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Recursive least-squares

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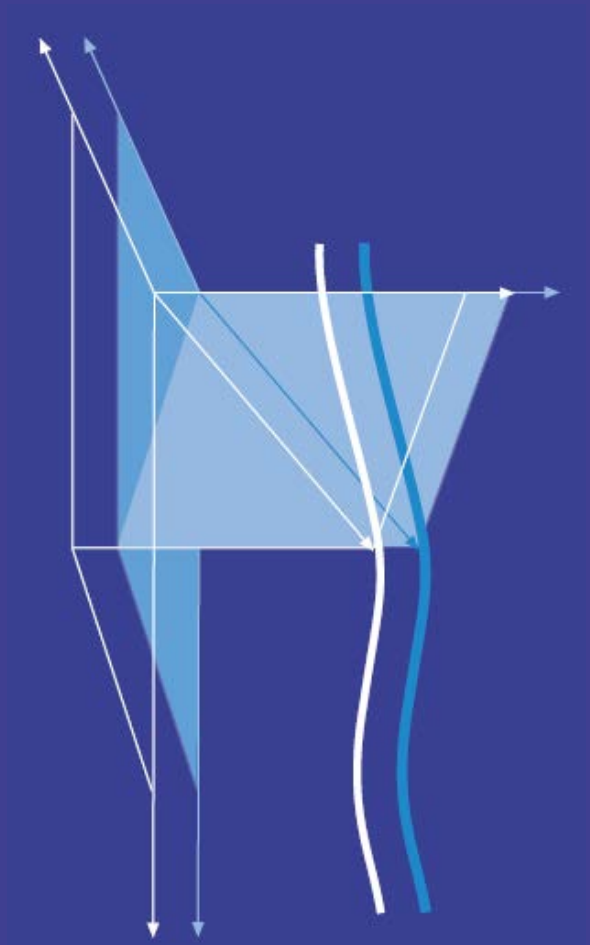
Peter J.G. Teunissen



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Delft University of Technology

Department of Mathematical Geodesy and Positioning

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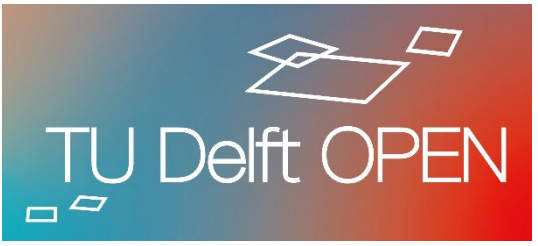

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Preface 2nd Edition

To promote open access, this new edition of *Dynamic Data Processing*, is published by TU Delft Open Publishing instead of Delft Academic Press. The book develops least-squares estimation theory for the case of time-varying parameters with an emphasis on their recursive determination. As such the book is a natural follow-on of *Adjustment Theory, An Introduction*, TU Delft Open Publishing, second edition 2024

August, 2024

Peter J.G. Teunissen

Foreword

This book is based on the lecture notes of the course *Dynamic data processing* as it has been given by the Department of Mathematical Geodesy and Positioning (MGP) of the Delft University of Technology since 1990. The prerequisites are a solid knowledge of adjustment theory and geodetic positioning, together with linear algebra, statistics and calculus. The theory and application of least-squares adjustment are treated in Adjustment theory (Delft University Press, 2000). The material of the present course extends the theory to the recursive estimation of time-varying or dynamic parameters. The time-varying parameters could for instance be geometric parameters such as position, attitude and shape, physical parameters such as temperature and humidity, or instrumental parameters such as clock drifts and biases. The time-varying parameters are said to be determined recursively when the method of determination enables sequential, rather than batch processing of the measurement data. The main goal is therefore to convey the knowledge necessary to be able to process sequentially collected measurement data in an optimal and efficient manner for the purpose of estimating time-varying parameters.

Following the *Introduction*, the basic theory of least-squares estimation is reviewed in *Chapter 1*. This is done for the model of observation equations and for the model of condition equations. In *Chapter 2* the principle of recursive least-squares estimation is introduced. The recursive principle allows one to update the least-squares solution for new observations without the need to store all past observations. Two different forms of the measurement-update equations are given. The results of Chapter 2, which hold true for time-invariant parameters, are generalized in *Chapter 3* to the case of time-varying parameters. The time-varying nature of the parameters is assumed captured by means of polynomial equations of motion. The recursive solution now consists of two types of update equations, the measurement-update equations and the time-update equations. Since there still exist many dynamic systems for which the rather simple polynomial model of Chapter 3 does not apply, a larger class of dynamic models is introduced in *Chapter 4*. These models are formulated using the state-space description of dynamic systems. In order to include randomness in the state-space description of dynamic systems, some of the elementary concepts of the theory of random functions are discussed in *Chapter 5*. This chapter also includes a description of the propagation laws for linear, time-varying systems. The results of Chapter 5 are used in *Chapter 6* to model possible uncertainties associated with the dynamic model. As a result the update equations are obtained for the recursive least-squares filtering and prediction of time-varying parameters.

Many colleagues of the Department of Mathematical Geodesy and Positioning whose assistance made the completion of this book possible are gratefully acknowledged. The typing of the book was done by Mrs. M.P.M. Scholtes, while C.D. de Jong took care of the editing. Various lecturers have taught the book's material over the past years. In particular the feedback and valuable recommendations of the lecturers H.M. de Heus, C.D. de Jong and C.C.J.M. Tiberius are acknowledged.

P.J.G. Teunissen
July, 2001

Contents

Introduction

As in other physical sciences, empirical data are used in geodesy to make inferences so as to describe physical reality. Many such problems involve the determination of unknown parameters from a set of redundant measurements. Measurements are said to be redundant when they exceed the minimum necessary for a unique determination of the parameters. There are two main reasons for collecting redundant measurements. First the requirement to be able to check for mistakes or errors. Second the wish to increase the accuracy of the results computed. As a consequence of measurement uncertainty (exact measurements do not exist), the redundant data are usually inconsistent in the sense that each sufficient subset yields results which will differ from the results obtained from another subset. To obtain a unique solution, consistency needs to be restored by applying corrections to the data. This computational process of making the measurement data consistent with the model such that the unknown parameters can be determined uniquely, is referred to as adjustment. Adjustment theory therefore deals with the optimal combination of redundant measurements together with the estimation of unknown parameters. An introductory course on adjustment was presented in *Adjustment theory* (Delft University Press, 2000). This theory is extended in this book to the case of *time-varying* or dynamic parameters with an emphasis on their recursive estimation.

Time-varying parameters occur in many geodetic models. They could be geometric parameters such as position, attitude and shape, physical parameters such as temperature and humidity, or instrumental parameters such as clock drifts and biases. When a body (e.g. satellite, aircraft, car, or ship) is in motion, its position changes as function of time. Being able to track the position of such a moving object is of importance, for instance for navigation and guidance. A moving body may also change its attitude as function of time. Attitude determination is sometimes needed as an aid to navigation and guidance, but it also applies, in case of earth rotation, to the Earth as a whole. Objects that are subject to deformation change their shape as a function of time. On a global scale, for instance, the earth deforms due to various geophysical processes. But the earth's surface may also change its shape on more local or regional scales. Subsidence due to gas extraction is one such example. Apart from time-varying geometric parameters, also physical and instrumental parameters may change as function of time. Atmospheric parameters such as those of the ionosphere and troposphere, change on an hourly, daily and even seasonal basis. Also the performance of instruments often displays a dependence on time. This is the reason why calibrations are carried out, so as to keep the time-varying instrumental parameters in control.

A parameter solution is said to be recursive when the method of determination enables sequential, rather than batch processing of the measurement data. The need for a recursive solution is usually driven by the efficiency with which such solutions can be computed. This holds true in particular for applications in which the time-varying parameters need to be determined instantly or in real-time. We speak of a (near) real-time determination when the time of determination (almost) coincides with the time the parameter takes on the value to be determined. Such applications can typically be found in the area of navigation and guidance. In the case of navigation, for instance, it does not make sense to determine one's position with a too long time delay. In these applications there is therefore a real need to have a computational cycle time of the position determination that is as short as possible. This is feasible when

recursive methods are used. But even in case real-time solutions are not an important issue, the use of recursive methods can still be attractive due to their computational efficiency.

When determining time-varying parameters from sequentially collected measurement data, one can discriminate between three types of estimation problems (see Figure 0.1). When the time at which a parameter estimate is required coincides with the time the last measurements are collected, the problem is referred to as *filtering*. When the time of interest falls within the time span of available measurement data, the problem is referred to as *smoothing*, and when the time of interest occurs after the time the last measurements are collected, the problem is called *prediction*. Thus filtering aims at the determination of current parameter values, while smoothing and prediction aim respectively at the determination of past and future parameter values. The emphasis in this book will be on recursive filtering.

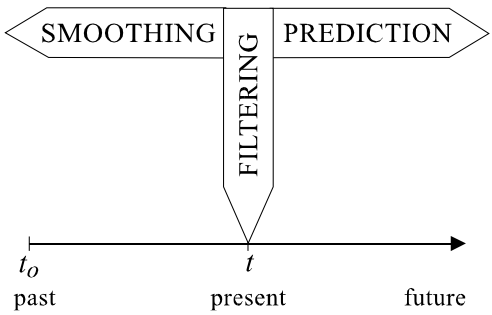


Figure 0.1: Prediction, filtering and smoothing.

The essence of a recursive method is that it enables one to update the parameter estimates for new measurements without the need to store all past measurements. Assume, for example, that one has collected at epoch *t-1* a redundant set of measurements y_{t-1} which bears a linear relationship with an unknown parameter vector x. The measurements y_{t-1} can then be used to obtain a linear least-squares estimate $\hat{x}_{t-1}$ of the unknown parameter vector x. Now assume that at the next epoch t a new set of measurements y_t becomes available which also bears a linear relationship to the same unknown parameter vector x. Since these additional measurements also contain information about the unknown parameter vector x, they can be used to improve the estimate $\hat{x}_{t-1}$ of x. One approach would be to use both y_{t-1} and y_t and to repeat the least-squares adjustment. As a result one obtains the improved least-squares estimate $\hat{x}_t$ of x. Although this approach is valid, it requires that one saves the past measurements y_{t-1}. In some cases this may be a too heavy computational burden, in particular if there are many past measurements or many epochs that precede the current epoch. Fortunately there is an alternative approach available, the recursive solution. It can be shown (under some mild restrictions) that the same improved least-squares estimate $\hat{x}_t$ of x, can also be computed from $\hat{x}_{t-1}$ and y_t instead of from y_{t-1} and y_t. The solution will then have the recursive structure:

$$\hat{x}_t = \hat{x}_{t-1} + K_t(y_t - \boldsymbol{A}_t\hat{x}_{t-1})$$

in which K_t and A_t are matrices. This recursive equation, which holds true for any epoch t, is referred to as the *measurement-update* equation: the new measurements y_t are used to update the previous parameter estimate $\hat{x}_{t-1}$ so as to obtain the current parameter estimate $\hat{x}_t$.

Some elements of recursive estimation were already briefly introduced in *Adjustment theory* (Chapter 6, Section 3). However, just as in the above example, this brief introduction only dealt with models in which the parameter vector remained constant in time. In this book we will extend the theory to the case of time-varying parameters. This implies that some additional modeling needs to be done, namely one that describes the time-dependence of the parameter vector. Depending on the application at hand, these equations of motion can be of a kinematic or of a dynamic nature. Kinematics is used to relate position, velocity, acceleration and time without reference to the cause of motion, whereas dynamics also includes an explicit description of the forces responsible for the motion. As a consequence of having incorporated the time-varying nature of the parameter vector into the model, the recursion will now consist of two different update equations, the *time-update* (TU) and the *measurement update* (MU):

$$\hat{x}_{t|t-1} = \mathbf{\Phi}_{t,t-1}\hat{x}_{t-1|t-1} \text{ (TU) and } \hat{x}_{t|t} = \hat{x}_{t|t-1} + K_t(y_t - \boldsymbol{A}_t\hat{x}_{t|t-1}) \text{ (MU)}$$

with $\Phi_{t,t-1}$ the transition matrix. The time-update uses the filtered estimate $\hat{x}_{t-1|t-1}$ of epoch *t-1* to predict the parameter vector of the next epoch, x_t, as $\hat{x}_{t|t-1}$. This predicted estimate together with the new measurements y_t are then combined in the measurement update to obtain the filtered estimate of x_t as $\hat{x}_{t|t}$.

1 Least-squares: a review

1.1 The linear A-model

1.1.1 Consistency and inconsistency

Assume that we want to determine n parameters $x_\alpha \in \mathbb{R}$, $\alpha = 1,...,n$. An m-number of measurements $y_i \in \mathbb{R}$, $i = 1,...,m$, are carried out to determine these parameters. If the measurements bear a known linear relationship with the unknown parameters, we may write the model of observation equations as:

(1) $$y_i = \sum_{\alpha=1}^{n} a_{i\alpha} x_\alpha \quad , \quad i = 1,...,m \, .$$

In this equation the known scalars $a_{i\alpha}$ model the assumed linear relationships between the measurements y_i and the parameters x_α. By introducing the matrix and vectors:

$$\underset{m\times n}{\boldsymbol{A}} = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} \quad \underset{m\times 1}{y} = \begin{pmatrix} y_1 \\ \vdots \\ y_m \end{pmatrix} \quad \underset{n\times 1}{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

equation (1) can be written in matrix-vector form as:

(2) $$\underset{m\times 1}{y} = \underset{m\times n}{\boldsymbol{A}} \; \underset{n\times 1}{x} \, .$$

This is a system of an m-number of linear equations in an n-number of unknown parameters. It is now of interest to know under what conditions a solution to the linear system (2) exists and if a solution exists, whether it is unique or not. It will be clear that a solution to (2) exists if and only if the vector y can be written as a linear combination of the column vectors of matrix A. If this is the case the vector y is an element of the column space or range space of matrix A. This space is denoted as $R(A)$. Thus a solution to (2) exists if and only if:

(3) $$y \in \boldsymbol{R}(\boldsymbol{A}) \, .$$

Systems for which this holds are called consistent systems. A system is said to be inconsistent if and only if:

(4) $$y \notin \boldsymbol{R}(\boldsymbol{A}) \, .$$

In this case the vector y cannot be written as a linear combination of the column vectors of matrix A and hence no vector x exists such that (2) holds. The difference between consistency and inconsistency is depicted geometrically in Figure 1.1.

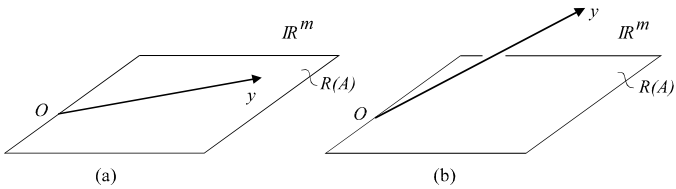


Figure 1.1: $\begin{cases}(a)\ \text{consistency} & : y \in R(A) \subset \mathbb{R}^m \\ (b)\ \text{inconsistency} & : y \notin R(A) \subset \mathbb{R}^m \end{cases}$.

Since $y \in \mathbb{R}^n$, it follows from (3) that consistency is guaranteed if $R(A) = \mathbb{R}^n$. But $R(A) = \mathbb{R}^n$ only holds if the dimension of $R(A)$ equals the dimension of $\mathbb{R}^n$. Hence, if dim $R(A) = m$. It follows therefore, since dim $R(A)$ equals the rank of matrix A (= number of linear independent columns or rows of A), that consistency is guaranteed if and only if:

(5) $$\text{rank } \boldsymbol{A} = m\,.$$

In all other cases, rank $A < m$, the linear system may or may not be consistent. Assuming consistency, the next question one can ask is whether the solution to (2) is unique or not. That is, whether the information content of the measurements collected in the vector y is sufficient for determining the parameter vector x uniquely. The solution is unique only if all the columns of matrix A are linearly independent. Hence, the solution is unique if the rank of matrix A equals the number of unknown parameters:

(6) $$\text{rank } \boldsymbol{A} = n\,.$$

To clarify this, assume x and $x' \neq x$ to be two different solutions of (2). Then $Ax = Ax'$ or $A(x - x') = 0$ must hold. But this can only be the case if some of the columns of matrix A are linearly dependent, which contradicts the assumption of full column rank (6). In all other cases, rank $A < n$, there will be more than one solution to a consistent system. In this book we will always assume that (6) holds. The case that rank $A < n$ is treated elsewhere [Teunissen, 1985a]. With rank $A = n$ and the fact that the rank of matrix A is always equal to or less than the number of rows or columns of A, it follows that two cases can be distinguished:

(7) $$m = n = \text{rank } \boldsymbol{A} \quad \text{or} \quad m > n = \text{rank } \boldsymbol{A}\,.$$

In the first case, both (5) and (6) are satisfied, implying that the linear system (2) is consistent and that a unique solution exists. The unique solution, denoted by $\hat{x}$, is found through an inversion of the matrix A:

(8) $$\left.\begin{array}{c} \underset{m\times 1}{y} = \underset{m\times n}{\boldsymbol{A}}\ \underset{n\times 1}{x} \\ m = n = \text{rank } \boldsymbol{A} \end{array}\right\} \Rightarrow \hat{x} = \boldsymbol{A}^{-1} y\,.$$

In the second case, only (6) is satisfied, implying that a unique solution to (2) exists provided that the system is consistent. Consistency in this case is however not guaranteed. But if we assume the system to be consistent, that is $y \in R(A)$, one way to obtain the unique solution is to invert n out of the $m > n$ linear equations:

$$(9)\qquad \left.\begin{array}{l} \underset{m\times 1}{y} = \underset{m\times n}{\boldsymbol{A}}\ \underset{n\times 1}{x} \\ m>n \ = \ \text{rank } \boldsymbol{A} \\ \text{consistent} \end{array}\right\} \Rightarrow \underbrace{\begin{array}{c} n \\ m-n \end{array}\begin{pmatrix} y_1 \\ \cdots \\ y_2 \end{pmatrix}}_{\underset{m\times 1}{y}} = \underbrace{\begin{array}{c} n \\ m-n \end{array}\overset{n}{\begin{pmatrix} \boldsymbol{A}_1 \\ \cdots \\ \boldsymbol{A}_2 \end{pmatrix}}}_{\underset{m\times n}{\boldsymbol{A}}} \Rightarrow \hat{x} = \boldsymbol{A}_1^{-1} y_1 .$$

Since the columns of matrix A are linearly independent, it is possible to find a matrix A_1 for which the columns are linearly independent as well, implying that the inverse of the square matrix A_1 exists. Note that y_2 is not used in computing $\hat{x}$. This is allowed in the present situation since y_2 is consistent with y_1 and hence does not contain any additional information.

Example 1

Consider the linear system:

$$(10)\qquad \underbrace{\begin{pmatrix} 2 \\ 1 \end{pmatrix}}_{y} = \underbrace{\begin{pmatrix} 1 & 3 \\ 2 & -1 \end{pmatrix}}_{\boldsymbol{A}} \underbrace{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}}_{x} .$$

In this case we have: $m = 2$, $n = 2$ and rank $A = 2$. Thus, the system is consistent since rank $A = m = 2$, and the system has a unique solution since rank $A = n = 2$. The unique solution of (10) reads: $\hat{x} = (5/7\ ,\ 3/7)^{*}$.

Example 2

A particle is moving with constant velocity along a straight line. If we denote the position of the particle as function of time as $u(t)$, we have:

$$u(t) = u(t_0) + \dot{u}(t_0)(t - t_0)$$

with $u(t_0)$ and $\dot{u}(t_0)$ being the initial position and initial velocity respectively of the particle at time t_0. It is assumed that the initial position and initial velocity of the particle are unknown. The unknown parameters $u(t_0)$ and $\dot{u}(t_0)$ can then be determined from two measurements of position at times t_1 and $t_2 \neq t_1$. This results in the following linear system (see Figure 1.2):

$$(11)\qquad \underbrace{\begin{pmatrix} u(t_1) \\ u(t_2) \end{pmatrix}}_{y} = \underbrace{\begin{pmatrix} 1 & (t_1 - t_0) \\ 1 & (t_2 - t_0) \end{pmatrix}}_{\boldsymbol{A}} \underbrace{\begin{pmatrix} u(t_0) \\ \dot{u}(t_0) \end{pmatrix}}_{x} .$$

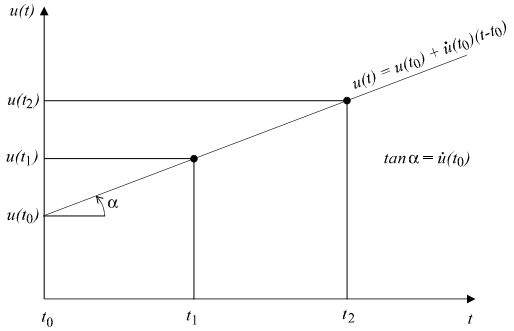


Figure 1.2: Position as function of time.

In this case we have: $m = 2$, $n = 2$ and rank $A = 2$. Thus, the system is consistent since rank $A = m = 2$, and the system has a unique solution since rank $A = n = 2$. With

$$\boldsymbol{A}^{-1} = \frac{1}{(t_2 - t_1)} \begin{pmatrix} (t_2 - t_0) & -(t_1 - t_0) \\ -1 & 1 \end{pmatrix}$$

the unique solution follows as:

$$\begin{pmatrix} \hat{u}(t_0) \\ \hat{\dot{u}}(t_0) \end{pmatrix} = \frac{1}{(t_2 - t_1)} \begin{pmatrix} (t_2 - t_0) & -(t_1 - t_0) \\ -1 & 1 \end{pmatrix} \begin{pmatrix} u(t_1) \\ u(t_2) \end{pmatrix}$$

or as:

(12)
$$\begin{cases} \hat{u}(t_0) = \dfrac{t_2 - t_0}{t_2 - t_1} u(t_1) - \dfrac{t_1 - t_0}{t_2 - t_1} u(t_2) = u(t_1) - \hat{\dot{u}}(t_0)(t_1 - t_0) \\ \hat{\dot{u}}(t_0) = \dfrac{u(t_2) - u(t_1)}{t_2 - t_1} \end{cases}.$$

Note that rank $A = 1$ if $t_2 = t_1$. In this case no unique solution exists.

Example 3

Consider the linear system:

(13)
$$\underbrace{\begin{pmatrix} -2 \\ 3 \\ -1 \end{pmatrix}}_{y} = \underbrace{\begin{pmatrix} 1 & 3 \\ 2 & -1 \\ 1 & 2 \end{pmatrix}}_{\boldsymbol{A}} \underbrace{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}}_{x}.$$

In this case we have: $m = 3$, $n = 2$ and rank $A = 2$. Since $m = 3 >$ rank $A = 2$, consistency of the system is not automatically guaranteed. A closer look at the measurement vector y of (13) shows however that:

$$\begin{pmatrix} -2 \\ 3 \\ -1 \end{pmatrix} = 1 \cdot \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} - 1 \cdot \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}.$$

This shows that y can be written as a linear combination of the column vectors of A. Therefore $y \in R(A)$, showing that the system is consistent. And since n = rank A = 2, its solution is also unique. If we partition (13) as:

$$\begin{pmatrix} -2 \\ 3 \\ .... \\ -1 \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 2 & -1 \\ ..... & \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

the unique solution follows as:

$$\text{(14)} \qquad \begin{pmatrix} \hat{x}_1 \\ \hat{x}_2 \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 2 & -1 \end{pmatrix}^{-1} \begin{pmatrix} -2 \\ 3 \end{pmatrix} = -\frac{1}{7} \begin{pmatrix} -1 & -3 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} -2 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

The system (13) may of course also be partitioned as:

$$\begin{pmatrix} -2 \\ ..... \\ 3 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ ..... & \\ 2 & -1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

The unique solution follows then as:

$$\text{(15)} \qquad \begin{pmatrix} \hat{x}_1 \\ \hat{x}_2 \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix}^{-1} \begin{pmatrix} 3 \\ -1 \end{pmatrix} = \frac{1}{5} \begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 3 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Example 4

Consider again the situation of a particle moving along a straight line with constant velocity. But now assume that three measurements of position are carried out at the three different times t_1, t_2 and t_3. The linear system reads then:

$$\text{(16)} \qquad \underbrace{\begin{pmatrix} u(t_1) \\ u(t_2) \\ u(t_3) \end{pmatrix}}_{y} = \underbrace{\begin{pmatrix} 1 & (t_1 - t_0) \\ 1 & (t_2 - t_0) \\ 1 & (t_3 - t_0) \end{pmatrix}}_{\boldsymbol{A}} \underbrace{\begin{pmatrix} u(t_0) \\ \dot{u}(t_0) \end{pmatrix}}_{x}.$$

With the times $t_0 = 0$, $t_1 = 1$, $t_2 = 2$ and $t_3 = 3$, and the position measurements $u(t_1 = 1) = 3$, $u(t_2 = 2) = 4$, $u(t_3 = 3) = 5$ system (16) becomes:

(17)
$$\begin{pmatrix}3\\4\\5\end{pmatrix} = \begin{pmatrix}1 & 1\\1 & 2\\1 & 3\end{pmatrix}\begin{pmatrix}u(t_0)\\\dot{u}(t_0)\end{pmatrix}.$$

In this case we have: $m = 3$, $n = 2$ and rank $A = 2$. Since $m = 3 >$ rank $A = 2$, consistency of the system is not automatically guaranteed. But a closer look at the measurement vector of (17) shows that:

$$\begin{pmatrix}3\\4\\5\end{pmatrix} = 2\cdot\begin{pmatrix}1\\1\\1\end{pmatrix} + 1\cdot\begin{pmatrix}1\\2\\3\end{pmatrix}.$$

Thus the measurement vector y can be written as a linear combination of the column vectors of matrix A. Therefore $y \in R(A)$, showing that (17) is consistent. And since n = rank $A = 2$ its solution is also unique. If we partition (17) as:

$$\begin{pmatrix}3\\4\\...\\5\end{pmatrix} = \begin{pmatrix}1 & 1\\1 & 2\\.....\\1 & 3\end{pmatrix}\begin{pmatrix}u(t_0)\\\dot{u}(t_0)\end{pmatrix}$$

the unique solution follows as:

(18)
$$\begin{pmatrix}\hat{u}(t_0)\\\hat{\dot{u}}(t_0)\end{pmatrix} = \begin{pmatrix}1 & 1\\1 & 2\end{pmatrix}^{-1}\begin{pmatrix}3\\4\end{pmatrix} = \begin{pmatrix}2 & -1\\-1 & 1\end{pmatrix}\begin{pmatrix}3\\4\end{pmatrix} = \begin{pmatrix}2\\1\end{pmatrix}.$$

In this case the solution for $u(t_0), \dot{u}(t_0)$ is found from fitting the line $u(t) = u(t_0) + \dot{u}(t_0)(t - t_0)$ through the two points $(t_1 = 1, u(t_1) = 3)$ and $(t_2 = 2,\ u(t_2) = 4)$. See Figure 1.3a. We may of course partition (17) also as:

$$\begin{pmatrix}3\\...\\4\\5\end{pmatrix} = \begin{pmatrix}1 & 1\\.....\\1 & 2\\1 & 3\end{pmatrix}\begin{pmatrix}u(t_0)\\\dot{u}(t_0)\end{pmatrix}$$

the unique solution follows then as:

(19)
$$\begin{pmatrix}\hat{u}(t_0)\\\hat{\dot{u}}(t_0)\end{pmatrix} = \begin{pmatrix}1 & 2\\1 & 3\end{pmatrix}^{-1}\begin{pmatrix}4\\5\end{pmatrix} = \begin{pmatrix}3 & -2\\-1 & 1\end{pmatrix}\begin{pmatrix}4\\5\end{pmatrix} = \begin{pmatrix}2\\1\end{pmatrix}.$$

In this case the solution for $u(t_0), \dot{u}(t_0)$ is found from fitting the line $u(t) = u(t_0) + \dot{u}(t_0)(t - t_0)$ through the two points $(t_2 = 2, u(t_2) = 4)$ and $(t_3 = 3, u(t_3) = 5)$. See Figure 1.3b. Since the linear system (17) is consistent, which means that all three points $(t_1, u(t_1))$, $(t_2, u(t_2))$ and $(t_3, u(t_3))$ lie on the same line (see Figure 1.3c), the two solutions (18) and (19) are of course identical.

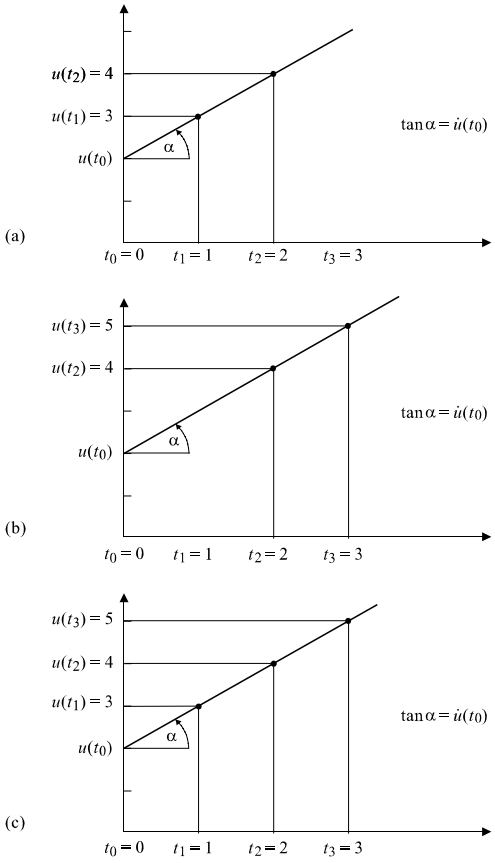


Figure 1.3: Fitting a straight line through consistent measurements.

Example 5

Consider the situation of Example 4. But now assume that the position measurements read:

$$u(t_1=1) = 3,\ u(t_2=2) = 5,\ u(t_3=3) = 6.$$

The linear system (16) then becomes:

$$\underbrace{\begin{pmatrix}3\\5\\6\end{pmatrix}}_{y} = \underbrace{\begin{pmatrix}1 & 1\\1 & 2\\1 & 3\end{pmatrix}}_{A} \underbrace{\begin{pmatrix}u(t_0)\\\dot{u}(t_0)\end{pmatrix}}_{x}. \tag{20}$$

In this case the measurement vector y cannot be written as a linear combination of the column vectors of matrix A. Hence $y \notin R(A)$, showing that the system is inconsistent. This inconsistency can clearly be seen from Figure 1.4a. Figure 1.4a shows clearly that no straight line

$u(t) = u(t_0) + \dot{u}(t_0)(t - t_0)$ exists that passes through all three measurement points. In order to find a solution one could disregard measurement $u(t_3 = 3) = 6$ and solve the system:

$$\begin{pmatrix} u(t_1) = 3 \\ u(t_2) = 5 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} u(t_0) \\ \dot{u}(t_0) \end{pmatrix}.$$

The solution of this system reads (see Figure 1.4b):

(21)
$$\begin{pmatrix} \hat{u}(t_0) \\ \hat{\dot{u}}(t_0) \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}^{-1} \begin{pmatrix} 3 \\ 5 \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 3 \\ 5 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

But instead of disregarding $u(t_3 = 3) = 6$ one could also opt for disregarding measurement $u(t_1 = 1) = 3$ or measurement $u(t_2 = 2) = 5$. In case of disregarding measurement $u(t_1 = 1) = 3$, one has to solve the system:

$$\begin{pmatrix} u(t_2) = 5 \\ u(t_3) = 6 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} u(t_0) \\ \dot{u}(t_0) \end{pmatrix}.$$

The solution of this system reads (see Figure 1.4b):

(22)
$$\begin{pmatrix} \hat{u}(t_0) \\ \hat{\dot{u}}(t_0) \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}^{-1} \begin{pmatrix} 5 \\ 6 \end{pmatrix} = \begin{pmatrix} 3 & -2 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 5 \\ 6 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}.$$

This solution differs however from solution (21) (see also Figure 1.4b). So, which solution should we accept? The problem with the above approach is the arbitrariness in disregarding measurements. Why should we disregard measurement $u(t_3 = 3) = 6$ and completely rely on the measurements $u(t_1 = 1) = 3$ and $u(t_2 = 2) = 5$? It seems more appropriate to have a solution method which somehow takes all measurements into account. In case of the present example one could for instance think of computing the line $u(t) = u(t_0) + \dot{u}(t_0)(t - t_0)$ such that it fits all three measurement points as closely as possible (see Figure 1.4c). A method that accomplishes this task in a predefined way, is the *method of least-squares*. This method will be introduced in the next section.

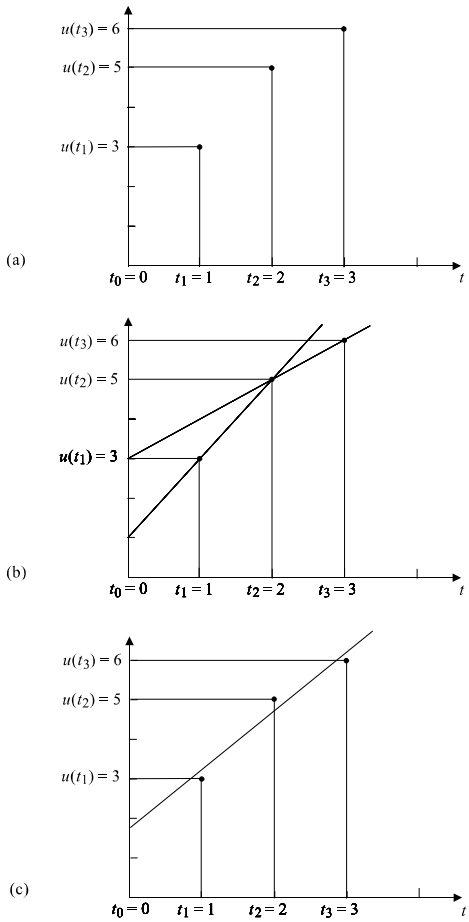


Figure 1.4: Fitting a straight line through inconsistent measurements.

1.1.2 Least-squares estimates

An inconsistent system, that is, a system for which $y \notin R(A)$ holds, can be made consistent by introducing an $m \times 1$ error vector e as (see Figure 1.5):

$$\underset{m\times 1}{y} = \underset{m\times n}{\boldsymbol{A}}\ \underset{n\times 1}{x} + \underset{m\times 1}{\boldsymbol{e}} \quad , \quad m > n = \operatorname{rank} \boldsymbol{A} . \tag{23}$$

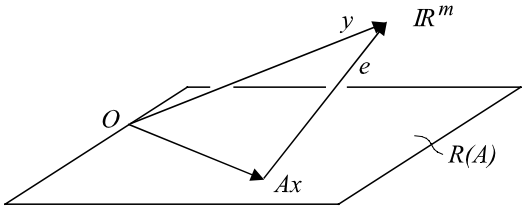


Figure 1.5: The geometry of $y = Ax + e$.

In (23), y and A are given, whereas x and e are unknown. From the geometry of Figure 1.5 it seems intuitively appealing to estimate x as $\hat{x}$ such that $A\hat{x}$ is as close as possible to the given measurement- or observation vector y. In other words, the idea is to find that value of x that minimizes the length of the vector $e = y - Ax$. This idea leads to the following minimization problem:

(24) $$\boxed{\underset{x}{\text{minimize}}\ (y - \boldsymbol{A}x)^*(y - \boldsymbol{A}x)}\ .$$

From calculus we know that $\hat{x}$ is a solution of (24) if $\hat{x}$ statisfies:

(25) $$\frac{\partial E}{\partial x}(\hat{x}) = 0 \quad \text{and} \quad \frac{\partial^2 E}{\partial x^2}(\hat{x})\ \text{positive-definite}$$

where $E(x)$ is given as:

(26) $$E(x) = (y - \boldsymbol{A}x)^*(y - \boldsymbol{A}x) = y^*y - 2y^*\boldsymbol{A}x + x^*\boldsymbol{A}^*\boldsymbol{A}x\ .$$

Taking the first-order and second-order partial derivatives of $E(x)$ gives:

(27) $$\frac{\partial E}{\partial x}(x) = -2\boldsymbol{A}^*y + 2\boldsymbol{A}^*\boldsymbol{A}x \quad \text{and} \quad \frac{\partial^2 E}{\partial x^2}(x) = 2\boldsymbol{A}^*\boldsymbol{A}\ .$$

Equating the first equation of (27) to zero shows that $\hat{x}$ satisfies the normal equations:

(28) $$\boldsymbol{A}^*\boldsymbol{A}\hat{x} = \boldsymbol{A}^*y\ .$$

Since rank A^*A = rank $A = n$, the system is consistent and has a unique solution. Through an inversion of the normal matrix A^*A the unique solution of (28) is found as:

(29) $$\boxed{\hat{x} = (\boldsymbol{A}^*\boldsymbol{A})^{-1}\boldsymbol{A}^*y}\ .$$

That this solution $\hat{x}$ is the minimizer of (26) follows from the fact that the matrix $\partial^2 E/\partial x^2$ of (27) is indeed positive-definite. The vector $\hat{x}$ is known as the least-squares estimate of x, since it produces the smallest possible value of the sum-of-squares function $E(x)$. From the normal equations (28) it follows that $A^*(y - A\hat{x}) = 0$. This shows that the vector $\hat{e} = y - A\hat{x}$, which is the least-squares estimate of e, is orthogonal to the range space of matrix A (see Figure 1.6):

(30) $$\boxed{\boldsymbol{A}^*\hat{\boldsymbol{e}} = 0,\ \text{with}\ \hat{\boldsymbol{e}} = y - \boldsymbol{A}\hat{x}}\ .$$

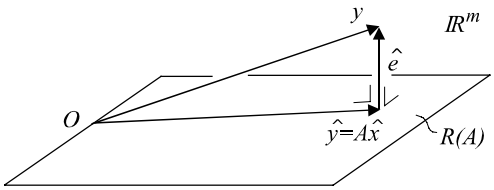


Figure 1.6: The geometry of least-squares: $y = A\hat{x} + \hat{e}$.

Example 6

Consider again the situation of a particle moving with constant velocity along a straight line. We assume that three observations of position are carried out at three different times t_1, t_2 and t_3: $u(t_1)$, $u(t_2)$ and $u(t_3)$. We also assume that the time instances are equidistant: $t_1 - t_0 = t_2 - t_1 = t_3 - t_2 = T$. The linear system reads then:

$$\begin{pmatrix} u(t_1) \\ u(t_2) \\ u(t_3) \end{pmatrix} = \begin{pmatrix} 1 & T \\ 1 & 2T \\ 1 & 3T \end{pmatrix} \begin{pmatrix} u(t_0) \\ \dot{u}(t_0) \end{pmatrix}.$$

If the system is inconsistent, it can be made consistent by introducing an error vector e as:

(31)
$$\underbrace{\begin{pmatrix} u(t_1) \\ u(t_2) \\ u(t_3) \end{pmatrix}}_{y} = \underbrace{\begin{pmatrix} 1 & T \\ 1 & 2T \\ 1 & 3T \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} u(t_0) \\ \dot{u}(t_0) \end{pmatrix}}_{x} + \underbrace{\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}}_{e}.$$

The least-squares solution of (31) follows now from minimizing $e^* e = \sum_{i=1}^{3} e_i^2$ as function of the parameters $u(t_0)$ and $\dot{u}(t_0)$. With

$$(A^* A)^{-1} = \begin{pmatrix} 3 & 6T \\ 6T & 14T^2 \end{pmatrix}^{-1} = \frac{1}{6T^2} \begin{pmatrix} 14T^2 & -6T \\ -6T & 3 \end{pmatrix}$$

and

$$A^* y = \begin{pmatrix} 1 & 1 & 1 \\ T & 2T & 3T \end{pmatrix} \begin{pmatrix} u(t_1) \\ u(t_2) \\ u(t_3) \end{pmatrix} = \begin{pmatrix} \sum_{i=1}^{3} u(t_i) \\ T\sum_{i=1}^{3} i u(t_i) \end{pmatrix}$$

the least-squares estimate of $x = (u(t_0),\ \dot{u}(t_0))^*$ follows as:

$$\begin{pmatrix} \hat{u}(t_0) \\ \hat{\dot{u}}(t_0) \end{pmatrix} = \frac{1}{6T^2} \begin{pmatrix} 14T^2 & -6T \\ -6T & 3 \end{pmatrix} \begin{pmatrix} \sum_{i=1}^{3} u(t_i) \\ T\sum_{i=1}^{3} iu(t_i) \end{pmatrix} = \begin{pmatrix} \frac{1}{3}(4u(t_1) + u(t_2) - 2u(t_3)) \\ \frac{1}{2T}(u(t_3) - u(t_1)) \end{pmatrix}.$$

This may be rearranged to give:

$$\text{(32)} \qquad \begin{cases} \hat{u}(t_0) = \frac{1}{3}\sum_{i=1}^{3} (u(t_i) - \hat{u}(t_0)iT) \\ \hat{\dot{u}}(t_0) = \frac{1}{2T}(u(t_3) - u(t_1)) \end{cases}.$$

This result shows that the slope of the straight line, $\dot{u}(t_0)$ (= velocity of particle), is determined from the two outer points $(t_1,\ u(t_1))$ and $(t_3,\ u(t_3))$, and that the intercept of the straight line, $u(t_0)$ (= initial position of particle), equals the average of $u(t_i) - \hat{\dot{u}}(t_0)iT,\ i = 1,\ 2,\ 3$.

So far we have discussed the unweighted least-squares principle. The least-squares principle can be generalized, however, by introducing a positive-definite $m \times m$ weight matrix W. This is done by replacing (24) by the following minimization problem:

$$\text{(33)} \qquad \boxed{\underset{x}{\text{minimize}}\ (y - Ax)^* W (y - Ax)}\ .$$

The solution of (33) can be derived along lines which are similar to the ones used for solving (24). The solution of (33) reads:

$$\text{(34)} \qquad \boxed{\hat{x} = (A^* W A)^{-1} A^* W y}\ .$$

This is the weighted least-squares estimate of x. In case of weighted least-squares the normal equations read: $A^* W A\hat{x} = A^* W y$. This shows that the vector $\hat{e} = y - A\hat{x}$, which is the weighted least-squares estimate of e, satisfies:

$$\text{(35)} \qquad \boxed{A^* W \hat{e} = 0, \text{ with } \hat{e} = y - A\hat{x}}\ .$$

If the inner product of the observation space $\mathbb{R}^m$ is defined as $(a,b) = a^* W b,\ \forall a,b \in \mathbb{R}^m$, (35) can also be written as $(Ax,\hat{e}) = 0,\ \forall x \in \mathbb{R}^n$. This shows that also in the case of weighted least-squares, the vector $\hat{e}$ can be considered to be orthogonal to the range space of A. A summary of the least-squares algorithm is given in Table 1.1.

Inconsistent linear system

$$y = Ax + e,\ y,\ e \in \mathbb{R}^m,\ x \in \mathbb{R}^n,\ m > n = \text{rank } A$$

Weighted least-squares principle

$$\underset{x}{\text{minimize}}\ (y - Ax)^* W (y - Ax),\ W = \text{positive-definite}$$

Weighted least-squares estimates

$$\begin{aligned} &\text{parameter vector} : \hat{x} = (A^* W A)^{-1} A^* W y \\ &\text{observation vector} : \hat{y} = A\hat{x} \\ &\text{error vector} : \hat{e} = y - \hat{y} \end{aligned}$$

Table 1.1: Weighted least-squares.

Example 7

The elements of the weight matrix W can be chosen to emphasize (or de-emphasize) the influence of specific observations upon the estimate $\hat{x}$. In this way different levels of importance may be attached to the different observations. This is of importance if one believes that some observations are more trustworthy than other observations. For instance, some observations may be more trustworthy than others if they are obtained from more accurate measurement instruments. In order to illustrate the influence of the weight matrix, we consider a stationary particle with unkown position $u(t_0)$. We assume that two observations of position are carried out at times t_1 and t_2. The linear system reads then:

$$\begin{pmatrix} u(t_1) \\ u(t_2) \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} u(t_0) + \begin{pmatrix} e_1 \\ e_2 \end{pmatrix}. \tag{36}$$

A diagonal matrix is taken as weight matrix:

$$W = \begin{pmatrix} w_{11} & 0 \\ 0 & w_{22} \end{pmatrix}. \tag{37}$$

Then

$$\hat{u}(t_0) = (A^* W A)^{-1} A^* W y = \frac{\sum_{i=1}^{2} w_{ii} u(t_i)}{\sum_{i=1}^{2} w_{ii}}$$

or

$$\hat{u}(t_0) = \frac{u(t_1)}{1 + \dfrac{w_{22}}{w_{11}}} + \frac{u(t_2)}{1 + \dfrac{w_{11}}{w_{22}}} . \tag{38}$$

This shows that $\hat{u}(t_0)$ equals the average of $u(t_i)$, $i = 1, 2$, if $w_{11} = w_{22}$ (see Figure 1.7a). In this case both observations have the same influence on $\hat{u}(t_0)$. However, if $w_{22} < w_{11}$ then less weight is attached to the second observation and $\hat{u}(t_0)$ is closer to $u(t_1)$, see Figure 1.7b.

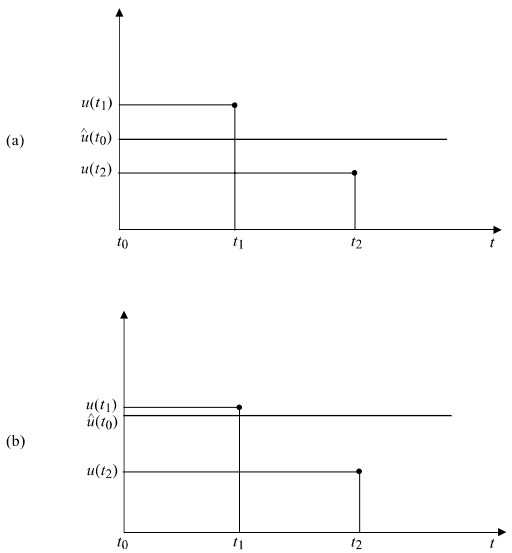


Figure 1.7: Weighted least-squares estimate of $u(t_0)$
(a) $w_{11} = w_{22}$; (b) $w_{11} > w_{22}$.

1.1.3 A stochastic model for the observations

In the previous section the principle of least-squares was introduced. The least-squares principle enables us, in case of inconsistent systems, to obtain an intuitively appealing estimate $\hat{x}$ of the parameter vector x. But although the least-squares estimate $\hat{x}$ is intuitively appealing, no quality measures as yet can be attached to the estimate. That is, we know how to compute the estimate $\hat{x}$, but we are not able yet to say how good the estimate really is. Of course, the numerical value of the sum of squares, $\hat{e}^* W \hat{e}$, does indicate something about the quality of $\hat{x}$. If $\hat{e}^* W \hat{e}$ is small one is inclined to have more confidence in the estimate $\hat{x}$, than if $\hat{e}^* W \hat{e}$ is large. But how small is small? Besides, $\hat{e}^* W \hat{e}$ is identically zero if the linear system is consistent. Would this then automatically imply that the estimate $\hat{x}$ has good quality? Not really, since the observations may still be subject to measurement errors. In order to obtain quality measures for the results of least-squares estimation, we start by introducing a qualitative description of the input, that is of the observations. This description will be of a probabilistic nature. The introduction of a probabilistic description is motivated by the experimental fact that the variability in the outcome of

measurements, when repeated under similar circumstances, can be described to a sufficient degree by stochastic or random variables. We will therefore assume that the observation vector y, which contains the numerical values of the measurements, constitutes a sample of the random vector of observables $\underline{y}$ (note: the underscore indicates that we are dealing with a random variable). It is furthermore assumed that the vector of observables $\underline{y}$ can be written as the sum of a deterministic functional part Ax and a random residual part $\underline{e}$:

$$\boxed{\underline{y} = \boldsymbol{A}x + \underline{\boldsymbol{e}}} \quad . \tag{39}$$

Although a random vector is completely described by its probability density function, we will restrict ourselves for the time being to the first two moments of random variables. That is, we will restrict ourselves to the mean and to the variance matrix. If we assume that $\underline{e}$ models the probabilistic nature of the variability in the measurements, it seems acceptable to assume that this variability is zero *on the average* and therefore that the mean of $\underline{e}$ is zero:

$$E\{\underline{\boldsymbol{e}}\} = \boldsymbol{0} \tag{40}$$

where $E\{.\}$ stands for the mathematical expectation operator. The measurement variability itself is modelled through the dispersion or variance matrix of $\underline{e}$. We will assume that this matrix is known and denote it by Q_y:

$$\boldsymbol{D}\{\underline{\boldsymbol{e}}\} = \boldsymbol{Q}_y \tag{41}$$

where $D\{.\}$ stands for the dispersion operator. It is defined in terms of $E\{.\}$ as $D\{.\} = E\{(.-E\{.\})(.-E\{.\})^*\}$. With (40) and (41) we are now in the position to determine the mean and variance matrix of the vector of observables $\underline{y}$. Application of the law of propagation of means and the law of propagation of variances to (39) gives with (40) and (41):

$$\boxed{E\{\underline{y}\} = \boldsymbol{A}x \quad ; \quad \boldsymbol{D}\{\underline{y}\} = \boldsymbol{Q}_y} \quad . \tag{42}$$

This will be our model of observation equations for the vector of observables $\underline{y}$. As the results of the next section show, model (42) enables us to describe the quality of the results of least-squares estimation in terms of the mean and the variance matrix.

1.1.4 Least-squares estimators

Functions of random variables are again random variables. It follows therefore, that if the vector of observables is assumed to be a random vector $\underline{y}$ and substituted for y in the formulae of Table 1.1 in Section 1.1.2 the results are again random variables:

$$\begin{cases} \underline{\hat{x}} = (\boldsymbol{A}^*\boldsymbol{W}\boldsymbol{A})^{-1}\boldsymbol{A}^*\boldsymbol{W}\underline{y} \\ \underline{\hat{y}} = \boldsymbol{A}\underline{\hat{x}} \\ \underline{\hat{\boldsymbol{e}}} = \underline{y} - \underline{\hat{y}} \end{cases} . \tag{43}$$

These random vectors will be called least-squares estimators. And if $\underline{y}$ is replaced by its sample or measurement value y, we speak of least-squares estimates. The quality of the above estimators can now be deduced from the first two moments of $\underline{y}$.

The first moment: the mean

Together with $E\{\underline{y}\} = Ax$, application of the propagation law of means to (43) gives:

$$\begin{cases} E\{\underline{\hat{x}}\} = x \\ E\{\underline{\hat{y}}\} = E\{\underline{y}\} \\ E\{\underline{\hat{e}}\} = E\{\underline{e}\} = 0 \ . \end{cases} \tag{44}$$

This important result shows that under the assumption that (42) holds, the least-squares estimators are unbiased estimators. Note that this property of unbiasedness is independent of the choice for the weight matrix W.

The second moment: the variance matrix and covariance matrix

Together with $D\{\underline{y}\} = Q_y$, application of the propagation law of variances and covariances to (43) gives:

$$\begin{cases} Q_{\hat{x}} = (A^*WA)^{-1}A^*WQ_yWA(A^*WA)^{-1} \\ Q_{\hat{y}} = AQ_{\hat{x}}A^* \\ Q_{\hat{e}} = [I - A(A^*WA)^{-1}A^*W]Q_y[I - A(A^*WA)^{-1}A^*W]^* \end{cases} \tag{45}$$

and

$$\begin{cases} Q_{\hat{x}\hat{y}} = Q_{\hat{x}}A^* \\ Q_{\hat{x}\hat{e}} = (A^*WA)^{-1}A^*WQ_y - Q_{\hat{x}}A^* \ . \\ Q_{\hat{y}\hat{e}} = AQ_{\hat{x}\hat{e}} \end{cases} \tag{46}$$

The above variance matrices enable us now to give a complete precision description of any arbitrary linear function of the estimators. Consider for instance the linear function $\underline{\hat{\theta}} = a^*\underline{\hat{x}}$. Application of the propagation law of variances gives then for the precision of $\underline{\hat{\theta}}$: $\sigma^2_{\hat{\theta}} = a^*Q_{\hat{x}}a$. The above results enable us to describe the quality of the results of least-squares estimation in terms of the mean and the variance matrix. The introduction of a stochastic model for the vector of observables $\underline{y}$ enables us however also to judge the merits of the least-squares principle itself. Recall that the least-squares principle was introduced on the basis of intuition and not on the basis of probabilistic reasoning. With the mathematical model (42) one could now however try to develop an estimation procedure that produces estimators with certain well-defined probabilistic optimality properties. One such procedure is based on the principle of Best Linear Unbiased Estimation (BLUE), [Teunissen, 2000]. Assume that we are interested in estimating a parameter θ which is a linear function of x:

$$\underset{1\times 1}{\theta} = \underset{1\times nn\times 1}{a^*\ x} \ . \tag{47}$$

The estimator of θ will be denoted as $\underline{\hat{\theta}}$. Then according to the BLUE's criteria, the estimator $\underline{\hat{\theta}}$ of θ has to be a linear function of $\underline{y}$:

(48) $$\underset{1\times 1}{\underline{\hat{\theta}}} = \underset{1\times m}{l^*}\ \underset{m\times 1}{\underline{y}}$$

such that it is unbiased:

(49) $$E\{\underline{\hat{\theta}}\} = \theta$$

and such that it is best in the sense of minimum variance:

(50) $$\sigma_{\hat{\theta}}^2 \rightarrow \text{minimum}\,.$$

The objective is thus to find a vector $l \in \mathbb{R}^m$ such that with (48), the conditions (49) and (50) are satisfied. From Adjustment theory, [Teunissen, 2000] we know that the solution to the above problem is given by:

$$l^* = a^*(A^*Q_y^{-1}A)^{-1}A^*Q_y^{-1}\,.$$

If we substitute this into (48) we get:

(51) $$\underline{\hat{\theta}} = a^*(A^*Q_y^{-1}A)^{-1}A^*Q_y^{-1}\underline{y}\,.$$

This is the best linear unbiased estimator of θ. The important result (51) shows that the best linear unbiased estimator of x is given by:

(52) $$\underline{\hat{x}} = (A^*Q_y^{-1}A)^{-1}A^*Q_y^{-1}\underline{y}\,.$$

A comparison between (43) and (52) shows that the BLUE of x is identical to the weighted least-squares estimator of x if the weight matrix W is taken to be equal to the inverse of the variance matrix of $\underline{y}$:

(53) $$W = Q_y^{-1}\,.$$

This is an important result, because it shows that the weighted least-squares estimators are best in the probabilistic sense of having minimal variance if (53) holds. The variances and covariances of these estimators follow if the weight matrix W is replaced in (45) and (46) by Q_y^{-1}. From now on we will always assume, unless stated otherwise, that the weight matrix W is chosen to be equal to Q_y^{-1}. Consequently no distinction will be made anymore in this book between weighted least-squares estimators and Best Linear Unbiased Estimators. Instead we will simply speak of least-squares estimators.

Example 8

Consider again the situation of a stationary particle. Assume that it is required to determine its position with a variance of $\sigma^2/10$. The position measurements are uncorrelated and all have a variance equal to σ^2. How many position measurements are then needed in order to estimate the particle's position with sufficient precision?

In order to answer this question we first introduce our linear A-model:

$$\underbrace{E\left\{\begin{pmatrix}\underline{u}(t_1)\\ \vdots \\ \underline{u}(t_m)\end{pmatrix}\right\}}_{\underline{y}} = \underbrace{\begin{pmatrix}1\\ \vdots \\ 1\end{pmatrix}}_{A} \underbrace{u(t_0)}_{x} \quad ; \quad Q_y = \sigma^2 I_m \quad .$$

The variance of the least-squares estimators $\underline{\hat{u}}(t_0)$ of $u(t_0)$ reads then:

$$\sigma^2_{\hat{u}(t_0)} = (A^* Q_y^{-1} A)^{-1} = \sigma^2/m \, .$$

This result shows that $m = 10$ position measurements are needed to satisfy the requirements.

Example 9

Consider the situation of Example 2. The position observables are assumed to be uncorrelated and to have the same variance σ^2. An interesting question is now how the times of measurement t_1 and t_2 should be chosen, in order to minimize the variances of the least-squares estimators of initial position $u(t_0)$ and initial velocity $\dot{u}(t_0)$. In order to answer this question we first introduce our linear A-model:

$$\underbrace{E\left\{\begin{pmatrix}\underline{u}(t_1)\\ \underline{u}(t_2)\end{pmatrix}\right\}}_{\underline{y}} = \underbrace{\begin{pmatrix}1 & (t_1 - t_0)\\ 1 & (t_2 - t_0)\end{pmatrix}}_{A} \underbrace{\begin{pmatrix}u(t_0)\\ \dot{u}(t_0)\end{pmatrix}}_{x} ; \quad Q_y = \sigma^2 I_2 \quad .$$

Note that this model consists of two equations in two unknowns. Hence the redundancy m-n equals zero, and matrix A is square and invertible (provided that $t_2 \neq t_1$). In this case the least-squares estimator and its variance matrix simply reduce to:

$$\begin{cases} \underline{\hat{x}} = (A^* Q_y^{-1} A)^{-1} A^* Q_y^{-1} \underline{y} = (A^{-1})(Q_y^{-1})^{-1}(A^*)^{-1} A^* Q_y^{-1} \underline{y} = A^{-1} \underline{y} \\ Q_{\hat{x}} = (A^* Q_y^{-1} A)^{-1} = A^{-1} Q_y A^{*-1} \end{cases} .$$

With

$$A^{-1} = \frac{1}{(t_2 - t_1)} \begin{pmatrix}(t_2 - t_0) & -(t_1 - t_0)\\ -1 & 1\end{pmatrix}$$

this gives for the variance matrix:

(54)
$$Q_{\hat{x}} = A^{-1} Q_y A^{*-1} = \frac{\sigma^2}{(t_2 - t_1)^2} \begin{pmatrix} \sum_{i=1}^{2} (t_i - t_0)^2 & -\sum_{i=1}^{2} (t_i - t_0) \\ -\sum_{i=1}^{2} (t_i - t_0) & 2 \end{pmatrix}.$$

Let us now first consider the variance of the velocity estimator $\underline{\hat{u}}(t_0)$. It reads:

$$\sigma^2_{\hat{u}(t_0)} = 2\sigma^2/(t_2 - t_1)^2 .$$

This result shows that the variance of $\underline{\hat{u}}(t_0)$ gets smaller, i.e. its precision gets better, if the time interval $t_2 - t_1$ gets larger. Thus one can improve the precision of the velocity estimator by increasing the time interval between the two position measurements. This is also quite understandable if one looks at Figure 1.2 in Section 1.1.1. A straight line can be fitted better through two points that are far apart than through two points that are close together. And in fact it becomes impossible to fit the line uniquely if the two points coincide, just like it is impossible to estimate the velocity if $t_2 = t_1$. Let us now consider the variance of the position estimator $\underline{\hat{u}}(t_0)$. It reads (see (54)):

$$\sigma^2_{\hat{u}(t_0)} = \sigma^2 \frac{\sum_{i=1}^{2} (t_i - t_0)^2}{(t_2 - t_1)^2} = \sigma^2 \left[1 + \frac{2(t_1 - t_0)}{(t_2 - t_1)} + \frac{2(t_1 - t_0)^2}{(t_2 - t_1)^2}\right].$$

Also this result shows that the variance of the initial position estimator gets smaller if the time interval $t_2 - t_1$ gets larger. Also note that in case $t_1 \neq t_0$, the variance of $\underline{\hat{u}}(t_0)$ is always larger than the variance of σ^2 of the position observables. The smallest value of $\sigma^2_{\hat{u}(t_0)}$ is obtained for $t_1 = t_0$. This shows that the precision of the initial position estimator is best if the first position measurement is taken at the initial time t_0.

1.1.5 Summary

In Table 1.2 an overview is given of the main characteristics of least-squares estimation.

The linear A-model

$$\underset{m\times 1}{E\{\underline{y}\}} = \underset{m\times n}{\boldsymbol{A}}\ \underset{n\times 1}{x}\ , \quad \underset{m\times m}{D\{\underline{y}\}} = \underset{m\times m}{Q_y}\ , \quad m \geq n = \operatorname{rank} \boldsymbol{A}$$

Least-squares estimators

$$\hat{\underline{x}} = (\boldsymbol{A}^* Q_y^{-1} \boldsymbol{A})^{-1} \boldsymbol{A}^* Q_y^{-1} \underline{y} \qquad \hat{\underline{e}} = \underline{y} - \hat{\underline{y}}$$

$$\hat{\underline{y}} = \boldsymbol{A}\hat{\underline{x}} \qquad \hat{\underline{\theta}} = \boldsymbol{a}^* \hat{\underline{x}}$$

Mean

$$E\{\hat{\underline{x}}\} = x \qquad E\{\hat{\underline{e}}\} = E\{\underline{e}\} = 0$$

$$E\{\hat{\underline{y}}\} = E\{\underline{y}\} = \boldsymbol{A}x \qquad E\{\hat{\underline{\theta}}\} = \theta = \boldsymbol{a}^* x$$

Variances and covariances

$$Q_{\hat{x}} = (\boldsymbol{A}^* Q_y^{-1} \boldsymbol{A})^{-1} \qquad Q_{\hat{e}} = Q_y - Q_{\hat{y}}$$

$$Q_{\hat{y}} = \boldsymbol{A} Q_{\hat{x}} \boldsymbol{A}^* \qquad \sigma^2_{\hat{\theta}} = \boldsymbol{a}^* Q_{\hat{x}} \boldsymbol{a}$$

$$Q_{\hat{x}\hat{y}} = Q_{\hat{x}} \boldsymbol{A}^*, \quad Q_{\hat{x}\hat{e}} = 0, \quad Q_{\hat{y}\hat{e}} = 0$$

Table 1.2: Least-squares estimation.

1.2 The nonlinear A-model

1.2.1 Nonlinear observation equations

Up to this point the development of our estimation theory was based on the assumption that the m-vector $E\{\underline{y}\}$ is linearly related to the n-vector of unknown parameters x. In geodetic applications there are however only a few cases where this assumption truly holds. A typical example is levelling. In the majority of applications, however, the m-vector $E\{\underline{y}\}$ is nonlinearly related to the n-vector of unknown parameters x. This implies that instead of the linear A-model (42), we are generally dealing with a nonlinear model of observation equations:

(55) $$\boxed{E\{\underline{y}\} = \boldsymbol{A}(x) \quad ; \quad \boldsymbol{D}\{\underline{y}\} = \boldsymbol{Q}_y}$$

where $A(.)$ is a nonlinear vector function from $\mathbb{R}^u$ into $\mathbb{R}^n$. The following two simple examples will make this clear.

Example 10

Consider the configuration of Figure 1.8a. The x, y coordinates of the three points 1, 2 and 3 are known and the coordinates x_4 and y_4 of point 4 are unknown. The observables consist of the three azimuth variates $\underline{\alpha}_{14}$, $\underline{\alpha}_{24}$ and $\underline{\alpha}_{34}$. Since azimuth and coordinates are related as (see Figure 1.8b):

$$\tan \alpha_{ij} = \frac{x_{ij}}{y_{ij}}$$

the model of observation equations for the configuration of Figure 1.8a reads:

$$E\left\{\begin{pmatrix} \underline{\alpha}_{14} \\ \underline{\alpha}_{24} \\ \underline{\alpha}_{34} \end{pmatrix}\right\} = \begin{pmatrix} \arctan[x_{14}/y_{14}] \\ \arctan[x_{24}/y_{24}] \\ \arctan[x_{34}/y_{34}] \end{pmatrix}.$$

This model consists of three nonlinear observation equations in the two unknown parameters x_4 and y_4.

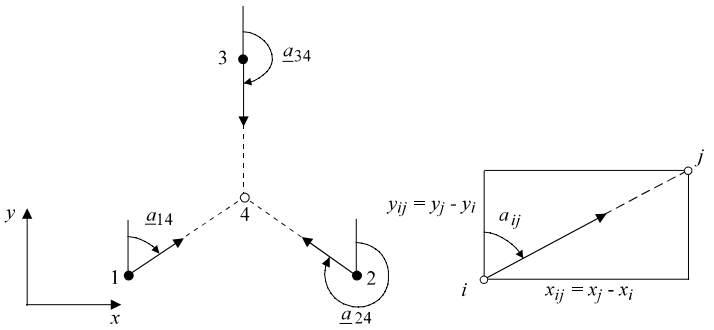


Figure 1.8: Azimuth resection.

Example 11

Consider the situation of Figure 1.9. It shows two cartesian coordinate systems: the x,y-system and the u,v-system. The two systems only differ in their orientation. This means that if the coordinates of a point i are given in the u,v-system, (u_i, v_i), a rotation by an angle α is needed to obtain the coordinates of the same point i in the x,y-system, (x_i, y_i):

(56)
$$\begin{pmatrix} x_i \\ y_i \end{pmatrix} = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} u_i \\ v_i \end{pmatrix}.$$

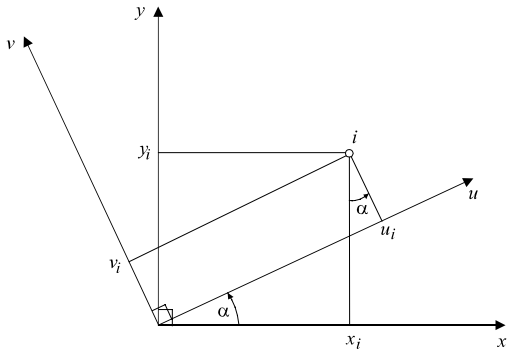


Figure 1.9: A coordinate transformation.

Let us now assume that we have at our disposal the coordinate observables of two points in both coordinate systems: $(\underline{x}_i, \underline{y}_i)$ and $(\underline{u}_i, \underline{v}_i)$, $i = 1, 2$. Using (56), our model reads then:

(57)
$$E\left\{\begin{pmatrix} \underline{x}_i \\ \underline{y}_i \end{pmatrix}\right\} = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} E\left\{\begin{pmatrix} \underline{u}_i \\ \underline{v}_i \end{pmatrix}\right\}, \quad i = 1, 2.$$

This model is however still not in the form of observation equations. If we consider the orientation angle α and the coordinates of the two points in the u,v-system as the unknown parameters, (57) can be written in terms of observation equations as:

(58)
$$E\left\{\begin{pmatrix} \underline{x}_1 \\ \underline{y}_1 \\ \underline{x}_2 \\ \underline{y}_2 \\ \underline{u}_1 \\ \underline{v}_1 \\ \underline{u}_2 \\ \underline{v}_2 \end{pmatrix}\right\} = \begin{pmatrix} u_1\cos\alpha - v_1\sin\alpha \\ u_1\sin\alpha + v_1\cos\alpha \\ u_2\cos\alpha - v_2\sin\alpha \\ u_2\sin\alpha + v_2\cos\alpha \\ u_1 \\ v_1 \\ u_2 \\ v_2 \end{pmatrix}.$$

This model consists of eight observations in five unknown parameters. Note that the first four observation equations are nonlinear.

1.2.2 The linearized A-model

We know how to compute least-squares estimators in case of a linear A-model. But how are we now going to compute least-squares estimators if the model of observation equations is non-linear? For the majority of nonlinear problems the solution is to approximate the originally nonlinear A-model with a linear one. In order to show how this can be done, we first recall the theorem of Taylor.

Taylor's Theorem

Let $f(x)$ be a function from $\mathbb{R}^n$ into $\mathbb{R}$. Let $x_0 \in \mathbb{R}^n$ be an approximation to $x \in \mathbb{R}^n$ and define $\Delta x = x - x^0$, and $\theta = x^0 + t(x - x^0)$ with $t \in \mathbb{R}$. Then a scalar $t \in (0,1)$ exists such that:

$$\begin{aligned} f(x) &= f(x^0) + \sum_{\alpha=1}^{n} \partial_\alpha f(x^0)\Delta x_\alpha + \frac{1}{2}\sum_{\alpha=1}^{n}\sum_{\beta=1}^{n} \partial^2_{\alpha\beta} f(x^0)\Delta x_\alpha \Delta x_\beta + \cdots \\ &\cdots + \frac{1}{(q-1)!}\sum_{\alpha_1=1}^{n}\cdots\sum_{\alpha_{q-1}=1}^{n} \partial^{q-1}_{\alpha_1\cdots\alpha_{q-1}} f(x^0)\Delta x_{\alpha_1}\cdots\Delta x_{\alpha_{q-1}} + \boldsymbol{R}_q(\theta,\Delta x) \end{aligned} \tag{59}$$

with the remainder:

$$\boldsymbol{R}_q(\theta,\ \Delta x) = \frac{1}{q!}\sum_{\alpha_1=1}^{n}\cdots\sum_{\alpha_q=1}^{n} \partial^q_{\alpha_1\cdots\alpha_q} f(\theta)\Delta x_{\alpha_1}\cdots\Delta x_{\alpha_q}. \tag{60}$$

In (59) and (60), $\partial^q_{\alpha_1\ldots\alpha_q} f(x)$ denotes the qth-order partial derivative of $f(x)$ evaluated at x. For the case $q = 2$, it follows from (59) and (60) that:

$$f(x) = f(x^0) + \sum_{\alpha=1}^{n} \partial_\alpha f(x^0)\Delta x_\alpha + \frac{1}{2}\sum_{\alpha=1}^{n}\sum_{\beta=1}^{n} \partial^2_{\alpha\beta} f(\theta)\Delta x_\alpha \Delta x_\beta. \tag{61}$$

If we introduce the gradient vector and Hessian matrix of $f(x)$ respectively as:

$$\partial_x f(x) = \begin{pmatrix} \partial_1 f(x) \\ \vdots \\ \partial_n f(x) \end{pmatrix} \quad \text{and} \quad \partial^2_{xx} f(x) = \begin{pmatrix} \partial^2_{11} f(x) & \cdots & \partial^2_{1n} f(x) \\ \vdots & & \vdots \\ \partial^2_{n1} f(x) & \cdots & \partial^2_{nn} f(x) \end{pmatrix}$$

then equation (61) may be written in the more compact matrix-vector form as:

$$\boxed{f(x) = f(x^0) + \partial_x f(x^0)^* \Delta x + \frac{1}{2}\Delta x^* \partial^2_{xx} f(\theta)\Delta x}. \tag{62}$$

This important result shows that a nonlinear function $f(x)$ can be written as a sum of three terms. The first term in this sum is the zero-order term $f(x^0)$. The zero-order term depends on x^0 but is independent of x. The second term in the sum is the first-order term $\partial_x f(x^0)^* \Delta x$. It depends on x^0 and is linearly dependent on x. Finally, the third term in the sum is the second-order remainder $R_2(\theta,\Delta x)$. An important consequence of Taylor's theorem is that the remainder $R_2(\theta,\Delta x)$ can be made arbitrarily small by choosing the approximation x^0 close enough to x. Now assume that

the approximation x^0 is chosen such that the second-order remainder can indeed be neglected. Then, instead of (62) we may write to a sufficient degree of approximation:

$$\boxed{f(x) = f(x^0) + \partial_x f(x^0)^* \Delta x} \quad . \tag{63}$$

Hence, if x^0 is sufficiently close to x, the nonlinear function $f(x)$ can be approximated to a sufficient degree by the function $f(x^0) + \partial_x f(x^0)^* \Delta x$ which is linear in x. This function is the linearized version of $f(x)$. A geometric interpretation of this linearization is given in Figure 1.10 for the case $n = 1$.

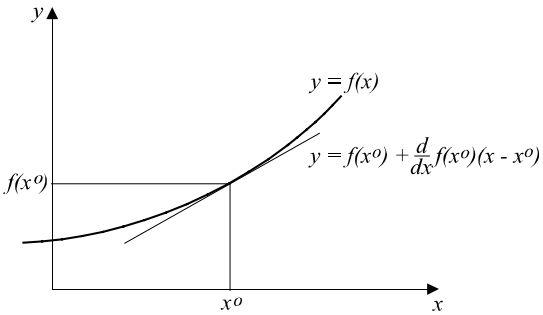


Figure 1.10: The nonlinear curve $y = f(x)$ and its linear tangent $y = f(x^0) + \frac{d}{dx} f(x^0)(x - x^0)$.

Let us now apply the above linearization to our nonlinear observation equations:

$$E\{\underline{y}\} = \boldsymbol{A}(x) = \begin{pmatrix} a_1(x) \\ \vdots \\ a_m(x) \end{pmatrix} . \tag{64}$$

Each nonlinear observation equation $a_i(x)$, $i = 1,\cdots,m$, can now be linearized according to (63). This gives:

$$\underset{m\times 1}{\begin{pmatrix} a_1(x) \\ \vdots \\ a_m(x) \end{pmatrix}} = \underset{m\times 1}{\begin{pmatrix} a_1(x^0) \\ \vdots \\ a_m(x^0) \end{pmatrix}} + \underset{m\times n}{\begin{pmatrix} \partial_x a_1(x^0)^* \\ \vdots \\ \partial_x a_m(x^0)^* \end{pmatrix}} \underset{n\times 1}{\Delta x} . \tag{65}$$

If we denote the $m\times n$ matrix of (65) as $\partial_x A(x^0)$, and substitute (65) into (64) we get:

$$E\{\underline{y}\} = \boldsymbol{A}(x^0) + \partial_x \boldsymbol{A}(x^0)\Delta x .$$

If we bring the constant m-vector $A(x^0)$ to the left-hand side of the equation and define $\Delta\underline{y} = \underline{y} - A(x^0)$, we finally obtain our linearized model of observation equations:

$$\boxed{E\{\Delta\underline{y}\} = \partial_x \boldsymbol{A}(x^0)\Delta x \quad ; \quad \boldsymbol{D}\{\Delta\underline{y}\} = \boldsymbol{Q}_y} \quad . \tag{66}$$

This is the linearized A-model. Compare (66) with (55) and (42). Note when comparing (66) with (42) that in the linearized A-model $\Delta\underline{y}$ takes the place of $\underline{y}$, $\partial_x A(x^0)$ takes the place of A

and Δx takes the place of x. Since the linearized A-model is linear in $\Delta x = x - x^0$ our standard formulae of least-squares can be applied again. This gives for the least-squares estimator $\underline{\hat{x}} = x^0 + \underline{\Delta\hat{x}}$ of x:

(67) $$\underline{\hat{x}} = x^0 + [\partial_x A(x^0)^* Q_y^{-1} \partial_x A(x^0)]^{-1} \partial_x A(x^0)^* Q_y^{-1} \underline{\Delta y} .$$

Application of the propagation law of variances to (67) gives:

(68) $$Q_{\hat{x}} = [\partial_x A(x^0)^* Q_y^{-1} \partial_x A(x^0)]^{-1} .$$

It will be clear that the above results, (67) and (68), are approximate in the sense that the second-order remainder is neglected. But these approximations are good enough if the second-order remainder can be neglected to a sufficient degree. In this case also the optimality conditions of least-squares (unbiasedness, minimal variance) hold to a sufficient degree. A summary of the linearized least-squares estimators is given in Table 1.3.

The nonlinear A-model

$$E\{\underline{y}\} = A(x) \quad ; \quad D\{\underline{y}\} = Q_y \quad ; \quad A(.) : \mathbb{R}^n \to \mathbb{R}^m$$

The linearized A-model

$$\underset{m\times 1}{E\{\underline{\Delta y}\}} = \underset{m\times n}{\partial_x A}\,(x^0)\underset{n\times 1}{\Delta x} \quad ; \quad \underset{m\times m}{D\{\underline{\Delta y}\}} = \underset{m\times m}{Q_y} \quad ; \quad m \geq n = \text{rank}\ \partial_x A(x^0)$$

Least-squares estimators

$$\begin{aligned} \underline{\hat{x}} &= x^0 + [\partial_x A(x^0)^* Q_y^{-1} \partial_x A(x^0)]^{-1} \partial_x A(x^0)^* Q_y^{-1} (\underline{y} - A(x^0)) \\ \underline{\hat{y}} &= A(\underline{\hat{x}}) \\ \underline{\hat{e}} &= \underline{y} - \underline{\hat{y}} \end{aligned}$$

Variances

$$\begin{aligned} Q_{\hat{x}} &= [\partial_x A(x^0)^* Q_y^{-1} \partial_x A(x^0)]^{-1} \\ Q_{\hat{y}} &= \partial_x A(x^0) Q_{\hat{x}} \partial_x A(x^0)^* \\ Q_{\hat{e}} &= Q_y - Q_{\hat{y}} \end{aligned}$$

Table 1.3: Linearized least-squares estimation.

Example 12

Consider the configuration of Figure 1.11a. The x, y coordinates of the three points 1, 2 and 3 are known and the two coordinates x_4 and y_4 of point 4 are unknown. The observables consist of the three distance variates $\underline{l}_{14}$, $\underline{l}_{24}$ and $\underline{l}_{34}$. Since distance and coordinates are related as (see Figure 1.11b):

$$l_{ij} = (x_{ij}^2 + y_{ij}^2)^{\frac{1}{2}}$$

the model of observation equations for the configuration of Figure 1.11a reads:

$$E\left\{\begin{pmatrix}\underline{l}_{14}\\ \underline{l}_{24}\\ \underline{l}_{34}\end{pmatrix}\right\} = \begin{pmatrix}(x_{14}^2 + y_{14}^2)^{\frac{1}{2}}\\ (x_{24}^2 + y_{24}^2)^{\frac{1}{2}}\\ (x_{34}^2 + y_{34}^2)^{\frac{1}{2}}\end{pmatrix}. \tag{69}$$

This model consists of three nonlinear observation equations in the two unknown parameters x_4 and y_4.

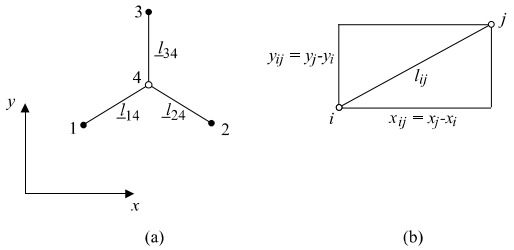


Figure 1.11: Distance resection.

In order to linearize (69) we need approximate values for the unknown coordinates x_4 and y_4. These approximate values will be denoted as x_4^0 and y_4^0. With these approximate values a linearization of (69) gives:

$$\underbrace{E\left\{\begin{pmatrix}\Delta\underline{l}_{14}\\ \Delta\underline{l}_{24}\\ \Delta\underline{l}_{34}\end{pmatrix}\right\}}_{\Delta\underline{y}} = \underbrace{\begin{pmatrix}(x_4^0 - x_1)/l_{14}^0 & (y_4^0 - y_1)/l_{14}^0\\ (x_4^0 - x_2)/l_{24}^0 & (y_4^0 - y_2)/l_{24}^0\\ (x_4^0 - x_3)/l_{34}^0 & (y_4^0 - y_3)/l_{34}^0\end{pmatrix}}_{\partial_x A(x_0)} \underbrace{\begin{pmatrix}\Delta x_4\\ \Delta y_4\end{pmatrix}}_{\Delta x} \tag{70}$$

where:

$$\begin{cases} \Delta l_{i4} = \underline{l}_{i4} - l_{i4}^0, \quad l_{i4}^0 = [(x_4^0 - x_i)^2 + (y_4^0 - y_i)^2]^{\frac{1}{2}}, \quad i = 1, 2, 3 \\ \Delta x_4 = x_4 - x_4^0, \quad \Delta y_4 = y_4 - y_4^0 \end{cases}.$$

Model (70) is the linearized version of the nonlinear A-model (69).

Example 13

Consider the nonlinear A-model (58) of Example 11. The unknown parameters are α and u_i, v_i for $i = 1,2$. The approximate values of these parameters will be denoted as α^0 and u_i^0, v_i^0 for $i = 1,2$. Linearization of (58) gives then:

(71)
$$E\{\underbrace{\begin{pmatrix} \Delta\underline{x}_1 \\ \Delta\underline{y}_1 \\ \Delta\underline{x}_2 \\ \Delta\underline{y}_2 \\ \Delta\underline{u}_1 \\ \Delta\underline{y}_1 \\ \Delta\underline{u}_2 \\ \Delta\underline{y}_2 \end{pmatrix}}_{\Delta\underline{y}}\} = \underbrace{\begin{pmatrix} -u_1^0\sin\alpha^0 - v_1^0\cos\alpha^0 & \cos\alpha^0 & -\sin\alpha^0 & 0 & 0 \\ u_1^0\cos\alpha^0 - v_1^0\sin\alpha^0 & \sin\alpha^0 & \cos\alpha^0 & 0 & 0 \\ -u_2^0\sin\alpha^0 - v_2^0\cos\alpha^0 & 0 & 0 & \cos\alpha^0 & -\sin\alpha^0 \\ u_2^0\cos\alpha^0 - v_2^0\sin\alpha^0 & 0 & 0 & \sin\alpha^0 & \cos\alpha^0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}}_{\partial_x A(x^0)} \underbrace{\begin{pmatrix} \Delta\alpha \\ \Delta u_1 \\ \Delta v_1 \\ \Delta u_2 \\ \Delta v_2 \end{pmatrix}}_{\Delta x}$$

where:

$$\begin{cases} \Delta\underline{x}_i = \underline{x}_i - x_i^0, \quad x_i^0 = u_i^0\cos\alpha^0 - v_i^0\sin\alpha^0 \\ \Delta\underline{y}_i = \underline{y}_i - y_i^0, \quad y_i^0 = u_i^0\sin\alpha^0 + v_i^0\cos\alpha^0 \\ \Delta\underline{u}_i = \underline{u}_i - u_i^0, \quad \Delta u_i = u_i - u_i^0 \\ \Delta\underline{v}_i = \underline{v}_i - v_i^0, \quad \Delta v_i = v_i - v_i^0, \quad \text{for } i = 1,2 \\ \Delta\alpha = \alpha - \alpha^0 \end{cases}.$$

Example 14

Consider the situation of Figure 1.12. A satellite orbiting the earth is assumed to have a circular orbit with unknown radius *R*. Distance measurements from two known points 1 and 2 on the

earth surface are carried out to the two satellite positions 3 and 4. It is assumed that the earth is a non-rotating body.

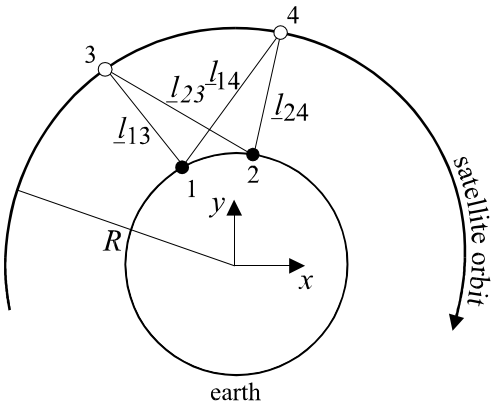


Figure 1.12: Distance measurement to a satellite orbiting the earth.

The distance l_{ij} between two points *i* and *j* can be parameterized in terms of cartesian coordinates as:

(72)
$$l_{ij} = (x_{ij}^2 + y_{ij}^2)^{\frac{1}{2}} .$$

The circular satellite orbit itself can be parameterized in terms of polar coordinates as:

(73)
$$\begin{cases} x_j = R\cos\phi_j \\ y_j = R\sin\phi_j \end{cases} .$$

With (72) and (73) the nonlinear model of observation equations becomes:

(74)
$$E\left\{\begin{pmatrix} l_{13} \\ l_{14} \\ l_{23} \\ l_{24} \end{pmatrix}\right\} = \begin{pmatrix} [(R\cos\phi_3 - x_1)^2 + (R\sin\phi_3 - y_1)^2]^{\frac{1}{2}} \\ [(R\cos\phi_4 - x_1)^2 + (R\sin\phi_4 - y_1)^2]^{\frac{1}{2}} \\ [(R\cos\phi_3 - x_2)^2 + (R\sin\phi_3 - y_2)^2]^{\frac{1}{2}} \\ [(R\cos\phi_4 - x_2)^2 + (R\sin\phi_4 - y_2)^2]^{\frac{1}{2}} \end{pmatrix} .$$

The unknowns in these observation equations are besides the orbital radius *R* also the coordinates ϕ_3 and ϕ_4. The approximate values of these parameters are denoted as R^0, ϕ_3^0 and ϕ_4^0. Linearization of (74) gives then:

$$\underbrace{E\begin{Bmatrix}\Delta \underline{l}_{13}\\ \Delta \underline{l}_{14}\\ \Delta \underline{l}_{23}\\ \Delta \underline{l}_{24}\end{Bmatrix}}_{\Delta \underline{y}} = \underbrace{\begin{pmatrix}\dfrac{R^0 - x_1\cos\phi_3^0 - y_1\sin\phi_3^0}{l_{13}^0} & \dfrac{R^0(x_1\sin\phi_3^0 - y_1\cos\phi_3^0)}{l_{13}^0} & 0\\ \dfrac{R^0 - x_1\cos\phi_4^0 - y_1\sin\phi_4^0}{l_{14}^0} & 0 & \dfrac{R^0(x_1\sin\phi_4^0 - y_1\cos\phi_4^0)}{l_{14}^0}\\ \dfrac{R^0 - x_2\cos\phi_3^0 - y_2\sin\phi_3^0}{l_{23}^0} & \dfrac{R^0(x_2\sin\phi_3^0 - y_2\cos\phi_3^0)}{l_{23}^0} & 0\\ \dfrac{R^0 - x_2\cos\phi_4^0 - y_2\sin\phi_4^0}{l_{24}^0} & 0 & \dfrac{R^0(x_2\sin\phi_4^0 - y_2\cos\phi_4^0)}{l_{24}^0}\end{pmatrix}}_{\partial_x A(x^0)} \underbrace{\begin{pmatrix}\Delta R\\ \Delta\phi_3\\ \Delta\phi_4\end{pmatrix}}_{\Delta x}.$$

(75)

1.2.3 Least-squares iteration

Up to this point it was assumed that the second-order remainder was sufficiently small and that x_0 was a good enough approximation for x. If this is not the case, then $\hat{x}$ as computed by (67) is not the least-squares estimate and hence an unacceptable error is made. In order to repair this situation, we need to improve upon the approximation x^0. It seems reasonable to expect that the estimate:

$$x^1 = x^0 + [\partial_x A(x^0)^* Q_y^{-1} \partial_x A(x^0)]^{-1} \partial_x A(x^0)^* Q_y^{-1}(y - A(x^0))$$

is a better approximation than x^0. That is, it seems reasonable to expect that x^1 is closer to the true least-squares estimate than x^0. In fact one can show that this is indeed the case for most practical applications. But if x^1 is a better approximation than x^0, a further improvement can be expected if we replace x^0 by x^1 in the linearization of the nonlinear model. The recomputed linearized least-squares estimate reads then:

$$x^2 = x^1 + [\partial_x A(x^1)^* Q_y^{-1} \partial_x A(x^1)]^{-1} \partial_x A(x^1)^* Q_y^{-1}(y - A(x^1)) .$$

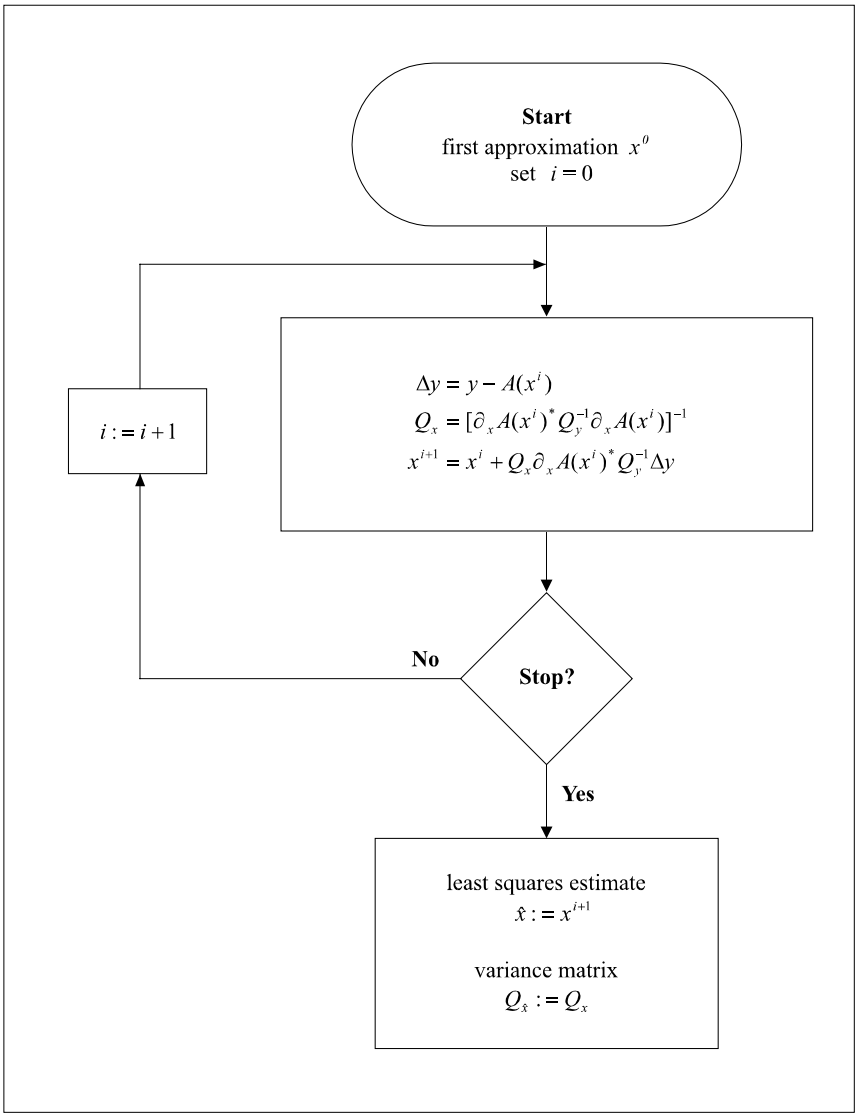


Table 1.4: Least-squares iteration.

By repeating this process a number of times, one can expect that finally the solution converges to the actual least-squares estimate $\hat{x}$. This is called the least-squares iteration process. The iteration is usually terminated if the difference between successive solutions is negligible. A flow diagram of the least-squares iteration process is shown in Table 1.4. For more details on the numerical properties of the iteration process and on the probabilistic properties of nonlinear least-squares estimators the reader is referred to the theory as developed in [Teunissen, 1985b].

1.3 The B-model

1.3.1 The linear B-model

In the previous sections we considered the model of observation equations. In this section and the next we briefly review the model of condition equations. For more details the reader is again referred to Adjustment theory, [Teunissen, 2000]. As our starting point we take the linear A-model:

$$E\{\underset{m\times 1}{\underline{y}}\} = \underset{m\times n}{\boldsymbol{A}}\ \underset{n\times 1}{x} \quad ; \quad D\{\underset{m\times m}{\underline{y}}\} = \underset{m\times m}{Q_y} \quad ; \quad m \geq n = \text{rank } \boldsymbol{A}\,. \tag{76}$$

This linear model is uniquely solvable if $m = n$, i.e. if the number of observables equals the number of unknown parameters. In this case A is a square matrix which is invertible because of rank $A = n$. If $m = n$, the redundancy equals zero, and no conditions can be imposed on the observables. If $m > n = \text{rank } A$, then more observables are available than strictly needed for the determination of the n unknown parameters. In this case an $(m-n)$-number of redundant observables exist. Each separate redundant observable gives rise to the possibility of formulating a condition equation. Thus the total number of independent condition equations that can be formulated equals:

$$\boxed{b = m - n}\,. \tag{77}$$

We will now show how one can construct the condition equations, given the linear A-model (76). Each of the column vectors of matrix A is an element of the observation space $\mathbb{R}^m$. Together the n-number of linearly independent column vectors of A span the range space of A. This range space has dimension n and it is a linear subspace of $\mathbb{R}^m$: $R(A) \subset \mathbb{R}^m$. Since $\dim R(A) = n$ and $\dim \mathbb{R}^m = m$, exactly $(m-n)$-number of linearly independent vectors can be found that are orthogonal to $R(A)$. Let us denote these vectors as: $b_i \in \mathbb{R}^m$, $i = 1,...,(m-n)$. Then:

$$b_i \perp R(A) \text{ or } A^* b_i = 0,\ i = 1,...,(m-n)\,.$$

From this it follows, if the $(m-n)$-number of linearly independent vectors b_i are collected in an $m\times(m-n)$ matrix B as:

$$\underset{m\times(m-n)}{\boldsymbol{B}} = (b_1,\ b_2,\ ...,\ b_{m-n})$$

that

$$\boxed{\underset{(m-n)\times m}{\boldsymbol{B}^*}\ \underset{m\times n}{\boldsymbol{A}} = \underset{(m-n)\times n}{0} \quad ; \quad \text{rank } \boldsymbol{B} = m - n}\,. \tag{78}$$

This result may now be used to obtain the model of condition equations from (76). Premultiplication of the linear system of observation equations in (76) by B^* gives together with (78) the following linear model of condition equations:

$$\boxed{\underset{b\times m}{\boldsymbol{B}^*}\ \underset{m\times 1}{E\{\underline{y}\}} = \underset{b\times 1}{0} \quad ; \quad D\{\underset{m\times m}{\underline{y}}\} = \underset{m\times m}{Q_y} \quad ; \quad \text{rank } \boldsymbol{B} = b = m - n}\,. \tag{79}$$

Example 15

Consider the following linear A-model:

$$E\{\begin{pmatrix} \underline{y}_1 \\ \underline{y}_2 \\ \underline{y}_3 \end{pmatrix}\} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} x \quad ; \quad D\{\underline{y}\} = Q_y \,. \tag{80}$$

Since $m = 3$, $n = 1$ and rank $A = 1 = n$, the redundancy equals $m-n = 2$. Hence two linearly independent condition equations can be formulated. The two vectors:

$$b_1 = (1, -1, 0)^* \text{ and } b_2 = (0, 1, -1)^*$$

are linearly independent and are both orthogonal to the single column vector of matrix A in (80). Hence the with (80) corresponding linear model of condition equations reads:

$$\underbrace{\begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \end{pmatrix}}_{B^*} \underbrace{E\{\begin{pmatrix} \underline{y}_1 \\ \underline{y}_2 \\ \underline{y}_3 \end{pmatrix}\}}_{\underline{y}} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad , \quad D\{\underline{y}\} = Q_y \,. \tag{81}$$

Example 16

Consider the linear A-model of Example 6 in Section 1.1.2:

$$\underbrace{E\{\begin{pmatrix} \underline{u}(t_1) \\ \underline{u}(t_2) \\ \underline{u}(t_3) \end{pmatrix}\}}_{\underline{y}} = \underbrace{\begin{pmatrix} 1 & T \\ 1 & 2T \\ 1 & 3T \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} u(t_0) \\ \dot{u}(t_0) \end{pmatrix}}_{x} \,. \tag{82}$$

Since $m = 3$, $n = 2$ and rank $A = 2 = n$, the redundancy equals $m-n = 1$. Hence, only one condition equations can be formulated. The with (82) corresponding linear B-model reads:

$$(83) \qquad \underbrace{(1 \quad -2 \quad 1)}_{\boldsymbol{B}^*} \quad E\{\underbrace{\begin{pmatrix} \underline{u}(t_1) \\ \underline{u}(t_2) \\ \underline{u}(t_3) \end{pmatrix}}_{\underline{y}}\} = 0 \quad .$$

Verify that $B^*A = 0$ holds.

Now that we have the linear B-model (79) at our disposal, how are we going to compute the corresponding least-squares estimators? We know how to compute the least-squares estimators for the linear A-model. The corresponding formulae are however all expressed in terms of the *A*-matrix. What is needed therefore is to transform these formulae such that they are expressed in terms of the *B*-matrix. This is possible with the following important matrix identity:

$$(84) \qquad \boxed{\boldsymbol{A}(\boldsymbol{A}^*\boldsymbol{Q}_y^{-1}\boldsymbol{A})^{-1}\boldsymbol{A}^*\boldsymbol{Q}_y^{-1} = \boldsymbol{I} - \boldsymbol{Q}_y\boldsymbol{B}(\boldsymbol{B}^*\boldsymbol{Q}_y\boldsymbol{B})^{-1}\boldsymbol{B}^*} \quad .$$

The proof of this matrix identity is as follows. We define two matrices *C* and $\overline{C}$ as:

$$(85) \qquad C \triangleq (\boldsymbol{A} \vdots \boldsymbol{Q}_y\boldsymbol{B}) \text{ and } \overline{C} \triangleq \begin{pmatrix} (\boldsymbol{A}^*\boldsymbol{Q}_y^{-1}\boldsymbol{A})^{-1}\boldsymbol{A}^*\boldsymbol{Q}_y^{-1} \\ \hline (\boldsymbol{B}^*\boldsymbol{Q}_y\boldsymbol{B})^{-1}\boldsymbol{B}^* \end{pmatrix} .$$

Since both matrices *C* and $\overline{C}$ are of dimension $m \times m$ and since both can be shown to be of full rank, it follows that they are invertible. From (85) it follows with the help of (78) that $\overline{C}C = I_m$. Hence $\overline{C} = C^{-1}$ and therefore $C\overline{C} = I_m$. Substitution of (85) into this last expression proves (84). With (84) and the least-squares results of Table 1.2 of Section 1.1.5 we are now in the position to derive the expressions for the least-squares estimators in terms of the matrix *B*. The results are summarized in Table 1.5.

The linear B-model

$$\underset{b\times m}{\boldsymbol{B}^{*}}\,\underset{m\times 1}{E\{\underline{y}\}} = \underset{b\times 1}{0} \quad ; \quad \underset{m\times m}{D\{\underline{y}\}} = \underset{m\times m}{Q_y} \quad ; \quad m \geq \operatorname{rank}\boldsymbol{B} = b$$

Least-squares estimators

$$\underline{\hat{y}} = [I - Q_y\boldsymbol{B}(\boldsymbol{B}^{*}Q_y\boldsymbol{B})^{-1}\boldsymbol{B}^{*}]\underline{y} \quad ; \quad \underline{\hat{e}} = \underline{y} - \underline{\hat{y}}$$

Variances and covariances

$$Q_{\hat{y}} = Q_y - Q_y\boldsymbol{B}(\boldsymbol{B}^{*}Q_y\boldsymbol{B})^{-1}\boldsymbol{B}^{*}Q_y \quad ; \quad Q_{\hat{e}} = Q_y - Q_{\hat{y}}$$
$$Q_{\hat{y}\hat{e}} = 0$$

Table 1.5: Least-squares estimation.

1.3.2 The nonlinear B-model

Just as in the case of the A-model, there are very few geodetic applications for which the model of condition equations is linear. In most cases the model of condition equations is nonlinear. The nonlinear B-model reads:

$$\boldsymbol{B}^{*}(E\{\underline{y}\}) = 0 \quad ; \quad D\{\underline{y}\} = Q_y \,. \tag{86}$$

Where $B^{*}(.)$ is a nonlinear vector function from $\mathbb{R}^n$ into $\mathbb{R}^{n-n}$. The relationship between the nonlinear B-model and the nonlinear A-model is given by:

$$\boldsymbol{B}^{*}(\boldsymbol{A}(x)) = 0 \quad ; \quad \forall x \in \mathbb{R}^n \,. \tag{87}$$

This is the nonlinear generalization of (78). If we take the partial derivative with respect to x of (87) and apply the chain rule, we get:

$$[\partial_y\boldsymbol{B}(y^0)]^{*}[\partial_x\boldsymbol{A}(x^0)] = 0 \quad ; \quad y^0 = \boldsymbol{A}(x^0) \,. \tag{88}$$

This is the linearized version of (87). Compare (88) with (78). With (88) we are now in the position to construct the linearized B-model from the linearized A-model (66). Premultiplication of (66) with the matrix $[\partial_y B(y^0)]^{*}$ gives together with (88) the result:

(89) $$\boxed{[\partial_y \boldsymbol{B}(y^0)]^* E\{\Delta \underline{y}\} = 0 \quad ; \quad \boldsymbol{D}\{\Delta \underline{y}\} = \boldsymbol{Q}_y} \; .$$

This is the linearized B-model. With (89) we are now in the position again to apply our standard least-squares estimation formulae.

Example 17

The cartesian coordinates of three points 1, 2 and 3 are measured. The observables are therefore: $\underline{x}_1$, $\underline{y}_1$, $\underline{x}_2$, $\underline{y}_2$, $\underline{x}_3$ and $\underline{y}_3$. The three points are assumed to lie on a circle with unknown radius R, see Figure 1.13. Since the circle can be parameterized as:

$$\begin{cases} x = R\cos\phi \\ y = R\sin\phi \end{cases}$$

the nonlinear A-model reads:

(90) $$E\left\{\begin{pmatrix} \underline{x}_1 \\ \underline{y}_1 \\ \underline{x}_2 \\ \underline{y}_2 \\ \underline{x}_3 \\ \underline{y}_3 \end{pmatrix}\right\} = \begin{pmatrix} R\cos\phi_1 \\ R\sin\phi_1 \\ R\cos\phi_2 \\ R\sin\phi_2 \\ R\cos\phi_3 \\ R\sin\phi_3 \end{pmatrix} .$$

This model consists of six nonlinear observation equations in the four unknown parameters R, ϕ_1, ϕ_2 and ϕ_3. The approximate values of these parameters are denoted as R^0, ϕ_1^0, ϕ_2^0 and ϕ_3^0. Linearization of (90) gives:

$$E\{\begin{pmatrix} \Delta\underline{x}_1 \\ \Delta\underline{y}_1 \\ \Delta\underline{x}_2 \\ \Delta\underline{y}_2 \\ \Delta\underline{x}_3 \\ \Delta\underline{y}_3 \end{pmatrix}\} = \begin{pmatrix} \cos\phi_1^0 & -R^0\sin\phi_1^0 & 0 & 0 \\ \sin\phi_1^0 & R^0\cos\phi_1^0 & 0 & 0 \\ \cos\phi_2^0 & 0 & -R^0\sin\phi_2^0 & 0 \\ \sin\phi_2^0 & 0 & R^0\cos\phi_2^0 & 0 \\ \cos\phi_3^0 & 0 & 0 & -R^0\sin\phi_3^0 \\ \sin\phi_3^0 & 0 & 0 & R^0\cos\phi_3^0 \end{pmatrix} \begin{pmatrix} \Delta R \\ \Delta\phi_1 \\ \Delta\phi_2 \\ \Delta\phi_3 \end{pmatrix} \tag{91}$$

$$\Delta\underline{y} \qquad \partial_x A(x^0) \qquad \Delta x$$

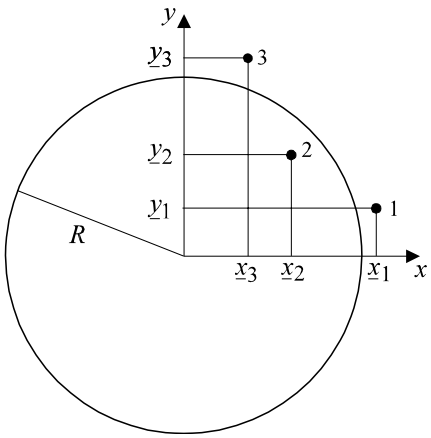


Figure 1.13: Circle with radius R.

Instead of parameterizing the circle, one can alternatively describe the circle implicitly as:

$$x^2 + y^2 = R^2.$$

This description leads to the following nonlinear model of condition equations:

$$\begin{pmatrix} (E\{\underline{x}_1\}^2 + E\{\underline{y}_1\}^2) - (E\{\underline{x}_2\}^2 + E\{\underline{y}_2\}^2) \\ (E\{\underline{x}_1\}^2 + E\{\underline{y}_1\}^2) - (E\{\underline{x}_3\}^2 + E\{\underline{y}_3\}^2) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{92}$$

The number of independent condition equations equals the redundancy, which is equal to $6-4 = 2$. If the approximate values are chosen such that:

$$(x_1^0)^2 + (y_1^0)^2 = (x_2^0)^2 + (y_2^0)^2 = (x_3^0)^2 + (y_3^0)^2$$

linearization of (92) gives:

(93)
$$\underbrace{\begin{pmatrix} 2x_1^0 & 2y_1^0 & -2x_2^0 & -2y_2^0 & 0 & 0 \\ 2x_1^0 & 2y_1^0 & 0 & 0 & -2x_3^0 & -2y_3^0 \end{pmatrix}}_{\partial_y \boldsymbol{B}(y^0)^*} E\{\underbrace{\begin{pmatrix} \Delta \underline{x}_1 \\ \Delta \underline{y}_1 \\ \Delta \underline{x}_2 \\ \Delta \underline{y}_2 \\ \Delta \underline{x}_3 \\ \Delta \underline{y}_3 \end{pmatrix}}_{\Delta y}\} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Verify yourself that $[\partial_y B(y^0)]^*[\partial_x A(x^0)] = 0$ holds for $y^0 = A(x^0)$.

2 Recursive least-squares: the static case

2.1 Introduction

In the previous chapter we reviewed the standard theory of least-squares estimation. In this chapter we will make a modest start with the development of the theory of *recursive* least-squares estimation. Recursive least-squares is a least-squares estimation procedure that enables us to update least-squares estimators for new observables without the need of having to save all past observables. As such the method is of great practical importance. The following example should make the practical relevance of such a recursive procedure clear.

Example 18

Figure 2.1a shows a levelling loop of four points. The heights of these points are denoted as x_0, x_1, x_2 and x_3. The height of point 0 is known and it is equal to zero: $x_0 = 0$.

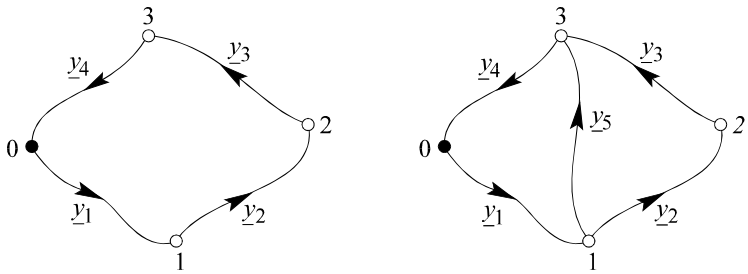


Figure 2.1: Two levelling networks.

The four height difference observables are denoted as $\underline{y}_1$, $\underline{y}_2$, $\underline{y}_3$ and $\underline{y}_4$. They are uncorrelated and they have all the same variance σ^2. The linear A-model reads therefore:

$$(1) \qquad E\left\{\begin{pmatrix}\underline{y}_1\\ \underline{y}_2\\ \underline{y}_3\\ \underline{y}_4\end{pmatrix}\right\} = \begin{pmatrix}1 & 0 & 0\\ -1 & 1 & 0\\ 0 & -1 & 1\\ 0 & 0 & -1\end{pmatrix}\begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix} \quad ; \quad Q_y = \sigma^2 I_4 .$$

Its least-squares solution reads (verify this yourself):

$$(2) \qquad \begin{pmatrix}\hat{\underline{x}}_1\\ \hat{\underline{x}}_2\\ \hat{\underline{x}}_3\end{pmatrix}_{(1)} = \frac{1}{4}\begin{pmatrix}3\underline{y}_1 - \underline{y}_2 - \underline{y}_3 - \underline{y}_4\\ 2\underline{y}_1 + 2\underline{y}_2 - 2\underline{y}_3 - 2\underline{y}_4\\ \underline{y}_1 + \underline{y}_2 + \underline{y}_3 - 3\underline{y}_4\end{pmatrix}$$

with corresponding variance matrix:

$$Q_{\hat{x}_{(1)}} = \frac{\sigma^2}{4}\begin{pmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{pmatrix}. \tag{3}$$

Now assume that an additional height difference observable $\underline{y}_5$, which has a variance of σ^2 and is uncorrelated with $\underline{y}_i$, $i = 1,...,4$, is included in the levelling network (see Figure 2.1b). What would then be an efficient way to compute the least-squares estimator of $x = (x_1, x_2, x_3)^*$? One could of course formulate again the A-model in terms of the original height difference observables:

$$E\{\begin{pmatrix} \underline{y}_1 \\ \underline{y}_2 \\ \underline{y}_3 \\ \underline{y}_4 \\ \underline{y}_5 \end{pmatrix}\} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \\ -1 & 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \quad ; \quad Q_y = \sigma^2 I_5 . \tag{4}$$

Its least-squares solution reads (verify this yourself):

$$\begin{pmatrix} \underline{\hat{x}}_1 \\ \underline{\hat{x}}_2 \\ \underline{\hat{x}}_3 \end{pmatrix}_{(2)} = \frac{1}{8}\begin{pmatrix} 5\underline{y}_1 - \underline{y}_2 - \underline{y}_3 - 3\underline{y}_4 - 2\underline{y}_5 \\ 4\underline{y}_1 + 4\underline{y}_2 - 4\underline{y}_3 - 4\underline{y}_4 \\ 3\underline{y}_1 + \underline{y}_2 + \underline{y}_3 - 5\underline{y}_4 + 2\underline{y}_5 \end{pmatrix} \tag{5}$$

with corresponding variance matrix:

$$Q_{\hat{x}_{(2)}} = \frac{\sigma^2}{8}\begin{pmatrix} 5 & 4 & 3 \\ 4 & 8 & 4 \\ 3 & 4 & 5 \end{pmatrix}. \tag{6}$$

Note that this approach requires that all original height difference observables are available. Also no use is made in this approach of the previous solution (2) and (3). The advantage of the recursive least-squares procedure is now that the same solution (5) and (6) can be obtained without the explicit need of having to store all original observables. Instead of using all original observables $\underline{y}_i$, $i = 1,...,4$, plus the new observables $\underline{y}_5$, the recursive least-squares procedure is based on an optimal combination of the previous solution $\underline{\hat{x}}_{(1)}$ with the new observable $\underline{y}_5$. The appropriate model for this combination reads:

(7)
$$E\left\{\begin{pmatrix}\begin{pmatrix}\underline{\hat{x}}_1\\ \underline{\hat{x}}_2\\ \underline{\hat{x}}_3\end{pmatrix}_{(1)}\\ \underline{y}_5\end{pmatrix}\right\} = \begin{pmatrix}1&0&0\\0&1&0\\0&0&1\\-1&0&1\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} \quad ; \quad D\left\{\begin{pmatrix}\begin{pmatrix}\underline{\hat{x}}_1\\ \underline{\hat{x}}_2\\ \underline{\hat{x}}_3\end{pmatrix}_{(1)}\\ \underline{y}_5\end{pmatrix}\right\} = \begin{pmatrix}\frac{\sigma^2}{4}\begin{pmatrix}3&2&1\\2&4&2\\1&2&3\end{pmatrix}&\begin{matrix}0\\0\\0\end{matrix}\\ \begin{matrix}0&0&0\end{matrix}&\sigma^2\end{pmatrix}.$$

As will be shown below, the solution of this model is identical to the solution (5) and (6).

Consider the *partitioned model*:

(8)
$$\underset{(m_1+m_2)\times 1}{E\left\{\begin{pmatrix}\underline{y}_1\\ \underline{y}_2\end{pmatrix}\right\}} = \underset{(m_1+m_2)\times n}{\begin{pmatrix}A_1\\A_2\end{pmatrix}} \underset{n\times 1}{x} \quad ; \quad \underset{(m_1+m_2)\times(m_1+m_2)}{D\left\{\begin{pmatrix}\underline{y}_1\\ \underline{y}_2\end{pmatrix}\right\}} = \begin{pmatrix}Q_1&0\\0&Q_2\end{pmatrix}.$$

Note that it is assumed that $\underline{y}_1$ and $\underline{y}_2$ are uncorrelated. The least-squares solution of (8) will be denoted as $\underline{\hat{x}}_{(2)}$. It reads:

(9)
$$\begin{cases}\underline{\hat{x}}_{(2)} = (A_1^* Q_1^{-1} A_1 + A_2^* Q_2^{-1} A_2)^{-1}(A^* Q_1^{-1}\underline{y}_1 + A_2^* Q_2^{-1}\underline{y}_2)\\ Q_{\hat{x}_{(2)}} = (A_1^* Q_1^{-1} A_1 + A_2^* Q_2^{-1} A_2)^{-1}\end{cases}.$$

Let us now consider the partial model:

(10)
$$\underset{m_1\times 1}{E\{\underline{y}_1\}} = \underset{m_1\times n}{A_1}\underset{n\times 1}{x} \quad ; \quad \underset{m_1\times m_1}{D\{\underline{y}_1\}} = Q_1.$$

Its solution will be denoted as $\underline{\hat{x}}_{(1)}$. It reads:

(11)
$$\begin{cases}\underline{\hat{x}}_{(1)} = (A_1^* Q_1^{-1} A_1)^{-1}(A_1^* Q_1^{-1}\underline{y}_1)\\ Q_{\hat{x}_{(1)}} = (A_1^* Q_1^{-1} A_1)^{-1}\end{cases}.$$

From this it follows that:

$$A_1^* Q_1^{-1} A_1 = Q_{\hat{x}_{(1)}}^{-1} \text{ and } A_1^* Q_1^{-1}\underline{y}_1 = Q_{\hat{x}_{(1)}}^{-1}\underline{\hat{x}}_{(1)}.$$

Substitution of this result into (9) shows that:

(12)
$$\begin{cases}\underline{\hat{x}}_{(2)} = (Q_{\hat{x}_{(1)}}^{-1} + A_2^* Q_2^{-1} A_2)^{-1}(Q_{\hat{x}_{(1)}}^{-1}\underline{\hat{x}}_{(1)} + A_2^* Q_2^{-1}\underline{y}_2)\\ Q_{\hat{x}_{(2)}} = (Q_{\hat{x}_{(1)}}^{-1} + A_2^* Q_2^{-1} A_2)^{-1}\end{cases}.$$

But this is exactly the solution of the model:

(13)
$$E\left\{\begin{pmatrix}\underline{\hat{x}}_{(1)}\\ \underline{y}_2\end{pmatrix}\right\} = \begin{pmatrix}I\\A_2\end{pmatrix}x \quad ; \quad D\left\{\begin{pmatrix}\underline{\hat{x}}_{(1)}\\ \underline{y}_2\end{pmatrix}\right\} = \begin{pmatrix}Q_{\hat{x}_{(1)}}&0\\0&Q_2\end{pmatrix}.$$

Hence, we have proven that the solution of the partitioned model (8) can be found in two steps. First one solves for the partial model (10). This gives $\underline{\hat{x}}_{(1)}$ and $Q_{\hat{x}_{(1)}}$. Then in a second step one uses $\underline{\hat{x}}_{(1)}$ and $Q_{\hat{x}_{(1)}}$ together with $\underline{y}_2$ and Q_2 to find, via model (13), the final solution $\underline{\hat{x}}_{(2)}$ and $Q_{\hat{x}_{(2)}}$. This result shows that there is no need to store past observables $\underline{y}_1$ for the purpose of computing present estimators $\underline{\hat{x}}_{(2)}$. This is the essence of recursive estimation.

2.2 Recursive least-squares: the A-form

The two-step procedure of the previous section can be generalized to more than two steps. Consider the partitioned model:

$$E\left\{\begin{pmatrix}\underline{y}_0\\ \underline{y}_1\\ \underline{y}_2\\ \vdots\\ \underline{y}_k\end{pmatrix}\right\} = \begin{pmatrix}A_0\\ A_1\\ A_2\\ \vdots\\ A_k\end{pmatrix} x \quad : \quad D\left\{\begin{pmatrix}\underline{y}_0\\ \underline{y}_1\\ \underline{y}_3\\ \vdots\\ \underline{y}_k\end{pmatrix}\right\} = \begin{pmatrix}Q_0 & & & & 0\\ & Q_1 & & & \\ & & Q_2 & & \\ & & & \ddots & \\ 0 & & & & Q_k\end{pmatrix}. \tag{14}$$

Note again that the $\underline{y}_i$, $i = 0,...,k$, are assumed to be mutually uncorrelated. The least-squares solution of model (14) will be denoted as $\underline{\hat{x}}_{(k)}$. It reads:

$$\begin{cases}\underline{\hat{x}}_{(k)} = (\sum_{i=0}^{k} A_i^* Q_i^{-1} A_i)^{-1} (\sum_{i=0}^{k} A_i^* Q_i^{-1} \underline{y}_i)\\ Q_{\hat{x}_{(k)}} = (\sum_{i=0}^{k} A_i^* Q_i^{-1} A_i)^{-1}\end{cases}. \tag{15}$$

Let us now consider model (14) with the exception of $\underline{y}_k$:

$$E\left\{\begin{pmatrix}\underline{y}_0\\ \underline{y}_1\\ \underline{y}_2\\ \vdots\\ \underline{y}_{k-1}\end{pmatrix}\right\} = \begin{pmatrix}A_0\\ A_1\\ A_2\\ \vdots\\ A_{k-1}\end{pmatrix} x \quad : \quad D\left\{\begin{pmatrix}\underline{y}_0\\ \underline{y}_1\\ \underline{y}_2\\ \vdots\\ \underline{y}_{k-1}\end{pmatrix}\right\} = \begin{pmatrix}Q_0 & & & & 0\\ & Q_1 & & & \\ & & Q_2 & & \\ & & & \ddots & \\ 0 & & & & Q_{k-1}\end{pmatrix}. \tag{16}$$

The least-squares solution of this model reads:

$$\begin{cases}\underline{\hat{x}}_{(k-1)} = (\sum_{i=0}^{k-1} A_i^* Q_i^{-1} A_i)^{-1} (\sum_{i=0}^{k-1} A_i^* Q_i^{-1} \underline{y}_i)\\ Q_{\hat{x}_{(k-1)}} = (\sum_{i=0}^{k-1} A_i^* Q_i^{-1} A_i)^{-1}\end{cases}. \tag{17}$$

From this it follows that:

$$\sum_{i=0}^{k-1} \boldsymbol{A}_i^{*} \boldsymbol{Q}_i^{-1} \boldsymbol{A}_i = \boldsymbol{Q}_{\hat{x}_{(k-1)}}^{-1} \text{ and } \sum_{i=0}^{k-1} \boldsymbol{A}_i^{*} \boldsymbol{Q}_i^{-1} \underline{y}_i = \boldsymbol{Q}_{\hat{x}_{(k-1)}}^{-1} \underline{\hat{x}}_{(k-1)} .$$

Using these equations, we may rewrite (15) as :

$$\text{(18)} \qquad \begin{cases} \underline{\hat{x}}_{(k)} = (\boldsymbol{Q}_{\hat{x}_{(k-1)}}^{-1} + \boldsymbol{A}_k^{*} \boldsymbol{Q}_k^{-1} \boldsymbol{A}_k)^{-1} (\boldsymbol{Q}_{\hat{x}_{(k-1)}}^{-1} \underline{\hat{x}}_{(k-1)} + \boldsymbol{A}_k^{*} \boldsymbol{Q}_k^{-1} \underline{y}_k) \\ \boldsymbol{Q}_{\hat{x}_{(k)}} = (\boldsymbol{Q}_{\hat{x}_{(k-1)}}^{-1} + \boldsymbol{A}_k^{*} \boldsymbol{Q}_k^{-1} \boldsymbol{A}_k)^{-1} \end{cases} .$$

But this is exactly the solution of the model:

$$\text{(19)} \qquad E\left\{ \begin{pmatrix} \underline{\hat{x}}_{(k-1)} \\ \underline{y}_k \end{pmatrix} \right\} = \begin{pmatrix} I \\ \boldsymbol{A}_k \end{pmatrix} x \quad ; \quad \boldsymbol{D}\left\{ \begin{pmatrix} \underline{\hat{x}}_{(k-1)} \\ \underline{y}_k \end{pmatrix} \right\} = \begin{pmatrix} \boldsymbol{Q}_{\hat{x}_{(k-1)}} & 0 \\ 0 & \boldsymbol{Q}_k \end{pmatrix} .$$

The above derivation shows that there is no need to store the previous observables $\underline{y}_i$, $i = 1,...,(k-1)$, for the purpose of computing the present least-squares estimator $\underline{\hat{x}}_{(k)}$. That is , the estimator $\underline{\hat{x}}_{(k)}$ can be computed directly from the previous estimator $\underline{\hat{x}}_{(k-1)}$ and the present observable $\underline{y}_k$. This, again, is the essence of recursive estimation. The recursive estimation procedure is initialized with the computation of the initial least-squares estimator $\underline{\hat{x}}_{(0)}$:

$$\begin{cases} \underline{\hat{x}}_{(0)} = (\boldsymbol{A}_0^{*} \boldsymbol{Q}_0^{-1} \boldsymbol{A}_0)^{-1} (\boldsymbol{A}_0^{*} \boldsymbol{Q}_0^{-1} \underline{y}_0) \\ \boldsymbol{Q}_{\hat{x}_{(0)}} = (\boldsymbol{A}_0^{*} \boldsymbol{Q}_0^{-1} \boldsymbol{A}_0)^{-1} \end{cases} .$$

Once $\underline{\hat{x}}_{(0)}$ is known, $\underline{\hat{x}}_{(k)}$ can be computed from $\underline{\hat{x}}_{(k-1)}$ and $\underline{y}_k$ for $k = 1,2,\cdots$ using (18). Note that by rearranging the right-hand side of the first equation of (18), the recursive estimator can alternatively be expressed as (verify this yourself):

$$\text{(20)} \qquad \boxed{\begin{aligned} \underline{\hat{x}}_{(k)} &= \underline{\hat{x}}_{(k-1)} + \boldsymbol{Q}_{\hat{x}_{(k)}} \boldsymbol{A}_k^{*} \boldsymbol{Q}_k^{-1} (\underline{y}_k - \boldsymbol{A}_k \underline{\hat{x}}_{(k-1)}) \\ \boldsymbol{Q}_{\hat{x}_{(k)}} &= (\boldsymbol{Q}_{\hat{x}_{(k-1)}}^{-1} + \boldsymbol{A}_k^{*} \boldsymbol{Q}_k^{-1} \boldsymbol{A}_k)^{-1} \end{aligned}} .$$

The first equation is called the *measurement update equation*. It clearly shows how to update the previous estimator $\underline{\hat{x}}_{(k-1)}$ in order to take care of the new observable $\underline{y}_k$. The second equation is called the *variance update equation*. A flow diagram of the above recursive least-squares procedure is shown in Table 2.1. Note the great resemblance in structure between the recursive least-squares process and the least-squares iteration process of Table 1.4 of Section 1.2.3. The resemblance is a consequence of the fact that both processes are based on the repeated application of the least-squares principle. Despite this resemblance, however, both processes solve two fundamentally different problems. In order to get a somewhat better understanding of the above recursive least-squares procedure, consider the correction term in the first equation of (20). In this term the vector $A_k \underline{\hat{x}}_{(k-1)}$ occurs. This vector depends on all previous observables $\underline{y}_i$, $i = 1,...,(k-1)$, but it is independent of the present observable $\underline{y}_k$.

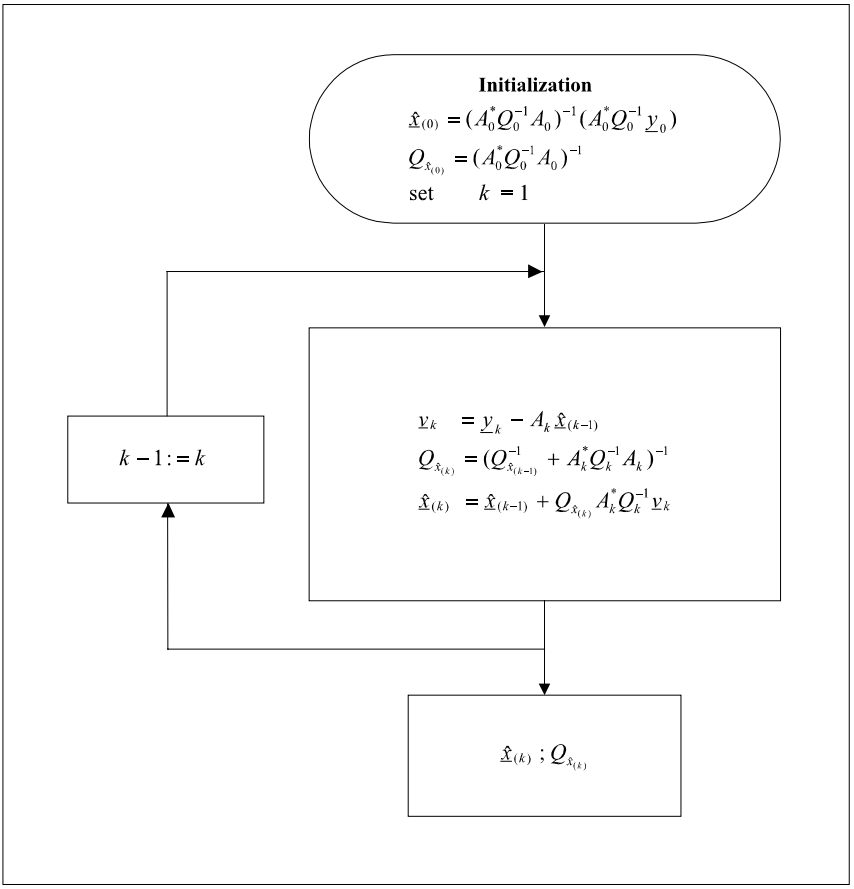


Table 2.1: Recursive least-squares estimation.

Since $\underline{\hat{x}}_{(k-1)}$ is unbiased, that is $E\{\underline{\hat{x}}_{(k-1)}\} = x$ holds, we have that $E\{A_k \underline{\hat{x}}_{(k-1)}\} = A_k E\{\underline{\hat{x}}_{(k-1)}\}$ $= A_k x$ also holds, showing that $A_k \underline{\hat{x}}_{(k-1)}$ is an unbiased estimator of $E\{\underline{y}_k\} = A_k x$. In fact, if one recalls the principle of Best Linear Unbiased Estimation, it will be clear that $A_k \underline{\hat{x}}_{(k-1)}$ is the Best Linear Unbiased Estimator of $E\{\underline{y}_k\}$ when the estimation is based on the model (16). This model contains only the past observables $\underline{y}_i$, $i = 0,...,(k-1)$. Hence, $A_k \underline{\hat{x}}_{(k-1)}$ can be interpreted as the prediction of the present observable $\underline{y}_k$. The difference $\underline{\hat{y}}_k - A_k \underline{\hat{x}}_{(k-1)}$ in (20) is therefore the residual value between the present observable and its prediction. It will be called the predicted residual and denoted as:

(21) $$\underline{v}_k = \underline{y}_k - \boldsymbol{A}_k \underline{\hat{x}}_{(k-1)}.$$

In (20), the predicted residual $\underline{v}_k$ is premultiplied by the matrix:

(22) $$K_k = (Q_{\hat{x}_{(k-1)}}^{-1} + \boldsymbol{A}_k^* \boldsymbol{Q}_k^{-1} \boldsymbol{A}_k)^{-1} \boldsymbol{A}_k^* \boldsymbol{Q}_k^{-1}$$

and the product is then added to the previous estimator $\underline{\hat{x}}_{(k-1)}$ to obtain the current estimation $\underline{\hat{x}}_{(k)}$. Hence the gain in estimation experienced by including the observable $\underline{y}_k$ is determined by the

product $K_k \underline{v}_k$. It depends both on the predicted residual $\underline{v}_k$ and on the matrix K_k, which will be called the gain matrix. Expression (20) shows that the correction to the estimate $\hat{x}_{(k-1)}$ is small if the predicted residual v_k is small. This is also what one would expect. Because if the predicted observation $A_k \hat{x}_{(k-1)}$ is close to the actual observation y_k there is no need really to change the estimate $\hat{x}_{(k-1)}$ by a large amount. Expression (20) also shows that the correction to the estimate $\hat{x}_{(k-1)}$ is small if the gain matrix K_k is small. Equation (22) shows that the gain matrix K_k depends on $Q_{\hat{x}_{(k-1)}}$, A_k and Q_k respectively. Hence, it depends on:

(23)
$$\begin{cases} \text{The precision of the previous estimator } \underline{\hat{x}}_{(k-1)} : Q_{\hat{x}_{(k-1)}} \\ \text{The type of observable } \underline{y}_k \text{ that is added} \quad : A_k \\ \text{The precision of the added observable } \underline{y}_k \quad : Q_k \ . \end{cases}$$

Equation (22) shows that the gain matrix is small if the variance matrix $Q_{\hat{x}_{(k-1)}}$ is small. This is also what one would expect. If $Q_{\hat{x}_{(k-1)}}$ is small, the confidence in the estimator $\underline{\hat{x}}_{(k-1)}$ is high and therefore the new estimator $\underline{\hat{x}}_k$ should not differ too much from the old estimator $\underline{\hat{x}}_{(k-1)}$. Equation (22) shows on the other hand that the gain matrix is large if the variance matrix Q_k is small. In this case one has high confidence in the new observable $\underline{y}_k$. The estimator $\underline{\hat{x}}_{(k)}$ should therefore gain considerably from the inclusion of the new observable $\underline{y}_k$. To conclude this section, a summary of the above discussed recursive least-squares procedure is given in Table 2.2.

Example 19

An unknown distance x is measured a k-number of times. The observables are denoted as $\underline{y}_i$, $i = 1,...,k$. They are uncorrelated and all have the same variance σ^2. The corresponding model of observation equations reads then:

(24)
$$E\left\{\begin{pmatrix} \underline{y}_1 \\ \vdots \\ \underline{y}_k \end{pmatrix}\right\} = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} x \quad ; \quad Q_y = \sigma^2 I_k \ .$$

Its batch solution reads:

(25)
$$\begin{cases} \underline{\hat{x}}_{(k)} = \dfrac{1}{k} \sum_{i=1}^{k} \underline{y}_i \\ \sigma^2_{\hat{x}_{(k)}} = \sigma^2/k \end{cases} .$$

In this simple case the estimator $\underline{\hat{x}}_{(k)}$ equals the average of the k-number of distance observables $\underline{y}_i$, $i = 1,...,k$. A plot of the variance $\sigma^2_{\hat{x}_{(k)}}$ as function of k is given in Figure 2.2. It shows that the variance $\sigma^2_{\hat{x}_{(k)}}$ decreases for increasing values of k. This is of course as it should be. When more distance observables are used for the estimation of the unknown distance x, more

Partitioned model

$$E\left\{\begin{pmatrix} \underline{y}_0 \\ \vdots \\ \underline{y}_k \end{pmatrix}\right\} = \begin{pmatrix} A_0 \\ \vdots \\ A_k \end{pmatrix} x \quad ; \quad D\left\{\begin{pmatrix} \underline{y}_0 \\ \vdots \\ \underline{y}_k \end{pmatrix}\right\} = \begin{pmatrix} Q_0 & & 0 \\ & \ddots & \\ 0 & & Q_k \end{pmatrix}$$

Batch least-squares solution

$$\underline{\hat{x}}_{(k)} = \left(\sum_{i=0}^{k} A_i^* Q_i^{-1} A_i\right)^{-1} \left(\sum_{i=0}^{k} A_i^* Q_i^{-1} \underline{y}_i\right)$$

$$Q_{\hat{x}_{(k)}} = \left(\sum_{i=0}^{k} A_i^* Q_i^{-1} A_i\right)^{-1}$$

Recursive least-squares solution

Initialization

$$\underline{\hat{x}}_{(0)} = (A_0^* Q_0^{-1} A_0)^{-1} A_0^{-1} \underline{y}_0$$

$$Q_{\hat{x}_{(0)}} = (A_0^* Q_0^{-1} A_0)^{-1}$$

Measurement update equation

$$\underline{\hat{x}}_{(k)} = \underline{\hat{x}}_{(k-1)} + K_k \underline{v}_k$$

$$K_k = Q_{\hat{x}_{(k)}} A_k^* Q_k^{-1}$$

$$\underline{v}_k = \underline{y}_k - A_k \underline{\hat{x}}_{(k-1)}$$

Variance update equation

$$Q_{\hat{x}_{(k)}} = (Q_{\hat{x}_{(k-1)}}^{-1} + A_k^* Q_k^{-1} A_k)^{-1}$$

Table 2.2: Batch and recursive least-squares estimation (the A-form).

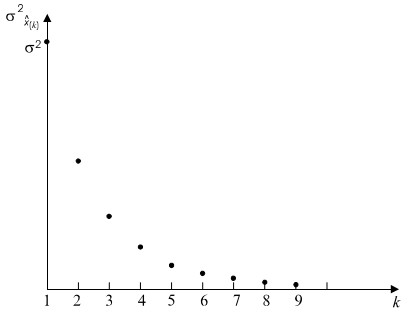


Figure 2.2: The variance $\sigma^2_{\hat{x}_{(k)}} = \sigma^2/k$.

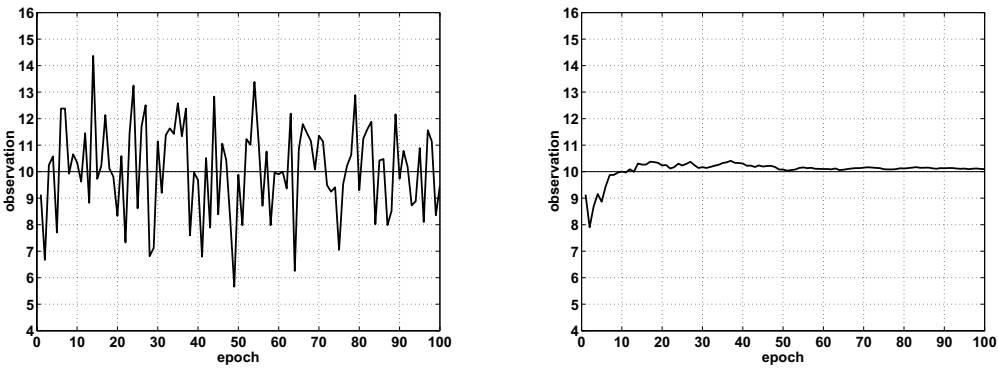
information is used and therefore a more precise estimator can be computed. In the limit we have:

$$\lim_{k\to\infty} \sigma^2_{\hat{x}_{(k)}} = 0 .$$

This implies that the estimator $\underline{\hat{x}}_{(k)}$ converges to a constant for $k\to\infty$. And since $\underline{\hat{x}}_{(k)}$ is an unbiased estimator of x, that is $E\{\underline{\hat{x}}_{(k)}\} = x$ holds, it follows that in the limit:

$$\lim_{k\to\infty} \underline{\hat{x}}_{(k)} = x .$$

Thus the larger k gets, the closer $\underline{\hat{x}}_{(k)}$ gets to x. This is shown in Figure 2.3 for a simulated example based on model (24). Figure 2.3a shows a plot of the simulated observations and Figure 2.3b shows a plot of the corresponding least-squares estimates as function of k. The true value of x used in the simulation was $x = 10$.



(a) (b)

Figure 2.3: (a) Observation plot; (b) Estimation plot.

Let us now consider the recursive least-squares estimator. It is readily derived from (25). From writing $\underline{\hat{x}}_{(k)}$ and $\sigma^2_{\hat{x}_{(k)}}$ as:

$$\begin{aligned}\underline{\hat{x}}_{(k)} &= \frac{k-1}{k}\left(\frac{1}{k-1}\sum_{i=1}^{k-1}\underline{y}_i + \frac{1}{k-1}\underline{y}_k\right)\\ &= \frac{1}{k-1}\sum_{i=1}^{k-1}\underline{y}_i + \frac{1}{k}\left(\underline{y}_k - \frac{1}{k-1}\sum_{i=1}^{k-1}\underline{y}_i\right)\end{aligned}$$

and

$$\sigma^2_{\hat{x}_{(k)}} = \left[\left(\frac{\sigma^2}{k-1}\right)^{-1} + (\sigma^2)^{-1}\right]^{-1}$$

it follows with $\underline{\hat{x}}_{(k-1)} = \frac{1}{k-1}\sum_{i=1}^{k-1}\underline{y}_i$ and $\sigma_{\hat{x}_{(k-1)}} = \sigma^2/(k-1)$ directly that:

(26)
$$\begin{cases}\underline{\hat{x}}_{(k)} = \underline{\hat{x}}_{(k-1)} + \frac{1}{k}\left(\underline{y}_k - \underline{\hat{x}}_{(k-1)}\right)\\ \sigma^2_{\hat{x}_{(k)}} = \left(\sigma^{-2}_{\hat{x}_{(k-1)}} + \sigma^{-2}\right)^{-1}\end{cases}.$$

This result shows that for the present case the gain matrix equals $K_k = 1/k$. It decreases for increasing values of k, showing that the gain in the estimator $\underline{\hat{x}}_{(k)}$ gets less for larger values of k. This is in agreement with the characteristics of Figure 2.3b.

Example 20

Consider the following model:

(27)
$$E\left\{\begin{pmatrix}\underline{y}_1\\ \vdots\\ \underline{y}_k\end{pmatrix}\right\} = \begin{pmatrix}a_1\\ \vdots\\ a_k\end{pmatrix}x \quad ; \quad Q_y = \sigma^2 I_k .$$

Its batch solution reads:

(28)
$$\begin{cases}\underline{\hat{x}}_{(k)} = \dfrac{1}{\sum_{i=1}^{k} a_i^2}\sum_{i=1}^{k} a_i\underline{y}_i\\ \sigma^2_{\hat{x}_{(k)}} = \dfrac{\sigma^2}{\sum_{i=1}^{k} a_i^2}\end{cases}.$$

Note that model (27) and its solution (28) reduce to that of (24) and (25) if $a_i = 1$ for $i = 1,...,k$. The above estimation problem can be viewed as the problem of estimating the unknown slope x of a straight line $y = a\,x$ that passes through the origin (see Figure 2.4a). The result (28) shows that the variance of the slope estimator, $\sigma^2_{\hat{x}_{(k)}}$, is large if all the a_i, $i = 1,...,k$ are close to zero. This is easy to understand if one looks at Figure 2.4b.

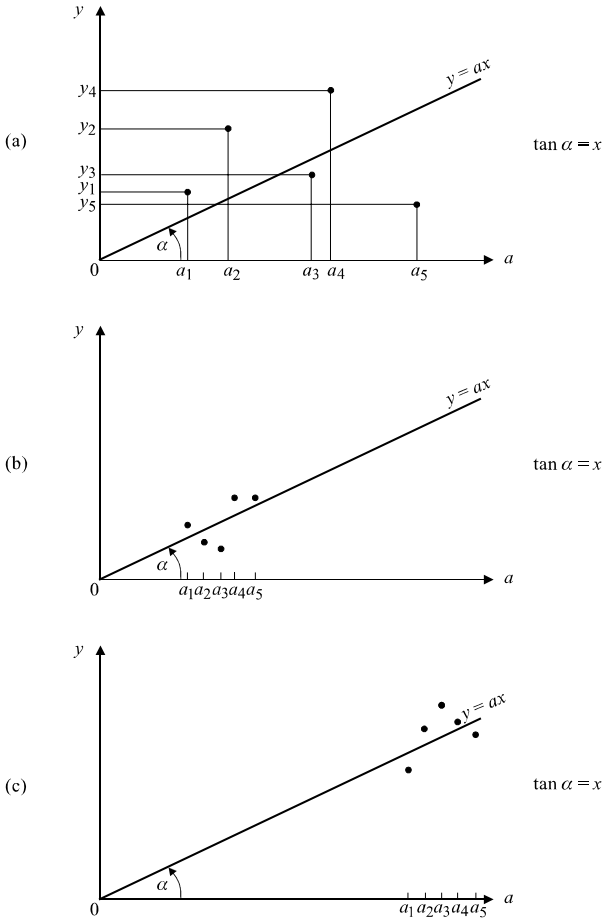


Figure 2.4: Estimation of slope.

The determination of the slope of a straight line becomes more difficult if the observation points are clustered together and close to the origin. The variance $\sigma^2_{\hat{x}_{(k)}}$ is small however if the values of a_i, $i = 1,...,k$ are large. See also Figure 2.4c. Let us now consider the recursive least-squares estimator of the slope. From writing $\hat{x}_{(k)}$ and $\sigma^2_{\hat{x}_{(k)}}$ as:

$$\hat{x}_{(k)} = \frac{\sum_{i=1}^{k-1} a_i^2}{\sum_{i=1}^{k} a_i^2} \left[\frac{\sum_{i=1}^{k-1} a_i y_i}{\sum_{i=1}^{k-1} a_i^2} + \frac{a_k y_k}{\sum_{i=1}^{k-1} a_i^2} \right]$$

$$= \frac{\sum_{i=1}^{k-1} a_i y_i}{\sum_{i=1}^{k-1} a_i^2} + \frac{a_k}{\sum_{i=1}^{k} a_i^2} \left[y_k - a_k \frac{\sum_{i=1}^{k-1} a_i y_i}{\sum_{i=1}^{k-1} a_i^2} \right]$$

and

$$\sigma^2_{\hat{x}_{(k)}} = [\left(\frac{\sigma^2}{\sum_{i=1}^{k-1} a_i^2}\right)^{-1} + a_k^2\sigma^{-2}]^{-1}$$

it follows directly that:

$$(29) \qquad \begin{cases} \underline{\hat{x}}_{(k)} = \underline{\hat{x}}_{(k-1)} + \dfrac{a_k}{\sum_{i=1}^{k} a_i^2}(y_k - a_k\underline{\hat{x}}_{(k-1)}) \\ \sigma^2_{\hat{x}_{(k)}} = (\sigma^{-2}_{\hat{x}_{(k-1)}} + a_k^2\sigma^{-2})^{-1} \end{cases} .$$

This is the recursive least-squares estimator of the slope x.

2.3 Recursive least-squares: the B-form

Expression (20) for the recursive estimator $\underline{\hat{x}}_{(k)}$ shows that a matrix of dimension n needs to be inverted. One can however also derive an expression for $\underline{\hat{x}}_{(k)}$ in which a matrix of dimension m_k needs to be inverted. This expression is found if we solve (19) via the model of condition equations. Model (19) reads in terms of condition equations as:

$$(30) \qquad \underbrace{\begin{pmatrix} -A_k & I_{m_k} \end{pmatrix}}_{B^*} \underbrace{E\left\{\begin{pmatrix} \underline{\hat{x}}_{(k-1)} \\ \underline{y}_k \end{pmatrix}\right\}}_{\underline{y}} = 0 \;;\; \underbrace{D\left\{\begin{pmatrix} \underline{\hat{x}}_{(k-1)} \\ \underline{y}_k \end{pmatrix}\right\}}_{\underline{y}} = \underbrace{\begin{pmatrix} Q_{\hat{x}_{(k-1)}} & 0 \\ 0 & Q_k \end{pmatrix}}_{Q_y} .$$

The solution of this model is identical to the solution of (19) and therefore also identical to the solution of the partitioned model (14). The solution of (30) follows by applying our standard least-squares algorithm for the linear B-model. It reads:

$$(31) \qquad \begin{cases} \begin{pmatrix} \underline{\hat{x}}_{(k)} \\ \underline{\hat{y}}_k \end{pmatrix} = [\begin{pmatrix} I_n & 0 \\ 0 & I_{m_k} \end{pmatrix} - \begin{pmatrix} -Q_{\hat{x}_{(k-1)}}A_k^* \\ Q_k \end{pmatrix}(A_kQ_{\hat{x}_{(k-1)}}A_k^* + Q_k)^{-1}(-A_k \;\; I_{m_k})] \begin{pmatrix} \underline{\hat{x}}_{(k-1)} \\ \underline{y}_k \end{pmatrix} \\ \begin{pmatrix} Q_{\hat{x}_{(k)}} & Q_{\hat{x}_{(k)}\hat{y}_k} \\ Q_{\hat{y}_k\hat{x}_{(k)}} & Q_{\hat{y}_k} \end{pmatrix} = \begin{pmatrix} Q_{\hat{x}_{(k-1)}} & 0 \\ 0 & Q_k \end{pmatrix} - \begin{pmatrix} -Q_{\hat{x}_{(k-1)}}A_k^* \\ Q_k \end{pmatrix}(A_kQ_{\hat{x}_{(k-1)}}A_k^* + Q_k)^{-1}(-A_kQ_{\hat{x}_{(k-1)}} \;\; Q_k) . \end{cases}$$

From the second equation of (31) it follows that:

(32) $$\underline{\hat{y}}_k = \underline{y}_k - \boldsymbol{Q}_k(\boldsymbol{Q}_k + \boldsymbol{A}_k\boldsymbol{Q}_{\hat{x}_{(k-1)}}\boldsymbol{A}_k^*)^{-1}(\underline{y}_k - \boldsymbol{A}_k\underline{\hat{x}}_{(k-1)}) .$$

In this expression we recognize the predicted residual $\underline{v}_k = \underline{y}_k - A_k\underline{\hat{x}}_{(k-1)}$ and its variance matrix:

(33) $$\boldsymbol{Q}_{v_k} = \boldsymbol{Q}_k + \boldsymbol{A}_k\boldsymbol{Q}_{\hat{x}_{(k-1)}}\boldsymbol{A}_k^* .$$

Since the least-squares residual is defined as $\underline{\hat{e}}_k = \underline{y}_k - \underline{\hat{y}}_k$, equation (32) shows how the least-squares residual $\underline{\hat{e}}_k$ and the predicted residual $\underline{v}_k$ are related:

(34) $$\underline{\hat{e}}_k = \boldsymbol{Q}_k\boldsymbol{Q}_{v_k}^{-1}\underline{v}_k .$$

This shows that the two residuals $\underline{\hat{e}}_k$ and $\underline{v}_k$ differ and that they should not be confused with one another. The least-squares residual $\underline{\hat{e}}_k$ is the difference between the actual observable and the estimated observable, whereas the predicted residual $\underline{v}_k$ is the difference between the actual observable and the predicted observable. The recursive least-squares estimator $\underline{\hat{x}}_{(k)}$ and its variance matrix $\boldsymbol{Q}_{\hat{x}_{(k)}}$ follow from (31) as:

(35) $$\boxed{\begin{aligned} \underline{\hat{x}}_{(k)} &= \underline{\hat{x}}_{(k-1)} + \boldsymbol{Q}_{\hat{x}_{(k-1)}}\boldsymbol{A}_k^*\boldsymbol{Q}_{v_k}^{-1}(\underline{y}_k - \boldsymbol{A}_k\underline{\hat{x}}_{(k-1)}) \\ \boldsymbol{Q}_{\hat{x}_{(k)}} &= \boldsymbol{Q}_{\hat{x}_{(k-1)}} - \boldsymbol{Q}_{\hat{x}_{(k-1)}}\boldsymbol{A}_k^*\boldsymbol{Q}_{v_k}^{-1}\boldsymbol{A}_k\boldsymbol{Q}_{\hat{x}_{(k-1)}} \end{aligned}} .$$

Compare this expression with (20) and note that the gain matrix K_k now takes the form:

(36) $$K_k = \boldsymbol{Q}_{\hat{x}_{(k-1)}}\boldsymbol{A}_k^*(\boldsymbol{Q}_k + \boldsymbol{A}_k\boldsymbol{Q}_{\hat{x}_{(k-1)}}\boldsymbol{A}_k^*)^{-1} .$$

Compare this with (22). Solution (35) is of course identical to (20). The principal difference between the two expressions (20) and (35) lies however in the number and dimension of the matrices that need to be inverted. In case of (20) three matrices of order n, m_k and n respectively need to be inverted, namely:

$$\boldsymbol{Q}_{\hat{x}_{(k-1)}} , \ \boldsymbol{Q}_k \text{ and } \boldsymbol{Q}_{\hat{x}_{(k-1)}}^{-1} + \boldsymbol{A}_k^*\boldsymbol{Q}_k^{-1}\boldsymbol{A}_k .$$

In case of (35) only one matrix of dimension m_k needs to be inverted, namely the variance matrix of the predicted residuals:

$$\boldsymbol{Q}_k + \boldsymbol{A}_k\boldsymbol{Q}_{\hat{x}_{(k-1)}}\boldsymbol{A}_k^* .$$

This indicates that for most practical applications expression (35) is to be preferred over (20). This is especially the case if $m_k < n$. Note that in case $m_k = 1$, the matrix inversion in (35) even reduces to a simple scalar division. To conclude this section, a summary of the above discussed recursive least-squares procedure is given in Table 2.3.

Partitioned model

$$E\left\{\begin{pmatrix} \underline{y}_0 \\ \vdots \\ \underline{y}_k \end{pmatrix}\right\} = \begin{pmatrix} A_0 \\ \vdots \\ A_k \end{pmatrix} x \quad ; \quad D\left\{\begin{pmatrix} \underline{y}_0 \\ \vdots \\ \underline{y}_k \end{pmatrix}\right\} = \begin{pmatrix} Q_0 & & 0 \\ & \ddots & \\ 0 & & Q_k \end{pmatrix}$$

Batch least-squares solution

$$\underline{\hat{x}}_{(k)} = \left(\sum_{i=0}^{k} A_i^* Q_i^{-1} A_i\right)^{-1} \left(\sum_{i=0}^{k} A_i^* Q_i^{-1} \underline{y}_i\right)$$

$$Q_{\hat{x}_{(k)}} = \left(\sum_{i=0}^{k} A_i^* Q_i^{-1} A_i\right)^{-1}$$

Recursive least-squares solution

Initialization

$$\underline{\hat{x}}_{(0)} = (A_0^* Q_0^{-1} A_0)^{-1} A_0^* Q_0^{-1} \underline{y}_0$$

$$Q_{\hat{x}_{(0)}} = (A_0^* Q_0^{-1} A_0)^{-1}$$

Measurement update equation

$$\underline{\hat{x}}_{(k)} = \underline{\hat{x}}_{(k-1)} + K_k \underline{v}_k$$

$$K_k = Q_{\hat{x}_{(k-1)}} A_k^* Q_{v_k}^{-1}$$

$$Q_{v_k} = Q_k + A_k Q_{\hat{x}_{(k-1)}} A_k^*$$

$$\underline{v}_k = \underline{y}_k - A_k \underline{\hat{x}}_{(k-1)}$$

Variance update equation

$$Q_{\hat{x}_{(k)}} = (I - K_k A_k) Q_{\hat{x}_{(k-1)}}$$

Table 2.3: Batch and recursive least-squares estimation (the B-form).

Example 21

Consider again Example 18 of Section 2.1. The result of the least-squares estimation after the first step was given as (see (2) and (3)):

$$\begin{pmatrix} \hat{\underline{x}}_1 \\ \hat{\underline{x}}_2 \\ \hat{\underline{x}}_3 \end{pmatrix}_{(1)} = \frac{1}{4}\begin{pmatrix} 3\underline{y}_1 - \underline{y}_2 - \underline{y}_3 - \underline{y}_4 \\ 2\underline{y}_1 + 2\underline{y}_2 - 2\underline{y}_3 - 2\underline{y}_4 \\ \underline{y}_1 + \underline{y}_2 + \underline{y}_3 - 3\underline{y}_4 \end{pmatrix} \tag{37}$$

with corresponding variance matrix:

$$\boldsymbol{Q}_{\hat{x}_{(1)}} = \frac{\sigma^2}{4}\begin{pmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{pmatrix}. \tag{38}$$

In order to take care of the second estimation step, we are now in the position to make use of the results of Table 2.3. The partial model for the new observable $\underline{y}_5$ reads:

$$E\{\underline{y}_5\} = \underbrace{(-1\ 0\ 1)}_{\boldsymbol{A}_2}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \quad \boldsymbol{D}\{\underline{y}_5\} = \sigma^2.$$

With $A_2 = (-1\ 0\ 1)$, $Q_2 = \sigma^2$ and (38) it follows that:

$$\begin{aligned} K_2 &= \boldsymbol{Q}_{\hat{x}_{(1)}}\boldsymbol{A}_2^*(\boldsymbol{Q}_2 + \boldsymbol{A}_2\boldsymbol{Q}_{\hat{x}_{(1)}}\boldsymbol{A}_2^*)^{-1} \\ &= \frac{\sigma^2}{4}\begin{pmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{pmatrix}\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}\left[\sigma^2 + (-1\ 0\ 1)\ \frac{\sigma^2}{4}\begin{pmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{pmatrix}\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}\right]^{-1} \\ &= \frac{1}{4}\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}. \end{aligned} \tag{39}$$

Hence, the updated estimator becomes:

$$\begin{pmatrix}\underline{\hat{x}}_1\\ \underline{\hat{x}}_2\\ \underline{\hat{x}}_3\end{pmatrix}_{(2)} = \begin{pmatrix}\underline{\hat{x}}_1\\ \underline{\hat{x}}_2\\ \underline{\hat{x}}_3\end{pmatrix}_{(1)} + \frac{1}{4}\begin{pmatrix}-1\\ 0\\ 1\end{pmatrix}\left[\underline{y}_5 - (-1\ 0\ 1)\begin{pmatrix}\underline{\hat{x}}_1\\ \underline{\hat{x}}_2\\ \underline{\hat{x}}_3\end{pmatrix}_{(1)}\right]. \tag{40}$$

Substitution of (37) into (40) will show that (40) is identical to (5) of Section 2.1 (verify this yourself). The variance matrix of (40) follows with $A_2 = (-1\ 0\ 1)$, (38) and (39) as:

$$\begin{aligned} Q_{\hat{x}_{(2)}} &= (I - K_2 A_2) Q_{\hat{x}_{(1)}} \\ &= \left[\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix} - \frac{1}{4}\begin{pmatrix}-1\\0\\1\end{pmatrix}(-1\ 0\ 1)\right]\frac{\sigma^2}{4}\begin{pmatrix}3&2&1\\2&4&2\\1&2&3\end{pmatrix} \\ &= \frac{\sigma^2}{8}\begin{pmatrix}5&4&3\\4&8&4\\3&4&5\end{pmatrix} \end{aligned} \tag{41}$$

which is identical to (6) of Section 2.1. Note that in the present example the matrix inversion is simply a division by a scalar.

Example 22

Consider the following model:

$$E\left\{\begin{pmatrix}\underline{y}_1\\ \vdots\\ \underline{y}_k\end{pmatrix}\right\} = \begin{pmatrix}1&a_1\\ \vdots&\vdots\\ 1&a_k\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix} \quad ; \quad Q_y = \sigma^2 I_k . \tag{42}$$

Note that this model reduces to that of (27) of Section 2.2 if $x_1 = 0$, and it reduces to that of (24) of Section 2.2 if $x_2 = 0$. The with model (42) corresponding estimation problem can be viewed as the problem of estimating the intercept x_1 and slope x_2 of a straight line $y = x_1 + ax_2$ (see Figure 2.5a).

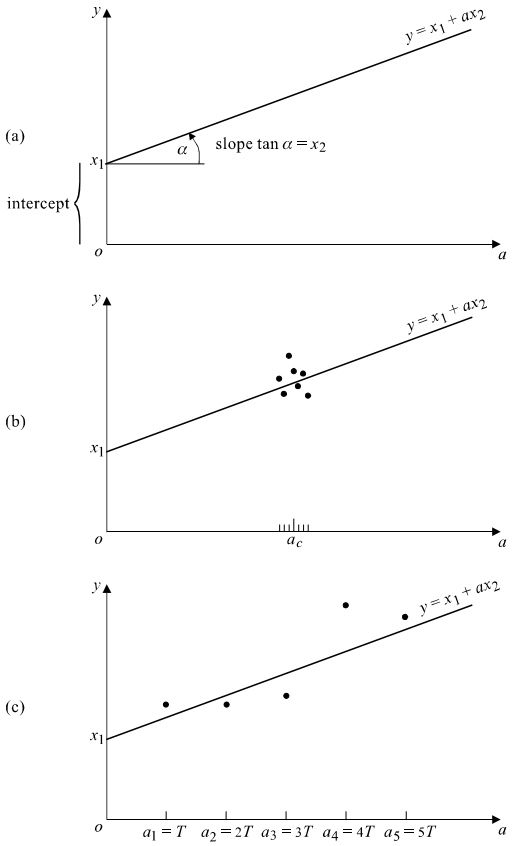


Figure 2.5: Estimation of intercept and slope.

Batch estimator

We will first consider the batch solution of (42). The normal equations read:

$$\begin{pmatrix} k & \sum_{i=1}^{k} a_i \\ \sum_{i=1}^{k} a_i & \sum_{i=1}^{k} a_i^2 \end{pmatrix} \begin{pmatrix} \hat{x}_{1(k)} \\ \hat{x}_{2(k)} \end{pmatrix} = \begin{pmatrix} \sum_{i=1}^{k} y_i \\ \sum_{i=1}^{k} a_i y_i \end{pmatrix}. \tag{43}$$

For the ease of deriving the solution of (43) we define:

$$\begin{cases} a_c = \dfrac{1}{k} \sum_{i=1}^{k} a_i & \text{(average value of } a_i,\ i = 1,\ldots,k) \\ \overline{a_i} = a_i - a_c & (a_i \text{ centred with respect to } a_c) \end{cases}. \tag{44}$$

Then

(45) $$\sum_{i=1}^{k} \bar{a}_i^2 = \sum_{i=1}^{k} a_i^2 - k a_c^2$$

and the normal equations (43) may be written as:

(46) $$\begin{pmatrix} k & k a_c \\ k a_c & (k a_c^2 + \sum_{i=1}^{k} \bar{a}_i^2) \end{pmatrix} \begin{pmatrix} \hat{x}_{1(k)} \\ \hat{x}_{2(k)} \end{pmatrix} = \begin{pmatrix} \sum_{i=1}^{k} y_i \\ \sum_{i=1}^{k} a_i y_i \end{pmatrix}.$$

If we subtract a_c times the first equation from the second we get:

$$\left(\sum_{i=1}^{k} \bar{a}_i^2 \right) \hat{x}_{2(k)} = \left(\sum_{i=1}^{k} \bar{a}_i y_i \right).$$

Hence, the least-squares estimator of the slope x_2 reads:

(47) $$\hat{x}_{2(k)} = \frac{1}{\sum_{i=1}^{k} \bar{a}_i^2} \sum_{i=1}^{k} \bar{a}_i y_i .$$

Compare this estimator with the slope estimator of (28) of Section 2.2. With the slope estimator (47) known, the intercept estimator follows from the first equation of (46) as:

(48) $$\hat{x}_{1(k)} = \frac{1}{k} \sum_{i=1}^{k} y_i - a_c \hat{x}_{2(k)} .$$

Compare this estimator with the intercept estimator of (25) of Section 2.2. The variance matrix of the estimator $(\hat{x}_{1(k)}, \hat{x}_{2(k)})^*$ follows from inverting the normal matrix of (43) and scaling by σ^2:

$$Q_{\hat{x}_{(k)}} = \sigma^2 (k \sum_{i=1}^{k} a_i^2 - (\sum_{i=1}^{k} a_i)^2)^{-1} \begin{pmatrix} \sum_{i=1}^{k} a_i^2 & -\sum_{i=1}^{k} a_i \\ -\sum_{i=1}^{k} a_i & k \end{pmatrix}.$$

With the use of (44) and (45), this expression can be simplified to:

(49) $$Q_{\hat{x}_{(k)}} = \sigma^2 \begin{pmatrix} (\frac{1}{k} + \frac{a_c^2}{\sum_{i=1}^{k} \bar{a}_i^2}) & \frac{-a_c}{\sum_{i=1}^{k} \bar{a}_i^2} \\ \frac{-a_c}{\sum_{i=1}^{k} \bar{a}_i^2} & \frac{1}{\sum_{i=1}^{k} \bar{a}_i^2} \end{pmatrix}.$$

Note that the variances of both the intercept estimator and the slope estimator are larger than the variances of the corresponding estimators in (25) and (28), respectively. This is of course due to the fact that in the present example the intercept x_1 and slope x_2 are estimated simultaneously, whereas in (24) and in (27) of Section 2.2 the intercept and slope were estimated separately. From the variance matrix (49) it follows that the precision of the estimators improves if either k increases or $\sum_{i=1}^{k} \bar{a}_i^2$ increases. The sum $\sum_{i=1}^{k} \bar{a}_i^2$ can be made large by having the measurements carried out such that the a_i, $i = 1,...,k$ cover a large interval. The clustering of observation points as shown in Figure 2.5b provides therefore a poor geometry for the estimation of the intercept and the slope. If the measurements are evenly distributed such that $a_i = iT$, with T = constant, then (see Figure 2.5c):

$$\begin{cases} a_c = \frac{1}{2}(k+1)T \\ \sum_{i=1}^{k} \bar{a}_i^2 = \frac{1}{12}k(k+1)(k-1)T^2 \end{cases} . \tag{50}$$

For this case the elements of the variance matrix (49) can be worked out to give:

$$Q_{\hat{x}_{(k)}} = \sigma^2 \begin{pmatrix} \frac{2(2k+1)}{k(k-1)} & \frac{-6}{k(k-1)T} \\ \frac{-6}{k(k-1)T} & \frac{12}{k(k+1)(k-1)T^2} \end{pmatrix} . \tag{51}$$

Note that the variances are infinite if $k = 1$. This is of course a consequence of the fact that it is impossible to determine intercept and slope simultaneously from only one measurement.

Recursive estimator

We will now consider the recursive solution of (42). We will assume that the measurements are evenly distributed such that $a_i = iT$, with T = constant, holds. The gain matrix K_k can then be computed from (51) for $k-1$, $Q_k = \sigma^2$ and $A_k = (1 \;\; kT)$ as:

$$K_k = Q_{\hat{x}_{(k-1)}} A_k^* (Q_k + A_k Q_{\hat{x}_{(k-1)}} A_k^*)^{-1}$$

$$= \sigma^2 \begin{pmatrix} \frac{2(2k-1)}{(k-1)(k-2)} & \frac{-6}{(k-1)(k-2)T} \\ \frac{-6}{(k-1)(k-2)T} & \frac{12}{(k-1)k(k-2)T^2} \end{pmatrix} \begin{pmatrix} 1 \\ kT \end{pmatrix} [\sigma^2 + (1 \;\; kT)\sigma^2 \begin{pmatrix} \frac{2(2k-1)}{(k-1)(k-2)} & \frac{-6}{(k-1)(k-2)T} \\ \frac{-6}{(k-1)(k-2)T} & \frac{12}{(k-1)k(k-2)T^2} \end{pmatrix} \begin{pmatrix} 1 \\ kT \end{pmatrix}]^{-1}$$

or as:

$$K_k = \begin{pmatrix} -\frac{2}{k} \\ \frac{6}{k(k+1)T} \end{pmatrix} . \tag{52}$$

The recursive estimator for the intercept and slope becomes therefore:

$$\begin{pmatrix} \hat{\underline{x}}_{1(k)} \\ \hat{\underline{x}}_{2(k)} \end{pmatrix} = \begin{pmatrix} \hat{\underline{x}}_{1(k-1)} \\ \hat{\underline{x}}_{2(k-1)} \end{pmatrix} + \begin{pmatrix} -2/k \\ 6/[k(k+1)T] \end{pmatrix} [\underline{y}_k - (1\ kT) \begin{pmatrix} \hat{\underline{x}}_{1(k-1)} \\ \hat{\underline{x}}_{2(k-1)} \end{pmatrix}] . \tag{53}$$

Note that the elements of the gain matrix decrease for increasing k.

Example 23

Consider the levelling network of Figure 2.6a (note that this is the network of Figure 2.1b of Example 18, extended with the observation $\underline{y}_6$). The heights of the four points are denoted as x_α, $\alpha = 0, 1, 2, 3$. The height of point 0 is known and equal to zero: $x_0 = 0$. The six height difference observables are denoted as $\underline{y}_i$, $i = 1,...,6$. They are uncorrelated and have the same variance σ^2.

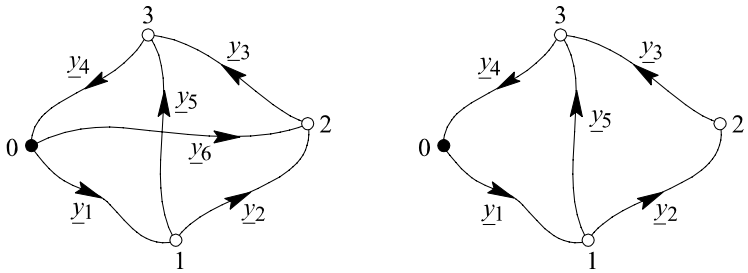


Figure 2.6: Levelling network.

The with the configuration of Figure 2.6a corresponding linear A-model reads:

$$E\{\begin{pmatrix} \underline{y}_1 \\ \underline{y}_2 \\ \underline{y}_3 \\ \underline{y}_4 \\ \underline{y}_5 \\ \underline{y}_6 \end{pmatrix}\} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \\ -1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \quad ; \quad Q_y = \sigma^2 I_6 . \tag{54}$$

Let us assume that it is required to determine the variance matrix $Q_{\hat{x}_{(3)}}$ of the least-squares estimator of $x = (x_1, x_2, x_3)^*$. Then clearly:

$$Q_{\hat{x}_{(3)}} = (A^* Q_y^{-1} A)^{-1} = \sigma^2 \begin{pmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{pmatrix}^{-1} . \tag{55}$$

In this case we need to invert a 3×3 matrix to find $Q_{\hat{x}_{(3)}}$. An easier way to determine $Q_{\hat{x}_{(3)}}$ would be to rely on the results of Example 18 of Section 2.1. In Example 18 it was shown that the variance matrix of the least-squares estimator of $x = (x_1, x_2, x_3)^*$ based on the configuration of Figure 2.6b, is given as:

(56) $$Q_{\hat{x}_{(2)}} = \frac{\sigma^2}{8}\begin{pmatrix} 5 & 4 & 3 \\ 4 & 8 & 4 \\ 3 & 4 & 5 \end{pmatrix}.$$

Since the two configurations of Figure 2.6 only differ in the observable $\underline{y}_6$, it follows that it is easier to compute $Q_{\hat{x}_{(3)}}$ from (56) using the variance update equation of the B-form. With (56), $Q_3 = \sigma^2$ and $A_3 = (0\ 1\ 0)$ it follows that:

$$Q_{\hat{x}_{(3)}} = Q_{\hat{x}_{(2)}} - Q_{\hat{x}_{(2)}} A_3^* (Q_3 + A_3 Q_{\hat{x}_{(2)}} A_3^*)^{-1} A_3 Q_{\hat{x}_{(2)}}$$

$$= \frac{\sigma^2}{8}\begin{pmatrix} 5 & 4 & 3 \\ 4 & 8 & 4 \\ 3 & 4 & 5 \end{pmatrix} - \frac{\sigma^2}{8}\begin{pmatrix} 4 \\ 8 \\ 4 \end{pmatrix} (\sigma^2 + \sigma^2)^{-1} \frac{\sigma^2}{8} (4\ 8\ 4)$$

or that:

(57) $$Q_{\hat{x}_{(3)}} = \frac{\sigma^2}{4}\begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}.$$

Note that only a scalar division was needed for the computation of (57). Verify yourself that (57) is indeed identical to (55).

2.4 Linearization, iteration and recursion

Up to this point the partitioned model was assumed to consist of linear observation equations. In this section we will consider the case that the partitioned model consists of nonlinear observation equations. As our starting point we take the following nonlinear model:

(58) $$E\{\begin{pmatrix} \underline{y}_0 \\ \underline{y}_1 \\ \vdots \\ \underline{y}_k \end{pmatrix}\} = \begin{pmatrix} A_0(x) \\ A_1(x) \\ \vdots \\ A_k(x) \end{pmatrix} \quad ; \quad D\{\begin{pmatrix} \underline{y}_0 \\ \underline{y}_1 \\ \vdots \\ \underline{y}_k \end{pmatrix}\} = \begin{pmatrix} Q_0 & & & 0 \\ & Q_1 & & \\ & & \ddots & \\ 0 & & & Q_k \end{pmatrix}.$$

Now assume that we have an approximation x^0 of x at our disposal, such that the second-order remainders of $A_i(x)$, $i = 0,...,k$, can be neglected. From the theory of Section 1.2.2 it follows then that we may replace the nonlinear model (58) by the linearized model:

$$(59) \qquad E\{\begin{pmatrix}\Delta\underline{y}_0\\ \Delta\underline{y}_1\\ \vdots\\ \Delta\underline{y}_k\end{pmatrix}\} = \begin{pmatrix}\partial_x A_0(x^0)\\ \partial_x A_1(x^0)\\ \vdots\\ \partial_x A_k(x^0)\end{pmatrix} \Delta x \quad ; \quad D\{\begin{pmatrix}\Delta\underline{y}_0\\ \Delta\underline{y}_1\\ \vdots\\ \Delta\underline{y}_k\end{pmatrix}\} = \begin{pmatrix}Q_0 & & & 0\\ & Q_1 & & \\ & & \ddots & \\ 0 & & & Q_k\end{pmatrix}.$$

in which $\Delta\underline{y}_i = \underline{y}_i - A_i(x^0)$, $\Delta x = x - x^0$ and $\partial_x A_i(x^0)$ is the matrix of partial derivatives of $A_i(x)$. Since the partitioned model (59) is linear in Δx, our results of the previous two sections may be applied again. An overview of the corresponding batch- and recursive estimators is given in Tables 2.4, 2.5 and 2.6. The results given in these tables hold as long as x^0 is a good approximation. Now assume that x^0 is not a good enough approximation. In that case we need an iteration to improve upon the approximation x^0 (see Section 1.2.3). For the batch solution of model (58) the iteration is straightforward. The corresponding flow diagram is given in Table 2.7. Compare this flow diagram with that of Table 1.4 of Section 1.2.3. The question is now how to iterate in case we want to compute the least-squares estimate of x recursively. It will be clear that the batch solution can be computed in recursive form in each iteration step. The disadvantage of this approach is however that, although recursion is achieved within each iteration step, the overall procedure, including the iteration, is not recursive. Each iteration cycle requires namely the availability of the whole batch of observables $\underline{y}_i$, $i = 0,\ldots,k$. Thus the iteration procedure of Table 2.7 does not lend itself for a recursive approach. Therefore an alternative approach needs to be developed. The idea is to have instead of a recursion step at each iteration step, an iteration at each recursion step. The procedure is as follows. Starting with the approximation x^0 one first solves the least-squares initialization in an iterative manner. This should take care of the nonlinearity in $A_0(x)$. The initialization provides the initial estimate $\hat{x}_{(0)}$. We are now in the position to commence with the first step of the recursion. For the first approximation to $\hat{x}_{(1)}$ we may either take x^0, which was also used as a first approximation for the initialization, or $\hat{x}_{(0)}$, the initial estimate. Since $\hat{x}_{(0)}$ is likely to be a better approximation, this value will be used as our first approximation to $\hat{x}_{(1)}$. Starting with $\hat{x}_{(0)}$ the first step of the recursion can now be solved in an iterative manner. This should take care of nonlinearity in $A_1(x)$. As a result of the first step of the recursion we get $\hat{x}_{(1)}$. This estimate may then be used in the second step of the recursion as the first approximation to $\hat{x}_{(2)}$, and the whole process is repeated again. By continuing in this manner it is possible to take care of the nonlinearity in the partitioned model and at the same time to obtain the estimates in recursive form. A flow diagram of the whole procedure is shown in Table 2.8.

Nonlinear partitioned model

$$E\left\{\begin{pmatrix}\underline{y}_0\\ \underline{y}_1\\ \vdots\\ \underline{y}_k\end{pmatrix}\right\} = \begin{pmatrix}A_0(x)\\ A_1(x)\\ \vdots\\ A_k(x)\end{pmatrix} \quad ; \quad D\left\{\begin{pmatrix}\underline{y}_0\\ \underline{y}_1\\ \vdots\\ \underline{y}_k\end{pmatrix}\right\} = \begin{pmatrix}Q_0 & & 0\\ & Q_1 & \\ & & \ddots\\ 0 & & Q_k\end{pmatrix}$$

Linearized partitioned model

$$E\left\{\begin{pmatrix}\Delta\underline{y}_0\\ \Delta\underline{y}_1\\ \vdots\\ \Delta\underline{y}_k\end{pmatrix}\right\} = \begin{pmatrix}\partial_x A_0(x^0)\\ \partial_x A_1(x^0)\\ \vdots\\ \partial_x A_k(x^0)\end{pmatrix} \Delta x \quad ; \quad D\left\{\begin{pmatrix}\Delta\underline{y}_0\\ \Delta\underline{y}_1\\ \vdots\\ \Delta\underline{y}_k\end{pmatrix}\right\} = \begin{pmatrix}Q_0 & & 0\\ & Q_1 & \\ & & \ddots\\ 0 & & Q_k\end{pmatrix}$$

Linearized batch least-squares

$$\Delta\underline{y}_i = \underline{y}_i - A_i(x^0),\ i = 0,\ldots,k$$

$$\Delta\hat{\underline{x}}_{(k)} = \left[\sum_{i=0}^{k} \partial_x A_i(x^0)^* Q_i^{-1} \partial_x A_i(x^0)\right]^{-1} \left(\sum_{i=1}^{k} \partial_x A_i(x^0)^* Q_i^{-1} \Delta\underline{y}_i\right)$$

$$\hat{\underline{x}}_{(k)} = x^0 + \Delta\hat{\underline{x}}_{(k)}$$

$$Q_{\hat{x}_{(k)}} = \left[\sum_{i=0}^{k} \partial_x A_i(x^0)^* Q_i^{-1} \partial_x A_i(x^0)\right]^{-1}$$

Table 2.4: Batch least-squares estimation.

Linearized recursive least-squares: the A-form

Initialization

$$\Delta \underline{y}_0 = \underline{y}_0 - \boldsymbol{A}_0(x^0)$$

$$\Delta \underline{\hat{x}}_{(0)} = [\partial_x \boldsymbol{A}_0(x^0)^* Q_0^{-1} \partial_x \boldsymbol{A}_0(x^0)]^{-1} [\partial_x \boldsymbol{A}_0(x^0)^* Q_0^{-1} \Delta \underline{y}_0]$$

$$\underline{\hat{x}}_{(0)} = x^0 + \Delta \underline{\hat{x}}_{(0)}$$

$$Q_{\hat{x}_{(0)}} = [\partial_x \boldsymbol{A}_0(x^0)^* Q_0^{-1} \partial_x \boldsymbol{A}_0(x^0)]^{-1}$$

Recursion

Measurement update

$$\Delta \underline{y}_k = \underline{y}_k - \boldsymbol{A}_k(x^0)$$

$$\Delta \underline{\hat{x}}_{(k-1)} = \underline{\hat{x}}_{(k-1)} - x^0$$

$$K_k = Q_{\hat{x}_{(k)}} \partial_x \boldsymbol{A}_k(x^0)^* Q_k^{-1}$$

$$\Delta \underline{\hat{x}}_{(k)} = \Delta \underline{\hat{x}}_{(k-1)} + K_k [\Delta \underline{y}_k - \partial_x \boldsymbol{A}_k(x^0) \Delta \underline{\hat{x}}_{(k-1)}]$$

$$\underline{\hat{x}}_{(k)} = x^0 + \Delta \underline{\hat{x}}_{(k)}$$

Variance update

$$Q_{\hat{x}_{(k)}} = [Q_{\hat{x}_{(k-1)}}^{-1} + \partial_x \boldsymbol{A}_k(x^0)^* Q_k^{-1} \partial_x \boldsymbol{A}_k(x^0)]^{-1}$$

Table 2.5: Recursive least-squares estimation: the A-form.

Linearized recursive least-squares: the B-form

Initialization

$$\Delta \underline{y}_0 = \underline{y}_0 - \boldsymbol{A}_0(x^0)$$

$$\Delta \underline{\hat{x}}_{(0)} = [\partial_x \boldsymbol{A}_0(x^0)^* Q_0^{-1} \partial_x \boldsymbol{A}_0(x^0)]^{-1} [\partial_x \boldsymbol{A}_0(x^0)^* Q_0^{-1} \Delta \underline{y}_0]$$

$$\underline{\hat{x}}_{(0)} = x^0 + \Delta \underline{\hat{x}}_{(0)}$$

$$Q_{\hat{x}_{(0)}} = [\partial_x \boldsymbol{A}_0(x^0)^* Q_0^{-1} \partial_x \boldsymbol{A}_0(x^0)]^{-1}$$

Recursion

Measurement update

$$\Delta \underline{y}_k = \underline{y}_k - \boldsymbol{A}_k(x^0)$$

$$\Delta \underline{\hat{x}}_{(k-1)} = \underline{\hat{x}}_{(k-1)} - x^0$$

$$Q_{v_k} = Q_k + \partial_x \boldsymbol{A}_k(x^0) Q_{\hat{x}_{(k-1)}} \partial_x \boldsymbol{A}_k(x^0)^*$$

$$K_k = Q_{\hat{x}_{(k-1)}} \partial_x \boldsymbol{A}_k(x^0)^* Q_{v_k}^{-1}$$

$$\Delta \underline{\hat{x}}_{(k)} = \Delta \underline{\hat{x}}_{(k-1)} + K_k [\Delta \underline{y}_k - \partial_x \boldsymbol{A}_k(x^0) \Delta \underline{\hat{x}}_{(k-1)}]$$

$$\underline{\hat{x}}_{(k)} = x^0 + \Delta \underline{\hat{x}}_{(k)}$$

Variance update

$$Q_{\hat{x}_{(k)}} = [I - K_k \partial_x \boldsymbol{A}_k(x^0)] Q_{\hat{x}_{(k-1)}}$$

Table 2.6: Recursive least-squares estimation: the B-form.

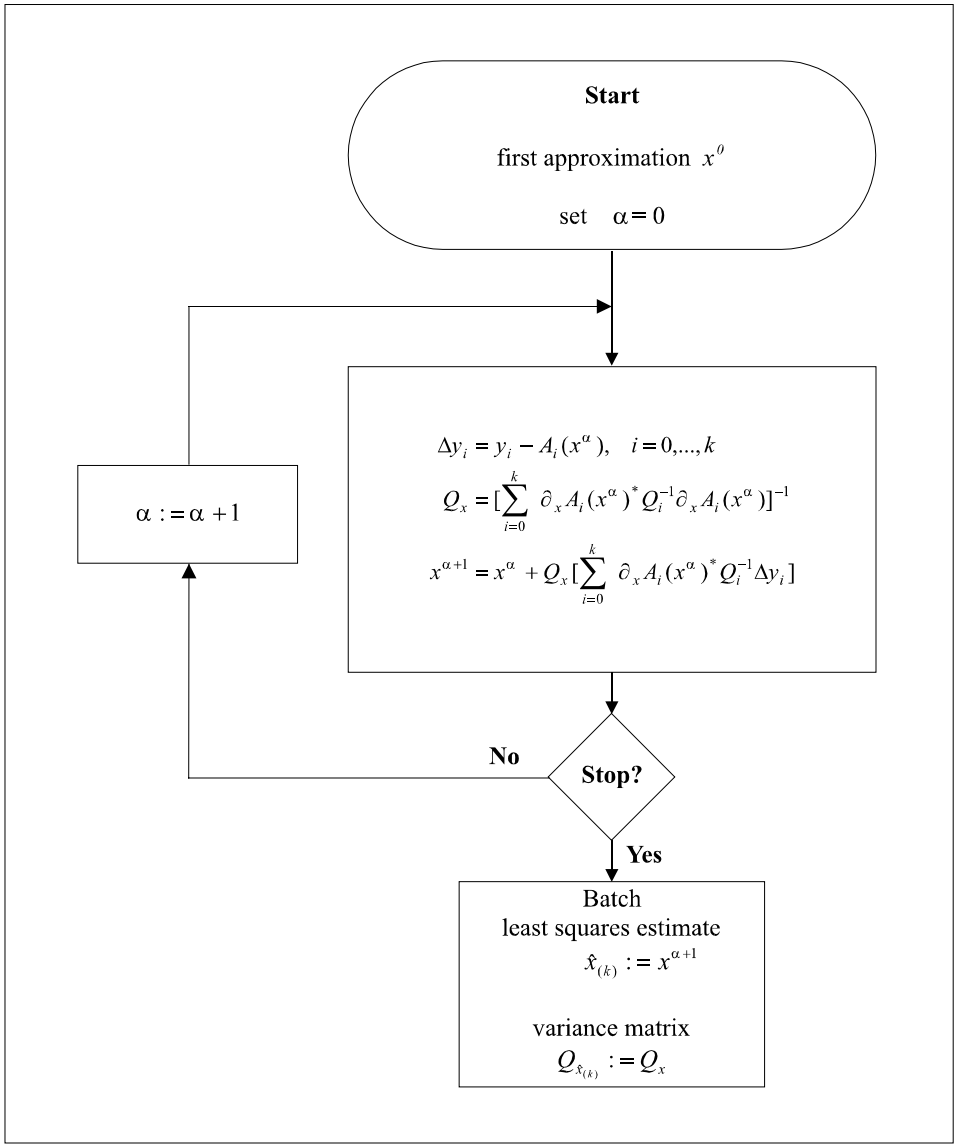


Table 2.7: Batch least-squares iteration.

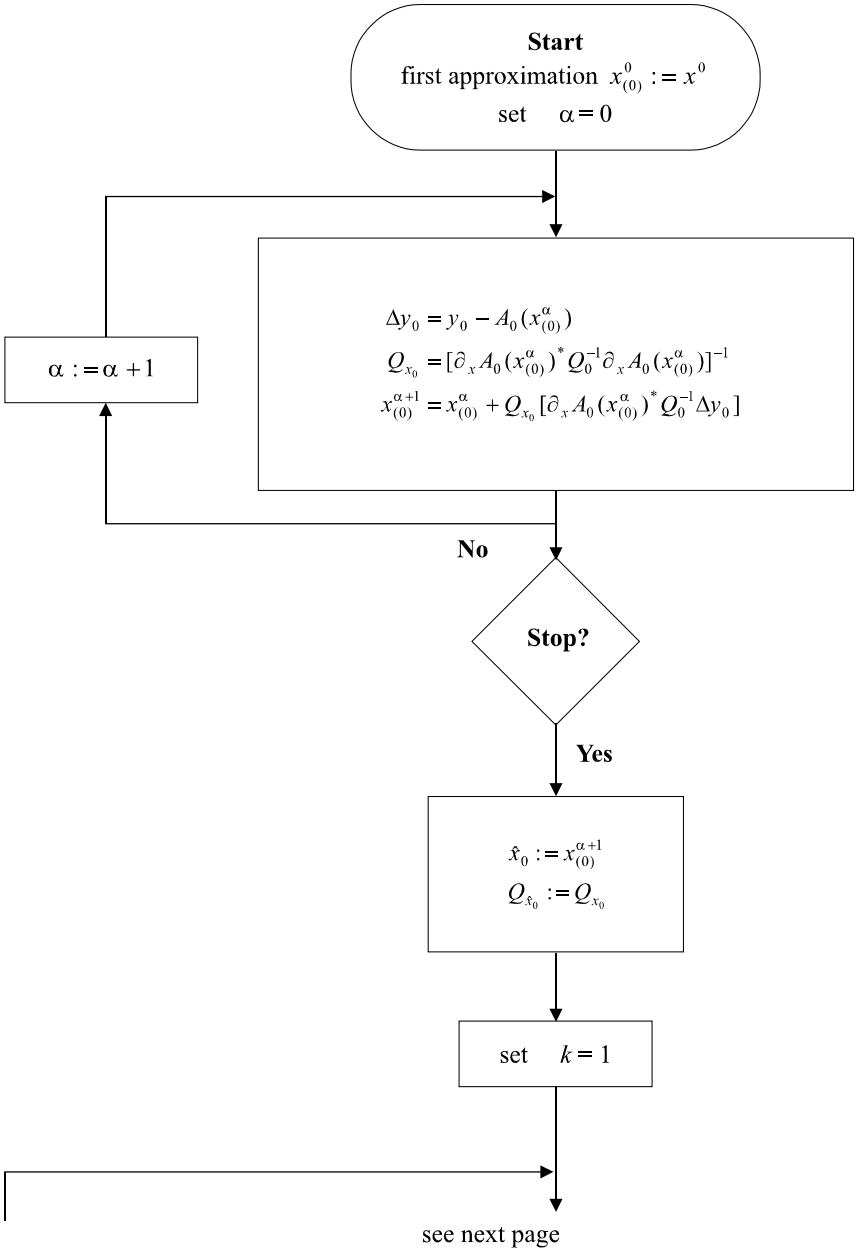


Table 2.8: Recursive least-squares iteration.

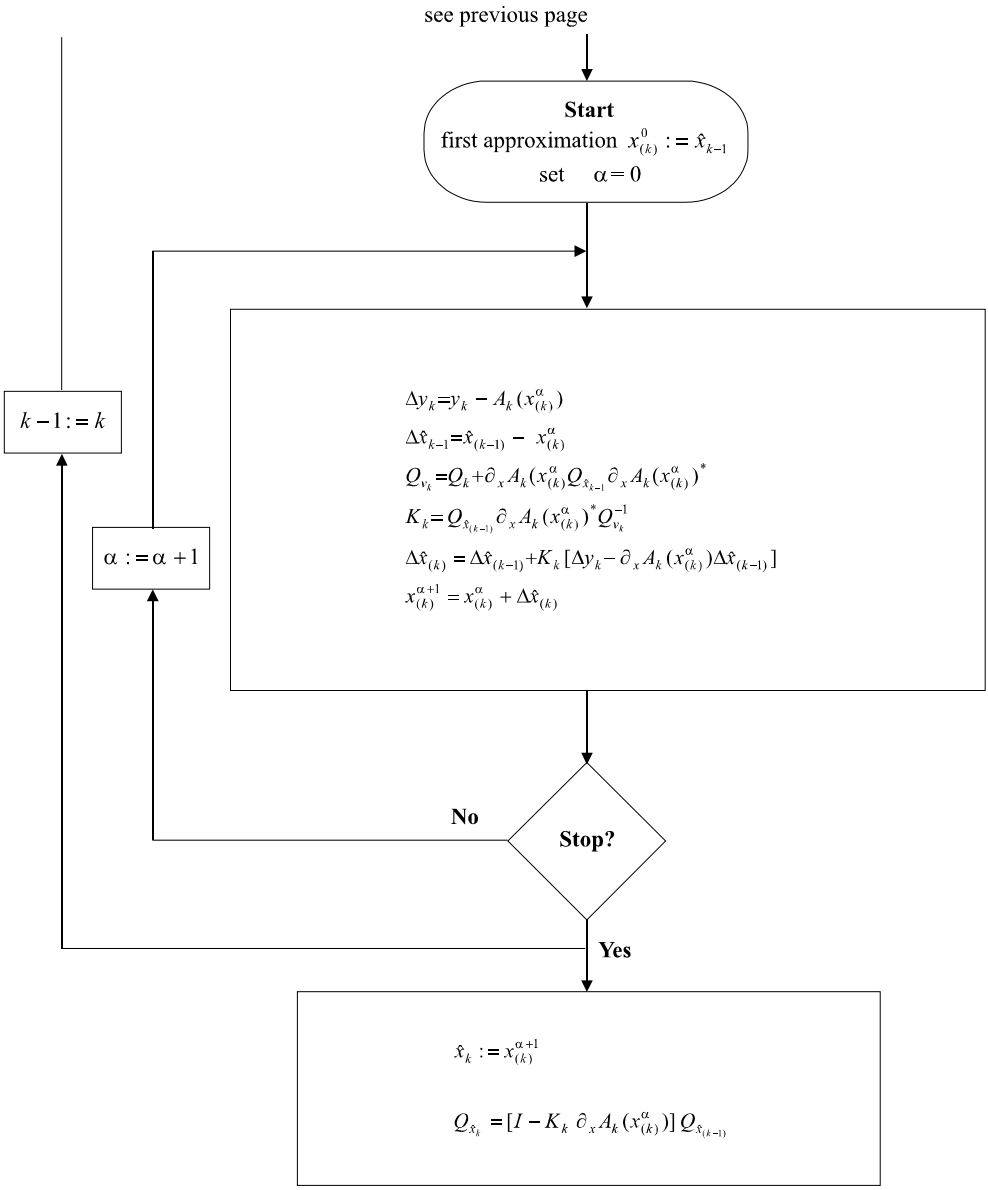


Table 2.8: continued.

Example 24

Consider the situation of Figure 2.7. It shows two cartesian coordinate systems: the x, y-system and the u, v-system. The two systems only differ in their orientation. Hence:

$$\begin{pmatrix} x_i \\ y_i \end{pmatrix} = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} u_i \\ v_i \end{pmatrix}.$$

It is assumed that the u, v-coordinates of a k-number of points are known. Of these same points we have at our disposal the x, y-coordinate observables $\underline{x}_i$, $\underline{y}_i$, $i = 1,...,k$. These observables are uncorrelated and all have the same variance σ^2. The corresponding nonlinear A-model reads then:

$$E\begin{Bmatrix} \underline{x}_1 \\ \underline{y}_1 \\ \vdots \\ \underline{x}_k \\ \underline{y}_k \end{Bmatrix} = \begin{pmatrix} u_1\cos\alpha - v_1\sin\alpha \\ u_1\sin\alpha + v_1\cos\alpha \\ \vdots \\ u_k\cos\alpha - v_k\sin\alpha \\ u_k\sin\alpha + v_k\cos\alpha \end{pmatrix} \quad , \quad Q_y = \sigma^2 I_{2k}. \tag{60}$$

This model consists of a $2k$-number of nonlinear observation equations in one unknown parameter α. We will assume that the difference in orientation between the two systems is small. The approximate value of α may therefore be taken to be equal to zero: $\alpha^0 = 0$. The linearized version of (60) reads then:

$$E\begin{Bmatrix} \Delta\underline{x}_1 \\ \Delta\underline{y}_1 \\ \vdots \\ \Delta\underline{x}_k \\ \Delta\underline{y}_k \end{Bmatrix} = \begin{pmatrix} -v_1 \\ u_1 \\ \vdots \\ -v_k \\ u_k \end{pmatrix} \Delta\alpha \quad , \quad Q_y = \sigma^2 I_{2k}. \tag{61}$$

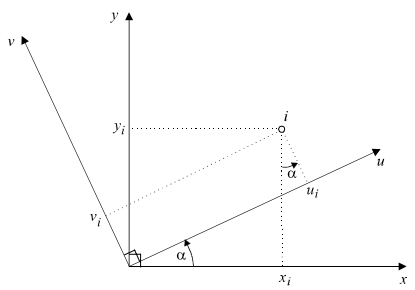


Figure 2.7: Coordinate transformation.

Let us assume that it is required to solve (61) in recursive form by taking one point at a time into account. In order to derive the corresponding recursive estimator, we first need an expression for the gain matrix K_k. The following two expressions for K_k are available to us:

$$(62) \qquad \begin{cases} a.\ K_k = [Q_{\hat{x}_{(k-1)}}^{-1} + \partial_x \boldsymbol{A}_k(x^0)^* Q_k^{-1} \partial_x \boldsymbol{A}_k(x^0)]^{-1} \partial_x \boldsymbol{A}_k(x^0) Q_k^{-1} \\ b.\ K_k = Q_{\hat{x}_{(k-1)}} \partial_x \boldsymbol{A}_k(x^0)^* [Q_k + \partial_x \boldsymbol{A}_k(x^0) Q_{\hat{x}_{(k-1)}} \partial_x \boldsymbol{A}_k(x^0)^*]^{-1} \end{cases}.$$

For the present example we have $m_k = 2$ and $n = 1$. The derivation of the gain matrix is therefore for the present example the easiest if we use expression (62a.) This gives, with

$$Q_{\hat{x}_{(k-1)}} = \sigma^2_{\Delta\hat{\boldsymbol{\alpha}}_{(k-1)}} = \frac{\sigma^2}{\sum_{i=1}^{k-1} (u_i^2 + v_i^2)}, \quad \partial_x \boldsymbol{A}_k(x^0) = (-v_k \quad u_k)^* \text{ and } Q_k = \sigma^2 I_2$$

the following expression for the gain matrix:

$$(63) \qquad K_k = \frac{1}{\sum_{i=1}^{k} (u_i^2 + v_i^2)} (-v_k \quad u_k).$$

Hence, the recursive least-squares estimator of $\Delta\alpha$ reads:

$$(64) \qquad \Delta\underline{\hat{\boldsymbol{\alpha}}}_{(k)} = \Delta\underline{\hat{\boldsymbol{\alpha}}}_{(k-1)} + \frac{1}{\sum_{i=1}^{k} (u_i^2 + v_i^2)} (-v_k \quad u_k) \left[\begin{pmatrix} \Delta\underline{x}_k \\ \Delta\underline{y}_k \end{pmatrix} - \begin{pmatrix} -v_k \\ u_k \end{pmatrix} \Delta\underline{\hat{\boldsymbol{\alpha}}}_{(k-1)} \right].$$

Note that the two elements of the gain matrix are small if point k is located close to the origin. This is understandable if one thinks of the point configuration that is needed for a good determination of a rotation. The rotation about a fixed origin is clearly better determined from points far away from the origin than from points that are close to it.

3 Recursive least-squares: the time-varying case

3.1 Introduction

In the previous chapter we considered the partitioned model:

$$(1) \qquad E\{\begin{pmatrix} \underline{y}_0 \\ \underline{y}_1 \\ \vdots \\ \underline{y}_k \end{pmatrix}\} = \begin{pmatrix} \boldsymbol{A}_0 \\ \boldsymbol{A}_1 \\ \vdots \\ \boldsymbol{A}_k \end{pmatrix} x \; ; \; \boldsymbol{D}\{\begin{pmatrix} \underline{y}_0 \\ \underline{y}_1 \\ \vdots \\ \underline{y}_k \end{pmatrix}\} = \begin{pmatrix} Q_0 & & & 0 \\ & Q_1 & & \\ & & \ddots & \\ 0 & & & Q_k \end{pmatrix}.$$

This model formed the basis of our development of a recursive least-squares procedure for the estimation of the parameter vector x. In this chapter we will generalize the theory of the previous chapter such that it becomes possible to estimate time-varying parameters in recursive form. This generalization is of great practical importance, since many geodetic applications exist in which time-varying parameters need to be determined. In navigation applications for instance, it is often the position and/or velocity of a moving vehicle, such as a car, ship, aircraft or satellite, that needs to be determined as function of time.

To start our development of the appropriate model, we will thus assume that the parameter vector x varies as time proceeds. Hence, instead of assuming x to be constant and thus time-independent, we assume in this chapter that the parameter vector x is a continuous function of time t: $x(t)$. In order to reconstruct the function $x(t)$ from measurements it is furthermore assumed that observables $\underline{y}_i$ are available at discrete time instances t_i, $i = 0,...,k$. These observables are uncorrelated from one time instance to another. The expected values of the observables $\underline{y}_i$ are assumed to bear a linear relationship with $x(t)$ at times $t = t_i$. This gives:

$$(2) \qquad E\{\begin{pmatrix} \underline{y}_0 \\ \underline{y}_1 \\ \vdots \\ \underline{y}_k \end{pmatrix}\} = \begin{pmatrix} \boldsymbol{A}_0 & & & 0 \\ & \boldsymbol{A}_1 & & \\ & & \ddots & \\ 0 & & & \boldsymbol{A}_k \end{pmatrix} \begin{pmatrix} x(t_o) \\ x(t_1) \\ \vdots \\ x(t_k) \end{pmatrix} ; \; \boldsymbol{D}\{\begin{pmatrix} \underline{y}_0 \\ \underline{y}_1 \\ \vdots \\ \underline{y}_k \end{pmatrix}\} = \begin{pmatrix} Q_0 & & & 0 \\ & Q_1 & & \\ & & \ddots & \\ 0 & & & Q_k \end{pmatrix}.$$

It will be clear that the assumptions underlying this discrete like model are not sufficient for estimating the continuous function $x(t)$. Model (2) enables us at the most the estimation of $x(t)$ at times t_i. Hence, additional information is needed in order to be able to estimate $x(t)$ for all t. In this chapter we will assume that this information is available in the form of the linear relationship:

$$(3) \qquad x(t) = \boldsymbol{\Phi}(t,t_0)\,x(t_0).$$

The $n \times n$ matrix $\Phi(t,t_0)$ in (3) will be called the *transition matrix*; it describes the transition from $x(t_0)$ to $x(t)$ for all t. The elements of the transition matrix $\boldsymbol{\Phi}(t,t_0)$ are assumed to be known functions of t and t_0. The transition matrix is also assumed to be invertible. This implies that knowledge of x at any one particular time instant is sufficient for the determination of the

complete function $x(t)$. It is admitted that this assumption seems to be a bit unrealistic for most practical applications. The assumption will therefore be relaxed somewhat in Chapter 5. For the moment, however, it is assumed that relationship (3) holds true. Model (2) together with the transition model (3) will constitute our basic model for the present chapter. Note that our previous model (1) can be interpreted as being a special case of (2) and (3). Because if the transition matrix is equal to the identity matrix, $\Phi(t,t_0) = I$, then $x(t) = x(t_0)$ for all t, and (2) reduces to (1). We will start our development of the recursive least-squares estimation algorithms in Section 3.3 and the sections following. First, however, we will introduce a class of models for the transition (3) that can be used in a fair amount of applications.

3.2 Equations of motion: a polynomial model

In many geodetic applications it is the position, velocity or acceleration of a moving object that needs to be determined as function of time. In many applications one also has some fair idea of the type of movement the object is subject to. For instance, whether the object is stationary or whether it moves with constant velocity, such as a ship or aircraft at cruising speed, or whether it moves with constant acceleration, such as an object in free fall in a uniform gravity field. It is this type of information that we will use in the following to derive an explicit expression for the transition model (3). Figure 3.1 shows the trajectory of a moving object P. The position of the object can be described with the coordinates u, v and w. Since the object moves as time proceeds, the coordinates are functions of time: $u(t)$, $v(t)$ and $w(t)$. The complete time history of the object's motion is known, once these coordinate functions are known. In the following we will develop the transition model for the single coordinate function $u(t)$. The development of the transition model for the other two coordinate functions $v(t)$ and $w(t)$ goes along similar lines. We will consider the following three cases: stationarity, constant velocity and constant acceleration.

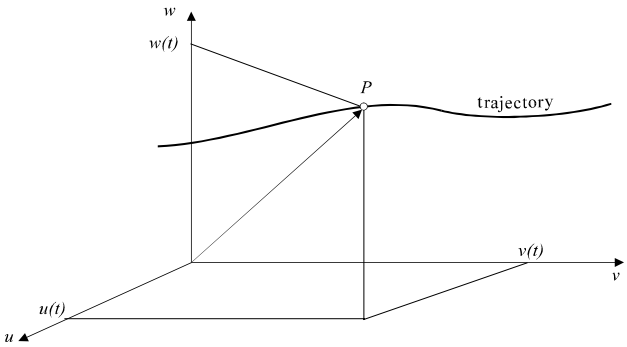


Figure 3.1: Trajectory of object P.

Stationarity

This case is trivial, but it is included since it illustrates in a straightforward manner the principal ideas involved. Assuming that the coordinate function $u(t)$ has a continuous time derivative, we may write:

$$u(t) = u(t_0) + \int_{t_0}^{t} \dot{u}(\tau)\,d\tau \,. \tag{4}$$

This expression shows that the coordinate function $u(t)$ is completely described once the initial position $u(t_0)$ and the velocity $\dot{u}(t)$ are known. Now assume that the object is stationary. Then $\dot{u}(t)$ vanishes and (4) reduces simply to:

$$u(t) = u(t_0) \,. \tag{5}$$

This is the transition model for the single coordinate function $u(t)$ in case of stationarity. The transition model for all three coordinate functions $u(t)$, $v(t)$ and $w(t)$ reads therefore in case of stationarity as:

$$\begin{pmatrix} u(t) \\ v(t) \\ w(t) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} u(t_0) \\ v(t_0) \\ w(t_0) \end{pmatrix} . \tag{6}$$

In this case the transition matrix is simply the identity matrix.

Constant velocity

Assuming that the function $\dot{u}(t)$ has a continuous time derivative, we may write:

$$\dot{u}(t) = \dot{u}(t_0) + \int_{t_0}^{t} \ddot{u}(\tau)\,d\tau \,. \tag{7}$$

If this expression is substituted into (4) we get:

$$u(t) = u(t_0) + \int_{t_0}^{t} \left(\dot{u}(t_0) + \int_{t_0}^{\tau} \ddot{u}(\rho)\,d\rho\right) d\tau$$

or

$$u(t) = u(t_0) + \dot{u}(t_0)(t - t_0) + \int_{t_0}^{t}\int_{t_0}^{\tau} \ddot{u}(\rho)\,d\rho\,d\tau \,. \tag{8}$$

The double integral in this expression may be written as a single integral if we use integration by parts. Since:

$$\int_{t_0}^{t} \dot{f}(\tau)\boldsymbol{g}(\tau)\boldsymbol{d}\tau = [\, f(\tau)\boldsymbol{g}(\tau)]_{t_0}^{t} - \int_{t_0}^{t} f(\tau)\dot{\boldsymbol{g}}(\tau)\boldsymbol{d}\tau$$

it follows by substituting:

$$f(\tau) = \tau \quad \textbf{and} \quad \boldsymbol{g}(\tau) = \int_{t_0}^{\tau} \ddot{u}(\rho)\mathrm{d}\rho$$

that

$$\int_{t_0}^{t}\int_{t_0}^{\tau} \ddot{u}(\rho)\mathrm{d}\rho\mathrm{d}\tau = [\tau \int_{t_0}^{\tau} \ddot{u}(\rho)\mathrm{d}\rho]_{t_0}^{t} - \int_{t_0}^{t} \tau\ddot{u}(\tau)\,\mathrm{d}\tau$$

$$= t\int_{t_0}^{t} \ddot{u}(\rho)\mathrm{d}\rho - \int_{t_0}^{t} \tau\ddot{u}(\tau)\,\mathrm{d}\tau$$

or that:

(9) $$\int_{t_0}^{t}\int_{t_0}^{\tau} \ddot{u}(\rho)\mathrm{d}\rho\mathrm{d}\tau = \int_{t_0}^{t} (t-\tau)\ddot{u}(\tau)\mathrm{d}\tau\,.$$

Substitution of (9) into (8) gives:

(10) $$u(t) = u(t_0) + \dot{u}(t_0)(t-t_0) + \int_{t_0}^{t} (t-\tau)\ddot{u}(\tau)\mathrm{d}\tau\,.$$

Expressions (7) and (10) combined read therefore in matrix vector form as:

(11) $$\boxed{\begin{pmatrix} u(t) \\ \dot{u}(t) \end{pmatrix} = \begin{pmatrix} 1 & (t-t_0) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} u(t_0) \\ \dot{u}(t_0) \end{pmatrix} + \int_{t_0}^{t} \begin{pmatrix} (t-\tau) \\ 1 \end{pmatrix} \ddot{u}(\tau)\mathrm{d}\tau}\,.$$

This expression shows that the coordinate function $u(t)$ and its time derivative $\dot{u}(t)$ are completely determined once the initial position $u(t_0)$, the initial velocity $\dot{u}(t_0)$ and acceleration $\ddot{u}(t)$ are known. Now assume that the object moves with constant velocity. Then $\ddot{u}(t)$ vanishes and (11) reduces to:

(12) $$\begin{pmatrix} u(t) \\ \dot{u}(t) \end{pmatrix} = \begin{pmatrix} 1 & (t-t_0) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} u(t_0) \\ \dot{u}(t_0) \end{pmatrix}.$$

This is the transition model for the single coordinate function $u(t)$ in case of constant velocity. The corresponding transition matrix reads therefore:

(13) $$\Phi(t,\ t_0) = \begin{pmatrix} 1 & (t-t_0) \\ 0 & 1 \end{pmatrix}.$$

Note that this transition matrix is indeed invertible.

Constant acceleration

Assuming that the function $\ddot{u}(t)$ has a continuous time derivative, we may write:

(14) $$\ddot{u}(t) = \ddot{u}(t_0) + \int_{t_0}^{t} \dddot{u}(\tau)\mathrm{d}\tau .$$

If this expression is substituted into (7) we get for velocity:

$$\dot{u}(t) = \dot{u}(t_0) + \int_{t_0}^{t} (\ddot{u}(t_0) + \int_{t_0}^{\tau} \dddot{u}(\rho)\mathrm{d}\rho)\mathrm{d}\tau$$

or

(15) $$\dot{u}(t) = \dot{u}(t_0) + \ddot{u}(t_0)(t-t_0) + \int_{t_0}^{t}\int_{t_0}^{\tau} \dddot{u}(\rho)\mathrm{d}\rho\mathrm{d}\tau .$$

Using (9) with $\ddot{u}(\rho)$ replaced by $\dddot{u}(\rho)$, the double integral of (15) can be replaced by a single integral, so that:

(16) $$\dot{u}(t) = \dot{u}(t_0) + \ddot{u}(t_0)(t-t_0) + \int_{t_0}^{t} (t-\tau)\dddot{u}(\tau)\mathrm{d}\tau .$$

If this expression is substituted into (4) we get for position:

$$u(t) = u(t_0) + \int_{t_0}^{t} [\dot{u}(t_0) + \ddot{u}(t_0)(\tau - t_0) + \int_{t_0}^{\tau} (\tau - \rho)\dddot{u}(\rho)\mathrm{d}\rho]\mathrm{d}\tau$$

or

(17) $$u(t) = u(t_0) + \dot{u}(t_0)(t-t_0) + \tfrac{1}{2}\ddot{u}(t_0)(t-t_0)^2 + \int_{t_0}^{t}\int_{t_0}^{\tau} (\tau - \rho)\dddot{u}(\rho)\mathrm{d}\rho\mathrm{d}\tau .$$

The double integral of (17) can be transformed into a single integral as follows. We have:

(18) $$\int_{t_0}^{t}\int_{t_0}^{\tau} (\tau - \rho)\dddot{u}(\rho)\mathrm{d}\rho\mathrm{d}\tau = \int_{t_0}^{t} \tau \int_{t_0}^{\tau} \dddot{u}(\rho)\mathrm{d}\rho\mathrm{d}\tau - \int_{t_0}^{t} 1 \int_{t_0}^{\tau} \rho\dddot{u}(\rho)\mathrm{d}\rho\mathrm{d}\tau .$$

Integration by parts gives:

(19) $$\int_{t_0}^{t}\int_{t_0}^{\tau} \ddot{u}(\rho)\mathrm{d}\rho\mathrm{d}\tau = \int_{t_0}^{t} \tfrac{1}{2}(t^2-\tau^2)\ddot{u}(\tau)\mathrm{d}\tau \text{ and } \int_{t_0}^{t}\int_{t_0}^{\tau} 1\rho\ddot{u}(\rho)\mathrm{d}\rho\mathrm{d}\tau = \int_{t_0}^{t}(t\tau-\tau^2)\ddot{u}(\tau)\mathrm{d}\tau .$$

From (17), (18) and (19) it follows then that $u(t)$ can be written as:

(20) $$u(t) = u(t_0) + \dot{u}(t_0)(t-t_0) + \tfrac{1}{2}\ddot{u}(t_0)(t-t_0)^2 + \int_{t_0}^{\tau} \tfrac{1}{2}(t-\tau)^2\ddot{u}(\tau)\mathrm{d}\tau .$$

Expression (14), (16) and (20) combined read therefore in matrix vector form as:

(21) $$\boxed{\begin{pmatrix} u(t) \\ \dot{u}(t) \\ \ddot{u}(t) \end{pmatrix} = \begin{pmatrix} 1 & (t-t_0) & \frac{1}{2}(t-t_0)^2 \\ 0 & 1 & (t-t_0) \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} u(t_0) \\ \dot{u}(t_0) \\ \ddot{u}(t_0) \end{pmatrix} + \int_{t_0}^{t} \begin{pmatrix} \frac{1}{2}(t-\tau)^2 \\ (t-\tau) \\ 1 \end{pmatrix} \ddot{u}(\tau)\mathrm{d}\tau}\ .$$

This expression shows that position, velocity and acceleration are completely determined once their initial values and $\ddot{u}(t)$ are known. Now assume that the object moves with constant acceleration. Then $\ddot{u}(t)$ vanishes and (21) reduces to:

(22) $$\begin{pmatrix} u(t) \\ \dot{u}(t) \\ \ddot{u}(t) \end{pmatrix} = \begin{pmatrix} 1 & (t-t_0) & \frac{1}{2}(t-t_0)^2 \\ 0 & 1 & (t-t_0) \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} u(t_0) \\ \dot{u}(t_0) \\ \ddot{u}(t_0) \end{pmatrix}.$$

This is the transition model for the single coordinate function $u(t)$ in case of constant acceleration. It will be clear by now that the above derivations of the transition model can be continued along similar lines to include time derivatives of the order higher than the second as well. From the above results (5), (12) and (22) there are three immediately obvious and very basic properties of the transition matrix. They are (verify yourself):

(23)
Initial value	$: \Phi(t_0,t_0) = I$
Transition property	$: \Phi(t,t_0) = \Phi(t,t_1)\Phi(t_1,t_0) \ \forall\ t,t_0,t_1$
Inverse property	$: \Phi(t,t_0)^{-1} = \Phi(t_0,t) \forall\ t,t_0$

These properties will be used repeatedly in the sequel.

Example 25

Consider an object which is being dropped with zero initial velocity from a building of height $h(t_0)$ (see Figure 3.2):

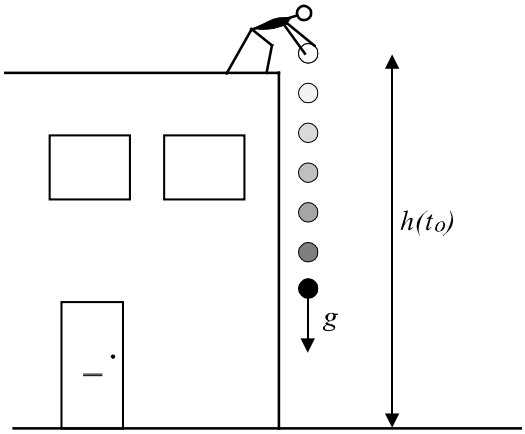


Figure 3.2: Free fall.

The gravity field is assumed to be uniform. The acceleration of the object is then constant and equal to the gravitational acceleration g. Hence, the transition model reads:

$$\begin{pmatrix} h(t) \\ \dot{h}(t) \\ \ddot{h}(t) \end{pmatrix} = \begin{pmatrix} 1 & (t-t_0) & \frac{1}{2}(t-t_0)^2 \\ 0 & 1 & (t-t_0) \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} h(t_0) \\ \dot{h}(t_0) = 0 \\ \ddot{h}(t_0) = -g \end{pmatrix}. \tag{24}$$

Example 26

A ship is sailing with constant velocity in the u-direction and with constant acceleration in the v-direction. Then $\ddot{u}(t) = 0$ and $\dddot{v}(t) = 0$. The corresponding transition model reads therefore:

$$\underbrace{\begin{pmatrix} u(t) \\ \dot{u}(t) \\ v(t) \\ \dot{v}(t) \\ \ddot{v}(t) \end{pmatrix}}_{x(t)} = \underbrace{\begin{pmatrix} 1 & (t-t_0) & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & (t-t_0) & \frac{1}{2}(t-t_0)^2 \\ 0 & 0 & 0 & 1 & (t-t_0) \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}}_{\Phi(t,t_0)} \underbrace{\begin{pmatrix} u(t_0) \\ \dot{u}(t_0) \\ v(t_0) \\ \dot{v}(t_0) \\ \ddot{v}(t_0) \end{pmatrix}}_{x(t_0)}. \tag{25}$$

3.3 Prediction, filtering and smoothing

In this section we will make a start with the estimation of time-varying parameters. But before we proceed, first a few words about the notation used. In order to simplify our notation somewhat, we will write instead of $x(t_i)$ from now on x_i. Also the continuous time argument of the vector function $x(t)$ will be written as an index. Thus instead of $x(t)$ we write x_t. And for the transition matrix we write, instead of $\Phi(t,t_0)$, simply Φ_{t,t_0}. Using this notation we may write the partitioned model (2) and the transition model (3) of Section 3.1 as:

$$E\{\begin{pmatrix} \underline{y}_0 \\ \underline{y}_1 \\ \vdots \\ \underline{y}_k \end{pmatrix}\} = \begin{pmatrix} \boldsymbol{A}_0 & & 0 \\ & \boldsymbol{A}_1 & \\ & & \ddots \\ 0 & & \boldsymbol{A}_k \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_k \end{pmatrix} \quad ; \quad D\{\begin{pmatrix} \underline{y}_0 \\ \underline{y}_1 \\ \vdots \\ \underline{y}_k \end{pmatrix}\} = \begin{pmatrix} \boldsymbol{Q}_0 & & 0 \\ & \boldsymbol{Q}_1 & \\ & & \ddots \\ 0 & & \boldsymbol{Q}_k \end{pmatrix} \tag{26}$$

and

$$x_t = \boldsymbol{\Phi}_{t,0} x_0 \ . \tag{27}$$

Now assume that we want to estimate x_t for an arbitrary time t. Then the parameter vectors x_i, $i = 0,...,k$ of (26) need to be replaced by x_t. This is achieved with the transition model (27). Inversion of (27) gives: $x_0 = \Phi_{t,0}^{-1} x_t$. With the inversion property of the transition matrix (see (23)), this may be written as: $x_0 = \Phi_{0,t} x_t$. Substitution of this expression into $x_i = \Phi_{i,0} x_0$ gives $x_i = \Phi_{i,0}\Phi_{0,t} x_t$. And with the transition property of the transition matrix, this may in turn be written as: $x_i = \Phi_{i,t} x_t$. This relation may now be used to replace all the n-vectors x_i of (26) such that only one parameter vector, namely x_t, remains:

$$E\{\begin{pmatrix} \underline{y}_0 \\ \underline{y}_1 \\ \vdots \\ \underline{y}_k \end{pmatrix}\} = \begin{pmatrix} \boldsymbol{A}_0\boldsymbol{\Phi}_{0,t} \\ \boldsymbol{A}_1\boldsymbol{\Phi}_{1,t} \\ \vdots \\ \boldsymbol{A}_k\boldsymbol{\Phi}_{k,t} \end{pmatrix} x_t \quad ; \quad \boldsymbol{D}\{\begin{pmatrix} \underline{y}_0 \\ \underline{y}_1 \\ \vdots \\ \underline{y}_k \end{pmatrix}\} = \begin{pmatrix} \boldsymbol{Q}_0 & & 0 \\ & \boldsymbol{Q}_1 & \\ & & \ddots \\ 0 & & \boldsymbol{Q}_k \end{pmatrix} . \tag{28}$$

This is our basic partitioned linear model for the estimation of time-varying parameters. Note the striking resemblance in structure between model (28) and model (1) in Section 3.1. The least-squares solution of model (28) will be denoted as $\underline{\hat{x}}_{t|k}$. The first index refers to the time to which the parameter vector corresponds and the second index indicates that the estimator is based on all observables $\underline{y}_i$, $i = 0,...,k$. The following three cases, depending on whether $t > t_k$, $t = t_k$ or $t < t_k$, may now be distinguished (see Figure 3.3).

Prediction $(t > t_k)$

One speaks of prediction if $t > t_k$. If t_k denotes the present time, that is, the time the last measurement has been carried out, $t > t_k$ implies that the parameter vector is being estimated for future times. In this case $\underline{\hat{x}}_{t|k}$ is the predicted estimator of x_t.

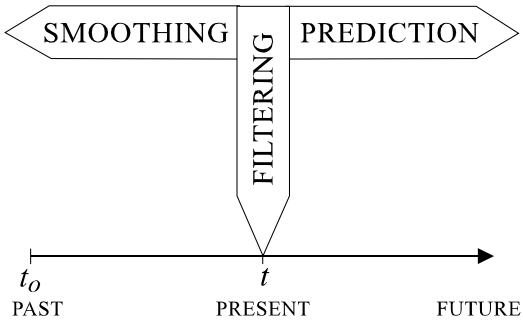


Figure 3.3: Prediction, filtering and smoothing.

Let us assume that we have the estimator $\underline{\hat{x}}_{k|k}$ at our disposal. Then, since $x_t = \Phi_{t,k} x_k$, the Best Linear Unbiased Estimator of x_t, which is the predicted estimator if $t > t_k$, reads in terms of $\underline{\hat{x}}_{k|k}$ as:

(29)
$$\boxed{\begin{aligned} \underline{\hat{x}}_{t|k} &= \Phi_{t,k} \underline{\hat{x}}_{k|k}, \; t > t_k \\ Q_{\hat{x}_{t|k}} &= \Phi_{t,k} Q_{\hat{x}_{k|k}} \Phi_{t,k}^* \end{aligned}} \; .$$

Filtering $(t = t_k)$

One speaks of filtering if $t = t_k$. In this case the parameter vector x_t is estimated for the present time t_k, that is, the time the last measurement, namely $\underline{y}_k$, has been carried out. The estimator $\underline{\hat{x}}_{k|k}$ is called the filtered estimator of x_k. The filtered estimator of x_k is the least-squares solution of model (28) for the case $t = t_k$. It reads:

(30)
$$\boxed{\begin{aligned} \underline{\hat{x}}_{k|k} &= \Big(\sum_{i=0}^{k} \Phi_{i,k}^* A_i^* Q_i^{-1} A_i \Phi_{i,k}\Big)^{-1} \Big(\sum_{i=0}^{k} \Phi_{i,k}^* A_i^* Q_i^{-1} \underline{y}_i\Big) \\ Q_{\hat{x}_{k|k}} &= \Big(\sum_{i=0}^{k} \Phi_{i,k}^* A_i^* Q_i^{-1} A_i \Phi_{i,k}\Big)^{-1} \end{aligned}} \; .$$

Smoothing $(t < t_k)$

One speaks of smoothing if $t < t_k$. In this case the parameter vector x_t is estimated for past times from measurements which have been carried out up to and including the present time t_k. In this case $\underline{\hat{x}}_{t|k}$ is the smoothed estimator of x_t. Let us assume that we have the filtered estimator $\underline{\hat{x}}_{k|k}$ at our disposal. Then, since $x_t = \Phi_{t,k} x_k$, the Best Linear Unbiased Estimator of x_t, which is the smoothed estimator if $t < t_k$, reads in terms of $\underline{\hat{x}}_{k|k}$ as:

(31)
$$\boxed{\begin{aligned} \underline{\hat{x}}_{t|k} &= \boldsymbol{\Phi}_{t,k}\ \underline{\hat{x}}_{k|k},\ t < t_k \\ Q_{\hat{x}_{t|k}} &= \boldsymbol{\Phi}_{t,k}\ Q_{\hat{x}_{k|k}} \boldsymbol{\Phi}^{*}_{t,k} \end{aligned}} \ .$$

When comparing (29) and (31), we note that the expression for the predicted and smoothed estimator, when written in terms of the filtered estimator, are identical in structure. In later chapters when we generalize our theory so as to include more realistic transition models, we will see that this will not be the case anymore.

Example 27

Consider a particle that moves with constant velocity along a straight line. Its transition model reads then:

(32)
$$\begin{pmatrix} u_t \\ \dot{u}_t \end{pmatrix} = \begin{pmatrix} 1 & (t-t_0) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} u_0 \\ \dot{u}_0 \end{pmatrix} \ .$$

The position of the particle is measured at successive time instances $t_1, \ldots, t_k$. The position observables are denoted as $\underline{u}_i$, $i = 1, \ldots, k$. They are uncorrelated and all have the same variance σ^2. The model of observation equations parameterized in terms of position u_i and velocity $\dot{u}_i$ reads then:

(33)
$$E\{\begin{pmatrix} \underline{u}_1 \\ \underline{u}_2 \\ \vdots \\ \underline{u}_k \end{pmatrix}\} = \begin{pmatrix} (1\ 0) & & & 0 \\ & (1\ 0) & & \\ & & \ddots & \\ 0 & & & (1\ 0) \end{pmatrix} \begin{pmatrix} \begin{pmatrix} u_1 \\ \dot{u}_1 \end{pmatrix} \\ \begin{pmatrix} u_2 \\ \dot{u}_2 \end{pmatrix} \\ \vdots \\ \begin{pmatrix} u_k \\ \dot{u}_k \end{pmatrix} \end{pmatrix} \quad ; \quad D\{\begin{pmatrix} \underline{u}_1 \\ \underline{u}_2 \\ \vdots \\ \underline{u}_k \end{pmatrix}\} = \sigma^2 I_k \ .$$

Now assume that we want to estimate $(u_t, \dot{u}_t)^*$ for an arbitrary time t. By combining (32) and (33) we may write the model parameterized in terms of $(u_t, \dot{u}_t)^*$ as:

(34)
$$E\{\begin{pmatrix}\underline{u}_1\\ \underline{u}_2\\ \vdots\\ \underline{u}_k\end{pmatrix}\} = \begin{pmatrix}(1\ 0)\begin{pmatrix}1 & (t_1-t)\\ 0 & 1\end{pmatrix}\\ (1\ 0)\begin{pmatrix}1 & (t_2-t)\\ 0 & 1\end{pmatrix}\\ \vdots\\ (1\ 0)\begin{pmatrix}1 & (t_k-t)\\ 0 & 1\end{pmatrix}\end{pmatrix}\begin{pmatrix}u_t\\ \dot{u}_t\end{pmatrix} \quad ; \quad D\{\begin{pmatrix}\underline{u}_1\\ \underline{u}_2\\ \vdots\\ \underline{u}_k\end{pmatrix}\} = \sigma^2 I_k \ .$$

or as:

(35)
$$E\{\begin{pmatrix}\underline{u}_1\\ \underline{u}_2\\ \vdots\\ \underline{u}_k\end{pmatrix}\} = \begin{pmatrix}1 & (t_1-t)\\ 1 & (t_2-t)\\ \vdots & \vdots\\ 1 & (t_k-t)\end{pmatrix}\begin{pmatrix}u_t\\ \dot{u}_t\end{pmatrix} \quad ; \quad D\{\begin{pmatrix}\underline{u}_1\\ \underline{u}_2\\ \vdots\\ \underline{u}_k\end{pmatrix}\} = \sigma^2 I_k \ .$$

This is the appropriate model in case position and velocity of the particle are unknown. For the present example we will assume however that the initial position u_0 of the particle is known to be zero: $u_0 = 0$. From the transition model (32) it follows then that:

(36)
$$\dot{u}_t = u_t/(t-t_0).$$

Substitution of (36) into (35) gives then:

(37)
$$E\{\begin{pmatrix}\underline{u}_1\\ \underline{u}_2\\ \vdots\\ \underline{u}_k\end{pmatrix}\} = \begin{pmatrix}(t_1-t_0)/(t-t_0)\\ (t_2-t_0)/(t-t_0)\\ \vdots\\ (t_k-t_0)/(t-t_0)\end{pmatrix} u_t \quad ; \quad D\{\begin{pmatrix}\underline{u}_1\\ \underline{u}_2\\ \vdots\\ \underline{u}_k\end{pmatrix}\} = \sigma^2 I_k \ .$$

Note that this model has the same structure as model (27) of Example 20 of Section 2.2. The least-squares solution of model (37) reads:

(38)
$$\begin{cases}\hat{\underline{u}}_{t|k} = \dfrac{(t-t_0)^2}{\sum_{i=1}^{k}(t_i-t_0)^2}\sum_{i=1}^{k}\dfrac{(t_i-t_0)\underline{u}_i}{(t-t_0)}\\ \sigma^2_{\hat{u}_{t|k}} = \sigma^2(t-t_0)^2 \,/\, \sum_{i=1}^{k}(t_i-t_0)^2\end{cases} .$$

Compare this result with (28) in Section 2.2. The variance $\sigma^2_{\hat{u}_{t|k}}$ of $\underline{\hat{u}}_{t|k}$ is a continuous function of time t, but a discrete function of the number of observables k. The result (38) shows that:

$$(39) \qquad \begin{cases} \sigma^2_{\hat{u}_{t'|k}} > \sigma^2_{u_{t|k}} \text{ for } t' > t \\ \sigma^2_{u_{t|k'}} < \sigma^2_{u_{t|k}} \text{ for } k' > k \end{cases}.$$

The first inequality tells us that for a fixed number of observables k, the precision of the position estimator $\underline{\hat{u}}_{t|k}$ deteriorates for increasing t. This is understandable if one thinks of the geometry of the straight line (see Figure 2.4 in Section 2.2). Since the line is hinged at the origin, small changes in its slope will result in changes in its position that are larger for points that are further away from the hinging point. The second inequality of (39) tells us that for time t fixed, the precision of the position estimator improves for increasing k. This is of course due to the additional position information brought in by the increasing number of position observables. In Figure 3.4 we have plotted the variance $\sigma^2_{\hat{u}_{t|k}}$ as function of time, for different values of k. The plot shows that the parabola $\sigma^2_{\hat{u}_{t|k}}$ increases as function of time (the first inequality of (39)), but that it gets more flattened if k increases (the second inequality of (39)).

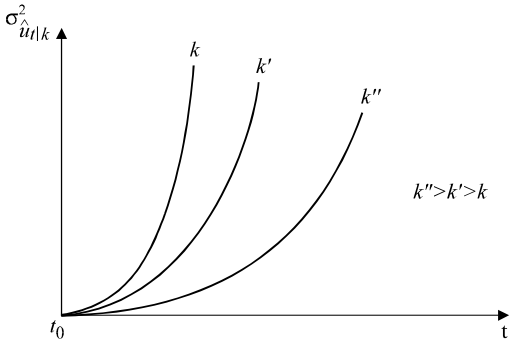


Figure 3.4: The parabola $\sigma^2_{\hat{u}_{t|k}}$ for different values of k.

Let us now consider the precision characteristics of the predicted, filtered and smoothed position estimators. We will first consider prediction and filtering. It is assumed that the present and future position estimates are computed every time that a new position measurement becomes available. Let us start at time t_k. At this time instant all the position observables $\underline{u}_i$, $i = 1, \ldots, k$ are available and the best (in the sense of minimal variance) position estimator $\underline{\hat{u}}_{k|k}$ can be derived. It has a variance of $\sigma^2_{\hat{u}_{k|k}}$. At time t_k also the best position estimators for future times $t > t_k$ can be derived. The variance of these predicted estimators is given as $\sigma^2_{\hat{u}_{t|k}}$ for $t > t_k$. The variances $\sigma^2_{\hat{u}_{k|k}}$ and $\sigma^2_{\hat{u}_{t|k}}$ for $t > t_k$ are plotted in Figure 3.5a. Now assume that the next position measurement is scheduled to be carried out at time t_{k+1}. The best position estimator for this time t_{k+1}, based on the position observables $\underline{u}_i$, $i = 1, \ldots, k$ is then given by the predicted position estimator $\underline{\hat{u}}_{k+1|k}$. Its variance is given by $\sigma^2_{\hat{u}_{k+1|k}}$. Once the new measurement at time t_{k+1} has been carried out, it can be used to improve upon the predicted estimate $\underline{\hat{u}}_{k+1|k}$. The position estimator for time t_{k+1}, which is based on all the past position observables $\underline{u}_i$, $i = 1, \ldots, k$ and the new position observable $\underline{u}_{k+1}$, reads $\underline{\hat{u}}_{k+1|k+1}$. It is the filtered position estimator for time t_{k+1}, and its

variance is of course smaller than the variance of the predicted position estimator for time t_{k+1}. That is, $\sigma^2_{\hat{u}_{k+1|k+1}} < \sigma^2_{\hat{u}_{k+1|k}}$. This is shown in Figure 3.5b.

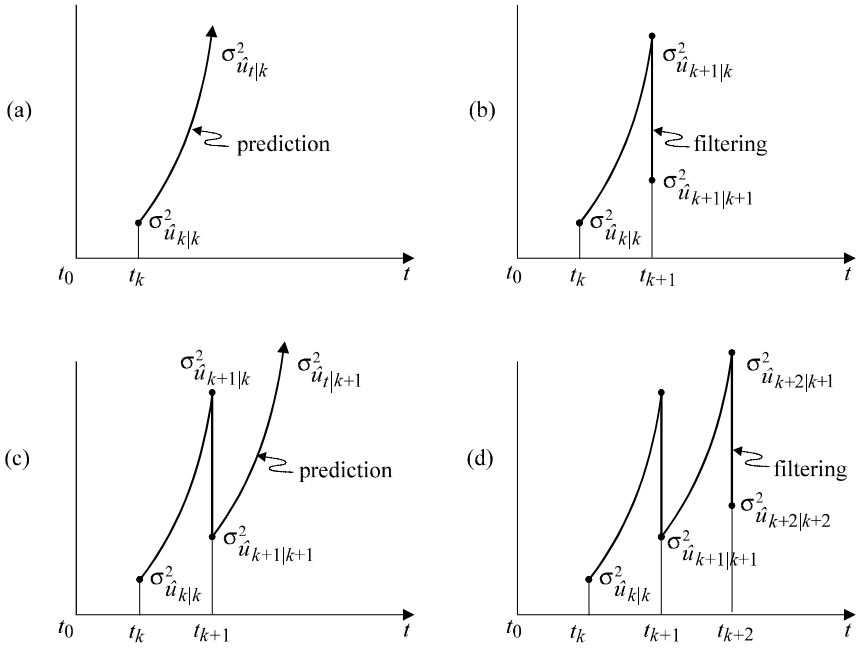


Figure 3.5: Variances of the predicted and filtered position estimator.

The whole process of predicting and filtering can now be repeated again. From time t_{k+1} one can predict the positions up to the time the next measurement is scheduled to be carried out, say t_{k+2} (see Figure 3.5c). Once the measurement at time t_{k+2} is available, it can be used to improve upon the predicted position. As a result one obtains the filtered position estimate for time t_{k+2} (see Figure 3.5d).

Let us now consider smoothing. Again we start at time t_k with the filtered position estimator $\underline{\hat{u}}_{k|k}$. At this time instant all the position observables $\underline{u}_i$, $i = 1,\ldots,k$ are available and based on them the best position estimators for all times $t < t_k$ can be derived. These are the smoothed position estimators with variances $\sigma_{\hat{u}_{t|k}}$, $t < t_k$. This is shown in Figure 3.6a. At the next time instant t_{k+1} when the new position observable $\underline{u}_{k+1}$ becomes available, the filtered position estimator $\underline{\hat{u}}_{k+1|k+1}$ can be derived. Since additional information is available in the form of the new position observable $\underline{u}_{k+1}$, the previous smoothed position estimators $\underline{\hat{u}}_{t|k}$, $t < t_k$, can be replaced by the improved smoothed position estimators $\underline{\hat{u}}_{t|k+1}$, $t < t_{k+1}$. This is shown in Figure 3.6b. This whole process can be repeated again when at t_{k+2} the next position observable becomes available (see Figure 3.6c).

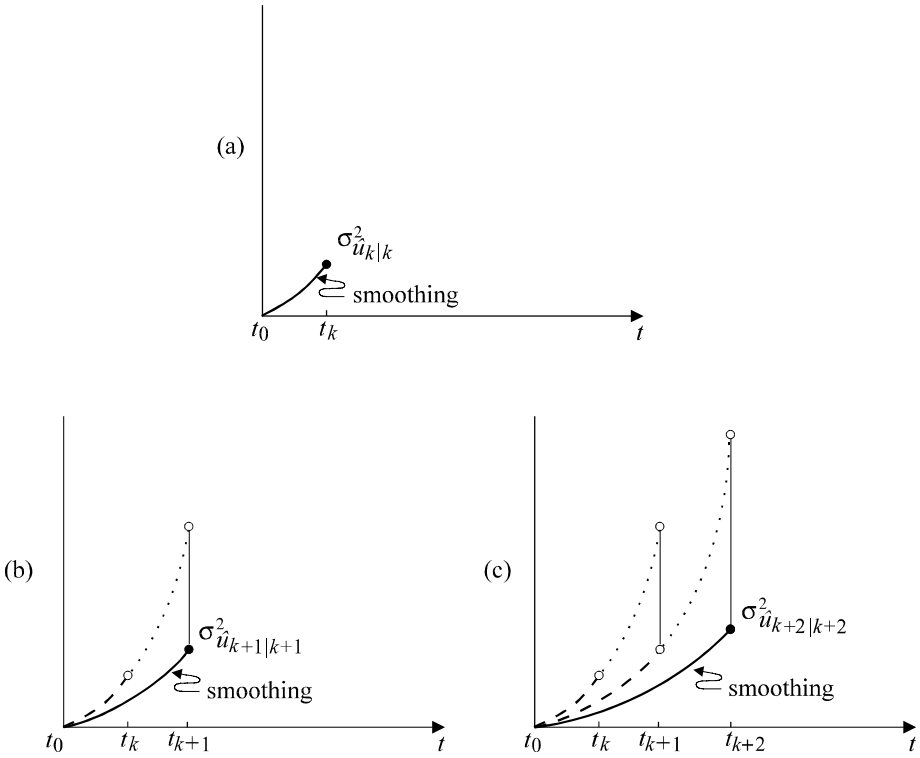


Figure 3.6: Variances of the smoothed position estimator.

So far in the whole discussion no restrictions were put on the intervals between the times that the measurements are carried out. These intervals were allowed to vary from one time to another. This has as a consequence that no statement can be made about the general behaviour of the variance of the filtered position estimator, $\sigma^2_{\hat{u}_{l|k}}$, as function of k. This changes, however, if we assume that the time instances t_i are equidistant in time (= uniform sampling). Then $t_i = t_0 + iT$, with T =constant, and $\sigma^2_{\hat{u}_{l|k}}$ (see (38)) reads for $t = t_0 + lT$:

$$\sigma^2_{\hat{u}_{l|k}} = \frac{\sigma^2 l^2 T^2}{\sum_{i=1}^{k} i^2 T^2} = \frac{6\sigma^2 l^2}{k(k+1)(2k+1)} . \tag{40}$$

This shows that in case of uniform sampling the variance of the filtered position estimator, $\sigma^2_{\hat{u}_{k|k}}$, decreases as k increases. This is shown in Figure 3.7a. The relationship between the variances in case of prediction, filtering and smoothing is shown in Figure 3.7b.

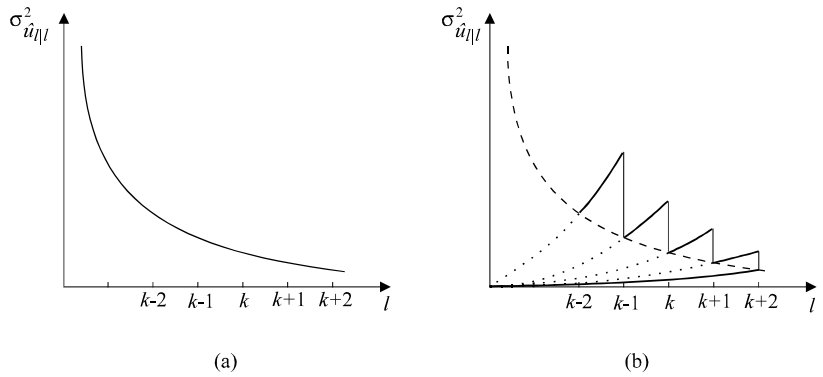


Figure 3.7: Variances in case of uniform sampling.

3.4 Recursive prediction and filtering: the A-form

In the previous section the concepts of prediction, filtering and smoothing were introduced. In this section we will develop the algorithm for recursive prediction and filtering. As was pointed out in the previous chapter, the essence of recursive estimation is that there is no need to store past measurements for the purpose of computing present least-squares estimates. Recursion enables one to keep track of the time process x_t by means of an efficient computation of the corresponding best estimates. Recursive estimation is therefore in particular of great importance for problems where it is required to estimate time-varying parameters in real-time. The central idea in our development of the theory of recursive prediction and filtering is to rely on the stepwise manner in which a partitioned model may be solved. As our starting point we take the linear model (28) parameterized in terms of x_{k-1}:

$$E\left\{\begin{pmatrix}\underline{y}_0\\ \vdots\\ \underline{y}_{k-2}\\ \underline{y}_{k-1}\\ \underline{y}_k\end{pmatrix}\right\} = \begin{pmatrix}\boldsymbol{A}_0\boldsymbol{\Phi}_{0,k-1}\\ \vdots\\ \boldsymbol{A}_{k-2}\boldsymbol{\Phi}_{k-2,k-1}\\ \boldsymbol{A}_{k-1}\\ \boldsymbol{A}_k\boldsymbol{\Phi}_{k,k-1}\end{pmatrix} x_{k-1} \quad ; \quad D\left\{\begin{pmatrix}\underline{y}_0\\ \vdots\\ \underline{y}_{k-2}\\ \underline{y}_{k-1}\\ \underline{y}_k\end{pmatrix}\right\} \begin{pmatrix}Q_0 & & & & \\ & \ddots & & 0 & \\ & & Q_{k-2} & & \\ & 0 & & Q_{k-1} & \\ & & & & Q_k\end{pmatrix}. \tag{41}$$

This model may be solved in two steps. In the first step we consider model (41) with the exception of $\underline{y}_k$. Its solution is then given by the filtered estimator $\underline{\hat{x}}_{k-1|k-1}$. The complete solution of model (41) then follows from solving the following model in a second step:

$$E\left\{\begin{pmatrix}\underline{\hat{x}}_{k-1|k-1}\\ \underline{y}_k\end{pmatrix}\right\} = \begin{pmatrix}I\\ \boldsymbol{A}_k\boldsymbol{\Phi}_{k,k-1}\end{pmatrix} x_{k-1} \quad ; \quad D\left\{\begin{pmatrix}\underline{\hat{x}}_{k-1|k-1}\\ \underline{y}_k\end{pmatrix}\right\} = \begin{pmatrix}Q_{\hat{x}_{k-1|k-1}} & O\\ O & Q_k\end{pmatrix}. \tag{42}$$

The solution of this model would give us $\underline{\hat{x}}_{k-1|\,k}$ in terms of the filtered estimator $\underline{\hat{x}}_{k-1|k-1}$. Instead however, we would like to obtain an expression where the filtered estimator $\underline{\hat{x}}_{k|k}$ is

expressed in terms of $\underline{\hat{x}}_{k-1|k-1}$. We therefore first parameterize model (42) in terms of x_k. Replacing x_{k-1} in (42) by $\Phi_{k-1,k}\, x_k$ gives:

$$(43) \qquad E\{\begin{pmatrix}\underline{\hat{x}}_{k-1|k-1}\\ \underline{y}_k\end{pmatrix}\} = \begin{pmatrix}\Phi_{k-1,k}\\ A_k\end{pmatrix} x_k \;\; ; \;\; D\{\begin{pmatrix}\underline{\hat{x}}_{k-1|k-1}\\ \underline{y}_k\end{pmatrix}\} = \begin{pmatrix}Q_{\hat{x}_{k-1|k-1}} & O\\ O & Q_k\end{pmatrix}.$$

This model may again be solved in two steps. In the first step we consider the partial model:

$$(44) \qquad E\{\underline{\hat{x}}_{k-1|k-1}\} = \Phi_{k-1,k}\, x_k \;\; ; \;\; D\{\underline{\hat{x}}_{k-1|k-1}\} = Q_{\hat{x}_{k-1|k-1}}.$$

Its solution would give us the predicted estimator $\underline{\hat{x}}_{k|k-1}$ in terms of the filtered estimator $\underline{\hat{x}}_{k-1|k-1}$. Note that, since the transition matrix $\Phi_{k-1,k}$ is square and invertible, the redundancy of the partial model (44) equals zero. The solution of (44) follows therefore simply from inverting the transition matrix:

$$(45) \qquad \boxed{\begin{aligned}\underline{\hat{x}}_{k|k-1} &= \Phi_{k,k-1}\underline{\hat{x}}_{k-1|k-1}\\ Q_{\hat{x}_{k|k-1}} &= \Phi_{k,k-1} Q_{\hat{x}_{k-1|k-1}} \Phi^*_{k,k-1}\end{aligned}}.$$

This result expresses the predicted estimator in terms of the filtered estimator. Both estimators are based on the same set of observables $\underline{y}_i$, $i = 0,\ldots,(k-1)$. They refer however to different time instances, namely t_k and t_{k-1} respectively. The equations of (45) will therefore be called the time-update equations. Note that the result (45) is in agreement with (29) in Section 3.3. The complete solution of model (43) follows now from solving in a second step the model:

$$(46) \qquad E\{\begin{pmatrix}\underline{\hat{x}}_{k|k-1}\\ \underline{y}_k\end{pmatrix}\} = \begin{pmatrix}I\\ A_k\end{pmatrix} x_k \;\; ; \;\; D\{\begin{pmatrix}\underline{\hat{x}}_{k|k-1}\\ \underline{y}_k\end{pmatrix}\} = \begin{pmatrix}Q_{\hat{x}_{k|k-1}} & O\\ O & Q_k\end{pmatrix}.$$

Its solution reads:

$$(47) \qquad \begin{cases}\underline{\hat{x}}_{k|k} &= (Q^{-1}_{\hat{x}_{k|k-1}} + A_k^* Q_k^{-1} A_k)^{-1}(Q^{-1}_{\hat{x}_{k|k-1}}\underline{\hat{x}}_{k|k-1} + A_k^* Q_k^{-1}\underline{y}_k)\\ Q_{\hat{x}_{k|k}} &= (Q^{-1}_{\hat{x}_{k|k-1}} + A_k^* Q_k^{-1} A_k)^{-1}\end{cases}.$$

Compare (46) and (47) with (19) and (18). Result (47) expresses the filtered estimator $\underline{\hat{x}}_{k|k}$ in terms of the predicted estimator $\underline{\hat{x}}_{k|k-1}$. It can be written in the more familiar form:

$$(48) \qquad \boxed{\begin{aligned}\hat{x}_{k|k} &= \hat{x}_{k|k-1} + Q_{\hat{x}_{k|k}} A_k^* Q_k^{-1}(\underline{y}_k - A_k\underline{\hat{x}}_{k|k-1})\\ Q_{\hat{x}_{k|k}} &= (Q^{-1}_{\hat{x}_{k|k-1}} + A_k^* Q_k^{-1} A_k)^{-1}\end{aligned}}.$$

These are the measurement-update equations. Compare (48) with (20) in Section 2.2 and note the striking resemblance. But also note that the measurement-update equations of (48) are not in a recursive form. This in contrast to the measurement-update equations of (20) in Secion 2.2. For the present time-varying case, it are the time-update equations (45) together with the measurement-update equations (48) that form a set of recursive equations. A flow diagram of the

above derived recursive prediction and filtering algorithm is shown in Table 3.1. Compare Table 3.1 with Table 2.1.

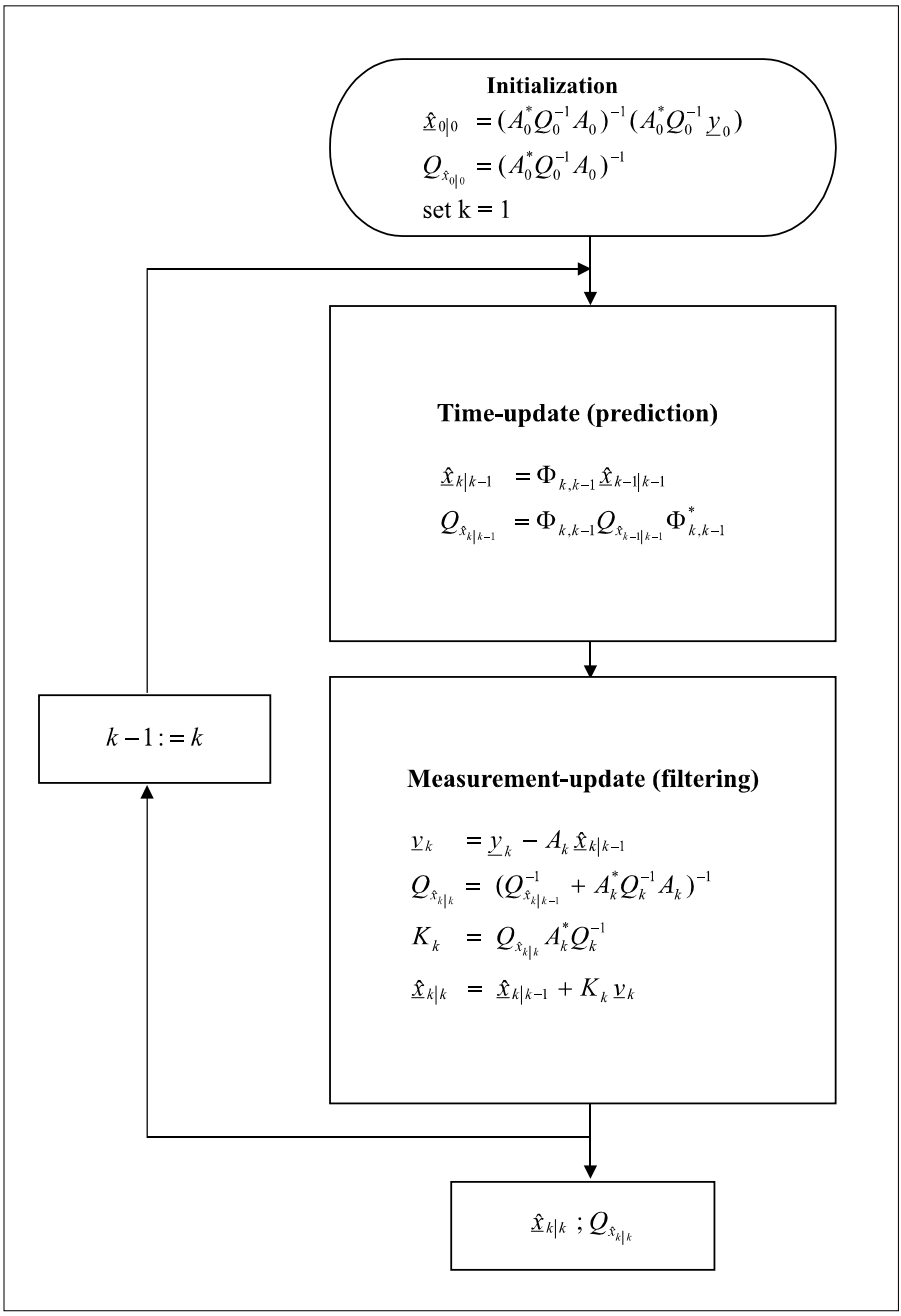


Table 3.1: Recursive prediction and filtering: the A-form.

3.5 Recursive prediction and filtering: the B-form

The B-form of the recursive prediction and filtering algorithm follows if we write model (46) in terms of condition-equations:

$$(-\boldsymbol{A}_k \; I)\; E\{\begin{pmatrix}\underline{\hat{x}}_{k|k-1}\\ \underline{y}_k\end{pmatrix}\} = 0 \quad ; \quad \boldsymbol{D}\{\begin{pmatrix}\underline{\hat{x}}_{k|k-1}\\ \underline{y}_k\end{pmatrix}\} = \begin{pmatrix}\boldsymbol{Q}_{\hat{x}_{k|k-1}} & 0\\ 0 & \boldsymbol{Q}_k\end{pmatrix}. \tag{49}$$

Solving this model in the standard way gives for the measurement-update equations:

$$\boxed{\begin{aligned}\underline{\hat{x}}_{k|k} &= \underline{\hat{x}}_{k|k-1} + \boldsymbol{Q}_{\hat{x}_{k|k-1}}\boldsymbol{A}_k^{*}\boldsymbol{Q}_{v_k}^{-1}[\underline{y}_k - \boldsymbol{A}_k\underline{\hat{x}}_{k|k-1}]\\ \boldsymbol{Q}_{\hat{x}_{k|k}} &= \boldsymbol{Q}_{\hat{x}_{k|k-1}} + \boldsymbol{Q}_{\hat{x}_{k|k-1}}\boldsymbol{A}_k^{*}\boldsymbol{Q}_{v_k}^{-1}\boldsymbol{A}_k\boldsymbol{Q}_{\hat{x}_{k|k-1}}\end{aligned}}\,. \tag{50}$$

in which $\boldsymbol{Q}_{v_k}$ is the variance matrix of the predicted residual $\underline{v}_k$:

$$\boldsymbol{Q}_{v_k} = \boldsymbol{Q}_k + \boldsymbol{A}_k\boldsymbol{Q}_{\underline{\hat{x}}_{k|k-1}}\boldsymbol{A}_k^{*}. \tag{51}$$

Compare (50) with (35) of Section 2.2. A flow diagram of the B-form of the recursive prediction and filtering algorithm is shown in Table 3.2. It should be noted that the time-update equations remain unchanged.

Example 28

We will start with Example 27 in Section 3.3 and consider model (35):

$$E\{\begin{pmatrix}\underline{u}_1\\ \underline{u}_2\\ \vdots\\ \underline{u}_k\end{pmatrix}\} = \begin{pmatrix}1 & (t_1 - t)\\ 1 & (t_2 - t)\\ \vdots & \vdots\\ 1 & (t_k - t)\end{pmatrix}\begin{pmatrix}u_t\\ \dot{u}_t\end{pmatrix} \quad ; \quad \boldsymbol{D}\{\begin{pmatrix}\underline{u}_1\\ \underline{u}_2\\ \vdots\\ \underline{u}_k\end{pmatrix}\} = \sigma^2 I_k\,. \tag{52}$$

Batch estimator

We will first consider the batch solution of (52). Note that the structure of model (52) is identical to that of model (42) of Section 2.2. Hence, the variance matrix of the estimator $\underline{\hat{x}}_{t|k} = (\underline{\hat{u}}_{t|k}\;\underline{\hat{u}}_{t|k})^{*}$ may be readily obtained from the result (49) in Section 2.2. With the definition:

$$t_{c_k} = \frac{1}{k}\sum_{i=1}^{k} t_i \tag{53}$$

this gives:

$$Q_{\hat{x}_{t|k}} = \begin{pmatrix} \sigma^2_{\hat{u}_{t|k}} & \sigma_{\hat{u}_{t|k},\hat{\dot{u}}_{t|k}} \\ \sigma_{\hat{u}_{t|k}\hat{\dot{u}}_{t|k}} & \sigma^2_{\hat{\dot{u}}_{t|k}} \end{pmatrix} = \begin{pmatrix} \frac{1}{k} + \frac{(t-t_{c_k})^2}{\sum_{i=1}^{k}(t_i-t_{c_k})^2} & \frac{t-t_{c_k}}{\sum_{i=1}^{k}(t_i-t_{c_k})^2} \\ \frac{t-t_{c_k}}{\sum_{i=1}^{k}(t_i-t_{c_k})^2} & \frac{1}{\sum_{i=1}^{k}(t_i-t_{c_k})^2} \end{pmatrix}.$$

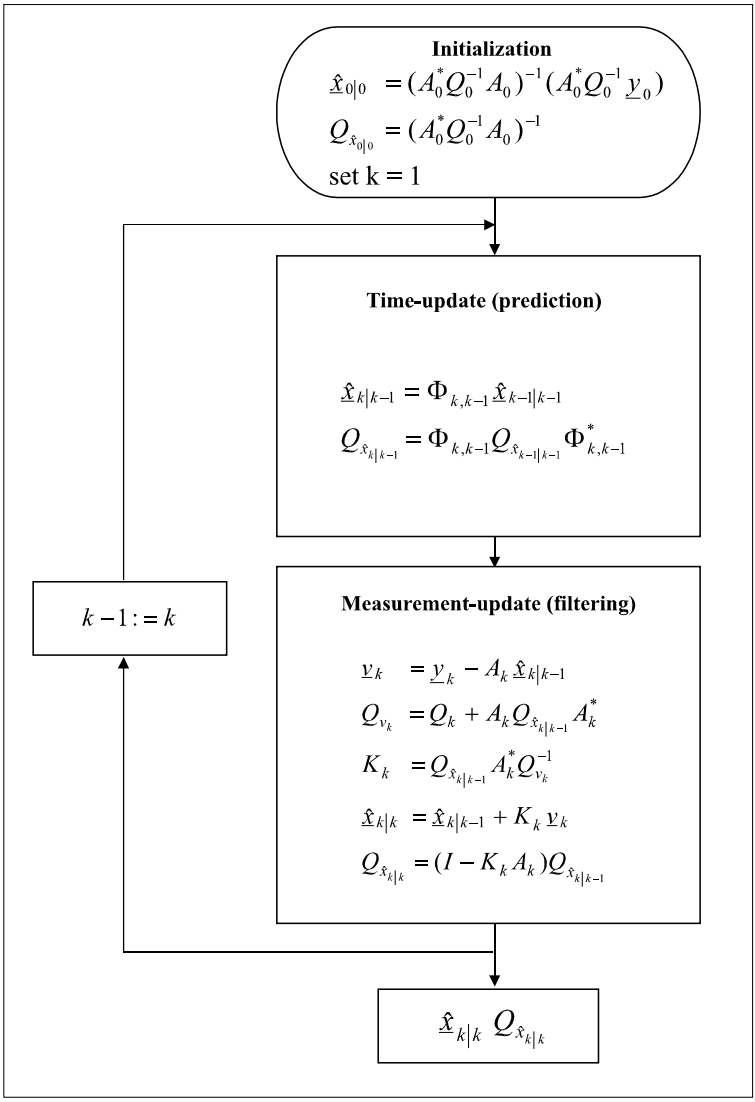


Table 3.2: Recursive prediction and filtering: the B form.

This shows that the variance of the position estimator $\underline{\hat{u}}_{\tau|k}$ reads:

$$\sigma^2_{\hat{u}_{t|k}} = \frac{1}{k} + \frac{(t - t_{c_k})^2}{\sum_{i=1}^{k} (t_i - t_{c_k})^2} . \tag{55}$$

Compare this result with (38) in Section 3.3. The following two inequalities hold:

$$\begin{cases} \sigma^2_{\hat{u}_{t'|k}} > \sigma_{\hat{u}_{t|k}} \text{ for } |t' - t_{c_k}| > |t - t_{c_k}| \\ \sigma^2_{\hat{u}_{t|k'}} < \sigma^2_{\hat{u}_{t|k}} \text{ for } k' > k \end{cases} . \tag{56}$$

The first inequality follows directly from (55), as this equation shows that the parabola $\sigma^2_{\hat{u}_{t|k}}$ attains its minimum of $1/k$ at $t = t_{c_k}$. The second inequality of (56) is, however, not so obvious when one considers expression (55). Still, it can be shown to hold true and the reason is of course that the precision of the estimator should improve when more observables are involved. In Figure 3.8 we have plotted the variance $\sigma^2_{\hat{u}_{t|k}}$ as function of time, for different values of *k*. The plot not only shows that $\sigma^2_{\hat{u}_{t|k}}$ attains its minimum of $1/k$ at $t = t_{c_k}$, but also that the parabola gets more flattened and that its minimum decreases in value, if more observables are involved, that is, if *k* increases. Compare Figure 3.8 with Figure 3.4 in Section 3.3.

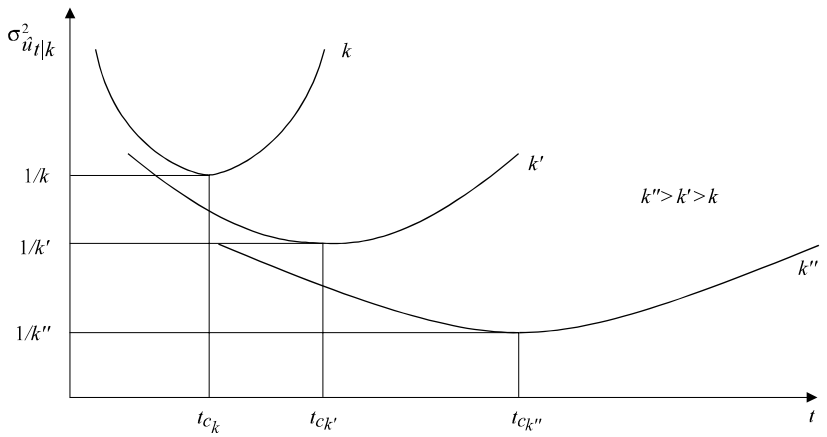


Figure 3.8: The parabola $\sigma^2_{\hat{u}_{t|k}}$ for different values of *k*.

Recursive prediction and filtering

The time-update of the estimator reads:

$$
(57)\qquad \begin{cases} \begin{pmatrix} \underline{\hat{u}}_{k\,|\,k-1} \\ \hat{\underline{u}}_{k\,|\,k-1} \end{pmatrix} = \begin{pmatrix} 1 & (t_k - t_{k-1}) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \underline{\hat{u}}_{k-1\,|\,k-1} \\ \hat{\underline{u}}_{k-1\,|\,k-1} \end{pmatrix} \\ Q_{\hat{x}_{k|k-1}} = \sigma^2 \begin{pmatrix} 1 & (t_k - t_{k-1}) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \dfrac{1}{k-1} + \dfrac{(t_{k-1} - t_{c_{k-1}})^2}{\sum_{i=1}^{k-1} (t_i - t_{c_{k-1}})^2} & \dfrac{t_{k-1} - t_{c_{k-1}}}{\sum_{i=1}^{k-1} (t_i - t_{c_{k-1}})^2} \\ \dfrac{t_{k-1} - t_{c_{k-1}}}{\sum_{i=1}^{k-1} (t_i - t_{c_{k-1}})^2} & \dfrac{1}{\sum_{i=1}^{k-1} (t_i - t_{c_{k-1}})^2} \end{pmatrix} \begin{pmatrix} 1 & (t_k - t_{k-1}) \\ 0 & 1 \end{pmatrix}^* \end{cases}
$$

For the measurement-update we may either use the A-form of (48) or the B-form of (50). From a numerical point of view it is more advantageous for the present example to use the B-form. However, since we already have the variance matrix (54) at our disposal, it is from an analytical point of view faster to use the A-form. The corresponding gain matrix $K_k = Q_{\hat{x}_{k|k}} A_k^* Q_k^{-1}$ follows then with (54) for $t = t_k$, $A_k = (1\ 0)$ and $Q_k = \sigma^2$ as:

$$
(58)\qquad K_k = \begin{pmatrix} \dfrac{1}{k} + \dfrac{(t_k - t_{c_k})^2}{\sum_{i=1}^{k} (t_i - t_{c_k})^2} \\ \dfrac{t_k - t_{c_k}}{\sum_{i=1}^{k} (t_i - t_{c_k})^2} \end{pmatrix}.
$$

The measurement-update of the estimator reads therefore:

$$
(59)\qquad \begin{cases} \begin{pmatrix} \underline{\hat{u}}_{k\,|\,k} \\ \hat{\underline{u}}_{k\,|\,k} \end{pmatrix} = \begin{pmatrix} \underline{\hat{u}}_{k|k-1} \\ \hat{\underline{u}}_{k|k-1} \end{pmatrix} + \begin{pmatrix} \dfrac{1}{k} + \dfrac{(t_k - t_{c_k})^2}{\sum_{i=1}^{k} (t_i - t_{c_k})^2} \\ \dfrac{t_k - t_{c_k}}{\sum_{i=1}^{k} (t_i - t_{c_k})^2} \end{pmatrix} (\underline{u}_k - (1\ 0) \begin{pmatrix} \underline{\hat{u}}_{k|k-1} \\ \hat{\underline{u}}_{k|k-1} \end{pmatrix}) \\ Q_{\hat{x}_{k|k}} = [Q_{\hat{x}_{k|k-1}}^{-1} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \sigma^{-2} (1\ 0)]^{-1} \end{cases}.
$$

The general precision characteristics of the predicted and filtered position estimator are shown in Figure 3.9. Compare Figure 3.9 with Figure 3.5.

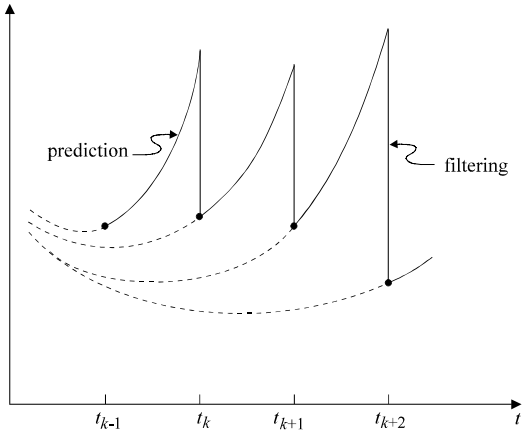


Figure 3.9: Variances of the predicted and filtered position estimator.

Uniform sampling

Let us now consider the case of uniform sampling. Then $t_i = t_0 + iT$ with T = constant. Using the identities:

(60)
$$\begin{cases} \sum_{i=1}^{k} i = \frac{1}{2}k(k+1) \\ \sum_{i=1}^{k} i^2 = \frac{1}{6}k(k+1)(2k+1) \end{cases}$$

it can be shown that in case of uniform sampling the variance matrix of (54) reduces to:

$$Q_{\hat{x}_{t|k}} = \sigma^2 \begin{pmatrix} \left(\frac{1}{k} + \frac{12\,(t-[t_0+\frac{1}{2}(k+1)T])^2}{k(k+1)(k-1)T^2}\right) & \frac{12\,(t-[t_0+\frac{1}{2}(k+1)T])}{k(k+1)(k-1)T^2} \\ \frac{12\,(t-[t_o+\frac{1}{2}(k+1)T])}{k(k+1)(k-1)T^2} & \frac{12}{k(k+1)(k-1)T^2} \end{pmatrix}.$$

Note that as a consequence of uniform sampling the minimum $(\frac{1}{2}(k+1)T, 1/k)$ of the parabola $\sigma^2_{\hat{u}_{t|k}}$ lies now in the centre of the measurement interval $[t_1 = T,\ t_k = kT]$.

The time-update equations read in case of uniform sampling as:

$$(62)\qquad \begin{cases} \begin{pmatrix} \underline{\hat{u}}_{k\,|\,k-1} \\ \underline{\hat{\dot{u}}}_{k\,|\,k-1} \end{pmatrix} = \begin{pmatrix} 1 & T \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \underline{\hat{u}}_{k-1\,|\,k-1} \\ \underline{\hat{\dot{u}}}_{k-1\,|\,k-1} \end{pmatrix} \\ Q_{\hat{x}_{k\,|\,k-1}} = \begin{pmatrix} 1 & T \\ 0 & 1 \end{pmatrix} \sigma^2 \begin{pmatrix} 2\dfrac{(2k-3)}{k(k-1)} & \dfrac{6}{k(k-1)T} \\ \dfrac{6}{k(k-1)T} & \dfrac{12}{k(k-1)(k-2)T^2} \end{pmatrix} \begin{pmatrix} 1 & T \\ 0 & 1 \end{pmatrix}^{*} . \end{cases}$$

And the corresponding measurement-update equations read:

$$(63)\qquad \begin{cases} \begin{pmatrix} \underline{\hat{u}}_{k|k} \\ \underline{\hat{\dot{u}}}_{k|k} \end{pmatrix} = \begin{pmatrix} \underline{\hat{u}}_{k|k-1} \\ \underline{\hat{\dot{u}}}_{k|k-1} \end{pmatrix} + \begin{pmatrix} \dfrac{2(2k-1)}{k(k+1)} \\ \dfrac{6}{k(k+1)T} \end{pmatrix} [\underline{u}_k - (1\ \ 0) \begin{pmatrix} \underline{\hat{u}}_{k|k-1} \\ \underline{\hat{\dot{u}}}_{k|k-1} \end{pmatrix}] \\ Q_{\hat{x}_{k|k}} = [Q^{-1}_{\hat{x}_{k|k-1}} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \sigma^{-2}\ (1\ \ 0)]^{-1} = \sigma^2 \begin{pmatrix} \dfrac{2(2k-1)}{k(k+1)} & \dfrac{6}{k(k+1)T} \\ \dfrac{6}{k(k+1)T} & \dfrac{12}{k(k+1)(k-1)T^2} \end{pmatrix} . \end{cases}$$

Note that the elements of the gain matrix reduce in value if the number of observables k increases. Also note that the variance of the filtered position estimator:

$$\sigma^2_{\hat{u}_{k|k}} = \sigma^2 2(2k-1)/k(k+1)$$

is a decreasing function of k. This implies that in case of uniform sampling the points $(t_k = kT, \sigma^2_{\hat{u}_{k|k}})$, $k = 1,2\ldots$, lie on a descending graph. The general precision characteristics of the predicted and filtered position estimator are in this case therefore as shown in Figure 3.10.

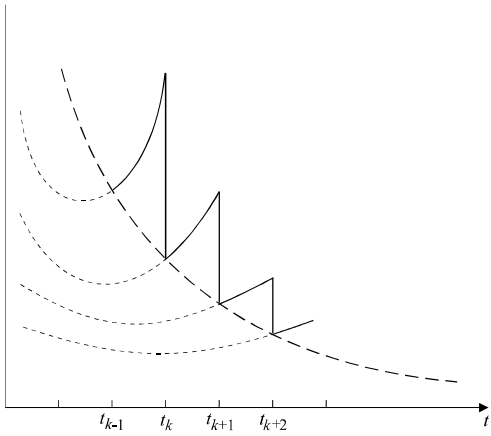


Figure 3.10: Variances of the predicted and filtered position estimator in case of uniform sampling.

Example 29

A satellite orbits the earth (see Figure 3.11). Position measurements are carried out for the determination of the satellite's position. The position observables $\underline{x}_i$, $\underline{y}_i, i = 1,...,k,$ are uncorrelated and all have the same variance σ^2. The model of observation equations parameterized in terms of the polar coordinates r_i and φ_i reads:

$$E\left\{\begin{pmatrix}\underline{x}_1\\ \underline{y}_1\\ \vdots\\ \underline{x}_k\\ \underline{y}_k\end{pmatrix}\right\} = \begin{pmatrix}r_1 \cos \varphi_1\\ r_1 \sin \varphi_1\\ \vdots\\ r_k \cos \varphi_k\\ r_k \sin \varphi_k\end{pmatrix} \quad ; \quad \boldsymbol{D}\left\{\begin{pmatrix}\underline{x}_1\\ \underline{y}_1\\ \vdots\\ \underline{x}_k\\ \underline{y}_k\end{pmatrix}\right\} = \sigma^2 I_{2k}. \tag{64}$$

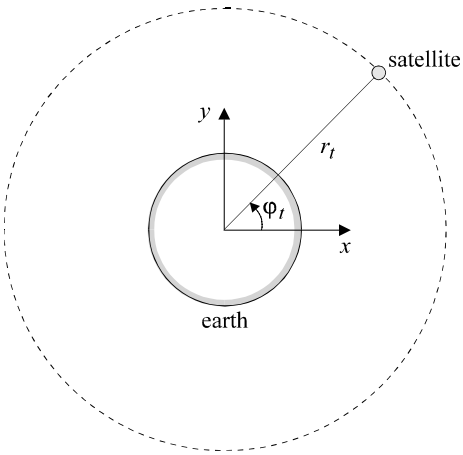


Figure 3.11: Earth orbiting satellite.

These observation equations are nonlinear. Linearization of (64) gives:

$$(65)\quad E\{\begin{pmatrix}\Delta \underline{x}_1\\ \Delta \underline{y}_1\\ \vdots\\ \Delta \underline{x}_k\\ \Delta \underline{y}_k\end{pmatrix}\} = \begin{pmatrix}\cos\varphi_1^0 & -r_1^0\sin\varphi_1^0 & & & \\ \sin\varphi_1^0 & r_1^0\cos\varphi_1^0 & & 0 & \\ & & \ddots & & \\ & & & \cos\varphi_k^0 & -r_k^0\sin\varphi_k^0\\ & 0 & & \sin\varphi_k^0 & r_k^0\cos\varphi_k^0\end{pmatrix}\begin{pmatrix}\Delta r_1\\ \Delta\varphi_1\\ \vdots\\ \Delta r_k\\ \Delta\varphi_k\end{pmatrix};\ D\{\begin{pmatrix}\Delta \underline{x}_1\\ \Delta \underline{y}_1\\ \vdots\\ \Delta \underline{x}_k\\ \Delta \underline{y}_k\end{pmatrix}\} = \sigma^2 I_{2k}.$$

Note that the redundancy of model (64) equals zero. It is assumed that the satellite orbit is a circle with unknown radius. It is furthermore assumed that the angular velocity of the satellite, $\dot{\varphi}_t$ is constant. The transition model of the satellite reads then:

$$(66)\quad \begin{pmatrix}r_t\\ \varphi_t\\ \dot{\varphi}_t\end{pmatrix} = \begin{pmatrix}1 & 0 & 0\\ 0 & 1 & (t-t_0)\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}r_0\\ \varphi_0\\ \dot{\varphi}_0\end{pmatrix}.$$

All three initial values r_0, φ_0 and $\dot{\varphi}_0$ are assumed to be unknown. In order to be able to combine (65) with (66) such that the combined model is parameterized in terms of the increments Δr_t, $\Delta\varphi_t$ and $\Delta\dot{\varphi}_t$, we first need to formulate the transition model with $(\Delta r_i, \Delta\varphi_i, \Delta\dot{\varphi}_i)^*$ expressed in terms of $(\Delta r_t, \Delta\varphi_t, \Delta\dot{\varphi}_t)^*$. This relation follows from (66) as:

$$(67)\quad \begin{pmatrix}\Delta r_i\\ \Delta\varphi_i\\ \Delta\dot{\varphi}_i\end{pmatrix} = \begin{pmatrix}1 & 0 & 0\\ 0 & 1 & (t_i-t)\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}\Delta r_t\\ \Delta\varphi_t\\ \Delta\dot{\varphi}_t\end{pmatrix}.$$

Substitution of (67) into (65) gives:

$$(68)\quad E\{\begin{pmatrix}\Delta \underline{x}_1\\ \Delta \underline{y}_1\\ \vdots\\ \Delta \underline{x}_k\\ \Delta \underline{y}_k\end{pmatrix}\} = \begin{pmatrix}\begin{pmatrix}\cos\varphi_1^0 & -r_1^0\sin\varphi_1^0 & 0\\ \sin\varphi_1^0 & r_1^0\cos\varphi_1^0 & 0\end{pmatrix}\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & (t_1-t)\\ 0 & 0 & 1\end{pmatrix}\\ \vdots\\ \begin{pmatrix}\cos\varphi_k^0 & -r_k^0\sin\varphi_k^0 & 0\\ \sin\varphi_k^0 & r_k^0\cos\varphi_k^0 & 0\end{pmatrix}\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & (t_k-t)\\ 0 & 0 & 1\end{pmatrix}\end{pmatrix}\begin{pmatrix}\Delta r_t\\ \Delta\varphi_t\\ \Delta\dot{\varphi}_t\end{pmatrix}.$$

At this point it is important to make some remarks about the approximate values in (68). Model (68) consists of only three unknown parameters, namely $\Delta r_t = r_t - r_t^0$, $\Delta\varphi_t = \varphi_t - \varphi_t^0$ and

$\Delta\dot{\varphi}_t = \dot{\varphi}_t - \dot{\varphi}_t^0$. This implies that all approximate values in (68) are functionally related to the approximate values of r_t, φ_t and $\dot{\varphi}_t$. Their relation follows from the transition model (66) as:

$$\begin{pmatrix} r_i^0 &= r_t^0 \\ \varphi_i^0 &= \varphi_t^0 + (t_i - t)\,\dot{\varphi}_t^0 \\ \dot{\varphi}_i^0 &= \dot{\varphi}_t^0 \end{pmatrix}. \tag{69}$$

Substitution of (69) into (68) gives:

$$E\left\{\begin{pmatrix} \Delta \underline{x}_1 \\ \Delta \underline{y}_1 \\ \vdots \\ \Delta \underline{x}_k \\ \Delta \underline{y}_k \end{pmatrix}\right\} = \begin{pmatrix} \cos(\varphi_t^0 + (t_1 - t)\dot{\varphi}_t^0) & -r_t^0 \sin(\varphi_t^0 + (t_1 - t)\dot{\varphi}_t^0) & -r_t^0 (t_1 - t)\sin(\varphi_t^0 + (t_1 - t)\dot{\varphi}_t^0) \\ \sin(\varphi_t^0 + (t_1 - t)\dot{\varphi}_t^0) & r_1^0 \cos(\varphi_t^0 + (t_1 - t)\dot{\varphi}_t^0) & r_t^0 (t_1 - t)\cos(\varphi_t^0 + (t_1 - t)\dot{\varphi}_t^0) \\ \vdots & \vdots & \vdots \\ \cos(\varphi_t^0 + (t_k - t)\dot{\varphi}_t^0) & -r_t^0 \sin(\varphi_t^0 + (t_k - t)\dot{\varphi}_t^0) & -r_t^0 (t_k - t)\sin(\varphi_t^0 + (t_k - t)\dot{\varphi}_t^0) \\ \sin(\varphi_t^0 + (t_k - t)\dot{\varphi}_t^0) & r_t^0 \cos(\varphi_t^0 + (t_k - t)\dot{\varphi}_t^0) & r_t^0 (t_k - t)\cos(\varphi_t^0 + (t_k - t)\dot{\varphi}_t^0) \end{pmatrix} \begin{pmatrix} \Delta r_t \\ \Delta \varphi_t \\ \Delta \dot{\varphi}_t \end{pmatrix}. \tag{70}$$

Note that the redundancy of this model equals $2k$-3.

Batch estimator

We will first consider the variance matrix of the batch estimator of (70). It reads:

$$Q_{\hat{x}_{t|k}} = \sigma^2 \begin{pmatrix} k & 0 & 0 \\ 0 & (r_t^o)^2 \begin{pmatrix} k & \sum_{i=1}^{k}(t_i - t) \\ \sum_{i=1}^{k}(t_i - t) & \sum_{i=1}^{k}(t_i - t)^2 \end{pmatrix} \end{pmatrix}^{-1}. \tag{71}$$

The 2x2 sub-matrix in (71) is a familiar one; see for instance (43) in Section 2.3. The variance matrix $Q_{\hat{x}_{t|k}}$ follows therefore readily as:

(72)
$$Q_{\hat{x}_{t|k}} = \sigma^2 \begin{pmatrix} \frac{1}{k} & 0 & 0 \\ 0 & \frac{1}{(r_t^0)^2} \begin{pmatrix} \frac{1}{k} + \frac{(t_c - t_{c_k})^2}{\sum_{i=1}^{k}(t_i - t_{c_k})^2} & \frac{t - t_{c_k}}{\sum_{i=1}^{k}(t_i - t_{c_k})^2} \\ \frac{t - t_{c_k}}{\sum_{i=1}^{k}(t_i - t_{c_k})^2} & \frac{1}{\sum_{i=1}^{k}(t_i - t_{c_k})^2} \end{pmatrix} \end{pmatrix}.$$

Recursive prediction and filtering

It is now rather straightforward to formulate the time-update and measurements-update equations. We therefore only give the gain matrix K_k. Since $K_k = Q_{\hat{x}_{k|k}} A_k^* Q_k^{-1}$, it follows with (72) for $t = t_k$ and

$$A_k = \begin{pmatrix} \cos\varphi_k^0 & -r_k^0 \sin\varphi_k^0 & 0 \\ \sin\varphi_k^0 & r_k^0 \cos\varphi_k^0 & 0 \end{pmatrix} \text{ and } Q_k = \sigma^2 I_2$$

that

(73)
$$K_k = \begin{pmatrix} \frac{\cos\varphi_k^0}{k} & \frac{\sin\varphi_k^0}{k} \\ \frac{-\sin\varphi_k^0}{r_k^0}\left(\frac{1}{k} + \frac{(t_k - t_{c_k})^2}{\sum_{i=1}^{k}(t_i - t_{c_k})^2}\right) & \frac{\cos\varphi_k^0}{r_k^0}\left(\frac{1}{k} + \frac{(t_k - t_{c_k})^2}{\sum_{i=1}^{k}(t_i - t_{c_k})^2}\right) \\ \frac{-\sin\varphi_k^0}{r_k^0}\left(\frac{t_k - t_{c_k}}{\sum_{i=1}^{k}(t_i - t_{c_k})^2}\right) & \frac{\cos\varphi_k^0}{r_k^0}\left(\frac{t_k - t_{c_k}}{\sum_{i=1}^{k}(t_i - t_{c_k})^2}\right) \end{pmatrix}.$$

Example 30

A ship sails along a straight line with constant velocity. Azimuth a_i and distance measurements l_i are carried out at successive time instances t_i, $i = 1,...,k$, from a known point with coordinates $u = 0$, $v = 0$, see Figure 3.12.

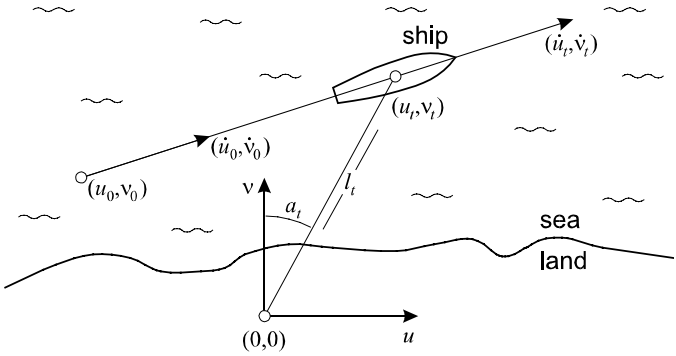


Figure 3.12: Positioning of a ship.

The observables $\underline{a}_i$ and $\underline{l}_i$ are uncorrelated and all have the same variance σ^2. The model of observation equations parameterized in terms of the cartesian coordinates u_i, v_i reads then:

$$E\begin{Bmatrix}\underline{a}_1\\ \underline{l}_1\\ \vdots\\ \underline{a}_k\\ \underline{l}_k\end{Bmatrix} = \begin{pmatrix}\arctan\,(u_1/v_1)\\ (u_1^2+v_1^2)^{1/2}\\ \vdots\\ \arctan\,(u_k/v_k)\\ (u_k^2+v_k^2)^{1/2}\end{pmatrix} \;;\; D\begin{Bmatrix}\underline{a}_1\\ \underline{l}_1\\ \vdots\\ \underline{a}_k\\ \underline{l}_k\end{Bmatrix} = \sigma^2 I_{2k}\,. \tag{74}$$

These observation equations are nonlinear. Linearization of (74) gives:

$$E\begin{Bmatrix}\Delta\underline{a}_1\\ \Delta\underline{l}_1\\ \vdots\\ \Delta\underline{a}_k\\ \Delta\underline{l}_k\end{Bmatrix} = \begin{pmatrix} v_1^0/l_1^{0^2} & -u_1^0/l_1^{0^2} & & & 0\\ u_1^0/l_1^0 & v_1^0/l_1^0 & & & \\ & & \ddots & & \\ 0 & & & v_k^0/l_k^{0^2} & -u_k^0/l_k^{0^2}\\ & & & u_k^0/l_k^0 & v_k^0/l_k^0\end{pmatrix}\begin{pmatrix}\Delta u_1\\ \Delta v_1\\ \vdots\\ \Delta u_k\\ \Delta v_k\end{pmatrix} \;;\; D\begin{Bmatrix}\Delta\underline{a}_1\\ \Delta\underline{l}_1\\ \vdots\\ \Delta\underline{a}_k\\ \Delta\underline{l}_k\end{Bmatrix} = \sigma^2 I_{2k}\,. \tag{75}$$

Note that the redundancy of this model equals zero. Let us now consider the transition model. Since it is assumed that the ship sails with constant velocity in both the *u*- and *v*-direction, the transition model reads:

(76)
$$\begin{pmatrix} u_t \\ v_t \\ \dot{u}_t \\ \dot{v}_t \end{pmatrix} = \begin{pmatrix} 1 & 0 & (t-t_0) & 0 \\ 0 & 1 & 0 & (t-t_0) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} u_0 \\ v_0 \\ \dot{u}_0 \\ \dot{v}_0 \end{pmatrix}.$$

All four initial values $u_0, v_0, \dot{u}_0$ and $\dot{v}_0$ are assumed to be unknown. In order to be able to combine (75) with (76) such that the combined model is parameterized in terms of the increments $\Delta u_t, \Delta v_t, \Delta \dot{u}_t$ and $\Delta \dot{v}_t$, we first need to formulate the transition model with $(\Delta u_i, \Delta v_i, \Delta \dot{u}_i, \Delta \dot{v}_i)^*$ expressed in terms of $(\Delta u_t, \Delta v_t, \Delta \dot{u}_t, \Delta \dot{v}_t)^*$. This relation follows from (76) as:

(77)
$$\begin{pmatrix} \Delta u_i \\ \Delta v_i \\ \Delta \dot{u}_i \\ \Delta \dot{v}_i \end{pmatrix} = \begin{pmatrix} 1 & 0 & (t_i-t) & 0 \\ 0 & 1 & 0 & (t_i-t) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \Delta u_t \\ \Delta v_t \\ \Delta \dot{u}_t \\ \Delta \dot{v}_t \end{pmatrix}.$$

Substitution of (77) into (75) gives:

$$E\{\begin{pmatrix} \Delta \underline{a}_1 \\ \Delta \underline{l}_1 \\ \vdots \\ \Delta \underline{a}_k \\ \Delta \underline{l}_k \end{pmatrix}\} = \begin{pmatrix} \begin{pmatrix} \frac{v_1^0}{l_1^{0^2}} & \frac{-u_1^0}{l_1^{0^2}} & 0 & 0 \\ \frac{u_1^0}{l_1^0} & \frac{v_1^0}{l_1^0} & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & (t_1-t) & 0 \\ 0 & 1 & 0 & (t_1-t) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\ \begin{pmatrix} \frac{v_k^0}{l_k^{0^2}} & \frac{-u_k^0}{l_k^{0^2}} & 0 & 0 \\ \frac{u_k^0}{l_k^0} & \frac{v_k^0}{l_k^0} & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & (t_k-t) & 0 \\ 0 & 1 & 0 & (t_k-t) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{pmatrix} \begin{pmatrix} \Delta u_t \\ \Delta v_t \\ \Delta \dot{u}_t \\ \Delta \dot{v}_t \end{pmatrix}.$$

or

$$(78)\qquad E\left\{\begin{pmatrix}\Delta \underline{a}_1\\ \Delta \underline{l}_1\\ \vdots\\ \Delta \underline{a}_k\\ \Delta \underline{l}_k\end{pmatrix}\right\} = \begin{pmatrix} \frac{v_1^0}{l_1^{0^2}} & \frac{-u_1^0}{l_1^{0^2}} & (t_1-t)\frac{v_1^0}{l_1^{0^2}} & -(t_1-t)\frac{u_1^0}{l_1^{0^2}}\\ \frac{u_1^0}{l_1^0} & \frac{v_1^0}{l_1^0} & (t_1-t)\frac{u_1^0}{l_1^0} & (t_1-t)\frac{v_1^0}{l_1^0}\\ \vdots & \vdots & \vdots & \vdots\\ \frac{v_k^0}{l_k^{0^2}} & \frac{-u_k^0}{l_k^{0^2}} & (t_k-t)\frac{v_k^0}{l_k^{0^2}} & -(t_k-t)\frac{u_k^0}{l_k^{0^2}}\\ \frac{u_k^0}{l_k^0} & \frac{v_k^0}{l_k^0} & (t_k-t)\frac{u_k^0}{l_k^0} & (t_k-t)\frac{u_k^0}{l_k^0} \end{pmatrix} \begin{pmatrix}\Delta u_t\\ \Delta v_t\\ \Delta \dot{u}_t\\ \Delta \dot{v}_t\end{pmatrix} \quad ; \quad \boldsymbol{D}\left\{\begin{pmatrix}\Delta \underline{a}_1\\ \Delta \underline{l}_1\\ \vdots\\ \Delta \underline{a}_k\\ \Delta \underline{l}_k\end{pmatrix}\right\} = \sigma^2 I_{2k}\,.$$

This is the basic model on which the recursive prediction and filtering of the ship's position at time t can be based. Note that the redundancy of model (78) equals $2k$-4. It should also be noted that the approximate values in (78) are functionally related to those of $u_t, v_t, \dot{u}_t$ and $\dot{v}_t$ as:

$$(79)\qquad \begin{cases} u_{\boldsymbol{i}}^0 = u_t^0 + (t_1 - t_{\boldsymbol{i}})\,\dot{u}_t^0\\ \dot{u}_{\boldsymbol{i}}^0 = \dot{u}_t^0\\ v_{\boldsymbol{i}}^0 = v_t^0 + (t_{\boldsymbol{i}} - t_{\boldsymbol{i}})\,\dot{v}_t^0\\ \dot{v}_{\boldsymbol{i}}^0 = \dot{v}_t^0 \end{cases}.$$

4 State-space models for dynamic systems

4.1 Introduction

In Chapter 3 a start was made with the development of the theory of recursive estimation of time-varying parameters $x(t)$. The development was based on a polynomial description of $x(t)$ which led to the transition model $x(t) = \Phi(t,t_0)x(t_0)$. Many dynamic systems exist, however, that cannot be described with such a simple linear transition model. In this chapter we will therefore have a closer look at a larger class of models that describe the time dependent behaviour of $x(t)$. The generalization from the simple polynomial model treated in the previous chapter to the dynamic models of the present chapter, becomes necessary if one wants to include a description of the forces that cause the time dependency of $x(t)$. The inclusion of forces in our model description implies that the reader needs to have some working knowledge of classical mechanics [Goldstein, 1980]. However, in order for the present book to be sufficiently self contained, we have included a separate section on the dynamics of motion. In that section the most elementary concepts of particle mechanics are introduced. The section on the dynamics of motion is preceded by Section 4.2 in which the kinematics of motion are discussed. Together with the material of the two Sections 4.2 and 4.3, the dynamic model can be formulated for a sufficiently large number of cases.

As it turns out these models consist in their most general form usually of nonlinear differential equations. In Sections 4.3 and following we will present ways of solving these models and show how their solution can be put into a form that is suitable for estimation purposes. In order to illustrate the theory we have again included a number of examples that are important from a geodetic point of view. Many of the examples are concerned with positioning and navigation.

4.2 Equations of motion: kinematics

The quantities required for the kinematic description of the motion of a particle are its position, velocity and acceleration. The form which the description of these vector quantities takes, depends on the coordinates in terms of which and the coordinate system with respect to which it is chosen to describe the motion of the particle. It should be clear that the choice of the coordinate system is quite arbitrary. In all the examples of the previous chapter the choice was made to describe motion with respect to a fixed coordinate system. Frequently, however, it is mathematically more convenient to employ the kinematic description of the motion of a particle with respect to a moving coordinate system. This is in particular the case when one wants to take care of the relation that exists between the motion of the particle, the mass of the particle and the forces acting on the particle. The fundamental laws of mechanics (see Section 4.3) are namely postulated with respect to an inertial coordinate system, which is a non-accelerating, non-rotating coordinate system. Hence, if one wants to take these laws into account and at the same time wants to describe the motion of a particle with respect to a coordinate system fixed for instance to the rotating and orbiting earth, a proper description of motion requires knowledge of the relationships that exist between coordinate systems which are in motion with respect to each other. The objective of this section is therefore to derive the relationships that exist between

position, velocity and acceleration of a particle, in coordinate systems which are in motion with respect to each other.

Position

Consider Figure 4.1. It shows two coordinate systems; the *i*-system and the *b*-system. The two systems or frames differ in their location and orientation. The cartesian coordinates of the origin 0_b of the *b*-frame with respect to the *i*-frame are denoted as $t_{i=1}$ and $t_{i=2}$. And the cartesian coordinates of point *P* when expressed in the *i*-frame or the *b*-frame are denoted as $r_{i=1}$, $r_{i=2}$ or $r_{b=1}$, $r_{b=2}$ respectively. From the geometry of Figure 4.1 it follows then that:

(1) $$\begin{pmatrix} r_{i=1} \\ r_{i=2} \end{pmatrix} = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} r_{b=1} \\ r_{b=2} \end{pmatrix} + \begin{pmatrix} t_{i=1} \\ t_{i=2} \end{pmatrix} \quad .$$

This expression shows how the coordinates of point *P* are related when expressed in the two different frames *i* and *b*. In order to transform the coordinates from the *b*-frame into the *i*-frame, one first needs to rotate the coordinates through an angle α and then add a translation vector to take care of the difference in origin of the two frames.

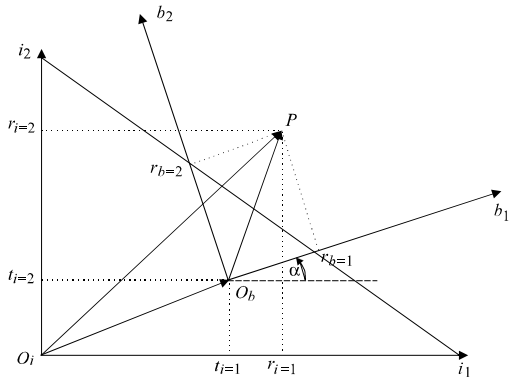


Figure 4.1: A coordinate transformation in two dimensions.

Expression (1) holds for the two-dimensional case. A similar expression can be derived for the three-dimensional case. It reads:

(2) $$\begin{pmatrix} r_{i=1} \\ r_{i=2} \\ r_{i=3} \end{pmatrix} = \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\beta & -\sin\beta \\ 0 & \sin\beta & \cos\beta \end{pmatrix} \begin{pmatrix} \cos\gamma & -\sin\gamma & 0 \\ \sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} r_{b=1} \\ r_{b=2} \\ r_{b=3} \end{pmatrix} + \begin{pmatrix} t_{i=1} \\ t_{i=2} \\ t_{i=3} \end{pmatrix} \quad .$$

The three angles of rotation α, β and γ of the three rotation matrices in (2) are defined in Figure 4.2. Note that if $\beta = \gamma = 0$, then transformation (2) reduces to that of (1) for the first two coordinates.

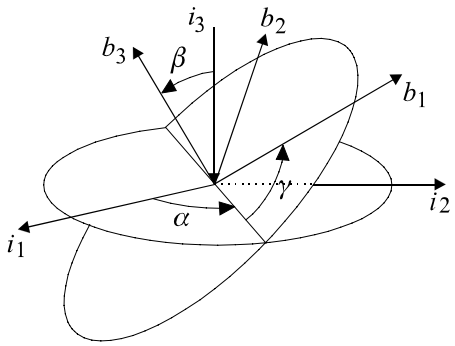


Figure 4.2: The rotation angles α, β, and γ defined.

Using a self-evident vector and matrix notation, expressions (1) and (2) can be written in the compact matrix-vector form:

(3) $$\boxed{r_i = R_{ib} r_b + t_i} \; .$$

In (3), r_i and r_b denote the vector of coordinates of point P when expressed in the i- and b-frame respectively, R_{ib} denotes the rotation matrix from the b-frame to the i-frame, and t_i denotes the translation vector expressed in the i-frame. Equation (3) thus relates the position of point P as seen by two observers fixed to the i- and b-frame respectively.

Velocity

We will assume that all quantities in (3) may depend on time. Hence, we not only assume that particle P may move with respect to the b-frame, but also that the b-frame itself may change its location and orientation with respect to the i-frame as function of time. The relation between the velocities of the particle as seen by the two observers fixed to the i- and b-frame respectively, follows then from taking the time-derivative of (3):

(4) $$\dot{r}_i = R_{ib}(R_{ib}^{-1}\dot{R}_{ib} r_b + \dot{r}_b) + \dot{t}_i \; .$$

The matrix $R_{ib}^{-1}\dot{R}_{ib}$ in (4) has a special significance in its own right. First of all it is easily demonstrated that this matrix is skew-symmetric:

(5) $$(R_{ib}^{-1}\dot{R}_{ib})^* = -(R_{ib}^{-1}\dot{R}_{ib}) \; .$$

This is shown as follows. First recall that R_{ib} is a rotation matrix and that the inverse of a rotation matrix equals its transpose: $R_{ib}^{-1} = R_{ib}^*$. Now, if we write $R_{ib}^{-1} R_{ib} = I$ as $R_{ib}^* R_{ib} = I$, and take the time derivative we get $\dot{R}_{ib}^* R_{ib} + R_{ib}^* \dot{R}_{ib} = 0$ or $(R_{ib}^* \dot{R}_{ib})^* = -R_{ib}^* \dot{R}_{ib}$. Replacing R_{ib}^* by R_{ib}^{-1} in this last expression proves (5). The matrix $R_{ib}^{-1}\dot{R}_{ib}$ will be denoted as Ω_b^{ib}:

(6) $$\Omega_b^{ib} = R_{ib}^{-1}\dot{R}_{ib}.$$

With this definition, expression (4) may be written as:

(7) $$\boxed{\dot{r}_i = R_{ib}(\Omega_b^{ib} r_b + \dot{r}_b) + \dot{t}_i} \; .$$

If the origin of the b-frame is stationary, then $\dot{t}_i = 0$ and v_b in $\dot{r}_i = R_{ib} v_b$ is the velocity of particle P referenced with respect to the i-frame, but expressed in the b-frame. It depends on $\dot{r}_b$, the velocity as seen by the observer fixed to the b-frame, and $\Omega_b^{ib} r_b$, the rate of change in orientation of the b-frame with respect to the i-frame. Because of the explicit form of the matrix Ω_b^{ib}, a coordinate vector ω_b^{ib} may be defined through:

(8) $$\Omega_b^{ib} = \begin{pmatrix} 0 & \omega_3^{ib} & \omega_2^{ib} \\ \omega_3^{ib} & 0 & -\omega_1^{ib} \\ -\omega_2^{ib} & \omega_1^{ib} & 0 \end{pmatrix} = \begin{pmatrix} \omega_1^{ib} \\ \omega_2^{ib} \\ \omega_3^{ib} \end{pmatrix}_b \times \; = \; \omega_b^{ib} \times$$

where "×" stands for the vector cross product. The meaning of the indices in Ω_b^{ib} will now be clear from (8). The lower index of Ω_b^{ib} indicates that the entries of matrix Ω_b^{ib} are expressed in the b-frame. And the two upper indices of Ω_b^{ib} indicate that the matrix describes the change in rotation from the b-frame to the i-frame. With (8), expression (7) may be written in the alternative form:

(9) $$\boxed{\dot{r}_i = R_{ib}(\omega_b^{ib} \times r_b + \dot{r}_b) + \dot{t}_i} \; .$$

The significance of introducing the coordinate vector ω_b^{ib} becomes clear if it is assumed that particle P is fixed to the b-frame and the origins O_i and O_b of the two frames coincide. Then $\dot{r}_b = 0$ and $t_i = 0$ and (9) reduces to:

(10) $$\dot{r}_i = R_{ib}(\omega_b^{ib} \times r_b) \; .$$

Through an application of the transformation rule:

(11) $$R_{ib}(\omega_b^{ib} \times r_b) = (R_{ib}\omega_b^{ib}) \times (R_{ib} r_b)$$

equation (10) can be written as:

(12) $$\dot{r}_i = \omega_i^{ib} \times r_i \; .$$

This result shows that the rate of change of a particle P fixed to a rotating b-frame with fixed origin, is seen by an observer fixed to the i-frame as an instantaneous rotation about the vector ω_i^{ib}, see Figure 4.3.

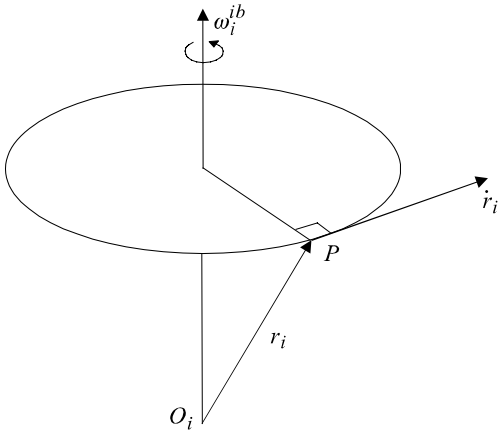


Figure 4.3: $\dot{r}_i = \omega_i^{ib} \times r_i$.

The vector ω_i^{ib} will therefore be referred to as the angular-velocity vector of the b-frame with respect to the i-frame expressed in the i-frame. The proof of the transformation rule (11) is based on the definition of a determinant of a 3×3 matrix and its invariance under rotations. We have for an arbitrary vector x_i:

$$
\begin{aligned}
x_i^* \boldsymbol{R}_{ib}(\omega_b^{ib} \times r_b) &= (\boldsymbol{R}_{ib}^* x_i)^* (\omega_b^{ib} \times r_b) \\
&= \det.(\boldsymbol{R}_{ib}^* x_i, \omega_b^{ib}, r_b) = \det.(\boldsymbol{R}_{ib}^* (x_i, \boldsymbol{R}_{ib}\omega_b^{ib}, \boldsymbol{R}_{ib} r_b)) \\
&= \det.(x_i, \boldsymbol{R}_{ib}\omega_b^{ib}, \boldsymbol{R}_{ib} r_b) \\
&= x_i^*((\boldsymbol{R}_{ib}\omega_b^{ib}) \times (\boldsymbol{R}_{ib} r_b)).
\end{aligned}
\tag{13}
$$

The second equality follows from the first by the definition of a determinant of a 3×3 matrix. The third equality follows from the second by taking the rotation matrix R_{ib}^* outside the brackets. The fourth equality follows from the third through the invariance of the determinant against rotations. Finally, the last equality is again a consequence of the definition of a determinant of a 3×3 matrix. Since (13) holds for any arbitrary vector x_i, transformation rule (11) must hold true. Transformation rule (11) can also be written for the skew-symmetric matrix Ω_b^{ib}. It reads (verify yourself):

$$
\Omega_i^{ib} = \boldsymbol{R}_{ib}\Omega_b^{ib}\boldsymbol{R}_{ib}^*. \tag{14}
$$

Acceleration

The relation between the accelerations as seen by the two observers fixed to the $i-$ and b-frame, respectively, follows from taking the time derivative of (7). Using (6) this gives:

$$
\boxed{\ddot{r}_i = \boldsymbol{R}_{ib}(\Omega_b^{ib}\Omega_b^{ib} r_b + \dot{\Omega}_b^{ib} r_b + 2\,\Omega_b^{ib}\dot{r}_b + \ddot{r}_b) + \ddot{t}_i}\ . \tag{15}
$$

With (8), equation (15) can be expressed alternatively as:

$$\boxed{\ddot{r}_i = \boldsymbol{R}_{ib}(\omega_b^{ib} \times (\omega_b^{ib} \times r_b) + \dot{\omega}_b^{ib} \times r_b + 2\omega_b^{ib} \times \dot{r}_b + \ddot{r}_b) + \ddot{t}_i} \quad . \tag{16}$$

If the particle P is fixed to the b-frame ($\dot{r}_b = 0$) and rotates about a fixed axis in space ($\dot{\omega}_b^{ib} = 0$) such that the origin of the b-frame remains stationary ($\dot{t}_i = 0$), then all terms, except the first, on the right-hand side of (15) vanish. The acceleration of the particle P is then directed towards the axis of rotation, see Figure 4.4.

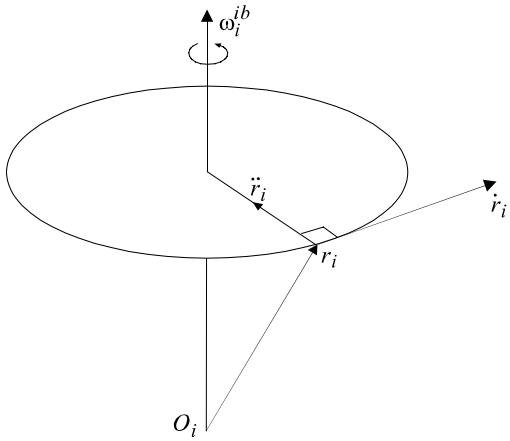


Figure 4.4: Centripetal acceleration.

The first term on the right-hand side of (15) is therefore called the *centripetal acceleration*, meaning towards the centre. The second term on the right-hand side of (15) is sometimes called the *tangential acceleration*, since it is tangential to the motion of particle P if the rotation is about a fixed axis with time-varying angular velocity. The third term is called *Coriolis acceleration*, after the French engineer C.G. Coriolis, who discovered it. It comes into play when particle P is moving with respect to the b-frame. Finally, $\ddot{r}_b$ is the acceleration of P as witnessed by an observer fixed to the b-frame and $\ddot{t}_i$ is the acceleration of the origin of the b-frame with respect to the i-frame. An overview of the existing relations between position, velocity and acceleration is given in Table 4.1.

Position	$r_i = R_{ib} r_b + t_i \; ; \; R_{ib}^{-1} = R_{ib}^* = R_{bi}$
Velocity	$\dot{r}_i = R_{ib}(\Omega_b^{ib} r_b + \dot{r}_b) + \dot{t}_i = R_{ib}(\omega_b^{ib} \times r_b + \dot{r}_b) + \dot{t}_i$ $\Omega_b^{ib} = R_{ib}^* \dot{R}_{ib} = \omega_b^{ib} \times$
Acceleration	$\ddot{r}_i = R_{ib}(\Omega_b^{ib}\Omega_b^{ib} r_b + \dot{\Omega}_b^{ib} r_b + 2\Omega_b^{ib}\dot{r}_b + \ddot{r}_b) + \ddot{t}_i$ $= R_{ib}(\omega_b^{ib} \times (\omega_b^{ib} \times r_b) + \dot{\omega}_b^{ib} \times r_b + 2\,\omega_b^{ib} \times \dot{r}_b + \ddot{r}_b) + \ddot{t}_i$
Transformation rule	$\Omega_i^{ib} = R_{ib}\Omega_b^{ib}R_{ib}^*$ $\omega_i^{ib} \times \cdot = R_{ib}(\omega_b^{ib} \times \cdot)$

Table 4.1: Position, velocity and acceleration.

Example 31

An observer *I* fixed to an *i*-frame monitors the motion of a particle P. Observer *I* comes to the conclusion that particle P moves with constant acceleration $\ddot{r}_i(t_0)$. The equation of motion of particle P for observer *I* reads then:

(17) $$r_i(t) = r_i(t_0) + \dot{r}_i(t_0)(t - t_0) + \tfrac{1}{2}\ddot{r}_i(t_0)(t - t_0)^2 \; .$$

A second observer *E* monitors the motion of the same particle P. Observer *E* is however fixed to an *e*-frame which itself is in motion with respect to the *i*-frame. Our objective is now to derive the equation of motion of particle *P* for observer *E*. The differential equation that governs the motion of particle P as seen by observer *E* follows readily from (15) as:

(18) $$\ddot{r}_e(t) + 2\Omega_e^{ie}(t)\dot{r}_e(t) + (\Omega_e^{ie}(t)\Omega_e^{ie}(t) + \dot{\Omega}_e^{ie}(t))\, r_e(t) = R_{ie}^*(t)\,(\ddot{r}_i(t_0) - \ddot{t}_i(t)) \; .$$

The solution of this second-order differential equation provides $r_e(t)$, the motion of the particle as seen by observer *E*. A block diagram of the computation steps involved in solving (18) is shown in Figure 4.5. It is not altogether obvious from equation (18) alone how the solution $r_e(t)$ looks like. But for the present case we do know how the solution looks like in the *i*-frame. The solution $r_e(t)$ of (18) follows therefore if we transform equation (17) into the *e*-frame. This gives with:

$$r_i = R_{ie} r_e + t_i \quad \text{and} \quad \dot{r}_i = R_{ie}(\Omega_e^{ie} r_e + \dot{r}_e) + \dot{t}_i$$

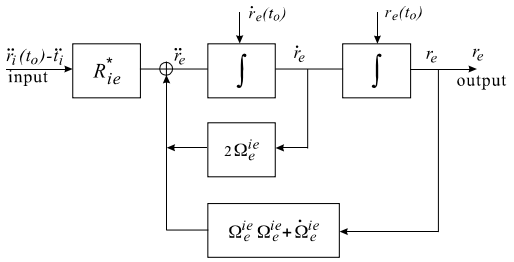


Figure 4.5: Block diagram of differential equation (18).

substituted into (17) the result:

$$
\begin{aligned}
r_e(t) \;=\; & \boldsymbol{R}^*_{ie}(t)\boldsymbol{R}_{ie}(t_0)\{[I+\Omega^{ie}_e(t_0)(t-t_0)]r_e(t_0)+\dot{r}_e(t_0)(t-t_0)\}+ \\
& +\boldsymbol{R}^*_{ie}(t)\{t_i(t_0)+\dot{t}_i(t_0)(t-t_0)+\tfrac{1}{2}\ddot{r}_i(t_0)(t-t_0)^2-t_i(t)\}.
\end{aligned}
\tag{19}
$$

Verify yourself that (19) is indeed the solution to (18). Let us now, in order to be more specific, consider a concrete example. We assume that the origins of the two frames coincide and that the *e*-frame rotates with respect to the *i*-frame with a constant angular velocity ω, see Figure 4.6. Then $t_i = 0$, and R_{ie} and Ω^{ie}_e read:

$$
\boldsymbol{R}_{ie} = \begin{pmatrix} \cos\omega t & -\sin\omega t \\ \sin\omega t & \cos\omega t \end{pmatrix}, \quad \Omega^{ie}_e = \boldsymbol{R}^*_{ie}\dot{\boldsymbol{R}}_{ie} = \begin{pmatrix} 0 & -\omega \\ \omega & 0 \end{pmatrix}.
\tag{20}
$$

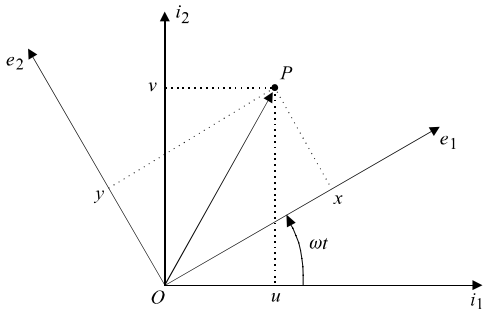


Figure 4.6: Rotating e-frame.

Hence, the differential equations (18) take the form:

$$\begin{cases} \ddot{x}(t) - 2\omega\dot{y}(t) - \omega^2 x(t) &= \ddot{u}(t_0)\cos\omega t + \ddot{v}(t_0)\sin\omega t \\ \ddot{y}(t) + 2\omega\dot{x}(t) - \omega^2 y(t) &= -\ddot{u}(t_0)\sin\omega t + \ddot{v}(t_0)\cos\omega t \end{cases} . \tag{21}$$

Their solution follows from (19) as:

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} \cos\omega(t-t_0) & \sin\omega(t-t_0) \\ -\sin\omega(t-t_0) & \cos\omega(t-t_0) \end{pmatrix} \left(\begin{pmatrix} 1 & -\omega(t-t_0) \\ \omega(t-t_0) & 1 \end{pmatrix} \begin{pmatrix} x(t_0) \\ y(t_0) \end{pmatrix} + \begin{pmatrix} \dot{x}(t_0) \\ \dot{y}(t_0) \end{pmatrix} (t-t_0) \right) + \\ + \frac{1}{2} \begin{pmatrix} \cos\omega t & \sin\omega t \\ -\sin\omega t & \cos\omega t \end{pmatrix} \begin{pmatrix} \ddot{u}(t_0) \\ \ddot{v}(t_0) \end{pmatrix} (t-t_0)^2 . \tag{22}$$

This is the equation of motion of particle P as seen by observer E. If the e-frame is fixed to the earth and lies in the equatorial plane with its origin at the earth's centre, then ω is the earth's rotation-rate. If we also assume that the gravitational field is uniform such that $\ddot{u}(t_0) = 0$ and $\ddot{v}(t_0) = -g$, then (22) is the equation of motion, for an earth-fixed observer, of an object that has been dropped from a building at the equator.

4.3 Equations of motion: dynamics

Kinematics, which is the study of the geometry of motion, was discussed in the previous section. Kinematics is used to relate position, velocity, acceleration and time, without reference to the cause of motion. Dynamics, on the other hand, is the study of the relation that exists between the forces acting on a body, the mass of the body, and the motion of the body. Dynamics is used to predict the motion caused by given forces or to determine the forces required to produce a given motion. In this section dynamics is discussed. In our presentation however, we will introduce only a few of the most elementary concepts. In order to gain a thorough introduction to the subject, the reader should consult [Goldstein, 1980].

Newton's formulation of classical mechanics given in the PRINCIPIA (1687)[1] forms the basis for the study of motion. Newton's three laws of motion can be stated in the following form:

Law I: Every particle remains in a state of rest or continues to move in a straight line with constant velocity unless compelled by external forces to change that state.

---

[1] "PHILOSOPHIAE NATURALIS PRINCIPIA MATHEMATICA", i.e. "Mathematical Principles of Natural Philosophy".

Law II: The acceleration of a particle is proportional to the resultant force acting on it and is in the direction of this force.

Law III: To every action there is always an equal and contrary reaction; or the forces of action and reaction between any two particles are equal in magnitude, opposite in direction, and collinear.

The first law, also known as the *law of inertia*, associates the motion of force with that of acceleration, and attaches significance to the state of zero acceleration. But before any statement can be made about the acceleration of a particle, a coordinate system or reference frame is required, since a particle may have zero acceleration relative to one set of axes, but not to another (see Section 4.2). The assumption behind the first law is therefore the existence of reference frames with respect to which every particle has a constant velocity vector when free from external forces. Such a frame is called inertial. Reference frames that accelerate or rotate with respect to such a frame are by this definition non-inertial. The axes of an inertial frame, though not arbitrary are also not unique, for axes moving with constant velocity relative to an inertial frame also constitute an inertial set. Although Newton's laws postulate the existence of inertial frames, they do not identify inertial frames and therefore do not explain how to choose such frames in practice. Here one must appeal to experience to decide what is reasonable. A particular reference frame can be considered a sufficient approximation to an inertial frame if the motion predicted by Newton's laws using this reference frame compares satisfactorily with observational data. A good approximation of an inertial frame of reference, and one from which Newton derived his concept, is an astronomic coordinate system with its origin in the bary-centre of the solar system and its axes fixed with respect to the distant galaxies. Thus an earth-fixed system of reference is by definition non-inertial in the Newtonian sense because the earth is rotating with respect to the distant galaxies and also has very small but measurable accelerations relative to an astronomic coordinate system.

Newton's second law, also known as the *law of acceleration*, is formulated for an inertial frame of reference. It defines force as being proportional to acceleration. The proportionality constant is the mass of the particle. Thus for a particle with mass m, subjected to a resultant force F_i the law may be stated as:

(23) $$\boxed{F_i = m\ddot{r}_i}$$

where both the force F_i and acceleration $\ddot{r}_i$ are expressed in an inertial frame i. Note that Newton's first law is a consequence of his second law, since there is no acceleration when the force is zero, and the particle is either at rest or is moving with constant velocity. Newton's second law yields three differential equations of second order, which when integrated, give the trajectory of the particle. However, for the integration to be useful, the forces acting on the particle need to be known. Therefore, before equation (23) can be used to make predictions, it is necessary to have further laws or measurement of force available which describe how forces vary when a particle is subject to the various mechanisms appearing in a given problem. There exist various forces in nature with their corresponding laws, such as for example contact forces, nuclear forces, electric and magnetic forces. Another law which is especially of importance for geodetic work, is Newton's universal law of gravitation. This law says that any two particles are

attracted to one another by a force that depends directly upon the product of their masses and inversely on the square of the distance separating them. Thus:

(24)
$$\boxed{F_i = \frac{kMm}{\|r_i\|^3} r_i}$$

where F_i is the force of attraction on mass m due to mass M, k is the universal gravitational constant, r_i is the vector separating M and m expressed in the i-frame and directed from m to M, and $\|r_i\|$ is the length of r_i. Newton arrived at his law of gravitation through a study of the three laws of planetary motion which were postulated by Kepler on the basis of careful astronomical observations of the Danish astronomer Tycho Brahe. Law (24), formulated for particles, forms the basis in the development of a model of the earth's gravitational field.

Newton's third law applies to *the interaction of two particles*. Note that Newton's universal law of gravitation (24) satisfies his third law. It should also be noted that Newton's first and second law define the concept of force in terms of mass times acceleration while Newton's second and third law define the ratio of masses of two interacting particles as the reciprocal ratio of their accelerations.

As was pointed out in Section 4.2, quite often the need arises in applications to introduce frames which are moving with respect to an inertial frame of reference. For instance, the description of motion of a point with respect to a coordinate system fixed to the earth involves quite naturally a coordinate system which is at the same time being translated and rotated in space. In order to see what form Newton's second law takes in a frame, say a b-frame, which is rotating and accelerating with respect to an inertial i-frame, we write equation (15) as:

$$\boldsymbol{R}_{bi}\ddot{r}_i - \Omega_b^{ib}\Omega_b^{ib} r_b - \dot{\Omega}_b^{ib} r_b - 2\Omega_b^{ib}\dot{r}_b - \boldsymbol{R}_{bi}\ddot{t}_i = \ddot{r}_b \ .$$

Together with Newton's second law (23) this gives:

(25)
$$\boxed{F_b + \sum_\alpha F_b^\alpha = m\ddot{r}_b}$$

where:

$$F_b = \boldsymbol{R}_{bi} F_i = \boldsymbol{R}_{bi}(m\ddot{r}_i)$$

$$\sum_\alpha F_b^\alpha = -m(\Omega_b^{ib}\Omega_b^{ib} r_b + \dot{\Omega}_b^{ib} r_b + 2\Omega_b^{ib}\dot{r}_b + \boldsymbol{R}_{bi}\ddot{t}_i) \ .$$

Note that (25) has the same form as Newton's original second law (23). The sum ΣF_b^α in (25) is called the sum of apparant forces or inertial forces, and it is a function of the motion of the selected b-frame with respect to the i-frame. It consists among other forces of the centripetal, tangential and Coriolis force. Equation (25) is the form of Newton's second law which is usually needed in practical applications.

Example 32

Consider a satellite orbiting planet earth. Both the earth and the satellite are assumed to be point masses with mass M and m respectively. It is also assumed that the centre of the earth is non-accelerating. It can therefore be taken as the origin of our inertial i-frame, see Figure 4.7.

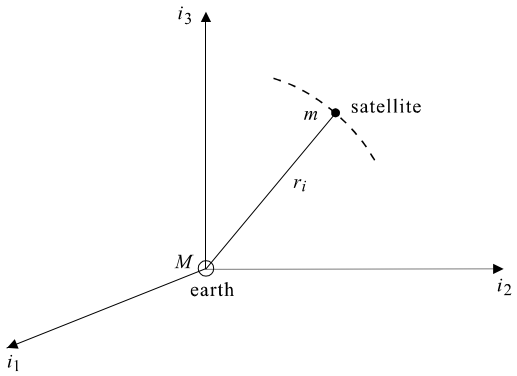


Figure 4.7: Earth orbiting satellite.

The satellite is assumed to be subject to two types of forces. The first force equals the gravitational force and reads:

$$F_i^1 = -\frac{kMm}{r^3} r_i \quad , \quad \text{with } r = \|r_i\| \ . \tag{26}$$

The second force is provided by the thrust of the satellite's rockets. It is denoted as F_i^2. According to Newton's second law we have:

$$F_i^1 + F_i^2 = m\ddot{r}_i \ .$$

Substitution of (26) and dividing by m, the mass of the satellite, gives:

$$\ddot{r}_i + \frac{kM}{r^3} r_i = f_i \tag{27}$$

where $f_i = F_i^2/m$ is the so-called specific force of the satellite's thrust. Equation (27) constitutes a set of three second-order nonlinear differential equations for the satellite's inertial cartesian coordinates. The nonlinearity is a consequence of the dependence of r on r_i. A block diagram of the computation steps involved in solving (27) is given in Figure 4.8. It shows how the inertial position vector r_i of the satellite, the output of the block diagram, can be computed from the input f_i.

The differential equations of (27) are expressed in inertial cartesian coordinates. We will now express them in a rotating and therefore non-inertial b-frame. For the sake of simplicity we will only consider the two-dimensional case. The b-frame is defined in Figure 4.9. In order to express (27) in the rotating b-frame we need the following result from Section 4.2:

(28)
$$\ddot{\boldsymbol{r}}_b = \boldsymbol{R}_{bi}(\ddot{\boldsymbol{r}}_i - \ddot{\boldsymbol{t}}_i) - \Omega_b^{ib}\Omega_b^{ib}\boldsymbol{r}_b - \dot{\Omega}_b^{ib}\boldsymbol{r}_b - 2\Omega_b^{ib}\dot{\boldsymbol{r}}_b \ .$$

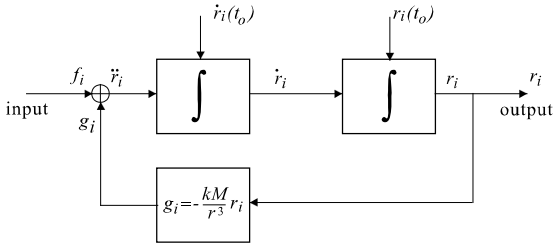


Figure 4.8 Block diagram of (27).

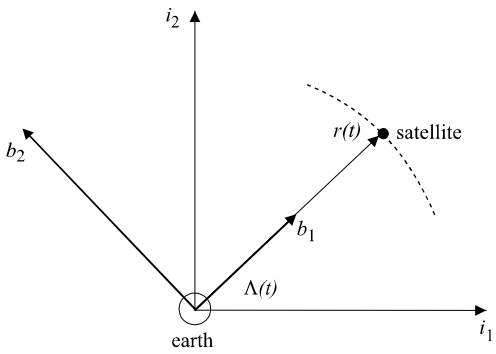


Figure 4.9 Definition of the *b*-frame.

Since:

(29)
$$\boldsymbol{R}_{ib} = \begin{pmatrix} \cos\Lambda & -\sin\Lambda \\ \sin\Lambda & \cos\Lambda \end{pmatrix}$$

we have:

(30)
$$\Omega_b^{ib} = \begin{pmatrix} 0 & -\dot{\Lambda} \\ \dot{\Lambda} & 0 \end{pmatrix} \quad \text{and} \quad \dot{\Omega}_b^{ib} = \begin{pmatrix} 0 & -\ddot{\Lambda} \\ \ddot{\Lambda} & 0 \end{pmatrix}$$

substitution of (29) and (30) into (28) gives together with $f_b = R_{bi} f_i$, $r_b = (r,0)^*$, $\ddot{t}_i = (0,0)^*$ and (27) the result:

(31)
$$\begin{cases} \ddot{r} + \dfrac{kM}{r^2} - r\dot{\Lambda} = f_{b=1} \\ \ddot{\Lambda} + 2\dfrac{\dot{r}\dot{\Lambda}}{r} = \dfrac{1}{r} f_{b=2} \end{cases} .$$

These two nonlinear second-order differential equations govern the motion of the satellite when expressed in the polar coordinates r and Λ.

Example 33

A linear mass-spring accelerometer is an instrument designed to measure the inertial result of translational motion. To analyze the performance of the instrument, consider Figure 4.10. This is a single-degree-of-freedom accelerometer since it has only one axis along which it is sensitive. Point C is the centre of mass of the sensitive element with mass m. Point O indicates the equilibrium position of the centre of mass of the sensitive element when there is no external force acting on the case along the sensitive axis.

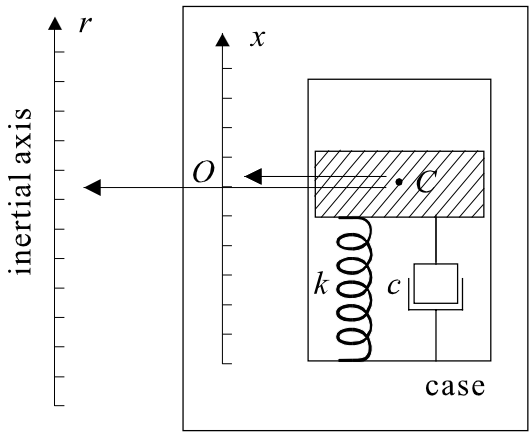


Figure 4.10: Linear mass-spring accelerometer.

Point O is fixed to the case. The output indication of the instrument, supplied by displacement-sensitive transducers, is made proportional to the displacement x of the sensitive element relative to the case. The sensitive element is supported by a spring which is attached to the case. The spring is assumed to be weightless and to develop a restoring force proportional to the deflection of the sensitive element from its equilibrium position. Thus if the spring has a stiffness coefficient of $k>0$, then by Hooke's law the restoring force is $-kx$. The sensitive element is also supported by a damper. Dampers are used to provide means of controlling the response of the instrument to dynamic inputs. The damper develops a viscous friction force $-c\dot{x}$, where $c>0$ is the damping coefficient.

Before developing the differential equation that governs the dynamics of the linear mass-spring accelerometer, we first consider what happens to the sensitive element under some different circumstances. Consider an accelerometer positioned on a flat platform such that its sensitive axis is parallel to the platform surface. If the platform is held horizontally and at rest, the system will be in equilibrium with zero indication of the accelerometer (see Figure 4.11a). Now if the case is tilted over an angle θ while the accelerometer remains switched on, it will indicate the

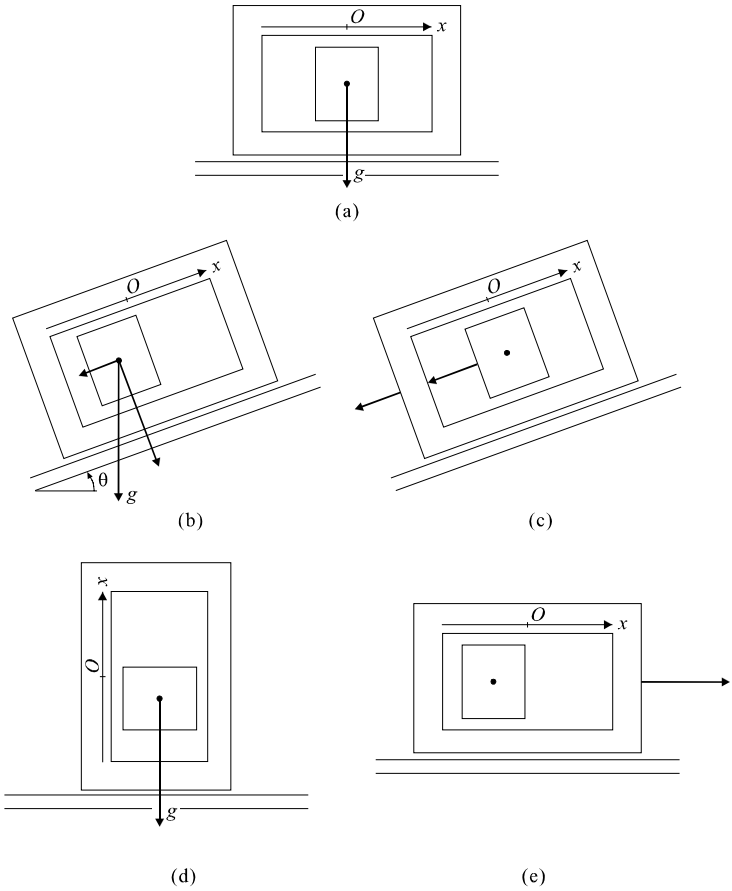


Figure 4.11: Accelerometer output.

component of gravity $g\sin\theta$, see Figure 4.11b. If the accelerometer is allowed to slide, in the absence of frictional contact forces, it will experience an acceleration of $g\sin\theta$ along the tilted platform, and indicate zero, see Figure 4.11c. This is due to the fact that the gravitational field influences simultaneously both the case and the sensitive element of the accelerometer and so by itself produces no deflection of the sensitive element. That is, the accelerometer does not measure any component of the acceleration due to the force of gravity unless an equal and opposite force is acting on its case. With friction the indication of the sliding accelerometer will be proportional to the friction force only. Now, if the accelerometer is placed on the platform so that its sensitive axis is vertical, mass attraction is acting along the sensitive axis of the accelerometer and pulls the sensitive element toward the platform until it is balanced by spring tension at which point the accelerometer indicates the value of gravity g, see Figure 4.11d. This output signal is, however, indistinguishable from the output of the same accelerometer when it is accelerated horizontally at a rate of g meters per seconds squared, see Figure 4.11e. The important conclusion is thus reached that an accelerometer cannot distinguish between inertial acceleration and gravitational acceleration. Hence, the effects of any component of gravity acting

along the accelerometer sensitive axis must be allowed for, if it is required to deduce the trajectory of the case.

To obtain the differential equation which governs the dynamic behaviour of the accelerometer, we apply Newton's second law to the sensitive element by equating all the forces acting on the element along the sensitive axis (spring force, friction force and gravity) to the product of its mass m and its acceleration relative to inertial space along the sensitive axis. Assuming that the x-axis is vertical and pointing upwards this gives:

$$-kx - c\dot{x} - m\boldsymbol{g} = m\ddot{\boldsymbol{r}} .$$

Substitution of $r = x + t$, where t is the position of point O on the case relative to inertial space, and dividing by m, gives:

$$\ddot{x} + \frac{c}{m}\dot{x} + \frac{k}{m}x = -(\boldsymbol{g} + \ddot{\boldsymbol{t}}) . \tag{32}$$

This equation shows that by measuring the displacement x and solving for the differential equation, the accelerometer determines the combination of the gravitational field intensity and the acceleration of the case relative to inertial space along the sensitive axis: $g + \ddot{t}$. Thus once g is known, $\ddot{t}$ is known and a double integration of $\ddot{t}$ enables us to compute the position t of the moving case. It is this principle which is used in inertial navigation.

4.4 State vector description of dynamic systems

In Section 4.3 we have seen how the appropriate equations of motion could be derived from Newton's laws of mechanics. Because of the nature of Newton's second law, these equations of motion consist of a set of scalar second-order differential equations. For the purpose of estimation it is, however, more convenient to transform these scalar second-order differential equations into a first-order vector form. The following two examples show how this can be done.

Example 34

According to (32) the differential equation representing a linear mass-spring system reads:

$$\ddot{x} + \frac{c}{m}\dot{x} + \frac{k}{m}x = f . \tag{33}$$

To put this scalar second-order differential equation into a first-order vector form, we define two variables x_1, and x_2 as:

$$\begin{cases} x_1 = x \\ x_2 = \dot{x} \end{cases} . \tag{34}$$

Then

(35) $$\dot{x}_1 = x_2$$

and substitution of (34) into (33) yields:

(36) $$\dot{x}_2 + \frac{c}{m}x_2 + \frac{k}{m}x_1 = f \ .$$

The two equations (35) and (36) can be arranged to give the first-order vector form:

(37) $$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -k/m & -c/m \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ f \end{pmatrix}.$$

Thus with (34) we have transformed the scalar second-order differential equation (33) into the first-order vector form (37). The vector $(x_1, x_2)^*$ is called state vector.

Example 35

According to (27) the equations of motion of an earth orbiting satellite expressed in inertial coordinates read:

(38) $$\begin{cases} \ddot{r}_1 + \dfrac{kM}{r^3}r_1 = f_1 \\ \ddot{r}_2 + \dfrac{kM}{r^3}r_2 = f_2 \\ \ddot{r}_3 + \dfrac{kM}{r^3}r_3 = f_3 \end{cases} .$$

To put these second-order differential equations into a first-order vector form, we define a 6-vector x as:

(39) $$x = (x_1,\ldots,x_6)^* = (r_1, r_2, r_3, \dot{r}_1, \dot{r}_2, \dot{r}_3)^* .$$

With this definition, the three second-order differential equations of (38) can be put into the following first-order vector form:

(40) $$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \\ \dot{x}_5 \\ \dot{x}_6 \end{pmatrix} = \begin{pmatrix} x_4 \\ x_5 \\ x_6 \\ -kMx_1/r^3 + f_1 \\ -kMx_2/r^3 + f_2 \\ -kMx_3/r^3 + f_3 \end{pmatrix}, \text{ with } r = (x_1^2 + x_2^2 + x_3^2)^{1/2} \ .$$

The device used in the above examples of transforming scalar differential equations involving second-order derivatives to first-order vector equations is an important one. This procedure is easily generalized to allow one to express an *n*th-order scalar differential equation by *n* first-order differential equations. Higher order scalar equations such as:

(41) $$u^{(n)}(t) = F(u(t), u^{(1)}(t), \ldots, u^{(n-1)}(t), z(t), t)$$

where $u^{(n)}(t) = d^n u(t)/dt^n$ can be put in a first-order vector form by letting $x_i(t) = u^{(i-1)}(t)$, $i = 1, \ldots, n$, to get:

(42) $$\begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \vdots \\ \dot{x}_n(t) \end{pmatrix} = \begin{pmatrix} x_2(t) \\ x_3(t) \\ \vdots \\ F(x_1(t), x_2(t), \ldots, x_n(t), z(t), t) \end{pmatrix}.$$

Clearly, this idea can be extended to simultaneous higher-order equations as well. The general form of a first-order vector differential equation is given by:

(43) $$\dot{x}(t) = f(x(t), z(t), t).$$

This equation will be referred to as the state equation. It is in a form which is general enough to describe the characteristics of many dynamic systems. The time-varying *n*-vector $x(t)$ will be referred to as the state vector of the system and the time-varying *l*-vector $z(t)$ indicates the input to the system. A block diagram of the state equation (43) is given in Figure 4.12.

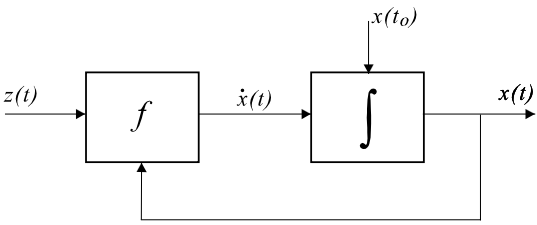


Figure 4.12: State equation (43).

It is implicit from the above representation that given the value of the state vector $x(t)$ at any time $t = t_0$, and given the time history of the input $z(\tau)$ over an interval of time $t_0 \leq \tau \leq t$, the solution of the state equation will yield $x(t)$. The state equations can be classified according to whether the vector function *f* is nonlinear or not, and whether it explicity depends on time or not. If the function *f* is nonlinear, one speaks of a nonlinear system. Equation (40) for example constitutes a nonlinear system. If the function *f* is linear and one or more parameters of the system vary with time, the system is called a linear, time-varying system. In this case the system equation takes the form:

(44) $$\boxed{\dot{x}(t) = F(t)x(t) + G(t)z(t)}$$

where $F(t)$ and $G(t)$ are time-varying matrices of appropriate dimensions. A matrix block diagram of a linear, time-varying system is given in Figure 4.13.

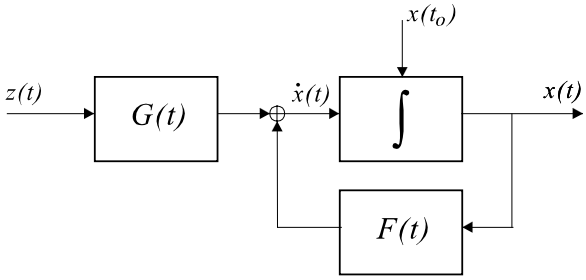


Figure 4.13: Matrix block diagram of (44).

If the system matrices F and G of (44) are constant, i.e. independent of time, the system is called a linear, time-invariant system:

(45) $$\boxed{\dot{x}(t) = Fx(t) + Gz(t)} \ .$$

Equation (37) for example constitutes a linear, time-invariant system.

For many systems the choice of the state vector follows naturally from the physical structure of the dynamic system. Similarly, the state equation usually follows directly from the physical laws that govern the system. For instance, in mechanical systems the state variables often correspond to position and velocity, and the inputs represent the action of several external forces. The precise nature of the differential equations is then deduced primarily through Newton's equations of motion. The following examples serve to demonstrate how a system may be placed in a state vector frame work. It should be pointed out, however, that the choice of the state variables is not unique. One may always transform a state vector $x(t)$ via a non-singular transformation into a new state vector $x'(t)$. The system of equations will then transform accordingly. It is clear that the transformed representation is completely equivalent to the original representation, since one can always reconstruct the behaviour of the dynamic system in terms of the original state by using the inverse transformation from $x'(t)$ to $x(t)$. This shows that the choice of the state can be adapted to suit various purposes.

Example 36

A mass m, considered a pointmass, is suspended from a massless rod of length l, as shown in Figure 4.14. The rod is connected to a rigid support at Q by means of a frictionless pinned arrangement. We shall assume the mass constrained so as to move in a plane. Consequently, only a single coordinate θ is required to describe completely the position of the point mass C at any time. There are two forces acting: the force of gravity F_i^1 pulling the point mass C toward the centre of the earth and the rod tension F_i^2 pulling the point mass toward the pin at Q. The force

of gravity will be assumed to be constant in magnitude and direction. A component of gravity serves as a restoring force, acting to return the point mass C to its lowest position at $\theta = 0$.

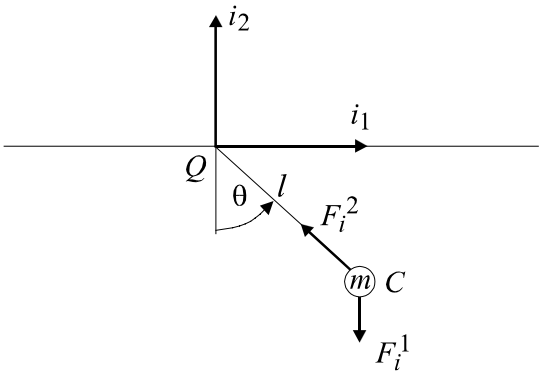


Figure 4.14: The mathematical pendulum.

According to Newton's second law we have:

(46) $$F_i^1 + F_i^2 = m\ddot{r}_i$$

with r_i the position of the point mass in the i-frame. Substitution of $r_i = (l\sin\theta, -l\cos\theta)^*$, $F_i^1 = (0, -mg)^*$ and $F_i^2 = (-F^2\sin\theta,\ F^2\cos\theta)^*$ into (46) gives, since l is constant:

$$\begin{pmatrix} 0 \\ -mg \end{pmatrix} + \begin{pmatrix} -F^2\sin\theta \\ F^2\cos\theta \end{pmatrix} = m \begin{pmatrix} -l\dot{\theta}^2\sin\theta + l\ddot{\theta}\cos\theta \\ l\dot{\theta}^2\cos\theta + l\ddot{\theta}\sin\theta \end{pmatrix}.$$

By eliminating the rod tension F^2, the equation of motion for the mathematical pendulum follows as:

(47) $$\ddot{\theta} + \frac{g}{l}\sin\theta = 0\ .$$

To put this equation into a state vector form, we define a two-dimensional state vector as $x = (\theta, \dot{\theta})^*$. The state vector form for (47) follows then as:

(48) $$\dot{x} = \begin{pmatrix} \dot{\theta} \\ \ddot{\theta} \end{pmatrix} = \begin{pmatrix} \dot{\theta} \\ \frac{-g}{l}\sin\theta \end{pmatrix}.$$

This is a zero-input nonlinear state equation.

Example 37

Figure 4.15 shows the meridian plane of a spherical non-rotating earth with radius R. A vehicle (car, ship, or airplane) is at time t located at position r,φ. A b-frame is attached to the vehicle. The vehicle is equipped with two single-degree-of-freedom accelerometers and one gyroscope. The sensitive axes of the two accelerometers are aligned with the b_1- and b_2-axis, respectively. They sense the resultant of acceleration along these two axes. The gyroscope senses the angular velocity of the b-frame with respect to the inertial i-frame.

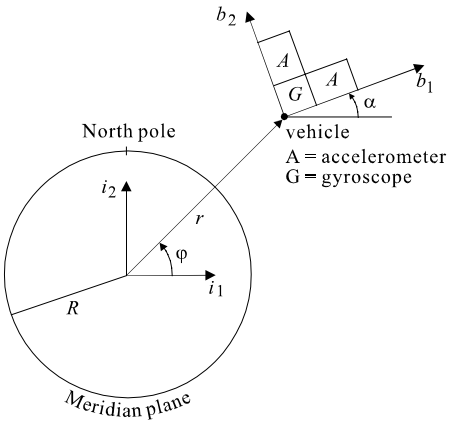


Figure 4.15: Spherical non-rotating earth with vehicle at position r,φ.

As we have seen in Example 33 of Section 4.3, accelerometers are not capable of separating vehicle acceleration $\ddot{r}_i$ from gravitational acceleration g_i. The sensed output of the accelerometers, denoted by f_i when expressed in the i-frame, is therefore equal to the difference of $\ddot{r}_i$ and g_i or:

(49) $$\ddot{\boldsymbol{r}}_{\boldsymbol{i}} = \boldsymbol{f}_{\boldsymbol{i}} + \boldsymbol{g}_{\boldsymbol{i}} \ .$$

A single integration of this equation gives the vehicle's velocity in the i-frame as:

(50) $$\dot{\boldsymbol{r}}_{\boldsymbol{i}}(t) = \dot{\boldsymbol{r}}_{\boldsymbol{i}}(t_0) + \int_{t_0}^{t} (\boldsymbol{f}_{\boldsymbol{i}}(\tau) + \boldsymbol{g}_{\boldsymbol{i}}(\tau))\,\boldsymbol{d}\tau \ .$$

And a second integration gives the vehicle's position in the i-frame as:

(51) $$\boldsymbol{r}_{\boldsymbol{i}}(t) = \boldsymbol{r}_{\boldsymbol{i}}(t_0) + \dot{\boldsymbol{r}}_{\boldsymbol{i}}(t_0)(t-t_0) + \int_{t_0}^{t}\int_{t_0}^{\tau} (\boldsymbol{f}_{\boldsymbol{i}}(\rho) + \boldsymbol{g}_{\boldsymbol{i}}(\rho))\,\boldsymbol{d}\rho\boldsymbol{d}\tau \ .$$

Thus positioning and navigation of the vehicle is possible once f_i, g_i, the initial position $r_i(t_0)$ and initial velocity $\dot{r}_i(t_0)$ are known. A model for the gravitational acceleration g_i is available if we assume that the earth can be considered to be a point mass with mass M. Then, according to Newton's universal law of gravitation:

(52) $$\boldsymbol{g_i} = -\frac{kM}{r^3}\, r_i.$$

In order to obtain f_i, we need to transform the sensed output of the accelerometers, f_b, to the inertial frame:

(53) $$f_i = \boldsymbol{R_{ib}}(\alpha) f_b$$

with

$$\boldsymbol{R_{ib}}(\alpha) = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}.$$

This shows that the angle α is needed for the transformation. Here is where the need of a gyroscope shows up. The gyroscope is capable of sensing the angular velocity of the *b*-frame with respect to the *i*-frame. If the output sensed by the gyroscope is denoted as ω^{ib}, then:

(54) $$\omega^{ib} = \dot{\alpha}\ .$$

Integration of (54) gives α, which when substituted into (53) enables us to compute f_i from the accelerometer output f_b. Summarizing, the appropriate set of differential equations for the equations of motion of the vehicle in inertial coordinates reads:

(55) $$\begin{cases} \ddot{r}_i + \dfrac{kM}{r^3} r_i = \boldsymbol{R_{ib}}(\alpha) f_b \\ \dot{\alpha} = \omega^{ib} \end{cases}.$$

The computations involved in solving (55) are shown in the block diagram of Figure 4.16. In order to put (55) into a first-order state vector form, we define the state vector x as $x = (r_1, r_2, \dot{r}_1, \dot{r}_2, \alpha)^*$. Then from (55) it follows that:

(56) $$\dot{x} = \begin{pmatrix} \dot{r}_1 \\ \dot{r}_2 \\ \dot{r}_1 \\ \dot{r}_2 \\ \alpha \end{pmatrix}^{\cdot} = \begin{pmatrix} \dot{r}_1 \\ \dot{r}_2 \\ -(kM/r^3) r_1 + f_1 \cos\alpha - f_2 \sin\alpha \\ -(kM/r^3) r_2 + f_1 \sin\alpha + f_2 \cos\alpha \\ \omega^{ib} \end{pmatrix}.$$

This is a nonlinear state-equation with input f_1, f_2 and ω^{ib}.

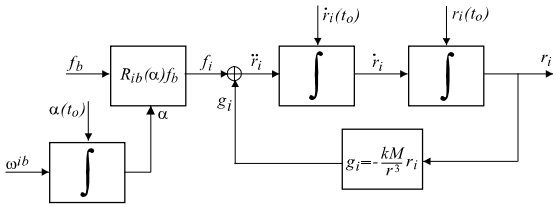


Figure 4.16: Block diagram of an inertial navigation system.

Example 38

In Example 32 the equations of motion of a satellite were expressed in the polar coordinates r and Λ (see (31) of Section 4.3). Assuming that the satellite orbit lies in the earth's equatorial plane, we will now express the satellite's equations of motion in terms of the geographic coordinates r (radius) and λ (longitude). This implies that we have to introduce an earth-fixed e-frame and take the constant angular velocity ω of the earth into account. The earth-fixed e-frame is defined in Figure 4.17a. Figure 4.17b shows the b-frame relative to the e-frame. The rotation matrices that relate the three frames, i, e and b, are given as:

$$\boldsymbol{R}_{ie} = \begin{pmatrix} \cos\omega t & -\sin\omega t \\ \sin\omega t & \cos\omega t \end{pmatrix} \quad \text{and} \quad \boldsymbol{R}_{eb} = \begin{pmatrix} \cos\lambda & -\sin\lambda \\ \sin\lambda & \cos\lambda \end{pmatrix}.$$

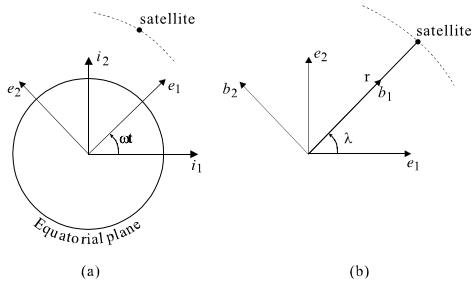


Figure 4.17: The earth-fixed e-frame.

From this it follows that:

$$\boldsymbol{R}_{ib} = \boldsymbol{R}_{ie}\boldsymbol{R}_{eb} = \begin{pmatrix} \cos(\lambda+\omega t) & -\sin(\lambda+\omega t) \\ \sin(\lambda+\omega t) & \cos(\lambda+\omega t) \end{pmatrix}. \tag{57}$$

A comparison of (57) with (29) shows that obviously $\Lambda = \lambda + \omega t$. Hence the equations of motion of the satellite in terms of the geographic coordinates r and λ, follow readily from (31) of Section 4.3 as:

(58)
$$\begin{cases} \ddot{r} + \dfrac{kM}{r^2} - r(\dot{\lambda} + \omega)^2 &= f_{b=1} \\ \ddot{\lambda} + 2\dfrac{\dot{r}(\dot{\lambda} + \omega)}{r} &= \dfrac{1}{r} f_{b=2} \end{cases} .$$

In order to put these equations into a first-order state vector form, we define the state vector x as $x = (r, \lambda, \dot{r}, \dot{\lambda})^*$. It follows then from (58) that:

(59)
$$\dot{x} = \begin{pmatrix} \dot{r} \\ \dot{\lambda} \\ \ddot{r} \\ \ddot{\lambda} \end{pmatrix} = \begin{pmatrix} \dot{r} \\ \dot{\lambda} \\ -kM/r^2 + r(\dot{\lambda} + \omega)^2 + f_1 \\ -2\dot{r}(\dot{\lambda} + \omega)/r + f_2/r \end{pmatrix} .$$

This is a nonlinear state equation with input f_1 and f_2.

4.5. Linearization of a nonlinear state equation

The main concern in the remaining sections of this chapter is to obtain solutions, analytical if possible, to the state equations of dynamic systems. For a nonlinear system, unfortunately, one cannot in general obtain an analytical closed-form formula which specifies $x(t)$ explicitly in terms of the initial state $x(t_0)$ and the input $z(t)$. This implies that the nonlinear state equation:

(60)
$$\dot{x}(t) = f(x(t), z(t), t)$$

has to be integrated numerically. For an overview of existing numerical integration techniques the reader is referred to e.g. [Stoer and Bulirsch, 1980].

If a solution of (60) is available (possibly after numerical integration) for an approximate initial state $x^0(t_0)$ and an approximate input $z^0(t)$, the actual solution of (60), $x(t)$, corresponding with the actual initial state $x(t_0)$ and actual input $z(t)$, can often be approximated to a sufficient degree by solving the linearized version of (60). In order to see how the linearized version of (60) can be derived, it is assumed that a solution $x^0(t)$ of the nonlinear state equation (60) is known for a given initial state $x^0(t_0)$ and input $z^0(t)$. Then:

(61)
$$\dot{x}^0(t) = f(x^0(t), z^0(t), t) .$$

The actual state $x(t)$ and actual input $z(t)$ can be written in terms of the approximate state $x^0(t)$ and approximate input $z^0(t)$ as:

(62)
$$\begin{cases} x(t) &= x^0(t) + \Delta x(t) \\ z(t) &= z^0(t) + \Delta z(t) \end{cases} .$$

Substitution of (62) into (60) gives:

$$\dot{x}^0(t) + \Delta\dot{x}(t) = f(x^0(t) + \Delta x(t),\ z^0(t) + \Delta z(t),\ t)\ .$$

If it is assumed that the vector function f is sufficiently smooth, a Taylor series expansion of f about $x^0(t)$ and $z^0(t)$ gives:

$$\begin{aligned}\dot{x}^0(t) + \Delta\dot{x}(t) &= f(x^0(t), z^0(t), t) + \partial_x f(x^0(t), z^0(t), t)\Delta x(t) + \\ &+ \partial_z f(x^0(t), z^0(t), t)\Delta z(t) + \text{higher order terms}\ .\end{aligned} \tag{63}$$

Here $\partial_x f(x^0(t), z^0(t), t)$ is the matrix of partial derivatives of f with respect to x evaluated at $x^0(t)$ and $z^0(t)$. Thus $\partial_x f$ is a matrix the (α,β)th element of which is:

$$[\partial_x f\ ]_{\alpha\beta} = \frac{\partial_\alpha f}{\partial x_\beta}$$

where f_α is the αth component of f and x_β is the βth component of x. The matrix $\partial_z f$ is similarly defined. From (61) and (63) it follows, after neglecting the higher order terms in (63), that:

$$\boxed{\Delta\dot{x}(t) = \partial_x f(x^0(t), z^0(t), t)\,\Delta x(t) + \partial_z f(x^0(t), z^0(t), t)\Delta z(t)}\ . \tag{64}$$

This is the linearized version of (60). Note that (64) represents a linear, time-varying system, with the state vector $\Delta x(t)$, the input vector $\Delta z(t)$ and the system matrices $\partial_x f(x^0(t), z^0(t), t)$ and $\partial_z f(x^0(t), z^0(t), t)$. Whether one is allowed to neglect the higher order terms in (63) depends on the smallness of $\Delta x(t)$ and $\Delta z(t)$ in relation to the nonlinearity of the vector function f. The following examples serve to demonstrate the linearization process.

Example 39

We will again consider the mathematical pendulum of Example 36. According to (48) of Section 4.4, its nonlinear state equation reads:

$$\begin{pmatrix}\dot{\theta}\\ \dot{\dot{\theta}}\end{pmatrix} = \begin{pmatrix}\dot{\theta}\\ -\dfrac{g}{l}\sin\theta\end{pmatrix}\ . \tag{65}$$

It is assumed that the angular displacement $\theta(t)$ and angular velocity $\dot{\theta}(t)$ of the pendulum remain small. A reasonable approximation to $\theta(t)$ and $\dot{\theta}(t)$ is then:

$$\begin{pmatrix}\theta^0(t)\\ \dot{\theta}^0(t)\end{pmatrix} = \begin{pmatrix}0\\ 0\end{pmatrix}\ . \tag{66}$$

Linearization of (65) about the approximate state (66) gives:

$$\begin{pmatrix} \Delta\theta \\ \Delta\dot{\theta} \end{pmatrix}^{\cdot} = \begin{pmatrix} 0 & 1 \\ -\frac{g}{l} & 0 \end{pmatrix} \begin{pmatrix} \Delta\theta \\ \Delta\dot{\theta} \end{pmatrix} . \tag{67}$$

This is a linear, time-invariant state equation with zero input. It describes the small angle behaviour of the mathematical pendulum.

Example 40

The nonlinear state equation for a satellite orbiting the earth in the equatorial plane reads in geographic coordinates r and λ (see (59) of Section 4.4):

$$\begin{pmatrix} r \\ \lambda \\ \dot{r} \\ \dot{\lambda} \end{pmatrix}^{\cdot} = \begin{pmatrix} \dot{r} \\ \dot{\lambda} \\ -kM/r^2 + r(\dot{\lambda} + \omega)^2 + f_1 \\ -2\dot{r}(\dot{\lambda} + \omega)/r + f_2/r \end{pmatrix} . \tag{68}$$

Linearization of this nonlinear state equation gives:

$$\begin{aligned} \begin{pmatrix} \Delta r \\ \Delta\lambda \\ \Delta\dot{r} \\ \Delta\dot{\lambda} \end{pmatrix}^{\cdot} = & \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{2kM}{r^0(t)^3} + (\dot{\lambda}^0(t) + \omega)^2 & 0 & 0 & 2r^0(t)(\dot{\lambda}^0(t) + \omega) \\ \frac{2\dot{r}^0(t)(\dot{\lambda}^0(t) + \omega) - f_2^0(t)}{r^0(t)^2} & 0 & \frac{-2(\dot{\lambda}^0(t) + \omega)}{r^0(t)} & \frac{-2\dot{r}^0(t)}{r^0(t)} \end{pmatrix} \begin{pmatrix} \Delta r \\ \Delta\lambda \\ \Delta\dot{r} \\ \Delta\dot{\lambda} \end{pmatrix} + \\ & + \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & \frac{1}{r^0(t)} \end{pmatrix} \begin{pmatrix} \Delta f_1 \\ \Delta f_2 \end{pmatrix} . \end{aligned} \tag{69}$$

This is a linear, time-varying state equation. Now assume that the satellite orbits the earth such that to a first approximation:

$$r^0(t) = r^0 = \text{constant and } \lambda^0(t) = \dot{\lambda}^0 t \text{ with } \dot{\lambda}^0 \text{ constant} . \tag{70}$$

Then

$$\dot{r}^0(t) = 0 \text{ and } \dot{\lambda}^0(t) = \dot{\lambda}^0 . \tag{71}$$

The with (70) and (71) corresponding input f_1^0 and f_2^0 reads then according to (68) as:

(72) $$f_1^0 = kM/(r^0)^2 - r^0(\dot{\lambda}^0 + \omega)^2 \quad \text{and} \quad f_2^0 = 0 \ .$$

Substitution of (70), (71) and (72) into (69) gives:

(73) $$\begin{pmatrix} \Delta r \\ \Delta\lambda \\ \Delta\dot{r} \\ \Delta\dot{\lambda} \end{pmatrix}^{\cdot} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{2kM}{(r^0)^3} + (\dot{\lambda}^0 + \omega)^2 & 0 & 0 & 2r^0(\dot{\lambda}^0 + \omega) \\ 0 & 0 & \frac{-2(\dot{\lambda}^0 + \omega)}{r^0} & 0 \end{pmatrix} \begin{pmatrix} \Delta r \\ \Delta\lambda \\ \Delta\dot{r} \\ \Delta\dot{\lambda} \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & \frac{1}{r^0} \end{pmatrix} \begin{pmatrix} \Delta f_1 \\ \Delta f_2 \end{pmatrix}.$$

This is a linear, time-invariant state equation. Thus if it is assumed that to a first approximation no radial displacement takes place and that the time rate of change of longitude is constant, the time-varying state equation (69) reduces to the time-invariant state equation (73).

Example 41

Consider the nonlinear state equation (56) of Section 4.4. It is expressed in terms of the inertial coordinates r_1 and r_2. Since we assume the earth to be non-rotating in the present example, we may use $r_1 = r\cos\varphi$ and $r_2 = r\sin\varphi$ to transform the state equation (56) such that it is expressed in the geographic coordinates r (radius) and φ (latitude). This gives:

(74) $$\begin{pmatrix} r \\ \varphi \\ \dot{r} \\ \dot{\varphi} \\ \alpha \end{pmatrix}^{\cdot} = \begin{pmatrix} \dot{r} \\ \dot{\varphi} \\ -kM/r^2 + r\dot{\varphi}^2 + f_1 c_{\varphi-\alpha} + f_2 s_{\varphi-\alpha} \\ -2\dot{r}\dot{\varphi}/r - f_1 s_{\varphi-\alpha}/r + f_2 c_{\varphi-\alpha}/r \\ \omega^{ib} \end{pmatrix}$$

where we have used the abbreviations $c_{\varphi-\alpha}$ and $s_{\varphi-\alpha}$ for $\cos(\varphi-\alpha)$ and $\sin(\varphi-\alpha)$, respectively. Linearization of the nonlinear state equation (74) gives:

$$
(75)\qquad
\begin{pmatrix} \Delta r \\ \Delta\varphi \\ \Delta\dot{r} \\ \Delta\dot{\varphi} \\ \Delta\alpha \end{pmatrix}^{\cdot}
=
\begin{pmatrix}
0 & 0 & 1 \\
0 & 0 & 0 \\
\dfrac{kM}{r^0(t)^3} + \dot{\varphi}^0(t)^2 & -f_1^0(t)s_{\varphi^0-\alpha^0} + f_2^0(t)c_{\varphi^0-\alpha^0} & 0 \\
\dfrac{2\dot{r}^0(t)\dot{\varphi}^0(t) + f_1^0(t)s_{\varphi^0-x^0} - f_2^0(t)c_{\varphi^0-\alpha^0}}{r^0(t)^2} & \dfrac{-f_1^0(t)c_{\varphi^0-\alpha^0} - f_2^{0,}(t)s_{\varphi^0-\alpha^0}}{r^0(t)} & \dfrac{-2\dot{\varphi}^0(t)}{r^0(t)} \\
0 & 0 & 0
\end{pmatrix}
$$

$$
\begin{pmatrix}
0 & 0 \\
1 & 0 \\
2r^0(t)\dot{\varphi}^0(t) & f_1^0(t)s_{\varphi^0-\alpha^0} - f_2^0(t)c_{\varphi^0-\alpha^0} \\
\dfrac{-2\dot{r}^0(t)}{r^0(t)} & \dfrac{f_1^0(t)c_{\varphi^0-\alpha^0} + f_2^0(t)s_{\varphi^0-\alpha^0}}{r^0(t)} \\
0 & 0
\end{pmatrix}
\begin{pmatrix} \Delta r \\ \Delta\varphi \\ \Delta\dot{r} \\ \Delta\dot{\varphi} \\ \Delta\alpha \end{pmatrix}
+
\begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
c_{\varphi^0-\alpha^0} & s_{\varphi^0-\alpha^0} & 0 \\
\dfrac{-s_{\varphi^0-\alpha^0}}{r^0(t)} & \dfrac{c_{\varphi^0-\alpha^0}}{r^0(t)} & 0 \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix} \Delta f_1 \\ \Delta f_2 \\ \Delta\omega^{ib} \end{pmatrix}.
$$

This is a linear, time-varying system. Now assume that the motion of the vehicle is such that:

$$
(76)\qquad r^0(t) = r^0 = \text{constant and } \varphi^0(t) = \dot{\varphi}^0 t, \text{ with } \dot{\varphi}^0 = \text{constant}\ .
$$

Then

$$
(77)\qquad \dot{r}^0(t) = 0,\quad \dot{\varphi}^0(t) = \dot{\varphi}^0,\quad f_1^0(t) = kM/(r^0)^2 - r^0(\dot{\varphi}^0)^2 \quad \text{and} \quad f_2^0(t) = 0\ .
$$

If we also assume that the orientation of the *b*-frame is such that to a first approximation:

$$
(78)\qquad \alpha^0(t) = \varphi^0(t)
$$

then

$$
(79)\qquad (\omega^{ib})^0(t) = \dot{\varphi}^0\ .
$$

With (76), (77), (78) and (79), the linear, time-varying state equation (75) reduces to the following linear, time-invariant state equation:

$$\begin{pmatrix} \Delta r \\ \Delta\varphi \\ \Delta\dot{r} \\ \Delta\dot{\varphi} \\ \Delta\alpha \end{pmatrix}^{\cdot} = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \frac{kM}{(r^0)^3} + (\dot{\varphi}^0)^2 & 0 & 0 & 2r^0\dot{\varphi}^0 & 0 \\ 0 & \frac{-kM}{(r^0)^3} + (\dot{\varphi}^0)^2 & \frac{-2\dot{\varphi}^0}{r^0} & 0 & \frac{kM}{(r^0)^3} - (\dot{\varphi}^0)^2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \Delta r \\ \Delta\varphi \\ \Delta\dot{r} \\ \Delta\dot{\varphi} \\ \Delta\alpha \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & \frac{1}{r^0} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \Delta f_1 \\ \Delta f_2 \\ \Delta\omega^{ib} \end{pmatrix}. \tag{80}$$

As was pointed out earlier, the choice of the state variables is not unique. One may always transform one state vector $x(t)$ via a non-singular transformation into a new state vector $x'(t)$. The state equations will then transform accordingly. In order to demonstrate this for the present example, we define new state variables as:

$$\begin{pmatrix} \Delta r \\ r^0\Delta\varphi \\ \Delta\dot{r} \\ r^0\Delta\dot{\varphi} \\ r^0\Delta\varepsilon \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & r^0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & r^0 & 0 \\ 0 & r^0 & 0 & 0 & -r^0 \end{pmatrix} \begin{pmatrix} \Delta r \\ \Delta\varphi \\ \Delta\dot{r} \\ \Delta\dot{\varphi} \\ \Delta\alpha \end{pmatrix}. \tag{81}$$

In terms of these new state variables, the state equation (80) reads:

$$\begin{pmatrix} \Delta r \\ r^0\Delta\varphi \\ \Delta\dot{r} \\ r^0\Delta\dot{\varphi} \\ r^0\Delta\varepsilon \end{pmatrix}^{\cdot} = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \frac{kM}{(r^0)^3} + (\dot{\varphi}^0)^2 & 0 & 0 & 2\dot{\varphi}^0 & 0 \\ 0 & 0 & -2\dot{\varphi}^0 & 0 & \frac{-kM}{(r^0)^3} + (\dot{\varphi}^0)^2 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \Delta r \\ r^0\Delta\varphi \\ \Delta\dot{r} \\ r^0\Delta\dot{\varphi} \\ r^0\Delta\varepsilon \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -r^0 \end{pmatrix} \begin{pmatrix} \Delta f_1 \\ \Delta f_2 \\ \Delta\omega^{ib} \end{pmatrix}. \tag{82}$$

Note that $\varepsilon = \varphi - \alpha$ is the misalignment angle between the b_1-axis of the b-frame and the local zenith (see Figure 4.15 of Section 4.4). By rearranging the order of the state variables such that the vertical and horizontal coordinates are separated, the state equation (82) may also be written as:

$$\begin{pmatrix} \Delta\dot{r} \\ \Delta r \\ \hline r^0\Delta\dot{\varphi} \\ r^0\Delta\varphi \\ r^0\Delta\varepsilon \end{pmatrix} = \left(\begin{array}{cc|ccc} 0 & 1 & 0 & 0 & 0 \\ \frac{kM}{(r^0)^3}+(\dot{\varphi}^0)^2 & 0 & 0 & 2\dot{\varphi}^0 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 \\ 0 & -2\dot{\varphi}^0 & 0 & 0 & \frac{-kM}{(r^0)^3}+(\dot{\varphi}^0)^2 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right) \begin{pmatrix} \Delta\dot{r} \\ \Delta r \\ \hline r^0\Delta\dot{\varphi} \\ r^0\Delta\varphi \\ r^0\Delta\varepsilon \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ \hline 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -r^0 \end{pmatrix} \begin{pmatrix} \Delta f_1 \\ \Delta f_2 \\ \Delta\omega^{ib} \end{pmatrix}$$

(83)

This shows that the coupling between the vertical and horizontal coordinates is provided by the term $2\dot{\varphi}^0$. The vertical and horizontal coordinates are decoupled if one assumes that the vehicle is approximately stationary, that is, if $\dot{\varphi}^0 = 0$. Now assume that the vehicle moves along the spherical surface of the earth (this is of course a stronger assumption then only $r^0(t) = \text{constant}$). Then $r(t) = R = \text{constant}$ and the first two equations of (83) vanish. As a result we get the state equation:

$$\begin{pmatrix} R\Delta\dot{\varphi} \\ R\Delta\varphi \\ R\Delta\varepsilon \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & \frac{-kM}{R^3}+(\dot{\varphi}^0)^2 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} R\Delta\dot{\varphi} \\ R\Delta\varphi \\ R\Delta\varepsilon \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & -R \end{pmatrix} \begin{pmatrix} \Delta f_2 \\ \Delta\omega^{ib} \end{pmatrix}. \tag{84}$$

Integration of this state equation gives the horizontal position and velocity of the vehicle as function of time. Note, since f_1 is absent, that in this case only one single-degree-of-freedom accelerometer is needed.

4.6 Linear time-varying state equations

In this section the solution and the properties of an n-dimensional linear, time-varying system represented by the state equation:

$$\boxed{\dot{x}(t) = F(t)x(t) + G(t)z(t)} \tag{85}$$

will be examined. First, attention is turned to the homogeneous part of the state equation which is obtained by setting the input $z(t)$ to zero: $z(t) = 0$. The homogeneous equation is studied first because its solution provides the solution for the general state equation (85). In other words, the solution to the homogeneous state equation is the difficult problem. As a start two distinct questions about the homogeneous equation $\dot{x}(t) = F(t)x(t)$ require attention:

(i) Given an initial state vector $x(t_0)$ at a time t_0, does there exist a solution that passes through $x(t_0)$ at time t_0?

(ii) If there exists a solution passing through $x(t_0)$ at time t_0, is it unique?

The following theorem states the condition under which a unique solution to the homogeneous equation can be shown to exist.

Theorem 1

If $F(t)$ is a matrix the elements of which are continuous functions of time in the interval $t_0 \le t \le t_1$ then there exists a solution of $\dot{x}(t) = F(t)x(t)$ which is defined in the interval $t_0 \le t \le t_1$ and takes on the value of $x(t_0)$ at $t = t_0$. Moreover, this solution is unique.

For a proof of this theorem the reader is referred to [Decarlo, 1989]. With the above existence and uniqueness theorem one can now show that the set of solutions of the homogeneous equation constitutes a linear vector space of a dimension equal to that of the underlying state space.

Theorem 2

Let $F(t)$ be an $n \times n$ matrix the elements of which are continuous functions of time t. Then the set of solutions of the homogeneous state equation:

$$\dot{x}(t) = F(t)x(t) \tag{86}$$

forms an n-dimensional linear vector space.

Proof

It is clear from Theorem 1 that the set of solutions of (86) is not empty. It is also clear by linearity of the time derivative and by linearity of matrix $F(t)$, that the set of solutions is closed under linear combinations. The set of solutions constitutes therefore a linear vector space. To determine the dimension of this vector space, it will first be shown that if the columns of the $n \times n$ matrix $X(t)$ satisfying the matrix initial value problem:

$$\dot{X}(t) = F(t)X(t), \quad X(t_0) = X_0 \tag{87}$$

are linearly independent at time t_0, they are linear independent for all times t. Suppose that some linear combination of the columns of matrix $X(t)$ is the zero function so that $X(t)a = 0$ for all t. Evaluation at t_0 yields $X(t_0)a = 0$. Hence, $a = 0$ if the columns of matrix $X(t_0)$ are linearly independent. But this implies, since $X(t)a = 0$ for all t, that the columns of matrix $X(t)$ are linearly independent for all t. The conclusion reads thus that if an $n \times n$ matrix $X(t)$ is nonsingular for some time t_0 and satisfies the homogeneous equation $\dot{X}(t) = F(t)X(t)$, it is nonsingular for all t. Any $n \times n$ matrix $X(t)$ which satisfies $\dot{X}(t) = F(t)X(t)$ and also has linearly independent columns at time t_0 is called a fundamental matrix of (86). Thus the columns of a fundamental matrix form a basis of $\mathbb{R}^n$ for all times t. To show that the columns of a fundamental matrix span the set of solutions of (86), let $x(t)$ be any solution of (86). Since $x(t_0) \in \mathbb{R}^n$ and the columns of $X(t_0)$ are linearly independent, a vector $b \in \mathbb{R}^n$ exists such that $x(t_0) = X(t_0)b$. Now consider $x'(t) = X(t)b$. It is a linear combination of solutions of (86) and thus itself a solution of (86). Its value at t_0 is $x(t_0)$. By uniqueness it follows then that $x'(t) = x(t)$. Hence, any solution of the homogeneous equation (86) can be written as a linear combination

of the n linearly independent columns of a fundamental matrix. This proves that the dimension of the linear vector space of solutions of (86) equals n.

End of proof.

The concept of a fundamental matrix can be used to formulate the solution of the homogeneous equation in terms of it. Let $X(t)$ be any fundamental matrix of (86). Since $X(t)$ satisfies (87) and is nonsingular for any t_0, the matrix $\Phi(t,t_0)$ defined by:

$$\Phi(t,t_0) = X(t)X(t_0)^{-1} \quad \forall\ t,t_0 \tag{88}$$

exists and satisfies:

$$\frac{\partial}{\partial t}\Phi(t,t_0) = F(t)\,\Phi(t,t_0) \tag{89}$$

and

$$\Phi(t_0,t_0) = I \tag{90}$$

for all t_0. This special fundamental matrix $\Phi(t,t_0)$ is called the state transition matrix of (86). Since the ith column of $\Phi(t,t_0)$ is a solution of the initial value problem $\dot{x}(t) = F(t)x(t)$, $x(t_0) = (0..0\ 1\ 0..0)^*$ with the "1" at the i-th position, the state transition matrix $\Phi(t,t_0)$ satisfies the matrix differential equation (88) and reduces to the identity matrix at $t = t_0$, one has the following result:

Theorem 3

The unique solution of:

$$\dot{x}(t) = F(t)x(t), \quad x(t_0) = x_0 \tag{91}$$

can be represented as:

$$x(t) = \Phi(t,t_0)x_0 \tag{92}$$

where $\Phi(t,t_0)$ is the unique matrix satisfying:

$$\frac{\partial}{\partial t}\Phi(t,t_0) = F(t)\Phi(t,t_0) \text{ and } \Phi(t_0,t_0) = I. \tag{93}$$

Proof

Differentiating $x(t) = \Phi(t,t_0)x_0$ yields:

$$\dot{x}(t) = \frac{\partial}{\partial t}\Phi(t,t_0)x_0 = F(t)\Phi(t,t_0)x_0 = F(t)x(t).$$

And evaluation at t_0 yields:

$$x(t_0) = \Phi(t_0,t_0)x_0 = x_0.$$

End of proof.

Equation (92) reveals the reason that $\Phi(t,t_0)$ is called the state transition matrix. It describes the zero-input motion of the state vector $x(t)$ of a linear, time-varying system and represents the linear transformation which maps the initial state $x(t_0)$ at time t_0 into the state at time t. The following theorem summarizes two important properties of the state transition matrix.

Theorem 4

(i) transition property: $\Phi(t_2,t_0) = \Phi(t_2,t_1)\Phi(t_1,t_0) \quad \forall\ t_0,t_1,t_2$

(ii) inversion property: $\Phi(t,t_0)^{-1} = \Phi(t_0,t) \quad \forall\ t_0,t$.

Proof

If $X(t)$ is any fundamental matrix of $\dot{x}(t) = F(t)x(t)$ then:

$$\Phi(t_2,t_0) = X(t_2)X(t_0)^{-1} = X(t_2)[X(t_1)^{-1}X(t_1)]X(t_0)^{-1} = \Phi(t_2,t_1)\Phi(t_1,t_0)$$

which is (i). Using (i) and letting $t_2 = t_0$:

$$I = \Phi(t_0,t_0) = \Phi(t_0,t_1)\Phi(t_1,t_0)$$

whence (ii) follows.

End of proof.

The transition and inversion properties are illustrated in Figure 4.18.

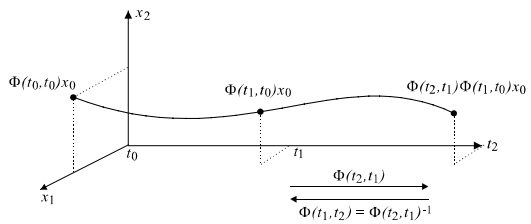


Figure 4.18: Two dimensional state space illustration of the transition and inversion property of $\Phi(t,t_0)$.

Having discussed the linear homogeneous equation in some detail, the machinery is now available to represent the solution $x(t)$ of the complete state equation (85) for some non-zero input $z(t)$.

Theorem 5

If $F(t)$ is continuous, and $\Phi(t,t_0)$ is the state transition matrix for $\dot{x}(t) = F(t)x(t)$ and $G(t)$ and $z(t)$ are piecewise continuous for all t, then the unique solution of the state equation:

(94) $$\dot{x}(t) = F(t)x(t) + G(t)z(t), \quad x(t_0) = x_0$$

is given by:

(95) $$\boxed{x(t) = \Phi(t,t_0)x(t_0) + \int_{t_0}^{t} \Phi(t,\tau)G(\tau)z(\tau)d\tau}\ .$$

Proof

Let $X(t)$ be a fundamental matrix of $\dot{x}(t) = F(t)x(t)$. From integrating:

$$\begin{aligned}
[X(t)^{-1}\dot{x}(t)] &= \dot{X}(t)^{-1}x(t) + X(t)^{-1}\dot{x}(t) \\
&= -X(t)^{-1}\dot{X}(t)X(t)^{-1}x(t) + X(t)^{-1}[F(t)x(t) + G(t)z(t)] \\
&= -X(t)^{-1}F(t)x(t) + X(t)^{-1}F(t)x(t) + X(t)^{-1}G(t)z(t) \\
&= X(t)^{-1}G(t)z(t)
\end{aligned}$$

follows:

$$X(t)^{-1}x(t) = X(t_0)^{-1}x(t_0) + \int_{t_0}^{t} X(\tau)^{-1}G(\tau)z(\tau)d\tau.$$

Premultiplication with $X(t)$ gives, with $\Phi(t,t_0) = X(t)X(t_0)^{-1}$, the desired result. Note that uniqueness of (95) is not a problem. For if $x_1(t)$ and $x_2(t)$ are two solutions of equation (94) satisfying the same initial condition, then $\dot{x}_1(t) - \dot{x}_2(t) = F(t)(x_1(t) - x_2(t))$ with $x_1(t_0) - x_2(t_0) = 0$, and thus $x_1(t) = x_2(t)$ for all t by Theorem 1.

End of proof.

Equation (95) expresses the state vector $x(t)$ in terms of the state transition matrix $\Phi(t,t_0)$. In general, it may be exceedingly difficult, or impossible, to find a closed form solution for the state transition matrix. This implies that in most cases one must resort to numerical integration techniques.

4.7 Linear time-invariant state equations

If the system matrices F and G in (85) of Section 4.6 are constant matrices, the linear system is time-invariant and the state equation becomes:

(96) $$\boxed{\dot{x}(t) = Fx(t) + Gz(t)}\ .$$

The state transition matrix $\Phi(t,t_0)$ of a linear, time-invariant system satisfies the matrix initial value problem:

(97) $$\dot{\Phi}(t,t_0) = F\Phi(t,t_0), \quad \Phi(t_0,t_0) = I.$$

In the previous section, matrix F was time dependent and no analytical expression for the solution of (97) could be derived. Note, however, that if F is time-independent one can obtain an explicit form of the state transition matrix. To see this, first consider the scalar version of (97):

$$\dot{\varphi}(t,t_0) = f\,\varphi(t,t_0), \quad \varphi(t_0,t_0) = 1.$$

The solution of this scalar first order differential equation is clearly:

$$\varphi(t,t_0) = e^{ft}$$

where the exponential function is defined as the infinite series:

(98) $$e^{ft} = \sum_{i=0}^{\infty} \frac{f^i t^i}{i!}.$$

This series expansion motivates to define the exponential of a square matrix by an infinite series identical in form to (98). Thus the matrix exponential for any square matrix F shall be defined as:

(99) $$\boxed{e^{Ft} = \sum_{i=0}^{\infty} \frac{F^i t^i}{i!}, \quad \text{with } F^i = \underbrace{F.F.\ldots F}_{i \text{ times}}}.$$

It can be shown that the infinite series of (99) is absolutely and uniformly convergent for all values of t. Consequently, the series can be differentiated and integrated term by term. Differentiating (99) term by term gives:

$$\frac{d}{dt}e^{Ft} = Fe^{Ft}.$$

But this shows that the unique solution of (97) is given by:

(100) $$\boxed{\Phi(t,t_0) = e^{F(t-t_0)}}.$$

Hence, the following theorem has been proven:

Theorem 6

The unique solution of the linear, time-invariant state equation:

(101) $$\dot{x}(t) = Fx(t) + Gz(t)$$

is given by:

(102)
$$\boxed{x(t) = e^{F(t-t_0)}x(t_0) + \int_{t_0}^{t} e^{F(t-\tau)}Gz(\tau)d\tau}\ .$$

Some important properties of the exponential of a square matrix are given in the next theorem.

Theorem 7

For any constant square matrix F, the matrix exponential e^{Ft} satisfies:

(*i*) $\dfrac{d}{dt}e^{Ft} = Fe^{Ft} = e^{Ft}F$

(*ii*) $e^{F\cdot 0} = I$

(*iii*) $(e^{Ft})^{-1} = e^{-Ft}$

(*iv*) $e^{F(t_1+t_2)} = e^{Ft_1}\cdot e^{Ft_2}$

(*v*) $e^{(F_1+F_2)t} = e^{F_1t}\cdot e^{F_2t}$ if and only if $F_1F_2 = F_2F_1$

(*vi*) $\int_{t_0}^{t} e^{F\tau}d\tau = F^{-1}(e^{Ft} - e^{Ft_0}) = (e^{Ft} - e^{Ft_0})F^{-1}$ if F^{-1} exists

(*vii*) $e^{(T^{-1}FT)t} = T^{-1}e^{Ft}T$ for any invertible constant matrix T.

The proof of the theorem is omitted since all above properties follow easily from the infinite series definition (99) of the matrix exponential. Note that there is a strong analogy between the matrix exponential and scalar exponential, although in some cases it breaks down slightly.

4.8 Evaluation of the matrix exponential

When the exponential of a matrix must be found, such as in the solution of a linear time-invariant system, a number of numerical calculation methods for evaluating the matrix exponential are available. In this section two methods will be discussed: the Taylor-series method and the Jordan canonical form method.

4.8.1 The Taylor-series method

One straightforward way to evaluate the matrix exponential is to use the infinite series by which the matrix exponential is defined. This method simply approximates e^{Ft} by evaluating only the first, say n, terms in the series expansion. In other words, one uses the approximation:

(103) $$e^{F\Delta t} \approx \sum_{i=0}^{n} \frac{F^i \Delta t^i}{i!} .$$

The larger the n and the smaller the time interval Δt, the better the approximation. For larger time intervals Δt, the transition property may be used:

(104) $$e^{F\Delta t} = (e^{F\Delta t/k})^k \approx (\sum_{i=0}^{n} \frac{F^i (\Delta t/k)^i}{i!})^k .$$

The Taylor-series method may also be used for approximating the state transition matrix of a linear time-varying system if the time interval $t - t_0$ is much less then the time required for significant changes in the system matrix $F(t)$. Thus in effect one uses the constant F approximation to replace the time-varying system $\dot{\Phi}(t,t_0) = F(t)\Phi(t,t_0)$ by the time-invariant system $\dot{\Phi}(t,t_0) = F(t_0)\Phi(t,t_0)$. The Taylor-series method is quite useful for numerical work since the repeated multiplication and addition are easily programmed and performed. The method also makes for flexible programming since it does not require major changes when the dimension of the state vector is changed. In general, the Taylor-series method is not preferred as an analytical approach. Only for small dimensions or simple structures of the system matrix F, the method may be used to write down the transition matrix explicity in terms of elementary functions. The following three examples illustrate this analytical approach.

Example 42

Consider a particle that moves with constant acceleration in the u-direction. Then $\ddot{u} = 0$. The state equation reads therefore:

(105) $$\begin{pmatrix} \dot{u} \\ \ddot{u} \\ \dddot{u} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} u \\ \dot{u} \\ \ddot{u} \end{pmatrix} .$$

This is a linear, time-invariant system with the system matrix:

(106) $$F = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} .$$

From this it follows that:

(107) $$F^2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ and } F^i = 0 \text{ for } i > 2.$$

Substitution of (106) and (107) into:

$$\Phi(t,t_0) = e^{F(t-t_0)} = \sum_{i=0}^{\infty} \frac{F^i (t-t_0)^i}{i!}$$

gives for the transition matrix:

(108) $$\Phi(t,t_0) = \begin{pmatrix} 1 & (t-t_0) & \frac{1}{2}(t-t_0)^2 \\ 0 & 1 & (t-t_0) \\ 0 & 0 & 1 \end{pmatrix}.$$

Compare this result with (22) of Section 3.2.

Example 43

From (67) of Section 4.5 it follows that the system matrix of the linearized state equation of the mathematical pendulum reads:

(109) $$F = \begin{pmatrix} 0 & 1 \\ -\omega^2 & 0 \end{pmatrix}, \quad \text{with} \quad \omega^2 = \frac{g}{l}.$$

The powers of F needed for the infinite series of the matrix exponential are:

$$F^2 = \begin{pmatrix} -\omega^2 & 0 \\ 0 & -\omega^2 \end{pmatrix}, \quad F^3 = \begin{pmatrix} 0 & -\omega^2 \\ \omega^4 & 0 \end{pmatrix}, \quad F^4 = \begin{pmatrix} \omega^4 & 0 \\ 0 & \omega^4 \end{pmatrix}, \ \ldots$$

Multipying by the appropriate powers of Δt and the factorials and summing the elements results in:

$$e^{F\Delta t} = \begin{pmatrix} 1 - \frac{1}{2!}\omega^2\Delta t^2 + \frac{1}{4!}\omega^4\Delta t^4 \ldots & \Delta t - \frac{1}{3!}\omega^2\Delta t^3 \ldots \\ -\omega^2\Delta t + \frac{1}{3!}\omega^4\Delta t^3 \ldots & 1 - \frac{1}{2!}\omega^2\Delta t^2 + \frac{1}{4!}\omega^4\Delta t^4 \ldots \end{pmatrix}.$$

Inspection of the terms in the above series reveals that:

(110) $$e^{F\Delta t} = \begin{pmatrix} \cos\omega\Delta t & \frac{1}{\omega}\sin\omega\Delta t \\ -\omega\sin\omega\Delta t & \cos\omega\Delta t \end{pmatrix}.$$

The solution of the linear, time-invariant homogeneous state equation (67) of Section 4.5 reads therefore:

(111) $$\begin{pmatrix} \Delta\theta(t) \\ \Delta\dot{\theta}(t) \end{pmatrix} = \begin{pmatrix} \Delta\theta(t_0)\cos\omega(t-t_0) + \frac{1}{\omega}\Delta\dot{\theta}(t_0)\sin\omega(t-t_0) \\ -\Delta\theta(t_0)\omega\sin\omega(t-t_0) + \Delta\dot{\theta}(t_0)\cos\omega(t-t_0) \end{pmatrix}.$$

Example 44

The system matrix of the linearized state equation (84) of Section 4.5 reads:

$$F = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -\omega^2 \\ 0 & 1 & 0 \end{pmatrix} \text{ with } \omega^2 = kM/R^3 - \dot{\varphi}_0^2 \,. \tag{112}$$

The powers of the system matrix are:

$$F^2 = \begin{pmatrix} 0 & 0 & -\omega^2 \\ 0 & -\omega^2 & 0 \\ 0 & 0 & -\omega^2 \end{pmatrix}, \quad F^3 = \begin{pmatrix} 0 & -\omega^2 & 0 \\ 0 & 0 & \omega^4 \\ 0 & -\omega^2 & 0 \end{pmatrix}, \quad F^4 = \begin{pmatrix} 0 & 0 & \omega^4 \\ 0 & \omega^4 & 0 \\ 0 & 0 & \omega^4 \end{pmatrix}, \ \ldots$$

Hence, the series expansion of the matrix exponential becomes:

$$e^{F\Delta t} = \begin{pmatrix} 1 & \Delta t - \frac{1}{3!}\omega^2\Delta t^3 \ldots & -\frac{1}{2!}\omega^2\Delta t^2 + \frac{1}{4!}\omega^4\Delta t^4 \ldots \\ 0 & 1 - \frac{1}{2!}\omega^2\Delta t^2 + \frac{1}{4!}\omega^4\Delta t^4 \ldots & -\omega^2\Delta t + \frac{1}{3!}\omega^4\Delta t^3 \ldots \\ 0 & \Delta t - \frac{1}{3!}\omega^2\Delta t^3 \ldots & 1 - \frac{1}{2!}\omega^2\Delta t^2 + \frac{1}{4!}\omega^4\Delta t^4 \ldots \end{pmatrix}.$$

Inspection of the terms in the above series reveals that:

$$e^{F\Delta t} = \begin{pmatrix} 1 & \frac{1}{\omega}\sin\omega\Delta t & (\cos\omega\Delta t - 1) \\ 0 & \cos\omega\Delta t & -\omega\sin\omega\Delta t \\ 0 & \frac{1}{\omega}\sin\omega\Delta t & \cos\omega\Delta t \end{pmatrix} . \tag{113}$$

The solution of the linear, time-invariant state equation (84) of Section 4.5 reads therefore:

$$\begin{pmatrix} R\Delta\varphi(t) \\ R\Delta\dot{\varphi}(t) \\ R\Delta\varepsilon(t) \end{pmatrix} = \begin{pmatrix} 1 & \frac{1}{\omega}\sin\omega(t-t_0) & (\cos\omega(t-t_0)-1) \\ 0 & \cos\omega(t-t_0) & -\omega\sin\omega(t-t_0) \\ 0 & \frac{1}{\omega}\sin\omega(t-t_0) & \cos\omega(t-t_0) \end{pmatrix} \begin{pmatrix} R\Delta\varphi(t_0) \\ R\Delta\dot{\varphi}(t_0) \\ R\Delta\varepsilon(t_0) \end{pmatrix} +$$

$$+ \int_{t_0}^{t} \begin{pmatrix} 1 & \frac{1}{\omega}\sin\omega(t-\tau) & (\cos\omega(t-\tau)-1) \\ 0 & \cos\omega(t-\tau) & -\omega\sin\omega(t-\tau) \\ 0 & \frac{1}{\omega}\sin\omega(t-\tau) & \cos\omega(t-\tau) \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & -R \end{pmatrix} \begin{pmatrix} \Delta f_2(\tau) \\ \Delta\omega^{ib}(\tau) \end{pmatrix} d\tau \,. \tag{114}$$

Note that if we take the limit $\omega \to 0$, the first two equations of (114) reduce to:

(115) $$\begin{pmatrix} R\Delta\varphi(t) \\ R\Delta\dot{\varphi}(t) \end{pmatrix} = \begin{pmatrix} 1 & (t-t_0) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} R\Delta\varphi(t_0) \\ R\Delta\dot{\varphi}(t_0) \end{pmatrix} + \int_{t_0}^{t} \begin{pmatrix} (t-\tau) \\ 1 \end{pmatrix} \Delta f_2(\tau) d\tau \ .$$

Compare this result with (11) of Section 3.2.

4.8.2 The Jordan canonical form method

This second method for computing the matrix exponential is based on the idea that the matrix exponential of a diagonal matrix is very easy to compute. For a diagonal matrix Λ, Λ = diag. $(\lambda_1,...,\lambda_n)$, the following equality holds:

(116) $$e^{\Lambda t} = \mathbf{diag}.(e^{\lambda_1 t},...,e^{\lambda_n t}).$$

It also holds, see Theorem 7 of Section 4.7, that for any invertible constant matrix T:

$$e^{T^{-1}FTt} = T^{-1}e^{Ft}T.$$

Hence, if one could determine an invertible matrix T such that:

(117) $$T^{-1}FT = \Lambda = \mathbf{diag}.(\lambda_1,...,\lambda_n)$$

the matrix exponential of F could be computed as:

(118) $$e^{Ft} = Te^{\Lambda t}T^{-1} \ .$$

Unfortunately not all square nxn matrices F can be diagonalized. Square matrices can be diagonalized, however, if they have n linearly independent eigenvectors t_i, i=1,...,n. This can be seen as follows. From the n relations:

$$Ft_i = \lambda_i t_i, \ i = 1,...,n$$

where λ_i are the eigenvalues of F, it follows that:

(119) $$FT = T\Lambda$$

where

$$T = (t_1,...,t_n) \text{ and } \Lambda = \mathbf{diag}.(\lambda_1,...,\lambda_n).$$

Then, since T is assumed to have n linearly independent columns, it has rank n and is thus invertible. Premultiplication of both sides of (119) with the inverse T^{-1} yields the desired result (117). Equation (118) is a valuable computational form for finding the matrix exponential e^{Ft}. It also clearly shows that the eigenvalues of F to a considerable extent determine the dynamic behaviour of the linear, time-invariant system. A form of (118) which shows this even more

clearly follows by expressing (118) explicitly in terms of the column vectors t_i, $i=1,...,n$, of T and row vectors u_i^*, $i=1,...,n$, of T^{-1}. Using:

$$T = (t_1,...,t_n) \text{ and } T^{-1} = \begin{pmatrix} u_1^* \\ \vdots \\ u_n^* \end{pmatrix}$$

in (118), gives the dyadic sum:

(120) $$e^{Ft} = \sum_{i=1}^{n} e^{\lambda_i t} t_i u_i^* \ .$$

This shows that the response of a system is a composition of motions along the eigenvectors of the system matrix F. A particular eigenvalue is excited if the initial state or input vector lies along the corresponding eigenvector. The following two examples illustrate the calculation of the matrix exponential using (118) with (116).

Example 45

From the state equation (37) of Section 4.4 the system matrix of a linear mass-spring accelerometer follows as:

(121) $$F = \begin{pmatrix} 0 & 1 \\ -\omega^2 & -2\gamma \end{pmatrix} \text{ with } \omega^2 = \frac{k}{m}, \quad \gamma = \frac{c}{2m} \ .$$

The characteristic polynomial of F is given by:

$$\det(F - \lambda I) = \lambda^2 + 2\gamma\lambda + \omega^2 = 0.$$

This gives the two eigenvalues:

(122) $$\lambda_{1,2} = -\gamma \pm (\gamma^2 - \omega^2)^{\frac{1}{2}} \ .$$

For the present example it will be assumed that $\gamma^2 > \omega^2$. Then, since γ is positive, both eigenvalues are negative. The eigenvectors are obtained by solving for $i=1,2$:

$$(F - \lambda_i I_2) t_i = 0 \ .$$

Ordering the eigenvectors by columns, the matrix T is obtained as:

(123) $$T = \begin{pmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{pmatrix}.$$

Its inverse reads:

(124) $$T^{-1} = (\lambda_2 - \lambda_1)^{-1} \begin{pmatrix} \lambda_2 & -1 \\ -\lambda_1 & 1 \end{pmatrix}.$$

With (123) and (124), application of (118) with (116) gives:

(125) $$e^{Ft} = \begin{pmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{pmatrix} \begin{pmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{pmatrix} \begin{pmatrix} \lambda_2 & -1 \\ -\lambda_1 & 1 \end{pmatrix} (\lambda_2 - \lambda_1)^{-1} .$$

The solution of the linear, time-invariant state equation (37) of Section 4.4 reads therefore:

(126) $$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} a_1 e^{\lambda_1 (t-t_0)} + a_2 e^{\lambda_2 (t-t_0)} \\ a_1 \lambda_1 e^{\lambda_1 (t-t_0)} + a_2 \lambda_2 e^{\lambda_2 (t-t_0)} \end{pmatrix} + \int_{t_0}^{t} \begin{pmatrix} e^{\lambda_1 (t-\tau)} - e^{\lambda_2 (t-\tau)} \\ \lambda_1 e^{\lambda_1 (t-\tau)} - \lambda_2 e^{\lambda_2 (t-\tau)} \end{pmatrix} (\lambda_2 - \lambda_1)^{-1} f(\tau) d\tau$$

where

$$\begin{cases} a_1 = (\lambda_2 - \lambda_1)^{-1} (\lambda_2 x_1(t_0) - x_2(t_0)) \\ a_2 = (\lambda_2 - \lambda_1)^{-1} (-\lambda_1 x_1(t_0) + x_2(t_0)) \end{cases}.$$

Example 46

Consider again the system matrix F of (112) of Section 4.8.1:

(127) $$F = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -\omega^2 \\ 0 & 1 & 0 \end{pmatrix}.$$

The characteristic polynomical of F reads:

$$\det(F - \lambda I_3) = -\lambda(\lambda^2 + \omega^2) = 0$$

which results in the three eigenvalues:

(128) $$\lambda_1 = 0 \text{ and } \lambda_{2,3} = \pm i\omega, \text{ with } i = (-1)^{1/2}.$$

The matrix of eigenvectors T and its inverse are given by:

(129) $$T = \begin{pmatrix} 1 & 1 & 1 \\ 0 & i\omega & -i\omega \\ 0 & 1 & 1 \end{pmatrix} \text{ and } T^{-1} = \begin{pmatrix} 1 & 0 & -1 \\ 0 & \frac{1}{2i\omega} & \frac{1}{2} \\ 0 & \frac{-1}{2i\omega} & \frac{1}{2} \end{pmatrix}.$$

With (128) and (129), application of (118) with (116) gives:

$$e^{Ft} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & i\omega & -i\omega \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{i\omega t} & 0 \\ 0 & 0 & e^{-i\omega t} \end{pmatrix} \begin{pmatrix} 1 & 0 & -1 \\ 0 & \frac{1}{2i\omega} & \frac{1}{2} \\ 0 & \frac{-1}{2i\omega} & \frac{1}{2} \end{pmatrix}$$

or

(130)
$$e^{Ft} = \begin{pmatrix} 1 & \frac{1}{\omega}\sin\omega t & (\cos\omega t - 1) \\ 0 & \cos\omega t & -\omega\sin\omega t \\ 0 & \frac{1}{\omega}\sin\omega t & \cos\omega t \end{pmatrix}.$$

where use was made of $e^{i\omega t} = \cos\omega t + i\sin\omega t$. Compare (130) with (113) of Section 4.8.1

It was already pointed out that not all square matrices can be diagonalized. For instance the system matrix of Example 45 cannot be diagonalized if $\gamma^2 = \omega^2$. This more complicated case is solved by transforming the systems matrix into an almost diagonal form so that the matrix exponential of this form is still easy to compute. This almost diagonal form is called the *Jordan canonical form*.

Theorem 8

Let F be any nxn matrix. Then it is always possible to find a nonsingular matrix T which can be partitioned as:

$$T = (T_1, T_2, \ldots, T_k)$$

such that:

$$F = TJT^{-1}$$

where

$$J = \mathbf{diag.}(J_1, J_2, \ldots, J_k) \ .$$

The block matrix J_i has dimension $m_i \times m_i$, $i = 1, \ldots, k$ and the partitioning of T matches that of J. The number k of blocks J_i equals the number of linearly independent eigenvectors of F. The block matrices J_i can be subpartitioned as:

$$J_i = \mathbf{diag.}(J_{i1}, J_{i2}, \ldots, J_{il_i})$$

where each subblock J_{ij} is of the form :

$$J_{ij} = \begin{pmatrix} \lambda_i & 1 & & 0 \\ & \lambda_i & 1 & \\ & & \ddots & \ddots \\ & 0 & & \ddots & 1 \\ & & & & \lambda_i \end{pmatrix}.$$

The number l_i of subblocks J_{ij} equals the number of linearly independent eigenvectors that correspond with eigenvalue λ_i. Matrix J is called the Jordan canonical form of F and matrix J_{ij} is called a Jordan block. For a proof of this theorem the reader is referred to [Noble, 1969]. The columns of matrix T can be computed as follows. From $FT=TJ$ and the form of J it follows that:

(131) $$Ft_i = \lambda t_i + \alpha_i t_{i-1}$$

where α_i is either 0 or 1, depending on J, and where λ is an eigenvalue of F. The number α_i is zero whenever the corresponding column vector t_i corresponds with the first column of a Jordan block. In this case t_i is an eigenvector of F. Once this column vector is found the remaining column vectors of T corresponding with the remaining columns of the Jordan block can be found from (131) with $\alpha_i=1$. These remaining column vectors t_i are known as generalized eigenvectors of F. After having computed the Jordan form J of F and the transformation matrix T and its inverse T^{-1}, the matrix exponential of F can be computed as shown in the next theorem.

Theorem 9

Let:

$$F = TJT^{-1}$$

where J is the Jordan canonical form of F. Then:

(i) $$e^{Ft} = Te^{Jt}T^{-1}$$

(ii) $$e^{Jt} = \mathbf{diag}.(e^{J_1 t}, e^{J_2 t}, \ldots, e^{J_k t})$$

(iii) $$e^{J_i t} = \mathbf{diag}.(e^{J_{i1} t}, e^{J_{i2} t}, \ldots, e^{J_{il_i} t})$$

(iv) $$e^{J_{ij}t} = \begin{pmatrix} 1 & t & \frac{1}{2!}t^2 & \ldots & \frac{1}{(p-1)!} & t^{p-1} \\ & 1 & t & \ldots & \frac{1}{(p-2)!} & t^{p-2} \\ & & 1 & \ddots & \vdots & \\ & & & \ddots & t & \\ & 0 & & & 1 & \end{pmatrix} . e^{\lambda_i t}$$

where p is the dimension of J_{ij} .

Proof

Only (iv) will be proven. With:

$$J_{ij} = \begin{pmatrix} \lambda_i & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda_i \end{pmatrix}$$

of dimension p, it follows that:

$$J_{ij}^k = \begin{pmatrix} \binom{k}{0}\lambda_i^k & \binom{k}{1}\lambda_i^{k-1} & \binom{k}{2}\lambda_i^{k-2} & \cdots & \binom{k}{p-1}\lambda_i^{k-p+1} \\ & \binom{k}{0}\lambda_i^k & \binom{k}{1}\lambda_i^{k-1} & \cdots & \binom{k}{p-2}\lambda_i^{k-p+2} \\ & & \ddots & \ddots & \vdots \\ & & & \ddots & \binom{k}{1}\lambda_i^{k-1} \\ & 0 & & & \binom{k}{0}\lambda_i^k \end{pmatrix}$$

where $\binom{k}{p-1} = 0$ if $k < p-1$.

This gives with $e^{J_{ij}t} = \sum_{k=0}^{\infty} \frac{J_{ij}^k t^k}{k!}$, the result:

$$e^{J_{ij}t} = \begin{pmatrix} \sum_{k=0}^{\infty}\frac{1}{k!}\lambda_i^k t^k & \sum_{k=1}^{\infty}\frac{1}{(k-1)!}\lambda_i^{k-1}t^k & \sum_{k=2}^{\infty}\frac{1}{2!}\frac{1}{(k-2)!}\lambda_i^{k-2}t^k & \cdots\cdots & \sum_{k=p-1}^{\infty}\frac{1}{(p-1)!}\frac{1}{(k-p+1)!}\lambda_i^{k-p+1}t^k \\ & \sum_{k=0}^{\infty}\frac{1}{k!}\lambda_i^k t^k & \sum_{k=1}^{\infty}\frac{1}{(k-1)!}\lambda_i^{k-1}t^k & \cdots\cdots & \sum_{k=p-2}^{\infty}\frac{1}{(p-2)!}\frac{1}{(k-p+2)!}\lambda_i^{k-p+2}t^k \\ & & \ddots & \ddots & \vdots \\ & & & \ddots & \\ & & & & \sum_{k=1}^{\infty}\frac{1}{(k-1)!}\lambda_i^{k-1}t^k \\ & 0 & & & \\ & & & & \sum_{k=0}^{\infty}\frac{1}{k!}\lambda_i^k t^k \end{pmatrix}$$

from which (iv) follows.

End of proof.

It is seen from this theorem that if the matrix F cannot be diagonalized, the matrix exponential of Ft contains besides the purely exponential terms of the form $e^{\lambda_i t}$ also terms of the form $te^{\lambda_i t}$, $t^2 e^{\lambda_i t}$ and so on. The dyadic sum of the matrix exponential of Ft will in this case therefore also contain the additional terms $te^{\lambda_i t}$, etc. Let the matrices T and T^{-1} be partitioned so that their blocks T_{ij} and U_{ij}^* respectively correspond with the Jordan blocks J_{ij}. Then, according to the above theorem one may write:

$$e^{Ft} = \sum_{i=1}^{k} \sum_{j=1}^{l_i} T_{ij}\, e^{J_{ij}t} U_{ij}^* \ .$$

To show explicitly the dependence on the terms $e^{\lambda_i t}$ and powers of t, one can take the exponential $e^{\lambda_i t}$ out of $e^{J_{ij}t}$ and write:

$$e^{Ft} = \sum_{i=1}^{k} e^{\lambda_i t} \sum_{j=1}^{\sigma_i} P_{ij} \frac{t^{j-1}}{(j-1)!} \tag{132}$$

where σ_i is the sum of the dimensions of the l_i-number of matrices J_{ij} and the matrices P_{ij} follow from the ordering per power of t. Expression (132) reduces to that of (120) of Section 4.8.2 if matrix F can be diagonalized. The following two examples illustrate the use of the Jordan canonical form.

Example 47

Consider a particle that moves with constant velocity in the u-direction. Then $\ddot{u} = 0$. The state equation reads therefore:

$$\begin{pmatrix} \dot{u} \\ \ddot{u} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} u \\ \dot{u} \end{pmatrix}. \tag{133}$$

This is a linear, time-invariant system with the system matrix:

$$F = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \tag{134}$$

This matrix has two identical eigenvalues: $\lambda_{1,2} = 0$. The number of linearly independent eigenvectors that correspond with this eigenvalue equals 1. This eigenvector reads:

$$t_1 = \begin{pmatrix} a \\ 0 \end{pmatrix} \text{ with } a \neq 0 .$$

Using (131) the corresponding generalized eigenvector t_2 follows from:

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} t_2 = 0 \cdot t_2 + 1 \cdot \begin{pmatrix} a \\ 0 \end{pmatrix}$$

as:

$$t_2 = \begin{pmatrix} 0 \\ a \end{pmatrix}.$$

Hence, the transformation matrix T and its inverse T^{-1} read as:

$$T = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \text{ and } T^{-1} = \begin{pmatrix} a^{-1} & 0 \\ 0 & a^{-1} \end{pmatrix}.$$

The Jordan canonical form of (134) reads therefore:

$$F = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} \begin{pmatrix} a^{-1} & 0 \\ 0 & a^{-1} \end{pmatrix}$$

with $\lambda = 0$. The matrix exponential of Ft follows with Theorem 9 then as:

$$e^{Ft} = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} e^{0 \cdot t} \begin{pmatrix} a^{-1} & 0 \\ 0 & a^{-1} \end{pmatrix}.$$

Hence, the solution of the state equation (133) reads:

$$\begin{pmatrix} u(t) \\ \dot{u}(t) \end{pmatrix} = \begin{pmatrix} 1 & (t - t_0) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} u(t_0) \\ \dot{u}(t_0) \end{pmatrix}.$$

Compare this result with (12) of Section 3.2

Example 48

In Example 45 it was assumed that $\gamma^2 > \omega^2$. This is known as the overdamped case. Now we will consider the critically damped case. This corresponds to the assumption that $\gamma^2 = \omega^2$. The eigenvalues of the system matrix:

$$F = \begin{pmatrix} 0 & 1 \\ -\omega^2 & -2\gamma \end{pmatrix}$$

are then:

$$\lambda_{1,2} = -\gamma.$$

One eigenvector of F is easily found from $(F + \gamma I_2)t_1 = 0$ as:

$$t_1 = \begin{pmatrix} 1 \\ -\gamma \end{pmatrix}.$$

Since the rank of the 2×2 matrix $(F + \gamma I_2)$ is one, there is only one linearly independent eigenvector which corresponds with the eigenvalue $-\gamma$. The vector t_2 must therefore be a generalized eigenvector of F. This vector is found from $(F + \gamma I_2)t_2 = t_1$ as:

$$t_2 = \begin{pmatrix} 1 \\ (1-\gamma) \end{pmatrix}.$$

The matrix T and its inverse T^{-1} become therefore:

$$T = \begin{pmatrix} 1 & 1 \\ -\gamma & (1-\gamma) \end{pmatrix} \text{ and } T^{-1} = \begin{pmatrix} (1-\gamma) & -1 \\ \gamma & 1 \end{pmatrix}.$$

With the Jordan canonical form of F given as:

$$J = \begin{pmatrix} -\gamma & 1 \\ 0 & -\gamma \end{pmatrix}$$

the matrix exponential of Ft follows as:

(135) $$e^{Ft} = \begin{pmatrix} 1 & 1 \\ -\gamma & (1-\gamma) \end{pmatrix} \begin{pmatrix} e^{-\gamma t} & te^{-\gamma t} \\ 0 & e^{-\gamma t} \end{pmatrix} \begin{pmatrix} (1-\gamma) & -1 \\ \gamma & 1 \end{pmatrix}.$$

The solution of the linear, time-invariant state equation (37) of Section 4.4 reads therefore for the critically damped case as:

(136) $$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} a_1 + a_2[(t-t_0)+1] \\ -\gamma a_1 + a_2[-\gamma[(t-t_0)+1]+1] \end{pmatrix} e^{-\gamma(t-t_0)} + \int_{t_0}^{t} \begin{pmatrix} (t-\tau)e^{-\gamma(t-\tau)} \\ [1-\gamma(t-\tau)]e^{-\gamma(t-\tau)} \end{pmatrix} f(\tau)d\tau$$

where :

$$\begin{cases} a_1 = (1-\gamma)x_1(t_0) - x_2(t_0) \\ a_2 = \gamma x_1(t_0) + x_2(t_0) \end{cases}.$$

4.9 Summary

In this chapter we introduced a class of models that enable us to describe the dynamic behaviour of a sufficient number of dynamic systems. In its most general form the dynamic model consists of a nonlinear first-order vector differential equation. An analytical closed form solution of it is generally not available. Linearization of the nonlinear system leads however to a linear, time-varying system of which the structure of the solution is known. The actual solution of this linear system is known once the state transition matrix is known. In most cases numerical integration is needed in order to find the state transition matrix. However if the linear system can be shown to be time-invariant, then an explicit formula for the state transition matrix is available. It is the matrix exponential of the constant system matrix. The chapter was concluded with a discussion of two methods for evaluating the matrix exponential. Table 4.2 gives an overview of the different types of state equations.

Nonlinear state equation

$$\dot{x}(t) = f(x(t),z(t),t)$$

Solution found through numerical integration

Linearized state equation

$$\Delta\dot{x}(t) = \partial_x f(t)\Delta x(t) + \partial_z f(t)\Delta z(t)$$

This is a linear, time-varying state equation

Linear time-varying state equation

$$\dot{x}(t) = F(t)x(t) + G(t)z(t)$$

Solution is of the form:

$$x(t) = \Phi(t,t_0)x(t_0) + \int_{t_0}^{t} \Phi(t,\tau)G(\tau)z(\tau)d\tau$$

State transition matrix is found through numerical integration of

$$\frac{\partial}{\partial t}\Phi(t,t_0) = F(t)\Phi(t,t_0), \quad \Phi(t_0,t_0) = I$$

Linear time-invariant state equation

$$\dot{x}(t) = Fx(t) + Gz(t)$$

Solution is

$$x(t) = e^{F(t-t_0)}x(t_0) + \int_{t_0}^{t} e^{F(t-\tau)}Gz(\tau)d\tau$$

Table 4.2: Different types of state equations.

5 Random functions

5.1 Introduction

In the previous chapter we developed the concept of a state-space model for time-varying parameters $x(t)$. The approach taken in the precious chapter was a purely deterministic one; no considerations of randomness were given. In practice however, it often happens that it does not suffice to base a study of the characteristics of a dynamic system solely on a deterministic description of the system. Often the inputs of the system are influenced by disturbances which are difficult or even impossible to describe deterministically. If a deterministic description of the input fails, it is fortunately often still possible to consider, to a sufficient degree of approximation, the disturbances as random. For instance, the acceleration imported to a vehicle when it travels over a road can be considered to be influenced by random disturbances, the statistical nature of which depends on the quality of the road. Also the forces acting upon a ship may sometimes be considered as random. And of course also the inherent uncertainty in measurements of for instance the system's input may be considered as random. In order to be able to include randomness in the description of dynamic systems, this chapter is concerned with some of the elementary concepts in the theory of random functions. Random functions are sometimes also called random processes or stochastic processes. As we will see, random functions are generalizations of random variables, and therefore much of the probability theory of random variables can be applied to random functions [Breiman, 1969, Papoulis, 1985, Peebles, 1987].

This chapter is organized as follows. In Section 5.2 we briefly discuss some of the statistical characteristics of random functions. For the purposes of this book the most important characteristics of random functions are their first two moments: the mean and variance. In Section 5.3 we present the extremely important propagation laws for the mean and variance of the output of dynamic systems. These laws generalize in a natural way the well-known propagation laws of random vectors. Section 5.4 deals with a special type of random function, namely the white noise random function. In Section 5.5 it is shown how the propagation laws of Section 5.3 simplify if the random inputs of the dynamic system consist of white noise. Finally, in the last section, Section 5.6, we introduce the random polynomial equations of motion.

5.2 The mean and covariance of random functions

The consideration of random variables which are functions of time, leads us to the study of random functions. A random function $\underline{x}(t)$ may be thought of as a family, or collection, of functions of time t, $x^i(t)$, $i=1,2,...$, any one of which might be observed on any trial of an experiment[1]. The functions $x^i(t)$, $i=1,2,...$, are called sample functions of the random function $\underline{x}(t)$. An interpretative picture of a random function $\underline{x}(t)$ in terms of its sample functions $x^i(t)$, $i=1,2,3$ is illustrated in Figure 5.1.

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[1] The parameter t will normally be interpreted as time, although this is not necessary.

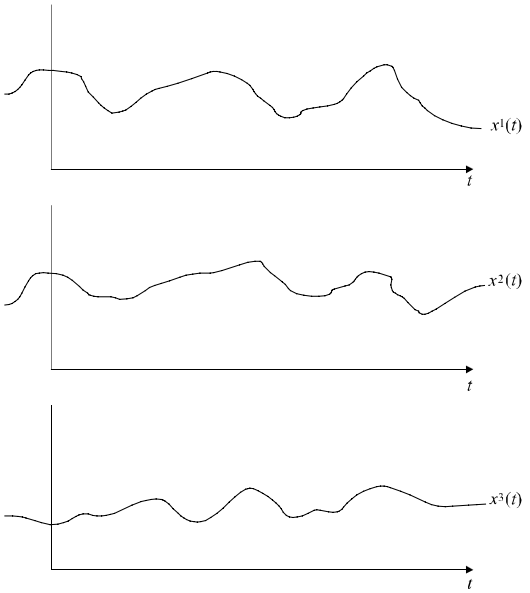


Figure 5.1: Three sample functions $x^1(t)$, $x^2(t)$, $x^3(t)$ of the random function $\underline{x}(t)$.

For time t fixed, say $t = t_1$, $\underline{x}(t_1)$ takes on repeated trials of the experiment different sample values at random. Thus $\underline{x}(t_1)$ is a random variable, and the probability that $\underline{x}(t_1)$ takes values in a certain range is given by the probability distribution function. In this case the dependence on time is shown explicitly in the notation of the probability distribution function:

$$P_{\underline{x}(t)}(x_1, t_1) = \text{Prob.}(\underline{x}(t_1) \leq x_1). \tag{1}$$

The corresponding probability density function is given as:

$$p_{\underline{x}(t)}(x_1, t_1) = \frac{dP_{\underline{x}(t)}(x_1, t_1)}{dx_1}. \tag{2}$$

By letting time t_1 vary in (1) and (2), we have in fact specified the first-order probability distribution function and the first-order probability density function of the random function $\underline{x}(t)$. For another fixed time instant, say t_2, $\underline{x}(t_2)$, is again a random variable. The probability of occurrence of a pair of sample values of $\underline{x}(t_1)$ and $\underline{x}(t_2)$ in certain ranges is then given by the joint probability distribution function:

$$P_{\underline{x}(t)}(x_1, t_1 \; ; \; x_2, t_2) = \text{Prob.}\ (\underline{x}(t_1) \leq x_1 \text{ and } \underline{x}(t_2) \leq x_2). \tag{3}$$

The corresponding joint probability density function is given as:

$$p_{\underline{x}(t)}(x_1,t_1;x_2,t_2) = \frac{\partial^2 P_{\underline{x}(t)}(x_1,t_1;x_2,t_2)}{\partial x_1 \partial x_2}. \tag{4}$$

By letting t_1 and t_2 vary in (3) and (4) we have specified the second-order probability distribution function and second-order probability density function of the random function $\underline{x}(t)$. Following this pattern, analogous definitions can be given for the higher-order distribution and density functions of $\underline{x}(t)$. As with random variables, we can define the mean and variance of a random function $\underline{x}(t)$. The mean of a random function $\underline{x}(t)$ is defined as:

$$E\{\underline{x}(t)\} = \int_{-\infty}^{+\infty} x\, p_{\underline{x}(t)}(x,t)\,dx. \tag{5}$$

Note that the mean of $\underline{x}(t)$ is, in general, a deterministic function of t. We shall say that a random function is *mean value stationary* if the mean is not a function of time t. The auto-covariance of a random function $\underline{x}(t)$ is defined as:

$$E\{(\underline{x}(t_1) - E\{\underline{x}(t_1)\})(\underline{x}(t_2) - E\{\underline{x}(t_2)\})\} = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} (x_1 - E\{\underline{x}(t_1)\})(x_2 - E\{\underline{x}(t_2)\}) p_{\underline{x}(t)}(x_1,t_1;x_2,t_2)\,dx_1\,dx_2. \tag{6}$$

The auto-variance is, in general, a function of times t_1 and t_2. We therefore speak of the auto-covariance function of $\underline{x}(t)$. The following short-hand notation will be used for the auto-covariance function:

$$\sigma_{xx}(t_1,t_2) = E\{(\underline{x}(t_1) - E\{\underline{x}(t_1)\})(\underline{x}(t_2) - E\{\underline{x}(t_2)\})\}. \tag{7}$$

Thus for t_1 and $t_2 \neq t_1$ fixed, $\sigma_{xx}(t_1,t_2)$ is the covariance between the two random variables $\underline{x}(t_1)$ and $\underline{x}(t_2)$. And for $t_2 = t_1$ fixed, $\sigma_{xx}(t_1,t_1)$ is the variance of the random variable $\underline{x}(t_1)$. We shall say that a random function $\underline{x}(t)$ is *covariance stationary*, if the auto-covariance function $\sigma_{xx}(t_1,t_2)$ is only dependent on the time difference $t_2 - t_1$. Thus if $\underline{x}(t)$ is covariance stationary, then $\sigma_{xx}(t,t+\tau)$ only depends on τ and not on t. If $\sigma_{xx}(t,t+\tau)$ is independent of t, we will write instead of $\sigma_{xx}(t,t+\tau)$ simply $\sigma_{xx}(\tau)$. Note that in this case, $\sigma_{xx}(0)$ is the variance of the random variable $x(t)$ for all t.

The cross-covariance between two random functions $\underline{x}(t)$ and $\underline{y}(t)$ is defined as:

$$E\{(\underline{x}(t_1) - E\{\underline{x}(t_1)\})(\underline{y}(t_2) - E\{\underline{y}(t_2)\})\} = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} (x - E\{\underline{x}(t_1)\})(y - E\{\underline{y}(t_2)\}) p_{\underline{x}(t)\underline{y}(t)}(x,t_1;y,t_2)\,dx\,dy. \tag{8}$$

Also the cross-covariance is, in general, a function of times t_1 and t_2. We speak therefore of the cross-covariance function between $x(t)$ and $y(t)$. The following short-hand notation will be used for the cross-covariance function:

(9) $$\sigma_{xy}(t_1,t_2) = E\{(\underline{x}(t_1) - E\{\underline{x}(t_1)\})(\underline{y}(t_2) - E\{\underline{y}(t_2)\})\}.$$

The above given definitions hold for scalar-valued random functions. The definitions can however be generalized quite naturally to the case of vector-valued random functions. If $\underline{x}(t)$ is a random vector function, the scalar auto-covariance of (7) generalizes to the auto-covariance matrix:

(10) $$Q_{xx}(t_1,t_2) = E\{(\underline{x}(t_1) - E\{\underline{x}(t_1)\})(\underline{x}(t_2) - E\{\underline{x}(t_2)\})^*\}.$$

Thus for t_1 and $t_2 \neq t_1$ fixed, $Q_{xx}(t_1,t_2)$ is the covariance matrix of the two random vectors $\underline{x}(t_1)$ and $\underline{x}(t_2)$. Note that the diagonal entries of the matrix $Q_{xx}(t_1,t_2)$ consist of auto-covariance functions, whereas the off-diagonal entries of this matrix consist of cross-covariance functions. Also note that:

(11) $$Q_{xx}(t_1,t_2)^* = Q_{xx}(t_2,t_1).$$

Thus the matrix $Q_{xx}(t_1,t_2)$ is square, but not symmetric. The matrix is only symmetric if $t_1 = t_2$. In this case, $Q_{xx}(t,t)$ is the variance matrix of $\underline{x}(t)$. The random vector function $\underline{x}(t)$ is said to be covariance stationary if $Q_{xx}(t_1,t_2)$ only depends on the time difference $t_2 - t_1$. If $\underline{x}(t)$ is covariance stationary, we will write instead of $Q_{xx}(t,t+\tau)$ simply $Q_{xx}(\tau)$. The auto-covariance of covariance stationary random functions has the following properties:

(12) $$\begin{cases} \text{a)} & Q_{xx}(\tau)^* = Q_{xx}(-\tau) \\ \text{b)} & |\sigma_{xx}(\tau)| \leq \sigma_{xx}(0) \end{cases}.$$

Proof

ad a) $$Q_{xx}(\tau)^* = Q_{xx}(t,t+\tau)^* = Q_{xx}(t+\tau,t) = Q_{xx}(-\tau).$$

This implies that the auto-covariance function of a covariance stationary scalar random function is an even function of τ:

$$\sigma_{xx}(\tau) = \sigma_{xx}(-\tau)$$

ad b)

Let $\underline{x}(t)$ be a scalar random function and define $\underline{u}(t)$ as $\underline{u}(t) = \underline{x}(t+\tau) + \underline{x}(t)$. Application of the propagation law of variances and covariances gives: $0 \leq \sigma_{uu}(t,t) = \sigma_{xx}(t+\tau,t+\tau) + \sigma_{xx}(t+\tau,t) + \sigma_{xx}(t,t+\tau) + \sigma_{xx}(t,t)$. Since $\sigma_{xx}(t+\tau,t+\tau) = \sigma_{xx}(t,t) = \sigma_{xx}(0)$ and $\sigma_{xx}(t+\tau,t) = \sigma_{xx}(t,t+\tau) = \sigma_{xx}(\tau)$, we get $0 \leq \sigma_{uu}(t,t) = 2(\sigma_{xx}(0) + \sigma_{xx}(\tau))$

or:

(13) $$-\sigma_{xx}(\tau) \leq \sigma_{xx}(0).$$

Now define $\underline{v}(t)$ as $\underline{v}(t) = \underline{x}(t+\tau) - \underline{x}(t)$. Then we find in a similar way as above that $0 \leq \sigma_{vv}(t,t) = 2(\sigma_{xx}(0) - \sigma_{xx}(\tau))$, or:

(14) $$+\sigma_{xx}(\tau) \leq \sigma_{xx}(0).$$

From (13) and (14) the result (12b) follows.

Let $\underline{x}(t)$ and $y(t)$ be two random vector functions. The scalar cross-covariance of (9) generalizes then to the cross-covariance matrix:

(15) $$Q_{xy}(t_1,t_2) = E\{(\underline{x}(t_1) - E\{\underline{x}(t_1)\})(\underline{y}(t_2) - E\{\underline{y}(t_2)\})^*\}.$$

The cross-covariance of two covariance stationary random functions has the following properties:

(16) $$\begin{cases} \text{a)} & Q_{xy}(\tau)^* = Q_{yx}(-\tau) \\ \text{b)} & |\sigma_{xy}(\tau)| \leq (\sigma_{xx}(0)\sigma_{yy}(0))^{1/2} \end{cases}.$$

Proof

ad a) $$Q_{xy}(\tau)^* = Q_{xy}(t,t+\tau)^* = Q_{yx}(t+\tau,t) = Q_{xy}(-\tau).$$

ad b) Define the random vector function $\underline{w}(t)$ in terms of the scalar random functions $\underline{x}(t)$ and $\underline{y}(t)$ as $\underline{w}(t) = (\underline{x}(t),\underline{y}(t+\tau))^*$. The variance matrix $Q_{ww}(t,t)$ of $\underline{w}(t)$ follows then as:

$$Q_{ww}(t,t) = \begin{pmatrix} \sigma_{xx}(t,t) & \sigma_{xy}(t,t+\tau) \\ \sigma_{yx}(t+\tau,t) & \sigma_{yy}(t+\tau,t+\tau) \end{pmatrix}.$$

Because of the covariance stationarity of $\underline{x}(t)$ and $\underline{y}(t)$, the entries of the matrix $Q_{ww}(t,t)$ are independent of t :

(17) $$Q_{ww}(t,t) = \begin{pmatrix} \sigma_{xx}(0) & \sigma_{xy}(\tau) \\ \sigma_{yx}(-\tau) = \sigma_{xy}(\tau) & \sigma_{yy}(0) \end{pmatrix}.$$

Matrix $Q_{ww}(t,t)$ is a variance matrix, and therefore always positive semi-definite. Its determinant is therefore always non-negative. From the non-negativeness of the determinant of (17) it follows then:

$$\sigma_{xx}(0)\sigma_{\underline{y}\underline{y}}(0) - \sigma_{\underline{x}\underline{y}}(\tau)^2 \geq 0.$$

And this proves (16b). It should be noted that (16) reduces to (12) if $\underline{y}(t) = \underline{x}(t)$. To conclude this section, some typical examples of auto-covariance functions are given in Table 5.1. Many of these auto-covariance functions will be seen to reappear in the remaining part of this book.

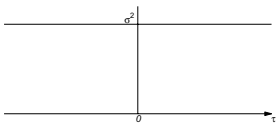
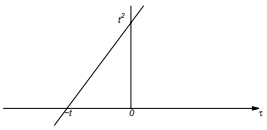
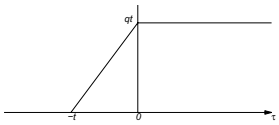
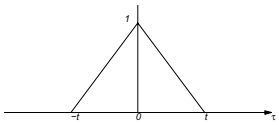
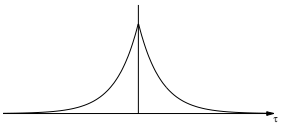
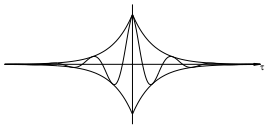
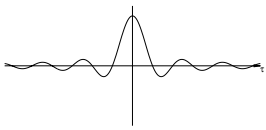
	$\sigma_{xx}(\tau) = \sigma^2$ The constant covariance function
	$\sigma_{xx}(t,\tau) = t(t+\tau)$ The linear covariance function
	$\sigma_{xx}(t,\tau) = \begin{cases} qt & \text{for } \tau \geq 0 \\ qt(t+\tau) & \text{for } \tau \leq 0 \end{cases}$
	$\sigma_{xx}(t,\tau) = \begin{cases} 1 - \dfrac{\lvert\tau\rvert}{T} & \text{for } \tau \leq T \\ 0 & \text{for } \tau \geq T \end{cases}$ The triangular covariance function
	$\sigma_{xx}(\tau) = e^{-\alpha\lvert\tau\rvert},\ \alpha > 0$ The exponential covariance function
	$\sigma_{xx}(\tau) = e^{-\alpha\lvert\tau\rvert}\cos w\tau,\ \alpha > 0$ The damped exponential covariance function
	$\sigma_{xx}(\tau) = \dfrac{\sin w\tau}{w\tau}$ The band limited covariance function

Table 5.1: Auto-covariance functions.

Example 49

Let $\underline{x}(t)$ be an $n\times 1$ random vector function with auto-covariance matrix $Q_{xx}(t,\tau)$. And let the $m\times 1$ random vector function $\underline{y}(t)$ be defined as:

(18) $$\underset{m\times 1}{\underline{y}(t)} = \underset{m\times n}{\boldsymbol{A}}\ \underset{n\times 1}{\underline{x}(t)} + \underset{m\times 1}{\boldsymbol{a}}$$

where A is a constant $m\times n$ matrix and a is a constant $m\times 1$ vector. We are asked to derive the auto-covariance matrix of $\underline{y}(t)$ and the cross-covariance matrix between $\underline{y}(t)$ and $\underline{x}(t)$. With (18), the auto-covariance matrix of $\underline{y}(t)$ follows as:

$$\begin{aligned} \boldsymbol{Q}_{yy}(t,\tau) &= E\{(\underline{y}(t) - E\{\underline{y}(t)\})(\underline{y}(\tau) - E\{\underline{y}(\tau)\})^*\} \\ &= E\{\boldsymbol{A}(\underline{x}(t) - E\{\underline{x}(t)\})(\underline{x}(\tau) - E\{\underline{x}(\tau)\})^*\boldsymbol{A}^*\} \\ &= \boldsymbol{A}\,E\{(\underline{x}(t) - E\{\underline{x}(t)\})(\underline{x}(\tau) - E\{\underline{x}(\tau)\})^*\}\boldsymbol{A}^* \end{aligned}$$

or as:

(19) $$\boldsymbol{Q}_{yy}(t,\tau) = \boldsymbol{A}\ \boldsymbol{Q}_{xx}(t,\tau)\boldsymbol{A}^*.$$

In a similar way the cross-covariance matrix between $\underline{y}(t)$ and $\underline{x}(t)$ follows as:

$$\begin{aligned} \boldsymbol{Q}_{yx}(t,\tau) &= E\{(\underline{y}(t) - E\{\underline{y}(t)\})(\underline{x}(\tau) - E\{\underline{x}(\tau)\})^*\} \\ &= E\{\boldsymbol{A}(\underline{x}(t) - E\{\underline{x}(t)\})(\underline{x}(\tau) - E\{\underline{x}(\tau)\})^*\} \\ &= \boldsymbol{A}\,E\{(\underline{x}(t) - E\{\underline{x}(t)\})(\underline{x}(\tau) - E\{\underline{x}(\tau)\})^*\} \end{aligned}$$

or as:

(20) $$\boldsymbol{Q}_{yx}(t,\tau) = \boldsymbol{A}\ \boldsymbol{Q}_{xx}(t,\tau).$$

Example 50

Let $\underline{x}_1(t)$ and $\underline{x}_2(t)$ be two scalar random functions with auto-covariance functions:

(21) $$\sigma_{x_1x_1}(\tau) = \boldsymbol{e}^{-|\tau|} \text{ and } \sigma_{x_2x_2}(\tau) = \cos 2\pi\tau.$$

Their cross-covariance function $\sigma_{x_1x_2}(t,\tau)$ is assumed to be identically zero. Hence, the two random functions are uncorrelated. A scalar random function $\underline{y}(t)$ is defined as the difference of $\underline{x}_1(t)$ and $\underline{x}_2(t)$:

(22) $$\underline{y}(t) = \underline{x}_1(t) - \underline{x}_2(t).$$

We are asked to derive the auto-covariance function $\sigma_{yy}(\tau)$, and the cross-covariance functions $\sigma_{yx_1}(\tau)$ and $\sigma_{yx_2}(\tau)$. In order to do so, we first write (22) in matrix-vector form as:

$$\underline{y}(t) = \underbrace{(1 \quad -1)}_{A} \underbrace{\begin{pmatrix} \underline{x}_1(t) \\ \underline{x}_2(t) \end{pmatrix}}_{\underline{x}(t)}.$$

With

$$Q_{xx}(\tau) = \begin{pmatrix} \sigma_{x_1x_1}(\tau) & \sigma_{x_1x_2}(\tau) \\ \sigma_{x_2x_1}(\tau) & \sigma_{x_2x_2}(\tau) \end{pmatrix} = \begin{pmatrix} e^{-|\tau|} & 0 \\ 0 & \cos 2\pi\tau \end{pmatrix} \tag{24}$$

application of (19) to (23) gives:

$$\sigma_{yy}(\tau) = (1 \quad -1) \begin{pmatrix} e^{-|\tau|} & 0 \\ 0 & \cos 2\pi\tau \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = e^{-|\tau|} + \cos 2\pi\tau \ . \tag{25}$$

In a similar way it follows from applying (20) to (23), with (24), that:

$$(\sigma_{yx_1}(\tau) \ \sigma_{yx_2}(\tau)) = (1 \quad -1) \begin{pmatrix} e^{-|\tau|} & 0 \\ 0 & \cos 2\pi\tau \end{pmatrix} = (e^{-|\tau|} \ -\cos 2\pi\tau) \ . \tag{26}$$

Example 51

Let $\underline{x}(t)$ be a random function with auto-covariance function:

$$\sigma_{xx}(\tau) = 25e^{-3|\tau|} - 16. \tag{27}$$

We are asked to derive the variances and covariances of the three random variables $\underline{x}(t_i)$, with $t_i = t_0 + (i-1)/2$ for $i = 1,2,3$. In order to obtain the required variances and covariances, we first define a 3×1 random vector y as:

$$\underline{y} = (\underline{x}(t_1), \ \underline{x}(t_2), \ \underline{x}(t_3))^*. \tag{28}$$

The variance matrix Q_{yy} of y reads then:

$$Q_{yy} = \begin{pmatrix} \sigma^2_{x(t_1)} & \sigma_{x(t_1)x(t_2)} & \sigma_{x(t_1)x(t_3)} \\ \sigma_{x(t_2)x(t_1)} & \sigma^2_{x(t_2)} & \sigma_{x(t_2)x(t_3)} \\ \sigma_{x(t_3)x(t_1)} & \sigma_{x(t_3)x(t_2)} & \sigma^2_{x(t_3)} \end{pmatrix} \tag{29}$$

$$= \begin{pmatrix} \sigma_{xx}(0) & \sigma_{xx}(t_2-t_1) & \sigma_{xx}(t_3-t_1) \\ \sigma_{xx}(t_1-t_2) & \sigma_{xx}(0) & \sigma_{xx}(t_3-t_2) \\ \sigma_{xx}(t_1-t_3) & \sigma_{xx}(t_2-t_3) & \sigma_{xx}(0) \end{pmatrix}.$$

With $t_i = t_0 + (i-1)/2$ and (27) this gives:

$$Q_{yy} = \begin{pmatrix} (25-16) & (25e^{3/2}-16) & (25e^{-3}-16) \\ (25e^{-3/2}-16) & (25-16) & (25e^{-3/2}-16) \\ (25e^{-3}-16) & (25e^{-3/2}-16) & (25-16) \end{pmatrix}.$$

The required variances and covariances of $x(t_i)$, $i = 1,2,3$ are given by the entries of Q_{yy}.

Example 52

Two scalar random functions $\underline{x}_1(t)$ and $\underline{x}_2(t)$ are defined as:

(31)
$$\begin{cases} \underline{x}_1(t) = \underline{a}\cos\omega t + \underline{b}\sin\omega t \\ \underline{x}_2(t) = -\underline{a}\sin\omega t + \underline{b}\cos\omega t \end{cases}$$

where $\underline{a}$ and $\underline{b}$ are random variables with the following variances and covariance:

(32)
$$\sigma_a^2 = \sigma^2,\ \sigma_b^2 = \sigma^2,\ \sigma_{ab} = 0.$$

We are asked to derive the auto-covariance functions and cross-covariance function of $\underline{x}_1(t)$ and $\underline{x}_2(t)$. In order to solve this problem, we first define a random vector function $\underline{x}(t)$ as $\underline{x}(t) = (\underline{x}_1(t), \underline{x}_2(t))^*$ and write (31) in matrix-vector form as:

(33)
$$\underline{x}(t) = \begin{pmatrix} \underline{x}_1(t) \\ \underline{x}_2(t) \end{pmatrix} = \begin{pmatrix} \cos\omega t & \sin\omega t \\ -\sin\omega t & \cos\omega t \end{pmatrix} \begin{pmatrix} \underline{a} \\ \underline{b} \end{pmatrix}.$$

Substitution of (33) into:

$$Q_{xx}(t,t+\tau) = E\{(\underline{x}(t) - E\{\underline{x}(t)\})(\underline{x}(t+\tau) - E\{\underline{x}(t+\tau)\})^*\}$$

gives:

$$Q_{xx}(t,t+\tau) = \begin{pmatrix} \cos\omega t & \sin\omega t \\ -\sin\omega t & \cos\omega t \end{pmatrix} E\left\{ \left(\begin{pmatrix} \underline{a} \\ \underline{b} \end{pmatrix} - E\left\{ \begin{pmatrix} \underline{a} \\ \underline{b} \end{pmatrix} \right\} \right) \left(\begin{pmatrix} \underline{a} \\ \underline{b} \end{pmatrix} - E\left\{ \begin{pmatrix} \underline{a} \\ \underline{b} \end{pmatrix} \right\} \right)^* \right\} \begin{pmatrix} \cos\omega(t+\tau) & \sin\omega(t+\tau) \\ -\sin\omega(t+\tau) & \cos\omega(t+\tau) \end{pmatrix}^*.$$

And with (32) this gives:

$$Q_{xx}(t,t+\tau) = \sigma^2 \begin{pmatrix} \cos\omega t & \sin\omega t \\ -\sin\omega t & \cos\omega t \end{pmatrix} \begin{pmatrix} \cos\omega(t+\tau) & -\sin\omega(t+\tau) \\ \sin\omega(t+\tau) & \cos\omega(t+\tau) \end{pmatrix}$$

or

(34)
$$Q_{xx}(t,t+\tau) = \sigma^2 \begin{pmatrix} \cos\omega\tau & -\sin\omega\tau \\ \sin\omega\tau & \cos\omega\tau \end{pmatrix}.$$

Hence, the required covariance functions follow as:

(35)
$$\begin{cases} \sigma_{x_1x_1}(\tau) = \sigma^2\cos\omega\tau \\ \sigma_{x_1x_2}(\tau) = -\sigma^2\sin\omega\tau \\ \sigma_{x_2x_2}(\tau) = \sigma^2\cos\omega\tau \end{cases}.$$

Note that both random functions $\underline{x}_1(t)$ and $\underline{x}_2(t)$ are covariance stationary.

5.3 Propagation laws for linear systems: the general case

In this section we will show how to compute the mean, the cross-covariance and the auto-covariance of the output of a linear, time-varying system:

(36)
$$\dot{x}(t) = F(t)x(t) + G(t)z(t)$$

when the initial state is a random vector and the input is a random function. According to Theorem 5 of Section 4.6 the solution of the state equation (36) reads:

(37)
$$x(t) = \Phi(t,t_0)x(t_0) + \int_{t_0}^{t} \Phi(t,\tau)G(\tau)z(\tau)d\tau.$$

It will be clear that when the initial state is a random vector, $\underline{x}(t_0)$, and the input is a random function, $\underline{z}(t)$, the output of the linear system becomes a random function:

(38)
$$\boxed{\underline{x}(t) = \Phi(t,t_0)\underline{x}(t_0) + \int_{t_0}^{t} \Phi(t,\tau)G(\tau)\underline{z}(\tau)d\tau}.$$

5.3.1 The mean of the output $\underline{x}(t)$

A formal derivation of the mean of $\underline{x}(t)$ will now be given.[2] From taking the expectation of (38) it follows that:

$$
\begin{aligned}
E\{\underline{x}(t)\} &= E\{\Phi(t,t_o)\underline{x}(t_o) + \int_{t_o}^{t} \Phi(t,\tau)G(\tau)\underline{z}\,d\tau\} \\
&= E\{\Phi(t,t_o)\underline{x}(t_o)\} + E\{\int_{t_o}^{t} \Phi(t,\tau)G(\tau)\underline{z}(\tau)\,d\tau\} \\
&= \Phi(t,t_o)E\{\underline{x}(t_o)\} + E\{\int_{t_o}^{t} \Phi(t,\tau)G(\tau)\underline{z}(\tau)\,d\tau\} \\
&= \Phi(t,t_o)E\{\underline{x}(t_o)\} + \int_{t_o}^{t} E\{\Phi(t,\tau)G(\tau)\underline{z}(\tau)\}\,d\tau \\
&= \Phi(t,t_o)E\{\underline{x}(t_o)\} + \int_{t_o}^{t} \Phi(t,\tau)G(\tau)E\{\underline{z}(\tau)\}\,d\tau
\end{aligned}
$$

The propagation law for the mean of the output of a linear system reads therefore:

(39)
$$
\boxed{E\{\underline{x}(t)\} = \Phi(t,t_o)E\{\underline{x}(t_0)\} + \int_{t_0}^{t} \Phi(t,\tau)G(\tau)E\{\underline{z}(\tau)\}\,d\tau} \quad .
$$

This shows that the mean of $\underline{x}(t)$ is a solution of the differential equation:

$$
E\{\dot{\underline{x}}(t)\} = F(t)E\{\underline{x}(t)\} + G(t)E\{\underline{z}(t)\}.
$$

---

[2] Derivatives and integrals of random functions are again random functions. As in the case of deterministic time functions, these derivative and integral operators are defined in terms of limits. However, since we wish to ensure convergence of the associated limits for the entire family of sample functions that contains the random function, the definitions of derivatives and integrals of random functions may differ from those of ordinary deterministic time functions. There is no difference if the associated limits exist for every sample function of the random function. But there is a difference if the ordinary limits of some of the sample functions fail to exist. This may for instance be due to the fact that some of the sample functions fail to be continuous. If the ordinary limits fail to exist for the entire family of sample functions, one can define the limits in an alternative, less stringent, way. These alternative definitions include convergence in probability, convergence with probability one and mean-square convergence, see e.g. [Stark and Woods, 1986]. If the ordinary limits do not exist for the entire family of sample functions, it is usually sufficient for most practical applications, to assume that the limits exist in the mean-square sense. And it can be shown that the same propagation laws, as derived in this section, follow if the derivatives and integrals are to be interpreted in the mean-square sense. For more details the reader is referred to e.g. [Melsa and Sage, 1973].

5.3.2 The cross-covariance between the output $\underline{x}(t)$ and input $\underline{z}(t)$

A formal derivation of the cross-covariance matrix $Q_{xz}(t_1,t_2)$ of the output $\underline{x}(t)$ and the input $\underline{z}(t)$ will now be given. With the definitions:

$$\begin{cases} \tilde{\underline{x}}(t) = \underline{x}(t) - E\{\underline{x}(t)\} \\ \tilde{\underline{z}}(t) = \underline{z}(t) - E\{\underline{z}(t)\} \end{cases} \tag{40}$$

the difference of (38) and (39) may be written as:

$$\tilde{\underline{x}}(t) = \boldsymbol{\Phi}(t,t_0)\tilde{\underline{x}}(t_0) + \int_{t_0}^{t} \boldsymbol{\Phi}(t,\tau) G(\tau) \tilde{\underline{z}}(\tau)\, d\tau. \tag{41}$$

Substitution of (41) into:

$$Q_{xz}(t_1,t_2) = E\{\tilde{\underline{x}}(t_1)\tilde{\underline{z}}(t_2)^*\}$$

gives:

$$\begin{aligned} Q_{xz}(t_1,t_2) &= E\{[\boldsymbol{\Phi}(t_1,t_o)\tilde{\underline{x}}(t_o) + \int_{t_o}^{t_1} \boldsymbol{\Phi}(t_1,\tau) G(\tau)\tilde{\underline{z}}(\tau)\, d\tau][\tilde{\underline{z}}(t_2)]^*\} \\ &= E\{\boldsymbol{\Phi}(t_1,t_o)\tilde{\underline{x}}(t_o)\tilde{\underline{z}}(t_2)^*\} + E\{\int_{t_o}^{t_1} \boldsymbol{\Phi}(t_1,\tau) G(\tau)\tilde{\underline{z}}(\tau)\tilde{\underline{z}}(t_2)^*\, d\tau\} \\ &= \boldsymbol{\Phi}(t_1,t_o) E\{\tilde{\underline{x}}(t_o)\tilde{\underline{z}}(t_2)^*\} + \int_{t_0}^{t_1} E\{\boldsymbol{\Phi}(t_1,\tau) G(\tau)\tilde{\underline{z}}(\tau)\tilde{\underline{z}}(t_2)^*\}\, d\tau \\ &= \boldsymbol{\Phi}(t_1,t_o) E\{\underline{x}(t_o)\tilde{\underline{z}}(t_2)^*\} + \int_{t_o}^{t} \boldsymbol{\Phi}(t_1,\tau) G(\tau) E\{\tilde{\underline{z}}(\tau)\tilde{\underline{z}}(t_2)^*\}\, d\tau \end{aligned}$$

The propagation law for the cross-covariance between the output $\underline{x}(t)$ and the input $\underline{z}(t)$ reads therefore:

$$\boxed{Q_{xz}(t_1,t_2) = \boldsymbol{\Phi}(t_1,t_0) Q_{xz}(t_0,t_2) + \int_{t_0}^{t_1} \boldsymbol{\Phi}(t_1,\tau) G(\tau) Q_{zz}(\tau,t_2)\, d\tau} \; . \tag{42}$$

Example 53

A random function $\underline{x}(t)$ is defined as the integral of the random function $\underline{z}(t)$:

$$\underline{x}(t) = \int_{t_0}^{t} \underline{z}(\tau)\, d\tau. \tag{43}$$

The auto-covariance function of $\underline{z}(t)$ is given as:

$$\sigma_{zz}(\tau) = \sigma^2. \tag{44}$$

We are asked to derive the cross-covariance function $\sigma_{xz}(t_1,t_2)$. According to (42) the cross-covariance function $\sigma_{xz}(t_1,t_2)$ between $x(t)$ and $z(t)$ of (43) satisfies:

$$\sigma_{xz}(t_1,t_2) = \int_{t_0}^{t_1} \sigma_{zz}(\tau_1,t_2)\,d\tau_1. \tag{45}$$

Since $\underline{z}(t)$ is covariance stationary, we have:

$$\sigma_{zz}(\tau_1,t_2) = \sigma_{zz}(t_2-\tau_1). \tag{46}$$

Substitution of (46) into (45) gives:

$$\sigma_{xz}(t_1,t_2) = \int_{t_0}^{t_1} \sigma_{zz}(t_2-\tau_1)\,d\tau_1. \tag{47}$$

If we apply the change of variable $\tau_1 = t_2 - \tau$, the integral (47) becomes:

$$\sigma_{xz}(t_1,t_2) = \int_{t_2-t_1}^{t_2-t_0} \sigma_{zz}(\tau)\,d\tau. \tag{48}$$

Substitution of (44) into (48) gives then after integration, for the required cross-covariance funtion:

$$\sigma_{xz}(t_1,t_2) = \sigma^2(t_1-t_0). \tag{49}$$

It is of interest to compare this result, with the result one would get in the discrete case. The discrete counterpart of the integral (43) is given by the sum:

$$\underline{x}_k = \sum_{i=k_0+1}^{k} \underline{z}_i. \tag{50}$$

Note that (44) implies that both the variance of $\underline{z}(t)$, as well as the covariance between $\underline{z}(t)$ and $\underline{z}(\tau)$ equal the constant σ^2 for all t,τ. For the discrete case this would mean that:

$$\begin{cases} \sigma_{z_i z_j} = \sigma^2 \text{ for } i = j \\ \sigma_{z_i z_j} = \sigma^2 \text{ for } i \neq j \end{cases}. \tag{51}$$

Application of the propagation law of covariances to (50) gives therefore with (51) for $l \geq k$:

$$\sigma_{x_k z_l} = \underset{1..(k-k_0)..l}{(1.....1\ \ 0..0)} \underset{l\times l}{\begin{pmatrix} \sigma^2 & \dots & \sigma^2 \\ \sigma^2 & \dots & \sigma^2 \\ \vdots & \ddots & \vdots \\ \sigma_2 & \dots & \sigma^2 \end{pmatrix}} \underset{l\times 1}{\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}} = \sigma^2(k-k_0) \tag{52}$$

The same result follows for $l \leq k$. Compare the discrete result (52) with its continuous counterpart (49).

Example 54

Consider again equation (43). Now however, it is assumed that the auto-covariance function of $\underline{z}(t)$ is of exponential form:

$$\sigma_{zz}(\tau) = \sigma^2 e^{-\alpha|\tau|},\ \alpha > 0. \tag{53}$$

Substitution of (53) into (48) gives:

$$\sigma_{xz}(t_1, t_2) = \int_{t_2-t_1}^{t_2-t_0} \sigma^2 e^{-\alpha|\tau|} d\tau. \tag{54}$$

In order to solve this integral we need to distinguish between two cases:

The case $t_0 \leq t_1 \leq t_2$

In this case (54) becomes:

$$\sigma_{xz}(t_1,t_2) = \int_{t_2-t_1}^{t_2-t_0} \sigma^2 e^{-\alpha\tau} d\tau = -\frac{\sigma^2}{\alpha}[e^{-\alpha\tau}]_{t_2-t_1}^{t_2-t_0}.$$

Hence:

$$\sigma_{xz}(t_1,t_2) = \frac{\sigma^2}{\alpha}(e^{-\alpha(t_2-t_1)} - e^{-\alpha(t_2-t_0)}). \tag{55}$$

The case $t_0 \leq t_2 \leq t_1$

In this case $t_2 - t_0 \geq 0$, but $t_2 - t_1 \leq 0$. We therefore write (54) as the sum of two integrals:

$$\sigma_{xz}(t_1,t_2) = \int_{t_2-t_1}^{0} \sigma^2 e^{\alpha\tau} d\tau + \int_{0}^{t_2-t_0} \sigma^2 e^{-\alpha\tau} d\tau.$$

Integration gives:

$$\sigma_{xz}(t_1,t_2) = \frac{\sigma^2}{\alpha}\left[e^{\alpha\tau}\right]_{t_2-t_1}^{0} + -\frac{\sigma^2}{\alpha}\left[e^{-\alpha\tau}\right]_{0}^{t_2-t_0}$$

$$= \frac{\sigma^2}{\alpha}\left(1-e^{\alpha(t_2-t_1)}\right) + \frac{\sigma^2}{\alpha}\left(1-e^{-\alpha(t_2-t_0)}\right)$$

or

(56) $$\sigma_{xz}(t_1,t_2) = \frac{\sigma^2}{\alpha}(2-e^{\alpha(t_2,t_1)}-e^{-\alpha(t_2-t_0)}).$$

From (55) and (56) the required cross-covariance function follows as:

(57) $$\sigma_{xz}(t_1,t_2) = \begin{cases} \frac{\sigma^2}{\alpha}\left(e^{\alpha(t_2-t_1)} - e^{-\alpha(t_2-t_0)}\right) & \text{for } t_2 \geq t_1 \\ \frac{\sigma^2}{\alpha}\left(2-e^{-\alpha(t_2-t_1)}-e^{-\alpha(t_2-t_0)}\right) & \text{for } t_2 \leq t_1 \end{cases}.$$

Let us now investigate what happens when the limit $\alpha\to 0$ of (57) is taken. Since (53) reduces to the constant σ^2 for $\alpha\to 0$, we expect the limit $\alpha\to 0$ of (57) to be identical to (49) of the previous example. In order to verify this, we make use of the expansion:

(58) $$e^{-\alpha t} = 1 - \alpha t + \tfrac{1}{2}\alpha^2 t^2 - \tfrac{1}{6}\alpha^3 t^3 + \tfrac{1}{24}\alpha^4 t^4 - \ldots$$

With (58), we may write (57) as:

(59) $$\sigma_{xz}(t_1,t_2) = \begin{cases} \frac{\sigma^2}{\alpha}\left(\alpha(t_1-t_0) + \frac{1}{2}\alpha^2(t_2-t_1)^2 - \frac{1}{2}\alpha^2(t_2-t_0)^2\ldots\right) & \text{for } t_2 \geq t_1 \\ \frac{\sigma^2}{\alpha}\left(\alpha(t_1-t_0) - \frac{1}{2}\alpha^2(t_2-t_1)^2 - \frac{1}{2}\alpha^2(t_2-t_0)^2\ldots\right) & \text{for } t_2 \leq t_1 \end{cases}.$$

By taking the limit $\alpha\to 0$ of (59) we get:

(60) $$\lim_{\alpha\to 0}\ \sigma_{xz}(t_1,t_2) = \sigma^2(t_1-t_0).$$

And this is indeed identical to (49).

Example 55

The random function $\underline{x}(t)$ is defined in terms of the random function $\underline{z}(t)$ as:

(61) $$\underline{x}(t) = \int_{t_0}^{t}(t-\tau)\underline{z}(\tau)d\tau.$$

The auto-covariance function of $\underline{z}(t)$ is assumed to be a damped exponential:

(62) $$\sigma_{zz}(\tau) = \sigma^2 e^{-\alpha|\tau|}\cos\omega\tau,\ a>0.$$

We are asked to derive the cross-covariance function between $\underline{x}(t)$ and $\underline{z}(t)$. According to (42) the cross-covariance function between $\underline{x}(t)$ and $\underline{z}(t)$ of (61) satisfies:

(63) $$\sigma_{xz}(t_1,t_2) = \int_{t_0}^{t_1} (t_1-\tau_1)\sigma_{zz}(t_2-\tau_1)d\tau_1.$$

With the change of variable $\tau_1 = t_2 - \tau$ this integral may also be written as:

(64) $$\sigma_{xz}(t_1,t_2) = \int_{t_2-t_1}^{t_2-t_0} (\tau + t_1 - t_2)\sigma_{zz}(\tau)d\tau.$$

Substitution of (62) into (64) gives:

(65) $$\sigma_{xz}(t_1,t_2) = \sigma^2 \int_{t_2-t_1}^{t_2-t_0} (\tau + t_1 - t_2)e^{-\alpha|\tau|}\cos\omega\tau d\tau.$$

In order to solve this integral we need to distinguish between two cases:

The case $t_0 \le t_1 \le t_2$

In this case (65) may be written as:

(66) $$\sigma_{xz}(t_1,t_2) = \sigma^2 \int_{t_2-t_1}^{t_2-t_0} \tau e^{-\alpha\tau}\cos\omega\tau d\tau + \sigma^2(t_1-t_2)\int_{t_2-t_1}^{t_2-t_0} e^{-\alpha\tau}\cos\omega\tau d\tau.$$

Using

(67) $$\int_a^b e^{-\alpha\tau}\cos\omega\tau d\tau = [\frac{1}{\alpha^2+\omega^2}e^{-\alpha\tau}(\omega\sin\omega\tau - \cos\omega\tau)]_a^b$$

and

(68) $$\int_a^b \tau e^{-\alpha\tau}\cos w\tau d\tau =$$
$$= [\frac{1}{\alpha^2+\omega^2}e^{-\alpha\tau}\{\tau(\omega\sin\omega\tau - \alpha\cos\omega\tau) - \frac{1}{\alpha^2+\omega^2}((\alpha^2-\omega^2)\cos\omega\tau - 2\alpha\omega\sin\omega\tau)\}]_a^b$$

we get for (66):

$$\sigma_{xz}(t_1,t_2) =$$

$$= \frac{\sigma^2}{\alpha^2+\omega^2}\,(e^{-\alpha(t_1-t_0)}\{(\omega(t_1-t_0)+\frac{2\alpha\omega}{\alpha^2+\omega^2})\sin\omega(t_2-t_0)-(\alpha(t_1-t_0)+\frac{\alpha^2-\omega^2}{\alpha^2+\omega^2})\cos\omega(t_2-t_0)\}$$

$$-e^{-\alpha(t_2-t_1)}\{\frac{2\alpha\omega}{\alpha^2+\omega^2}\sin\omega(t_2-t_1)-\frac{\alpha^2-\omega^2}{\alpha^2+\omega^2}\cos\omega(t_2-t_1)\})$$

(69)

The case $t_0 \le t_2 \le t_1$

In this case (65) may be written as:

(70) $$\sigma_{xz}(t_1,t_2) = \sigma^2\int_{t_2-t_1}^{0}(\tau+t_1-t_2)e^{\alpha\tau}\cos\omega\tau d\tau+\sigma^2\int_{0}^{t_2-t_0}(\tau+t_1-t_2)e^{-\alpha\tau}\cos\omega\tau d\tau.$$

Again using (67) and (68), we get for (70):

$$\sigma_{xz}(t_1,t_2) = \frac{\sigma^2}{\alpha^2+\omega^2}(2\alpha(t_1-t_2)+$$

(71) $$e^{-\alpha(t_2-t_0)}\{(\omega(t_1-t_0)+\frac{2\alpha\omega}{\alpha^2+\omega^2})\sin\omega(t_2-t_0)-(\alpha(t_1-t_0)+\frac{\alpha^2-\omega^2}{\alpha^2+\omega^2})\cos\omega(t_2-t_0)\}.$$

$$-e^{-\alpha(t_2-t_1)}\{\frac{-2\alpha\omega}{\alpha^2+\omega^2}\sin\omega(t_2-t_1)-\frac{\alpha^2-\omega^2}{\alpha^2+\omega^2}\cos\omega(t_2-t_1)\})$$

Thus the required cross-covariance function is given by (69) and (71).

Let us now investigate what happens to the cross-covariance function if we take the limit $\omega\to 0$, or the limit $\alpha\to 0$, or both these limits. It follows from (69) and (71) that:

(72) $$\lim_{\omega\to 0}\sigma_{xz}(t_1,t_2) = \begin{cases}\frac{\sigma^2}{\alpha^2}(e^{-\alpha|t_2-t_1|}-(1+\alpha(t_1-t_0))e^{-\alpha(t_2-t_0)}) & \text{for } t_2\ge t_1\\ \frac{\sigma^2}{\alpha^2}(e^{-\alpha|t_2-t_1|}-(1+\alpha(t_1-t_0))e^{-\alpha(t_2-t_0)}+2\alpha|t_2-t_1|) & \text{for } t_2\le t_1\end{cases}.$$

This is the cross-covariance function one would get when the auto-covariance function of the input $\underline{z}(t)$ is of exponential form (verify this yourself). If we take the limit $\alpha\to 0$, it follows from (69) and (71) that:

(73) $$\lim_{\alpha\to 0}\sigma_{xz}(t_1,t_2) = \frac{\sigma^2}{\omega^2}(\omega(t_1-t_0)\sin\omega(t_2-t_0)+\cos\omega(t_2-t_0)-\cos\omega(t_2-t_1)).$$

This is the cross-covariance function one would get when the auto-covariance of the input $\underline{z}(t)$ is of cosine form (verify this yourself). Finally, if we take the limit $\omega \to 0$ of (73), it follows that:

$$\lim_{\substack{\alpha \to 0 \\ \omega \to 0}} \sigma_{xz}(t_1,t_2) = \frac{1}{2}\sigma^2 (t_1 - t_0)^2. \tag{74}$$

This is the cross-covariance function one would get when the auto-covariance function of the input $\underline{z}(t)$ equals a constant (verify this yourself).

5.3.3 The auto-covariance of the output $\underline{x}(t)$

A formal derivation of the auto-covariance matrix $Q_{xx}(t_1,t_2)$ of the output $\underline{x}(t)$ will now be given. Substitution of (41) into:

$$Q_{xx}(t_1,t_2) = E\{\tilde{\underline{x}}(t_1)\tilde{\underline{x}}(t_2)^*\}$$

gives:

$$\begin{aligned}
Q_{xx}(t_1,t_2) = {} & E\{\Phi(t_1,t_0)\tilde{\underline{x}}(t_0)\tilde{\underline{x}}(t_0)^*\Phi(t_2,t_0)^*\} + \\
& E\{\int_{t_0}^{t_1}\Phi(t_1,\tau_1)G(\tau_1)\tilde{\underline{z}}(\tau_1)d\tau_1\tilde{\underline{x}}(t_0)^*\Phi(t_2,t_0)^*\} + \\
& E\{\Phi(t_1,t_0)\tilde{\underline{x}}(t_0)\int_{t_0}^{t_2}\tilde{\underline{z}}(\tau_2)^*G(\tau_2)^*\Phi(t_2,\tau_2)^*d\tau_2\} + \\
& E\{\int_{t_0}^{t_1}\Phi(t_1,\tau_1)G(\tau_1)\tilde{\underline{z}}(\tau_1)d\tau_1\int_{t_0}^{t_2}\tilde{\underline{z}}(\tau_2)^*G(\tau_2)^*\Phi(t_2,\tau_2)^*d\tau_2\} \\
= {} & \Phi(t_1,t_0)E\{\tilde{\underline{x}}(t_0)\tilde{\underline{z}}(t_0)^*\}\Phi(t_2,t_0)^* + \\
& \int_{t_0}^{t_1}\Phi(t_1,\tau_1)G(\tau_1)E\{\tilde{\underline{z}}(\tau_1)\tilde{\underline{x}}(t_0)^*\}\Phi(t_2,t_0)^*d\tau_1 + \\
& \int_{t_0}^{t_2}\Phi(t_1,t_0)E\{\tilde{\underline{x}}(t_0)\tilde{\underline{z}}(\tau_2)^*\}G(\tau_2)^*\Phi(t_2,\tau_2)^*d\tau_2 + \\
& \int_{t_0}^{t_1}\int_{t_0}^{t_2}\Phi(t_1,\tau_1)G(\tau_1)E\{\tilde{\underline{z}}(\tau_1)\tilde{\underline{z}}(\tau_2)^*\}G(\tau_2)^*\Phi(t_2,\tau_2)^*d\tau_1 d\tau_2 \ .
\end{aligned}$$

The propagation law for the auto-covariance of the output $\underline{x}(t)$ follows therefore as:

(75)
$$\begin{aligned} Q_{xx}(t_1,t_2) \;=\; & \boldsymbol{\Phi}(t_1,t_0)\,Q_{xx}(t_0,t_0)\,\boldsymbol{\Phi}(t_2,t_0)^* + \\ & \int_{t_0}^{t_1} \boldsymbol{\Phi}(t_1,\tau_1)\,G(\tau_1)\,Q_{zx}(\tau_1,t_0)\,\boldsymbol{\Phi}(t_2,t_0)^*\,d\tau_1 + \\ & \int_{t_0}^{t_2} \boldsymbol{\Phi}(t_1,t_0)\,Q_{xz}(t_0,\tau_2)\,G(\tau_2)^*\,\boldsymbol{\Phi}(t_2,\tau_2)^*\,d\tau_2 + \\ & \int_{t_0}^{t_1}\int_{t_0}^{t_2} \boldsymbol{\Phi}(t_1,\tau_1)\,G(\tau_1)\,Q_{zz}(\tau_1,\tau_2)\,G(\tau_2)^*\,\boldsymbol{\Phi}(t_2,\tau_2)^*\,d\tau_1\,d\tau_2 \end{aligned} \quad .$$

This result shows that the auto-covariance matrix of $\underline{x}(t)$ is composed of four terms. The first term depends on the variance matrix $Q_{xx}(t_0,t_0)$ of the initial state vector $\underline{x}(t_0)$. The second and third term are dependent on the cross-covariance matrix $Q_{xz}(t_0,t) = Q_{zx}(t,t_0)^*$. These two terms are absent if the input $\underline{z}(t)$ is uncorrelated with the initial state vector $\underline{x}(t_0)$. And the fourth term is dependent on the auto-covariance matrix $Q_{zz}(t_1,t_2)$ of the input vector $\underline{z}(t)$. It happens that in many practical applications $\underline{x}(t_0)$ is uncorrelated with $\underline{z}(t)$. Then:

$$Q_{xz}(t_0,t) \;=\; Q_{zx}(t,t_0)^* \;=\; 0$$

and (75) simplifies to:

(76)
$$\begin{aligned} Q_{xx}(t_1,t_2) \;=\; & \boldsymbol{\Phi}(t_1,t_0)\,Q_{xx}(t_0,t_0)\,\boldsymbol{\Phi}(t_2,t_0)^* + \\ & \int_{t_0}^{t_1}\int_{t_0}^{t_2} \boldsymbol{\Phi}(t_1,\tau_2)\,G(\tau_1)\,Q_{zz}(\tau_1,\tau_2)\,G(\tau_2)^*\,\boldsymbol{\Phi}(t_2,\tau_2)^*\,d\tau_1 d\tau_2 \end{aligned} \quad .$$

Note that a double integration is needed for the computation of $Q_{xx}(t_1,t_2)$. However, if the cross-covariance matrix $Q_{zx}(t_1,t_2)$ is known, one can do with just a single integration. It follows namely with (42) that (76) may be expressed in terms of $Q_{zx}(t_1,t_2) = Q_{xz}(t_2,t_1)^*$ as:

(77)
$$\begin{aligned} Q_{xx}(t_1,t_2) \;=\; & \boldsymbol{\Phi}(t_1,t_0)\,Q_{xx}(t_0,t_0)\,\boldsymbol{\Phi}(t_2,t_0)^* + \\ & \int_{t_0}^{t_1} \boldsymbol{\Phi}(t_1,\tau_1)\,G(\tau_1)\,Q_{zx}(\tau_1,t_2)\,d\tau_1 \\ \;=\; & \boldsymbol{\Phi}(t_1,t_0)\,Q_{xx}(t_0,t_0)\,\boldsymbol{\Phi}(t_2,t_0)^* + \\ & \int_{t_0}^{t_2} Q_{xz}(t_1,\tau_2)\,G(\tau_2)^*\,\boldsymbol{\Phi}(t_2,\tau_2)^*\,d\tau_2 \end{aligned} \quad .$$

Example 56

The random function $\underline{x}(t)$ is defined as the integral of $\underline{z}(t)$:

$$\underline{x}(t) = \int_{t_0}^{t} \underline{z}(\tau)\,d\tau. \tag{78}$$

The auto-covariance function of $\underline{z}(t)$ is assumed to be a constant:

$$\sigma_{zz}(\tau) = \sigma^2. \tag{79}$$

We are asked to derive the auto-covariance function of $\underline{x}(t)$. According to (76) the auto-covariance function of $\underline{x}(t)$ satisfies:

$$\sigma_{xx}(t_1,t_2) = \int_{t_o}^{t_1}\int_{t_o}^{t_2} \sigma_{zz}(\tau_1,\tau_2)\,d\tau_1\,d\tau_2. \tag{80}$$

Since $\underline{z}(t)$ is covariance stationary we have:

$$\sigma_{zz}(\tau_1,\tau_2) = \sigma_{zz}(\tau_2-\tau_1). \tag{81}$$

Substitution of (81) into (80) gives with the change of variable $\tau_1 = \tau_2 - \tau$:

$$\sigma_{xx}(t_1,t_2) = \int_{t_0}^{t_2}\int_{\tau_2-t_1}^{\tau_2-t_0} \sigma_{zz}(\tau)\,d\tau\,d\tau_2. \tag{82}$$

Substitution of (79) into (82) gives then:

$$\sigma_{xx}(t_1,t_2) = \sigma^2\int_{t_0}^{t_2}\int_{\tau_2-t_1}^{\tau_2-t_0} d\tau\,d\tau_2$$

$$= \sigma^2\int_{t_0}^{t_2}(t_1-t_0)\,d\tau_2$$

or

$$\sigma_{xx}(t_1,t_2) = \sigma^2(t_1-t_0)(t_2-t_0). \tag{83}$$

Note that the variance of $\underline{x}(t)$, $\sigma_{xx}(t,t) = \sigma^2(t-t_0)^2$, increases quadratically with time t, whereas the covariance between $\underline{x}(t)$ and $\underline{x}(t+\tau)$, $\sigma_{xx}(t,t+\tau) = \sigma^2(t-t_0)^2 + \sigma^2(t-t_0)\tau$, increases linearly with τ. A plot of the auto-covariance function (83) is shown in Figure 5.2 for $\sigma^2 = 1$ and $t_0 = 0$.

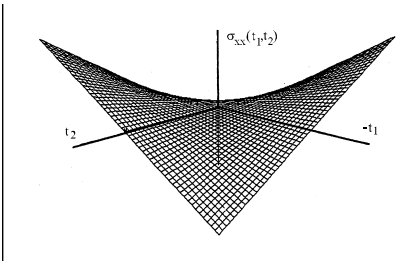


Figure 5.2: The auto-covariance function $\sigma_{xx}(t_1,t_2) = \sigma^2(t_1 - t_0)(t_2 - t_0)$, with $\sigma^2 = 1$ and $t_0 = 0$.

Instead of using (76) for the derivation of (83), we could also have used (77). According to (77), the auto-covariance function of $\underline{x}(t)$ satisfies:

$$\sigma_{xx}(t_1,t_2) = \int_{t_0}^{t_2} \sigma_{xz}(t_1,\tau_2)d\tau_2. \tag{84}$$

Since the cross-covariance function $\sigma_{xz}(t_1,t_2)$ has already been derived in Example 53, we may use also (49) in Section 5.4.3. Substitution of (49) into (84) gives then after integration indeed (83).

Example 57

The random function $\underline{x}(t)$ is defined as the average of $\underline{z}(t)$:

$$\underline{x}(t) = \frac{1}{t - t_0}\int_{t_0}^{t} \underline{z}(\tau)d\tau. \tag{85}$$

The auto-covariance function of $\underline{z}(t)$ is given as:

$$\sigma_{zz}(\tau) = \sigma^2 \tag{86}$$

The auto-covariance function of the output $\underline{x}(t)$ follows then with (76) as:

$$\sigma_{xx}(t_1,t_2) = \sigma^2 \tag{87}$$

Hence, the variance of $\underline{x}(t)$ is constant and independent of t:

$$\sigma_{xx}(t,t) = \sigma^2 \tag{88}$$

At first sight it may seem strange that the variance of the average $\underline{x}(t)$ is constant. Would we not expect that the variance of $\underline{x}(t)$ decreases for increasing t? This is at least what we encountered so many times in the discrete case. See for instance equation (25) in Section 2.2. In order to understand this difference, we consider the discrete counterpart of (85):

(89) $$\underline{x}_k = \frac{1}{k-k_0} \sum_{i=k_0+1}^{k} \underline{z}_i$$

Let us assume that the variances and covariances of $\underline{z}_i$ are given as:

(90) $$\sigma_{z_i z_j} = \sigma^2 \delta_{ij}$$

where δ_{ij} is the Kronecker symbol, which is defined as:

(91) $$\delta_{ij} = \begin{cases} 1 & \text{for } i = j \\ 0 & \text{for } i \neq j \end{cases}$$

If we apply the propagation law for variances and covariances to (89) we get with (90) and (91):

$$\sigma_{x_k x_l} = \frac{1}{k-k_0} \underbrace{(1 \ldots 1}_{1 \ldots (k-k_0)} \underbrace{0 \ldots 0)}_{} \begin{pmatrix} \sigma^2 & & 0 \\ & \sigma^2 & \\ & & \ddots \\ 0 & & \sigma^2 \end{pmatrix} \begin{pmatrix} 1 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \frac{1}{l-k_0}$$

or

(92) $$\sigma_{x_k x_l} = \begin{cases} \dfrac{\sigma^2}{k-k_0} & \text{for } k \geq l \\ \dfrac{\sigma^2}{l-k_0} & \text{for } l \geq k \end{cases}.$$

This result indeed shows that the variance of the average $\underline{x}_k$ decreases as k increases. How does this result then compare with (88)? The answer lies in our assumptions about the variances and covariances of $\underline{z}_i$. Equation (90) is namely not the discrete counterpart of (86). With (90) we have $\sigma_{z_i z_j} = 0$ for $i \neq j$, whereas with (86) we have $\sigma_{zz}(t_1,t_2) = \sigma^2 \neq 0$ for $t_1 \neq t_2$. The correct discrete counterpart of (86) is therefore:

(93) $$\sigma_{z_i z_j} = \begin{cases} \sigma^2 & \text{for } i = j \\ \sigma^2 & \text{for } i \neq j \end{cases}.$$

If we now apply the propagation law of variances and covariances to (89) we get with (93):

$$\sigma_{x_k x_l} = \frac{1}{k-k_0}(1\ldots1\ \ 0\ldots0)\begin{pmatrix}\sigma^2 & \ldots & \sigma^2\\ \sigma^2 & \ldots & \sigma^2\\ \vdots & \ddots & \vdots\\ \sigma^2 & \ldots & \sigma^2\end{pmatrix}\begin{pmatrix}1\\ \vdots\\ 1\\ 0\\ \vdots\\ 0\end{pmatrix}\frac{1}{l-k_0}$$

or

$$\sigma_{x_k x_l} = \sigma^2. \tag{94}$$

And this result indeed corresponds with (87). A random function $\underline{z}(t)$ having as auto-covariance function the constant (86), is called a *random constant*. This terminology is best explained if we look at the discrete case. We define a $(k-k_0)\times 1$ vector $\underline{z}$ as:

$$\underline{z} = (\underline{z}_{k_0+1},\ldots,\underline{z}_k)^*. \tag{95}$$

Its variance matrix reads according to (93) as:

$$\underset{(k-k_0)\times(k-k_0)}{\boldsymbol{Q}_{zz}} = \begin{pmatrix}\sigma^2 & \ldots & \sigma^2\\ \sigma^2 & \ldots & \sigma^2\\ \vdots & \ddots & \vdots\\ \sigma^2 & \ldots & \sigma^2\end{pmatrix}. \tag{96}$$

Note that the rank of this matrix equals 1. Hence, there exist a $(k-k_0-1)$-number of linearly independent functions of $\underline{z}$ that have a zero-variance. Since:

$$\boldsymbol{B}^*\boldsymbol{Q}_{zz}\boldsymbol{B} = 0$$

if

$$\underset{(k-k_0-1)\times(k-k_0)}{\boldsymbol{B}^*} = \begin{pmatrix}1 & -1 & & & & \\ & 1 & -1 & & 0 & \\ & & 1 & -1 & & \\ & 0 & & \ddots & & \\ & & & & 1 & -1\end{pmatrix}$$

the vector $\underline{w} = B^*\underline{z}$ has a zero variance matrix and is therefore constant. This shows that the successive differences of the $\underline{z}_i$ are constant:

$$\begin{cases} z_{k_0+1} - z_{k_0+2} = w_1 \\ z_{k_0+2} - z_{k_0+3} = w_2 \\ \vdots \qquad\qquad \vdots \\ z_{k-1} - z_k = w_{k-k_0-1} \end{cases}.$$

Hence:

$$z_{k_0+i} = z_{k_0+1} - \sum_{j=1}^{i-1} w_j$$

which shows that the random characteristics of the z_i are constant for all i.

Example 58

Let $x(t)$ be defined as:

$$x(t) = \int_{t_0}^{t} z(\tau)\,d\tau. \tag{97}$$

The auto-covariance function of $z(t)$ is assumed to be of exponential form:

$$\sigma_{zz}(\tau) = \sigma^2 e^{-\alpha|\tau|}, \quad \alpha > 0. \tag{98}$$

According to (76), the auto-covariance function of $x(t)$ satisfies:

$$\sigma_{xx}(t_1,t_2) = \sigma^2 \int_{t_0}^{t_1}\int_{t_0}^{t_2} e^{-\alpha|\tau_2-\tau_1|}\,d\tau_1 d\tau_2. \tag{99}$$

This gives with the change of variable $\tau_2 = \tau_1 + \tau$:

$$\sigma_{xx}(t_1,t_2) = \sigma^2 \int_{t_0}^{t_1}\int_{t_0-\tau_1}^{t_2-\tau_1} e^{-\alpha|\tau|}\,d\tau\,d\tau_1. \tag{100}$$

Since $\sigma_{xx}(t_1,t_2) = \sigma_{xx}(t_2,t_1)$ we only need to consider the case $t_0 \le t_1 \le t_2$. For this case (100) may be written as:

$$\sigma_{xx}(t_1,t_2) = \sigma^2 \int_{t_0}^{t_1} \left(\int_{t_0-\tau_1}^{0} e^{\alpha\tau}\,d\tau + \int_{0}^{t_2-\tau_1} e^{-\alpha\tau}\,d\tau \right) d\tau_1.$$

Performing the integration between brackets gives:

$$\sigma_{xx}(t_1,t_2) = \sigma^2 \int_{t_0}^{t_1} \{ [\frac{1}{\alpha} e^{\alpha\tau}]_{t_0-\tau_1}^{0} + [-\frac{1}{\alpha} e^{-\alpha\tau}]_{0}^{t_2-\tau_1} \} d\tau_1$$

or

$$\sigma_{xx}(t_1,t_2) = \frac{\sigma^2}{\alpha}\int_{t_0}^{t_1}(2 - e^{\alpha(t_0-\tau_1)} - e^{-\alpha(t_2-\tau_1)})\,d\tau_1.$$

And a second integration gives:

$$\sigma_{xx}(t_1,t_2) = \frac{\sigma^2}{\alpha}[2\tau_1 + \frac{1}{\alpha}e^{\alpha(t_0-\tau_1)} - \frac{1}{\alpha}e^{-\alpha(t_2-\tau_1)}]_{t_0}^{t_1}$$

or

(101) $$\sigma_{xx}(t_1,t_2) = \frac{2\sigma^2}{\alpha}(t_1 - t_0) - \frac{\sigma^2}{\alpha^2}(1 + e^{-\alpha(t_2-t_1)} - e^{-\alpha(t_1,t_0)} - e^{-\alpha(t_2-t_0)}).$$

The result that corresponds to the case that $t_0 \leq t_2 \leq t_1$ follows from interchanging t_1 and t_2 in (101). The auto-covariance function of $\underline{x}(t)$ reads therefore:

(102) $$\sigma_{xx}(t_1,t_2) = \begin{cases} \frac{2\sigma^2}{\alpha}(t_1 - t_0) - \frac{\sigma^2}{\alpha^2}(1 + e^{-\alpha(t_2-t_1)} - e^{-\alpha(t_1-t_0)} - e^{-\alpha(t_2-t_0)}) \text{ for } t_2 \geq t_1 \\ \frac{2\sigma^2}{\alpha}(t_2 - t_0) - \frac{\sigma^2}{\alpha^2}(1 + e^{\alpha(t_2-t_1)} - e^{-\alpha(t_1-t_0)} - e^{-\alpha(t_2-t_0)}) \text{ for } t_1 \geq t_2 \end{cases}.$$

Let us now investigate what happens to the auto-covariance function if the limit $\alpha \to 0$ is taken. Using:

$$e^{-\alpha t} = 1 - \alpha t + \tfrac{1}{2}\alpha^2 t^2 - \tfrac{1}{6}\alpha^3 t^3 + \ldots$$

we may expand (102) as:

(103) $$\sigma_{xx}(t_1,t_2) = \begin{cases} \sigma^2(t_1 - t_0)(t_2 - t_0) + \sigma^2\alpha(\frac{1}{6}(t_2 - t_1)^3 - \frac{1}{6}(t_1 - t_0)^3 - \frac{1}{6}(t_2 - t_0)^3 \ldots) \text{ for } t_2 \geq t_1 \\ \sigma^2(t_1 - t_0)(t_2 - t_0) + \sigma^2\alpha(-\frac{1}{6}(t_2 - t_1)^3 - \frac{1}{6}(t_1 - t_0)^3 - \frac{1}{6}(t_2 - t_0)^3 \ldots) \text{ for } t_1 \geq t_2 \end{cases}.$$

This shows that:

(104) $$\lim_{\alpha \to 0} \sigma_{xx}(t_1,t_2) = \sigma^2(t_1 - t_0)(t_2 - t_0).$$

Compare this result with (83).

Example 59

Let $\underline{x}(t)$ be defined as the average of $\underline{z}(t)$:

(105) $$\underline{x}(t) = \frac{1}{t - t_0}\int_{t_0}^{t}\underline{z}(\tau)\,d\tau.$$

The auto-covariance function of $\underline{z}(t)$ is assumed to be of the following exponential form:

(106) $$\sigma_{zz}(\tau) = \tfrac{1}{2}\alpha\sigma^2 e^{-\alpha|\tau|}, \alpha > 0.$$

The auto-covariance function of the average $\underline{x}(t)$ follows then from multiplying (102) with $\frac{1}{2}\alpha/((t_1 - t_0)(t_2 - t_0))$. This gives:

(107) $$\sigma_{xx}(t_1,t_2) = \begin{cases} \dfrac{\sigma^2}{t_2 - t_0} - \dfrac{\sigma^2}{2\alpha}(1 + e^{-\alpha(t_2 - t_1)} - e^{-\alpha(t_1 - t_0)} - e^{-\alpha(t_2 - t_0)})\dfrac{1}{(t_1 - t_0)(t_2 - t_0)} & \text{for } t_2 \geq t_1 \\ \dfrac{\sigma^2}{t_1 - t_0} - \dfrac{\sigma^2}{2\alpha}(1 + e^{\alpha(t_2 - t_1)} - e^{-\alpha(t_1 - t_0)} - e^{-\alpha(t_2 - t_0)})\dfrac{1}{(t_1 - t_0)(t_2 - t_0)} & \text{for } t_1 \geq t_2 \end{cases}.$$

If we take the limit $\alpha \to \infty$ of (107) we get:

(108) $$\lim_{\alpha \to \infty} \sigma_{xx}(t_1,t_2) = \begin{cases} \dfrac{\sigma^2}{(t_2 - t_0)} & \text{for } t_2 \geq t_1 \\ \dfrac{\sigma^2}{(t_1 - t_0)} & \text{for } t_1 \geq t_2 \end{cases}.$$

This result is the continuous counterpart of (92) in Section 5.3.3. It seems therefore natural to believe that (106) is the continuous counterpart of (90) in Section 5.3.3 for the case that $\alpha \to \infty$. Note however, that although the limit (108) exists, the corresponding limit of $\sigma_{zz}(\tau)$ of (106) does not exist. It follows therefore that (108) should be seen as an approximation to the auto-covariance function of the average $\underline{x}(t)$, for the case the input $\underline{z}(t)$ has (106) as auto-covariance function with large, but finite, value of α. The auto-covariance function (106) is plotted in Figure 5.3 for some different values of α. This figure shows that the covariance between $\underline{z}(t)$ and $\underline{z}(t+\tau)$ decreases as x decreases. And for α large enough one may consider the covariance between $\underline{z}(t)$ and $\underline{z}(t+\tau)$ as practically absent. This shows that in practical applications, one may use (106) with large, but finite, α as an approximation to the auto-covariance function of a random function for which $\underline{z}(t)$ and $\underline{z}(t+\tau)$, $\tau \neq 0$, are uncorrelated. We will have more to say about uncorrelated random functions in the next section.

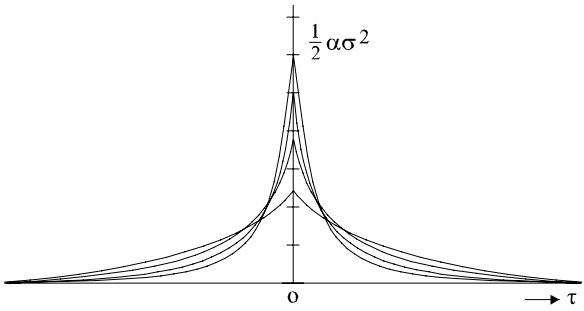


Figure 5.3: The auto-covariance function $\sigma_{zz}(\tau) = \frac{1}{2}\alpha\sigma^2 e^{-\alpha|\tau|}$ for increasing values of α.

Example 60

Let $\underline{x}(t)$ be defined as the moving average of $\underline{z}(t)$:

$$\underline{x}(t) = \frac{1}{T}\int_{t-T}^{t} \underline{z}(\tau)d\tau. \tag{109}$$

The auto-covariance function of $\underline{z}(t)$ is of exponential form:

$$\sigma_{zz}(\tau) = \sigma^2 e^{-\alpha|\tau|}, \ \alpha > 0. \tag{110}$$

Then

$$\sigma_{xx}(t_1,t_2) = \frac{\sigma^2}{T^2}\int_{t_1-T}^{t_1}\int_{t_2-T}^{t_2} e^{-\alpha|\tau_2-\tau_1|}d\tau_2 d\tau_1.$$

This gives with the change of variable $\tau_1 = \tau_2 - \tau$:

$$\sigma_{xx}(t_1,t_2) = \frac{\sigma^2}{T^2}\int_{t_2-T}^{t_2}\int_{\tau_2-t_1}^{\tau_2-t_1+T} e^{-\alpha|\tau|}d\tau d\tau_2. \tag{111}$$

We now need to distinguish between the case $|t_2 - t_1| \le T$ and the case $|t_2 - t_1| \ge T$:

The case $|t_2 - t_1| \le T$

If $t_2 \ge t_1$, then $\tau_2 - t_1 \le 0$ for $\tau_2 \in [t_2 - T, t_1]$, $\tau_2 - t_1 \ge 0$ for $\tau_2 \in [t_1, t_2]$ and $\tau_2 - t_1 + T \ge 0$ for $\tau_2 \in [t_2 - T, t_2]$. We therefore write (111) as the sum of two double integrals:

$$\sigma_{xx}(t_1,t_2)=\frac{\sigma^2}{T^2}\int_{t_2-T}^{t_1}\int_{\tau_2-t_1}^{\tau_2-t_1+T} e^{-\alpha|\tau|}d\tau d\tau_2+\frac{\sigma^2}{T^2}\int_{t_1}^{t_2}\int_{\tau_2-t_1}^{\tau_2-t_1+T} e^{-\alpha|\tau|}d\tau d\tau_2.$$

This may be written as:

$$\sigma_{xx}(t_1,t_2)=\frac{\sigma^2}{T^2}\int_{t_2-T}^{t_1}\Big(\int_{\tau_2-t_1}^{0} e^{\alpha\tau}d\tau+\int_{0}^{\tau_2-t_1+T} e^{-\alpha\tau}d\tau\Big)d\tau_2+\frac{\sigma^2}{T^2}\int_{t_1}^{t_2}\int_{\tau_2-t_1}^{\tau_2-t_1+T} e^{-\alpha\tau}d\tau d\tau_2.$$

Solving for the inner integrals gives:

$$\sigma_{xx}(t_1,t_2)=\frac{\sigma^2}{T^2}\int_{t_2-T}^{t_1}\Big\{\Big[\frac{1}{\alpha}e^{\alpha\tau}\Big]_{\tau_2-t_1}^{0}+\Big[-\frac{1}{\alpha}e^{-\alpha\tau}\Big]_{0}^{\tau_2-t_1+T}\Big\}\,d\tau_2+\frac{\sigma^2}{T^2}\int_{t_1}^{t_2}\Big[-\frac{1}{\alpha}e^{-\alpha\tau}\Big]_{\tau_2-t_1}^{\tau_2-t_1+T}d\tau_2$$

or

$$\sigma_{xx}(t_1,t_2)=\frac{\sigma^2}{T^2}\int_{t_2-T}^{t_1}\Big(\frac{2}{\alpha}-\frac{1}{\alpha}e^{\alpha(\tau_2-t_1)}-\frac{1}{\alpha}e^{-\alpha(\tau_2-t_1+T)}\Big)d\tau_2+\frac{\sigma^2}{T^2}\int_{t_1}^{t_2}\Big(\frac{1}{\alpha}e^{-\alpha(\tau_2-t_1)}-\frac{1}{\alpha}e^{-\alpha(\tau_2-t_1+T)}\Big)d\tau_2.$$

And a second integration gives:

$$\sigma_{xx}(t_1,t_2)=\frac{\sigma^2}{T^2}\Big[\frac{2}{\alpha}\tau_2-\frac{1}{\alpha^2}e^{\alpha(\tau_2-t_1)}+\frac{1}{\alpha^2}e^{-\alpha(\tau_2-t_1+T)}\Big]_{t_2-T}^{t_1}+$$

$$\frac{\sigma^2}{T^2}\Big[-\frac{1}{\alpha^2}e^{-\alpha(\tau_2-t_1)}+\frac{1}{\alpha^2}e^{-\alpha(\tau_2-t_1+T)}\Big]_{t_1}^{t_2}$$

or finally

(112) $$\sigma_{xx}(t_1,t_2)=\frac{2\sigma^2}{\alpha T^2}(t_1-t_2+T)-\frac{2\sigma^2}{\alpha^2T^2}e^{-\alpha(t_2-t_1)}+\frac{\sigma^2}{\alpha^2T^2}e^{\alpha(t_2-t_1-T)}+\frac{\sigma^2}{\alpha^2T^2}e^{-\alpha(t_2-t_1+T)}.$$

The case $|t_2-t_1|\geq T$

If $t_2\geq t_1$, then $\tau_2-t_1\geq 0$ and $\tau_2-t_1+T\geq 0$ for $\tau_2\in[t_2-T,t_2]$. We may therefore write (111) as:

$$\sigma_{xx}(t_1,t_2)=\frac{\sigma^2}{T^2}\int_{t_2-T}^{t_2}\int_{\tau_2-t_1}^{\tau_2-t_1+T} e^{-\alpha\tau}d\tau d\tau_2.$$

Solving for the inner integral gives:

$$\sigma_{xx}(t_1,t_2)=\frac{\sigma^2}{T^2}\int_{t_2-T}^{t_2}\Big(\frac{1}{\alpha}e^{-\alpha(\tau_2-t_1)}-\frac{1}{\alpha}e^{-\alpha(\tau_2-t_1+T)}\Big)d\tau_2.$$

And a second integration gives:

(113) $$\sigma_{xx}(t_1,t_2) = -\frac{2\sigma^2}{\alpha^2 T^2} e^{-\alpha(t_2-t_1)} + \frac{\sigma^2}{\alpha^2 T^2} e^{-\alpha(t_2-t_1-T)} + \frac{\sigma^2}{\alpha^2 T^2} e^{-\alpha(t_2-t_1+T)}.$$

Both the equations (112) and (113) hold for $t_2 \geq t_1$. The corresponding result for $t_1 \geq t_2$ follows, since $\sigma_{xx}(t_1,t_2) = \sigma_{xx}(t_2,t_1)$, from interchanging t_1 and t_2 in the right-hand sides of (112) and (113). The auto-covariance function of the moving average $\underline{x}(t)$ follows therefore, with $\tau = t_2 - t_1$ as:

$$\sigma_{xx}(t_1,t_2) = \begin{cases} \frac{-2\sigma^2}{\alpha T^2}(|\tau| - T) - \frac{2\sigma^2}{\alpha^2 T^2} e^{-\alpha|\tau|} + \frac{\sigma^2}{\alpha^2 T^2} e^{\alpha(|\tau|-T)} + \frac{\sigma^2}{\alpha^2 T^2} e^{-\alpha(|\tau|+T)} & \text{for } |\tau| \leq T \\ -\frac{2\sigma^2}{\alpha^2 T^2} e^{-\alpha|\tau|} + \frac{\sigma^2}{\alpha^2 T^2} e^{-\alpha(|\tau|-T)} + \frac{\sigma^2}{\alpha^2 T^2} e^{-\alpha(|\tau|+T)} & \text{for } |\tau| \geq T \end{cases}.$$

(114)

Let us now investigate what happens to the auto-covariance function (114) if the limit $\alpha \to 0$ is taken. Using the expansion:

$$e^{\mp\alpha t} = 1 \mp \alpha t + \frac{1}{2}\alpha^2 t^2 \mp \frac{1}{6}\alpha^3 t^3 \ldots$$

it follows from (114) that:

(115) $$\lim_{\alpha \to 0} \sigma_{xx}(\tau) = \sigma^2.$$

This agrees with the auto-covariance function of the average, as derived in equation (87) in Section 5.3.3. In a similar way it follows from (114) that:

(116) $$\lim_{\alpha \to \infty} \tfrac{1}{2}\alpha \sigma_{xx}(\tau) = \begin{cases} \sigma^2 \left(\frac{1}{T} - \frac{|\tau|}{T^2}\right) & \text{for } |\tau| \leq T \\ 0 & \text{for } |\tau| \geq T \end{cases}.$$

This shows that if the auto-covariance function of the input $\underline{z}(t)$ is given by $\sigma_{zz}(\tau) = \frac{1}{2}\alpha\sigma^2 e^{-\alpha|\tau|}$, the auto-covariance function of the moving average $\underline{x}(t)$ approaches, as α goes to infinity, the triangular function (116), see Figure 5.4.

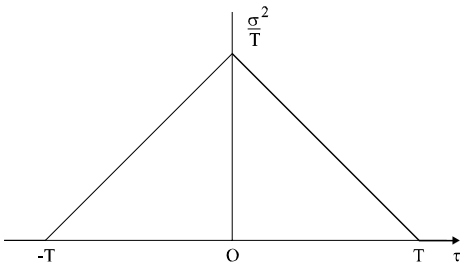


Figure 5.4: The triangular auto-covariance function (116).

Example 61

Consider a particle that moves along a straight line. It is assumed that the acceleration of the particle is measured continuously with an accelerometer. The initial position and initial velocity of the particle are assumed known and zero. The position $\underline{x}(t)$ of the particle follows then from the observed acceleration $\underline{\ddot{x}}(t)$ as (see e.g. equation (11) in Section 3.2):

$$\underline{x}(t) = \int_{t_0}^{t} (t-\tau)\,\underline{\ddot{x}}(\tau)\,d\tau. \tag{117}$$

The auto-covariance function of the observed acceleration is assumed to be given as:

$$\sigma_{\ddot{x}\ddot{x}}(\tau) = \sigma^2 e^{-\alpha|\tau|},\ \alpha \geq 0. \tag{118}$$

We are asked to compute the variance $\sigma_{xx}(t,t)$ of the position $\underline{x}(t)$. From (117) and (118) it follows with (76) that:

$$\sigma_{xx}(t,t) = \sigma^2 \int_{t_0}^{t}\int_{t_0}^{t} (t-\tau_1)(t-\tau_2) e^{-\alpha|\tau_2-\tau_1|}\,d\tau_1 d\tau_2.$$

This may be written, with the change of variable $\tau_1 = \tau_2 - \tau$, as:

$$\sigma_{xx}(t,t) = \sigma^2 \int_{t_0}^{t}\int_{\tau_2-t}^{\tau_2-t_0} (t-\tau_2+\tau)(t-\tau_2) e^{-\alpha|\tau|}\,d\tau d\tau_2$$

or as:

$$\sigma_{xx}(t,t) = \sigma^2 \int_{t_0}^{t} \left(\int_{\tau_2-t}^{0} (t-\tau_2+\tau)(t-\tau_2) e^{\alpha\tau} d\tau + \int_{0}^{\tau_2-t_0} (t-\tau_2+\tau)(t-\tau) e^{-\alpha\tau} d\tau \right) d\tau_2. \tag{119}$$

Integration of the inner integrals between brackets gives:

$$\begin{cases} \displaystyle\int_{\tau_2-t}^{0} (t-\tau_2+\tau)(t-\tau_2) e^{\alpha\tau} d\tau = \frac{1}{\alpha}(t-\tau_2)\left((t-\tau_2-\frac{1}{\alpha}) + \frac{1}{\alpha} e^{\alpha(\tau_2-\tau)}\right), \text{ and} \\ \displaystyle\int_{0}^{\tau_2-t_0} (t-\tau_2+\tau)(t-\tau_2) e^{-\alpha\tau} d\tau = \frac{1}{\alpha}(t-\tau_2)\left((t-\tau_2+\frac{1}{\alpha}) - (t-t_0+\frac{1}{\alpha}) e^{-\alpha(\tau_2-t_0)}\right) \end{cases}. \tag{120}$$

Substitution of (120) into (119) gives:

$$\sigma_{xx}(t,t) = \sigma^2 \int_{t_0}^{t} \frac{1}{\alpha}(t-\tau_2)\left(2(t-\tau_2) + \frac{1}{\alpha} e^{\alpha(\tau_2-t)} - (t-t_0+\frac{1}{\alpha}) e^{-\alpha(\tau_2-t_0)}\right) d\tau_2.$$

After integration, the variance function of position $\underline{x}(t)$ follows as:

(121) $$\sigma_{xx}(t,t) = \frac{2\sigma^2}{\alpha}\left(\frac{1}{\alpha^3} - \frac{1}{2\alpha}(t-t_0)^2 + \frac{1}{3}(t-t_0)^3 - \frac{1}{\alpha^2}(t-t_0+\frac{1}{\alpha})e^{-\alpha(t-t_0)}\right).$$

If we substitute the expansion:

$$e^{-\alpha(t-t_0)} = 1 - \alpha(t-t_0) + \frac{1}{2}\alpha^2(t-t_0)^2 - \frac{1}{6}\alpha^3(t-t_0)^3 + \frac{1}{24}\alpha^4(t-t_0)^4 \ldots$$

into (121) and rearrange terms we get:

(122) $$\sigma_{xx}(t,t) = \sigma^2[\frac{1}{4}(t-t_0)^4 - \frac{1}{15}\alpha(t-t_0)^5 \ldots].$$

This shows that:

(123) $$\lim_{\alpha \to 0} \sigma_{xx}(t,t) = \frac{1}{4}\sigma^2(t-t_0)^4.$$

This would be the variance of position, if the acceleration is treated as a random constant, having as its auto-covariance function the constant $\sigma_{\ddot{x}\ddot{x}}(\tau) = \sigma^2$. It also follows from (121) that:

(124) $$\lim_{\alpha \to \infty} \frac{1}{2}\alpha\sigma_{xx}(t,t) = \frac{1}{3}\sigma^2(t-t_0)^3.$$

This would for large, but finite, α approximate the variance of position, if the continuously observed accelerations have the auto-covariance function $\sigma_{\ddot{x}\ddot{x}}(\tau) = \frac{1}{2}\alpha\sigma^2 e^{-\alpha|\tau|}$.

5.4 White noise

Up to this point we have been dealing with random functions $\underline{z}(t)$ of which the auto-covariance functions $\sigma_{zz}(t,t+\tau)$ satisfy for any t $\sigma_{zz}(t,t+\tau) \neq 0$ if $\tau \neq 0$. That is, we have so far considered only random functions $\underline{z}(t)$ of which the random variables of any pair $\underline{z}(t_1)$, $\underline{z}(t_2)$ are correlated for $t_1 \neq t_2$. The interesting question now arises as to how to define a random function $\underline{z}(t)$ of which the random variables of any pair $\underline{z}(t_1)$, $\underline{z}(t_2)$ are uncorrelated for $t_1 \neq t_2$. Such a random function will be called a white noise random function. By drawing the parallel with the discrete case, one is tempted to define a white noise random function as one for which the auto-covariance function satisfies $\sigma_{zz}(t,t+\tau) = 0$ for $\tau \neq 0$ and $\sigma_{zz}(t,t+\tau) \neq 0$ otherwise. However, as it turns out, no such random function exists, except in a highly degenerate sense. To gain some insight into this rather surprising result, let us start with a brief review of the discrete case. A random variable $\underline{x}_k$ is defined in terms of a sequence of random variables $\underline{z}_i$, $i = 1,2,\ldots$, as:

(125) $$\underline{x}_k = \sum_{i=1}^{k} a_i \underline{z}_i .$$

The a_i are non-random constants. For the variances and covariances of the $\underline{z}_i$ we assume that:

(126) $$\sigma_{z_j z_j} = \sigma_i^2 \delta_{ij}.$$

The symbol δ_{ij} denotes the Kronecker symbol. It is defined as:

(127) $$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}.$$

Thus it is assumed that the sequence $\underline{z}_i$ consists of uncorrelated random variables. If we apply the propagation law of variances and covariances to (125) we get:

(128) $$\sigma_{x_k x_l} = \sum_{i=1}^{k} \sum_{j=1}^{l} a_i \sigma_{z_i z_j} a_j .$$

With (126) this gives:

(129) $$\sigma_{x_k x_l} = \sum_{i=1}^{k} \sum_{j=1}^{l} \sigma_i^2 a_i \delta_{ij} a_j$$

and with property (127) of the Kronecker symbol, equation (129) reduces to:

(130) $$\sigma_{x_k x_l} = \sum_{i=1}^{\min.[k,l]} \sigma_i^2 a_i^2$$

with $\min.[k,l]$ defined as:

(131) $$\min.[k,l] = \begin{cases} k & \text{if } k \leq l \\ l & \text{if } l \leq k \end{cases}.$$

Note that the above derivation shows that it is the property of uncorrelatedness of $\underline{z}_i$, that makes the double sum (128) transform into the single sum (130). Let us now consider the continuous case. The continuous counterpart of the sum (125) is given by the integral:

(132) $$\underline{x}(t) = \int_{t_0}^{t} a(\tau)\underline{z}(\tau)d\tau.$$

The function $a(t)$ is non-random. Equation (126) suggests to define the auto-covariance function of an uncorrelated random function $\underline{z}(t)$ as:

(133) $$\sigma_{zz}(t,t+\tau) = \begin{cases} \sigma^2(t) & \text{if } \tau = 0 \\ 0 & \text{if } \tau \neq 0 \end{cases}.$$

Application of the propagation law (76) to (132) gives:

(134) $$\sigma_{xx}(t_1,t_2) = \int_{t_0}^{t_1}\int_{t_0}^{t_2} a(\tau_1)\sigma_{zz}(\tau_1,\tau_2)a(\tau_2)d\tau_1 d\tau_2.$$

This result is the continuous counterpart of the discrete result (128). We may therefore expect to find the continuous counterpart of (130) when (133) is substituted into (134). However, if we substitute (133) into (134), we find that the double integral equals zero! This is because the function (133) vanishes everywhere except along the line $\tau = 0$. Hence, the volume of $\sigma_{zz}(\tau_1,\tau_2)$ is zero. Thus we find, that although (133) seems to be the continuous counterpart of (126), the double integral (134), being zero for this case, does not correspond with the non-zero sum (130). It therefore follows that (133) is not the way to define an uncorrelated random function. Another approach needs therefore be taken. We have seen above in the discrete case that it is the property of uncorrelatedness that enables us to write the double sum (128) as a single sum (130). This shows that instead of defining uncorrelatedness in terms of the zero covariances of the input $\underline{z}_i$, we may also define uncorrelatedness in terms of the variances and covariances of the output $\underline{x}_k$. That is, we may define the sequence $\underline{z}_i$ to be uncorrelated if the variances and covariances of the output $\underline{x}_k$ satisfy the single sum (130). Let us now try to generalize this approach to the continuous case. A random function is then said to be uncorrelated if the auto-covariance of the output of (132) satisfies:

(135) $$\sigma_{xx}(t_1,t_2) = \int_{t_0}^{\min.[t_1,t_2]} \sigma^2(\tau)a(\tau)^2 d\tau.$$

This integral is the continuous counterpart of the sum (130). The question now arises what type of auto-covariance function of the input $\underline{z}(t)$ will make the auto-covariance function of the output look like (135). The answer is that the auto-covariance function of an uncorrelated random function $\underline{z}(t)$ must be a scaled version of the so-called *Dirac delta function* or *impulse function*. That is, the required auto-covariance function of an uncorrelated random function $\underline{z}(t)$ is given by:

(136) $$\boxed{\sigma_{zz}(t_1,t_2) = \sigma^2(t_1)\delta(t_2 - t_1)}\ .$$

where $\delta(t)$ is the Dirac delta function or impulse function. The impulse function $\delta(t)$ is defined by its property that it isolates or reproduces the function value $f(t)$ of any function $f(\tau)$, which is continuous at t, according to the following integral formula:

(137) $$\boxed{\int_{-\infty}^{+\infty} f(t-\tau)\delta(\tau)d\tau = f(t)}\ .$$

Let us now verify that we indeed get (135) when (136) is substituted into (134). Substitution of (136) into (134) gives with the change of variable $\tau_1 = \tau_2 - \tau$:

(138) $$\sigma_{xx}(t_1,t_2) = \int_{t_0}^{t_2} \Big(\int_{\tau_2 - t_1}^{\tau_2 - t_0} \sigma^2(\tau_2 - \tau)\, a(\tau_2 - \tau)\, \delta(\tau)\, d\tau \Big)\; a(\tau_2)\, d\tau_2.$$

if $t_1 \geq t_2$, then $\tau_2 - t_1 \leq 0$ and $\tau_2 - t_0 \geq 0$ for $\tau_2 \in [t_0, t_2]$. The inner integral of (138) may then be considered as a special case of (137). Indeed, defining:

(139) $$f(\tau_2 - \tau) = \begin{cases} \sigma^2(\tau_2 - \tau)\, a(\tau_2 - \tau) & \text{for } \tau_2 - t_1 \leq \tau \leq \tau_2 - t_0 \\ 0 & \text{otherwise} \end{cases}$$

we obtain with (137):

(140) $$\int_{\tau_2 - t_1 \leq 0}^{\tau_2 - t_0 \geq 0} \sigma^2(\tau_2 - \tau)\, a(\tau_2 - \tau)\, \delta(\tau)\, d\tau = \int_{-\infty}^{+\infty} f(\tau_2 - \tau)\, \delta(\tau)\, d\tau = f(\tau_2) = \sigma^2(\tau_2)\, a(\tau_2).$$

Substitution of (140) into (138) gives:

(141) $$\sigma_{xx}(t_1,t_2) = \int_{t_0}^{t_2} \sigma^2(\tau_2)\, a(\tau_2)^2\, d\tau_2$$

which is indeed identical to (135) for $t_2 \leq t_1$. The corresponding result for the case $t_1 \leq t_2$ follows by replacing t_2 in the right-hand side of (141) by t_1. Hence, we have shown that (134) reduces to (135), if the auto-covariance function of $\underline{z}(t)$ satisfies (136). Thus, it seems that a white noise random function $\underline{z}(t)$ may be defined as a random function having (136) as its auto-covariance function. There is however one problem with this definition. The impulse function $\delta(t)$ does not exist as an ordinary function. It will be clear that the essential feature of the above derivation is given by the defining property (137) of the impulse function $\delta(\tau)$. Unfortunately, no ordinary function $\delta(\tau)$ exists, that has the reproducing property (137) for any function $f(t-\tau)$ continuous at t. This can be seen as follows. Let us define a function $\bar{f}(t-\tau)$ in terms of $f(t-\tau)$ as:

(142) $$\bar{f}(t-\tau) = \begin{cases} f(t-\tau) & \text{if } |\tau| \leq T \\ 0 & \text{if } |\tau| > T \end{cases}.$$

It follows then with (137) that:

(143) $$\int_{-T}^{+T} f(t-\tau)\, \delta(\tau)\, d\tau = \int_{-\infty}^{+\infty} f(t-\tau)\, \delta(\tau)\, d\tau = \bar{f}(t) = f(t).$$

But this would imply with (137) that:

(144) $$\int_{-T}^{+T} f(t-\tau)\delta(\tau)d\tau = \int_{-\infty}^{+\infty} f(t-\tau)\delta(\tau)d\tau.$$

And clearly no ordinary function $\delta(\tau)$ exists such that (144) holds true for any function $f(t-\tau)$ continuous at t. That is, no function $\delta(\tau)$ exists that is zero in any interval not containing the origin and non-zero at the origin. Thus, strictly speaking (137), and therefore also (136), make no sense. A way out of this dilemma is however to consider instead of $\delta(\tau)$, a family of functions $\delta_\alpha(\tau)$, parametrized with the parameter α, such that (137) holds true approximately. Consider for instance the following family of functions:

(145) $$\delta_\alpha(\tau) = \begin{cases} \dfrac{\alpha}{2} & \text{if} \quad |\tau| \le \dfrac{1}{\alpha} \\ 0 & \text{if} \quad |\tau| \ge \dfrac{1}{\alpha} \end{cases}.$$

Then:

(146) $$\int_{-\infty}^{+\infty} f(t-\tau)\delta_\alpha(\tau)d\tau = \frac{\alpha}{2}\int_{-\frac{1}{\alpha}}^{\frac{1}{\alpha}} f(t-\tau)d\tau.$$

This is the average of $f(t-\tau)$ over the interval $[-\frac{1}{\alpha},\frac{1}{\alpha}]$, see Figure 5.5 which is a good approximation to $f(t)$ for large values of α.

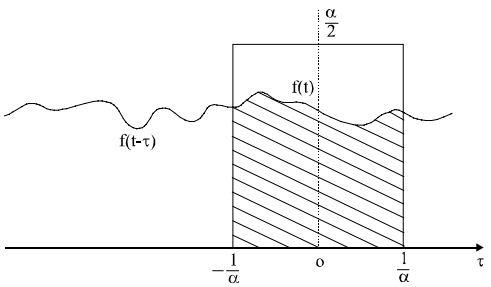


Figure 5.5: The average of $f(t-\tau)$ for $\tau \in [-\frac{1}{\alpha},\frac{1}{\alpha}]$.

In fact, this average reduces exactly to $f(t)$ if the limit $\alpha\to 0$ of the integral (146) is taken. Hence, it follows that although no ordinary function $\delta(\tau)$ exists such that (137) holds true, there do exist functions $\delta_\alpha(\tau)$ for which:

(147) $$\lim_{\alpha\to\infty}\int_{-\infty}^{+\infty} f(t-\tau)\delta_\alpha(\tau)d\tau = f(t).$$

For α large enough, the corresponding class of functions:

(148) $$\sigma_{zz}(t,t+\tau) = \sigma^2(t)\delta_\alpha(\tau)$$

may therefore be considered to approximate to a sufficient degree the auto-covariance function of practically uncorrelated random functions $\underline{z}(t)$. Note namely that for increasing values of α, the auto-covariance function (148) behaves more and more like an impulse, see also Figure 5.6.

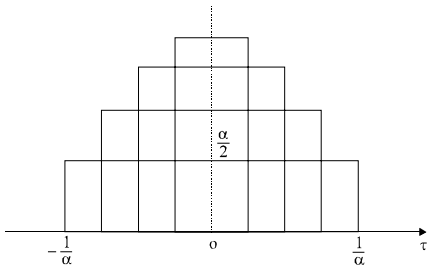


Figure 5.6: Graph of $\delta_\alpha(\tau)$ for different values of α.

The limit of (148) as $\alpha\to\infty$ does, however, not exist. This stipulates, that strictly speaking white noise random functions do not exist. Hence, in order to model the auto-covariance function of a practically uncorrelated random function $\underline{z}(t)$, we should strictly speaking make use of (148) with α large but finite. The finite value of α has however the disadvantage that no use can be made of the integral property (147). As a consequence the simplification from the double integral (134) to the single integral (135) is strictly speaking impossible if α is finite. Practically however, the single integral may be considered a good enough approximation to the double integral for α large enough. Thus for practically uncorrelated random functions, the simplification from (134) to (135) can be done with a sufficient degree of approximation. We will therefore agree from now on to define white noise random functions as those inputs to (132) that result in an output $\underline{x}(t)$ having (135) as auto-covariance function. The name white noise arises out of the fact that $\delta_\alpha(\tau)$ contains in the limit as $\alpha\to\infty$, just as white light, all frequencies in the same amounts. The notation used for the auto-covariance function of a white noise random function will be that of (136), with the impulse function $\delta(\tau)$ satisfying the integral equation (137). And the interpretation given to the integral equation (137) will be that of the limit (147). As a final remark, we note (see 137) that since the impulse function $\delta(\tau)$ has the units of 1/time, the function $\sigma^2(t)$ of (136) cannot have the same units as $\sigma_{zz}(t_1,t_2)$. The function $\sigma^2(t)$ is therefore not a variance function. It is called a spectral density function. And it has the units of the auto-covariance function multiplied with the unit of time.

Example 62

In this example we will show that a random function $\underline{z}(t)$ with auto-covariance function:

(149) $$\sigma_{zz}(\tau) = \frac{1}{2}\alpha\sigma^2 e^{-\alpha|\tau|},\ \alpha > 0$$

can be considered as white noise when $\alpha \to \infty$. The auto-covariance function (149) is plotted in Figure 5.7 for different values of α. Note the impulse-like behaviour of this auto-covariance function as α increases. We consider the following integral:

(150) $$\underline{x}(t) = \int_{t_0}^{t} a(\tau)\underline{z}(\tau)d\tau$$

with $a(t)$ a non-random function.

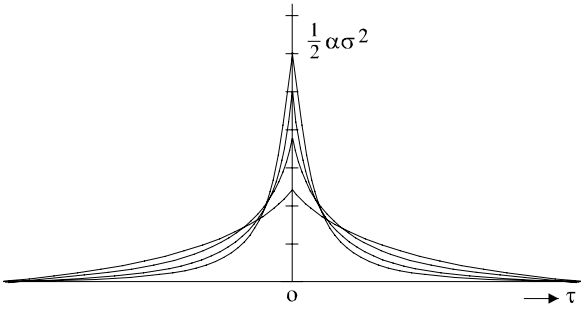


Figure 5.7: The auto-covariance function $\frac{1}{2}\alpha\sigma^2 e^{-\alpha|\tau|}$ for increasing values of α.

Application of propagation law (76) to (150) gives, with the change of variable $\tau_1 = \tau_2 - \tau$:

(151) $$\sigma_{xx}(t_1,t_2) = \sigma^2 \int_{t_0}^{t_2} \left(\int_{\tau_2 - t_1}^{\tau_2 - t_0} \frac{1}{2}\alpha e^{-\alpha|\tau|} a(\tau_2 - \tau)d\tau \right) a(\tau_2)d\tau_2 .$$

Let us first evaluate the inner integral of (151):

(152) $$f_\alpha(\tau_2) = \int_{\tau_2 - t_1}^{\tau_2 - t_o} \frac{1}{2}\alpha e^{-\alpha|\tau|} a(\tau_2 - \tau)d\tau .$$

If $t_1 \geq t_2$, then $\tau_2 - t_1 \leq 0$ and $\tau_2 - t_0 \geq 0$ for $\tau_2 \in [t_0, t_2]$. Hence, we may write (152) as:

(153) $$f_\alpha(\tau_2) = \int_{\tau_2 - t_1}^{0} \frac{1}{2}\alpha e^{\alpha\tau} a(\tau_2 - \tau)d\tau + \int_{0}^{\tau_2 - t_0} \frac{1}{2}\alpha e^{-\alpha\tau} a(\tau_2 - \tau)d\tau .$$

Integration by parts gives for the first integral:

$$\int_{\tau_2-t_1}^{o} \tfrac{1}{2}\alpha e^{\alpha\tau} a(\tau_2-\tau)d\tau = [\tfrac{1}{2}e^{\alpha\tau}a(\tau_2-\tau)]_{\tau_2-t_1}^{0} + \int_{\tau_2-t_1}^{0} \tfrac{1}{2}e^{\alpha\tau}\frac{da}{d\tau}(\tau_2-\tau)d\tau$$

(154)

$$= \tfrac{1}{2}a(\tau_2) - \tfrac{1}{2}a(t_1)e^{\alpha(\tau_2-t_1)} + \int_{\tau_2-t_1}^{0} \tfrac{1}{2}e^{\alpha\tau}\frac{da}{d\tau}(\tau_2-\tau)d\tau$$

and for the second integral:

$$\int_{0}^{\tau_2-t_0} \tfrac{1}{2}\alpha e^{-\alpha\tau} a(\tau_2-\tau)d\tau = [-\tfrac{1}{2}e^{-\alpha\tau}a(\tau_2-\tau)]_{0}^{\tau_2-t_0} - \int_{0}^{\tau_2-t_0} \tfrac{1}{2}e^{-\alpha\tau}\frac{da}{d\tau}(\tau_2-\tau)d\tau$$

(155)

$$= \tfrac{1}{2}a(\tau_2) - \tfrac{1}{2}a(t_0)e^{-\alpha(\tau_2-t_0)} - \int_{0}^{\tau_2-t_0} \tfrac{1}{2}e^{-\alpha\tau}\frac{da}{d\tau}(\tau_2-\tau)d\tau .$$

Since $\tau_2 - t_1 \leq 0$ and $\tau_2 - t_0 \geq 0$, it follows that the limits of (154) and (155) as $\alpha\to\infty$ are both equal to $\frac{1}{2}a(\tau_2)$. Hence (152), which is the sum of (154) and (155), becomes after taking the limit $\alpha\to\infty$, equal to $a(\tau_2)$:

$$\lim_{\alpha\to\infty} \int_{\tau_2-t_1}^{\tau_2-t_1} \tfrac{1}{2}\alpha e^{-\alpha|\tau|} a(\tau_2-\tau)d\tau = a(\tau_2).$$

(156)

We have therefore shown that for $t_1 \geq t_2$, the limit of (151) as $\alpha\to\infty$ equals:

$$\lim_{\alpha\to\infty} \sigma_{xx}(t_1,t_2) = \sigma^2 \int_{t_0}^{t_2} a(\tau_2)^2 d\tau_2 .$$

(157)

The corresponding result for the case $t_2 \geq t_1$ follows by replacing t_2 on the right-hand side of (157) by t_1. As a general result we therefore have:

$$\lim_{\alpha\to\infty} \sigma_{xx}(t_1,t_2) = \sigma^2 \int_{t_0}^{\min.[t_1,t_2]} a(\tau)^2 d\tau .$$

(158)

And this proves that the random function $\underline{z}(t)$ can indeed be considered to be white noise for $\alpha\to\infty$.

5.5 The auto-covariance of the output of a linear system: the white noise case.

In this section we will show how to compute the auto-covariance of the output:

(159) $$\underline{x}(t) = \Phi(t,t_0)\underline{x}(t_0) + \int_{t_0}^{t} \Phi(t,\tau)G(\tau)\underline{z}(\tau)\,d\tau$$

when the input $\underline{z}(t)$ is a white noise random function with auto-covariance matrix:

(160) $$\boxed{Q_{zz}(t_1,t_2) = S_{zz}(t_1)\delta(t_2 - t_1)}\ .$$

The matrix $S_{zz}(t)$ is called the *spectral density matrix* of the random function $\underline{z}(t)$. It will be assumed that the input $\underline{z}(t)$ is uncorrelated with the initial state vector $\underline{x}(t_0)$:

(161) $$Q_{xz}(t_0,t) = Q_{zx}(t,t_0)^* = 0 \text{ for all } t.$$

Then, according to (76), the following propagation law applies:

(162) $$\begin{aligned} Q_{xx}(t_1,t_2) &= \Phi(t_1,t_0)Q_{xx}(t_0,t_0)\Phi(t_2,t_0)^* + \\ &\int_{t_0}^{t_1}\int_{t_0}^{t_2} \Phi(t_1,\tau_1)G(\tau_1)Q_{zz}(\tau_1,\tau_2)G(\tau_2)^*\Phi(t_2,\tau_2)^* d\tau_1 d\tau_2 \end{aligned}$$

Substitution of (160) into (162) gives after the change of variable $\tau_1 = \tau_2 - \tau$:

(163) $$\begin{aligned} Q_{xx}(t_1,t_2) &= \Phi(t_1,t_0)Q_{xx}(t_0,t_0)\Phi(t_2,t_o)^* + \\ &\int_{t_0}^{t_2}\Big(\int_{\tau_2 - t_1}^{\tau_2 - t_0} \Phi(t_1,\tau_2 - \tau)G(\tau_2 - \tau)S_{zz}(\tau_2 - \tau)\delta(\tau)d\tau\Big)G(\tau_2)^*\Phi(t_2,\tau_2)^* d\tau_2 \end{aligned}$$

If $t_1 \geq t_2$, then $\tau_2 - t_1 \leq 0$ and $\tau_2 - t_0 \geq 0$ for $\tau_2 \in [t_0,t_2]$. In this case the inner integral of (163) can be evaluated using property (137) of the impulse function. The result reads $\Phi(t_1,\tau_2)G(\tau_2)S_{zz}(\tau_2)$. We therefore have for $t_1 \geq t_2$:

(164) $$\begin{aligned} Q_{xx}(t_1,t_2) &= \Phi(t_1,t_0)Q_{xx}(t_0,t_0)\Phi(t_2,t_0)^* + \\ &\int_{t_0}^{t_2} \Phi(t_1,\tau_2)G(\tau_2)S_{zz}(\tau_2)G(\tau_2)^*\Phi(t_2,\tau_2)^* d\tau_2. \end{aligned}$$

The corresponding result for the case $t_2 \geq t_1$ follows by replacing t_2 in the integral of (164) by t_1. Hence, the propagation law for the auto-covariance of the output of a linear system driven by white noise, reads:

(165) $$\boxed{\begin{aligned} Q_{xx}(t_1,t_2) &= \Phi(t_1,t_0)Q_{xx}(t_0,t_0)\Phi(t_2,t_0)^* + \\ &\int_{t_0}^{\min.[t_1,t_2]} \Phi(t_1,\tau)G(\tau)S_{zz}(\tau)G(\tau)^*\Phi(t_2,\tau)^* d\tau \end{aligned}}\ .$$

Example 63

Let $\underline{x}(t)$ be defined as:

$$\underline{x}(t) = \int_{t_0}^{t} \underline{z}(\tau) d\tau \tag{166}$$

with the auto-covariance function of the white noise input $\underline{z}(t)$ given as:

$$\sigma_{zz}(\tau) = \sigma^2 \delta(\tau). \tag{167}$$

If we apply the propagation law (165) to (166) we get with (167) for the auto-covariance function of the output $\underline{x}(t)$:

$$\sigma_{xx}(t_1,t_2) = \int_{t_0}^{\min.[t_1,t_2]} \sigma^2 d\tau = \sigma^2 \min.[t_1 - t_0, t_2 - t_0]. \tag{168}$$

Note that this result is identical to the result one would get if the limit $\alpha \to \infty$ were taken of $\frac{1}{2}\alpha$ times the auto-covariance function $\sigma_{xx}(t_1,t_2)$ of (102) in Section 5.3.3.

Example 64

Let $\underline{x}(t)$ be defined as:

$$\underline{x}(t) = \int_{t_0}^{t} (t - \tau)\underline{z}(\tau) d\tau \tag{169}$$

with the auto-covariance function of the white noise input $\underline{z}(t)$ given as:

$$\sigma_{zz}(\tau) = \sigma^2 \delta(\tau). \tag{170}$$

Application of the propagation law (165) to (169) gives with (170):

$$\begin{aligned} \sigma_{xx}(t_1,t_2) &= \sigma^2 \int_{t_0}^{\min.[t_1,t_2]} (t_1 - \tau)(t_2 - \tau) d\tau \\ &= \sigma^2 [t_1 t_2 \tau - \tfrac{1}{2}(t_1 + t_2)\tau^2 + \tfrac{1}{3}\tau^3]_{t_0}^{\min.[t_1,\tau_2]} \end{aligned} \tag{171}$$

Hence, the variance of $\underline{x}(t)$ reads:

(172) $$\sigma_{xx}(t,t) = \frac{1}{3}\sigma^2(t-t_0)^3.$$

Compare this result with (124) in Section 5.3.3.

Example 65

Let $\underline{x}(t)$ be defined through the differential equation:

(173) $$\dot{\underline{x}}(t) = -\alpha\underline{x}(t) + \underline{z}(t),\ \alpha>0.$$

The output $\underline{x}(t)$ reads then:

(174) $$\underline{x}(t) = e^{-\alpha(t-t_0)}\underline{x}(t_0) + \int_{t_0}^{t} e^{-\alpha(t-\tau)}\underline{z}(\tau)d\tau.$$

It is assumed that the input $\underline{z}(t)$ is uncorrelated with the initial state $\underline{x}(t_0)$. If we apply propagation law (76) to (174), the auto-covariance function of the output $\underline{x}(t)$ follows as:

(175) $$\begin{aligned}\sigma_{xx}(t_1,t_2) &= e^{-\alpha(t_1-t_0)}\sigma_{xx}(t_0,t_0)e^{-\alpha(t_2-t_0)} + \\ &\int_{t_0}^{t_1}\int_{t_0}^{t_2} e^{-\alpha(t_1-\tau_1)}\sigma_{zz}(\tau_1,\tau_2)e^{-\alpha(t_2-\tau_2)}d\tau_1 d\tau_2\end{aligned}.$$

The double integral of (175) transforms into a single integral if $\underline{z}(t)$ is a white noise random function. Let us assume that the input $\underline{z}(t)$ is a white noise random function with auto-covariance function:

(176) $$\sigma_{zz}(\tau) = 2\alpha\sigma^2\delta(\tau).$$

Then (175) simplifies, according to (165), to:

(177) $$\begin{aligned}\sigma_{xx}(t_1,t_2) &= e^{-\alpha(t_1-t_0)}\sigma_{xx}(t_0,t_0)e^{-\alpha(t_2-t_0)} + \\ &2\alpha\sigma^2\int_{t_0}^{\min.[t_1,t_2]} e^{-\alpha(t_1-\tau)}e^{-\alpha(t_2-\tau)}d\tau\end{aligned}.$$

If $t_2 \geq t_1$, equation (177) may be written as:

$$\sigma_{xx}(t_1,t_2) = e^{-\alpha(t_2-t_1)}(e^{-2\alpha(t_1-t_0)}\sigma_{xx}(t_0,t_0) + 2\alpha\sigma^2\int_{t_0}^{t_1} e^{-2\alpha(t_1-\tau)}d\tau).$$

Solving for the integral gives:

(178) $$\sigma_{xx}(t_1,t_2) = e^{-\alpha(t_2-t_1)}(e^{-2\alpha(t_1-t_0)}\sigma_{xx}(t_0,t_0) + \sigma^2(1-e^{-2\alpha(t_1-t_0)})).$$

This shows that if the variance of the initial state $\underline{x}(t_0)$ equals:

$$\sigma_{xx}(t_0,t_0) = \sigma^2 \tag{179}$$

the auto-covariance function of the output $\underline{x}(t)$ becomes:

$$\sigma_{xx}(t_1,t_2) = \sigma^2 e^{-\alpha(t_2-t_1)}. \tag{180}$$

The corresponding result for the case $t_1 \geq t_2$ follows by interchanging t_1 and t_2 in the right-hand side of (180). Hence, we have shown that if $\underline{x}(t)$ satifies (173) and (179), and if the input $\underline{z}(t)$ is uncorrelated with the initial state and satisfies (176), then the auto-covariance function of the output $\underline{x}(t)$ equals the exponential:

$$\sigma_{xx}(t_1,t_2) = \sigma^2 e^{-\alpha|t_2-t_1|}. \tag{181}$$

This is an important result. It shows that a random function $\underline{x}(t)$ having an exponential as its auto-covariance function, can be thought of as being generated by the first-order differential equation (173) with a white noise input $\underline{z}(t)$.

The propagation law for the auto-covariance of the output of a linear system driven by white noise is given by (165). This law is written as an integral equation. It is also possible, however, to write this propagation law as a linear matrix differential equation. In order to show this, we first consider the variance matrix of $\underline{x}(t)$. The variance matrix of $\underline{x}(t)$ follows by setting t_1 and t_2 in (165) equals to t:

$$Q_{xx}(t,t) = \Phi(t,t_0)Q_{xx}(t_0,t_0)\Phi(t,t_0)^* + \int_{t_0}^{t} \Phi(t,\tau)G(\tau)S_{zz}G(\tau)^*\Phi(t,\tau)^* d\tau\,. \tag{182}$$

We will now transform this integral equation into a linear matrix differential equation. Taking the time-derivative of (182) gives:

$$\begin{aligned}\frac{d}{dt}Q_{xx}(t,t) = {} & \frac{\partial}{\partial t}\Phi(t,t_0)Q_{xx}(t_0,t_0)\Phi(t,t_0)^* + \Phi(t,t_0)Q_{xx}(t_0,t_0)\frac{\partial}{\partial t}\Phi(t,t_0)^* + \Phi(t,t)G(t)S_{zz}(t)G(t)^*\Phi(t,t) + \\ & \int_{t_0}^{t}\frac{\partial}{\partial t}\Phi(t,\tau)G(\tau)S_{zz}(\tau)G(\tau)^*\Phi(t,\tau)^* d\tau + \int_{t_0}^{t}\Phi(t,\tau)G(\tau)S_{zz}(\tau)G(\tau)^*\frac{\partial}{\partial t}\Phi(t,\tau)^* d\tau.\end{aligned} \tag{183}$$

Recall that the transition matrix satisfies (see equation (93) in Section 4.6):

$$\frac{\partial}{\partial t}\Phi(t,t_0) = F(t)\Phi(t,t_0) \text{ and } \Phi(t_0,t_0) = I.$$

Equation (183) may therefore be written as:

$$\begin{aligned}\frac{d}{dt}Q_{xx}(t,t) &= F(t)\left(\Phi(t,t_0)Q_{xx}(t_0,t_0)\Phi(t,t_0)^* + \int_{t_0}^{t}\Phi(t,\tau)G(\tau)S_{zz}(\tau)G(\tau)^*\Phi(t,\tau)^*d\tau\right)\\ &+ \left(\Phi(t,t_0)Q_{xx}(t_0,t_0)\Phi(t,t_0)^* + \int_{t_0}^{t}\Phi(t,\tau)G(\tau)S_{zz}(\tau)G(\tau)^*\Phi(t,\tau)^*d\tau\right)F(t)\\ &+ G(t)S_{zz}(t)G(t)^* \ .\end{aligned} \tag{184}$$

And by substituting (182) into (184) we obtain:

$$\boxed{\dot{Q}_{xx}(t,t) = F(t)Q_{xx}(t,t) + Q_{xx}(t,t)F(t) + G(t)S_{zz}(t)G(t)^*} \ . \tag{185}$$

This is a linear matrix differential equation for the variance matrix $Q_{xx}(t,t)$ of $\underline{x}(t)$. Once (185) has been solved for $Q_{xx}(t,t)$, the auto-covariance matrix of $\underline{x}(t)$ can be computed from $Q_{xx}(t,t)$ as:

$$\boxed{Q_{xx}(t_1,t_2) = \begin{cases}\Phi(t_1,t_2)Q_{xx}(t_2,t_2) & \text{for } t_1 \geq t_2\\ Q_{xx}(t_1,t_1)\Phi(t_2,t_1)^* & \text{for } t_2 \geq t_1\end{cases}} \ . \tag{186}$$

Example 66

Let $\underline{x}(t)$ be defined through the differential equation:

$$\dot{\underline{x}}(t) = -\alpha\underline{x}(t) + \underline{z}(t), \ \alpha > 0. \tag{187}$$

The input $\underline{z}(t)$ is assumed to be white noise, which is uncorrelated with the initial state $\underline{x}(t_0)$. The auto-covariance of $\underline{z}(t)$ is given as:

$$\sigma_{zz}(\tau) = 2\alpha\sigma^2\delta(\tau). \tag{188}$$

According to (185) the variance $\sigma_x^2(t)$ of $\underline{x}(t)$ satisfies:

$$\begin{aligned}\dot{\sigma}_x^2(t) &= (-\alpha)\sigma_x^2(t) + \sigma_x^2(t)(-\alpha) + 1\cdot 2\alpha\sigma^2\cdot 1\\ &= -2\alpha\sigma_x^2(t) + 2\alpha\sigma^2.\end{aligned} \tag{189}$$

The solution of this differential equation reads:

$$\begin{aligned}\sigma_x^2(t) &= e^{-2\alpha(t-t_0)}\sigma_x^2(t_0) + \int_{t_0}^{t} e^{-2\alpha(t-\tau)} 2\alpha\sigma^2 d\tau \\ &= e^{-2\alpha(t-t_0)}\sigma_x^2(t_0) + \sigma^2[e^{-2\alpha(t-\tau)}]_{t_0}^{t} \\ &= \sigma^2 + (\sigma_x^2(t_0) - \sigma^2)e^{-2\alpha(t-t_0)}.\end{aligned} \tag{190}$$

For the auto-covariance function $\sigma_{xx}(t_1,t_2)$ we have according to (186):

$$\sigma_{xx}(t_1,t_2) = \begin{cases} e^{-\alpha(t_2-t_1)}\sigma_x^2(t_2) & \text{for } t_1 \geq t_2 \\ \sigma_x^2(t_1)e^{-\alpha(t_1-t_2)} & \text{for } t_2 \geq t_1 \end{cases}. \tag{191}$$

Substitution of (190) into (191) gives then:

$$\sigma_{xx}(t_1,t_2) = \begin{cases} e^{-\alpha(t_2-t_1)}(\sigma^2 + (\sigma_x^2(t_0) - \sigma^2)e^{-2\alpha(t_2-t_0)}) & \text{for } t_1 \geq t_2 \\ e^{-\alpha(t_1-t_2)}(\sigma^2 + (\sigma_x^2(t_0) - \sigma^2)e^{-2\alpha(t_1-t_0)}) & \text{for } t_2 \geq t_1 \end{cases}. \tag{192}$$

This shows that if $\sigma_x^2(t_0) = \sigma^2$, then:

$$\sigma_{xx}(t_1,t_2) = \sigma^2 e^{-\alpha|t_2-t_1|}. \tag{193}$$

Compare this result with (181) in Section 5.4.

Example 67

Let $\underline{x}(t)$ be defined through the second-order differential equation:

$$\ddot{\underline{x}}(t) = -\underline{x}(t) + \underline{z}(t). \tag{194}$$

The input $\underline{z}(t)$ is assumed to be white noise, with auto-covariance function:

$$\sigma_{zz}(\tau) = \sigma^2\delta(\tau). \tag{195}$$

The initial conditions of (194) are assumed to be constant. Hence:

$$\sigma_x^2(t_0) = 0,\ \sigma_{\dot{x}}^2(t_0) = 0,\ \sigma_{x\dot{x}}(t_0,t_0) = 0. \tag{196}$$

We are asked to derive the variance $\sigma_{xx}(t,t)$ of the output $\underline{x}(t)$. First, we write the second-order scalar differential equation (194) as a first-order vector differential equation. To put (194) into a state vector form, we define the two-dimensional state vector as:

(197) $$(\underline{x}_1(t),\ \underline{x}_2(t))^* = (\underline{x}(t),\ \dot{\underline{x}}(t))^*.$$

The state vector form of (194) follows then as:

(198) $$\begin{pmatrix}\underline{x}_1(t)\\ \underline{x}_2(t)\end{pmatrix} = \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\begin{pmatrix}\underline{x}_1(t)\\ \underline{x}_2(t)\end{pmatrix} + \begin{pmatrix}0\\ 1\end{pmatrix}\underline{z}(t).$$

Two methods can now be used for deriving the auto-covariance of the output of (198). The first method makes use of the integral form (165) of the propagation law for the auto-covariance, the second is based on the linear matrix differential equation (185). We will consider both methods.

First method

In order to make use of (165), we first need to find the matrix exponential of the system matrix:

(199) $$F = \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}.$$

Since:

$$e^{Ft} = \sum_{i=0}^{\infty}\frac{F^i t^i}{i!} = \begin{pmatrix}(1-\frac{1}{2}t^2+\frac{1}{24}t^4\ldots) & (t-\frac{1}{6}t^3\ldots)\\ (-t+\frac{1}{6}t^3\ldots) & (1-\frac{1}{2}t^2+\frac{1}{24}t^4\ldots)\end{pmatrix}.$$

it follows that:

(200) $$\Phi(t,t_0) = e^{F(t-t_0)} = \begin{pmatrix}\cos(t-t_0) & \sin(t-t_0)\\ -\sin(t-t_0) & \cos(t-t_0)\end{pmatrix}.$$

With the initial state vector of (198) being constant, the variance matrix of the output follows according to (165) as:

$$\begin{aligned}Q_{xx}(t,t) &= \int_{t_0}^{t}\Phi(t,\tau)G(\tau)S_{zz}(\tau)G(\tau)^*\Phi(t,\tau)^*d\tau\\ &= \int_{t_0}^{t}\begin{pmatrix}\cos(t-\tau) & \sin(t-\tau)\\ -\sin(t-\tau) & \cos(t-\tau)\end{pmatrix}\begin{pmatrix}0 & 0\\ 0 & \sigma^2\end{pmatrix}\begin{pmatrix}\cos(t-\tau) & -\sin(t-\tau)\\ \sin(t-\tau) & \cos(t-\tau)\end{pmatrix}d\tau\\ &= \sigma^2\int_{t_0}^{t}\begin{pmatrix}\sin^2(t-\tau) & \sin(t-\tau)\cos(t-\tau)\\ \sin(t-\tau)\cos(t-\tau) & \cos^2(t-\tau)\end{pmatrix}d\tau.\end{aligned}$$

Integration gives:

$$Q_{xx}(t,t) = \sigma^2 \begin{pmatrix} [\frac{1}{2}\tau + \frac{1}{4}\sin 2(t-\tau)]_{t_0}^{t} & [\frac{1}{4}\cos^2(t-\tau)]_{t_0}^{t} \\ [\frac{1}{4}\cos^2(t-\tau)]_{t_0}^{t} & [\frac{1}{2}\tau - \frac{1}{4}\sin 2(t-\tau)]_{t_0}^{t} \end{pmatrix}$$

or

$$Q_{xx}(t,t) = \sigma^2 \begin{pmatrix} \frac{1}{2}(t-t_0) - \frac{1}{4}\sin 2(t-t_0) & \frac{1}{4} - \frac{1}{4}\cos^2(t-t_0) \\ \frac{1}{4} - \frac{1}{4}\cos^2(t-t_0) & \frac{1}{2}(t-t_0) + \frac{1}{4}\sin 2(t-t_0) \end{pmatrix}. \tag{201}$$

Hence, the required variance-function of the output $\underline{x}(t)$ of (194) reads:

$$\sigma_{xx}(t,t) = \frac{1}{2}\sigma^2((t-t_0) - \frac{1}{2}\sin 2(t-t_0)). \tag{202}$$

Second method

According to (185), the linear matrix differential equation for the variance matrix of the output of the state equation (198) reads:

$$\frac{d}{dt}\begin{pmatrix} \sigma_{x_1}^2(t) & \sigma_{x_1x_2}(t,t) \\ \sigma_{x_2x_1}(t,t) & \sigma_{x_2}^2(t) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} \sigma_{x_1}^2(t) & \sigma_{x_1x_2}(t,t) \\ \sigma_{x_2x_1}(t,t) & \sigma_{x_2}^2(t) \end{pmatrix} + \begin{pmatrix} \sigma_{x_1}^2(t) & \sigma_{x_1x_2}^2(t,t) \\ \sigma_{x_2x_1}(t,t) & \sigma_{x_2}^2(t) \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}\sigma^2\begin{pmatrix} 0 \\ 1 \end{pmatrix}^*. \tag{203}$$

This matrix equation contains three independent scalar equations:

$$\begin{cases} \dot{\sigma}_{x_1}^2(t) = \sigma_{x_2x_1}(t,t) + \sigma_{x_1x_2}(t,t) = 2\sigma_{x_1x_2}(t,t) \\ \dot{\sigma}_{x_1x_2}(t,t) = \sigma_{x_2}^2(t) - \sigma_{x_1}^2(t) \\ \dot{\sigma}_{x_2}^2(t) = -\sigma_{x_1x_2}(t,t) - \sigma_{x_2x_1}(t,t) + \sigma^2 = -2\sigma_{x_1x_2}(t,t) + \sigma^2. \end{cases} \tag{204}$$

These equations may be put into state vector form as:

$$\begin{pmatrix} \dot{\sigma}^2_{x_1}(t) \\ \dot{\sigma}_{x_1x_2}(t,t) \\ \dot{\sigma}^2_{x_2}(t) \end{pmatrix} = \begin{pmatrix} 0 & 2 & 0 \\ -1 & 0 & 1 \\ 0 & -2 & 0 \end{pmatrix} \begin{pmatrix} \sigma^2_{x_1}(t) \\ \sigma_{x_1x_2}(t,t) \\ \sigma^2_{x_2}(t) \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \sigma^2. \tag{205}$$

In order to solve this state equation, we first need to find the matrix exponential of the system matrix:

$$F = \begin{pmatrix} 0 & 2 & 0 \\ -1 & 0 & 1 \\ 0 & -2 & 0 \end{pmatrix}. \tag{206}$$

We will use the Jordan canonical form (see Section 4.8.2) for the derivation of the matrix exponential of F. The eigenvalues of the system matrix F follow from solving the characteristic equation:

$$\det(F - \lambda I_3) = \begin{pmatrix} -\lambda & 2 & 0 \\ -1 & -\lambda & 1 \\ 0 & -2 & -\lambda \end{pmatrix} = 0.$$

This gives:

$$\begin{pmatrix} -\lambda & 2 & 0 \\ -1 & -\lambda & 1 \\ 0 & -2 & -\lambda \end{pmatrix} = -\lambda \cdot \begin{pmatrix} -\lambda & 1 \\ -2 & -\lambda \end{pmatrix} + 1 \cdot \begin{pmatrix} 2 & 0 \\ -2 & -\lambda \end{pmatrix} + 0 \cdot \begin{pmatrix} 2 & 0 \\ -\lambda & 1 \end{pmatrix} = 0$$

or

$$-\lambda(\lambda^2 + 4) = 0.$$

Hence, the three distinct eigenvalues of the system matrix F are:

$$\lambda_1 = 0,\ \lambda_2 = 2i,\ \lambda_3 = -2i\ \ (i^2 = -1). \tag{207}$$

The corresponding eigenvectors t_α are obtained by solving $(F - \lambda_\alpha I_3)t_\alpha = 0$, $\alpha = 1,2,3$. Ordering the eigenvectors so obtained by columns, the matrix T of eigenvectors becomes:

$$T = \begin{pmatrix} 1 & -i & i \\ 0 & 1 & 1 \\ 1 & i & -i \end{pmatrix}. \tag{208}$$

Its inverse reads:

$$T^{-1} = \begin{pmatrix} 1/2 & 0 & 1/2 \\ i/4 & 1/2 & -i/4 \\ -i/4 & 1/2 & i/4 \end{pmatrix}. \tag{209}$$

Hence, the matrix exponential of F follows from (207), (208) and (209) as:

$$e^{Ft} = \begin{pmatrix} 1 & -i & i \\ 0 & 1 & 1 \\ 1 & i & -i \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{2it} & 0 \\ 0 & 0 & e^{-2it} \end{pmatrix} \begin{pmatrix} 1/2 & 0 & 1/2 \\ i/4 & 1/2 & -i/4 \\ -i/4 & 1/2 & i/4 \end{pmatrix}. \tag{210}$$

Since the initial state vector of (205) equals zero (see (196)), it follows with (210) that the solution of the state equation (205) satisfies:

$$\begin{pmatrix} \sigma^2_{x_1}(t) \\ \sigma_{x_1x_2}(t,t) \\ \sigma^2_{x_2}(t) \end{pmatrix} = \int_{t_0}^{t} e^{F(t-\tau)} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \sigma^2 d\tau$$

$$= \sigma^2 \int_{t_0}^{t} \begin{pmatrix} \frac{1}{2} - \frac{1}{2}\cos 2(t-\tau) \\ \frac{1}{2}\sin 2(t-\tau) \\ \frac{1}{2} + \frac{1}{2}\cos 2(t-\tau) \end{pmatrix} d\tau .$$

Integration gives:

$$\begin{pmatrix} \sigma^2_{x_1}(t) \\ \sigma_{x_1x_2}(t,t) \\ \sigma^2_{x_2}(t) \end{pmatrix} = \sigma^2 \begin{pmatrix} [\frac{1}{2}\tau + \frac{1}{4}\sin 2(t-\tau)]_{t_0}^{t} \\ [\frac{1}{4}\cos 2(t-\tau)]_{t_0}^{t} \\ [\frac{1}{2}\tau - \frac{1}{4}\sin 2(t-\tau)]_{t_0}^{t} \end{pmatrix}$$

or finally:

$$\begin{pmatrix} \sigma^2_{x_1}(t) \\ \sigma_{x_1x_2}(t,t) \\ \sigma^2_{x_2}(t) \end{pmatrix} = \sigma^2 \begin{pmatrix} \frac{1}{2}(t-t_0) - \frac{1}{4}\sin 2(t-t_0) \\ \frac{1}{4} - \frac{1}{4}\cos 2(t-t_0) \\ \frac{1}{2}(t-t_0) + \frac{1}{4}\sin 2(t-t_0) \end{pmatrix}. \tag{211}$$

Compare this result with that of (201).

5.6 Random polynomial equations of motion

In Section 3.2 polynomial equations of motion were derived for respectively: a stationary object, an object that moves with constant velocity and an object that moves with constant acceleration. As a result we were able to derive a one-to-one, linear relationship between the state vector $x(t)$ at an arbitrary time instant t and the state vector $x(t_0)$ at the initial epoch t_0. The fact that this relationship is one-to-one implies that knowledge of x at any one particular time-instant is sufficient for the exact determination of the complete time history of $x(t)$. But one will agree that this is not very realistic for most practical applications. It is for instance highly unlikely that a ship sailing at cruising speed, will indeed move with an exactly constant velocity. It is more likely that the ship sails with a velocity that can be considered constant on the average. But the velocity will nevertheless be subject to random changes from time-instant to time-instant. For a ship sailing at cruising speed, it seems therefore more realistic to model the ship's acceleration as a random function with zero mean. The expected erratic behaviour of acceleration should then be reflected in the choice of the auto-covariance function of acceleration.

In this section we will develop the random polynomial equations of motion that correspond with, respectively, zero-mean velocity and zero-mean acceleration. As in Section 3.2 we will develop the random polynomial equations of motion only for the single coordinate function $u(t)$. The development for the other coordinate functions is similar.

5.6.1 Random constants as input

Zero-mean velocity

If we assume the initial position to be a random variable and velocity to be a random function, the random position variable follows as (compare with equation (4) in Section 3.2):

(212) $$\boxed{\underline{u}(t) = \underline{u}(t_0) + \int_{t_0}^{t} \dot{\underline{u}}(\tau)\,d\tau}\ .$$

If we model velocity as a zero-mean random function:

(213) $$E\{\dot{\underline{u}}(t)\} = 0$$

the mean of the random position variable becomes:

(214) $$E\{\underline{u}(t)\} = E\{\underline{u}(t_0)\} = \text{constant}.$$

Note that the deterministic relation (5) in Section 3.2 is now replaced by the expectation of $\underline{u}(t)$. We will assume that the velocity input of (212) is a random constant:

(215) $$\boxed{\sigma_{\dot{u}\dot{u}}(t_1,t_2) = \sigma^2}\ .$$

We will also assume that velocity is uncorrelated with the initial position $\underline{u}(t_0)$. Application of propagation law (76) to (212) gives then, with (215), for the variance of $\underline{u}(t)$:

$$\sigma_{uu}(t,t) = \sigma_{uu}(t_0,t_0) + \int_{t_0}^{t}\int_{t_0}^{t} \sigma^2 d\tau_1 d\tau_2$$

or

(216) $$\boxed{\sigma_{uu}(t,t) = \sigma_{uu}(t_0,t_0) + \sigma^2(t-t_0)^2} \ .$$

Zero-mean acceleration

If we assume the initial position and the initial velocity to be random variables, and acceleration to be a random function, the random position and velocity variables follow as (compare with equation (11) in Section 3.2):

(217) $$\boxed{\begin{pmatrix}\underline{u}(t)\\ \dot{\underline{u}}(t)\end{pmatrix} = \begin{pmatrix}1 & (t-t_0)\\ 0 & 1\end{pmatrix}\begin{pmatrix}\underline{u}(t_0)\\ \dot{\underline{u}}(t_0)\end{pmatrix} + \int_{t_0}^{t}\begin{pmatrix}(t-\tau)\\ 1\end{pmatrix}\ddot{\underline{u}}(\tau)d\tau}\ .$$

If we model acceleration as a zero-mean random function:

(218) $$E\{\ddot{\underline{u}}(t)\} = 0$$

the mean of position and velocity becomes:

(219) $$E\left\{\begin{pmatrix}\underline{u}(t)\\ \dot{\underline{u}}(t)\end{pmatrix}\right\} = \begin{pmatrix}1 & (t-t_0)\\ 0 & 1\end{pmatrix} E\left\{\begin{pmatrix}\underline{u}(t_0)\\ \dot{\underline{u}}(t_0)\end{pmatrix}\right\}.$$

Compare this with equation (12) in Section 3.2.

We will assume that the acceleration input to (217) is a random constant:

(220) $$\boxed{\sigma_{\ddot{u}\ddot{u}}(t_1,t_2) = \sigma^2}\ .$$

We will also assume that acceleration is uncorrelated with $\underline{u}(t_0)$ and $\dot{\underline{u}}(t_0)$. Application of propagation law (76) to (217) gives then, with (220), for the variance matrix of $(\underline{u}(t),\dot{\underline{u}}(t))^*$:

$$\begin{pmatrix} \sigma_u^2(t) & \sigma_{u\dot{u}}(t,t) \\ \sigma_{\dot{u}u}(t,t) & \sigma_{\dot{u}}^2(t) \end{pmatrix} =$$

$$\begin{pmatrix} 1 & (t-t_0) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \sigma_u^2(t_0) & \sigma_{u\dot{u}}(t_0,t_0) \\ \sigma_{\dot{u}u}(t_0,t_0) & \sigma_{\dot{u}}^2(t_0) \end{pmatrix} \begin{pmatrix} 1 & (t-t_0) \\ 0 & 1 \end{pmatrix}^* + \int_{t_0}^{t}\int_{t_0}^{t} \sigma^2 \begin{pmatrix} (t-\tau_1)(t-\tau_2) & (t-\tau_1) \\ (t-\tau_2) & 1 \end{pmatrix} d\tau_1 d\tau_2$$

or:

(221)
$$\boxed{\begin{pmatrix} \sigma_u^2(t) & \sigma_{u\dot{u}}(t,t) \\ \sigma_{\dot{u}u}(t,t) & \sigma_{\dot{u}}^2(t) \end{pmatrix} = \begin{pmatrix} 1 & (t-t_0) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \sigma_u^2(t_0) & \sigma_{u\dot{u}}(t_0,t_0) \\ \sigma_{\dot{u}u}(t_0,t_0) & \sigma_{\dot{u}}^2(t_0) \end{pmatrix} \begin{pmatrix} 1 & (t-t_0) \\ 0 & 1 \end{pmatrix}^* + \sigma^2 \begin{pmatrix} \frac{1}{4}(t-t_0)^4 & \frac{1}{2}(t-t_0)^3 \\ \frac{1}{2}(t-t_0)^3 & (t-t_0)^2 \end{pmatrix}}$$

Note that the last matrix of (221) is singular: it has rank 1. This is a direct consequence of the random constant assumption (220).

5.6.2 White noise as input

Velocity as white noise

We will now assume that velocity is a white-noise random function with spectral density $q_{\dot{u}}$:

(222)
$$\boxed{\sigma_{\dot{u}\dot{u}}(\tau) = q_{\dot{u}}\delta(\tau), \quad q_{\dot{u}} \text{ has the dimension } m^2/s}\ .$$

We also assume that velocity is uncorrelated with the initial position $\underline{u}(t_0)$. Application of propagation law (165) to (212) gives then, with (222), for the variance of $\underline{u}(t)$:

$$\sigma_{uu}(t,t) = \sigma_{uu}(t_0,t_0) + \int_{t_0}^{t} q_{\dot{u}} d\tau$$

or:

(223)
$$\boxed{\sigma_{uu}(t,t) = \sigma_{uu}(t_0,t_0) + q_{\dot{u}}(t-t_0)}\ .$$

Compare this result with (216) and note the difference.

Acceleration as white noise

We assume that acceleration is a white noise random function with spectral density $q_{\ddot{u}}$:

(224) $$\boxed{\sigma_{\ddot{u}\ddot{u}}(\tau) = q_{\ddot{u}}\delta(\tau),\ q_{\ddot{u}} \text{ has the dimension } m^2/s^3} .$$

We also assume that acceleration is uncorrelated with $\underline{u}(t_0)$ and $\underline{\dot{u}}(t_0)$. Application of propagation law (165) to (217) gives then, with (224), for the variance matrix of $(\underline{u}(t),\underline{\dot{u}}(t))^*$:

$$\begin{pmatrix} \sigma_u^2(t) & \sigma_{u\dot{u}}(t,t) \\ \sigma_{\dot{u}u}(t,t) & \sigma_{\dot{u}}^2(t) \end{pmatrix} = \begin{pmatrix} 1 & (t-t_0) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \sigma_u^2(t_0) & \sigma_{u\dot{u}}(t_0,t_0) \\ \sigma_{\dot{u}u}(t_0,t_0) & \sigma_{\dot{u}}^2(t_0) \end{pmatrix} \begin{pmatrix} 1 & (t-t_0) \\ 0 & 1 \end{pmatrix}^*$$

$$+ \int_{t_0}^{t} q_{\ddot{u}} \begin{pmatrix} (t-\tau)^2 & (t-\tau) \\ (t-\tau) & 1 \end{pmatrix} d\tau$$

or:

(225) $$\boxed{\begin{pmatrix} \sigma_u^2(t) & \sigma_{u\dot{u}}(t,t) \\ \sigma_{\dot{u}u}(t,t) & \sigma_{\dot{u}}^2(t) \end{pmatrix} = \begin{pmatrix} 1 & (t-t_0) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \sigma_u^2(t_0) & \sigma_{u\dot{u}}(t_0,t_0) \\ \sigma_{\dot{u}u}(t_0,t_0) & \sigma_{\dot{u}}^2(t_0) \end{pmatrix} \begin{pmatrix} 1 & (t-t_0) \\ 0 & 1 \end{pmatrix}^* + q_{\ddot{u}} \begin{pmatrix} \frac{1}{3}(t-t_0)^3 & \frac{1}{2}(t-t_0)^2 \\ \frac{1}{2}(t-t_0)^2 & (t-t_0) \end{pmatrix}} .$$

Compare this result with (221) and note the difference.

5.6.3 Exponentially correlated noise as input

In Section 5.6.2 the random functions were modelled as white noise. This is a realistic assumption if for a particular application it is believed that velocity or acceleration behave in a highly erratic manner. There are applications, however, for which the white noise assumption is not realistic. For instance, an object navigating a straight-line constant velocity course may be acted upon by zero-mean random forces that are correlated in time (e.g., surge and sway, atmospheric turbulence). As a result the acceleration at time t will be correlated with the acceleration at time $t+\tau$ for sufficiently small τ. In this section we will therefore consider velocity and acceleration as being correlated in time. We will assume that the exponential can be used as a representative model for the correlations in time.

Exponentially correlated velocity

We assume that velocity is a random function with auto-covariance function:

(226) $$\boxed{\sigma_{\dot{u}\dot{u}}(\tau) = \sigma^2 e^{-\alpha|\tau|},\ \alpha > 0}\ .$$

We also assume that velocity is uncorrelated with the initial position $\underline{u}(t_0)$. Application of propagation law (76) to (212) gives then, with (226), for the variance of $\underline{u}(t)$:

$$\sigma_{uu}(t,t) = \sigma_{uu}(t_0,t_0) + \int_{t_0}^{t}\int_{t_0}^{t} \sigma^2 e^{-\alpha|\tau_2-\tau_1|} d\tau_1 d\tau_2$$

or

(227) $$\boxed{\sigma_{uu}(t,t) = \sigma_{uu}(t_0,t_0) + \frac{2\sigma^2}{\alpha}((t-t_0) - \frac{1}{\alpha}(1-e^{-\alpha(t-t_0)}))}\ .$$

There are two limiting cases of interest. First, we consider long time intervals:

(228) $$(t-t_0) \gg \frac{1}{\alpha} \text{ or } \alpha(t-t_0) \to \infty.$$

Then, since:

$$\lim_{\alpha(t-t_0)\to\infty} \frac{2\sigma^2}{\alpha}(t-t_0)(1 - \frac{1}{\alpha(t-t_0)}(1-e^{-\alpha(t-t_0)}) = \frac{2\sigma^2}{\alpha}(t-t_0)$$

the variance function of $\underline{u}(t)$ becomes:

(229) $$\boxed{\lim_{\alpha(t-t_0)\to\infty} \sigma_{uu}(t,t) = \sigma_{uu}(t_0,t_0) + \frac{2\sigma^2}{\alpha}(t-t_0)}\ .$$

This shows, when we compare (229) with (223), that for time intervals $(t-t_0)$ that are large compared with the correlation length $1/\alpha$, the contribution of random velocity is essentially that of white noise. This is also clear from Figure 5.8. This figure shows the exponential auto-covariance function on three different time scales. In Figure 5.8b the time scale has been stretched considerably with respect to that of Figure 5.8a. As a result the auto-covariance function exhibits an impulse-like behaviour.

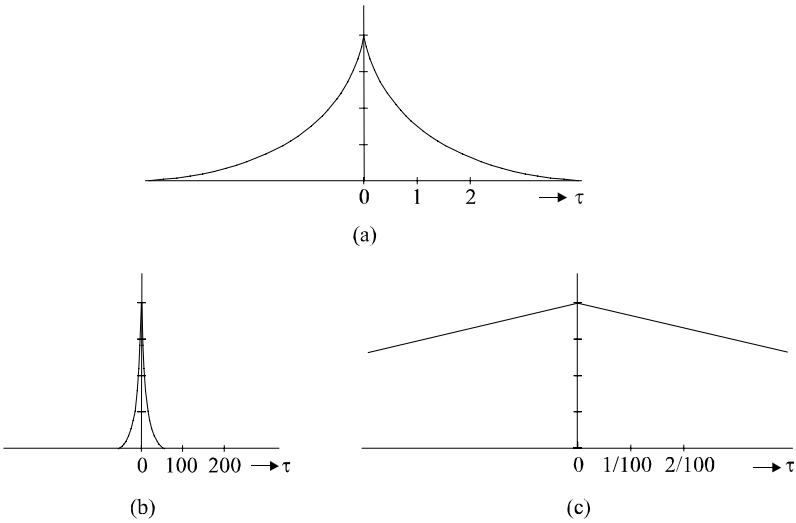


Figure 5.8: The exponential auto-covariance function.

Figure 5.9 shows the relationship between (227) and (223) for some decreasing values of the correlation length $1/\alpha$. It was assumed that $t_0 = 0$, $\sigma_{uu}(t_0,t_0) = 0$ and $\sigma^2 = \frac{1}{2}\alpha q_u$. Note that the position variance in case of correlated velocity is always smaller than the position variance in case of white noise velocity.

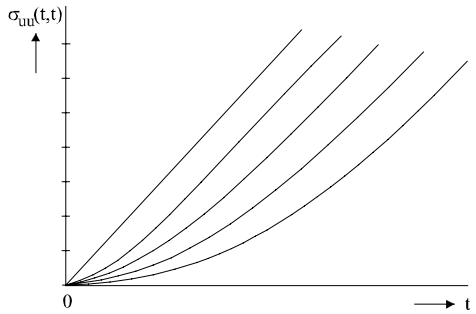


Figure 5.9: Graph of (223) and graphs of (227) for different values of α.

The second limiting case of interest concerns short time intervals:

(230) $$(t - t_0) \ll \frac{1}{\alpha} \text{ or } \alpha(t - t_0) \to 0.$$

Then, using the expansion:

$$e^{-\alpha(t-t_0)} = 1 - \alpha(t-t_0) + \tfrac{1}{2}\alpha^2(t-t_0)^2 - \tfrac{1}{6}\alpha^3(t-t_0)^3 + \ldots$$

it follows from (227) that:

(231) $$\boxed{\lim_{\alpha(t-t_0)\to 0} \sigma_{uu}(t,t) = \sigma_{uu}(t_0,t_0) + \sigma^2(t-t_0)^2} .$$

This shows that for time intervals $(t-t_0)$ that are small compared to the correlation length $1/\alpha$, the contribution of random velocity is essentially that of a random constant. Compare (231) with (216), and see also Figure 5.8c. Hence, this result reflects the fact that for sufficiently short time intervals, the object moves essentially with a random, but constant, velocity.

Exponentially correlated acceleration

We assume that acceleration is a random function with auto-covariance function:

(232) $$\boxed{\sigma_{\ddot{u}\ddot{u}}(\tau) = \sigma^2 e^{-\alpha|\tau|},\ \alpha>0} .$$

We also assume that acceleration is uncorrelated with $\underline{u}(t_0)$ and $\underline{\dot{u}}(t_0)$. Application of propagation law (76) to (217) gives then with (232) for the variance matrix of $(\underline{u}(t),\underline{\dot{u}}(t))^*$:

$$\begin{pmatrix} \sigma_u^2(t) & \sigma_{u\dot{u}}(t,t) \\ \sigma_{\dot{u}u}(t,t) & \sigma_{\dot{u}}^2(t) \end{pmatrix} = \begin{pmatrix} 1 & (t-t_0) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \sigma_u^2(t_0) & \sigma_{u\dot{u}}(t_0,t_0) \\ \sigma_{\dot{u}u}(t_0,t_0) & \sigma_{\dot{u}}^2(t_0) \end{pmatrix} \begin{pmatrix} 1 & (t-t_0) \\ 0 & 1 \end{pmatrix}^*$$

$$+ \int_{t_0}^{t}\int_{t_0}^{t} \begin{pmatrix} t-\tau_1 \\ 1 \end{pmatrix} \sigma^2 e^{-\alpha|t_2-t_1|} \begin{pmatrix} t-\tau_2 \\ 1 \end{pmatrix}^* d\tau_1 d\tau_2$$

or (verify this yourself):

$$\underset{Q_{xx}(t,t)}{\begin{pmatrix} \sigma_u^2(t) & \sigma_{u\dot{u}}(t,t) \\ \sigma_{\dot{u}u}(t,t) & \sigma_{\dot{u}}^2(t) \end{pmatrix}} = \underset{\Phi(t,t_0)}{\begin{pmatrix} 1 & (t-t_0) \\ 0 & 1 \end{pmatrix}} \underset{Q_{xx}(t_0,t_0)}{\begin{pmatrix} \sigma_u^2(t_0) & \sigma_{u\dot{u}}(t_0,t_0) \\ \sigma_{\dot{u}u}(t_0,t_0) & \sigma_{\dot{u}}^2(t_0) \end{pmatrix}} \underset{\Phi(t,t_0)^*}{\begin{pmatrix} 1 & (t-t_0) \\ 0 & 1 \end{pmatrix}^*} + \underset{Q}{\begin{pmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{pmatrix}}$$

with:

$$\begin{cases} q_{11} = 2\dfrac{\sigma^2}{\alpha}\left(\frac{1}{3}(t-t_o)^3 - \frac{1}{2\alpha}(t-t_0)^2 - \frac{1}{\alpha^2}(t-t_0)e^{-\alpha(t-t_0)} + \frac{1}{\alpha^3}(1-e^{-\alpha(t-t_0)})\right) \\ q_{12} = q_{21} = \dfrac{\sigma^2}{\alpha}(t-t_0)\left((t-t_0) - \frac{1}{\alpha}(1-e^{-\alpha(t-t_0)})\right) \\ q_{22} = 2\dfrac{\sigma^2}{\alpha}\left((t-t_0) - \frac{1}{\alpha}(1-e^{-\alpha(t-t_0)})\right) \end{cases}$$

Again two limiting cases are of interest. First, for long time intervals $(t-t_0) \gg 1/\alpha$, the variance contribution of acceleration becomes:

(234)
$$\lim_{\alpha(t-t_0)\to\infty} Q = \frac{2\sigma^2}{\alpha} \begin{pmatrix} \frac{1}{3}(t-t_0)^3 & \frac{1}{2}(t-t_0)^2 \\ \frac{1}{2}(t-t_0)^2 & (t-t_0) \end{pmatrix}.$$

Compare this result with (225). Second, for short time intervals, $(t-t_0) \ll 1/\alpha$, the variance contribution of acceleration becomes:

(235)
$$\lim_{\alpha(t-t_0)\to 0} Q = \sigma^2 \begin{pmatrix} \frac{1}{4}(t-t_0)^4 & \frac{1}{2}(t-t_0)^3 \\ \frac{1}{2}(t-t_0)^3 & (t-t_0)^2 \end{pmatrix}.$$

Compare this result with (221).

5.7 Summary

In this chapter we introduced some elementary concepts of the theory of random functions. In particular, attention was given to the extremely important propagation laws for the cross-covariance and auto-covariance of the output of a linear, time-varying state-equation. These propagation laws for random functions generalize in a natural way the well-known propagation laws for random vectors. We have seen that in its most general form, the propagation law for the auto-covariance consists of a double integral. This double integral, however, simplifies to a single integral if the input can be considered to be a white noise random function. It was shown that the white-noise random function can be seen as the continuous counterpart of a discrete sequence of uncorrelated random variables. As a preparation for the next chapter, we also introduced the random polynomial equations of motion. As inputs we considered random constants, white noise random functions and exponentially correlated random functons. It was shown how the model of exponentially correlated random functions can be used to approximate white noise inputs and inputs that are random constants. This is achieved through a suitable choice of the correlation length $1/\alpha$ in relation to the time interval considered. This is an important result, which shows the flexibility of the model of exponentially correlated random functions.

6 Recursive least-squares: the dynamic case

6.1 Introduction: filter divergence

In Chapter 3 we considered the partitioned model:

$$(1) \qquad E\left\{\begin{pmatrix}\underline{y}_0\\ \underline{y}_1\\ \vdots\\ \underline{y}_k\end{pmatrix}\right\} = \begin{pmatrix}A_0 & & 0\\ & A_1 & \\ & & \ddots\\ 0 & & A_k\end{pmatrix}\begin{pmatrix}x(t_0)\\ x(t_1)\\ \vdots\\ x(t_k)\end{pmatrix} \;;\; D\left\{\begin{pmatrix}\underline{y}_0\\ \underline{y}_1\\ \vdots\\ \underline{y}_k\end{pmatrix}\right\} = \begin{pmatrix}Q_{y_0} & & 0\\ & Q_{y_1} & \\ & & \vdots\\ 0 & & Q_{y_k}\end{pmatrix}$$

together with the transition equation:

$$(2) \qquad x(t) = \Phi(t,t_0)x(t_0).$$

The combination of (1) and (2) led to the partitioned model (see (28) in Section 3.3):

$$(3) \qquad E\left\{\begin{pmatrix}\underline{y}_0\\ \underline{y}_1\\ \vdots\\ \underline{y}_k\end{pmatrix}\right\} = \begin{pmatrix}A_0\Phi_{0,t}\\ A_1\Phi_{1,t}\\ \vdots\\ A_k\Phi_{k,t}\end{pmatrix}x_t \;;\; D\left\{\begin{pmatrix}\underline{y}_0\\ \underline{y}_1\\ \vdots\\ \underline{y}_k\end{pmatrix}\right\} = \begin{pmatrix}Q_{y_0} & & 0\\ & Q_{y_1} & \\ & & \ddots\\ 0 & & Q_{y_k}\end{pmatrix}.$$

It was on the basis of this model, that we developed in Chapter 3 two methods for recursively predicting and filtering the time-varying state vector x_t. The essence of recursive estimation is that there is no need to store past measurements for the purpose of computing present least-squares estimates. Hence, recursion enables us to keep track of the time process x_t by means of an efficient computation of the corresponding best estimates. As was pointed out in Chapter 3, the transition equation (2) implies that knowledge of x at any particular time instant is sufficient for the determination of the complete time history of $x(t)$. In other words, equation (2) implies that the time-varying state vector $x(t)$ can be parameterized in terms of one single vector $x(t_0)$, for all times t. It will be clear that this is a rather stringent assumption, which will not be realistic for most practical applications. In the present chapter this assumption will therefore be relaxed considerably. It will be assumed that the dynamics of $x(t)$ can be modelled through the linear, time-varying state equation:

$$(4) \qquad \dot{x}(t) = F(t)x(t) + G(t)z(t).$$

This implies that the transition equation (2) will be replaced by the solution of (4):

$$(5) \qquad x(t) = \Phi(t,t_0)x(t_0) + \int_{t_0}^{t}\Phi(t,\tau)G(\tau)z(\tau)d\tau.$$

Note that the difference between (5) and our earlier transition equation (2) is given by:

(6) $$\boldsymbol{d}(t,t_0) = \int_{t_0}^{t} \boldsymbol{\Phi}(t,\tau) G(\tau) z(\tau) \boldsymbol{d}\tau .$$

See also Figure 6.1.

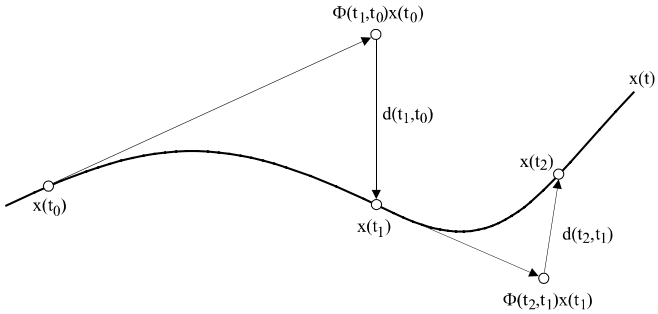


Figure 6.1: Trajectory of $x(t) = \phi(t,t_0)x(t_0) + d(t,t_0)$.

Now, if the transition equation (2) is replaced by (5), then the partitioned model (3) gets replaced by [1]):

(7) $$E\{\begin{pmatrix} \underline{y}_0 \\ \underline{y}_1 \\ \vdots \\ \underline{y}_k \end{pmatrix}\} = \begin{pmatrix} \boldsymbol{A}_0\boldsymbol{\Phi}_{0,t} \\ \boldsymbol{A}_1\boldsymbol{\Phi}_{1,t} \\ \vdots \\ \boldsymbol{A}_k\boldsymbol{\Phi}_{k,t} \end{pmatrix} x_t + \begin{pmatrix} \boldsymbol{A}_0\boldsymbol{d}_{0,t} \\ \boldsymbol{A}_1\boldsymbol{d}_{1,t} \\ \vdots \\ \boldsymbol{A}_k\boldsymbol{d}_{k,t} \end{pmatrix} \quad ; \quad \boldsymbol{D}\{\begin{pmatrix} \underline{y}_0 \\ \underline{y}_1 \\ \vdots \\ \underline{y}_k \end{pmatrix}\} = \begin{pmatrix} Q_{y_0} & & & 0 \\ & Q_{y_1} & & \\ & & \ddots & \\ 0 & & & Q_{y_k} \end{pmatrix}.$$

It will be clear that the information content of the discrete set of observables $\underline{y}$, $i = 0,1,\ldots$, is not enough for the simultaneous estimation of both x_t and the $d_{i,t}$, $i = 0,1,\ldots$. What we need is information about the difference vectors $d_{i,t}$. That is, we need to have some information available about the input $z(t)$ of (6), in order to be able to relate the state vectors of the various epochs with one another. One way to tackle this problem is to assume the input $z(t)$ to be identically zero. Then also the difference vectors $d_{i,t}$ are zero and (7) reduces to (3). But as was pointed out earlier, the zero-input assumption is too stringent for most practical applications. In fact, an unwarranted zero-input assumption will result in what is known as filter divergence. One speaks of filter divergence, when the error in the estimated state vector grows without bound. In order to explain the phenomenon of filter divergence, let us assume that the least-squares estimator of x_t is computed on the basis of model (3). The estimator reads then:

(8) $$\hat{\underline{x}}_{t|k} = (\sum_{i=0}^{k} \boldsymbol{\Phi}_{i,t}^{*}\, \boldsymbol{A}_i^{*}\, Q_{y_i}^{-1}\, \boldsymbol{A}_i\, \boldsymbol{\Phi}_{i,t})^{-1}\, (\sum_{i=0}^{k} \boldsymbol{\Phi}_{i,t}^{*}\, \boldsymbol{A}_i^{*}\, Q_{y_i}^{-1}\, \underline{y}_i).$$

---

[1] We have used the notation $d_{i,t}$ instead of $d(t_i,t)$.

This estimator is unbiased if model (3) holds true. It is, however, a biased estimator if $d_{i,t} \neq 0$ and model (7) holds true. If the expectation is taken of (8), we get with (7):

$$E\{\underline{\hat{x}}_{t|k}\} = (\sum_{i=0}^{k} \boldsymbol{\Phi}_{i,t}^{*} \boldsymbol{A}_i^{*} \boldsymbol{Q}_{y_i}^{-1} \boldsymbol{A}_i \boldsymbol{\Phi}_{i,t})^{-1} (\sum_{i=0}^{k} \boldsymbol{\Phi}_{i,t}^{*} \boldsymbol{A}_i^{*} \boldsymbol{Q}_{y_i}^{-1} (\boldsymbol{A}_i \boldsymbol{Q}_{i,t} x_t + \boldsymbol{A}_i \boldsymbol{d}_{i,t}))$$

or

$$E\{\underline{\hat{x}}_{t|k}\} = x_t + (\sum_{i=0}^{k} \boldsymbol{\Phi}_{i,t}^{*} \boldsymbol{A}_i^{*} \boldsymbol{Q}_{y_i}^{-1} \boldsymbol{A}_i \boldsymbol{\Phi}_{i,t})^{-1} (\sum_{i=0}^{k} \boldsymbol{\Phi}_{i,t}^{*} \boldsymbol{A}_i^{*} \boldsymbol{Q}_{y_i}^{-1} \boldsymbol{A}_i \boldsymbol{d}_{i,t}). \tag{9}$$

This shows that $E\{\underline{\hat{x}}_{t|k}\} \neq x_t$ if $d_{i,t} \neq 0$. The second term on the right-hand side of (9) describes the bias of the estimator $\underline{\hat{x}}_{t|k}$. It is the bias in the filtered estimator if $t = t_k$. If the bias in the filtered estimator has the tendency to grow as k gets larger, the separation between $E\{\underline{\hat{x}}_{k|k}\}$ and x_k increases and the filtered estimator is said to diverge. The following example illustrates this phenomenon of divergence.

Example 67

We are asked to estimate the one-dimensional position of a particle. Position measurements are carried out at times t_i, $i = 0, 1, 2, \ldots, k$. The position observables $\underline{u}_i$ are uncorrelated and all have the same variance σ^2. The position of the particle at time t_k is related to its position at time t_{k-1} and its velocity as:

$$u_k = u_{k-1} + \boldsymbol{d}_{k,k-1}, \text{ with } \boldsymbol{d}_{k,k-1} = \int_{t_{k-1}}^{t_k} \dot{u}_\tau \boldsymbol{d}\tau. \tag{10}$$

Let us assume that the velocity $\dot{u}_t$ of the particle is so small that we decide to ignore it in (10). The corresponding model of our choice reads then:

$$E\left\{\begin{pmatrix} \underline{u}_0 \\ \underline{u}_1 \\ \vdots \\ \underline{u}_k \end{pmatrix}\right\} = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} u_k \quad ; \quad \boldsymbol{D}\left\{\begin{pmatrix} \underline{u}_0 \\ \underline{u}_1 \\ \vdots \\ \underline{u}_k \end{pmatrix}\right\} = \sigma^2 I_{k+1}. \tag{11}$$

Based on this model, we obtain the filtered position estimator as:

$$\underline{\hat{u}}_{k|k} = \frac{1}{k+1} \sum_{i=0}^{k} \underline{u}_i. \tag{12}$$

This estimator is unbiased if model (11) holds true. It is however a biased estimator if the velocity $\dot{u}_t$ of the particle cannot be ignored. Let us now investigate the effect of the neglected velocity on the filtered position estimator (12). If the expectation is taken of (12), we get with:

$$E\{\underline{u}_i\} = u_k + \int_{t_k}^{t_i} \dot{u}_\tau \boldsymbol{d}\tau$$

that:

(13) $$E\{\underline{\hat{u}}_{k|k}\} = u_k - \frac{1}{k+1}\sum_{i=0}^{k}\int_{t_i}^{t_k} \dot{u}_\tau \, d\tau.$$

Now assume that the velocity $\dot{u}_t$ is indeed small, but constant. Then we have in case of uniform sampling, $t_i = t_0 + i\ T$, that:

(14) $$\sum_{i=0}^{k}\int_{t_i}^{t_k} \dot{u}_\tau \, d\tau = \dot{u}\sum_{i=0}^{k}(t_k - t_i) = \frac{1}{2}\dot{u}k(k+1)T.$$

Substitution of (14) into (13) gives:

(15) $$E\{\underline{\hat{u}}_{k|k}\} = u_k - \frac{1}{2}\dot{u}kT.$$

This result shows that the bias in the filtered position estimator increases as k gets larger. Hence, the mean of $\underline{\hat{u}}_{k|k}$ diverges from the actual value u_k. In order to remedy this problem of divergence, one might be inclined to include velocity as an unknown but constant parameter in model (11). In this case, however, one may end up with the same problem of divergence due to unmodelled constant accelerations.

We have seen that the zero-input assumption is too stringent for most practical applications, and that an unwarranted zero-input assumption leads to the serious problem of filter divergence. So what to do? It will be clear that the solution to the problem depends on the information we believe to have available about the input $z(t)$. Let us therefore assume for the moment that the continuous input $z(t)$ is observable. If the continuous input $z(t)$ is observable, we may also consider the difference vector $d(t,t_0)$ of (6) observable, and construct with (5) and (6) for all t, the observation equations:

(16) $$E\{\underline{d}(t,t_0)\} = x(t) - \Phi(t,t_0)x(t_0).$$

These observation equations together with (1) will allow us then to estimate the state vector $x(t)$ for all times t. The supposition of this approach is of course that one has sensors at one's disposal that observe the input $z(t)$ on a continuous basis. In some applications this is indeed the case. For instance, if the input $z(t)$ equals velocity or acceleration, speedometers or accelerometers could be available for observing $z(t)$ on a continuous basis. Still, in a majority of applications, no sensors are available for actually observing the input $z(t)$. Hence, it seems that we are confronted with a dilemma. It seems that either we have to assume that the input $z(t)$ is identically zero and fall back on the transition equation (2). Or, that we have to assume that the input $z(t)$ is unknown and rely safely on the partitioned model (1). Both approaches have their drawbacks. In the first approach filter divergence is likely to occur, especially over longer time spans. And in the second approach, the drawback is that we have no means to estimate $x(t)$ other than at the discrete time instants t_i, $i = 0,1,\ldots$. In this chapter, our solution to the above described dilemma will be the following. We will assume that, even in the absence of sensors that actually observe $z(t)$, we can still treat the input $z(t)$ as being observable. This approach is motivated by the fact that in most applications the input $z(t)$ can be modelled as a zero-mean

random function. For a ship sailing at cruising speed for instance, one may model the ship's acceleration as a random function with zero mean. For this case we will then consider the zero-mean value of acceleration as the sample value of the input $z(t)$ and use the auto-variance function of acceleration for the modelling of the dispersion of the input. With this approach equation (16) is interpreted as an observation equation and used together with (1) for the least-squares estimation of the state vector $x(t)$ for all times t.

6.2 The dynamic model of observation equations

In this section we will formulate the observation equations for the dynamic model. As a start we will assume that the time-varying state vector x_t can be modelled as:

(17) $$x_i = \Phi_{i,i-1}\, x_{i-1} + \int_{t_{i-1}}^{t_i} \Phi_{i,\tau} G_\tau z_\tau d\tau.$$

We also assume that the input can be treated as being observable and that its auto-covariance matrix is given as:

(18) $$Q_{zz}(t_1, t_2) = S_{zz}(t_1)\,\delta(t_2 - t_1).$$

This implies that the difference vector:

(19) $$\underline{d}_i = \int_{t_{i-1}}^{t_i} \Phi_{i,\tau} G_\tau \underline{z}_\tau d\tau$$

is also observable, and that its variance matrix is given as [2]:

(20) $$Q_{d_i} = \int_{t_{i-1}}^{t_i} \Phi_{i,\tau} G_\tau S_{zz,\tau} G_\tau^* \Phi_{i,\tau}^* d\tau.$$

It should be noted that the $\underline{d}_i$, $i = 1,2,\ldots$, are mutually uncorrelated:

(21) $$E\{(\underline{d}_i - E\{\underline{d}_i\})(\underline{d}_j - E\{\underline{d}_j\})^*\} = 0 \text{ for } i \neq j.$$

This can bee seen as follows. If we apply the propagation law for the auto-covariance to (19) we get:

$$E\{(\underline{d}_i - E\{\underline{d}_i\})(\underline{d}_j - E\{\underline{d}_j\})^*\} = \int_{t_{i-1}}^{t_i} \int_{t_{j-1}}^{t_j} \Phi_{i,\tau_1} G_{\tau_1} Q_{zz}(\tau_1, \tau_2) G_{\tau_2}^* \Phi_{j,\tau_2}^* d\tau_1 d\tau_2.$$

With the change of variable $\tau = \tau_2 - \tau_1$, this may be written as:

---

[2] We have used the notation Q_{d_i} instead of $Q_{dd}(t_i, t_i)$.

$$E\{(\underline{d}_i - E\{\underline{d}_i\})(\underline{d}_i - E\{\underline{d}_j\})^*\} = \int_{t_{j-1}}^{t_j} \left(\int_{\tau_2 - t_i}^{\tau_2 - t_{i-1}} \Phi_{i,\tau-\tau_2} G_{\tau-\tau_2} Q_{zz}(\tau-\tau_2,\tau_2) d\tau \right) G\tau_2^* \Phi_{j,\tau_2}^* d\tau_2$$

or after substitution of (18) as:

$$E\{(\underline{d}_i - E\{\underline{d}_i\})(\underline{d}_j - E\{\underline{d}_j\})^*\} = \int_{t_{j-1}}^{t_j} \left(\int_{\tau_2 - t_i}^{\tau_2 - t_{i-1}} \Phi_{i,\tau_1-\tau_2} G_{\tau_2-\tau_1} S_{zz}(\tau_1-\tau_2)\delta(\tau) d\tau \right) G_{\tau_2}^* \Phi_{j,\tau_2}^* d\tau_2. \tag{22}$$

But if the two intervals $[t_{i-1}, t_i]$ and $[t_{j-1}, t_j]$ are disjunct, then $\tau_2 - t_i$ and $\tau_2 - t_{i-1}$ are both positive or both negative. This implies that the inner integral of (22) vanishes due to the impulse function $\delta(\tau)$ and that (21) indeed holds true. We are now in the position to formulate the appropriate observation equations for the dynamic model. The observation equations themselves follow from combining (17) and (19). And the corresponding dispersion follows from (20) and (21). The dynamic model of observation equations reads therefore:

$$\boxed{\begin{array}{c} E\{\underline{d}_i\} = x_i - \Phi_{i,i-1}\, x_{i-1} \;;\; E\{(\underline{d}_i - E\{\underline{d}_i\})(\underline{d}_j - E\{\underline{d}_j\})^*\} = Q_{d_i}\delta_{ij} \\ \text{with: } Q_{d_i} = \int_{t_{i-1}}^{t_i} \Phi_{i,\tau} G_\tau S_{zz,\tau} G_\tau^* \Phi_{i,\tau}^* d\tau \end{array}} \tag{23}$$

6.3 Recursive prediction and filtering

In this section we will generalize the results of Sections 3.4 and 3.5. This will be done by taking into account the dynamic model of observation equations (23). The complete partitioned model of observation equations, on the basis of which the state vector will be estimated, reads therefore:

$$E\left\{\begin{pmatrix} \underline{y}_0 \\ \underline{d}_1 \\ \underline{y}_1 \\ \vdots \\ \underline{d}_k \\ \underline{y}_k \end{pmatrix}\right\} = \begin{pmatrix} A_0 & & & & \\ -\Phi_{1,0} & I & 0 & & \\ & A_1 & & & \\ & & \ddots & & \\ 0 & & & -\Phi_{k,k-1} & I \\ & & & & A_k \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_{k-1} \\ x_k \end{pmatrix} ; \quad D\left\{\begin{pmatrix} \underline{y}_0 \\ \underline{d}_1 \\ \underline{y}_1 \\ \vdots \\ \underline{d}_k \\ \underline{y}_k \end{pmatrix}\right\} = \begin{pmatrix} Q_{y_0} & & & & \\ & Q_{d_1} & 0 & & \\ & & Q_{y_1} & & \\ & & & \ddots & \\ & 0 & & Q_{d_k} & \\ & & & & Q_{y_k} \end{pmatrix}. \tag{24}$$

This model is based on (1) and (23). Our objective is now to derive an algorithm for the recursive computation of the best estimate $\hat{x}_{k|k}$ of x_k. We therefore consider as a first step model (24) with the exception of $\underline{d}_k$ and $\underline{y}_k$. The solution of this model is then given by the vector of estimates $(\underline{\hat{x}}^*_{0|k-1}, \underline{\hat{x}}^*_{1|k-1}, \ldots, \underline{\hat{x}}^*_{k-1|k-1})^*$. The solution of the complete model (24) follows from solving in a second step the model:

$$E\left\{\begin{pmatrix} \underline{\hat{x}}_{0|k-1} \\ \underline{\hat{x}}_{1|k-1} \\ \vdots \\ \underline{\hat{x}}_{k-2|k-1} \\ \hline \underline{\hat{x}}_{k-1|k-1} \\ \underline{d}_k \\ \underline{y}_k \end{pmatrix}\right\} = \left(\begin{array}{ccccc|cc} I & & 0 & & & & \\ & I & & & & 0 & \\ & & \ddots & & & & \\ 0 & & & I & & & \\ \hline & 0 & & & & I & 0 \\ & & & & & -\Phi_{k,k-1} & I \\ & & & & & 0 & A_k \end{array}\right)\begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ \vdots \\ x_{k-1} \\ x_k \end{pmatrix}$$

(25)

$$D\left\{\begin{pmatrix} \underline{\hat{x}}_{0|k-1} \\ \underline{\hat{x}}_{1|k-1} \\ \vdots \\ \underline{\hat{x}}_{k-2|k-1} \\ \hline \underline{\hat{x}}_{k-1|k-1} \\ \underline{d}_k \\ \underline{y}_k \end{pmatrix}\right\} = \left(\begin{array}{ccccc} & & & 0 & \\ & & & & \\ & & & & \\ & & & & \\ \hline & & & 0 & 0 \\ 0 & 0 & Q_{d_k} & 0 & \\ & 0 & 0 & Q_{y_k} & \end{array}\right)$$

The solution of this model is given by the vector of estimators $(\underline{\hat{x}}_{0|k}^{*}, \underline{\hat{x}}_{1|k}^{*}, \ldots, \underline{\hat{x}}_{k-1|k}^{*}, \underline{\hat{x}}_{k|k}^{*})'$. The structure of the design matrix of (25) reveals that the $\underline{\hat{x}}_{i|k-1}$, $i = 0, 1 \ldots, k-2$, do not contribute to the solution of x_{k-1} and x_k. That is, the estimators $\underline{\hat{x}}_{k-1|k}$ and $\underline{\hat{x}}_{k|k}$ of x_{k-1} and x_k only depend on $\underline{\hat{x}}_{k-1|k-1}$, $\underline{d}_k$ and $\underline{y}_k$. Hence, the estimators $\underline{\hat{x}}_{k-1|k}$ and $\underline{\hat{x}}_{k|k}$ can be derived from solving the model:

(26)
$$E\left\{\begin{pmatrix} \underline{\hat{x}}_{k-1|k-1} \\ \underline{d}_k \\ \hline \underline{y}_k \end{pmatrix}\right\} = \begin{pmatrix} I & 0 \\ \Phi_{k,k-1} & I \\ -- & -- \\ 0 & A_k \end{pmatrix}\begin{pmatrix} x_{k-1} \\ x_k \end{pmatrix} \quad ; \quad D\left\{\begin{pmatrix} \underline{\hat{x}}_{k-1|k-1} \\ \underline{d}_k \\ \hline \underline{y}_k \end{pmatrix}\right\} = \left(\begin{array}{ccc|c} Q_{\hat{x}_{k-1|k-1}} & 0 & & \\ 0 & Q_{d_k} & & 0 \\ \hline & & & \\ & 0 & & Q_{y_k} \end{array}\right).$$

But the solution of this model can again be obtained in two steps. We first consider:

(27)
$$E\left\{\begin{pmatrix} \underline{\hat{x}}_{k-1|k-1} \\ \underline{d}_k \end{pmatrix}\right\} = \begin{pmatrix} I & 0 \\ -\Phi_{k,k-1} & I \end{pmatrix}\begin{pmatrix} x_{k-1} \\ x_k \end{pmatrix} \quad ; \quad D\left\{\begin{pmatrix} \underline{\hat{x}}_{k-1|k-1} \\ \underline{d}_k \end{pmatrix}\right\} = \begin{pmatrix} Q_{\hat{x}_{k-1|k-1}} & 0 \\ 0 & Q_{d_k} \end{pmatrix}.$$

Note that the design matrix of this model is square and of full rank. The redundancy equals therefore zero, and (27) may be solved by simply inverting the design matrix. Since:

$$\begin{pmatrix} I & 0 \\ -\Phi_{k,k-1} & I \end{pmatrix}^{-1} = \begin{pmatrix} I & 0 \\ \Phi_{k,k-1} & I \end{pmatrix}.$$

The solution of (27) follows as:

(28)
$$\begin{cases} \begin{pmatrix} \underline{\hat{x}}_{k-1|k-1} \\ \underline{\hat{x}}_{k|k-1} \end{pmatrix} = \begin{pmatrix} I & 0 \\ \Phi_{k,k-1} & I \end{pmatrix} \begin{pmatrix} \underline{\hat{x}}_{k-1|k-1} \\ \underline{d}_k \end{pmatrix} \\ \begin{pmatrix} Q_{\hat{x}_{k-1|k-1}} & Q_{\hat{x}_{k-1|k-1}\hat{x}_{k|k-1}} \\ Q_{\hat{x}_{k|k-1}\hat{x}_{k-1|k-1}} & Q_{\hat{x}_{k|k-1}} \end{pmatrix} = \begin{pmatrix} Q_{\hat{x}_{k-1|k-1}} & Q_{\hat{x}_{k-1|k-1}} \Phi^*_{k,k-1} \\ \Phi_{k,k-1} Q_{\hat{x}_{k-1|k-1}} & (\Phi_{k,k-1} Q_{\hat{x}_{k-1|k-1}} \Phi^*_{k,k-1} + Q_{d_k}) \end{pmatrix}. \end{cases}$$

The result shows how the predicted estimator $\underline{\hat{x}}_{k|k-1}$ can be expressed in terms of the filtered estimator $\underline{\hat{x}}_{k-1|k-1}$ and the vector $\underline{d}_k$. The time-update equations read therefore:

(29)
$$\boxed{\begin{aligned} \underline{\hat{x}}_{k|k-1} &= \Phi_{k,k-1} \underline{\hat{x}}_{k-1|k-1} + \underline{d}_k \\ Q_{\hat{x}_{k|k-1}} &= \Phi_{k,k-1} Q_{\hat{x}_{k-1|k-1}} \Phi^*_{k,k-1} + Q_{d_k} \end{aligned}}.$$

Compare this result with the time-update equations of (45) in Section 3.4. Note that the above result reduces to that of (45) in Section 3.4, if $\underline{d}_k$ is identically zero. This corresponds to the case that the input $\underline{z}_t$ is assumed to be identically zero. The time-update equations (29) are formulated for the discrete time instant t_k. This is the time instant that the next observable $\underline{y}_k$ becomes available. But in fact one may formulate the time-update equations for the complete time interval (t_{k-1}, t_k). The continuous time-update equations read therefore:

(30)
$$\boxed{\begin{aligned} \underline{\hat{x}}_{t|k-1} &= \Phi_{t,k-1} \underline{\hat{x}}_{k-1|k-1} + \int_{t_{k-1}}^{t} \Phi_{t,\tau} G_\tau \underline{z}_\tau \, d\tau \\ Q_{\hat{x}_{t|k-1}} &= \Phi_{t,k-1} Q_{\hat{x}_{k-1|k-1}} \Phi^*_{t,k-1} + \int_{t_{k-1}}^{t} \Phi_{t,\tau} G_\tau S_{zz} G^*_\tau \Phi^*_{t,\tau} \, d\tau \quad \text{for } t \in (t_{k-1}, t_k) \end{aligned}}.$$

See also Figure 6.2.

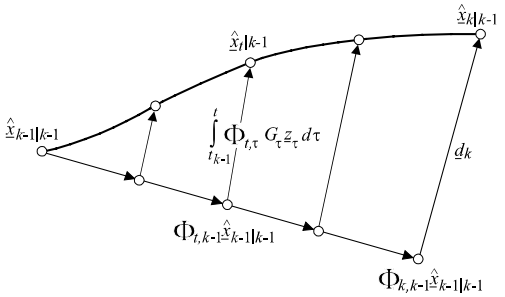


Figure 6.2: A continuous time-update $\Phi_{k,k-1}\hat{\underline{x}}_{k-1|k-1}$.

The above time-update equations followed from solving model (27). The complete solution of model (26) follows now from solving in a second step the model:

$$E\{\begin{pmatrix} \hat{\underline{x}}_{k-1|k-1} \\ \hline \hat{\underline{x}}_{k|k-1} \\ \underline{y}_k \end{pmatrix}\} = \left(\begin{array}{c|c} I & 0 \\ \hline 0 & I \\ 0 & A_k \end{array}\right) \begin{pmatrix} \underline{x}_{k-1} \\ \underline{x}_k \end{pmatrix} ; \; D\{\begin{pmatrix} \hat{\underline{x}}_{k-1|k-1} \\ \hline \hat{\underline{x}}_{k|k-1} \\ \underline{y}_k \end{pmatrix}\} = \left(\begin{array}{c|cc} & 0 & \\ \hline & Q_{\hat{x}_{k|k-1}} & 0 \\ 0 & 0 & Q_{y_k} \end{array}\right).$$

The solution of this model is given by the vector of estimators $(\hat{\underline{x}}^*_{k-1|k}, \hat{\underline{x}}^*_{k|k})^*$. Again we note that the structure of the design matrix of (31) is such that the estimator $\hat{\underline{x}}_{k|k}$ of x_k will only depend on $\hat{\underline{x}}_{k|k-1}$ and $\underline{y}_k$. Hence the estimator $\hat{\underline{x}}_{k|k}$ can be derived from solving the model:

(32)
$$E\{\begin{pmatrix} \hat{\underline{x}}_{k|k-1} \\ \underline{y}_k \end{pmatrix}\} = \begin{pmatrix} I \\ A_k \end{pmatrix} x_k \; ; \; D\{\begin{pmatrix} \underline{x}_{k|k-1} \\ \underline{y}_k \end{pmatrix}\} = \begin{pmatrix} Q_{\hat{x}_{k|k-1}} & 0 \\ 0 & Q_{y_k} \end{pmatrix}.$$

But this model is identical to model (46) in Section 3.4. Hence, the solution of (32) is identical to (48) in Section 3.4 and reads:

(33)
$$\boxed{\begin{aligned} \hat{\underline{x}}_{k|k} &= \hat{\underline{x}}_{k|k-1} + Q_{\hat{x}_{k|k}} A_k^* Q_{y_k}^{-1} (\underline{y}_k - A_k \hat{\underline{x}}_{k|k-1}) \\ Q_{\hat{x}_{k|k}} &= (Q_{\hat{x}_{k|k-1}}^{-1} + A_k^* Q_{y_k}^{-1} A_k)^{-1} \end{aligned}}.$$

These are the measurement-update equations. If we write model (32) in terms of condition equations and solve this model, we get as in (50) in Section 3.5 the alternative measurement-update equations:

$$\boxed{\begin{aligned} \hat{\underline{x}}_{k|k} &= \hat{\underline{x}}_{k|k-1} + Q_{\hat{x}_{k|k}} A_k^* Q_{v_k}^{-1} (\underline{y}_k - A_k \hat{\underline{x}}_{k|k-1}) \\ Q_{\hat{x}_{k|k}} &= Q_{\hat{x}_{k|k-1}} - Q_{\hat{x}_{k|k-1}} A_k^* Q_{v_k}^{-1} A_k Q_{\hat{x}_{k|k-1}} \end{aligned}}$$

in which Q_{v_k} is the variance matrix of the predicted residuals. When we compare the recursive prediction and filtering results of this section with the results of Section 3.4 and 3.5, we note that the only difference lies in the way the time-update computations are performed.

Example 68

We are asked to determine the one-dimensioned position of a particle. Position measurements are carried out at the discrete time instants t_i, i=0,1,2,.... The observables $\underline{x}_i$ are assumed to be uncorrelated and all have the same variance σ_x^2. We will start with the assumption that the particle has a zero-velocity, i.e., that its position is constant. The corresponding model reads then:

$$E\left\{\begin{pmatrix}\underline{x}_0\\ \underline{x}_1\\ \vdots\\ \underline{x}_k\end{pmatrix}\right\} = \begin{pmatrix}1\\ 1\\ \vdots\\ 1\end{pmatrix} x_k \;;\; D\left\{\begin{pmatrix}\underline{x}_0\\ \underline{x}_1\\ \vdots\\ \underline{x}_k\end{pmatrix}\right\} = \sigma_x{}^2 I_k. \tag{35}$$

From this the filtered position estimator $\hat{\underline{x}}_{k|k}$ and its variance follow as:

$$\begin{cases}\hat{\underline{x}}_{k|k} = \hat{\underline{x}}_{k|k-1} + \sigma^2_{\hat{x}_{k|k}} \cdot 1 \cdot \sigma_x^{-2}(\underline{x}_k - \hat{\underline{x}}_{k|k-1}) \\ \sigma^2_{\hat{x}_{k|k}} = \sigma_x^2/(k+1)\end{cases}.$$

Note that the variance of the filtered position estimator gets smaller as k gets larger. And in the limit we have:

$$\lim_{k\to\infty} \sigma^2_{\hat{x}_{k|k}} = 0\ . \tag{37}$$

See also Figure 6.3.

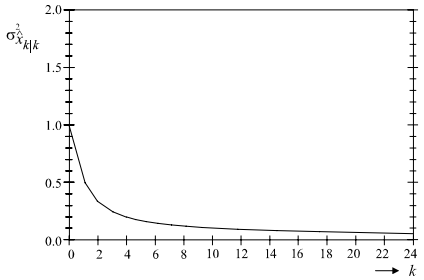


Fig. 6.3.: The variance $\sigma^2_{\hat{x}_{k|k}} = \sigma_x^2/(k+1)$ for $\sigma_x^2 = 1$.

Also the gain $K_k = 1/(k+1)$ gets smaller as k gets larger. And in the limit we have:

$$\lim_{k\to\infty} K_k = 0\,. \tag{38}$$

This implies that, as k gets larger less weight is given to the new position observable $\underline{x}_k$. That is, the contribution of the new position observable $\underline{x}_k$ to the filtered position estimator $\underline{\hat{x}}_{k|k}$ diminishes for increasing k. This is of course completely justified as long as the model (35) is valid. Let us now assume that the velocity of the particle is not zero, but zero on the average. The position of the particle is then constant on the average, although it may nevertheless change from time instant to time instant. Would we now still base our filter computations on model (35), we would end up for large k with a filtered position-estimator that is insensitive to the position observable $\underline{x}_k$. Hence, the filtered position estimate $\hat{x}_{k|k}$ will then be unable to follow any position changes of the particle. As a consequence filter divergence is likely to occur. We therefore have to modify model (35) in order to make the filtered position estimator $\underline{\hat{x}}_{k|k}$ more responsive to position changes of the particle. If we believe that the particle moves with a velocity that is zero on the average, it seems realistic to model velocity as a zero-mean random function. Let us therefore model velocity $\dot{x}_k$ as a zero-mean random function, with the following auto-covariance function:

(39) $$\sigma_{\dot{x}\dot{x}}(\tau) = q_{\dot{x}}\,\delta(\tau) \quad q_{\dot{x}} \text{ has the dimension } m^2/s \ .$$

The spectral density $q_{\dot{x}}$ should be chosen such that it reflects the expected erratic behaviour of velocity. The variance contribution of velocity to position follows from applying with (39) the propagation law to:

as: $$\underline{d}_i = \int_{t_{i-1}}^{t_i} \underline{\dot{x}}_\tau \, d\tau$$

(40) $$\sigma_{d_i d_j} = q_{\dot{x}}(t_i - t_{i-1})\,\delta_{ij}.$$

Our modified model follows now from combining the observation equations $E\{\underline{d}_i\} = x_i - x_{i-1}$ and (40) with (35). The result reads:

(41) $$E\left\{\begin{pmatrix}\underline{x}_0\\ \underline{d}_1\\ \underline{x}_1\\ \vdots\\ \underline{d}_k\\ \underline{x}_k\end{pmatrix}\right\} = \begin{pmatrix}1 & & & & & 0\\ -1 & 1 & & & & \\ & 1 & & & & \\ & & \ddots & & & \\ 0 & & & -1 & 1 & \\ & & & & 1 & \end{pmatrix}\begin{pmatrix}x_0\\ x_1\\ \vdots\\ x_{k-1}\\ x_k\end{pmatrix} \quad ; \quad D\left\{\begin{pmatrix}\underline{x}_0\\ \underline{d}_1\\ \underline{x}_1\\ \vdots\\ \underline{d}_k\\ \underline{x}_k\end{pmatrix}\right\} = \begin{pmatrix}\sigma_x^2 & & & & & \\ & \sigma_{d_1}^2 & & & 0 & \\ & & \sigma_x^2 & & & \\ & & & \ddots & & \\ & 0 & & & \sigma_{d_k}^2 & \\ & & & & & \sigma_x^2\end{pmatrix}.$$

Unfortunately, no analytical solution of model (41) can be given. The computer was therefore used for the computation of the variances of respectively $\underline{\hat{x}}_{k|k-1}$ and $\underline{\hat{x}}_{k|k}$. They are given in Table 6.1 for the case $\sigma_x^2 = 1$, $q_{\dot{x}} = 1/4$ and $t_i - t_{i-1} = 1$.

k	$\sigma^2_{\hat{x}_{k\|k-1}}$	$\sigma^2_{\hat{x}_{k\|k}}$		k	$\sigma^2_{\hat{x}_{k\|k-1}}$	$\sigma^2_{\hat{x}_{k\|k}}$
0	-	1		9	0.641	0.391
1	1.25	0.556		8	0.641	0.390
2	0.806	0.446		10	0.640	0.390
3	0.696	0.410		11	0.640	0.390
4	0.660	0.398		12	0.640	0.390
5	0.648	0.393		13	0.640	0.390
6	0.643	0.391		14	0.640	0.390
7	0.641	0.391		15	0.640	0.390

Table 6.1: The variances $\sigma^2_{\hat{x}_{k|k-1}}$ and $\sigma^2_{\hat{x}_{k|k}}$ for $\sigma^2_x = 1$, $q_{\hat{x}} = 1/4$, $t_i - t_{i-1} = 1$.

A plot of the results of Table 6.1 is given in Figure 6.4.

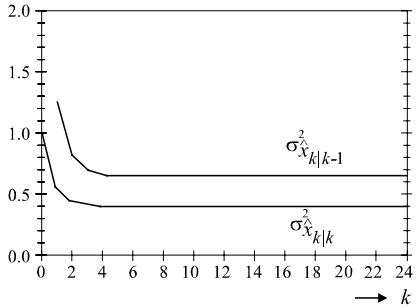


Figure 6.4: The variances $\sigma^2_{\hat{x}_{k|k-1}}$ and $\sigma^2_{\hat{x}_{k|k}}$ for $\sigma^2_x = 1$, $\sigma^2_d = 1/4$.

When we look at the results of Table 6.1 or at the plot of Fig. 6.4, we note that the variance of the filtered position estimator $\hat{x}_{k|k}$ does not seem to go to zero for increasing k. This in contrast to (37). In fact it seems that for $k \to \infty$ both $\sigma^2_{\hat{x}_{k|k-1}}$ and $\sigma^2_{\hat{x}_{k|k}}$ converge to a non-zero constant value:

$$\lim_{k\to\infty} \sigma^2_{\hat{x}_{k|k-1}} = 0.64; \quad \lim_{k\to\infty} \sigma^2_{\hat{x}_{k|k}} = 0.39. \tag{42}$$

This would imply that in the limit the loss of precision due to prediction is exactly equal to the gain in precision due to filtering:

$$\lim_{k\to\infty} (\sigma^2_{\hat{x}_{k|k-1}} - \sigma^2_{\hat{x}_{k|k}}) = 0.25. \tag{43}$$

We will now prove analytically that (42) and (43) indeed hold true. We will first consider the variance of the filtered estimator. From:

$$\begin{cases} Q_{\hat{x}_{k|k}} = (Q_{\hat{x}_{k|k-1}}^{-1} + \boldsymbol{A}_k^* Q_{y_k}^{-1} \boldsymbol{A}_k)^{-1} \\ Q_{\hat{x}_{k|k-1}} = \boldsymbol{\Phi}_{k,k-1} Q_{\hat{x}_{k-1|k-1}} \boldsymbol{\Phi}_{k,k-1}^* + Q_{\boldsymbol{d}_k} \end{cases}$$

see (29) and (30) it follows that:

(44) $$Q_{\hat{x}_{k|k}} = ((\boldsymbol{\Phi}_{k,k-1} Q_{\hat{x}_{k-1|k-1}} \boldsymbol{\Phi}_{k,k-1}^* + Q_{\boldsymbol{d}_k})^{-1} + \boldsymbol{A}_k^* Q_{y_k}^{-1} \boldsymbol{A}_k)^{-1}.$$

Now, if the variance matrix of the filtered estimator is constant in the limit, then:

$$Q_{\hat{x}_{k|k}} = Q_{\hat{x}_{k-1|k-1}} \text{ for } k \to \infty$$

and (44) may be written as:

(45) $$Q_{\hat{x}_{k|k}} = ((\boldsymbol{\Phi}_{k,k-1} Q_{\hat{x}_{k|k}} \boldsymbol{\Phi}_{k,k-1}^* + Q_{\boldsymbol{d}_k})^{-1} + \boldsymbol{A}_k^* Q_{y_k}^{-1} \boldsymbol{A}_k)^{-1} \text{ for } k \to \infty.$$

For our present example this implies:

(46) $$\sigma_{\hat{x}_{k|k}}^2 = ((1 \cdot \sigma_{\hat{x}_{k|k}}^2 \cdot 1 + \sigma_{\boldsymbol{d}}^2)^{-1} + 1 \cdot \sigma_x^{-2} \cdot 1)^{-1} \text{ for } k \to \infty.$$

This equation can be rewritten as the quadratic equation:

(47) $$(\sigma_{\hat{x}_{k|k}}^2)^2 + \sigma_{\boldsymbol{d}}^2 (\sigma_{\hat{x}_{k|k}}^2) - \sigma_{\boldsymbol{d}}^2 \sigma_x^2 = 0 \text{ for } k \to \infty.$$

This equation has two roots, a positive one and a negative one. Since $\sigma_{\hat{x}_{k|k}}^2 \geq 0$ only the positive root is valid. It reads:

(48) $$\boxed{\lim_{k \to \infty} \sigma_{\hat{x}_{k|k}}^2 = \tfrac{1}{2} \sigma_{\boldsymbol{d}}^2 (-1 + (1 + 4\sigma_x^2 / \sigma_{\boldsymbol{d}}^2)^{1/2})}.$$

The limiting value of the variance of the predicted estimator follows with (48) from $\sigma_{\hat{x}_{k|k-1}}^2 = 1 \cdot \sigma_{\hat{x}_{k-1|k-1}}^2 \cdot 1 + \sigma_d^2$ as:

(49) $$\boxed{\lim_{k \to \infty} \sigma_{\hat{x}_{k|k-1}}^2 = \tfrac{1}{2} \sigma_{\boldsymbol{d}}^2 (1 + (1 + 4\sigma_x^2 / \sigma_{\boldsymbol{d}}^2)^{1/2})}.$$

And from (48) and (49) it follows that:

(50) $$\boxed{\lim_{k \to \infty} (\sigma_{\hat{x}_{k|k-1}}^2 - \sigma_{\hat{x}_{k|k}}^2) = \sigma_{\boldsymbol{d}}^2}.$$

The numerical results (42) and (43) are now easily verified with (48), (49) and (50) (do this yourself). Another two examples of (48), (49) and (50) are given in Figure 6.5. Note that as σ_d^2 gets larger, the limiting values of both $\sigma_{x_{k|k}}^2$ and $\sigma_{x_{k|k-1}}^2$, and the difference $(\sigma_{x_{k|k-1}}^2 - \sigma_{\hat{x}_{k|k-1}}^2)$ get larger.

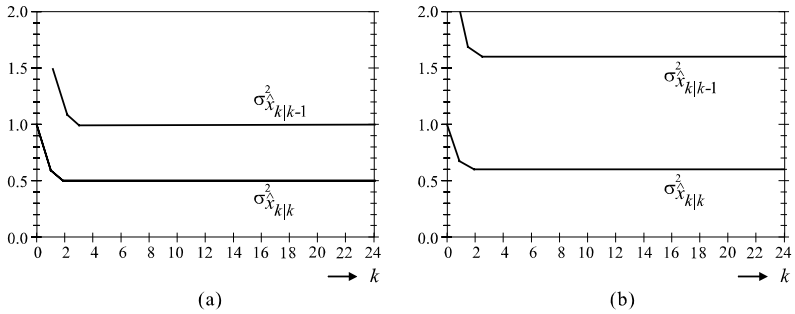


Figure 6.5: The variances $\sigma^2_{\hat{x}_{k|k-1}}$ and $\sigma^2_{\hat{x}_{k|k}}$ for (a) $\sigma_x^2 = 1, \sigma_d^2 = 0.5$; and (b) $\sigma_x^2 = 1, \sigma_d^2 = 1$.

It is also of interest to find the limiting value of the gain. Since:

$$K_k = Q_{\hat{x}_{k|k}} A_k^* Q_{y_k}^{-1} = \sigma^2_{\hat{x}_{k|k}} \cdot 1 \cdot \sigma_x^{-2}$$

it follows with (48) that:

(51)
$$\boxed{\lim_{k \to \infty} K_k = \frac{(-1 + (1 + 4\sigma_x^2/\sigma_d^2)^{1/2})}{2\sigma_x^2/\sigma_d^2}} .$$

This shows that the limiting value of the gain is non-zero (compare with (38)) and that it is uniquely determined by the ratio σ_x^2/σ_d^2. Note that the limiting value of the gain gets smaller if the ratio σ_x^2/σ_d^2 gets larger. This can be explained as follows. If σ_x^2 gets larger or σ_d^2 gets smaller, then there is less confidence in the position observable $\underline{x}_k$ but more confidence in the dynamic model. In this case the filtered position estimator needs to be less responsive to the new position observable and thus the gain will be smaller.

Model (41) has been formulated for the case that velocity is assumed to be zero on the average. In that case the sample values of the $\underline{d}_i$ are taken to be zero and the spectral density $q_{\dot{x}}$ is chosen so as to reflect the expected erratic behaviour of velocity. We will now consider two different but related situations. First, we consider the situation where we really have a sensor available that measures velocity on a continuous basis. Our observables are the discrete position observables $\underline{x}_i$, $i = 0,1,...$, and the continuous velocity observable $\underline{\dot{x}}_t$. In this case our estimation can again be based on model (41). The assumptions on the basis of which model (41) is formulated are now, however, fundamentally different. First of all, no assumption needs to be made about the mean value of velocity. Since velocity is measured on a continuous basis, model (41) is now valid for every velocity profile of the particle. Also, the spectral density $q_{\dot{x}}$ should now describe the precision of the velocity sensor and not the expected erratic behaviour of velocity. And finally, the sample values of the $\underline{d}_i$ are now not set to zero, but of course given as the integrated velocity measurements.

Let us now consider the situation where we have a sensor available that measures velocity on a discrete-time basis. Our observables are then the discrete position observables $\underline{x}_i$, $i=0,1,...$ and the discrete velocity observables $\underline{\dot{x}}_i$, $i=1,2,...$. It will be clear that these observables are not sufficient for the estimation of position for all times t. We therefore assume in addition that the unknown velocity of the particle is constant. Then:

(52)
$$\underline{\boldsymbol{d}}_{\boldsymbol{i}} = \int_{t_{\boldsymbol{i}-1}}^{t_{\boldsymbol{i}}} \underline{\dot{x}}_{\tau}\, \boldsymbol{d}\tau = \underline{\dot{x}}_{\boldsymbol{i}}(t_{\boldsymbol{i}}-t_{\boldsymbol{i}-1})$$

and

(53)
$$\begin{pmatrix} \underline{x}_0 \\ \underline{x}_1 \\ \vdots \\ \underline{x}_k \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & (t_1-t_0) \\ \vdots & \vdots \\ 1 & (t_k-t_0) \end{pmatrix} \begin{pmatrix} x_0 \\ \dot{x}_0 \end{pmatrix}.$$

Substitution of (52) and (53) into (41) gives:

$$E\left\{ \begin{pmatrix} \underline{x}_0 \\ \underline{\dot{x}}_1(t_1-t_0) \\ \underline{x}_1 \\ \underline{\dot{x}}_2(t_2-t_1) \\ \vdots \\ \underline{\dot{x}}_k(t_k-t_{k-1}) \\ \underline{x}_k \end{pmatrix} \right\} = \begin{pmatrix} 1 & 0 \\ 0 & (t_1-t_0) \\ 1 & (t_1-t_0) \\ 0 & (t_2-t_1) \\ \vdots & \vdots \\ 0 & (t_k-t_{k-1}) \\ 1 & (t_k-t_{k-1}) \end{pmatrix} \begin{pmatrix} x_0 \\ \dot{x}_0 \end{pmatrix}.$$

or

(54)
$$E\left\{ \begin{pmatrix} \underline{x}_0 \\ \underline{\dot{x}}_1 \\ \underline{x}_1 \\ \underline{\dot{x}}_2 \\ \vdots \\ \underline{\dot{x}}_k \\ \underline{x}_k \end{pmatrix} \right\} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & (t_1-t_0) \\ 0 & 1 \\ \vdots & \vdots \\ 0 & 1 \\ 1 & (t_k-t_{k-1}) \end{pmatrix} \begin{pmatrix} x_0 \\ \dot{x}_0 \end{pmatrix}.$$

This shows that the constant, but unknown, velocity model can be obtained from (41).

Example 69

Consider Figure 6.6. A ship follows an unknown trajectory. Our objective is to compute, on a real-time basis, the best estimate of the ships's position u_t, v_t. For this purpose we assume that the ship is equipped with two accelerometers that observe on a continuous basis the accelerations $\ddot{u}_t$ and $\ddot{v}_t$. The autocovariance functions of acceleration are given as:

$$(55) \qquad \begin{cases} \sigma_{\ddot{u}\ddot{u}}(\tau) &= q_{\ddot{u}}\delta(\tau),\ q_{\ddot{u}} \text{ has the dimension } m^2/s^3 \\ \sigma_{\ddot{v}\ddot{v}}(\tau) &= q_{\ddot{v}}\delta(\tau),\ q_{\ddot{v}} \text{ has the dimension } m^2/s^3 \\ \sigma_{\ddot{u}\ddot{v}}(t_1,t_2) &= 0 \end{cases}.$$

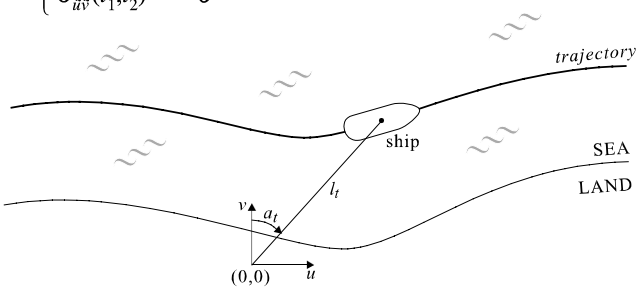


Figure 6.6: Positioning of a ship.

Thus, we assume the measurement errors of the two accelerometers to behave as white noise and $\underline{\ddot{u}}_t$ to be uncorrelated with $\underline{\ddot{v}}_t$. It will be clear that acceleration information alone is not sufficient for the recovery of the ship's trajectory. We therefore also assume that at successive discrete time instances t_i, $i=0,1,...$ azimuth a_i and distance measurements l_i are carried out. These measurements are carried out from a known point with coordinates u=0, v=0. The variances and covariances of the azimuth and distance observables are given as:

$$(56) \qquad \begin{cases} \sigma_{a_i a_j} &= \sigma^2_{a_i}\delta_{ij} \\ \sigma_{l_i l_j} &= \sigma^2_{l_i}\delta_{ij} \\ \sigma_{a_i l_j} &= 0 \end{cases}.$$

In order to formulate the complete partitioned model of observation equations, we first start with the discrete observation equations for azimuth and distance. These nonlinear observation equations parameterized in terms of the cartesian coordinates u_i, v_i read:

$$\underbrace{E\{\begin{pmatrix}\underline{a}_i\\ \underline{l}_i\end{pmatrix}\}}_{\underline{y}_i} = \begin{pmatrix}\arctan(u_i/v_i)\\ (u_i^2+v_i^2)^{1/2}\end{pmatrix};\ Q_{y_iy_j} = \begin{pmatrix}\sigma_{a_i}^2 & 0\\ 0 & \sigma_{l_i}^2\end{pmatrix}\delta_{ij}. \tag{57}$$

The corresponding linearized observation equations read:

$$\underbrace{E\{\begin{pmatrix}\Delta\underline{a}_i\\ \Delta\underline{l}_i\end{pmatrix}\}}_{\Delta\underline{y}_i} = \begin{pmatrix}v_i^0/(l_i^0)^2 & -u_i^0/(l_i^0)^2\\ u_i^0/l_i^0 & v_i^0/l_i^0\end{pmatrix}\begin{pmatrix}\Delta\underline{u}_i\\ \Delta\underline{v}_i\end{pmatrix};\ Q_{\Delta y_i\Delta y_j} = \begin{pmatrix}\sigma_{a_i}^2 & 0\\ 0 & \sigma_{l_i}^2\end{pmatrix}\delta_{ij}. \tag{58}$$

This may also be written as:

$$\boxed{\underbrace{E\{\begin{pmatrix}\Delta\underline{a}_i\\ \Delta\underline{l}_i\end{pmatrix}\}}_{\Delta\underline{y}_i} = \begin{pmatrix}\cos a_i^0/l_i^0 & -\sin a_i^0/l_i^0\\ \sin a_i^0 & \cos a_i^0\end{pmatrix}\begin{pmatrix}\Delta u_i\\ \Delta v_i\end{pmatrix};\ Q_{\Delta y_i\Delta y_j} = \begin{pmatrix}\sigma_{a_i}^2 & 0\\ 0 & \sigma_{l_i}^2\end{pmatrix}\delta_{ij}}. \tag{59}$$

Let us now consider the observation equations for the observed accelerations. We will only consider the accelerations along the *u*-axis. The development for the acceleration along the *v*-axis goes along identical lines. The position u_i and velocity $\dot{u}_i$ follow from acceleration $\ddot{u}_i$ as:

$$\begin{pmatrix}u_i\\ \dot{u}_i\end{pmatrix} = \begin{pmatrix}1 & (t_i-t_{i-1})\\ 0 & 1\end{pmatrix}\begin{pmatrix}u_{i-1}\\ \dot{u}_{i-1}\end{pmatrix} + \int_{t_{i-1}}^{t_i}\begin{pmatrix}t_i-\tau\\ 1\end{pmatrix}\ddot{u}_\tau d\tau. \tag{60}$$

If we define the observable random vector $\underline{d}_i = (\underline{d}_{u,i}, \underline{d}_{\dot{u},i})^*$ of integrated acceleration as:

$$\begin{pmatrix}\underline{d}_{u,i}\\ \underline{d}_{\dot{u},i}\end{pmatrix} = \int_{t_{i-1}}^{t_i}\begin{pmatrix}t_i-\tau\\ 1\end{pmatrix}\underline{\ddot{u}}_\tau d\tau \tag{61}$$

the corresponding linear observation equations follow with (55) from (60) and (61) as:

$$\boxed{\underbrace{E\{\begin{pmatrix}\underline{d}_{u,i}\\ \underline{d}_{\dot{u},i}\end{pmatrix}\}}_{\underline{d}_i} = \left(-\begin{pmatrix}1 & (t_i-t_{i-1})\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\right)\begin{pmatrix}u_{i-1}\\ \dot{u}_{i-1}\\ u_i\\ \dot{u}_i\end{pmatrix};\ Q_{d_id_j} = q_{\ddot{u}}\begin{pmatrix}\frac{1}{3}(t_i-t_{i-1})^3 & \frac{1}{2}(t_i-t_{i-1})^2\\ \frac{1}{2}(t_i-t_{i-1})^2 & (t_i-t_{i-1})\end{pmatrix}\delta_{ij}}. \tag{62}$$

$$
E\left\{\begin{pmatrix}
\Delta\underline{\boldsymbol{a}}_0\\ \Delta\underline{l}_0\\ \Delta\underline{\boldsymbol{d}}_{u,1}\\ \Delta\underline{\boldsymbol{d}}_{\dot{u},1}\\ \Delta\underline{\boldsymbol{d}}_{v,1}\\ \Delta\underline{\boldsymbol{d}}_{\dot{v},1}\\ \Delta\underline{\boldsymbol{a}}_1\\ \Delta\underline{l}_1\\ \vdots\\ \vdots\\ \vdots\\ \Delta\underline{\boldsymbol{d}}_{u,k}\\ \Delta\underline{\boldsymbol{d}}_{\dot{u},k}\\ \Delta\underline{\boldsymbol{d}}_{v,k}\\ \Delta\underline{\boldsymbol{d}}_{\dot{v},k}\\ \Delta\underline{\boldsymbol{a}}_k\\ \Delta\underline{l}_k
\end{pmatrix}\right\} = \begin{pmatrix}
\begin{pmatrix}\cos\alpha_0^0/l_0^0 & 0\\ \sin\alpha_0^0 & 0\end{pmatrix} & \begin{pmatrix}-\sin\alpha_0^0/l_0^0 & 0\\ \cos\alpha_0^0 & 0\end{pmatrix} & & & \cdots & \cdots\\
-\begin{pmatrix}1 & (t_1-t_0)\\ 0 & 1\end{pmatrix} & \begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix} & \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} & \begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix} & \cdots & \cdots\\
\begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix} & -\begin{pmatrix}1 & (t_1-t_0)\\ 0 & 1\end{pmatrix} & \begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix} & \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} & \cdots & \cdots\\
 & & \begin{pmatrix}\cos\alpha_1^0/l_1^0 & 0\\ \sin\alpha_1^0 & 0\end{pmatrix} & \begin{pmatrix}-\sin\alpha_1^0/l_1^0 & 0\\ \cos\alpha_1^0 & 0\end{pmatrix} & \cdots & \cdots\\
 & & \ddots & \ddots & & \\
\cdots & \cdots & -\begin{pmatrix}1 & (t_k-t_{k-1})\\ 0 & 1\end{pmatrix} & \begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix} & \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} & \begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}\\
\cdots & \cdots & \begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix} & -\begin{pmatrix}1 & (t_k-t_{k-1})\\ 0 & 1\end{pmatrix} & \begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix} & \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\\
\cdots & \cdots & & & \begin{pmatrix}\cos\alpha_k^0/l_k^0 & 0\\ \sin\alpha_k^0 & 0\end{pmatrix} & \begin{pmatrix}-\sin\alpha_k^0/l_k^0 & 0\\ \cos\alpha_k^0 & 0\end{pmatrix}
\end{pmatrix}\begin{pmatrix}
\Delta u_0\\ \Delta\dot{u}_0\\ \Delta v_0\\ \Delta\dot{v}_0\\ \Delta u_1\\ \Delta\dot{u}_1\\ \Delta v_1\\ \Delta\dot{v}_1\\ \vdots\\ \vdots\\ \vdots\\ \Delta u_{k-1}\\ \Delta\dot{u}_{K-1}\\ \Delta v_{k-1}\\ \Delta\dot{v}_{k-1}\\ \Delta u_k\\ \Delta\dot{u}_k\\ \Delta v_k\\ \Delta\dot{v}_k
\end{pmatrix}
$$

(63)

$$
D\left\{\begin{pmatrix}
\Delta\underline{\boldsymbol{a}}_0\\
\Delta\underline{l}_0\\
\Delta\underline{\boldsymbol{d}}_{u,1}\\
\Delta\underline{\boldsymbol{d}}_{\dot{u},1}\\
\Delta\underline{\boldsymbol{d}}_{v,1}\\
\Delta\underline{\boldsymbol{d}}_{\dot{v},1}\\
\Delta\underline{\boldsymbol{a}}_1\\
\Delta\underline{l}_1\\
\vdots\\
\vdots\\
\vdots\\
\Delta\underline{\boldsymbol{d}}_{u,k}\\
\Delta\underline{\boldsymbol{d}}_{\dot{u},k}\\
\Delta\underline{\boldsymbol{d}}_{v,k}\\
\Delta\underline{\boldsymbol{d}}_{\dot{v},k}\\
\Delta\underline{\boldsymbol{a}}_k\\
\Delta\underline{l}_k
\end{pmatrix}\right\} = \begin{pmatrix}
\begin{pmatrix}\sigma^2_{\boldsymbol{a}_0} & 0\\ 0 & \sigma^2_{l_0}\end{pmatrix} & & & & & & & & \\
& \boldsymbol{q}_{\ddot{u}}\begin{pmatrix}\frac{1}{3}(t_1-t_0)^3 & \frac{1}{2}(t_1-t_0)^2\\ \frac{1}{2}(t_1-t_0)^2 & (t_1-t_0)\end{pmatrix} & & & & & & & \\
& & \boldsymbol{q}_{\ddot{v}}\begin{pmatrix}\frac{1}{3}(t_1-t_0)^2 & \frac{1}{2}(t_1-t_0)^2\\ \frac{1}{2}(t_1-t_0)^2 & (t_1-t_0)\end{pmatrix} & & & & 0 & & \\
& & & \begin{pmatrix}\sigma^2_{\boldsymbol{a}_1} & 0\\ 0 & \sigma^2_{l_1}\end{pmatrix} & & & & & \\
& & & & \ddots & & & & \\
& & & & & \boldsymbol{q}_{\ddot{u}}\begin{pmatrix}\frac{1}{3}(t_k-t_{k-1})^3 & \frac{1}{2}(t_k-t_{k-1})^2\\ \frac{1}{2}(t_k-t_{k-1})^2 & (t_k-t_{k-1})\end{pmatrix} & & & \\
& & & & & & \boldsymbol{q}_{\ddot{v}}\begin{pmatrix}\frac{1}{3}(t_k-t_{k-1})^3 & \frac{1}{2}(t_k-t_{k-1})^2\\ \frac{1}{2}(t_k-t_{k-1})^2 & (t_k-t_{k-1})\end{pmatrix} & & \\
& & 0 & & & & & & \begin{pmatrix}\sigma^2_{\boldsymbol{a}_k} & 0\\ 0 & \sigma^2_{l_k}\end{pmatrix}
\end{pmatrix} \tag{64}
$$

With (59) and (62) we are now in the position to formulate the complete partitioned model of linearized observation equations for all times t_i. The complete model is given on the pages 226 and 227. It is on the basis of this model that the recursive estimation of position and velocity can be carried out.

The recursive algorithm has to be initialized. Note that in the present example, initialization cannot be based on only the azimuth and distance observable of time t_0. Their information content is namely not sufficient for the determination of both position and velocity at time t_0. For the present example, initialization has therefore to be based on the observables:

$$\underline{\alpha}_0,\ \underline{l}_0,\ \underline{d}_{u,1},\ \underline{d}_{\dot{u},1},\ \underline{d}_{v,1},\ \underline{d}_{\dot{v},1},\ \underline{\alpha}_1 \text{ and } \underline{l}_1 . \tag{65}$$

There are two ways for using these eight observables for the initialization. In the first approach one solves for the eight state vector elements:

$$u_0,\ \dot{u}_0,\ v_0,\ \dot{v}_0,\ u_1,\ \dot{u}_1,\ v_1,\ \dot{v}_1 \tag{66}$$

by using the first eight observation equations of model (63). Since these observation equations are linearized, an iteration has to be performed to solve for the corresponding state vector elements. After iteration we obtain as a result of the initialization the state vector estimates, $\hat{u}_{0|1},\ \hat{\dot{u}}_{0|1},\ \hat{v}_{0|1},\ \hat{\dot{v}}_{0|1},\ \hat{u}_{1|1},\ \hat{\dot{u}}_{1|1},\ \hat{v}_{1|1},\ \hat{\dot{v}}_{1|1}$ with their corresponding variance-covariance matrix.

The alternative approach to initialization is based on the fact that there exists a one-to-one relationship between the first eight observables of (65) and the eight state vector elements of (66) (note that the redundancy of the first eight observation equations of (63) equals zero). The position estimators may therefore be computed directly from the azimuth and distance observables as:

$$\begin{cases} \hat{\underline{u}}_{0|1} = \underline{l}_0 \sin\underline{\alpha}_0, & \hat{\underline{u}}_{1|1} = \underline{l}_1 \sin\underline{\alpha}_1 \\ \hat{\underline{v}}_{0|1} = \underline{l}_0 \cos\underline{\alpha}_0, & \hat{\underline{v}}_{1|1} = \underline{l}_1 \cos\underline{\alpha}_1 \end{cases} . \tag{67}$$

For the first velocity estimators we can make use of (60) and (61), This gives:

$$\begin{cases} \hat{\underline{\dot{u}}}_{0|1} = (\hat{\underline{u}}_{1|1} - \hat{\underline{u}}_{0|1} - \underline{d}_{u,1})/(t_1 - t_0), & \hat{\underline{\dot{u}}}_{1|1} = \hat{\underline{\dot{u}}}_{0|1} + \underline{d}_{\dot{u},1} \\ \hat{\underline{\dot{v}}}_{0|1} = (\hat{\underline{v}}_{1|1} - \hat{\underline{v}}_{0|1}, -\underline{d}_{v,1})/(t_1 - t_0), & \hat{\underline{\dot{v}}}_{1|1} = \hat{\underline{\dot{v}}}_{0|1} + \underline{d}_{\dot{v},1} \end{cases} . \tag{68}$$

or after substitution of (67):

(69)
$$\begin{cases} \hat{\underline{u}}_{0|1} &= (\underline{l}_1 \sin\underline{\alpha}_1 - \underline{l}_0 \sin\underline{\alpha}_0 - \underline{\delta}_{u,1})/(t_1 - t_0) \\ \hat{\underline{u}}_{1|1} &= (\underline{l}_1 \sin\underline{\alpha}_1 - \underline{l}_0 \sin\underline{\alpha}_0 - \underline{\delta}_{u,1})/(t_1 - t_0) + \underline{\delta}_{\dot{u},1} \\ \hat{\underline{v}}_{0|1} &= (\underline{l}_1 \cos\underline{\alpha}_1 - \underline{l}_0 \cos\underline{\alpha}_0 - \underline{\delta}_{v,1})/(t_1 - t_0) \\ \hat{\underline{v}}_{1|1} &= (\underline{l}_1 \cos\underline{\alpha}_1 - \underline{l}_0 \cos\underline{\alpha}_0 - \underline{\delta}_{v,1})/(t_1 - t_0) + \underline{\delta}_{\dot{v},1} \end{cases}.$$

In order to obtain the variances and covariances of the position and velocity estimators of (67) and (69), we first have to linearize (67) and (69). this gives:

(70)
$$\begin{pmatrix} \Delta\hat{\underline{u}}_{0|1} \\ \Delta\hat{\underline{\dot{u}}}_{0|1} \\ \Delta\hat{\underline{v}}_{0|1} \\ \Delta\hat{\underline{\dot{v}}}_{0|1} \\ \Delta\hat{\underline{u}}_{1|1} \\ \Delta\hat{\underline{\dot{u}}}_{1|1} \\ \Delta\hat{\underline{v}}_{1|1} \\ \Delta\hat{\underline{\dot{v}}}_{1|1} \end{pmatrix} = \begin{pmatrix} l_0^0\cos\alpha_0^0 & \sin\alpha_0^0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{-l_0^0\cos\alpha_0^0}{t_1-t_0} & \frac{-\sin\alpha_0^0}{t_1-t_0} & \frac{-1}{t_1-t_0} & 0 & 0 & 0 & \frac{l_1^0\cos\alpha_1^0}{t_1-t_0} & \frac{\sin\alpha_1^0}{t_1-t_0} \\ -l_0^0\sin\alpha_0^0 & \cos\alpha_0^0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{l_0^0\sin\alpha_0^0}{t_1-t_0} & \frac{-\cos\alpha_0^0}{t_1-t_0} & 0 & 0 & \frac{-1}{t_1-t_0} & 0 & \frac{-l_1^0\sin\alpha_1^0}{t_1-t_0} & \frac{\cos\alpha_1^0}{t_1-t_0} \\ 0 & 0 & 0 & 0 & 0 & 0 & l_1^0\cos\alpha_1^0 & \sin\alpha_1^0 \\ \frac{-l_0^0\cos\alpha_0^0}{t_1-t_0} & \frac{-\sin\alpha_0^0}{t_1-t_0} & \frac{-1}{t_1-t_0} & 1 & 0 & 0 & \frac{l_1^0\cos\alpha_1^0}{t_1-t_0} & \frac{\sin\alpha_1^0}{t_1-t_0} \\ 0 & 0 & 0 & 0 & 0 & 0 & -l_1^0\sin\alpha_1^0 & \cos\alpha_1^0 \\ \frac{l_0^0\sin\alpha_0^0}{t_1-t_0} & \frac{-\cos\alpha_0^0}{t_1-t_0} & 0 & 0 & \frac{-1}{t_1-t_0} & 1 & \frac{-l_1^0\sin\alpha_1^0}{t_1-t_0} & \frac{\cos\alpha_1^0}{t_1-t_0} \end{pmatrix} \begin{pmatrix} \Delta\underline{\alpha}_0 \\ \Delta\underline{l}_0 \\ \Delta\underline{\delta}_{u,1} \\ \Delta\underline{\delta}_{\dot{u},1} \\ \Delta\underline{\delta}_{v,1} \\ \Delta\underline{\delta}_{\dot{v},1} \\ \Delta\underline{\alpha}_1 \\ \Delta\underline{l}_1 \end{pmatrix}$$

Note that the matrix of (70) is the inverse of the design matrix of the first eight observation equations of (63). For the approximate values needed in (70) one may take the sample values of $\underline{\alpha}_0$, $\underline{l}_0$, $\underline{\alpha}_1$ and $\underline{l}_1$. The variances and covariances of the position and velocity estimators follow now from applying with (56) the propagation law to (70). This concludes the initialization phase. The first step after initialization is prediction. For the present example this implies that we have to time-propagate the filtered state vector $(\hat{\underline{u}}_{1|1}, \hat{\underline{\dot{u}}}_{1|1}, \hat{\underline{v}}_{1|1}, \hat{\underline{\dot{v}}}_{1|1})^*$ to the next time instant t_2. This is done with the linear time-update equations:

(71)
$$\underbrace{\begin{pmatrix} \hat{\underline{u}}_{2|1} \\ \hat{\dot{\underline{u}}}_{2|1} \\ \hat{\underline{v}}_{2|1} \\ \hat{\dot{\underline{v}}}_{2|1} \end{pmatrix}}_{\hat{\underline{x}}_{2|1}} = \underbrace{\begin{pmatrix} \begin{pmatrix} 1 & (t_2-t_1) \\ 0 & 1 \end{pmatrix} & 0 \\ 0 & \begin{pmatrix} 1 & (t_2-t_1) \\ 0 & \end{pmatrix} \end{pmatrix}}_{\boldsymbol{\Phi}_{2|1}} \underbrace{\begin{pmatrix} \hat{\underline{u}}_{1|1} \\ \hat{\dot{\underline{u}}}_{1|1} \\ \hat{\underline{v}}_{1|1} \\ \hat{\dot{\underline{v}}}_{1|1} \end{pmatrix}}_{\hat{\underline{x}}_{1|1}} + \underbrace{\begin{pmatrix} \underline{d}_{u,2} \\ \underline{d}_{\dot{u},2} \\ \underline{d}_{v,2} \\ \underline{d}_{\dot{v},2} \end{pmatrix}}_{\underline{d}_2} .$$

The variance matrix of the predicted estimator follows from applying the propagation law to (71) as:

(72)
$$Q_{\hat{x}_{2|1}} = \boldsymbol{\Phi}_{2,1}\, Q_{\hat{x}_{1|1}}\, \boldsymbol{\Phi}^{*}_{2,1} + Q_{d_2} .$$

The variance matrix of the filtered estimator $\hat{\underline{x}}_{1|1}$ was computed in the initialization phase as:

$$Q_{\hat{x}_{1|1}} = \begin{pmatrix} (\sigma^2_{a_1}(l_1^0)^2\cos^2 a_1^0 + \sigma^2_{l_1}\sin^2 a_1^0) & (\dfrac{\sigma^2_{a_1}(l_1^0)^2\cos^2 a_1^0 + \sigma^2_{l_1}\sin^2 a_1^0}{(t_1-t_0)}) \\ & (\dfrac{\sum_{i=0}^{1}(\sigma^2_{a_i}(l_i^0)^2\cos^2 a_i^0 + \sigma^2_{l_i}\sin^2 a_i^0)}{(t_1-t_0)^2} + \frac{1}{3}q_{\ddot{u}}(t_1-t_0)) \\ \textit{symmetric} & \end{pmatrix}$$

(73)
$$\begin{pmatrix} ((\sigma^2_{l_1} - \sigma^2_{a_1}(l_1^0)^2)\cos a_1^0 \sin a_1^0) & (\dfrac{\sigma^2_{l_1} - \sigma^2_{a_1}(l_1^0)^2)\cos a_1^0 \sin a_1^0}{(t_1-t_0)}) \\ (\dfrac{(\sigma^2_{l_1} - \sigma^2_{a_1} l_1^0)\cos a_1^0 \sin a_1^0}{(t_1-t_0)}) & (\dfrac{\sum_{i=0}^{1}(\sigma^2_{l_i} - \sigma^2_{a_i}(l_i^0)^2)\cos a_i^0 \sin a_i^0}{(t_1-t_0)^2}) \\ (\sigma^2_{a_1}(l_1^0)^2\sin^2 a_1^0 + \sigma^{0^2}_{l_1}\cos^2 a_1^0) & (\dfrac{\sigma^2_{a_1}(l_1^0)^2\sin^2 a_1^0 + \sigma^2_{l_1}\cos^2 a_1^0}{(t_1-t_0)}) \\ & (\dfrac{\sum_{i=0}^{1}(\sigma^2_{a_i}(l_i^0)^2\sin^2 a_i^0 + \sigma^2_{l_i}\cos^2 a_i^0)}{(t_1-t_0)^2} + \frac{1}{3}q_{\ddot{v}}(t_1-t_0)) \end{pmatrix}$$

and Q_{d_2} is given as:

$$Q_{d_2} = \begin{pmatrix} q_{\ddot{u}} \begin{pmatrix} \frac{1}{3}(t_2-t_1)^3 & \frac{1}{2}(t_2-t_1)^2 \\ \frac{1}{2}(t_2-t_1)^2 & (t_2-t_1) \end{pmatrix} & 0 \\ 0 & q_{\ddot{v}} \begin{pmatrix} \frac{1}{3}(t_2-t_1)^3 & \frac{1}{2}(t_2-t_1)^2 \\ \frac{1}{2}(t_2-t_1)^2 & (t_2-t_1) \end{pmatrix} \end{pmatrix}. \tag{74}$$

Note that if $\sigma_{l_i}^2 = (l_i^0)^2\sigma_{a_i}^2$, then the variance matrix $Q_{\hat{x}_{1|1}}$ of (73) simplifies to:

$$Q_{\hat{x}_{1|1}} = \begin{pmatrix} \begin{pmatrix} \sigma_{l_1}^2 & \frac{\sigma_{l_1}^2}{(t_1-t_0)} \\ * & \left(\frac{\sum_{i=0}^{1} \sigma_{l_i}^2}{(t_1-t_0)^2} + \frac{1}{3} q_{\ddot{u}} (t_1-t_0) \right) \end{pmatrix} & 0 \\ 0 & \begin{pmatrix} \sigma_{l_1}^2 & \frac{\sigma_{l_1}^2}{(t_1-t_0)} \\ * & \left(\frac{\sum \sigma_{l_1}^2}{\frac{i=0}{(t_1-t_0)^2}} + \frac{1}{3} q_{\ddot{v}} (t_1-t_0) \right) \end{pmatrix} \end{pmatrix}. \tag{75}$$

In that case the position and velocity estimators along the *u*-axis are and remain uncorrelated with the correponding position and velocity estimators along the *v*-axis. The next step after prediction is filtering. This implies for the present example that we have to solve the linearized model:

$$\begin{pmatrix} \Delta\underline{\hat{u}}_{2|1} \\ \Delta\underline{\hat{\dot{u}}}_{2|1} \\ \Delta\underline{\hat{v}}_{2|1} \\ \Delta\underline{\hat{\dot{v}}}_{2|1} \\ \Delta\underline{a}_2 \\ \Delta\underline{l}_2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ v_2^0/(l_2^0)^2 & 0 & -u_2^0/(l_2^0)^2 & 0 \\ u_2^0/l_2^0 & 0 & v_2^0/l_2^0 & 0 \end{pmatrix} \begin{pmatrix} \Delta u_2 \\ \Delta\dot{u}_2 \\ \Delta v_2 \\ \Delta\dot{v}_2 \end{pmatrix} \boldsymbol{D} \left\{ \begin{pmatrix} \Delta\underline{\hat{u}}_{2|1} \\ \Delta\underline{\hat{\dot{u}}}_{2|1} \\ \Delta\underline{\hat{v}}_{2|1} \\ \Delta\underline{\hat{\dot{v}}}_{2|1} \\ \Delta\underline{a}_2 \\ \Delta\underline{l}_2 \end{pmatrix} \right\} = \begin{pmatrix} (Q_{\hat{x}_{2|1}}) & 0 \\ 0 & \begin{pmatrix} \sigma_{a_2}^2 & 0 \\ 0 & \sigma_{l_2}^2 \end{pmatrix} \end{pmatrix}. \tag{76}$$

The corresponding measurement-update equation gives then the solution for the filtered estimate $\hat{x}_{2|2} = (\hat{u}_{2|2}, \hat{\dot{u}}_{2|2}, \hat{v}_{2|2}, \hat{\dot{v}}_{2|2})^*$ with corresponding variance matrix $Q_{\hat{x}_{2|2}}$. Note that since (76) is a

linearized model one has to perform an iteration. As starting values for the approximate values one may take (compare with our earlier discussion in Section 2.4):

$$u_2^0 := \hat{u}_{2|1}, \quad v_2^0 := \hat{v}_{2|1}, \quad l_2^0 := (\hat{u}_{2|1}^2 + \hat{v}_{2|1}^2)^{1/2}. \tag{77}$$

After model (76) has been solved, the whole cycle of prediction and filtering is performed again. To conclude this example one final remark is in order. Up to now it was assumed that the ship was equipped with two accelerometers. In that case the trajectory of the ship is allowed to take any form. Now assume that no accelerometers are available. In that case the same recursive algorithm as described above can be applied, provided that some additional assumptions are satisfied. First of all, one will have to assume that the ship moves with an acceleration that is zero on the average. The sample values of the $\underline{d}_i$ are then taken to be zero. And secondly, the spectral densities $q_{\ddot{u}}$ and $q_{\ddot{v}}$ will now have to correspond with the expected erratic behaviour of the ship's acceleration and not with the measurement precision of the accelerometers.

6.4 State vector augmentation

In the previous section the general time-update and measurement-update equations were derived for the recursive prediction and filtering of the time-varying state vector *x(t)*. One of the starting assumptions in the development of the resursive estimation algorithms was that the variance matrix of the vector of observables has to be of a block-diagonal form, see the dispersion of model (24) in Section 6.3. In particular was assumed that:

$$E\{(\underline{\boldsymbol{d}}_i - E\{\underline{\boldsymbol{d}}_i\})(\underline{\boldsymbol{d}}_j - E\{\underline{\boldsymbol{d}}_j\})^*\} = \begin{cases} Q_{d_i} \text{ for } i = j \\ 0 \text{ for } i \neq j \end{cases}. \tag{78}$$

In Section 6.2 it was shown that (78) holds true if the input $\underline{z}_t$ is modelled as a white-noise random function, see (18) in Section 6.2. But as was pointed out in Section 5.6.3 there do exist particular applications for which the white-noise assumption is not realistic. In some applications it is more realistic to model the input as being correlated in time. Correlation in time, however, will affect the structure of the variance matrix of the vector of observables. That is, it will then not be in a block-diagonal form anymore. And this has as a consequence that the recursive algorithms of the previous section will not be applicable anymore. It thus seems that recursive estimation is impossible if correlations in time are present. Fortunately, there is one way out of this dilemma, namely when the correlations in time are of exponential form. In this section we will show that if the input is exponentially correlated in time, a modification of the dynamic model through an augmentation of the state vector is possible, such that (78) still holds true but now for an augmented dynamics vector $\underline{d}$. The idea is based on the result of Example 65 in Section 5.5. In this example it was shown that an exponentially correlated random function can be thought of as being generated by a first-order differential equation driven by white-noise. We will consider the two cases of exponentially correlated zero-mean velocity and exponentially correlated zero-mean acceleration.

6.4.1 Exponentially correlated zero-mean velocity

Consider a particle that moves with an exponentially correlated velocity $\dot{u}(t)$ that is zero on the average. Velocity will then be modelled as a zero-mean random function with auto-covariance function:

(79) $$\sigma_{\dot{u}\dot{u}}(\tau) = \sigma^2 e^{-\alpha|\tau|}, \alpha > 0.$$

The corresponding dynamic model of observation equations reads then:

(80) $$E\{\underline{d}_i\} = u_i - u_{i-1};\ E\{(\underline{d}_i - E\{\underline{d}_i\})(\underline{d}_j - E\{\underline{d}_j\})^*\} = \int_{t_{i-1}}^{t_i}\int_{t_{j-1}}^{t_j} \sigma_{\dot{u}\dot{u}}(\tau_1,\tau_2)d\tau_1 d\tau_2.$$

In this case, however, (78) will fail to hold true. Based on the result of Example 65 in Section 5.5, we therefore model the zero-mean random velocity $\underline{\dot{u}}(t)$ as:

(81) $$\underline{\ddot{u}}(t) = -\alpha\underline{\dot{u}}(t) + \underline{z}(t).$$

with

(82) $$\begin{cases} \sigma_{zz}(\tau) = 2\alpha\sigma^2\delta(\tau) \text{ (white noise)} \\ \sigma^2_{\dot{u}}(t_0) = \sigma^2,\ \sigma_{\dot{u}\dot{u}}(t_0,t_0) = 0,\ \sigma_{\dot{u}z}(t_0,t) = 0\ \forall\ t \\ E\{\underline{\dot{u}}(t_0)\} = 0,\ E\{\underline{z}(t)\} = 0\ \forall\ t. \end{cases}$$

With (81) and (82) we are now in a position to formulate a dynamic model for which (78) indeed holds true. In order to show this we first put (81) into a first-order state vector form:

(83) $$\underbrace{\begin{pmatrix} \underline{u}(t) \\ \dot{\underline{u}}(t) \end{pmatrix}^{\cdot}}_{\dot{\underline{x}}(t)} = \underbrace{\begin{pmatrix} 0 & 1 \\ 0 & -\alpha \end{pmatrix}}_{F} \underbrace{\begin{pmatrix} \underline{u}(t) \\ \dot{\underline{u}}(t) \end{pmatrix}}_{\underline{x}(t)} + \underbrace{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{G\underline{z}(t)} \underline{z}(t).$$

This is a linear, time-invariant state equation. The matrix-exponential of the system matrix F reads (verify this yourself):

(84) $$e^{Ft} = \begin{pmatrix} 1 & \frac{1}{\alpha}(1 - e^{-\alpha t}) \\ 0 & e^{-\alpha t} \end{pmatrix}.$$

The solution of the linear, time-invariant state equation (83) reads therefore:

(85) $$\boxed{\begin{pmatrix} \underline{u}(t) \\ \dot{\underline{u}}(t) \end{pmatrix} = \begin{pmatrix} 1 & \frac{1}{\alpha}(1 - e^{-\alpha(t-t_0)}) \\ 0 & e^{-\alpha(t-t_0)} \end{pmatrix} \begin{pmatrix} \underline{u}(t_0) \\ \dot{\underline{u}}(t_0) \end{pmatrix} + \int_{t_0}^{t} \begin{pmatrix} \frac{1}{\alpha}(1 - e^{-\alpha(t-\tau)}) \\ e^{-\alpha(t-\tau)} \end{pmatrix} \underline{z}(\tau)d\tau}.$$

It is this relation which replaces (212) in Section 5.6.1 when the zero-mean velocity is exponentially corrrelated. Note that with $E\{\underline{\dot{u}}(t_0)\}=0$ and $E\{\underline{z}(t)\}=0$ (see (82)), the expectation

of (85) indeed satisfies (213) and (214) in Section 5.6.1. The variance matrix of position and velocity follows with (82) from (85) as:

$$\begin{pmatrix} \sigma_u^2(t) & \sigma_{u\dot{u}}(t,t) \\ \sigma_{\dot{u}u}(t,t) & \sigma_{\dot{u}}^2(t) \end{pmatrix} = \begin{pmatrix} 1 & \frac{1}{\alpha}(1-e^{-\alpha(t-t_0)}) \\ 0 & e^{-\alpha(t-t_0)} \end{pmatrix} \begin{pmatrix} \sigma_u^2(t_0) & 0 \\ 0 & \sigma^2 \end{pmatrix} \begin{pmatrix} 1 & \frac{1}{\alpha}(1-e^{-\alpha(t-t_0)}) \\ 0 & e^{-\alpha(t-t_0)} \end{pmatrix}^* +$$

$$+ \int_{t_0}^{t} 2\alpha\sigma^2 \begin{pmatrix} \frac{1}{\alpha}(1-e^{-\alpha(t-\tau)}) \\ e^{-\alpha(t-\tau)} \end{pmatrix} \begin{pmatrix} \frac{1}{\alpha}(1-e^{-\alpha(t-\tau)}) \\ e^{-\alpha(t-\tau)} \end{pmatrix}^* d\tau$$

or as:

$$\begin{pmatrix} \sigma_u^2(t) & \sigma_{u\dot{u}}(t,t) \\ \sigma_{\dot{u}u}(t,t) & \sigma_{\dot{u}}^2(t) \end{pmatrix} = \begin{pmatrix} \sigma_u^2(t_0) + \frac{\sigma^2}{\alpha^2}(1-e^{-\alpha(t-t_0)})^2 & \frac{\sigma^2}{\alpha}e^{-\alpha(t-t_0)}(1-e^{-\alpha(t-t_0)}) \\ \frac{\sigma^2}{\alpha}e^{-\alpha(t-t_0)}(1-e^{-\alpha(t-t_0)}) & \sigma^2 e^{-2\alpha(t-t_0)} \end{pmatrix} + \underbrace{\begin{pmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{pmatrix}}_{Q_{xx}(t,t)}$$

with

$$\begin{pmatrix} q_{11} = 2\frac{\sigma^2}{\alpha}[(t-t_0) - \frac{2}{\alpha}(1-e^{-\alpha(t-t_0)}) + \frac{1}{2\alpha}(1-e^{-\alpha(t-t_0)})] \\ q_{12} = q_{21} = 2\sigma^2[\frac{1}{\alpha}(1-e^{-\alpha(t-t_0)}) - \frac{1}{2\alpha}(1-e^{-2\alpha(t-t_0)})] \\ q_{22} = \sigma^2[1-e^{-2\alpha(t-t_0)}] \end{pmatrix}$$

(86)

Note that the variance function $\sigma_u^2(t)$ of (86) is indeed identical to (227) in Section 5.6.3. Again there are two limiting cases of interest. First, for long time-intervals $(t-t_0) >> \frac{1}{\alpha}$, we have:

(87)
$$\lim_{\alpha(t-t_0)\to\infty} Q_{xx}(t,t) = \begin{pmatrix} \sigma_u^2(t_0) + 2\frac{\sigma^2}{\alpha}(t-t_0) & \frac{\sigma^2}{\alpha} \\ \frac{\sigma^2}{\alpha} & \sigma^2 \end{pmatrix}.$$

This result agrees with (229) in Section 5.6.3. Secondly, for short time-intervals $(t-t_0) << \frac{1}{\alpha}$, we have:

(88)
$$\lim_{\alpha(t-t_0)\to 0} Q_{xx}(t,t) = \begin{pmatrix} \sigma_u^2(t_0) + \sigma^2(t-t_0)^2 & 0 \\ 0 & 0 \end{pmatrix}.$$

And this result agrees with (231) in Section 5.6.3.

6.4.2 Exponentially correlated zero-mean acceleration

Consider a particle that moves with an exponentially correlated acceleration $\ddot{u}(t)$ that is zero on the average. In order to obtain a dynamic model such that (78) holds true, we model the zero-mean random acceleration $\underline{\ddot{u}}(t)$ as:

(89)
$$\dot{\underline{\ddot{u}}}(t) = -\alpha\underline{\ddot{u}}(t) + \underline{z}(t)$$

with

(90)
$$\begin{cases} \sigma_{zz}(\tau) = 2\alpha\sigma^2\delta(\tau) \text{ (white noise)} \\ \sigma^2_{\ddot{u}}(t_0) = \sigma^2,\ \sigma_{u\ddot{u}}(t_0t_0) = 0,\ \sigma_{\dot{u}\ddot{u}}(t_0,t_0) = 0,\quad \sigma_{\ddot{u}z}(t_0,t) = 0,\ \forall\ t \\ E\{\underline{\ddot{u}}(t_0)\} = 0,\ E\{\underline{z}(t)\} = 0,\ \forall\ t. \end{cases}$$

The first-order state vector form of (89) reads:

(91)
$$\underbrace{\begin{pmatrix} \dot{\underline{u}}(t) \\ \dot{\underline{\dot{u}}}(t) \\ \dot{\underline{\ddot{u}}}(t) \end{pmatrix}}_{\dot{\underline{x}}(t)} = \underbrace{\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -\alpha \end{pmatrix}}_{F} \underbrace{\begin{pmatrix} \underline{u}(t) \\ \dot{\underline{u}}(t) \\ \ddot{\underline{u}}(t) \end{pmatrix}}_{\underline{x}(t)} + \underbrace{\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}}_{G} \underline{z}(t).$$

And the matrix exponential of the system matrix F reads (verify this yourself):

(92)
$$e^{Ft} = \begin{pmatrix} 1 & t & \frac{1}{\alpha^2}(-1+\alpha t+e^{-\alpha t}) \\ 0 & 1 & \frac{1}{\alpha}(1-e^{-\alpha t}) \\ 0 & 0 & e^{-\alpha t} \end{pmatrix}.$$

The solution of the linear, time-invariant state equation (91) reads therefore:

(93)
$$\boxed{\begin{pmatrix} \underline{u}(t) \\ \dot{\underline{u}}(t) \\ \ddot{\underline{u}}(t) \end{pmatrix} = \begin{pmatrix} 1 & (t-t_0) & \frac{1}{\alpha^2}[-1+\alpha(t-t_0)+e^{-\alpha(t-t_0)}] \\ 0 & 1 & \frac{1}{\alpha}[e^{-\alpha(t-t_0)}] \\ 0 & 0 & e^{-\alpha(t-t_0)} \end{pmatrix} \begin{pmatrix} \underline{u}(t_0) \\ \dot{\underline{u}}(t_0) \\ \ddot{\underline{u}}(t_0) \end{pmatrix} + \int_{t_0}^{t} \begin{pmatrix} \frac{1}{\alpha^2}[-1+\alpha(t-\tau)+e^{-\alpha(t-\tau)}] \\ \frac{1}{\alpha}[1-e^{-\alpha(t-\tau)}] \\ e^{-\alpha(t-\tau)} \end{pmatrix} \underline{z}(\tau)d\tau}$$

It is this relation which replaces (217) in Section 5.6.1 when the zero-mean acceleration is exponentially correlated. Note that with $E\{\underline{\dot{u}}(t_0)\}=0$ and $E\{\underline{z}(t)\}=0$, see (90), the expectation of (93) indeed satisfies (218) and (219) in Section 5.6.1. The variance matrix of position, velocity and acceleration follows with (90) and (93) as (verify this yourself), formula (94).

$$\begin{pmatrix} \sigma_u^2(t) & \sigma_{u\dot{u}}(t,t) & \sigma_{u\ddot{u}}(t,t) \\ & \sigma_{\dot{u}}^2(t) & \sigma_{\dot{u}\ddot{u}}(t,t) \\ symm. & & \sigma_{\ddot{u}}^2(t) \end{pmatrix} =$$

$$\begin{pmatrix} 1 & (t-t_0) & \dfrac{-1+\alpha(t-t_0)+e^{-\alpha(t-t_0)}}{\alpha^2} \\ 0 & 1 & \frac{1}{\alpha}[1-e^{-\alpha(t-t_0)}] \\ 0 & 0 & e^{-\alpha(t-t_0)} \end{pmatrix} \begin{pmatrix} \sigma_u^2(t_0) & \sigma_{u\dot{u}}(t_0,t_0) & 0 \\ \sigma_{\dot{u}u}(t_0,t_0) & \sigma_{\dot{u}}^2(t_0) & 0 \\ 0 & 0 & \sigma^2 \end{pmatrix} \begin{pmatrix} 1 & (t-t_0) & \dfrac{-1+\alpha(t-t_0)+e^{-\alpha(t-t_0)}}{\alpha^2} \\ 0 & 1 & \frac{1}{\alpha}[1-e^{-\alpha(t-t_0)}] \\ 0 & 0 & e^{-\alpha(t-t_0)} \end{pmatrix}^* +$$

$$+ \begin{pmatrix} q_{11} & q_{12} & q_{13} \\ & q_{22} & q_{23} \\ symm. & & q_{33} \end{pmatrix}$$

(94)

with

$$\begin{cases} q_{11} = 2\dfrac{\sigma^2}{\alpha^3}[(t-t_0)-\alpha(t-t_0)^2+\dfrac{\alpha^2}{3}(t-t_0)^3-2(t-t_0)e^{-\alpha(t-t_0)}+\dfrac{1}{2\alpha}(1-e^{-2\alpha(t-t_0)})] \\ q_{12} = 2\dfrac{\sigma^2}{\alpha^2}[-(t-t_0)+\dfrac{\alpha}{2}(t-t_0)^2+(t-t_0)e^{-\alpha(t-t_0)}-\dfrac{1}{\alpha}e^{-\alpha(t-t_0)}+\dfrac{1}{2\alpha}(1+e^{-2\alpha(t-t_0)})] \\ q_{22} = \dfrac{2\sigma^2}{\alpha}[(t-t_0)-\dfrac{3}{2\alpha}+\dfrac{2}{\alpha}e^{-\alpha(t-t_0)}-\dfrac{1}{2\alpha}e^{-2\alpha(t-t_0)}] \\ q_{13} = \dfrac{2\sigma^2}{\alpha}[-(t-t_0)e^{-\alpha(t-t_0)}+\dfrac{1}{2\alpha}(1-e^{-2\alpha(t-t_0)})] \\ q_{23} = \dfrac{2\sigma^2}{\alpha}[-e^{-\alpha(t-t_0)}+\dfrac{1}{2}(1+e^{-2\alpha(t-t_0)})] \\ q_{33} = \sigma^2[1-e^{-2\alpha(t-t_0)}] \end{cases}$$

Verify yourself that the variance matrix of position and velocity of (94) is identical to (233) in Section 5.6.3.

Example 70

Consider again the situation of Example 69. But now assume that the ship sails with an exponentially correlated acceleration that is zero on the average, Then the dynamic model (62) in Section 6.3 needs to be modified using the results of (93) and (94). The augmented dynamic model reads then:

$$E\left\{\begin{pmatrix} \underline{d}_{u,i} \\ \underline{d}_{\dot{u},i} \\ \underline{d}_{\ddot{u},i} \end{pmatrix}\right\} = \left(- \begin{pmatrix} 1 & (t_i - t_{i-1}) & \frac{1}{\alpha^2}(-1 + \alpha(t_i - t_{i-1}) + e^{-\alpha(t_i - t_{i-1})}) \\ 0 & 1 & \frac{1}{\alpha}(1 - e^{-\alpha(t_i - t_{i-1})}) \\ 0 & 0 & e^{-\alpha(t_i - t_{i-1})} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right) \begin{pmatrix} u_{i-1} \\ \dot{u}_{i-1} \\ \ddot{u}_{i-1} \\ u_i \\ \dot{u}_i \\ \ddot{u}_i \end{pmatrix}$$

$$\text{with } Q_{\underline{d}_i \underline{d}_j} = \begin{pmatrix} q_{11} & q_{12} & q_{13} \\ & q_{22} & q_{23} \\ sym. & & q_{33} \end{pmatrix} \delta_{ij} \text{ of (94) for the time-interval } t_i - t_{i-1}.$$

In this case the sample values of the $\underline{d}_i = (\underline{d}_{u,i}, \underline{d}_{\dot{u},i}, \underline{d}_{\ddot{u},i})^*$ are taken to be zero. And at initialization $\underline{\ddot{u}}(t_0)$ is treated as an observable with a sample value of zero and variance σ^2 (see (90)).

Example 71

In Example 69 the choice was made to parameterize the models in terms of cartesian coordinates see (63). This resulted in nonlinear observation equations for the azimuth and distance observables (see (57)), but in a linear dynamic model (see (62)). In the present example a different parameterization will be used. Instead of using a fixed *i*-frame, we will now parameterize with respect to a rotating *b*-frame such that the observation equations for the azimuth and distance observables will become linear. This will have as a consequence that now the dynamic model will take a nonlinear form.

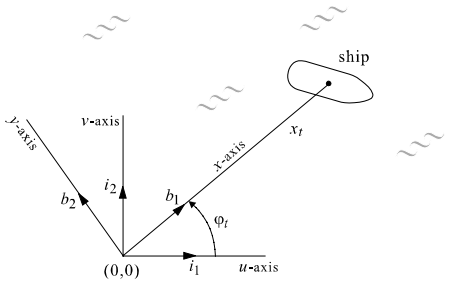


Figure 6.7: The *i*-frame and *b*-frame.

Consider Figure 6.7. The position, velocity and acceleration of the ship in the rotating *b*-frame read:

(96) $$\begin{pmatrix} x \\ 0 \end{pmatrix} = \boldsymbol{R}_{bi} r_i, \quad \begin{pmatrix} v_x \\ v_y \end{pmatrix} = \boldsymbol{R}_{bi} \dot{r}_i, \quad \begin{pmatrix} a_x \\ a_y \end{pmatrix} = \boldsymbol{R}_{bi} \ddot{r}_i$$

where

(97) $$\boldsymbol{R}_{bi} = \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix}, \quad r_i = \begin{pmatrix} u \\ v \end{pmatrix}.$$

As state vector elements we will take: x, v_x, φ and v_y. The observation equations for azimuth and distance will then take the following linear form:

(98) $$E\left\{ \begin{pmatrix} \frac{1}{2}\pi - \underline{a}_i \\ \underline{l}_i \end{pmatrix} \right\} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \varphi_i \\ x_i \end{pmatrix}.$$

Let us now consider the corresponding state equation. We have according to Table 4.1 for velocity and acceleration:

$$\begin{aligned} \boldsymbol{R}_{bi}\dot{r}_i &= \Omega_b^{ib} r_b + \dot{r}_b \\ &= \Omega_b^{ib}(\boldsymbol{R}_{bi} r_i) + (\dot{\boldsymbol{R}}_{bi} r_i) \end{aligned}$$

and

$$\begin{aligned} \boldsymbol{R}_{bi}\ddot{r}_i &= \Omega_b^{ib}\Omega_b^{ib} r_b + \dot{\Omega}_b^{ib} r_b + 2\Omega_b^{ib}\dot{r}_b + \ddot{r}_b \\ &= \Omega_b^{ib}(\boldsymbol{R}_{bi}\dot{r}_i) + (\dot{\boldsymbol{R}}_{bi}\dot{r}_i). \end{aligned}$$

This shows that if the acceleration $R_{bi}\ddot{r}_i$ is considered as input, the first-order state equation may be written as:

(99)
$$\begin{pmatrix} \dot{\boldsymbol{R}}_{bi}\boldsymbol{r}_i \\ \dot{\boldsymbol{R}}_{bi}\dot{\boldsymbol{r}}_i \end{pmatrix} = \begin{pmatrix} -\Omega_b^{ib} & I \\ 0 & -\Omega_b^{ib} \end{pmatrix} \begin{pmatrix} \boldsymbol{R}_{bi}\boldsymbol{r}_i \\ \boldsymbol{R}_{bi}\dot{\boldsymbol{r}}_i \end{pmatrix} + \begin{pmatrix} 0 \\ \boldsymbol{R}_{bi}\ddot{\boldsymbol{r}}_i \end{pmatrix}$$

or in the components x, v_x, φ and v_y as:

(100)
$$\begin{pmatrix} \dot{x} \\ \dot{v}_x \\ \dot{\varphi} \\ \dot{v}_y \end{pmatrix} = \begin{pmatrix} v_x \\ v_y^2/x \\ v_y/x \\ -v_x v_y/x \end{pmatrix} + \begin{pmatrix} 0 \\ a_x \\ 0 \\ a_y \end{pmatrix}.$$

This is a nonlinear, time-invariant state equation.

6.5 Summary

In this chapter we developed the general least-squares prediction and filtering formulae for the recursive estimation of a time-varying state vector *x(t)*. It was shown that the measurement-update equations are identical to the ones developed in Chapter 3. The time-update equations differ, however, from those of Chapter 3. This difference is due to the way the dynamics of the system are modelled. In the present chapter the input $\underline{z}(t)$ was included in the dynamic model. It was shown that an unwarranted zero-input assumption may lead to filter divergence. It is therefore expedient to model inputs that can be considered to be zero on the average, as zero-mean random functions. This leads to a more flexible dynamic model, with the zero-input assumption as special case. It was also shown how to augment the dynamic model in case the input cannot be treated as white noise. Augmentation is possible if the input is exponentially correlated in time.

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Index

Dynamic data processing :
Recursive least-squares

Peter J.G. Teunissen

This book is a follow-up on Adjustment theory. It extends the theory to the case of time-varying parameters with an emphasis on their recursive determination. Least-squares estimation will be the leading principle used. A least-squares solution is said to be recursive when the method of computation enables sequential, rather than batch, processing of the measurement data. The recursive equations enable the updating of parameter estimates for new observations without the need to store all past observations. Methods of recursive least-squares estimation are therefore particularly useful for applications in which the time-varying parameters need to be instantly determined. Important examples of such applications can be found in the fields of real-time kinematic positioning, navigation and guidance, or multivariate time series analysis. The goal of this book is therefore to convey the necessary knowledge to be able to process sequentially collected measurements for the purpose of estimating time-varying parameters.

When determining time-varying parameters from sequentially collected measurement data, one can discriminate between three types of estimation problems: filtering, prediction and smoothing. Filtering aims at the determination of current parameter values, while smoothing and prediction aim at the determination of respectively past and future parameter values. The emphasis in this book will be on recursive least-squares filtering. The theory is worked out for the important case of linear(ized) models. The measurement-update and time-update equations of recursive least-squares are discussed in detail. Models with sequentially collected data, but time-invariant parameters are treated first.

In this case only the measurement-update equations apply. State-space models for dynamic systems are discussed so as to include time-varying parameters. This includes their linearization and the construction of the state trans matrix. Elements from the theory of random functions are used to describe the propagation laws for linear dynamic systems. The theory is illustrated by means of many worked out examples. They are drawn from applications such as kinematic positioning, satellite orbit determination and inertial navigation

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