d$ write: $x = (\mathrm{x}_1, \mathrm{x}_2, \ldots, \mathrm{x}_d)$. We choose the hyperplane $\mathrm{x}_d = 0$ which corresponds to taking $\theta = (0, \ldots, 0, 1) \in \mathbb{S}^{d-1}$ and $s = 0$ in Lemma 7. In view of Lemma 7, consider the function which maps a pair $(x, h) \in \Psi$ to the corresponding (x', h'). Hence, this function is given by $(\mathrm{x}_1, \ldots, \mathrm{x}_d, h) \mapsto (\mathrm{x}_1, \ldots, \mathrm{x}_{d-1}, 0, h + \mathrm{x}_d^2)$. Naturally, the d-th component of the resulting vector is always zero. We identify the hyperplane $\mathrm{x}_d = 0$ with $\mathbb{R}^{d-1}$ and therefore we consider the function $\tau : \mathbb{R}^d \times (0, \infty) \to \mathbb{R}^{d-1} \times (0, \infty)$ which is defined via: $\tau(x, h) = (\mathrm{x}_1, \ldots, \mathrm{x}_{d-1}, h + \mathrm{x}_d^2)$. Hence, the point process $\eta := \tau(\Psi)$ generates the sectional tessellation. By the mapping theorem (see Theorem 5.1 in Last and Penrose (2017)) η is again a Poisson process on $\mathbb{R}^{d-1} \times (0, \infty)$ with intensity measure: $\mathbb{E}(\Psi(\tau^{-1}(\cdot)))$. Let $B \subset \mathbb{R}^{d-1}$ be a Borel set and let $z \ge 0$. Note that $h + \mathrm{x}_d^2 \le z$ if and only if $h \le z$ and $\mathrm{x}_d \in [-\sqrt{z - h}, \sqrt{z - h}]$. As a result:

$$\tau(x, h) \in B \times (0, z] \Leftrightarrow x \in B \times \left[-\sqrt{z - h}, \sqrt{z - h}\right] \text{ and } h \le z. \tag{32}$$

Via the Campbell formula and (32) we find:

$$\begin{aligned}\mathbb{E}\big(\Psi(\tau^{-1}(B \times (0, z]))\big) &= \int_{\mathbb{R}^d} \int_0^\infty \mathbb{1}\{\tau(x, h) \in B \times (0, z]\} \mathrm{d}H(h)\mathrm{d}x \\ &= \int_{\mathbb{R}^d} \int_0^z \mathbb{1}\left\{x \in B \times \left[-\sqrt{z - h}, \sqrt{z - h}\right]\right\} \mathrm{d}H(h)\mathrm{d}x \\ &= \nu_{d-1}(B) 2 \int_0^z \sqrt{z - h}\, \mathrm{d}H(h).\end{aligned}$$

Hence, we obtain:

$$F(z) = 2 \int_0^z \sqrt{z - h}\, \mathrm{d}H(h).$$

Let us discuss some properties of this function F. First of all, F is not a bounded function. Indeed, choose $z_0 > 0$ such that $H(z_0) > 0$, and let $z > z_0$, via integration by parts we observe:

$$F(z) = \int_0^z H(h) \frac{1}{\sqrt{z - h}} \mathrm{d}h \ge \int_{z_0}^z H(h) \frac{1}{\sqrt{z}} \mathrm{d}h \ge H(z_0) \frac{z - z_0}{\sqrt{z}}. \tag{33}$$

It immediately follows that $\lim_{z\to\infty} F(z) = \infty$. Another property of F is that it is absolutely continuous, and has a Lebesgue density f given by:

$$f(z) = \int_0^z \frac{1}{\sqrt{z-t}} \mathrm{d}H(t).$$

Indeed, via Fubini's theorem, we can verify that f is a density of F:

$$\int_0^z f(s)\mathrm{d}s = \int_0^z \int_0^s \frac{1}{\sqrt{s-t}} \mathrm{d}H(t)\mathrm{d}s = \int_0^z \int_t^z \frac{1}{\sqrt{s-t}} \mathrm{d}s\mathrm{d}H(t) = 2\int_0^z \sqrt{z-t}\mathrm{d}H(t) = F(z).$$

It is possible to express H in terms of F, because this is an Abel integral equation. For a direct derivation of the inversion formula, see, e.g., Srivastav (1963). Here, we simply show that the following expression is indeed an inversion formula for $H(z)$:

$$\begin{aligned}\frac{1}{\pi}\int_0^z \frac{1}{\sqrt{z-t}} \mathrm{d}F(t) &= \frac{1}{\pi}\int_0^z \frac{1}{\sqrt{z-t}} \int_0^t \frac{1}{\sqrt{t-s}} \mathrm{d}H(s)\mathrm{d}t \\ &= \int_0^z \int_s^z \frac{1}{\pi} \frac{1}{\sqrt{z-t}\sqrt{t-s}} \mathrm{d}t\mathrm{d}H(s) \\ &= \int_0^z \int_0^1 \frac{1}{\pi}(1-u)^{-\frac{1}{2}} u^{-\frac{1}{2}} \mathrm{d}u\mathrm{d}H(s) \qquad (34) \\ &= H(z). \qquad (35)\end{aligned}$$

In Equation (34), we substituted $u = (t-s)/(z-s)$. Finally, (35) follows from the fact that the inner integral in Equation (34) is equal to one, since this integral represents the Beta function evaluated in $(1/2, 1/2)$. A plugin estimator for $H(z)$ is therefore given by:

$$H_n(z) := \frac{1}{\pi}\int_0^z \frac{1}{\sqrt{z-t}} \mathrm{d}\overline{F}_n(t), \qquad (36)$$

where $\overline{F}_n$ is a piecewise constant, strongly consistent estimator for F. This estimator is, however, rather ill-behaved. While H is a monotone function, H_n is not. Because $\overline{F}_n$ is piecewise constant, H_n is decreasing between jump locations of $\overline{F}_n$. Moreover, if z_0 is a jump location of $\overline{F}_n$, then $\lim_{z\downarrow z_0} H_n(z) = \infty$. Therefore, we use isotonization to obtain an estimator for H which is monotone, and show that it is consistent. We note that our estimator is similarly defined as the isotonic estimator in Groeneboom and Jongbloed (1995). For the remainder of this section, let $k = k(n)$ be the number of jump locations of $\overline{F}_n$. Let $h_1, h_2, \ldots, h_k$ with $0 < h_1 < h_2 < \cdots < h_k < \infty$ be the jump locations of $\overline{F}_n$. To introduce the isotonic estimator, we define for $z \geq 0$:

$$U_n(z) := \int_0^z H_n(t)\mathrm{d}t = \frac{2}{\pi}\int_0^z \sqrt{z-t}\mathrm{d}\overline{F}_n(t). \qquad (37)$$

Choose (a large) $M > 0$ and write $z_M := \min\{h_k, M\}$. Let U_n^M be the greatest convex minorant of U_n on $[0, z_M]$. That is, U_n^M is the greatest convex function on $[0, z_M]$ which lies below U_n. Then, define:

$$\hat{H}_n^M(z) := \begin{cases} U_n^{M,r}(z) & \text{if } z \in [0, z_M) \\ U_n^{M,l}(z_M) & \text{if } z \geq z_M, \end{cases} \qquad (38)$$

where $U_n^{M,l}$, $U_n^{M,r}$ denote the left- and right-derivative of U_n^M, respectively. The reason we cannot simply extend the definition of U_n^M to the whole of $[0, \infty)$ is due to the fact that U_n is concave on $[h_k, \infty)$. As a result, the greatest convex minorant of U_n on $[0, \infty)$ is the zero function. Because of the convexity of U_n^M on $[0, z_M]$, $\hat{H}_n^M$ is guaranteed to be non-decreasing, and is referred to as an isotonic estimator. Analogously to (37) we define for $z \geq 0$:

$$U(z) := \int_0^z H(t)\mathrm{d}t = \frac{2}{\pi}\int_0^z \sqrt{z-t}\,\mathrm{d}F(t). \tag{39}$$

Note that $U^r(z) = H(z)$, so indeed, the right-derivative of U_n^M is a natural choice for an estimator of H. In the next theorem, we prove the consistency of $\hat{H}_n^M$. Currently, it is not known whether $\hat{H}_n := \hat{H}_n^\infty$ is a globally consistent estimator.

Theorem 9 (Consistency of $\hat{H}_n^M$). *Let $M > 0$ and let $\hat{H}_n^M$ be as in Equation (38). Let $z \in [0, M)$, then with probability one*:

$$H(z-) \leq \liminf_{n\to\infty} \hat{H}_n^M(z) \leq \limsup_{n\to\infty} \hat{H}_n^M(z) \leq H(z).$$

In particular, if z is a continuity point of H: $\lim_{n\to\infty} \hat{H}_n^M(z) = H(z)$ almost surely.

Proof. Let $z \in [0, M)$. Because $\overline{F}_n$ is piecewise constant and a consistent estimator of the unbounded function F (recall Equation (33)), it follows that $\lim_{n\to\infty} h_{k(n)} = \infty$ almost surely. Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space supporting a Poisson process η, with intensity measure $\nu_{d-1} \times \mathbb{F}$. Choose $\Omega_0 \in \mathcal{A}$ with $\mathbb{P}(\Omega_0) = 1$ such that for all $\omega \in \Omega_0$ we have $\lim_{n\to\infty} h_{k(n)}(\omega) = \infty$ and $\lim_{n\to\infty} \overline{F}_n(h; \omega) = F(h)$ for all $h \geq 0$. For the remainder of the proof, take n sufficiently large such that $h_{k(n)}(\omega) > M$. Note how U_n and U depend on $\overline{F}_n$ and F respectively, see Equations (37) and (39). As a consequence, the pointwise convergence of $\overline{F}_n(\cdot; \omega)$ to F implies: $\lim_{n\to\infty} U_n(x; \omega) = U(x)$ for all $x \geq 0$. Note that U is non-decreasing and continuous, therefore the convergence is also uniform on $[0, M]$. That is, $\lim_{n\to\infty} \sup_{x\in[0,M]} |U_n(x; \omega) - U(x)| = 0$. Because U is defined as the integral of a non-decreasing function, it is convex. A variant of Marshall's lemma (the convex analogue of 7.2.3. on p. 329 in Robertson et al. (1988)) directly yields:

$$\sup_{x\in[0,M]} \left|U_n^M(x; \omega) - U(x)\right| \leq \sup_{x\in[0,M]} |U_n(x; \omega) - U(x)|.$$

Therefore, we also have $\lim_{n\to\infty} \sup_{x\in[0,M]} \left|U_n^M(x; \omega) - U(x)\right| = 0$. Take $\delta > 0$ such that $z + \delta < M$. Then, for each $0 < h < \delta$ we have by the convexity of U_n^M:

$$\frac{U_n^M(z; \omega) - U_n^M(z-h; \omega)}{h} \leq U_n^{M,l}(z; \omega) \leq U_n^{M,r}(z; \omega) \leq \frac{U_n^M(z; \omega) - U_n^M(z+h; \omega)}{h}.$$

By using $\lim_{n\to\infty} \sup_{x\in[0,M]} \left|U_n^M(x; \omega) - U(x)\right| = 0$, the following holds:

$$\frac{U(z) - U(z-h)}{h} \leq \liminf_{n\to\infty} U_n^{M,r}(z; \omega) \leq \limsup_{n\to\infty} U_n^{M,r}(z; \omega) \leq \frac{U(z) - U(z+h)}{h}.$$

The result follows from letting $h \downarrow 0$ and by recognizing that $U^l(z) = H(z-)$ and $U^r(z) = H(z)$. ∎

Remark 4. By choosing $M > 0$ very large, the estimators $\hat{H}_n := \hat{H}_n^\infty$ and $\hat{H}_n^M$ will in practice often coincide, since we will typically observe $h_k < M$. Therefore, in the remainder of this paper, we will only consider computational aspects and simulation performance of the estimator $\hat{H}_n$.

We now show that computing the isotonic estimator $\hat{H}_n$ is equivalent to solving an isotonic regression problem. This is achieved via the following lemma, which is a straightforward modification of Lemma 2 in Groeneboom and Jongbloed (1995).

Lemma 8. *Let $M > 0$, and let φ be an a.e. continuous non-negative function on $[0, M]$. Define the function Φ, for $z \geq 0$ as:*

$$\Phi(z) = \int_0^z \varphi(x)\mathrm{d}x.$$

Let Φ^* be the greatest convex minorant of Φ on $[0, M]$. Let $\Phi^{*,r}$ be the right-derivative of Φ^*, then:

$$\int_0^M (\varphi(x) - \psi(x))^2\mathrm{d}x \geq \int_0^M (\varphi(x) - \Phi^{*,r}(x))^2\mathrm{d}x + \int_0^M (\Phi^{*,r} - \psi(x))^2\mathrm{d}x,$$

for all functions ψ in the set:

$$\mathcal{F}_M := \left\{\psi : [0, M] \to [0, \infty) : \psi \text{ is non-decreasing and right-continuous}\right\}.$$

We use Lemma 8 to show that $\hat{H}_n$ may be interpreted as the L^2-projection of H_n on the space of monotone functions. Recall that $h_1, h_2, \ldots, h_k$ are the unique jump locations of $\overline{F}_n$. Additionally, let $h_0 = 0$. Define $\tilde{H}_n$ to be the piece-wise constant function on $[0, h_k)$ which is given by:

$$\tilde{H}_n(z) = \frac{U_n(h_{i+1}) - U_n(h_i)}{h_{i+1} - h_i},\ z \in [h_i, h_{i+1}),\ i \in \{0, 1, \ldots, k-1\}.$$

For $z \in [0, h_k]$, let: $\tilde{U}_n(z) = \int_0^z \tilde{H}_n(t)\mathrm{d}t$. Then, $U_n(h_i) = \tilde{U}_n(h_i)$ for all $i \in \{0, 1, \ldots, k\}$. While U_n is concave between successive jump locations (due to the square root), $\tilde{U}_n$ is linear between successive jump locations. As a consequence, U_n and $\tilde{U}_n$ have the same greatest convex minorant. Hence, $\hat{H}_n(z) = U_n^{*,r}(z) = \tilde{U}_n^{*,r}(z)$, for $z \in [0, h_k)$. Finally, by taking $\varphi = \tilde{H}_n$ (and $\varphi = H_n$) and $M = h_k$ in Lemma 8 we see that:

$$U_n^{*,r} = \underset{H \in \mathcal{F}_{h_k}}{\arg\min} \int_0^{h_k} (H(x) - H_n(x))^2\mathrm{d}x = \underset{H \in \mathcal{F}_{h_k}}{\arg\min} \int_0^{h_k} (H(x) - \tilde{H}_n(x))^2\mathrm{d}x. \tag{40}$$

Because $\tilde{H}_n$ is piece-wise constant, in Equation (40) we may even minimize over all functions in $\mathcal{F}_{h_k}$ which are also piece-wise constant with jump locations at $h_1, h_2, \ldots, h_k$. Hence, $\hat{H}_n$ is piece-wise constant, and when solving the minimization problem in Equation (40), we only seek to determine the values $\hat{H}_n$ attains at these jump locations. Let $y_i = \tilde{H}_n(h_i)$, and $w_i = h_{i+1} - h_i$.

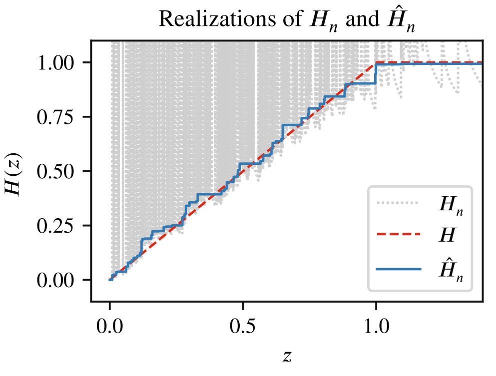

FIGURE 4 A comparison of the plugin estimator H_n and the isotonic estimator $\hat{H}_n$. The actual underlying H is equal to the CDF of a uniform distribution on $(0, 1)$.

Then, by setting $\hat{\beta} = (\hat{H}_n(h_1), \hat{H}_n(h_2), \dots, \hat{H}_n(h_{k-1}))$, (40) may be written as:

$$\hat{\beta} = \arg\min_{\beta \in C_+} \sum_{i=1}^{k-1} (\beta_i - y_i)^2 w_i, \tag{41}$$

where the closed convex cone C_+ is given by: $C_+ := \{\beta \in \mathbb{R}^{k-1} : 0 \leq \beta_1 \leq \beta_2 \leq \cdots \leq \beta_{k-1}\}$. Finally, observe that $\hat{H}_n(h_k) = \hat{H}_n(h_{k-1})$. The optimization problem in Equation (41) is indeed an isotonic regression problem. We note that implementations for solving this problem are widely available. In Figure 4, a realization of H_n and the corresponding realization of $\hat{H}_n$ is shown.

7 | SIMULATIONS

In previous sections, we have derived consistent estimators for the distribution function corresponding to the underlying Poisson process η, both in the direct setting ($\mathbb{R}^d$) and in the stereological setting ($\mathbb{R}^{d-1}$). Additionally, we have shown how to compute these estimators. In this section, we perform some simulations such that we can assess their performance. We note that edge effects, which occur in practice, were not taken into account for the simulations in this section. We discuss practical issues arising from edge effects in Section 8. For the simulations, we compute Laguerre tessellations using the Voro++ software Rycroft (2009). For the estimators $\hat{F}_n^0$ and $\hat{F}_n$ we focus on the case $d = 2$. Let $M > 0$ and $z \geq 0$, we consider the following choices for the underlying F.

$$F_1(z; M) = z \cdot \mathbb{1}\{z < M\} + M \cdot \mathbb{1}\{z \geq M\} \tag{42}$$

$$F_2(z) = 0.01 \cdot \mathbb{1}\{z \geq 1\} + 0.04 \cdot \mathbb{1}\{z \geq 8\} + 0.95 \cdot \mathbb{1}\{z \geq 10\}. \tag{43}$$

Note that F_2 corresponds to the F in Example 1. For both choices of F, it is simple to simulate a corresponding Poisson process, because these Poisson processes can be recognized as independently marked homogeneous Poisson processes. Throughout this section we write $P_n := \mathbb{E}(\eta^0(W_n \times (0, \infty))$. We choose a square observation window W_n such that $P_n = 1000$. In words, we choose a square W_n with an area such that the expected number of observed points of η^0 in W_n is equal to 1,000. First, we consider F_1 as the underlying truth. For $M = 1$ and $M = 3$ we repeat

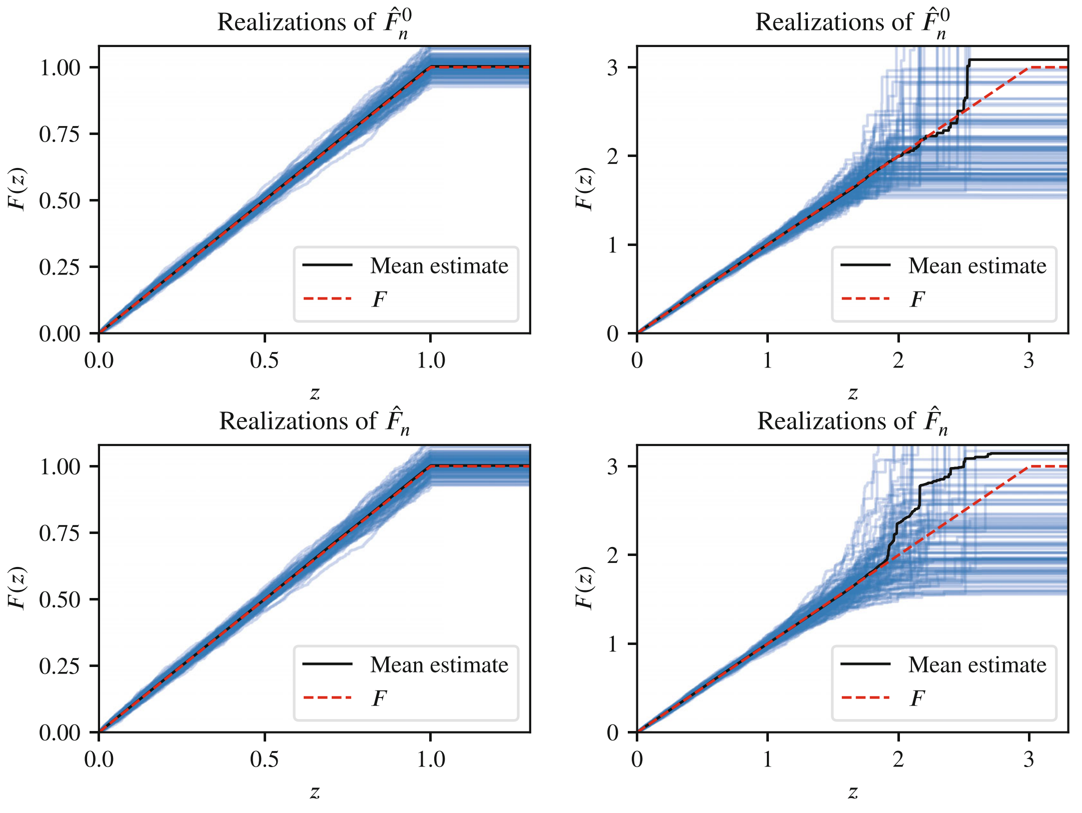

FIGURE 5 Simulation results for $\hat{F}_n^0$ and $\hat{F}_n$, where F is given by (42), with $M = 1$ (Left) and $M = 3$ (Right).

the simulation procedure with this F 100 times, such that we obtain 100 realizations of $\hat{F}_n^0$ and $\hat{F}_n$ for each value of M. For each estimator and each choice of M, we also compute the pointwise average of all realizations. The results are shown in Figure 5. A blue line is a realization of an estimator, and a black line is a pointwise average. We can clearly see that estimates of $F(z)$ for z close to zero are much more accurate than estimates of $F(z)$ for large values of z. This is especially evident for the results corresponding to $M = 3$. This is not too surprising in view of the crystallization interpretation of a Laguerre tessellation as described in Section 3. We expect that points with large weights are less likely to generate non-empty cells. As a result, we sample points with large weights less often, which makes estimation of $F(z)$ for large values of z more difficult. This also means that we expect that the accuracy of an estimate of F near zero is much more important if we wish to use this estimate to simulate a Poisson-Laguerre tessellation, which is similar to the observed tessellation. From Figure 5, it is not very clear whether there are significant differences between $\hat{F}_n^0$ and $\hat{F}_n$, though it does seem that $\hat{F}_n^0$ performs slightly better on average when z is large.

Now, we consider F_2 as the underlying truth. For this choice of F, we only observe points with weights in the set $\{1, 8, 10\}$. As a result, realizations of $\hat{F}_n^0$ and $\hat{F}_n$ will only have jumps at these values. We can therefore easily quantify the error of $\hat{F}_n^0$ by computing $F(z) - \hat{F}_n^0(z)$ for $z \in \{1, 8, 10\}$. Of course, we can do the same for $\hat{F}_n$. Again, we repeat the simulation procedure 100 times. This time, however, we also repeat this for multiple choices of observation windows. We choose W_n such that P_n is equal to 500, 1,000, 2,000, or 5,000. The simulation results are shown in

TABLE 1 Simulation results for $\hat{F}_n^0$, where F is given by (43).

	$F(1) - \hat{F}_n^0(1)$		$F(8) - \hat{F}_n^0(8)$		$F(10) - \hat{F}_n^0(10)$	
P_n	mean (| · |)	(2.5%, 97.5%)	mean (| · |)	(2.5%, 97.5%)	mean (| · |)	(2.5%, 97.5%)
500	0.002 84	(−0.0057, 0.0052)	0.007 34	(−0.018, 0.018)	0.0430	(−0.11, 0.10)
1000	0.001 97	(−0.0042, 0.0046)	0.004 24	(−0.010, 0.0099)	0.0327	(−0.065, 0.091)
2000	0.001 45	(−0.0034, 0.0029)	0.003 49	(−0.0079, 0.0081)	0.0192	(−0.042, 0.046)
5000	0.000 845	(−0.0019, 0.0020)	0.002 06	(−0.0046, 0.0052)	0.0146	(−0.030, 0.031)

TABLE 2 Simulation results for $\hat{F}_n$, where F is given by (43).

	$F(1) - \hat{F}_n(1)$		$F(8) - \hat{F}_n(8)$		$F(10) - \hat{F}_n(10)$	
P_n	mean (| · |)	(2.5%, 97.5%)	mean (| · |)	(2.5%, 97.5%)	mean (| · |)	(2.5%, 97.5%)
500	0.002 94	(−0.0069, 0.0057)	0.007 62	(−0.019, 0.017)	0.386	(−1.9, 0.47)
1000	0.001 98	(−0.0039, 0.0050)	0.004 42	(−0.010, 0.010)	0.267	(−1.1, 0.33)
2000	0.001 51	(−0.0037, 0.0031)	0.003 54	(−0.0074, 0.0083)	0.170	(−0.61, 0.25)
5000	0.000 885	(−0.0020, 0.0020)	0.002 13	(−0.0048, 0.0045)	0.107	(−0.30, 0.21)

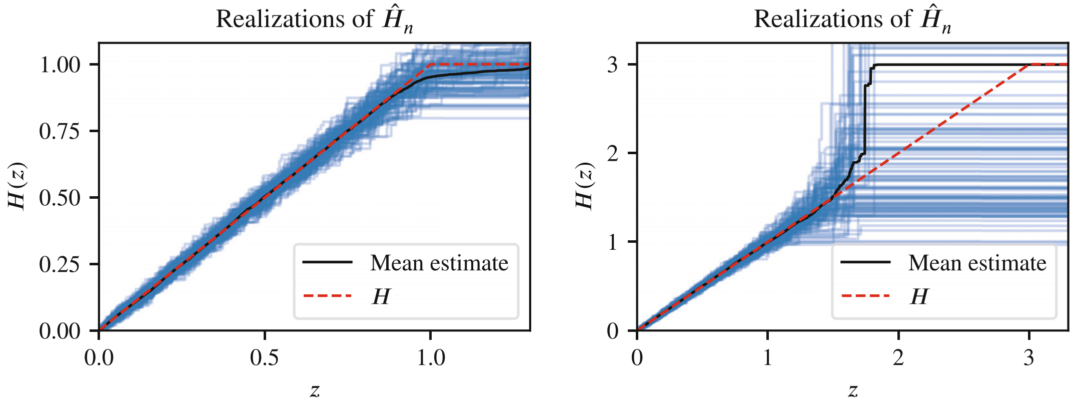

FIGURE 6 Simulation results for $\hat{H}_n$ where H is given by (42), with $M = 1$ (Left) and $M = 3$ (Right).

Tables 1 and 2. This table contains the mean over all 100 absolute errors for each choice of W_n and for each choice of z. We also include the 2.5% and 97.5% quantiles of these 100 errors. We can see that at $z = 1$ and $z = 8$ the performance of the estimators $\hat{F}_n^0$ and $\hat{F}_n$ is quite similar. However, at $z = 10$ it is clear that $\hat{F}_n^0$ performs much better. This is somewhat surprising, after all, $\hat{F}_n$ takes into account more information than $\hat{F}_n^0$. We do not yet know whether the difference in performance is due to differences in numerical stability of the inversion procedures or due to different rates of convergence of the estimators. We also should point out that for a single realization of $\hat{F}_n$ corresponding to $P_n = 500$, we observed a numerical overflow. That is, we observed: $\hat{F}_n(10) < \hat{F}_n(8)$. It may therefore be of future interest to study whether there are more numerically stable ways to compute $\hat{F}_n$.

Finally, we show some simulation results for $\hat{H}_n$. In the previous simulations we observed that $\hat{F}_n^0$ performs better than $\hat{F}_n$. Therefore, we compute $\hat{H}_n$ via $\overline{F}_n = \hat{F}_n^0$. We consider $d = 3$ such that we observe a 2D sectional tessellation. We take the underlying H equal to F_1 as in Equation (42). Again, we choose W_n such that $P_n = 1000$ and perform 100 repeated simulations for both $M = 1$ and $M = 3$. The results are shown in Figure 6. As expected, in the stereological setting, we observe a bigger variance in realizations of $\hat{H}_n$ compared to the realizations shown in Figure 5. Overall, all estimators seem to perform satisfactorily.

8 | DISCUSSION

In this paper, we have defined two estimators for the distribution function F, which describes the distribution of the arrival times of the generators. For these estimators, we have established their consistency and studied their performance in simulations. We have also considered statistical inference in a stereological setting. When computing the proposed estimators in practice, one has to deal with edge effects, and we now briefly discuss some of the challenges caused by this phenomenon. Throughout this paper, all points of η^* in $W_n \times (0, \infty)$ are considered to be known, as well as the Laguerre cells corresponding to these points. In practice, it is often the case that some of the cells near the boundary of W_n are only partially observed. Then, one could follow the suggestion in Remark 2 and compute the estimator based on a smaller window $W_n' \subset W_n$. Here, one should choose $W_n' \subset W_n$ as large as possible such that $C((x, h), \eta) \subset W_n$ for all $(x, h) \in \eta^* \cap (W_n' \times (0, \infty))$. The estimator of interest can then be computed, replacing W_n by W_n' in the definition of the estimator.

In a practical setting, a Laguerre tessellation may have been fitted to some image data using a reconstruction algorithm. It could be the case that a generator point in the observation window cannot be reconstructed because its cell is located outside of the observation window. Because there are various ways of reconstructing Laguerre tessellations and their extreme points, detailed simulations that consider the constraints induced by those reconstruction methods are out of the scope of this paper. We believe it will be useful to perform those kinds of simulations in future research, to better understand how well the estimators perform when all practical considerations are taken into account. For now, it should also be possible to mitigate these edge effects caused by the reconstruction approach by computing the estimators based on a smaller observation window $W_n' \subset W_n$.

Besides challenges arising from edge effects, we would also like to discuss a few possible directions that may be pursued in future research. There are various important properties of the estimators $\hat{F}_n^0$ and $\hat{F}_n$ which are still unknown. For instance, at present we do not know which of the estimators $\hat{F}_n^0$ or $\hat{F}_n$ should, in general, be preferred in practice. Hence, it may be of interest to study the rates of convergence of these estimators. Another important challenge for future research is the derivation of the asymptotic distributions of $\hat{F}_n^0$ and $\hat{F}_n$. Knowledge of these asymptotic distributions is essential for deriving (asymptotic) confidence intervals for $\hat{F}_n^0$ and $\hat{F}_n$. This information may then also be used to determine guidelines for required observation window sizes. Finally, from a practical point of view, it is also of interest to know how the estimators $\hat{F}_n^0$ and $\hat{F}_n$ perform when the underlying point process is not necessarily a Poisson process.

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CONFLICT OF INTEREST STATEMENT

None of the authors have a conflict of interest to disclose.

DATA AVAILABILITY STATEMENT

The data that support the findings of this study are available from the corresponding author upon reasonable request.

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APPENDIX A. PROOFS

Proof of Lemma 2. Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space supporting the sequence $(F_n)_{n\geq 1}$. For $z \in \mathbb{R}$, there exists by assumption a set $\Omega_z \in \mathcal{A}$ be such that $\lim_{n\to\infty} F_n(z;\omega) = F(z)$ for all $\omega \in \Omega_z$ and $\mathbb{P}(\Omega_z) = 1$. Define $D := \{z \in \mathbb{R} : z \in \mathbb{Q}$ or z is a discontinuity point of $F\}$. Because monotone functions have at most countably many discontinuity points and because the rationals are countable, it follows that D is countable. Letting $\Omega' := \cap_{z\in D}\Omega_z$ we obtain $\mathbb{P}(\Omega') = 1$. Let $z \in \mathbb{R}$ and $\omega \in \Omega'$, we show that $\lim_{n\to\infty} F_n(z;\omega) = F(z)$. If $z \in \mathbb{Q}$ or if z is a discontinuity point of F, then the result is immediate. Suppose that $z \in \mathbb{R} \setminus \mathbb{Q}$ is a continuity point of F. For $m \in \mathbb{N}$ choose $\delta_m > 0$ such that $|F(z) - F(x)| < 1/m$ whenever $|z - x| < \delta_m$. Choose $r_m, s_m \in \mathbb{Q}$ such that $r_m \leq z \leq s_m$ and $|r_m - z| < \delta_m$ and $|s_m - z| < \delta_m$. By the monotonicity of each F_n, and since $\omega \in \Omega'$ we have for all $m \in \mathbb{N}$:

$$F(r_m) = \lim_{n\to\infty} F_n(r_m;\omega) \leq \lim_{n\to\infty} F_n(z;\omega) \leq \lim_{n\to\infty} F_n(s_m;\omega) = F(s_m).$$

Due to the choice of r_m and s_m we obtain:

$$-\frac{1}{m} < F(r_m) - F(z) \leq \lim_{n\to\infty} F_n(z;\omega) - F(z) \leq F(s_m) - F(z) < \frac{1}{m}.$$

The result now follows since $|\lim_{n\to\infty} F_n(z;\omega) - F(z)| < 1/m$ for all $m \in \mathbb{N}$. ■

Proof of Lemma 3. Let $R > 0$ and assume $\lim_{n\to\infty} F_n(z) = F(z)$ for all $z \in [0, R)$. Fix $z \in [0, R)$. We introduce the following shorthand notation, for $h \in [0, z]$:

$$\phi_n(h) := \int_0^h (h-t)^{\frac{d}{2}} \mathrm{d}F_n(t), \text{ and } \phi(h) := \int_0^h (h-t)^{\frac{d}{2}} \mathrm{d}F(t).$$

The triangle inequality yields the following bound:

$$\begin{aligned}
&\left|G_{F_n}(z) - G_F(z)\right| \\
&\leq \left|\int_0^z \exp\left(-\kappa_d \int_0^h (h-t)^{\frac{d}{2}} \mathrm{d}F_n(t)\right) - \exp\left(-\kappa_d \int_0^h (h-t)^{\frac{d}{2}} \mathrm{d}F(t)\right) \mathrm{d}F_n(h)\right| + \\
&\quad + \left|\int_0^z \exp\left(-\kappa_d \int_0^h (h-t)^{\frac{d}{2}} \mathrm{d}F(t)\right) \mathrm{d}(F_n - F)(h)\right| \\
&\leq \sup_{h\in[0,z]} \left|\exp(-\kappa_d \phi_n(h)) - \exp(-\kappa_d \phi(h))\right| F_n(z) + \\
&\quad + \left|\int_0^z \exp(-\kappa_d \phi(h)) \mathrm{d}(F_n - F)(h)\right|. \qquad \text{(A1)}
\end{aligned}$$

Let us consider the first term of (A1). Fix $h \in [0, z]$. Since F_n converges pointwise to F on $[0, z]$ and $t \mapsto (h-t)^{\frac{d}{2}} \mathbb{1}\{h \geq t\}$ is continuous and bounded on $[0, z]$ it follows that $\lim_{n\to\infty} \phi_n(h) = \phi(h)$. Hence, the sequence of monotone functions $\exp(-\kappa_d \phi_n(\cdot))$ converges pointwise to the monotone function $\exp(-\kappa_d \phi(\cdot))$ on $[0, z]$. Because ϕ is (absolutely) continuous, the limit function $\exp(-\kappa_d \phi(\cdot))$ is continuous. The convergence is therefore uniform on $[0, z]$, and we obtain:

$$\lim_{n\to\infty} \sup_{h\in[0,z]} \left|\exp(-\kappa_d \phi_n(h)) - \exp(-\kappa_d \phi(h))\right| F_n(z) = 0 \cdot F(z) = 0.$$

Let us now consider the second term of (A1). Because $\exp(-\kappa_d \phi(\cdot))$ is continuous and bounded, it immediately follows from the pointwise convergence of F_n to F on $[0, z]$ that this second term vanishes as $n \to \infty$. This proves that $\lim_{n\to\infty} G_{F_n}(z) = G_F(z)$. The proof remains valid when $R = \infty$. ■

Proof of Theorem 7. Let $z \geq 0$. For $i \in \{1, 2\}$ write $F_i^V := V(\cdot; F_i, m_{F_i})$. From Equation (18), it can be seen that the (Lebesgue-Stieltjes) measures associated with F_i^V and F_i are mutually absolutely continuous. The corresponding Radon-Nikodym derivative is given by:

$$\frac{\mathrm{d}F_i^V}{\mathrm{d}F_i}(z) = \pi \int_z^\infty \exp\left(-\pi \int_0^u F_i(t)\mathrm{d}t\right) \mathrm{d}u =: p_i(z).$$

Hence, we may also write:

$$F_i(z) = \int_0^z \frac{\mathrm{d}F_i}{\mathrm{d}F_i^V}(h)\mathrm{d}F_i^V(h) = \int_0^z \frac{1}{p_i(h)} \mathrm{d}F_i^V(h).$$

Since $m_{F_i} < \infty$ (this is shown in the proof of Lemma 4), we have $p_i(0) < \infty$, and from its definition, it is clear that p_i is a decreasing function. Because $x \mapsto 1/x$ is Lipschitz on (c, ∞) for $c > 0$ with Lipschitz constant $1/c^2$ we have for $h \in [0, z]$:

$$\left|\frac{1}{p_1(h)} - \frac{1}{p_2(h)}\right| \leq \max\left\{\frac{1}{p_1(h)^2}, \frac{1}{p_2(h)^2}\right\} |p_1(h) - p_2(h)| \leq C(h)|p_1(h) - p_2(h)|.$$

Here we have defined $C(h) := \max\{1/p_1(h)^2, 1/p_2(h)^2\}$, which is increasing. As a consequence, we obtain the following upper bound for $|F_1(z) - F_2(z)|$:

$$\begin{aligned} |F_1(z) - F_2(z)| &= \left| \int_0^z \frac{1}{p_1(h)} - \frac{1}{p_2(h)} \mathrm{d}F_1^V(h) + \int_0^z \frac{1}{p_2(h)} \mathrm{d}(F_1^V - F_2^V)(h) \right| \\ &\le C(z) \int_0^z |p_1(h) - p_2(h)| \mathrm{d}F_1^V(h) + \left| \int_0^z \frac{1}{p_2(h)} \mathrm{d}(F_1^V - F_2^V)(h) \right|. \end{aligned} \tag{A2}$$

We consider the two terms in Equation(A2) separately. The first term of (A2) is bounded by:

$$\begin{aligned} &\pi C(z) \int_0^z \left| \int_0^\infty \exp\left(-\pi \int_0^u F_1(t)\mathrm{d}t\right) - \exp\left(-\pi \int_0^u F_2(t)\mathrm{d}t\right) \mathrm{d}u \right| \mathrm{d}F_1^V(h) + \\ &+ \pi C(z) \int_0^z \left| \int_0^h \exp\left(-\pi \int_0^u F_1(t)\mathrm{d}t\right) - \exp\left(-\pi \int_0^u F_2(t)\mathrm{d}t\right) \mathrm{d}u \right| \mathrm{d}F_1^V(h). \end{aligned} \tag{A3}$$

The first term of (A3) is equal to $\pi C(z) F_1^V(z) \left| m_{F_1} - m_{F_2} \right|$ and the second term of (A3) is bounded by:

$$\begin{aligned} &\pi C(z) \int_0^z \int_0^h \left| \exp\left(-\pi \int_0^u F_1(t)\mathrm{d}t\right) - \exp\left(-\pi \int_0^u F_2(t)\mathrm{d}t\right) \right| \mathrm{d}u \mathrm{d}F_1^V(h) \\ &\le \pi^2 C(z) \int_0^z \int_0^h \int_0^u |F_1(t) - F_2(t)| \mathrm{d}t \mathrm{d}u \mathrm{d}F_1^V(h) \\ &\le \pi^2 C(z) F_1^V(z) z \int_0^z |F_1(t) - F_2(t)| \mathrm{d}t. \end{aligned} \tag{A4}$$

In Equation (A4), we used the fact $|e^{-x} - e^{-y}| \le |x - y|$ for $x, y \ge 0$. Via the integration by parts formula, the second term of (A2) is bounded by:

$$\begin{aligned} &\left| \left(F_1^V(z) - F_2^V(z)\right) \frac{1}{p_2(z)} - \int_0^z F_1^V(h) - F_2^V(h) \mathrm{d}\left(\frac{1}{p_2(h)}\right)(h) \right| \\ &\le \frac{1}{p_2(z)} \left| F_1^V(z) - F_2^V(z) \right| + \sup_{h \in [0,z]} \left| F_1^V(h) - F_2^V(h) \right| \left| \int_0^z \mathrm{d}\left(\frac{1}{p_2(h)}\right)(h) \right| \\ &\le \frac{2}{p_2(z)} \sup_{h \in [0,z]} \left| F_1^V(h) - F_2^V(h) \right|. \end{aligned}$$

Collecting all results, we obtain:

$$\begin{aligned} |F_1(z) - F_2(z)| \le \pi C(z) F_1^V(z) \left| m_{F_1} - m_{F_2} \right| + \pi^2 C(z) F_1^V(z) z \int_0^z |F_1(t) - F_2(t)| \mathrm{d}t + \\ + \frac{2}{p_2(z)} \sup_{h \in [0,z]} \left| F_1^V(h) - F_2^V(h) \right|. \end{aligned}$$

Applying Theorem 3 and (9) yields:

$$|F_1(z) - F_2(z)| \le K(z) \left(\pi C(z) F_1^V(z) \left| m_{F_1} - m_{F_2} \right| + \frac{2}{p_2(z)} \sup_{h \in [0,z]} \left| F_1^V(h) - F_2^V(h) \right| \right). \tag{A5}$$

Here, $K(z)$ is given by:

$$K(z) := \left(1 + \pi^2 C(z) F_1^V(z) z^2 \exp\left(\pi^2 C(z) F_1^V(z) z^2\right)\right)$$

The statement of the theorem immediately follows from Equation (A5). ■

Proof of Lemma 4. We first note that we may assume without loss of generality that $(F_n)_{n\geq 1}$ is a sequence of functions not containing the zero function. Indeed, we could take an arbitrary subsequence $(n_l)_{l\geq 1} \subset (n)_{n\geq 1}$, and then use the pointwise convergence of F_n to F to choose a further subsequence $(n_k)_{k\geq 1} \subset (n_l)_{l\geq 1}$ such that $(F_{n_k})_{k\geq 1}$ is a sequence which does not contain the zero function. If we then show $\lim_{k\to\infty} m_{F_{n_k}} = m_F$ then the whole sequence also converges: $\lim_{n\to\infty} m_{F_n} = m_F$.

We introduce the following notation: For $u \geq 0$ let:

$$p_n(u) := \exp\left(-\pi \int_0^u F_n(t)\mathrm{d}t\right),\quad p(u) := \exp\left(-\pi \int_0^u F(t)\mathrm{d}t\right).$$

Via the inequality $|e^{-x} - e^{-y}| \leq |x - y|$ for $x, y \geq 0$ and (21) we obtain the following upper bound for $|p_n(u) - p(u)|$:

$$|p_n(u) - p(u)| \leq \pi \left| \int_0^u F(t) - F_n(t)\mathrm{d}t \right| = \pi \left| \int_0^u (u - t)\mathrm{d}(F - F_n)(t) \right|. \tag{A6}$$

Due to the pointwise convergence of F_n to F, we obtain that p_n converges pointwise to p as $n \to \infty$. The triangle inequality yields:

$$\left| m_{F_n} - m_F \right| \leq \int_0^z |p_n(u) - p(u)|\mathrm{d}u + \int_z^\infty |p_n(u) - p(u)|\mathrm{d}u. \tag{A7}$$

The first term of (A7) vanishes as $n \to \infty$. Indeed, p_n converges pointwise to p as $n \to \infty$, and since $|p_n(u) - p(u)| \leq 1$ the dominated convergence theorem may be applied. The dominated convergence theorem can also be used to show that the second term of (A7) vanishes as $n \to \infty$. We now show which dominating function g may be used. Choose $z \geq 0$ large enough such that $F(z) > 0$ and set $c := F(z)$. We show that $m_F < \infty$:

$$\begin{aligned} m_F &= \int_0^z p(u)\mathrm{d}u + \int_z^\infty p(u)\mathrm{d}u \\ &\leq z + \exp\left(-\pi \int_0^z F(t)\mathrm{d}t\right) \int_z^\infty \exp\left(-\pi \int_z^u F(t)\mathrm{d}t\right)\mathrm{d}u \\ &\leq z + \int_z^\infty \exp\left(-\pi c \int_z^u \mathrm{d}t\right)\mathrm{d}u = z + \frac{1}{\pi c}. \end{aligned} \tag{A8}$$

Choose $N \in \mathbb{N}$ large enough such that $F_n(z) \geq c/2$ for all $n \geq N$. This can be done since F_n converges pointwise to F. Applying the same bound as in Equation (A8) yields $|p_n(u)| \leq \exp(-\pi c(u - z)/2)$ for all $u \geq z$ and all $n \geq N$. Hence, we may define the dominating function $g : [z, \infty) \to [0, \infty)$ as:

$$g(u) := p(u) + \max\left\{ \max_{k\in\{1,..,N\}} p_k(u), \exp\left(-\pi \frac{c}{2}(u - z)\right) \right\}.$$

Note that $m_F < \infty$ and $m_{F_k} < \infty$ for all $k \in \{1, \ldots, N\}$ by (A8), applied to F and F_k, respectively. As a consequence, g is integrable on $[z, \infty)$. Because $|p_n(u) - p(u)| \leq g(u)$ for all $u \geq z$ the proof is finished. ■

Proof of Lemma 6. Let $z \geq 0$, we readily obtain the following bound:

$$\begin{aligned}
&\left|V(z; F_n, m_n) - V(z; F, m)\right| \\
&\quad \leq \left|\exp\left(-\pi \int_0^z F(t)\mathrm{d}t\right) - \exp\left(-\pi \int_0^z F_n(t)\mathrm{d}t\right)\right| + \\
&\quad + \pi|F(z) - F_n(z)|\left|m - \int_0^z \exp\left(-\pi \int_0^u F(t)\mathrm{d}t\right)\mathrm{d}u\right| + \\
&\quad + \pi F_n(z)\left(|m_n - m| + \left|\int_0^z \exp\left(-\pi \int_0^u F_n(t)\mathrm{d}t\right) - \exp\left(-\pi \int_0^u F(t)\mathrm{d}t\right)\mathrm{d}u\right|\right)
\end{aligned} \tag{A9}$$

Each of the three terms of (A9) vanishes as $n \to \infty$, and each of the terms appearing here also appears in the proof of Lemma 4. The fact that the first term vanishes follows from Equation (A6). The second term vanishes due to the pointwise convergence of F_n to F. The third term vanishes since $\lim_{n\to\infty} F_n(z) = F(z)$, $\lim_{n\to\infty} m_n = m$, and by using the same argument as for the first term in Equation (A7). ■

APPENDIX B. COMPUTATIONAL FORMULA

First of all, note that:

$$\hat{m}_n = \int_0^{h_1} \exp\left(-\pi \int_0^u \hat{F}_n^0(t)\mathrm{d}t\right)\mathrm{d}u + \int_{h_1}^{\infty} \exp\left(-\pi \int_0^u \hat{F}_n^0(t)\mathrm{d}t\right)\mathrm{d}u \tag{B1}$$

The first integral of (B1) is equal to h_1, since $\hat{F}_n^0$ is zero on $[0, h_1)$. Let $h_{k+1} > h_k$, via a direct computation we obtain:

$$\begin{aligned}
&\int_{h_1}^{h_{k+1}} \exp\left(-\pi \int_0^u \hat{F}_n^0(t)\mathrm{d}t\right)\mathrm{d}u = \\
&\quad = \sum_{i=2}^{k+1} \int_{h_{i-1}}^{h_i} \exp\left(-\pi \int_0^u \hat{F}_n^0(t)\mathrm{d}t\right)\mathrm{d}u \\
&\quad = \sum_{i=2}^{k+1} \exp\left(-\pi \int_0^{h_{i-1}} \hat{F}_n^0(t)\mathrm{d}t\right) \int_{h_{i-1}}^{h_i} \exp\left(-\pi \int_{h_{i-1}}^u \hat{F}_n^0(t)\mathrm{d}t\right)\mathrm{d}u \\
&\quad = \sum_{i=2}^{k+1} \exp\left(-\pi \int_0^{h_{i-1}} \hat{F}_n^0(t)\mathrm{d}t\right) \int_{h_{i-1}}^{h_i} \exp\left(-\pi(u - h_{i-1})\hat{F}_n^0(h_{i-1})\right)\mathrm{d}u \\
&\quad = \sum_{i=2}^{k+1} \exp\left(-\pi \sum_{j=1}^{i-1} \hat{F}_n^0(h_j)(h_j - h_{j-1})\right) \frac{1}{\pi \hat{F}_n^0(h_{i-1})}\left(1 - \exp\left(-\pi \hat{F}_n^0(h_{i-1})(h_i - h_{i-1})\right)\right).
\end{aligned}$$

Letting $h_{k+1} \to \infty$ we obtain:

$$\hat{m}_n = h_1 + \exp\left(-\pi \sum_{j=1}^{k} \hat{F}_n^0(h_j)(h_j - h_{j-1})\right) \frac{1}{\pi \hat{F}_n^0(h_k)} +$$
$$+ \sum_{i=2}^{k} \exp\left(-\pi \sum_{j=1}^{i-1} \hat{F}_n^0(h_j)(h_j - h_{j-1})\right) \frac{1}{\pi \hat{F}_n^0(h_{i-1})} \left(1 - \exp\left(-\pi \hat{F}_n^0(h_{i-1})(h_i - h_{i-1})\right)\right).$$