

**ON ASPECTS OF VIBRATIONS OF
AXIALLY MOVING CONTINUA**

Svetlana HAGERAATS-PONOMAREVA

On aspects of vibrations of axially moving continua

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Svetlana Viktorovna HAGERAATS-PONOMAREVA

Master of Science in Mechanics
Polytechnische Staatsuniversiteit Sint-Petersburg, Rusland
geboren te Leningrad, Sovjet Unie

Dit proefschrift is goedgekeurd door de promotor:

Prof.dr.ir. A.W. Heemink

Copromotor:

Dr.ir. W.T. van Horssen

Samenstelling promotiecommissie:

Rector Magnificus	voorzitter
Prof.dr.ir. A.W. Heemink,	Technische Universiteit Delft, promotor
Dr.ir. W.T. van Horssen,	Technische Universiteit Delft, copromotor
Prof.dr. A.K. Abramyan,	Russian Academy of Sciences, Institut for Problems of Mechanical Engineering
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Prof.dr.ir. G. Lodewijks,	Technische Universiteit Delft
Prof.dr. A.V. Metrikine,	Technische Universiteit Delft

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On the cover: Eva, by D. Potapov

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To my parents

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CHAPTER 1

Introduction

1.1 Background

Vibrations are everywhere around us, many occur in nature, and many in man-made devices, structures and machines. Very often vibrations are not a desirable state of a structure, and the main goal of engineers is to avoid or to reduce these vibrations. Mathematicians, mechanical engineers, and physicists studied vibrations for many years by means of numerical and analytical methods, and these phenomena are still of interest today. A wide range of scientific journal papers and books are devoted to this problem. In order to solve the challenging problems for vibrations, a number of techniques and methods has been developed (see, for instance [1], [2]). Mathematical models of vibrating systems are usually divided into two classes: discrete, or lumped-parameter models, and continuous, or distributed-parameter models. In reality, however, systems can contain both discrete and continuous parts. Because exact solutions are possible to find only for a limited number of problems, formal approximations of the solution are usually constructed by means of asymptotic or numerical methods. Many books have been written about asymptotic and numerical methods. An introduction to asymptotic methods with the basic principles of asymptotics and its applications, and an overview of traditional and modern approaches can be found in [3], [4], [5], and [6].

The goal of the present thesis is to study vibrations of axially moving continua, which are described by distributed-parameter or continuous mathematical models. Objects like, for instance, conveyor belts, magnetic tapes,

tramways, power transmission belts, band saw blades, oil pipelines, etc, are often referred to as axially moving materials or axially moving continua. Axially moving continua can be mathematically modelled as strings, rods or beams, depending on the bending stiffness. But they all have something in common, namely, the dimension in so-called “axial” direction is much larger than the dimensions in the other two directions. Studying the dynamical behaviour of axially moving materials as well as problems of interaction of structures with flows are both of great practical and theoretical interest (see, for instance [7] and [8]). In Figure 1.1 an example of a moving cable system is given.



Figure 1.1: An example of a moving cable system.

A schematic model of a conveyor belt system which moves with velocity V_b is shown in Figure 1.2.

Moving conveyor belt systems can be divided into two kinds: belt systems which move with constant velocity and belt systems which move with time-varying velocities. A constant belt speed case was studied in [9], [10], and [11]. It was found that the natural frequencies of the moving belt system depend on the axial speed V_b and decrease with increasing belt speed. It was also found that instabilities may occur for certain critical velocity values.

In reality, however, a moving conveyor belt system has some imperfections such as pulley eccentricities and so on, and correspondingly experiences accelerating and decelerating motions. More realistic models with the low and time-dependent belt velocity V_b were recently studied in [12], [13], and [14]. The two time-scales perturbation method was used for a beam-like equation

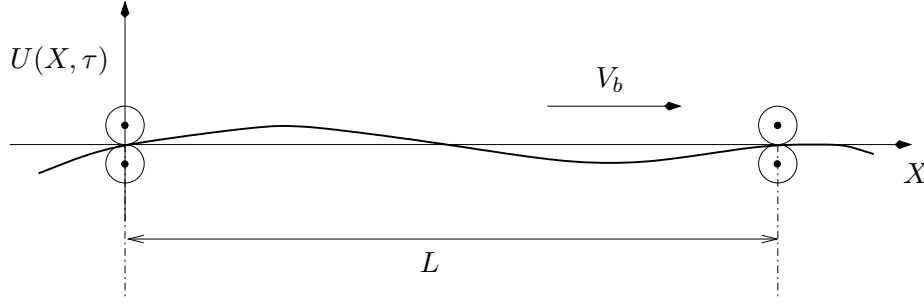


Figure 1.2: Schematic model of a moving conveyor belt system with velocity V_b , length L , transversal displacement $U(X, \tau)$. The coordinate in axial direction is X , and τ represents the time.

(in [12]), for a string-like equation (in [13]), and for a nonlinear beam-like equation (in [14]). Transversal vibrations of a moving string with a high and time-dependent velocity were studied in [15]. To find approximate solutions of these four problems a truncation to just one vibration mode was applied in [12]-[15]. On the other hand, the solution of a partial differential equation consists of infinitely many modes and application of the truncation method (without justification) may lead to inaccurate results. Transversal vibrations of an axially moving string with a low and time-varying velocity were studied in [16], [17]. It was found that there are infinitely many interaction between vibration modes, and that the truncation method can not be applied to obtain asymptotic results on long-time scales in this case. An associated beam-like equation was studied in [18]. It was shown that there are less problems in applying of the truncation method in this case. Transversal vibration of a moving beam with viscous damping were also studied recently in [19]. A number of papers was also devoted to (weakly) nonlinear transversal vibrations of axially moving belts (see, for instance, [20]). The effect of an end mass on stability properties of a translating string and beam with variable length was studied in [21]. More studies of string and beam equations in application to deep mine hoisting and elevators cables can be found in [22], [23]. A coupled belt-pulley vibration was studied in [24].

1.2 Motivation

It has already been indicated that the truncation method can not be applied to a moving string-like equation for the case of a low and time-dependent belt velocity. The purpose of the present work is to study the applicability of the truncation method in case of a relatively high and time-dependent belt velocity

(that is, of the same order of magnitude as the wave speed in the string).

On the other hand, in [18] it has been shown that the truncation method can be applied to the beam-string-like equations when the belt velocity is low. From experiments and from theoretical investigations (see, for example [25]) it is known that the real dynamic behaviour of conveyer belt systems with relatively small bending stiffness is some sort of combination of both string-like and beam-string-like models. The first low vibration modes look more like string modes and the higher order modes (when the bending stiffness becomes more important) look more like beam-string or beam modes. It is not only interesting but also important from the applicational point of view to investigate such phenomena of transient “from string to beam-string (or beam)” behaviour.

It is also of great practical interest to study the case of a sudden stop of a moving belt system. A corresponding system of coupled nonlinear partial differential equations (in transversal and in longitudinal directions) will be studied in this thesis.

1.3 Mathematical model

In the classical analysis of axially moving continua the vibrations are usually classified into two types, i.e. whether it is of a string-like type or of a beam-like type, depending on the bending stiffness of the belt. If the bending stiffness can be neglected then the system is classified as string (wave)-like, otherwise it is classified as beam-like. The belt under consideration is assumed to be rather elastic. Equations of motions for a moving conveyer belt system can be derived by using Hamilton’s principle. (see, for instance [26]). The transverse vibrations of a belt system (with time-varying velocity $V(\tau)$) can be modelled mathematically as:

- string-like by

$$U_{\tau\tau} + 2VU_{X\tau} + V_{\tau}U_X + (V^2 - c^2)U_{XX} = 0, \text{ and} \quad (1.1)$$

- beam-like (with a string effect) by

$$U_{\tau\tau} + 2VU_{X\tau} + V_{\tau}U_X + (V^2 - c^2)U_{XX} + \frac{EI}{\rho A}U_{XXXX} = 0, \quad (1.2)$$

where

$U(X, \tau)$	: the displacement of the string in vertical direction,
$V(\tau)$	: the time-varying belt speed,
$c > 0$	: the wave speed due to a pretension of the belt,
X	: the coordinate in horizontal direction,
E	: the modulus of elasticity,
I	: the second moment of inertia with respect to the belt middle plane,
ρ	: the mass density of the belt,
A	: the area of the cross section of the belt,
τ	: the time, and
L	: the distance between the pulleys,

and where $c = \sqrt{\frac{T_0}{A\rho}}$, in which T_0 is assumed to be the constant non zero tension of the belt. The time-varying belt velocity $V(\tau)$ is given by $V(\tau) = V_0 + \varepsilon\alpha \sin(\omega\tau)$ or by $V(\tau) = \varepsilon(V_0 + \alpha \sin(\omega\tau))$, where V_0 , ω , and α are some positive constants with $V_0 > 0$, and where ε is a small parameter with $0 < \varepsilon \ll 1$. The small parameter ε indicates that there are perturbations of the main belt speed V_0 , due to different kinds of imperfections of the belt system. When $V(\tau) = \varepsilon(V_0 + \alpha \sin(\omega\tau))$, it is assumed that the belt always moves forward in one direction (the condition $V_0 > |\alpha|$ is applied). Equations (1.1) and (1.2) are linearized equations of motion, derived from the more complicated nonlinear equations of motion. Although nonlinear equations are more accurate, it is not meaningless to study first weakly perturbed linear equations. Nonlinear equations of motion for the moving belt system will be studied in Chapter 5.

It is assumed that the belt is simply supported, and that the boundary and initial conditions are given by

$$\begin{aligned} U(0, \tau; \varepsilon) &= U(L, \tau; \varepsilon) = 0, \quad \tau \geq 0, \\ U(X, 0; \varepsilon) &= f(X), \text{ and } U_\tau(X, 0; \varepsilon) = r(X), \quad 0 < X < L, \end{aligned} \quad (1.3)$$

for the string-like equation (1.1), and

$$\begin{aligned} U(0, \tau; \varepsilon) &= U(L, \tau; \varepsilon) = U_{XX}(0, \tau; \varepsilon) = U_{XX}(L, \tau; \varepsilon) = 0, \quad \tau \geq 0, \\ U(X, 0; \varepsilon) &= f(X), \text{ and } U_\tau(X, 0; \varepsilon) = r(X), \quad 0 < X < L, \end{aligned} \quad (1.4)$$

for the beam-like equation (1.2). In (1.3) and (1.4) $f(X)$ represents the initial displacement and $r(X)$ is the initial velocity of the belt.

1.4 Methods

In mathematics analytical solutions can be found only for a limited number of differential equations. Real physical problems are subjected to an essential influence of variable parameters, nonlinearities, and complicated boundary conditions. To find an analytical solution in these cases is usually not possible. In order to obtain information about behaviour of such physical systems, approximated solutions can sometimes be constructed. In this thesis formal approximation of the solution of the initial-boundary value problems (1.1),(1.3) and (1.2),(1.4) will be constructed. These formal approximations satisfy the differential equations and the boundary conditions up to some order in ε . Since a straight-forward expansion in ε may have secular terms (terms which are unbounded in τ) in the approximations of the solution, the two time-scales perturbation method will be used in this thesis to avoid these unbounded terms. To apply this method two time-scales are introduced: $\tau_0 = \tau$ and $\tau_1 = \varepsilon\tau$, so that $U(X, \tau; \varepsilon) = V(X, \tau_0, \tau_1; \varepsilon)$. The introduction of these two time-scales defines the following transformations for the time derivatives:

$$\begin{aligned}\frac{\partial(\cdot)}{\partial\tau} &= \frac{\partial(\cdot)}{\partial\tau_0} + \varepsilon \frac{\partial(\cdot)}{\partial\tau_1}, \\ \frac{\partial^2(\cdot)}{\partial\tau^2} &= \frac{\partial^2(\cdot)}{\partial\tau_0^2} + 2\varepsilon \frac{\partial^2(\cdot)}{\partial\tau_0\partial\tau_1} + \varepsilon^2 \frac{\partial^2(\cdot)}{\partial\tau_1^2}.\end{aligned}\tag{1.5}$$

Assuming that $V(X, \tau_0, \tau_1; \varepsilon) = V_0(X, \tau_0, \tau_1) + \varepsilon V_1(X, \tau_0, \tau_1) + \dots$, a set of differential equations for each order in ε will be obtained, which can be solved one by one. Secular terms should be eliminated in each step while solving. For a more complete overview of this perturbation method (i.e. the method of multiple time-scales) the reader is referred to [4] or [5].

1.5 Outline of the thesis

The thesis is organized as follows. In this Chapter 1 a brief introduction to the subject has been given.

In Chapter 2 an initial-boundary value problem for a linear, non-homogeneous axially moving string equation will be considered. The velocity of the string is assumed to be constant (i.e. $V_b = V_0$, and V_0 is of the same order of magnitude as the wave speed c), and the non-homogeneous terms in the string equation are due to external forces acting on the string. The Laplace transform method will be used to construct the solution of the problem. It will turn out that the method has considerable, computational advantages compared to the usually applied method of modal analysis based on eigenfunction expansions.

In Chapter 3 an initial-boundary value problem for a linear equation describing an axially moving string will be considered for which the bending stiffness will be neglected. The velocity of the string is assumed to be time-varying and to be of the same order of magnitude as the wave speed. A two time-scales perturbation method and the Laplace transform method will be used to construct formal asymptotic approximations of the solutions. It will be shown that the linear axially moving string model already has complicated dynamical behaviour and that the truncation method can not be applied to this problem in order to obtain approximations which are valid on long time-scales.

In Chapter 4 a new model approach describing the transient “from string to beam-string (or beam)” behaviour, based on the computations of the natural frequencies will be proposed. The influence of the bending stiffness on the time behaviour of an approximate solution of the problem will be studied. The regions of applicability of the simplified models will be found for different values of the bending stiffness parameter and different values of the relative error. It turns out that there are infinitely many values of ω that give rise to internal resonances in the axially moving belt system. In fact, this happens when ω is equal to any sum or difference combination of the natural frequencies of the string model and (or) the stretched beam model. It will also turn out that the two-time scales perturbation method defines a correction in the string model frequency which represents the effect of the small bending stiffness for the lower oscillation modes. Formal approximations of the solution will be constructed, and the stability analysis in some resonant cases will be studied for four different values of the relative errors: 5%, 3%, 1% and 0.1%. It will turn out that the properties of the solutions remain more or less the same, and correspond to the properties of the exact solution of the problem.

In Chapter 5 the energy transfer between the longitudinal motion and the transversal motion of a conveyor belt system after stoppage will be studied. Nonlinear equations of motion for the moving belt system will be studied. It will turn out that complicated internal resonances can occur leading to interesting energy transfers between longitudinal and transversal motions. For certain parameter values it will turn out that relatively high mode longitudinal vibrations can induce low mode transversal vibrations with relatively large amplitudes.

In Chapter 6 some conclusions will be given and some possibilities for future research related to this thesis will be discussed.

Most of the chapters have been submitted to different journals and therefore they are self-contained.

CHAPTER 2

On the construction of the solution of an equation describing an axially moving string

Abstract: In this chapter an initial-boundary value problem for a linear, non-homogeneous axially moving string equation will be considered. The velocity of the string is assumed to be constant, and the non-homogeneous terms in the string equation are due to external force acting on the string. The Laplace transform method will be used to construct the solution of the problem. It will turn out that the method has considerable, computational advantages compared to the usually applied method of modal analysis based on eigenfunction expansions.

2.1 Formulation of the problem

The dynamics of axially moving materials have been studied by many researchers due to their technological importance. Transversal vibrations of belt systems have been investigated for many years. A lot of literature is devoted to this problem (see the reference lists in [12]-[28]). The main goal of this chapter is to investigate a linear, non-homogeneous equation for a moving string which

This chapter is a slightly revised version of [27]: W.T. van Horssen and S.V. Ponomareva: On the construction of the solution of an equation describing an axially moving string, *Journal of Sound and Vibration*, 287, (2005), 359-366.

is a starting point to study later weakly perturbed or weakly nonlinear problems for moving string equations. First the the displacement of the string in vertical direction will be determined by using the Laplace transform method. In the second part of this chapter the solution will be constructed by using the method of eigenfunction expansions. This last method was developed for this problems by Meirovitch in [29], [30] and by Wickert and Mote in [31]. Both methods will be compared. The conditions under which bounded solutions exist will be derived, and it will be shown for what kind of external forces internal resonances in the system will occur.

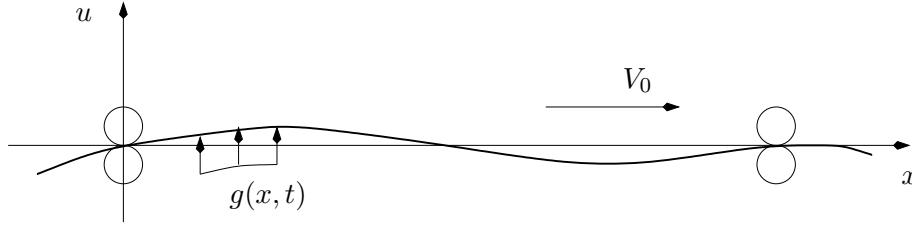


Figure 2.1: An axially moving string.

The following linear equation of motion for the string with initial tension (moving in one direction with a constant velocity V_0) will be considered in this chapter:

$$u_{tt} + 2V_0u_{xt} + (V_0^2 - c^2)u_{xx} = g(x,t), \quad 0 < x < l, \quad t > 0, \quad (2.1)$$

where

- $u(x,t)$: the displacement of the string in the vertical direction,
- V_0 : the string speed,
- $c > 0$: the wave speed,
- x : the coordinate in horizontal direction,
- $g(x,t)$: the external force,
- t : the time, and
- l : the distance between the pulleys.

In this chapter the case $V_0 < c$ is considered. At the pulleys it is assumed that there is no displacement of the string in vertical direction. Equation (2.1) can also be found in [16], but now it is assumed that V_0 is not necessarily small. The boundary and initial conditions for $u(x,t)$ are given by

$$\begin{aligned} u(0,t) &= u(l,t) = 0, \quad t \geq 0, \\ u(x,0) &= f(x), \quad \text{and} \quad u_t(x,0) = r(x), \quad 0 < x < l, \end{aligned} \quad (2.2)$$

where $f(x)$ and $r(x)$ represent the initial displacement and the initial velocity of the string, respectively. It is assumed that the functions $f(x)$ and $r(x)$ are sufficiently smooth such that a two times continuously differentiable solution for the initial-boundary value problem (2.1)-(2.2) exists.

2.2 Application of the Laplace transform method

The initial-boundary value problem (2.1)-(2.2) for $u(x, t)$ can readily be solved by applying the Laplace transform method (with respect to time t) to (2.1)-(2.2), yielding

$$s^2 U(x, s) - su(x, 0) - u_t(x, 0) + 2V_0(sU_x(x, s) - u_x(x, 0)) + U_{xx}(x, s)(V_0^2 - c^2) = G_1(x, s), \quad (2.3)$$

$$U(0, s) = U(l, s) = 0, \quad (2.4)$$

where $U(x, s)$ and $G_1(x, s)$ are the Laplace transforms of $u(x, t)$ and $g(x, t)$ respectively.

By dividing (2.3) by $(V_0^2 - c^2)$ and by rearranging terms in (2.3) it follows that

$$U_{xx} + \frac{2V_0 s}{V_0^2 - c^2} U_x + \frac{s^2}{V_0^2 - c^2} U = G(x, s), \quad (2.5)$$

where

$$G(x, s) = \frac{G_1(x, s) + sf(x) + r(x) + 2V_0 f_x(x)}{V_0^2 - c^2}. \quad (2.6)$$

The general solution of the homogeneous equation (that is, (2.5) with $G \equiv 0$) is given by

$$U(x, s) = C_1(s) \exp\left(\frac{-sx}{V_0 + c}\right) + C_2(s) \exp\left(\frac{-sx}{V_0 - c}\right), \quad (2.7)$$

where $C_1(s)$ and $C_2(s)$ are still arbitrary functions. The method of variation of parameters can be used to find the particular solution of the non-homogeneous equation (2.5). To apply this method the solution is written in the form

$$U(x, s) = C_1(x, s) \exp\left(\frac{-sx}{V_0 + c}\right) + C_2(x, s) \exp\left(\frac{-sx}{V_0 - c}\right). \quad (2.8)$$

The functions $C_1(x, s)$ and $C_2(x, s)$ then follow from

$$\begin{cases} C_1'(x, s) \exp\left(\frac{-sx}{V_0 + c}\right) + C_2'(x, s) \exp\left(\frac{-sx}{V_0 - c}\right) = 0, \\ C_1'(x, s) \left(\frac{-s}{V_0 + c}\right) \exp\left(\frac{-sx}{V_0 + c}\right) \\ \quad + C_2'(x, s) \left(\frac{-s}{V_0 - c}\right) \exp\left(\frac{-sx}{V_0 - c}\right) = G(x, s), \end{cases} \quad (2.9)$$

or, equivalently, from

$$\begin{cases} C_1'(x, s) = G(x, s) \frac{(V_0^2 - c^2)}{2sc} \exp\left(\frac{sx}{V_0 + c}\right), \\ C_2'(x, s) = -G(x, s) \frac{(V_0^2 - c^2)}{2sc} \exp\left(\frac{sx}{V_0 - c}\right), \end{cases} \quad (2.10)$$

where the prime ' denotes differentiation with respect to x . By integrating (2.10) with respect to x it follows that

$$\begin{cases} C_1(x, s) = \frac{(V_0^2 - c^2)}{2sc} \int_0^x G(x^*, s) \exp\left(\frac{x^*s}{V_0 + c}\right) dx^* + K_1(s), \\ C_2(x, s) = -\frac{(V_0^2 - c^2)}{2sc} \int_0^x G(x^*, s) \exp\left(\frac{x^*s}{V_0 - c}\right) dx^* + K_2(s), \end{cases} \quad (2.11)$$

where $K_1(s)$ and $K_2(s)$ are still arbitrary functions. The solution of the non-homogeneous equation (2.3) or (2.5) is given by

$$\begin{aligned} U(x, s) &= K_1(s) \exp\left(\frac{-sx}{V_0 + c}\right) + K_2(s) \exp\left(\frac{-sx}{V_0 - c}\right) \\ &\quad + \frac{(V_0^2 - c^2)}{2sc} \int_0^x G(x^*, s) \left(\exp\left(\frac{-s(x - x^*)}{V_0 + c}\right) - \exp\left(\frac{-s(x - x^*)}{V_0 - c}\right) \right) dx^*, \end{aligned} \quad (2.12)$$

where $K_1(s)$ and $K_2(s)$ can be determined from the boundary conditions (2.4).

So, finally, the following expression for $U(x, s)$ is found:

$$\begin{aligned}
 & U(x, s) \\
 &= \frac{(V_0^2 - c^2)}{2sc} \left(\frac{\int_0^l G(x^*, s) \left(\exp\left(\frac{-s(l-x^*)}{V_0+c}\right) - \exp\left(\frac{-s(l-x^*)}{V_0-c}\right) \right) dx^*}{\exp\left(\frac{-sl}{V_0+c}\right) - \exp\left(\frac{-sl}{V_0-c}\right)} \right. \\
 &\quad \times \left(-\exp\left(\frac{-sx}{V_0+c}\right) + \exp\left(\frac{-sx}{V_0-c}\right) \right) \\
 &\quad \left. + \int_0^x G(x^*, s) \left(\exp\left(\frac{-s(x-x^*)}{V_0+c}\right) - \exp\left(\frac{-s(x-x^*)}{V_0-c}\right) \right) dx^* \right). \quad (2.13)
 \end{aligned}$$

The inverse Laplace transform of $U(x, s)$ is given by

$$u(x, t) = \frac{1}{2\pi i} \int_{\nu-i\infty}^{\nu+i\infty} U(x, s) e^{st} ds = \sum_n \text{Res}(s_n, x, t), \text{ for some } \nu > 0, \quad (2.14)$$

and where Res stands for the residue at $s = s_n$. To evaluate the inverse Laplace transform (2.14) the poles of $U(x, s)$ and the order of these poles have to be determined in the complex s -plane. As long as $g(x, t)$ is not specified the poles due to $G_1(x, s)$ will be unknown. For that reason three cases will be considered: the case 2.2.1: $g(x, t) = 0$, the case 2.2.2: $g(x, t) = \varphi(x) \sin(\omega t)$ with $\omega = \frac{\pi n^*}{lc}(V_0^2 - c^2)$ for some fixed $n^* \in \mathbb{Z}$, and the case 2.2.3: $g(x, t) = \varphi(x) \sin(\omega t)$ with ω is not in the neighbourhood of $\frac{\pi n}{lc}(V_0^2 - c^2)$ for all $n \in \mathbb{Z}$.

2.2.1 The case: $g(x, t) = 0$

In this case the poles of $U(x, s)$ follow from (see (2.13))

$$s \left(\exp\left(\frac{-sl}{V_0+c}\right) - \exp\left(\frac{-sl}{V_0-c}\right) \right) = 0. \quad (2.15)$$

Now it should be observed that $s = 0$ is not a pole of $U(x, s)$ since $\lim_{s \rightarrow 0} U(x, s)$ exists. All other poles of $U(x, s)$ now follow from

$$\exp\left(\frac{-sl}{V_0+c}\right) - \exp\left(\frac{-sl}{V_0-c}\right) = 0 \quad (2.16)$$

and are given by

$$s_n = \frac{\pi n}{lc} (V_0^2 - c^2) i, \quad (2.17)$$

with $n \in \mathbb{Z} \setminus \{0\}$. It should be observed that these poles are all simple. The solution of the initial-boundary value problem (2.1)-(2.2) with $g(x, t) = 0$ now easily follows from (2.14), yielding

$$\begin{aligned} u(x, t) = & \sum_{n=1}^{\infty} \left\{ a_n \left(\cos \left(\frac{\pi n (V_0^2 - c^2) t}{lc} \right) \left(\cos \left(\frac{\pi n (V_0 + c)x}{lc} \right) - \cos \left(\frac{\pi n (V_0 - c)x}{lc} \right) \right) \right. \right. \\ & + \sin \left(\frac{\pi n (V_0^2 - c^2) t}{lc} \right) \left(\sin \left(\frac{\pi n (V_0 + c)x}{lc} \right) - \sin \left(\frac{\pi n (V_0 - c)x}{lc} \right) \right) \\ & + b_n \left(\cos \left(\frac{\pi n (V_0^2 - c^2) t}{lc} \right) \left(\sin \left(\frac{\pi n (V_0 + c)x}{lc} \right) - \sin \left(\frac{\pi n (V_0 - c)x}{lc} \right) \right) \right. \\ & \left. \left. - \sin \left(\frac{\pi n (V_0^2 - c^2) t}{lc} \right) \left(\cos \left(\frac{\pi n (V_0 + c)x}{lc} \right) - \cos \left(\frac{\pi n (V_0 - c)x}{lc} \right) \right) \right) \right\}, \end{aligned} \quad (2.18)$$

where

$$\begin{aligned} a_n = & \frac{(V_0^2 - c^2)}{c} \left(\frac{1}{2lc} \int_0^l f(x^*) \left(\cos \left(\frac{(V_0 - c)\pi n x^*}{lc} \right) - \cos \left(\frac{(V_0 + c)\pi n x^*}{lc} \right) \right) dx^* \right. \\ & \left. + \frac{1}{2} \int_0^l \frac{r(x^*) + 2V_0 f'_x(x^*)}{\pi n (V_0^2 - c^2)} \left(\sin \left(\frac{(V_0 - c)\pi n x^*}{lc} \right) - \sin \left(\frac{(V_0 + c)\pi n x^*}{lc} \right) \right) dx^* \right), \end{aligned} \quad (2.19)$$

$$\begin{aligned} b_n = & \frac{(V_0^2 - c^2)}{c} \left(\frac{1}{2lc} \int_0^l f(x^*) \left(\sin \left(\frac{(V_0 - c)\pi n x^*}{lc} \right) - \sin \left(\frac{(V_0 + c)\pi n x^*}{lc} \right) \right) dx^* \right. \\ & \left. - \frac{1}{2} \int_0^l \frac{r(x^*) + 2V_0 f'_x(x^*)}{\pi n (V_0^2 - c^2)} \left(\cos \left(\frac{(V_0 - c)\pi n x^*}{lc} \right) - \cos \left(\frac{(V_0 + c)\pi n x^*}{lc} \right) \right) dx^* \right). \end{aligned} \quad (2.20)$$

2.2.2 The case: $g(x, t) = \varphi(x) \sin(\omega t)$, the resonant case

In this case it is assumed that ω is equal to a frequency of the axially moving string, that is, $\omega = \frac{\pi n^*}{lc} (V_0^2 - c^2)$ for some fixed $n^* \in \mathbb{Z}$. The Laplace transform of $g(x, t)$ in this case is

$$G_1(x, s) = \frac{\omega}{s^2 + \omega^2} \varphi(x). \quad (2.21)$$

In (2.13) $G(x, s)$ is now given by $G(x^*, s) = \frac{\omega \varphi(x^*)}{(s^2 + \omega^2)(V_0^2 - c^2)} + h(x^*, s)$, where $h(x^*, s) = \frac{s f(x^*) + r(x^*) + 2V_0 f'_{x^*}(x^*)}{V_0^2 - c^2}$. And so, $U(x, s)$ can be written as $U(x, s) = A(x, s) + B(x, s) + D(x, s) + E(x, s)$, where

$$\begin{aligned} A(x, s) = & \frac{(V_0^2 - c^2)}{2sc} \frac{\omega \int_0^l \varphi(x^*) \left(\exp \left(\frac{-s(l - x^*)}{V_0 + c} \right) - \exp \left(\frac{-s(l - x^*)}{V_0 - c} \right) \right) dx^*}{(s^2 + \omega^2)(V_0^2 - c^2) \left(\exp \left(\frac{-sl}{V_0 + c} \right) - \exp \left(\frac{-sl}{V_0 - c} \right) \right)} \\ & \times \left(-\exp \left(\frac{-sx}{V_0 + c} \right) + \exp \left(\frac{-sx}{V_0 - c} \right) \right), \end{aligned} \quad (2.22)$$

$$B(x, s) = \frac{(V_0^2 - c^2)}{2sc} \frac{\int_0^l h(x^*, s) \left(\exp\left(\frac{-s(l-x^*)}{V_0+c}\right) - \exp\left(\frac{-s(l-x^*)}{V_0-c}\right) \right) dx^*}{\left(\exp\left(\frac{-sl}{V_0+c}\right) - \exp\left(\frac{-sl}{V_0-c}\right) \right)} \quad (2.23)$$

$$\times \left(-\exp\left(\frac{-sx}{V_0+c}\right) + \exp\left(\frac{-sx}{V_0-c}\right) \right),$$

$$D(x, s) = \frac{\omega}{2sc(s^2 + \omega^2)} \int_0^x \varphi(x^*) \left(\exp\left(\frac{-s(x-x^*)}{V_0+c}\right) - \exp\left(\frac{-s(x-x^*)}{V_0-c}\right) \right) dx^*, \quad (2.24)$$

$$E(x, s) = \frac{(V_0^2 - c^2)}{2sc} \int_0^x h(x^*, s) \left(\exp\left(\frac{-s(x-x^*)}{V_0+c}\right) - \exp\left(\frac{-s(x-x^*)}{V_0-c}\right) \right) dx^*. \quad (2.25)$$

The inverse Laplace transform of $U(x, s)$ is given by

$$L^{inv}(U(x, s)) = L^{inv}(A(x, s)) + L^{inv}(B(x, s)) + L^{inv}(D(x, s)) + L^{inv}(E(x, s)). \quad (2.26)$$

In (2.26) $L^{inv}(B(x, s))$ and $L^{inv}(E(x, s))$ only depend on the initial values $f(x)$ and $r(x)$ and already have been determined in the case 2.2.1 with $g(x, t) = 0$. So, only $L^{inv}(A(x, s))$ and $L^{inv}(D(x, s))$ have to be calculated. It should be observed that these inverse Laplace transforms only depend on $g(x, t)$. Furthermore, it should be observed that $A(x, s)$ is the product of $\frac{\omega}{s^2 + \omega^2}$ and another term (following from (2.22)), and so the inverse Laplace transform of $A(x, s)$ can be determined by using the convolution integral, that is

$$L^{inv}(A(x, s)) = \int_0^t \sin(\omega(t - \tau)) \sum_n Res(s_n, x, \tau) d\tau, \quad (2.27)$$

where s_n is given by (2.17). Finally, if $\omega = \frac{\pi n^*}{lc} (V_0^2 - c^2)$ for a fixed $n^* \in \mathbb{Z}$ it follows that $L^{inv}(A(x, s))$ is

$$\begin{aligned} L^{inv}(A(x, s)) &= \frac{F_{1n^*}(x)}{2} t \sin(\omega_{n^*} t) + \frac{F_{2n^*}(x)}{2} \left(\frac{\sin(\omega_{n^*} t)}{\omega_{n^*}} - t \cos(\omega_{n^*} t) \right) \\ &+ \sum_{n=1, n \neq |n^*|}^{\infty} \left\{ \frac{F_{1n}(x)}{2} \left(\frac{1}{\omega_{n^*} - \omega_n} (\cos(\omega_n t) - \cos(\omega_{n^*} t)) \right. \right. \\ &+ \left. \frac{1}{\omega_{n^*} + \omega_n} (\cos(\omega_n t) - \cos(\omega_{n^*} t)) \right) \\ &+ \left. \frac{F_{2n}(x)}{2} \left(\frac{1}{\omega_{n^*} + \omega_n} (\sin(\omega_n t) + \sin(\omega_{n^*} t)) + \frac{1}{\omega_{n^*} - \omega_n} (\sin(\omega_n t) - \sin(\omega_{n^*} t)) \right) \right\}, \end{aligned} \quad (2.28)$$

where

$$\begin{aligned} F_{1n}(x) &= w_n \left(\cos\left(\frac{\pi n(V_0 + c)x}{lc}\right) - \cos\left(\frac{\pi n(V_0 - c)x}{lc}\right) \right) \\ &+ p_n \left(\sin\left(\frac{\pi n(V_0 + c)x}{lc}\right) - \sin\left(\frac{\pi n(V_0 - c)x}{lc}\right) \right), \end{aligned} \quad (2.29)$$

$$F_{2n}(x) = w_n \left(\sin \left(\frac{\pi n(V_0 + c)x}{lc} \right) - \sin \left(\frac{\pi n(V_0 - c)x}{lc} \right) \right) - p_n \left(\cos \left(\frac{\pi n(V_0 + c)x}{lc} \right) - \cos \left(\frac{\pi n(V_0 - c)x}{lc} \right) \right), \quad (2.30)$$

$$w_n = \frac{1}{2c} \int_0^l \frac{\varphi(x^*)}{\pi n} \left(\sin \left(\frac{(V_0 - c)\pi n x^*}{lc} \right) - \sin \left(\frac{(V_0 + c)\pi n x^*}{lc} \right) \right) dx^*, \quad (2.31)$$

$$p_n = -\frac{1}{2c} \int_0^l \frac{\varphi(x^*)}{\pi n} \left(\cos \left(\frac{(V_0 - c)\pi n x^*}{lc} \right) - \cos \left(\frac{(V_0 + c)\pi n x^*}{lc} \right) \right) dx^*. \quad (2.32)$$

It can be seen from (2.22) that if $n = n^*$ then there are two poles of order two (one in $s = i\omega_{n^*}$ and one in $s = -i\omega_{n^*}$). To calculate $L^{inv}(D(x, s))$ it should be observed that $s = 0$ is not a pole as $\lim_{s \rightarrow 0} D(x, s)$ exists. So the inverse Laplace transform for $D(x, s)$ is

$$\begin{aligned} L^{inv}(D(x, s)) &= \frac{(V_0^2 - c^2)}{c\omega} \left(\frac{1}{2} \cos(\omega t) \int_0^x \varphi(x^*) \left(\cos \left(\frac{\omega(x - x^*)}{V_0 + c} \right) - \cos \left(\frac{\omega(x - x^*)}{V_0 - c} \right) \right) dx^* \right. \\ &\quad \left. + \frac{1}{2} \sin(\omega t) \int_0^x \varphi(x^*) \left(\sin \left(\frac{\omega(x - x^*)}{V_0 + c} \right) - \sin \left(\frac{\omega(x - x^*)}{V_0 - c} \right) \right) dx^* \right). \end{aligned} \quad (2.33)$$

It can be seen that only $L^{inv}(A(x, s))$ contributes to unbounded terms in the solution.

2.2.3 The case: $g(x, t) = \varphi(x) \sin(\omega t)$, the non-resonant case

Let $\omega_n = \frac{\pi n}{lc}(V_0^2 - c^2)$, $n \in \mathbb{Z}$ be the natural frequencies of an axially moving string and let ω be not in a neighbourhood of any of these frequencies ω_n .

For $L^{inv}(A(x, s))$ it then follows that

$$\begin{aligned} L^{inv}(A(x, s)) &= \sum_{n=1}^{\infty} \left(\frac{F_{1n}(x)}{2} \left(\frac{1}{\omega - \omega_n} (\cos(\omega_n t) - \cos(\omega t)) + \frac{1}{\omega + \omega_n} (\cos(\omega_n t) - \cos(\omega t)) \right) \right. \\ &\quad \left. + \frac{F_{2n}(x)}{2} \left(\frac{1}{\omega + \omega_n} (\sin(\omega_n t) + \sin(\omega t)) + \frac{1}{\omega - \omega_n} (\sin(\omega_n t) - \sin(\omega t)) \right) \right), \end{aligned} \quad (2.34)$$

where $F_{1n}(x)$ and $F_{2n}(x)$ are given by (2.29) and (2.30), respectively. $L^{inv}(D(x, s))$ is again given by (2.33). In this case there are no unbounded terms in the solution $u(x, t)$. Obviously, unbounded solutions will occur when $g(x, t)$ contains terms $\varphi_1(x) \sin(\omega t)$ and (or) terms $\varphi_2(x) \cos(\omega t)$ for which ω is equal to an eigenfrequency ω_n .

2.3 An eigenfunction approach

Following Wickert and Mote [31] equation (2.1) can be rewritten in the following dimensionless form:

$$u_{tt} + 2\nu u_{xt} - (1 - \nu^2)u_{xx} = g(x, t), \quad 0 < x < 1, \quad t > 0, \quad (2.35)$$

where $\nu = \frac{V_0}{c}$, other parameters can be found in [31]. Introducing the operators $M = 1$, $G = 2\nu \frac{\partial}{\partial x}$ and $K = -(1 - \nu^2) \frac{\partial^2}{\partial x^2}$ in (2.35) it follows that

$$Mu_{tt} + Gu_t + Ku = g(x, t). \quad (2.36)$$

Now the following vectors and matrices are defined:

$$\mathbf{w}(x, t) = \begin{pmatrix} u_t(x, t) \\ u(x, t) \end{pmatrix}, \quad \mathbf{q}(x, t) = \begin{pmatrix} g(x, t) \\ 0 \end{pmatrix}, \quad (2.37)$$

$$\mathbf{A} = \begin{pmatrix} M & 0 \\ 0 & K \end{pmatrix}, \quad \text{and} \quad \mathbf{B} = \begin{pmatrix} G & K \\ -K & 0 \end{pmatrix}. \quad (2.38)$$

Equation (2.36) then becomes

$$\mathbf{A}\mathbf{w}_t + \mathbf{B}\mathbf{w} = \mathbf{q}. \quad (2.39)$$

The solution of equation (2.39) has to satisfy the initial and the boundary conditions

$$\mathbf{w}(x, 0) = \mathbf{w}_0(x) = \begin{pmatrix} r(x) \\ f(x) \end{pmatrix}, \quad \text{and} \quad \mathbf{w}(x, t) = \mathbf{0} \quad \text{at} \quad x = 0 \quad \text{and} \quad x = 1. \quad (2.40)$$

The inner product of two vectors $\mathbf{w}_1$ and $\mathbf{w}_2$ is defined to be

$$\langle \mathbf{w}_1, \mathbf{w}_2 \rangle = \int_0^1 \mathbf{w}_1^T \mathbf{w}_2^* dx, \quad (2.41)$$

where the $*$ means the complex conjugate, and where the superscript T means the transpose. With respect to this inner product the operators $\mathbf{A}$ and $\mathbf{B}$ have the following properties:

$$\langle \mathbf{A}\mathbf{w}_1, \mathbf{w}_2 \rangle = \langle \mathbf{w}_1, \mathbf{A}\mathbf{w}_2 \rangle, \quad \langle \mathbf{B}\mathbf{w}_1, \mathbf{w}_2 \rangle = -\langle \mathbf{w}_1, \mathbf{B}\mathbf{w}_2 \rangle, \quad (2.42)$$

where both $\mathbf{w}_1(x, t)$ and $\mathbf{w}_2(x, t)$ satisfy the Dirichlet boundary conditions (2.40) at $x = 0$ and $x = 1$. The general solution of the equation (2.39) will be found in an expansion of eigenfunctions. First the homogeneous equation (2.39) with $\mathbf{q} = 0$ is solved by using the method of separation of variables, that is, it is assumed that (2.39) with $\mathbf{q} = 0$ admits solution of the form $\boldsymbol{\phi}(x) \exp(\lambda t)$, where $\boldsymbol{\phi}(x)$ is a complex vector function and where λ is a complex-valued number. This leads to the eigenvalue problem

$$\lambda \mathbf{A} \boldsymbol{\phi} + \mathbf{B} \boldsymbol{\phi} = 0, \quad (2.43)$$

where the eigenvalues λ_n are equal to $i\omega_n$ as has been shown by Meirovitch in [29]. The complex-valued eigenfunctions $\boldsymbol{\phi}_n$ have the structure $\boldsymbol{\phi}_n(x) = \boldsymbol{\phi}_n^R(x) + i\boldsymbol{\phi}_n^I(x)$, where

$$\boldsymbol{\phi}_n^R(x) = \begin{Bmatrix} -\omega_n \psi_n^I(x) \\ \psi_n^R(x) \end{Bmatrix} \quad \text{and} \quad \boldsymbol{\phi}_n^I(x) = \begin{Bmatrix} \omega_n \psi_n^R(x) \\ \psi_n^I(x) \end{Bmatrix}. \quad (2.44)$$

In (2.44) $\psi_n(x)$ is the complex normalized scalar eigenfunction and $\psi_n(x) = \psi_n^R(x) + i\psi_n^I(x)$. After solving the eigenvalue problem (2.43) and after normalization the following expressions can be found:

$$\omega_n = n\pi(1 - \nu^2), \quad (2.45)$$

$$\begin{aligned} \psi_n^R(x) &= \frac{1}{n\pi} \sqrt{\frac{2}{1 - \nu^2}} \sin(n\pi x) \cos(n\pi\nu x), \\ \psi_n^I(x) &= \frac{1}{n\pi} \sqrt{\frac{2}{1 - \nu^2}} \sin(n\pi x) \sin(n\pi\nu x). \end{aligned} \quad (2.46)$$

The normalization requirements can be found in [31]. The $\boldsymbol{\phi}_n(x)$ satisfy the orthonormality relations

$$\begin{aligned} \langle \mathbf{A} \boldsymbol{\phi}_n^R, \boldsymbol{\phi}_m^R \rangle &= \delta_{nm}, \quad \langle \mathbf{A} \boldsymbol{\phi}_n^I, \boldsymbol{\phi}_m^I \rangle = \delta_{nm}, \quad \langle \mathbf{A} \boldsymbol{\phi}_n^R, \boldsymbol{\phi}_m^I \rangle = 0, \\ \langle \mathbf{B} \boldsymbol{\phi}_n^R, \boldsymbol{\phi}_m^R \rangle &= 0, \quad \langle \mathbf{B} \boldsymbol{\phi}_n^I, \boldsymbol{\phi}_m^I \rangle = 0, \quad \langle \mathbf{B} \boldsymbol{\phi}_n^R, \boldsymbol{\phi}_m^I \rangle = \omega_n \delta_{nm}, \end{aligned} \quad (2.47)$$

where δ_{nm} is the Kronecker delta symbol, that is, $\delta_{nm} = 1$ for $n = m$ and zero otherwise. It was shown by Meirovitch in [30] that the solution of the non-homogeneous equation (2.39) can be found in the following form:

$$\mathbf{w}(x, t) = \sum_{n=1}^{\infty} \xi_n^R(t) \boldsymbol{\phi}_n^R(x) + \xi_n^I(t) \boldsymbol{\phi}_n^I(x), \quad (2.48)$$

where $\xi_n^I(t)$ and $\xi_n^R(t)$ are the components of the so-called generalized coordinates. Substituting this expansion into equation (2.39) it follows that

$$\mathbf{A} \frac{\partial}{\partial t} \sum_{n=1}^{\infty} [\xi_n^R(t) \phi_n^R(x) + \xi_n^I(t) \phi_n^I(x)] + \mathbf{B} \sum_{n=1}^{\infty} [\xi_n^R(t) \phi_n^R(x) + \xi_n^I(t) \phi_n^I(x)] = \mathbf{q}. \quad (2.49)$$

By multiplying (2.49) consecutively first with ϕ_m^R and then with ϕ_m^I and by integrating the so-obtained equations with respect to x from $x = 0$ to $x = 1$ in the sense of the defined inner product (2.41) and with the help of the orthonormality relations (2.47) the following system is obtained:

$$\begin{cases} \frac{d\xi_m^R}{dt} - \omega_m \xi_m^I = -\omega_m \int_0^1 \psi_m^I g(x, t) dx, \\ \frac{d\xi_m^I}{dt} + \omega_m \xi_m^R = \omega_m \int_0^1 \psi_m^R g(x, t) dx. \end{cases} \quad (2.50)$$

Solving this system with respect to the initial values $\xi_m^R(0) = \langle \mathbf{A} \mathbf{w}_0, \phi_m^R \rangle$ and $\xi_m^I(0) = \langle \mathbf{A} \mathbf{w}_0, \phi_m^I \rangle$ it follows that

$$\begin{aligned} \xi_m^R(t) = & \int_0^t (g_m^R(s) \cos(\omega_m(t-s)) + g_m^I(s) \sin(\omega_m(t-s))) ds \\ & + \xi_m^R(0) \cos(\omega_m t) + \xi_m^I(0) \sin(\omega_m t), \end{aligned} \quad (2.51)$$

$$\begin{aligned} \xi_m^I(t) = & \int_0^t (g_m^I(s) \cos(\omega_m(t-s)) - g_m^R(s) \sin(\omega_m(t-s))) ds \\ & + \xi_m^I(0) \cos(\omega_m t) - \xi_m^R(0) \sin(\omega_m t), \end{aligned} \quad (2.52)$$

where

$$\begin{aligned} g_m^R(t) &= -\sqrt{2(1-\nu^2)} \int_0^1 g(x, t) \sin(m\pi x) \sin(m\pi\nu x) dx \text{ and} \\ g_m^I(t) &= \sqrt{2(1-\nu^2)} \int_0^1 g(x, t) \sin(m\pi x) \cos(m\pi\nu x) dx \end{aligned} \quad (2.53)$$

are defined to be the generalized forces. And $\xi_m^R(0)$, $\xi_m^I(0)$ are given by

$$\begin{aligned} \xi_m^R(0) = & m\pi \sqrt{2(1-\nu^2)} \int_0^1 f(x) ((1+\nu^2) \sin(m\pi x) \cos(m\pi\nu x) \\ & + 2\nu \cos(m\pi x) \sin(m\pi\nu x)) dx \\ & - \sqrt{2(1-\nu^2)} \int_0^1 r(x) \sin(m\pi x) \sin(m\pi\nu x) dx, \end{aligned} \quad (2.54)$$

$$\begin{aligned}\xi_m^I(0) = & m\pi\sqrt{2(1-\nu^2)}\int_0^1 f(x) ((1+\nu^2)\sin(m\pi x)\sin(m\pi\nu x) \\ & - 2\nu\cos(m\pi x)\cos(m\pi\nu x)) dx \\ & + \sqrt{2(1-\nu^2)}\int_0^1 r(x)\sin(m\pi x)\cos(m\pi\nu x)dx.\end{aligned}\quad (2.55)$$

Using the definition (2.37) the expansion for $u(x, t)$ can be written in the form

$$u(x, t) = \sum_{m=1}^{\infty} \xi_m^R(t)\psi_m^R(x) + \xi_m^I(t)\psi_m^I(x), \quad (2.56)$$

or, equivalently,

$$\begin{aligned}u(x, t) = & \sum_{m=1}^{\infty} \frac{1}{m\pi} \sqrt{\frac{2}{1-\nu^2}} (\xi_m^R(t)\sin(m\pi x)\cos(m\pi\nu x) \\ & + \xi_m^I(t)\sin(m\pi x)\sin(m\pi\nu x)).\end{aligned}\quad (2.57)$$

And, finally, the solution of the problem has the form

$$\begin{aligned}u(x, t) = & \sum_{m=1}^{\infty} \frac{1}{m\pi} \sqrt{\frac{2}{1-\nu^2}} \\ & \times \left\{ \left[\int_0^t \left\{ \left(-\sqrt{2(1-\nu^2)} \int_0^1 g(x, s)\sin(m\pi x)\sin(m\pi\nu x)dx \right) \cos(\omega_m(t-s)) \right. \right. \right. \\ & + \left. \left(\sqrt{2(1-\nu^2)} \int_0^1 g(x, s)\sin(m\pi x)\cos(m\pi\nu x)dx \right) \sin(\omega_m(t-s)) \right\} ds \\ & + \xi_m^R(0)\cos(\omega_m t) + \xi_m^I(0)\sin(\omega_m t) \Big] \sin(m\pi x)\cos(m\pi\nu x) \\ & + \left[\int_0^t \left\{ \left(-\sqrt{2(1-\nu^2)} \int_0^1 g(x, s)\sin(m\pi x)\cos(m\pi\nu x)dx \right) \cos(\omega_m(t-s)) \right. \right. \\ & + \left. \left(\sqrt{2(1-\nu^2)} \int_0^1 g(x, s)\sin(m\pi x)\sin(m\pi\nu x)dx \right) \sin(\omega_m(t-s)) \right\} ds \\ & + \left. \left. \xi_m^I(0)\cos(\omega_m t) - \xi_m^R(0)\sin(\omega_m t) \right] \sin(m\pi x)\sin(m\pi\nu x) \right\}.\end{aligned}\quad (2.58)$$

From the structure of the solution as given by (2.58) it can be seen that if $g(x, t) = \varphi(x)\sin(\omega t)$ with $\omega = \omega_{m^*}$ for a fixed $m^* \in \mathbb{Z}$ resonance will occur (that is, unbounded solutions will occur). To show this two cases will be studied: the case 2.3.1 $\omega = m^*\pi(1-\nu^2)$ and the case 2.3.2 ω is not in a neighbourhood of $\omega = m\pi(1-\nu^2)$ for all $m \in \mathbb{Z}$.

2.3.1 The case: $\omega = m^* \pi(1 - \nu^2)$ for a fixed $m^* \in \mathbb{Z}$, the resonant case

The solution of the problem in this case is given by

$$\begin{aligned}
u(x, t) = & \frac{1}{m^* \pi} \left\{ t \sin(\omega_{m^*} t) \left(- \int_0^1 H_{1m^*}(x) dx \sin(m^* \pi x) \cos(m^* \pi \nu x) \right. \right. \\
& + \int_0^1 H_{2m^*}(x) dx \sin(m^* \pi x) \sin(m^* \pi \nu x) \Big) + \left(\frac{\sin(\omega_{m^*} t)}{\omega_{m^*}} - t \cos(\omega_{m^*} t) \right) \\
& \times \left(\int_0^1 H_{2m^*}(x) dx \sin(m^* \pi x) \cos(m^* \pi \nu x) + \int_0^1 H_{1m^*}(x) dx \sin(m^* \pi x) \sin(m^* \pi \nu x) \right) \Big\} \\
& + \sum_{m=1, m \neq |m^*|}^{\infty} \left\{ \frac{2}{m \pi} \left[\int_0^1 H_{1m^*}(x) dx \left(\frac{\omega(\cos(\omega t) - \cos(\omega_m t))}{\omega^2 - \omega_m^2} \right) \right. \right. \\
& + \int_0^1 H_{2m^*}(x) dx \left(\frac{\omega \sin(\omega_m t) - \omega_m \sin(\omega t)}{\omega^2 - \omega_m^2} \right) + \xi_m^R(0) \cos(\omega_m t) + \xi_m^I(0) \sin(\omega_m t) \Big] \\
& \times \sin(m \pi x) \sin(m \pi \nu x) + \left[\int_0^1 H_{2m^*}(x) dx \left(\frac{\omega(\cos(\omega_m t) - \cos(\omega t))}{\omega^2 - \omega_m^2} \right) \right. \\
& + \int_0^1 H_{1m^*}(x) dx \left(\frac{\omega \sin(\omega_m t) - \omega_m \sin(\omega t)}{\omega^2 - \omega_m^2} \right) + \xi_m^I(0) \cos(\omega_m t) - \xi_m^R(0) \sin(\omega_m t) \Big] \\
& \times \sin(m \pi x) \sin(m \pi \nu x) \Big\}, \tag{2.59}
\end{aligned}$$

where

$$\begin{aligned}
H_{1m}(x) &= \varphi(x) \sin(m \pi x) \sin(m \pi \nu x), \\
H_{2m}(x) &= \varphi(x) \sin(m \pi x) \cos(m \pi \nu x), \tag{2.60}
\end{aligned}$$

and $\xi_m^R(0)$, $\xi_m^I(0)$ are given by (2.54) and (2.55).

2.3.2 The case: $g(x, t) = \varphi(x) \sin(\omega t)$, the non-resonant case

It can easily be seen from (2.58) that if ω is not in a neighbourhood of a natural frequency ω_m for all $m \in \mathbb{Z}$ then there are no unbounded in time terms in the solution. Finally it should be observed (by using trigonometric relations) that the solution as obtained in Section 2.2 of this chapter and in Section 2.3 are identical.

2.4 Conclusions and remarks

In this chapter an initial-boundary value problem for a linear, non-homogeneous axially moving string equation has been studied. The velocity of the string is assumed to be constant, and the non-homogeneous terms in

the string equation are due to external forces acting on the string. To solve the initial-boundary value problem two methods are presented. One of these methods is the method of modal analysis based on eigenfunction expansions. This method has been introduced in [29], [30] and in [31], and is used nowadays frequently for these types of problems (see for instance [12], [15]). To apply this method an operator notation has to be introduced, an inner product has to be defined, an eigenvalue problem has to be solved, and orthonormality relations have to be determined. Altogether this method is rather complicated to apply to these types of problems. For that reason in this chapter it is proposed to apply the Laplace transform method to these types of problems. When this method is applied poles (and the order of the poles) have to be determined, residues have to be calculated, and Cauchy's theorem has to be used (that is, integrals have to be evaluated by using the theory of complex variables). To construct the solution by using the Laplace transform method is rather straightforward and more easy than the use of the method of modal analysis based on eigenfunction expansions. Moreover, the Laplace transform method is nowadays well-described in elementary textbooks on partial differential equations (see for instance [32]). In the following chapters it will be shown that the Laplace transform method can efficiently and easily be applied to weakly perturbed or weakly nonlinear (axially moving) string equations.

CHAPTER 3

On the transversal vibrations of a conveyor belt with time-varying velocity. The string-like case

Abstract: In this chapter an initial-boundary value problem for a linear equation describing an axially moving string will be considered for which the bending stiffness will be neglected. The velocity of the string is assumed to be time-varying and to be of the same order of magnitude as the wave speed. A two time-scales perturbation method and the Laplace transform method will be used to construct formal asymptotic approximations of the solutions. It will be shown that the linear axially moving string model already has complicated dynamical behaviour and that the truncation method can not be applied to this problem in order to obtain approximations which are valid on long time-scales.

3.1 Formulation of the problem

In this section the dynamic behaviour of a conveyor belt which is modelled by a moving string will be studied (see Figure 3.1).

The following linear equation of motion for a string will be considered in this chapter:

This chapter is a slightly revised version of [33]: S.V. Ponomareva and W.T. van Horssen: On the transversal vibrations of an axially moving string with a time-varying velocity, *Non-linear Dynamics*, 50, (2007), 315-323.

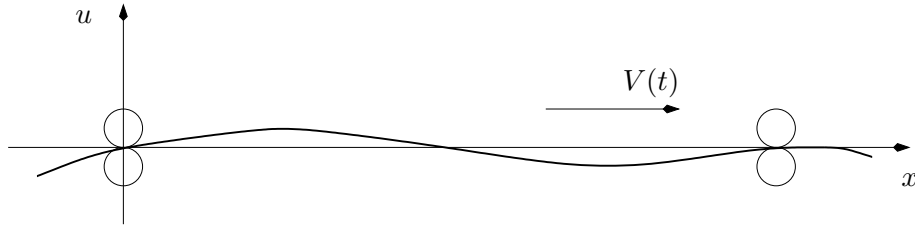


Figure 3.1: Conveyor belt system.

$$c^2 u_{xx} = u_{tt} + 2V u_{xt} + V^2 u_{xx} + V_t u_x, \quad 0 < x < l, \quad t > 0, \quad (3.1)$$

where

- $u(x, t)$: the displacement of the string in the vertical direction,
- $V(t)$: the string speed,
- c : the wave speed,
- x : the coordinate in horizontal direction,
- t : the time, and
- l : the distance between the pulleys,

where $c = \sqrt{\frac{T_0}{\rho}}$, in which T_0 and ρ are assumed to be the constant tension and the constant mass density of the string, respectively.

In this chapter the case $V_0 < c$ is considered and it is assumed that $V(t) = V_0 + \varepsilon \alpha \sin(\omega t)$, where V_0 , ω , and α are some positive constants, and where ε is a small parameter with $0 < \varepsilon \ll 1$. The term $\varepsilon \alpha \sin(\omega t)$ can be seen as a small perturbation of the main belt speed V_0 due to different kinds of imperfections of the belt system. At the pulleys it is assumed that there is no displacement of the string in vertical direction. Equation (3.1) can also be found in [16], but now it is assumed that V_0 is not necessarily small. This belt velocity should be interpreted as the value of the same order as the wave speed c . Consequently (3.1) becomes

$$\begin{aligned} u_{tt} + 2V_0 u_{xt} + (V_0^2 - c^2) u_{xx} \\ = \varepsilon(-2\alpha \sin(\omega t) u_{xt} - 2V_0 \alpha \sin(\omega t) u_{xx} - \alpha \omega \cos(\omega t) u_x) \\ - \varepsilon^2(\alpha^2 \sin^2(\omega t) u_{xx}), \end{aligned} \quad (3.2)$$

where the boundary and initial conditions are giving by

$$\begin{aligned} u(0, t; \varepsilon) = u(l, t; \varepsilon) = 0, \quad t \geq 0, \\ u(x, 0; \varepsilon) = f(x), \quad \text{and} \quad u_t(x, 0; \varepsilon) = r(x), \quad 0 < x < l, \end{aligned} \quad (3.3)$$

where $f(x)$ and $r(x)$ represent the initial displacement and the initial velocity of the string, respectively. It is assumed that the functions $f(x)$ and $r(x)$ are sufficiently smooth such that a two times continuously differentiable solution for the initial-boundary value problem (3.2)-(3.3) exists. Moreover, it is assumed that all series representations for the solution u (and its derivatives), and for the functions f and r are convergent. In the following section the asymptotic approximations of the solution of the initial-boundary value problem (3.2)-(3.3) will be constructed using two time-scales perturbation method. To study nonlinear, transversal vibrations of conveyor belt problems the solution of related linear problems always play an important role. In this chapter not only approximations of these linear problems will be constructed, but also the (non-) applicability of the truncation method will be discussed. For a recent overview of the literature on axially moving linear and nonlinear strings the reader is referred to [34], [35], [36], and [31].

3.2 Application of the two time-scales perturbation method

The approximations of the solution of the initial-boundary value problem (3.2)-(3.3) which are constructed by means of a straight-forward expansion method become unbounded on long time-scales due to the occurrence of so-called secular terms. To avoid these terms two time-scales are introduced $t_0 = t$ and $t_1 = \varepsilon t$, so that $u(x, t; \varepsilon) = v(x, t_0, t_1; \varepsilon)$. The introduction of these two time-scales defines the following transformations for the time derivatives:

$$\begin{aligned}\frac{\partial u(x, t; \varepsilon)}{\partial t} &= \frac{\partial v}{\partial t_0} + \varepsilon \frac{\partial v}{\partial t_1}, \\ \frac{\partial^2 u(x, t; \varepsilon)}{\partial t^2} &= \frac{\partial^2 v}{\partial t_0^2} + 2\varepsilon \frac{\partial^2 v}{\partial t_0 \partial t_1} + \varepsilon^2 \frac{\partial^2 v}{\partial t_1^2}.\end{aligned}\tag{3.4}$$

Considering transformations (3.4), equation (3.2) becomes

$$\begin{aligned}\frac{\partial^2 v}{\partial t_0^2} + 2V_0 \frac{\partial^2 v}{\partial t_0 \partial x} + (V_0^2 - c^2) \frac{\partial^2 v}{\partial x^2} &= \varepsilon \left(-2 \frac{\partial^2 v}{\partial t_0 \partial t_1} - 2V_0 \frac{\partial^2 v}{\partial t_1 \partial x} \right. \\ &\quad \left. - 2\alpha \sin(\omega t) \frac{\partial^2 v}{\partial t_0 \partial x} - 2V_0 \alpha \sin(\omega t) \frac{\partial^2 v}{\partial x^2} - \alpha \omega \cos(\omega t) \frac{\partial v}{\partial x} \right) + \mathcal{O}(\varepsilon^2)\end{aligned}\tag{3.5}$$

Assuming that $v(x, t_0, t_1; \varepsilon) = v_0(x, t_0, t_1) + \varepsilon v_1(x, t_0, t_1) + \dots$, the following problems have to be solved in order to remove secular terms up to $\mathcal{O}(\varepsilon)$:

$$\begin{aligned} \mathcal{O}(1) : \quad & \frac{\partial^2 v_0}{\partial t_0^2} + 2V_0 \frac{\partial^2 v_0}{\partial t_0 \partial x} + (V_0^2 - c^2) \frac{\partial^2 v_0}{\partial x^2} = 0 \\ \mathcal{O}(\varepsilon) : \quad & \frac{\partial^2 v_1}{\partial t_0^2} + 2V_0 \frac{\partial^2 v_1}{\partial t_0 \partial x} + (V_0^2 - c^2) \frac{\partial^2 v_1}{\partial x^2} = -2 \frac{\partial^2 v_0}{\partial t_0 \partial t_1} - 2V_0 \frac{\partial^2 v_0}{\partial t_1 \partial x} \\ & - 2\alpha \sin(\omega t) \frac{\partial^2 v_0}{\partial t_0 \partial x} - 2V_0 \alpha \sin(\omega t) \frac{\partial^2 v_0}{\partial x^2} - \alpha \omega \cos(\omega t) \frac{\partial v_0}{\partial x} \end{aligned} \quad (3.6)$$

The solution of the $\mathcal{O}(1)$ -problem can be found by means of the Laplace transform method. For more detailed analyses of that equation the readers are referred to [27] or [37]. The solution is given by

$$\begin{aligned} v_0(x, t_0, t_1) = \sum_{n=1}^{\infty} \{ & F_{[1]n}(x) (A_{n0}(t_1) \cos(\Omega_n t_0) - B_{n0}(t_1) \sin(\Omega_n t_0)) \\ & + F_{[2]n}(x) (A_{n0}(t_1) \sin(\Omega_n t_0) + B_{n0}(t_1) \cos(\Omega_n t_0)) \}, \end{aligned} \quad (3.7)$$

where

$$\begin{aligned} F_{[1]n}(x) &= \cos\left(\frac{\pi n(V_0 + c)x}{lc}\right) - \cos\left(\frac{\pi n(V_0 - c)x}{lc}\right), \\ F_{[2]n}(x) &= \sin\left(\frac{\pi n(V_0 + c)x}{lc}\right) - \sin\left(\frac{\pi n(V_0 - c)x}{lc}\right), \end{aligned} \quad (3.8)$$

and $\Omega_n = \frac{\pi n(V_0^2 - c^2)}{lc}$, $n \in \mathbb{Z}^+$ are the natural frequencies of the conveyor belt system. In equation (3.7) $A_{n0}(t_1)$ and $B_{n0}(t_1)$ are still arbitrary functions that can be used in order to avoid secular terms in the solution of the $\mathcal{O}(\varepsilon)$ -problem.

By substituting (3.7) into $\mathcal{O}(\varepsilon)$ -problem it follows that

$$\begin{aligned} \mathcal{O}(\varepsilon) : \quad & \frac{\partial^2 v_1}{\partial t_0^2} + 2V_0 \frac{\partial^2 v_1}{\partial t_0 \partial x} + (V_0^2 - c^2) \frac{\partial^2 v_1}{\partial x^2} \\ & = \sum_{n=1}^{\infty} \{ \sin(\Omega_n t_0) \varphi_n(x, t_1) + \cos(\Omega_n t_0) \tilde{\varphi}_n(x, t_1) \} \\ & + \sum_{n=1}^{\infty} \left\{ \sin(\Omega_n t_0) \sin(\omega_n t_0) \psi_n(x, t_1) + \sin(\Omega_n t_0) \cos(\omega t) \tilde{\psi}_n(x, t_1) \right. \\ & \left. + \cos(\Omega_n t_0) \sin(\omega_n t_0) \theta_n(x, t_1) + \cos(\Omega_n t_0) \cos(\omega t) \tilde{\theta}_n(x, t_1) \right\}, \end{aligned} \quad (3.9)$$

where

$$\begin{aligned}
 \varphi_n(x, t_1) &= 2 \left(\frac{\partial A_{n0}}{\partial t_1} \left(F_{[1]n}(x) \Omega_n - V_0 \frac{\partial F_{[2]n}(x)}{\partial x} \right) \right. \\
 &\quad \left. + \frac{\partial B_{n0}}{\partial t_1} \left(F_{[2]n}(x) \Omega_n + V_0 \frac{\partial F_{[1]n}(x)}{\partial x} \right) \right), \\
 \tilde{\varphi}_n(x, t_1) &= 2 \left(\frac{\partial A_{n0}}{\partial t_1} \left(-F_{[2]n}(x) \Omega_n - V_0 \frac{\partial F_{[1]n}(x)}{\partial x} \right) \right. \\
 &\quad \left. + \frac{\partial B_{n0}}{\partial t_1} \left(F_{[1]n}(x) \Omega_n - V_0 \frac{\partial F_{[2]n}(x)}{\partial x} \right) \right), \\
 \psi_n(x, t_1) &= 2\alpha \left(A_{n0} \left(\frac{\partial F_{[1]n}(x)}{\partial x} \Omega_n - V_0 \frac{\partial^2 F_{[2]n}(x)}{\partial x^2} \right) \right. \\
 &\quad \left. + B_{n0} \left(\frac{\partial F_{[2]n}(x)}{\partial x} \Omega_n + V_0 \frac{\partial^2 F_{[1]n}(x)}{\partial x^2} \right) \right), \\
 \tilde{\psi}_n(x, t_1) &= \alpha\omega \left(-A_{n0} \frac{\partial F_{[2]n}(x)}{\partial x} + B_{n0} \frac{\partial F_{[1]n}(x)}{\partial x} \right), \\
 \theta_n(x, t_1) &= 2\alpha \left(A_{n0} \left(-\frac{\partial F_{[2]n}(x)}{\partial x} \Omega_n - V_0 \frac{\partial^2 F_{[1]n}(x)}{\partial x^2} \right) \right. \\
 &\quad \left. + B_{n0} \left(\frac{\partial F_{[1]n}(x)}{\partial x} \Omega_n - V_0 \frac{\partial^2 F_{[2]n}(x)}{\partial x^2} \right) \right), \\
 \tilde{\theta}_n(x, t_1) &= \alpha\omega \left(-A_{n0} \frac{\partial F_{[1]n}(x)}{\partial x} - B_{n0} \frac{\partial F_{[2]n}(x)}{\partial x} \right).
 \end{aligned} \tag{3.10}$$

Following the method which has been presented in the previous chapter for the equations of the type (3.9), it follows that four cases have to be considered to eliminate the secular terms in the solution of the equation (3.9) : the case 3.2.1 when ω is not in a neighbourhood of any Ω_n (the non-resonant case), the case 3.2.2 for $\omega = \Omega_1$, the case 3.2.3 for $\omega = \Omega_2$, and the case 3.2.4 $\omega = \Omega_{m^*}$, where $m^* \in \mathbb{Z}^+$ and fixed.

3.2.1 The case: ω is not in a neighbourhood of any Ω_n , the non-resonant case

In this case it is assumed that the frequency of the velocity-fluctuations of the axially moving string is not equal to any of its natural frequencies. In that case only terms under first sum in the right part of the (3.9) will lead to unbounded behaviour in the solution of the $\mathcal{O}(\varepsilon)$ -problem. After applying the Laplace transform method to (3.9), calculating the poles, and then the residues, and then after applying the convolution integral theorem to find the

inverse Laplace transform, one obtains

$$\begin{aligned} v_1(x, t_0, t_1) = & \frac{1}{2} \sum_{n=1}^{\infty} \left\{ t \sin(\Omega_n t) (f_{[1]n}(x, t_1) + \tilde{f}_{[2]n}(x, t_1)) \right. \\ & + t \cos(\Omega_n t) (-f_{[2]n}(x, t_1) + \tilde{f}_{[1]n}(x, t_1)) \Big\} \\ & + \text{"terms with non-secular behaviour",} \end{aligned} \quad (3.11)$$

where

$$\begin{aligned} f_{[1]n}(x, t_1) &= w_n(x, t_1) F_{[1]n}(x) + p_n(x, t_1) F_{[2]n}(x), \\ f_{[2]n}(x, t_1) &= w_n(x, t_1) F_{[2]n}(x) - p_n(x, t_1) F_{[1]n}(x), \\ \tilde{f}_{[1]n}(x, t_1) &= \tilde{w}_n(x, t_1) F_{[1]n}(x) + \tilde{p}_n(x, t_1) F_{[2]n}(x), \\ \tilde{f}_{[2]n}(x, t_1) &= \tilde{w}_n(x, t_1) F_{[2]n}(x) - \tilde{p}_n(x, t_1) F_{[1]n}(x), \end{aligned} \quad (3.12)$$

and

$$\begin{aligned} w_n(x, t_1) &= \frac{1}{2c} \int_0^l \frac{\varphi_n(x, t_1)}{\pi n} (-F_{[2]n}(x)) dx, \\ p_n(x, t_1) &= -\frac{1}{2c} \int_0^l \frac{\varphi_n(x, t_1)}{\pi n} (-F_{[1]n}(x)) dx, \\ \tilde{w}_n(x, t_1) &= \frac{1}{2c} \int_0^l \frac{\tilde{\varphi}_n(x, t_1)}{\pi n} (-F_{[2]n}(x)) dx, \\ \tilde{p}_n(x, t_1) &= -\frac{1}{2c} \int_0^l \frac{\tilde{\varphi}_n(x, t_1)}{\pi n} (-F_{[1]n}(x)) dx. \end{aligned} \quad (3.13)$$

In (3.12) and (3.13) $F_{[1]n}(x)$ and $F_{[2]n}(x)$ are given by (3.8). To eliminate the secular terms in the solution (3.11) it is necessary to put $(f_{[1]n}(x, t_1) + \tilde{f}_{[2]n}(x, t_1))$ and $(-f_{[2]n}(x, t_1) + \tilde{f}_{[1]n}(x, t_1))$ equal to zero, or, equivalently,

$$\begin{cases} -\frac{dA_{n0}}{dt_1} \cos\left(\frac{\pi n x V_0}{lc}\right) \sin\left(\frac{\pi n x}{l}\right) - \frac{dB_{n0}}{dt_1} \sin\left(\frac{\pi n x V_0}{lc}\right) \sin\left(\frac{\pi n x}{l}\right) = 0, \\ \frac{dA_{n0}}{dt_1} \sin\left(\frac{\pi n x V_0}{lc}\right) \sin\left(\frac{\pi n x}{l}\right) - \frac{dB_{n0}}{dt_1} \cos\left(\frac{\pi n x V_0}{lc}\right) \sin\left(\frac{\pi n x}{l}\right) = 0. \end{cases} \quad (3.14)$$

System (3.14) can be seen as a system for the two unknowns $\frac{dA_{n0}}{dt_1}$ and $\frac{dB_{n0}}{dt_1}$. The determinant of this system is non-zero for all $x \in (0, l)$. So the only solution is the trivial one that is, $\frac{dA_{n0}}{dt_1} = 0$ and $\frac{dB_{n0}}{dt_1} = 0$. It then follows that $A_{n0}(t_1)$ and $B_{n0}(t_1)$ are constants. A similar result has been obtained in [15] for a non-resonant case.

3.2.2 The case: $\omega = \frac{\pi(V_0^2 - c^2)}{lc}$, the first resonance case

In this section it is assumed that ω is equal to the first natural frequency of the traveling string, that is, $\omega = \frac{\pi(V_0^2 - c^2)}{lc}$. In this case terms in both sums in the right hand side of (3.9) will lead to unbounded behaviour in the solution of the $\mathcal{O}(\varepsilon)$ -problem. After introducing $\omega = \frac{\pi(V_0^2 - c^2)}{lc}$ into (3.9), equation (3.9) becomes

$$\begin{aligned} \mathcal{O}(\varepsilon) : & \frac{\partial^2 v_1}{\partial t_0^2} + 2V_0 \frac{\partial^2 v_1}{\partial t_0 \partial x} + (V_0^2 - c^2) \frac{\partial^2 v_1}{\partial x^2} \\ &= \sum_{n=1}^{\infty} \left\{ \sin(\Omega_n t_0) \varphi_n(x, t_1) + \cos(\Omega_n t_0) \tilde{\varphi}_n(x, t_1) \right\} \\ &+ \frac{1}{2} \sum_{n=1}^{\infty} \left\{ \cos(\Omega_{n-1} t_0) (\psi_n(x, t_1) + \tilde{\theta}_n(x, t_1)) \right. \\ &+ \cos(\Omega_{n+1} t_0) (-\psi_n(x, t_1) + \tilde{\theta}_n(x, t_1)) \\ &+ \sin(\Omega_{n-1} t_0) (\tilde{\psi}_n(x, t_1) - \theta_n(x, t_1)) + \sin(\Omega_{n+1} t_0) (\tilde{\psi}_n(x, t_1) + \theta_n(x, t_1)) \left. \right\}, \end{aligned} \quad (3.15)$$

where functions $\varphi_n(x, t_1), \tilde{\varphi}_n(x, t_1), \psi_n(x, t_1), \tilde{\psi}_n(x, t_1), \theta_n(x, t_1), \tilde{\theta}_n(x, t_1)$ are given by (3.10). Following the same procedure as in the non-resonant case (3.2.1) one finally obtains for v_1 :

$$\begin{aligned} v_1(x, t_0, t_1) &= \left\{ \frac{1}{2} \left(f_{[1]1}(x, t_1) + \tilde{f}_{[2]1}(x, t_1) \right) + \frac{1}{4} \left(f_{[2]1}^{[1]}(x, t_1) + f_{[1]1}^{[3]}(x, t_1) \right) \right\} t_0 \sin(\Omega_1 t_0) \\ &+ \left\{ \frac{1}{2} \left(\tilde{f}_{[1]1}(x, t_1) - f_{[2]1}(x, t_1) \right) + \frac{1}{4} \left(f_{[1]1}^{[1]}(x, t_1) - f_{[2]1}^{[3]}(x, t_1) \right) \right\} t_0 \cos(\Omega_1 t_0) \\ &+ \sum_{n=2}^{\infty} \left[\left\{ \frac{1}{2} \left(f_{[1]n}(x, t_1) + \tilde{f}_{[2]n}(x, t_1) \right) \right. \right. \\ &+ \frac{1}{4} \left(f_{[2]n}^{[1]}(x, t_1) + f_{[1]n}^{[3]}(x, t_1) + f_{[2]n}^{[2]}(x, t_1) + f_{[1]n}^{[4]}(x, t_1) \right) \left. \right\} t_0 \sin(\Omega_n t_0) \\ &+ \left\{ \frac{1}{2} \left(\tilde{f}_{[1]n}(x, t_1) - f_{[2]n}(x, t_1) \right) \right. \\ &+ \frac{1}{4} \left(f_{[1]n}^{[1]}(x, t_1) - f_{[2]n}^{[3]}(x, t_1) + f_{[1]n}^{[2]}(x, t_1) - f_{[2]n}^{[4]}(x, t_1) \right) \left. \right\} t_0 \cos(\Omega_n t_0) \left. \right] \\ &+ \text{"terms with non-secular behaviour"}, \end{aligned} \quad (3.16)$$

where the functions $f_{[1]n}(x, t_1)$, $f_{[2]n}(x, t_1)$, $\tilde{f}_{[1]n}(x, t_1)$ and $\tilde{f}_{[2]n}(x, t_1)$ are given by (3.12), and the functions $f_{[1]n}^{[k]}(x, t_1)$ and $f_{[2]n}^{[k]}(x, t_1)$ are given by the following formulas:

$$\begin{aligned} f_{[1]n}^k(x, t_1) &= w_n^k(x, t_1)F_{[1]n}(x) + p_n^k(x, t_1)F_{[2]n}(x), \\ f_{[2]n}^k(x, t_1) &= w_n^k(x, t_1)F_{[2]n}(x) - p_n^k(x, t_1)F_{[1]n}(x), \end{aligned} \quad (3.17)$$

with the index $k = 1, 2, 3, 4$, respectively, where $F_{[1]n}(x)$ and $F_{[2]n}(x)$ are given by (3.8) and where $w_n^{[k]}(t_1)$, $p_n^{[k]}(t_1)$ are given by

$$\begin{aligned} w_n^{[1]}(t_1) &= \frac{1}{2c} \int_0^l \frac{\psi_{n+1}(x, t_1) + \tilde{\theta}_{n+1}(x, t_1)}{\pi n} (-F_{[2]n}(x)) dx, \\ p_n^{[1]}(t_1) &= -\frac{1}{2c} \int_0^l \frac{\psi_{n+1}(x, t_1) + \tilde{\theta}_{n+1}(x, t_1)}{\pi n} (-F_{[1]n}(x)) dx, \\ w_n^{[2]}(t_1) &= \frac{1}{2c} \int_0^l \frac{-\psi_{n-1}(x, t_1) + \tilde{\theta}_{n-1}(x, t_1)}{\pi n} (-F_{[2]n}(x)) dx, \\ p_n^{[2]}(t_1) &= -\frac{1}{2c} \int_0^l \frac{-\psi_{n-1}(x, t_1) + \tilde{\theta}_{n-1}(x, t_1)}{\pi n} (-F_{[1]n}(x)) dx, \\ w_n^{[3]}(t_1) &= \frac{1}{2c} \int_0^l \frac{\tilde{\psi}_{n+1}(x, t_1) - \theta_{n+1}(x, t_1)}{\pi n} (-F_{[2]n}(x)) dx, \\ p_n^{[3]}(t_1) &= -\frac{1}{2c} \int_0^l \frac{\tilde{\psi}_{n+1}(x, t_1) - \theta_{n+1}(x, t_1)}{\pi n} (-F_{[1]n}(x)) dx, \\ w_n^{[4]}(t_1) &= \frac{1}{2c} \int_0^l \frac{\tilde{\psi}_{n-1}(x, t_1) + \theta_{n-1}(x, t_1)}{\pi n} (-F_{[2]n}(x)) dx, \\ p_n^{[4]}(t_1) &= -\frac{1}{2c} \int_0^l \frac{\tilde{\psi}_{n-1}(x, t_1) + \theta_{n-1}(x, t_1)}{\pi n} (-F_{[1]n}(x)) dx, \end{aligned} \quad (3.18)$$

where $\psi_n(x, t_1)$, $\tilde{\psi}_n(x, t_1)$, $\tilde{\theta}_n(x, t_1)$, $\theta_n(x, t_1)$ are given by (3.10). It follows from (3.16) that the solution of the $\mathcal{O}(\varepsilon)$ -problem does not contain secular terms if and only if

$$\begin{cases} f_{[1]1}(x, t_1) + \tilde{f}_{[2]1}(x, t_1) + \frac{1}{2} \left(f_{[2]1}^{[1]}(x, t_1) + f_{[1]1}^{[3]}(x, t_1) \right) = 0, \\ \tilde{f}_{[1]1}(x, t_1) - f_{[2]1}(x, t_1) + \frac{1}{2} \left(f_{[1]1}^{[1]}(x, t_1) - f_{[2]1}^{[3]}(x, t_1) \right) = 0, \end{cases} \quad (3.19)$$

and for $n \geq 2$:

$$\begin{cases} f_{[1]n}(x, t_1) + \tilde{f}_{[2]n}(x, t_1) \\ + \frac{1}{2} \left(f_{[2]n}^{[1]}(x, t_1) + f_{[1]n}^{[3]}(x, t_1) + f_{[2]n}^{[2]}(x, t_1) + f_{[1]n}^{[4]}(x, t_1) \right) = 0, \\ \tilde{f}_{[1]n}(x, t_1) - f_{[2]n}(x, t_1) \\ + \frac{1}{2} \left(f_{[1]n}^{[1]}(x, t_1) - f_{[2]n}^{[3]}(x, t_1) + f_{[1]n}^{[2]}(x, t_1) - f_{[2]n}^{[4]}(x, t_1) \right) = 0. \end{cases} \quad (3.20)$$

Defining $A_{00}(t_1) \equiv 0$ and $B_{00}(t_1) \equiv 0$ it follows that for all $n = 1, 2, \dots$

$$\begin{cases} F_{[1]n}(x)\sigma_{[1]n}(t_1) + F_{[2]n}(x)\sigma_{[2]n}(t_1) = 0, \\ F_{[1]n}(x)\sigma_{[2]n}(t_1) - F_{[2]n}(x)\sigma_{[1]n}(t_1) = 0, \end{cases} \quad (3.21)$$

where

$$\begin{aligned} \sigma_{[1]n}(t_1) &= w_n(t_1) - \tilde{p}_n(t_1) + \frac{1}{2} \left(w_n^{[3]}(t_1) - p_n^{[1]}(t_1) + w_n^{[4]}(t_1) - p_n^{[2]}(t_1) \right), \\ \sigma_{[2]n}(t_1) &= \tilde{w}_n(t_1) + p_n(t_1) + \frac{1}{2} \left(w_n^{[1]}(t_1) + p_n^{[3]}(t_1) + w_n^{[2]}(t_1) + p_n^{[4]}(t_1) \right). \end{aligned} \quad (3.22)$$

System (3.21) can be seen as a system for two unknown $\sigma_{[1]n}(t_1)$ and $\sigma_{[2]n}(t_1)$. For $x \in (0, l)$ the determinant of this system is equal to $(F_{[1]n})^2 + (F_{[2]n})^2 \neq 0$. From this it follows that $\sigma_{[1]n}(t_1) = 0$ and $\sigma_{[2]n}(t_1) = 0$, or, equivalently,

$$\begin{cases} w_n(t_1) - \tilde{p}_n(t_1) + \frac{1}{2} \left(w_n^{[3]}(t_1) - p_n^{[1]}(t_1) + w_n^{[4]}(t_1) - p_n^{[2]}(t_1) \right) = 0, \\ \tilde{w}_n(t_1) + p_n(t_1) + \frac{1}{2} \left(w_n^{[1]}(t_1) + p_n^{[3]}(t_1) + w_n^{[2]}(t_1) + p_n^{[4]}(t_1) \right) = 0. \end{cases} \quad (3.23)$$

System (3.23) involves the functions $\frac{dA_{n0}}{dt_1}$, $\frac{dB_{n0}}{dt_1}$, $A_{n0}(t_1)$ and $B_{n0}(t_1)$. Solving this system for $\frac{dA_{n0}}{dt_1}$, $\frac{dB_{n0}}{dt_1}$ finally yields for $n = 1, 2, 3 \dots$:

$$\left\{ \begin{array}{l} \frac{dA_{n0}}{dt_1} = \frac{\alpha \sin\left(\frac{\pi V_0}{c}\right)}{2l} (n+1) A_{(n+1)0} - \frac{\left(\cos\left(\frac{\pi V_0}{c}\right) + 1\right) \alpha}{2l} (n+1) B_{(n+1)0} \\ \quad - \frac{\alpha \sin\left(\frac{\pi V_0}{c}\right)}{2l} (n-1) A_{(n-1)0} - \frac{\left(\cos\left(\frac{\pi V_0}{c}\right) + 1\right) \alpha}{2l} (n-1) B_{(n-1)0}, \\ \frac{dB_{n0}}{dt_1} = \frac{\left(\cos\left(\frac{\pi V_0}{c}\right) + 1\right) \alpha}{2l} (n+1) A_{(n+1)0} + \frac{\alpha \sin\left(\frac{\pi V_0}{c}\right)}{2l} (n+1) B_{(n+1)0} \\ \quad + \frac{\left(\cos\left(\frac{\pi V_0}{c}\right) + 1\right) \alpha}{2l} (n-1) A_{(n-1)0} - \frac{\alpha \sin\left(\frac{\pi V_0}{c}\right)}{2l} (n-1) B_{(n-1)0}. \end{array} \right. \quad (3.24)$$

This system is an infinite dimensional system of ordinary differential equations. It can clearly be seen that for $\omega = \Omega_1$ all vibration modes are interacting, and it will be difficult to solve the system analytically. It can also be seen that in the limit case $V_0 = 0$ system (3.24) coincides with the system as studied in [16]. System (3.24) can be rewritten in the following way:

$$\left\{ \begin{array}{l} \frac{dA_{n0}}{d\bar{t}_1} = \gamma(n+1) A_{(n+1)0} - \sigma(n+1) B_{(n+1)0} \\ \quad - \gamma(n-1) A_{(n-1)0} - \sigma(n-1) B_{(n-1)0}, \\ \frac{dB_{n0}}{d\bar{t}_1} = \sigma(n+1) A_{(n+1)0} + \gamma(n+1) B_{(n+1)0} \\ \quad + \sigma(n-1) A_{(n-1)0} - \gamma(n-1) B_{(n-1)0}, \end{array} \right. \quad (3.25)$$

where $\bar{t}_1 = \frac{\alpha}{2l} t_1$, $\gamma = \sin\left(\frac{\pi V_0}{c}\right)$ and $\sigma = 1 + \cos\left(\frac{\pi V_0}{c}\right)$. If the truncation method is applied to system (3.25), so only some first modes are taken into account and higher order modes are being neglected, the following system has to be solved:

$$\dot{X} = AX, \quad (3.26)$$

where vector $\dot{X}$ represents the derivatives of A_n and B_n with respect to $\bar{t}_1$ and the dimension of the square matrix A is two times the number of modes which are considered. Table 3.1 represents the eigenvalues of the truncated system (3.25) up to 10 modes, which have been calculated by using the computer

software package Maple. From this table it can be seen that the eigenvalues of the truncated system are always purely imaginary or zero. It is well known in mathematics that in this case no conclusions can be drawn for the infinite dimensional system.

No. of modes	Eigenvalues of matrix A (all multiplicity 2)	Dimension eigen-space of A
1	0	2
2	$\pm\sqrt{2(\gamma^2 + \sigma^2)}i$	4
3	$0, \pm 2\sqrt{2(\gamma^2 + \sigma^2)}i$	6
4	$\pm 1.13\sqrt{(\gamma^2 + \sigma^2)}i, \pm 4.33\sqrt{(\gamma^2 + \sigma^2)}i$	8
5	$0, \pm 2.30\sqrt{(\gamma^2 + \sigma^2)}i, \pm 5.89\sqrt{(\gamma^2 + \sigma^2)}i$	10
6	$\pm 7.50\sqrt{(\gamma^2 + \sigma^2)}i, \pm 1.00\sqrt{(\gamma^2 + \sigma^2)}i, \pm 3.56\sqrt{(\gamma^2 + \sigma^2)}i$	12
7	$0, \pm 9.15\sqrt{(\gamma^2 + \sigma^2)}i, \pm 2.05\sqrt{(\gamma^2 + \sigma^2)}i, \pm 4.90\sqrt{(\gamma^2 + \sigma^2)}i$	14
8	$\pm 10.83\sqrt{(\gamma^2 + \sigma^2)}i, \pm 0.93\sqrt{(\gamma^2 + \sigma^2)}i, \pm 3.18\sqrt{(\gamma^2 + \sigma^2)}i, \pm 6.30\sqrt{(\gamma^2 + \sigma^2)}i$	16
9	$0, \pm 12.54\sqrt{(\gamma^2 + \sigma^2)}i, \pm 1.89\sqrt{(\gamma^2 + \sigma^2)}i, \pm 4.38\sqrt{(\gamma^2 + \sigma^2)}i, \pm 7.74\sqrt{(\gamma^2 + \sigma^2)}i$	18
10	$\pm 14.26\sqrt{(\gamma^2 + \sigma^2)}i, \pm 0.87\sqrt{(\gamma^2 + \sigma^2)}i, \pm 5.65\sqrt{(\gamma^2 + \sigma^2)}i, \pm 9.23\sqrt{(\gamma^2 + \sigma^2)}i, \pm 2.93\sqrt{(\gamma^2 + \sigma^2)}i$	20

Table 3.1: Approximations of the eigenvalues of the truncated system (3.25).

Analysis of the infinite dimensional system (3.25).

By introducing $X_{n0}(\bar{t}_1) = nA_{n0}(\bar{t}_1)$ and $Y_{n0}(\bar{t}_1) = nB_{n0}(\bar{t}_1)$ system (3.25) becomes

$$\begin{cases} \frac{dX_{n0}}{d\bar{t}_1} = n(\gamma X_{(n+1)0} - \sigma Y_{(n+1)0} - \gamma X_{(n-1)0} - \sigma Y_{(n-1)0}), \\ \frac{dY_{n0}}{d\bar{t}_1} = n(\sigma X_{(n+1)0} + \gamma Y_{(n+1)0} + \sigma X_{(n-1)0} - \gamma Y_{(n-1)0}), \end{cases} \quad (3.27)$$

for $n = 1, 2, \dots$, and $X_{00} = Y_{00} = 0$. Then it can be deduced that

$$\begin{cases} X_{n0}\dot{X}_{n0} = n(\gamma X_{(n+1)0}X_{n0} - \sigma Y_{(n+1)0}X_{n0} - \gamma X_{(n-1)0}X_{n0} - \sigma Y_{(n-1)0}X_{n0}), \\ Y_{n0}\dot{Y}_{n0} = n(\sigma X_{(n+1)0}Y_{n0} + \gamma Y_{(n+1)0}Y_{n0} + \sigma X_{(n-1)0}Y_{n0} - \gamma Y_{(n-1)0}Y_{n0}). \end{cases} \quad (3.28)$$

By adding both equations in (3.28) and by taking the sum from $n = 1$ to ∞ it follows that

$$\begin{aligned} & \frac{1}{2} \sum_{n=1}^{\infty} \frac{d}{dt_1} (X_{n0}^2 + Y_{n0}^2) \\ &= \sigma \sum_{n=1}^{\infty} \{Y_{(n+1)0}X_{n0} - X_{(n+1)0}Y_{n0}\} - \gamma \sum_{n=1}^{\infty} \{X_{(n+1)0}X_{n0} + Y_{(n+1)0}Y_{n0}\}. \end{aligned} \quad (3.29)$$

By differentiating (3.29) with respect to $\bar{t}_1$ one gets (for details readers are referred to [16])

$$\frac{1}{2} \sum_{n=1}^{\infty} \frac{d^2}{d\bar{t}_1^2} (X_{n0}^2 + Y_{n0}^2) = 2(\sigma^2 + \gamma^2) \sum_{n=1}^{\infty} (X_{n0}^2 + Y_{n0}^2), \quad (3.30)$$

and then by putting $\sum_{n=1}^{\infty} (X_{n0}^2 + Y_{n0}^2) = W(\bar{t}_1)$ it follows that

$$\frac{d^2 W(\bar{t}_1)}{d\bar{t}_1^2} - 4(\sigma^2 + \gamma^2)W(\bar{t}_1) = 0. \quad (3.31)$$

The solution of (3.31) is $W(\bar{t}_1) = K_1 e^{2\sqrt{(\sigma^2 + \gamma^2)}\bar{t}_1} + K_2 e^{-2\sqrt{(\sigma^2 + \gamma^2)}\bar{t}_1}$, or, equivalently,

$$\begin{aligned} W(t_1) = & K_1 \exp \left(\frac{\alpha}{l} \sqrt{\left(\cos \left(\frac{\pi V_0}{c} \right) + 1 \right)^2 + \sin^2 \left(\frac{\pi V_0}{c} \right) t_1} \right) \\ & + K_2 \exp \left(-\frac{\alpha}{l} \sqrt{\left(\cos \left(\frac{\pi V_0}{c} \right) + 1 \right)^2 + \sin^2 \left(\frac{\pi V_0}{c} \right) t_1} \right), \end{aligned} \quad (3.32)$$

where K_1 and K_2 are both constants of integration. Now it should be observed that for $K_1 \neq 0$ $W(t_1)$ increases if t_1 increases. As it has been shown before the application of the truncation method to system (3.25) (that is, by considering only a finite number of vibration modes) that only purely imaginary eigenvalues or zero eigenvalues will be found. This implies that only oscillatory behaviour will be found by applying the truncation method, whereas

(3.32) clearly indicates that also exponential behaviour should be included. Because of that the approximations obtained by the truncation method are not accurate on time-scales of order ε^{-1} . For example, the approximation of the solution that was obtained in [15] is not valid on long time-scales of order ε^{-1} .

3.2.3 The case: $\omega = \frac{\pi 2(V_0^2 - c^2)}{lc}$, the second resonance case

In this section it is assumed that ω is equal to the second natural frequency of the traveling string, that is $\omega = \frac{\pi 2(V_0^2 - c^2)}{lc}$. In this case, as in the case 3.2.2 both sums in the right hand side of (3.9) will lead to unbounded behaviour in the solution of the $\mathcal{O}(\varepsilon)$ -problem. After introducing $\omega = \frac{\pi 2(V_0^2 - c^2)}{lc}$ into (3.9), equation (3.9) becomes

$$\begin{aligned} \mathcal{O}(\varepsilon) : & \frac{\partial^2 v_1}{\partial t_0^2} + 2V_0 \frac{\partial^2 v_1}{\partial t_0 \partial x} + (V_0^2 - c^2) \frac{\partial^2 v_1}{\partial x^2} \\ &= \sum_{n=1}^{\infty} \left\{ \sin(\Omega_n t_0) \varphi_n(x, t_1) + \cos(\Omega_n t_0) \tilde{\varphi}_n(x, t_1) \right\} \\ &+ \frac{1}{2} \sum_{n=1}^{\infty} \left\{ \cos(\Omega_{n-2} t_0) (\psi_n(x, t_1) + \tilde{\theta}_n(x, t_1)) \right. \\ &+ \cos(\Omega_{n+2} t_0) (-\psi_n(x, t_1) + \tilde{\theta}_n(x, t_1)) \\ &+ \sin(\Omega_{n-2} t_0) (\tilde{\psi}_n(x, t_1) - \theta_n(x, t_1)) + \sin(\Omega_{n+2} t_0) (\tilde{\psi}_n(x, t_1) + \theta_n(x, t_1)) \left. \right\}, \end{aligned} \quad (3.33)$$

where functions $\varphi_n(x, t_1), \tilde{\varphi}_n(x, t_1), \psi_n(x, t_1), \tilde{\psi}_n(x, t_1), \theta_n(x, t_1), \tilde{\theta}_n(x, t_1)$ are still given by (3.10). In (3.33) it should be observed that $\Omega_{-n} = \Omega_n$ and $\Omega_0 = 0$.

Following the same procedure, as in the case 3.2.2 to avoid secular terms in the solution of the $\mathcal{O}(\varepsilon)$ -problem, one obtains after some lengthy, but elementary calculations the following systems for $A_{n0}(t_1)$ and $B_{n0}(t_1)$:

$$\begin{cases} \frac{dA_{(1)0}}{dt_1} = -\frac{3\alpha}{l} \sin\left(\frac{\pi V_0}{c}\right) \cos\left(\frac{\pi V_0}{c}\right) A_{(3)0} + \frac{3\alpha}{l} \left(\cos\left(\frac{\pi V_0}{c}\right) - 1\right)^2 B_{(3)0}, \\ \frac{dB_{(1)0}}{dt_1} = -\frac{3\alpha}{l} \left(\cos\left(\frac{\pi V_0}{c}\right) - 1\right)^2 A_{(3)0} - \frac{3\alpha}{l} \sin\left(\frac{\pi V_0}{c}\right) \cos\left(\frac{\pi V_0}{c}\right) B_{(3)0}, \end{cases} \quad (3.34)$$

for $n = 1$, and

$$\left\{ \begin{array}{l} \frac{dA_{n0}}{dt_1} = -\frac{\alpha\mu}{2l}(n+2)A_{(n+2)0} + \frac{\alpha\eta}{2l}(n+2)B_{(n+2)0} \\ \quad + \frac{\alpha\mu}{2l}(n-2)A_{(n-2)0} + \frac{\alpha\eta}{2l}(n-2)B_{(n-2)0}, \\ \frac{dB_{n0}}{dt_1} = -\frac{\alpha\eta}{2l}(n+2)A_{(n+2)0} - \frac{\alpha\mu}{2l}(n+2)B_{(n+2)0} \\ \quad - \frac{\alpha\eta}{2l}(n-2)A_{(n-2)0} + \frac{\alpha\mu}{2l}(n-2)B_{(n-2)0}, \end{array} \right. \quad (3.35)$$

for $n = 2, 3, \dots$ where

$$\mu = \sin\left(\frac{\pi V_0}{c}\right) \cos\left(\frac{\pi V_0}{c}\right), \quad \eta = \left(\cos\left(\frac{\pi V_0}{c}\right) - 1\right)^2. \quad (3.36)$$

In system (3.34)-(3.35) A_{00} and $B_{00} = 0$ are defined to be identically equal to zero. The system (3.34)-(3.35) is an infinite dimensional system of coupled, ordinary differential equations. From the structure of the equations it can easily be seen that there are infinitely many interactions between the vibration modes, namely between $n-2$, n and $n+2$ modes. So, to apply the truncation method to system (3.34)-(3.35) (to find approximations of the solution which are valid on long time-scales of order ε^{-1}) can be a wrong procedure as has been shown for the case $m^* = 1$. How to obtain more information out of system (3.34)-(3.35) for arbitrary m^* is still an open subject for future research.

3.2.4 The case: $\omega = \frac{\pi m^*(V_0^2 - c^2)}{lc}$, a general resonance case

By considering the cases 3.2.1, 3.2.2, and 3.2.3, and by taking into account the results that have been obtained in [27] it follows, that if an external frequency is equal to a natural frequency of the moving string or equal to the sum or difference of those natural frequencies it will cause resonance in the conveyor belt system. Now it will be assumed that $\omega = \Omega_{m^*}$, that is, $\omega = \frac{\pi m^*(V_0^2 - c^2)}{lc}$, where $m^* \in \mathbb{Z}^+$ and fixed. In this section the system of ordinary differential equations will be derived, which describes the interactions between the different oscillation modes.

Substituting $\omega = \frac{\pi m^*(V_0^2 - c^2)}{lc}$ into (3.9) yields

$$\begin{aligned}
\mathcal{O}(\varepsilon) : & \frac{\partial^2 v_1}{\partial t_0^2} + 2V_0 \frac{\partial^2 v_1}{\partial t_0 \partial x} + (V_0^2 - c^2) \frac{\partial^2 v_1}{\partial x^2} \\
= & \sum_{n=1}^{\infty} \left\{ \sin(\Omega_n t_0) \varphi_n(x, t_1) + \cos(\Omega_n t_0) \tilde{\varphi}_n(x, t_1) \right\} \\
+ & \frac{1}{2} \sum_{n=1}^{\infty} \left\{ \cos(\Omega_{n-m^*} t_0) (\psi_n(x, t_1) + \tilde{\theta}_n(x, t_1)) \right. \\
+ & \cos(\Omega_{n+m^*} t_0) (-\psi_n(x, t_1) + \tilde{\theta}_n(x, t_1)) \\
+ & \left. \sin(\Omega_{n-m^*} t_0) (\tilde{\psi}_n(x, t_1) - \theta_n(x, t_1)) + \sin(\Omega_{n+m^*} t_0) (\tilde{\psi}_n(x, t_1) + \theta_n(x, t_1)) \right\}, \tag{3.37}
\end{aligned}$$

where functions $\varphi_n(x, t_1), \tilde{\varphi}_n(x, t_1), \psi_n(x, t_1), \tilde{\psi}_n(x, t_1), \theta_n(x, t_1), \tilde{\theta}_n(x, t_1)$ are given again by (3.10). In (3.37) it should be observed that $\Omega_{-n} = \Omega_n$ and $\Omega_0 = 0$.

Following the same procedure, as in the previous cases to avoid secular terms in the solution of the $\mathcal{O}(\varepsilon)$ -problem, one obtains after some lengthy, but elementary calculations the following systems for $A_{n0}(t_1)$ and $B_{n0}(t_1)$:

$$\begin{cases} \frac{dA_{n0}}{dt_1} = \frac{\alpha\mu_{m^*}}{m^*l} (n+m^*)A_{(n+m^*)0} + \frac{\alpha\eta_{m^*}}{m^*l} (n+m^*)B_{(n+m^*)0}, \\ \frac{dB_{n0}}{dt_1} = -\frac{\alpha\eta_{m^*}}{m^*l} (n+m^*)A_{(n+m^*)0} + \frac{\alpha\mu_{m^*}}{m^*l} (n+m^*)B_{(n+m^*)0}, \end{cases} \tag{3.38}$$

for $n = 1, 2, \dots, m^* - 1$, and

$$\begin{cases} \frac{dA_{n0}}{dt_1} = \frac{\alpha\mu_{m^*}}{2m^*l} (n+m^*)A_{(n+m^*)0} + \frac{\alpha\eta_{m^*}}{2m^*l} (n+m^*)B_{(n+m^*)0} \\ \quad - \frac{\alpha\mu_{m^*}}{2m^*l} (n-m^*)A_{(n-m^*)0} + \frac{\alpha\eta_{m^*}}{2m^*l} (n-m^*)B_{(n-m^*)0}, \\ \frac{dB_{n0}}{dt_1} = -\frac{\alpha\eta_{m^*}}{2m^*l} (n+m^*)A_{(n+m^*)0} + \frac{\alpha\mu_{m^*}}{2m^*l} (n+m^*)B_{(n+m^*)0} \\ \quad - \frac{\alpha\eta_{m^*}}{2m^*l} (n-m^*)A_{(n-m^*)0} - \frac{\alpha\mu_{m^*}}{2m^*l} (n-m^*)B_{(n-m^*)0}, \end{cases} \tag{3.39}$$

for $n = m^*, m^* + 1, \dots$, where

$$\mu_{m^*} = (-1)^{1+m^*} \sin\left(\frac{\pi m^* V_0}{c}\right), \quad \eta_{m^*} = (-1)^{m^*} \left(\cos\left(\frac{\pi m^* V_0}{c}\right) - 1 \right). \tag{3.40}$$

In system (3.38)-(3.39) A_{00} and B_{00} are defined to be identically equal to zero. The system (3.38)-(3.39) is an infinite dimensional system of coupled, ordinary differential equations. From the structure of the equations it can easily be seen that there are infinitely many interactions between the vibration modes, namely between $n - m^*$, n and $n + m^*$ modes. So, to apply the truncation method to system (3.38)-(3.39) (to find approximations of the solution which are valid on long time-scales of order ε^{-1}) can be a wrong procedure as it has been shown for the case $m^* = 1$ and for the case $m^* = 2$. How to obtain more information out of system (3.38)-(3.39) for arbitrary m^* is still an open subject for the future research.

3.3 Conclusions and remarks

In this chapter an initial-boundary value problem for a linear equation, describing an axially moving string has been studied. This equation can be used as a model for the lower frequency, transversal vibrations of a conveyor belt system. The axially moving string is assumed to move in one direction with a non-constant speed $V(t)$, that is, $V(t) = V_0 + \varepsilon\alpha \sin(\omega t)$, where $0 < \varepsilon \ll 1$ and where V_0, α and ω are positive constants. For V_0 it is assumed that $V_0 < c$, where c is the wave speed. Formal asymptotic approximations of the solution of the initial-boundary value problem have been constructed by using a combination of a two time-scales perturbation method and a Laplace transform method (see also [27] and [37]). It turns out that there are infinitely many values of ω that give rise to internal resonances in the axially moving string system. In fact, that happens when ω is equal to any natural frequency of the moving string, that is, $\omega = \Omega_n = \frac{\pi n(V_0^2 - c^2)}{lc}$, where $n = 1, 2, \dots$. It turned out for $\omega = \Omega_1$ that only a one-mode approximation, as for instance has been used in [15] is not accurate on time-scales of order ε^{-1} , as the solution of the boundary-value problem (3.2)-(3.3) consists of infinitely many, interacting vibration modes. Moreover in [15] due to the application of the truncation method, the odd numbered resonance frequencies Ω_n were not found.

Four cases have been studied in this chapter: ω is not in a neighbourhood of any Ω_n , $\omega = \Omega_1$, $\omega = \Omega_2$ and $\omega = \Omega_{m^*}$, where $m^* \in \mathbb{Z}^+$ and fixed. For the second case (that is when ω is equal to the lowest natural frequency of the moving string) a first integral has been found and it has been shown that the truncation method does not give accurate results on long time-scales. All approximations which are obtained by the method as introduced in this chapter are valid on long time-scales, that is, on time-scales of order ε^{-1} . Moreover, the results as obtained in [16] are a special case of the results as obtained in this chapter.

CHAPTER 4

On the transversal vibrations of an axially moving continuum with a time-varying velocity: transient from string to beam behaviour

Abstract: In this chapter an initial boundary value problem for a linear equation describing an axially moving stretched beam will be considered. The velocity of the beam is assumed to be time-varying. Since the order of magnitude of the bending stiffness terms depends on the vibration modes and the frequencies involved a combination of two simplified models (that is, a string equation for the lower frequencies, and a beam with string effect equation for the higher frequencies) will be used to describe the transversal vibrations of the system accurately. Based on the calculations of the natural frequencies the regions of applicability of these sub-models will be determined. A two time-scales perturbation method will be used to construct formal asymptotic approximations of the solutions. Non-resonant and some resonant cases will be studied for four different values of the relative errors. An important implication of the earlier results in the literature is that for these types of axially moving continua problems the use of only string-like models is not appropriate. To describe the dynamics of these types of axially moving continua problems correctly one has to include (small) bending stiffness in the model. In this paper it is explicitly shown how to work with a combined model that is a string model at the low frequencies and a tensioned beam model at the higher frequencies.

4.1 Introduction

Axially moving systems are present in a vast class of engineering problems which arise in industrial, civil, aerospace, mechanical, electronic, medical, and automotive applications. Serpentine belts, aerial cables, tram and train ways, oil pipelines, magnetic tapes, power transmission belts, band saw blades, chair lifts in skiing resorts, and even models of human DNA are examples of real objects where axial transport of mass can be associated with transverse vibrations. Investigating transverse vibrations of a belt system is a challenging subject which has been studied for many years by many researches and still is of interest today (see the references for a recent overview). In the classical analysis of axially moving continua the vibrations are usually classified into two types, i.e. whether it is of a string-like type or of a beam-like type, depending on the bending stiffness of the belt. If the bending stiffness can be neglected then the system is classified as string (wave)-like, otherwise it is classified as beam-like. The transverse vibrations of a belt system (with time-varying velocity $V(\tau)$) can be modelled mathematically as:

- string-like by

$$U_{\tau\tau} + 2VU_{X\tau} + V_{\tau}U_X + (V^2 - c^2)U_{XX} = 0, \text{ and} \quad (4.1)$$

- beam-like (with a string effect) by

$$U_{\tau\tau} + 2VU_{X\tau} + V_{\tau}U_X + (V^2 - c^2)U_{XX} + \frac{EI}{\rho A}U_{XXXX} = 0, \quad (4.2)$$

This chapter is a slightly revised version of [38]: S.V. Ponomareva and W.T. van Horssen: On the transversal vibrations of an axially moving continuum with time-varying velocity: transient from string to beam behaviour, *Journal of Sound and Vibration*, accepted for publication, (2009).

where

A	: the area of the cross section of the belt,
X	: the coordinate in horizontal direction,
$U(X, \tau)$	: the displacement of the belt in vertical direction,
πL	: the distance between the pulleys,
ρ	: the mass density of the belt,
E	: the modulus of elasticity,
I	: the moment of inertia with respect to the belt middle plane,
τ	: the time,
$V(\tau)$	: the time-varying belt speed, and
$c > 0$	: the wave speed, due to a pretension of the belt.

The time-varying belt velocity $V(\tau)$ is given by $V(\tau) = \varepsilon(\bar{V}_0 + \bar{\alpha} \sin(\bar{\omega}\tau))$, where $\bar{V}_0$, $\bar{\omega}$, and $\bar{\alpha}$ are some positive constants with $\bar{V}_0 > 0$ and $\bar{V}_0 > |\bar{\alpha}|$, and where ε is a small parameter with $0 < \varepsilon \ll 1$. The term $\varepsilon \bar{\alpha} \sin(\bar{\omega}\tau)$ can be seen as a small perturbation of the main belt speed $\varepsilon \bar{V}_0$, due to different kinds of imperfections of the belt system. The small parameter ε indicates that the belt speed $V(\tau)$ is small compared to the wave speed c . The condition $\bar{V}_0 > |\bar{\alpha}|$ guarantees that the belt always moves forward in one direction.

Due to different kinds of imperfections of the belt system such as roll eccentricities and varying belt speed, severe transversal vibrations (due to internal resonances) can occur. The occurrence of resonances should be prevented since they can cause operational and maintenance problems including excessive wear of the belt and the support components, and an increase of energy consumption of the belt system. By knowing the natural frequencies of the belt, so called resonance-free belt system can be designed. Although the nonlinear models can be more informative, and describe the real conveyer belt systems usually better, it is not meaningless to investigate the linear equations. (4.1) and (4.2) first.

Equation (4.1) with $V(\tau) = \varepsilon(\bar{V}_0 + \bar{\alpha} \sin(\bar{\omega}\tau))$ was studied in [16]. It was found that there are infinitely many values of $\bar{\omega}$ giving rise to internal resonances in the belt system. It was also shown that the truncation method can not be applied in order to obtain asymptotic results on long time-scales (that is, on time-scales of order ε^{-1}). Equation (4.1) with $V(\tau) = \bar{V}_0 + \varepsilon \bar{\alpha} \sin(\bar{\omega}\tau)$ was studied in [33]. It was shown that in this case for a high velocity the truncation method also can not be applied. On the other hand it was also shown in [18] that for the beam equation (4.2) the truncation method can be applied, but the dynamic behaviour of the belt system is still very complicated. The stability conditions for the belt system were also derived in [18]. From experiments and from the theoretical investigations (see, for

example [25]) it is known that the real dynamic behaviour of conveyer belt systems with relatively small bending stiffness is some sort of combination of both models (4.1)-(4.2). The low frequency vibration modes look more like string modes and the higher order modes (when the bending stiffness becomes more important) look more like beam modes. It is not only interesting but also important from the applicational point of view to investigate such phenomena of transient “from string to beam” behaviour. In [25] the effect of bending stiffness on higher order modes was discussed. In recent papers [39] and [40] the following attempts to describe these phenomena can be found. When the belt speed is high and has the same order of magnitude as the wave speed c (that is, $V(\tau) = \bar{V}_0 + \varepsilon \bar{\alpha} \sin(\bar{\omega}\tau)$) a case has been studied in [39] for which the bending stiffness is of order ε , and an approximate analytical expression for the natural frequency and stability regions has been presented. In [41] the same assumptions have been used and boundary layer solutions have been constructed. Approximations of the eigenvalues of the belt system with a small bending stiffness were also presented in [40]. In [42] it has been found that small bending stiffness term becomes very important near the boundaries. All the authors of the aforementioned papers found that the natural frequencies change due to the presence of a small bending stiffness, but did not include the fact that for the higher order modes the bending stiffness terms are not of order ε anymore, and actually should be included in the $\mathcal{O}(1)$ -problem. Namely, the assumption that most axially moving belt systems have small bending stiffness relative to their tension and can be modelled as an axially moving beam with small dimensionless bending stiffness (as it was done in [40]) is only valid for lower order vibration modes. Moreover, the natural frequencies of the beam model (4.2) with $V(\tau) = \bar{V}_0 + \varepsilon \bar{\alpha} \sin(\bar{\omega}\tau)$ can not be found exactly (see for instance [31] and [8]). In this chapter it will be assumed that $V(\tau) = \varepsilon(\bar{V}_0 + \bar{\alpha} \sin(\bar{\omega}\tau))$. The idea how and when in this case different simplified models may be applied to construct a more realistic model of the travelling belt system was proposed in [36]. Usually it is not possible to calculate the natural frequencies of a real belt system exactly. The bending stiffness, however, is not important for the lower modes of vibrations. And for the higher modes of vibration the bending stiffness terms become more important than the string terms.

Let us assume that $\bar{\omega}$ is given, and that initially at $T = 0$ the first N modes are present. If N is not too large the bending stiffness terms in the problem equation are small, and can be neglected. A string equation (see equation (4.1)) is then obtained. The problem for the string equation, however, has a serious drawback: the truncation method can not be applied due to the presence of internal resonances for which all modes are interacting. And as a consequence, no good approximations of the exact solution on a long time-

scale can be found. Moreover, the resonance frequencies (which are found in this way) might not correspond to the exact resonance frequencies. Now, there are at least three simplified models depending on the vibration modes and the corresponding frequencies: a string model for the lower frequencies, a beam-string model (that is, the exact model) for the intermediate frequencies, and a pure beam model for the higher frequencies. A combination of these models can improve the results of the existing models and methods. The proposed method is based on calculating the natural frequencies of each sub-model, and determining the relative errors in it. In this way one can define intervals of applicability of these simplified models with a predefined, desired accuracy. It should be observed that a pure beam equation can not be truncated either (see [43]), whereas for a beam-string model there is no problem with truncation. In this paper a combination of two simplified models will be proposed: a string model for the lower frequencies, and a beam-string model for the intermediate and the higher frequencies. It will turn out that this combination model has the following advantages:

1. The model allows to truncate a string equation correctly.
2. The natural frequencies of the lower vibration modes can be found more accurately, as the small bending stiffness effect has to be included as a small perturbation into the problem.

When a string-like model is used it has been shown in [16] and [33] that the infinite series representation for the solution can not be truncated from the mathematical point of view (else one can obtain wrong internal resonances, and incorrect modal interactions). So, using only a string-like model for which the number of modes is truncated to a finite number can or will lead to wrong mathematical results. On the other hand an infinite mode representation for the solution of a string-like model is physically irrelevant since for the higher order modes the bending stiffness becomes important, and so the beam-string-like equation has to be studied. In this chapter it will be demonstrated how to work with the combined model that is a string model at the low frequencies and a tensioned beam model at the higher frequencies, such that mathematically and physically correct results are obtained.

The chapter is organized as follows. In Section 4.2 the formulation of the problem will be given. The regions of applicability of the simplified models will be determined. In Section 4.3 the two time-scales perturbation method will be applied to construct approximate solutions of the problems. Values of ω that give rise to internal resonances will be presented. The non-resonant case and some resonant cases will be studied for four different values of the relative error in the Sections 4.4, 4.5, 4.6, and 4.7. Stability properties of the

solution for all cases will be given in Section 4.8. Finally, in Section 4.9 some conclusions will be drawn and some remarks will be made.

4.2 Formulation of the problem

In this section a new approach will be proposed to construct a model for an axially moving continuum, which includes both string type and beam type dynamic behaviour. The simplest mechanical model for a travelling belt is a simply supported tensioned Euler-Bernoulli beam (see Figure 4.1). The equation for this model is given by (see also (4.2))

$$U_{\tau\tau} + 2VU_{X\tau} + V_{\tau}U_X + (V^2 - c^2)U_{XX} + \frac{EI}{\rho A}U_{XXXX} = 0. \quad (4.3)$$

The speed of the belt is assumed to be time-varying and to be given by $V(\tau) = \varepsilon(\bar{V}_0 + \bar{\alpha}\sin(\bar{\omega}\tau))$. The boundary conditions and the initial conditions for (4.3) are given by

$$\begin{aligned} U(0, \tau; \varepsilon) = U(\pi L, \tau; \varepsilon) = U_{XX}(0, \tau; \varepsilon) = U_{XX}(\pi L, \tau; \varepsilon) = 0, \quad \tau \geq 0, \\ U(X, 0; \varepsilon) = \bar{f}(X), \text{ and } U_{\tau}(X, 0; \varepsilon) = \bar{r}(X), \quad 0 < X < \pi L, \end{aligned} \quad (4.4)$$

where $\bar{f}(X)$ represents the initial displacement of the belt, $\bar{r}(X)$ is the initial velocity of the belt, and where πL is the distance between the pulleys. For simplicity it is assumed that the cross section of the belt has a rectangular shape, so that $A = hb$ and $I = \frac{bh^3}{12}$, where h is the thickness and b is the width of the belt cross-section, respectively (see Figure 4.2). Following the 3D theory of elasticity additional conditions have to be imposed to the stretched beam equation (4.3), that is: $\frac{\pi L}{k} \gg h$ and $b \gg h$, where k is the mode number.

Equation (4.3) in non-dimensional form becomes

$$\begin{aligned} u_{tt} - u_{xx} + \mu u_{xxxx} \\ = \varepsilon(-\alpha\omega \cos(\omega t)u_x - 2(V_0 + \alpha \sin(\omega t))u_{xt}) - \varepsilon^2(V_0 + \alpha \sin(\omega t))^2 u_{xx}, \end{aligned} \quad (4.5)$$

where $x = \frac{X}{L}$, $V_0 = \frac{\bar{V}_0}{c}$, $t = \frac{c}{L}\tau$, $u = \frac{U}{L}$, $\omega = \frac{L}{c}\bar{\omega}$, $\alpha = \frac{\bar{\alpha}}{c}$ and $\mu = \frac{EI}{\rho Ac^2 L^2} = \frac{Eh^2}{12\rho c^2 L^2}$. The boundary conditions and the initial conditions for (4.5) are given by

$$u(0, t; \varepsilon) = u(\pi, t; \varepsilon) = u_{xx}(0, t; \varepsilon) = u_{xx}(\pi, t; \varepsilon) = 0, \quad t \geq 0, \quad (4.6)$$

$$u(x, 0; \varepsilon) = f(x), \text{ and } u_t(x, 0; \varepsilon) = r(x), \quad 0 < x < \pi, \quad (4.7)$$

where $f(x) = \frac{\bar{f}(X)}{L}$ and $r(x) = \frac{\bar{r}(X)}{c}$.

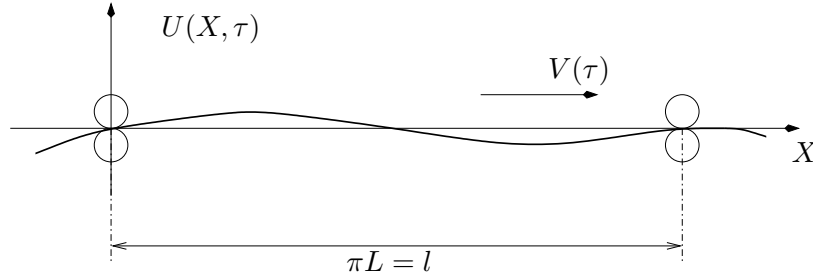


Figure 4.1: The travelling belt system.

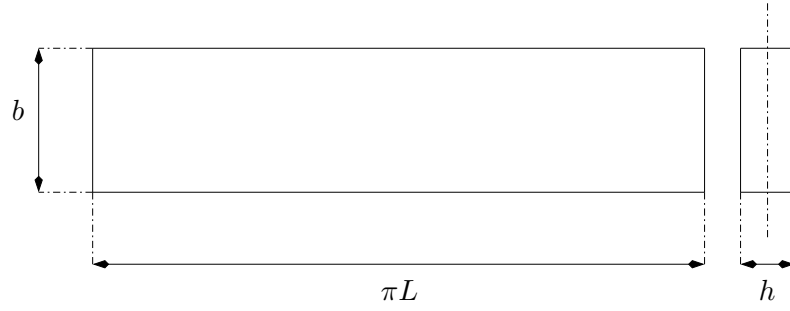


Figure 4.2: The travelling belt configuration.

As it was explained in the Section 4.1, the natural frequencies can usually not be calculated exactly for real problems of travelling belts due to the presence of complicated boundary conditions, different sorts of (non-)linearities, variable stiffness, and so on. It is not meaningless to consider only string behaviour for the lower vibration modes of the belt (when the influence of bending stiffness is very small and can be neglected in the $\mathcal{O}(1)$ -problem), and the following approach shows how one can define the regions of applicability. Let us first consider the equation

$$u_{tt} - u_{xx} + \mu u_{xxxx} = 0, \quad (4.8)$$

subjected to the boundary conditions (4.6). The parameter μ (as defined in (4.5)) is usually a small parameter for belt systems. To determine the natural frequencies of this problem the method of separation of variables can be used, giving as non-trivial solutions for $k = 1, 2, 3, \dots$

$$e^{i\Omega_k t} \sin(kx), \quad (4.9)$$

where $i = \sqrt{-1}$, and

$$\Omega_k = k\sqrt{1 + \mu k^2}. \quad (4.10)$$

Equation (4.10) gives us exact natural frequencies for equation (4.8) subjected to the boundary conditions (4.6). For the string model (i.e. equation (4.8) without bending stiffness) and for the beam model (i.e. equation (4.8) without string effect) the natural frequencies also can be found, so that,

$$\begin{aligned} \Omega_k^{(1)} &= k && \text{for the string model,} \\ \Omega_k^{(2)} &= k\sqrt{1 + \mu k^2} && \text{for the stretched beam model, and} \\ \Omega_k^{(3)} &= k\sqrt{\mu} && \text{for the beam model.} \end{aligned} \quad (4.11)$$

It is possible now to find intervals of applicability of these simplified models (for k), based on the natural frequencies (4.11), with a desired or required accuracy. In Table 4.1 these regions for k are given (where the simplified models, i.e. the string model and the beam model, can be used) for μ equal to 0.0001, 0.002, 0.01, 0.1, 1, and 10, and with relative errors of at most 0.1 %, 1 %, 3 %, and 5 % in the frequencies respectively.

Let us consider a real moving belt, fabricated from material, with the following mechanical properties: $E=1.8$ GPa, $\rho=1.5$ g cm⁻³, $h = 0.8$ cm, $l=100$ m, $T_0=5$ N mm⁻¹, and $b < 3000$ mm. This implies that $\mu = 0.002$. From Table 4.1 it can be seen, that with a relative error of 5 % in the frequencies the following model can be derived (the original initial-boundary value problem for (4.5) can now be split up by assuming that $u(x, t) = \sum_{k=1}^{\infty} u_k(t) \sin(kx)$).

For $1 \leq k \leq 7$ - the string model:

$$\begin{aligned} u_{tt} - u_{xx} = & \varepsilon(-\alpha\omega \cos(\omega t)u_x - 2(V_0 + \alpha \sin(\omega t))u_{xt} - c_1 u_{xxxx}) \\ & - \varepsilon^2(V_0 + \alpha \sin(\omega t))^2 u_{xx}, \end{aligned} \quad (4.12)$$

where it is assumed in (4.12) that $\mu u_{xxxx} = \mathcal{O}(\varepsilon)$, so that $\mu u_{xxxx} = \varepsilon c_1 u_{xxxx}$.

For $8 \leq k \leq 68$ - the beam with string effect model:

$$\begin{aligned} u_{tt} - u_{xx} + \mu u_{xxxx} = & \varepsilon(-\alpha\omega \cos(\omega t)u_x - 2(V_0 + \alpha \sin(\omega t))u_{xt}) \\ & - \varepsilon^2(V_0 + \alpha \sin(\omega t))^2 u_{xx}, \end{aligned} \quad (4.13)$$

where it should be observed that the terms in the left hand side of (4.13) are of leading order, and are of the same order of magnitude.

μ	0.0001	0.002	0.01
rel error; model			
string	$1 \leq k \leq 4$	$k = 1$	—
0,1%; beam-string	$5 \leq k \leq 2234$	$2 \leq k \leq 691$	$1 \leq k \leq 223$
beam	$2235 \leq k < \infty$	$692 \leq k < \infty$	$224 \leq k < \infty$
string	$1 \leq k \leq 14$	$1 \leq k \leq 3$	$k = 1$
1%; beam-string	$15 \leq k \leq 702$	$4 \leq k \leq 161$	$2 \leq k \leq 70$
beam	$703 \leq k < \infty$	$162 \leq k < \infty$	$71 \leq k < \infty$
string	$1 \leq k \leq 25$	$1 \leq k \leq 5$	$1 \leq k \leq 2$
3%; beam-string	$26 \leq k \leq 399$	$6 \leq k \leq 89$	$3 \leq k \leq 39$
beam	$400 \leq k < \infty$	$90 \leq k < \infty$	$40 \leq k < \infty$
string	$1 \leq k \leq 32$	$1 \leq k \leq 7$	$1 \leq k \leq 3$
5%; beam-string	$33 \leq k \leq 304$	$8 \leq k \leq 68$	$4 \leq k \leq 30$
beam	$305 \leq k < \infty$	$69 \leq k < \infty$	$31 \leq k < \infty$
μ	0.1	1	10
rel error; model			
string	—	—	—
0,1%; beam-string	$1 \leq k \leq 70$	$1 \leq k \leq 22$	$1 \leq k \leq 7$
beam	$71 \leq k < \infty$	$23 \leq k < \infty$	$8 \leq k < \infty$
string	—	—	—
1%; beam-string	$1 \leq k \leq 23$	$1 \leq k \leq 7$	$1 \leq k \leq 2$
beam	$24 \leq k < \infty$	$8 \leq k < \infty$	$3 \leq k < \infty$
string	—	—	—
3%; beam-string	$1 \leq k \leq 12$	$1 \leq k \leq 3$	$k = 1$
beam	$13 \leq k < \infty$	$4 \leq k < \infty$	$2 \leq k < \infty$
string	$k = 1$	—	—
5%; beam-string	$2 \leq k \leq 9$	$1 \leq k \leq 3$	—
beam	$10 \leq k < \infty$	$4 \leq k < \infty$	$1 \leq k < \infty$

Table 4.1: Applicability regions for the simplified models: string, beam with string effect, and beam equations.

For $69 \leq k < \infty$ - the beam model:

$$\begin{aligned}
u_{tt} + \mu u_{xxxx} = & \varepsilon(-\alpha \omega \cos(\omega t) u_x - 2(V_0 + \alpha \sin(\omega t)) u_{xt} + \frac{1}{\varepsilon} u_{xx}) \\
& - \varepsilon^2 (V_0 + \alpha \sin(\omega t))^2 u_{xx},
\end{aligned} \tag{4.14}$$

where it should be observed that the terms in the left-hand side of (4.14) are at least an order of magnitude larger than those terms in the right-hand side

of (4.14).

As it was shown in [16] the truncation method can not be applied to the string equation (4.12), but for the beam with string effect equation (4.13) the method can be applied (see [18]) when the internal resonances are taken into account. In [43] it was shown that the truncation method also can not be applied to the beam equation (4.14) when one wants to obtain accurate approximations of the solution on long time-scale. Based on these observations it is assumed now for simplicity, that the original problem (4.5) is split up into 2 models: for $1 \leq k \leq 7$ - the string model (4.12), and for $8 \leq k < \infty$ - the beam with string effect model (4.13). It was shown in [18] that by substituting $u(x, t) = \sum_{n=1}^{\infty} u_n(t; \varepsilon) \sin(nx)$ into (4.5), by multiplying both sides of the so-obtained equation with $\sin(kx)$, and then by integrating with respect to x from $x = 0$ to $x = \pi$ it follows that

$$\begin{aligned} & \ddot{u}_k + (\mu k^4 + k^2)u_k \\ &= \varepsilon \sum_{n=1}^{\infty *} \frac{kn}{(n^2 - k^2)\pi} \left(4\alpha\omega \cos(\omega t)u_n + 8(V_0 + \alpha \sin(\omega t))\dot{u}_n \right) + \mathcal{O}(\varepsilon^2), \end{aligned} \quad (4.15)$$

where the $*$ in $\sum_{n=1}^{\infty *}$ indicates that the summation is only carried out for $n \pm k$ is odd. For $t = 0$ $u_k(t)$ satisfies: $u_k(0; \varepsilon) = \frac{2}{L\pi} \int_0^\pi f(x) \sin(kx) dx$, and $\dot{u}_k(0; \varepsilon) = \frac{2}{c\pi} \int_0^\pi r(x) \sin(kx) dx$. In the next Section 4.3 a two time-scales perturbation method will be applied to approximate the solution of (4.15).

4.3 Application of the two time-scales perturbation method

To avoid secular terms in the approximate solution of (4.5) and (4.15) a two time-scales perturbation method is used. The two new time scales are $t_0 = t$ and $t_1 = \varepsilon t$, implying that $u_k(t; \varepsilon) = v_k(t_0, t_1; \varepsilon)$. The following transformations are needed for the time derivatives:

$$\begin{aligned} \frac{du_k}{dt} &= \frac{\partial v_k}{\partial t_0} + \varepsilon \frac{\partial v_k}{\partial t_1}, \\ \frac{d^2 u_k}{dt^2} &= \frac{\partial^2 v_k}{\partial t_0^2} + 2\varepsilon \frac{\partial^2 v_k}{\partial t_0 \partial t_1} + \varepsilon^2 \frac{\partial^2 v_k}{\partial t_1^2}. \end{aligned} \quad (4.16)$$

It is assumed that $v_k = v_{k0} + \varepsilon v_{k1} + \dots$, and $v_k(t_0, t_1; \varepsilon) = v_k^{(1)}(t_0, t_1; \varepsilon)$ for $1 \leq k \leq 7$ (i.e. the time behaviour is mainly determined by the string model), and $v_k(t_0, t_1; \varepsilon) = v_k^{(2)}(t_0, t_1; \varepsilon)$ for $8 \leq k < \infty$ (i.e. the time behaviour is mainly determined by the beam with string effect model). So that, in fact there are two sets of $\mathcal{O}(1)$ problems, two sets of $\mathcal{O}(\varepsilon)$ problems, and so on. By substituting (4.16) into (4.15), by substituting the expansion for v_k into (4.15), and by taking together terms of equal powers in ε for $v_k^{(1)}$ and $v_k^{(2)}$ it follows that for $1 \leq k \leq 7$:

$$\begin{aligned} \mathcal{O}(1)^{(1)} : \quad & \frac{\partial^2 v_{k0}^{(1)}}{\partial t_0^2} + k^2 v_{k0}^{(1)} = 0, \\ \mathcal{O}(\varepsilon)^{(1)} : \quad & \frac{\partial^2 v_{k1}^{(1)}}{\partial t_0^2} + k^2 v_{k1}^{(1)} = -2 \frac{\partial^2 v_{k0}^{(1)}}{\partial t_0 \partial t_1} - c_1 k^4 v_{k0}^{(1)} \\ & + \sum_{n=1}^{7*} \left(\frac{kn}{(n^2 - k^2)\pi} 4\omega\alpha \cos(\omega t) v_{n0}^{(1)} + 8(V_0 + \alpha \sin(\omega t)) \frac{\partial v_{n0}^{(1)}}{\partial t_0} \right) \\ & + \sum_{n=8}^{\infty*} \left(\frac{kn}{(n^2 - k^2)\pi} 4\omega\alpha \cos(\omega t) v_{n0}^{(2)} + 8(V_0 + \alpha \sin(\omega t)) \frac{\partial v_{n0}^{(2)}}{\partial t_0} \right), \end{aligned} \quad (4.17)$$

and for $8 \leq k < \infty$:

$$\begin{aligned} \mathcal{O}(1)^{(2)} : \quad & \frac{\partial^2 v_{k0}^{(2)}}{\partial t_0^2} + (k^2 + \mu k^4) v_{k0}^{(2)} = 0, \\ \mathcal{O}(\varepsilon)^{(2)} : \quad & \frac{\partial^2 v_{k1}^{(2)}}{\partial t_0^2} + (k^2 + \mu k^4) v_{k1}^{(2)} = -2 \frac{\partial^2 v_{k0}^{(2)}}{\partial t_0 \partial t_1} \\ & + \sum_{n=1}^{7*} \left(\frac{kn}{(n^2 - k^2)\pi} 4\omega\alpha \cos(\omega t) v_{n0}^{(1)} + 8(V_0 + \alpha \sin(\omega t)) \frac{\partial v_{n0}^{(1)}}{\partial t_0} \right) \\ & + \sum_{n=8}^{\infty*} \left(\frac{kn}{(n^2 - k^2)\pi} 4\omega\alpha \cos(\omega t) v_{n0}^{(2)} + 8(V_0 + \alpha \sin(\omega t)) \frac{\partial v_{n0}^{(2)}}{\partial t_0} \right). \end{aligned} \quad (4.18)$$

Equation (4.17) represents the time behaviour of the main equation (4.5) for the first seven modes with small bending stiffness terms. The second sum in equation (4.17) represents the influence of the beam with string effect model (i.e. the influence of the higher order modes ($k \geq 8$) on the lower order modes ($k < 8$)). In the first sum in (4.18) there still is the influence of the string modes on the higher order modes. So, there is an interaction between the two models. In (4.18) the bending stiffness terms are now of leading order. The

solution of the $\mathcal{O}(1)^{(1)}$ -problem is given by

$$v_{k0}^{(1)} = A_{k0}(t_1) \sin(\Omega_k^{(1)} t_0) + B_{k0}(t_1) \cos(\Omega_k^{(1)} t_0), \quad k = 1, 2, \dots, 7. \quad (4.19)$$

The solution of the $\mathcal{O}(1)^{(2)}$ -problem is given by

$$v_{k0}^{(2)} = A_{k0}(t_1) \sin(\Omega_k^{(2)} t_0) + B_{k0}(t_1) \cos(\Omega_k^{(2)} t_0), \quad k = 8, 9, \dots. \quad (4.20)$$

In (4.19) and (4.20) $\Omega_k^{(1)}$ and $\Omega_k^{(2)}$ are given by (4.11). $A_{k0}(t_1)$ and $B_{k0}(t_1)$ are still arbitrary functions and can be used to avoid secular terms in the solutions of the $\mathcal{O}(\varepsilon)^{(1)}$ -problem and the $\mathcal{O}(\varepsilon)^{(2)}$ -problem. The $\mathcal{O}(\varepsilon)^{(1)}$ equation (for $k = 1, 2, \dots, 7$) now becomes

$$\begin{aligned} \frac{\partial^2 v_{k1}^{(1)}}{\partial t_0^2} + (\Omega_k^{(1)})^2 v_{k1}^{(1)} = & -2\Omega_k^{(1)} \left(\frac{\partial A_{k0}}{\partial t_1} \cos(\Omega_k^{(1)} t_0) - \frac{\partial B_{k0}}{\partial t_1} \sin(\Omega_k^{(1)} t_0) \right) \\ & - c_1 k^4 \left(A_{k0}(t_1) \sin(\Omega_k^{(1)} t_0) + B_{k0}(t_1) \cos(\Omega_k^{(1)} t_0) \right) \\ & + \sum_{n=1}^{7*} \left\{ \frac{kn}{(n^2 - k^2)\pi} 4\omega\alpha \cos(\omega t) A_{n0}(t_1) \sin(\Omega_n^{(1)} t_0) + B_{n0}(t_1) \cos(\Omega_n^{(1)} t_0) \right. \\ & \left. + 8(V_0 + \alpha \sin(\omega t)) \Omega_n^{(1)} \left(A_{n0}(t_1) \cos(\Omega_n^{(1)} t_0) - B_{n0}(t_1) \sin(\Omega_n^{(1)} t_0) \right) \right\} \\ & + \sum_{n=8}^{\infty*} \left\{ \frac{kn}{(n^2 - k^2)\pi} 4\omega\alpha \cos(\omega t) A_{n0}(t_1) \sin(\Omega_n^{(2)} t_0) + B_{n0}(t_1) \cos(\Omega_n^{(2)} t_0) \right. \\ & \left. + 8(V_0 + \alpha \sin(\omega t)) \Omega_n^{(2)} \left(A_{n0}(t_1) \cos(\Omega_n^{(2)} t_0) - B_{n0}(t_1) \sin(\Omega_n^{(2)} t_0) \right) \right\}, \end{aligned} \quad (4.21)$$

and the $\mathcal{O}(\varepsilon)^{(2)}$ equation (for $k = 8, 9, \dots$) is given by

$$\begin{aligned} \frac{\partial^2 v_{k1}^{(2)}}{\partial t_0^2} + (\Omega_k^{(2)})^2 v_{k1}^{(2)} = & -2\Omega_k^{(2)} \left(\frac{\partial A_{k0}}{\partial t_1} \cos(\Omega_k^{(2)} t_0) - \frac{\partial B_{k0}}{\partial t_1} \sin(\Omega_k^{(2)} t_0) \right) \\ & + \sum_{n=1}^{7*} \left\{ \frac{kn}{(n^2 - k^2)\pi} 4\omega\alpha \cos(\omega t) A_{n0}(t_1) \sin(\Omega_n^{(1)} t_0) + B_{n0}(t_1) \cos(\Omega_n^{(1)} t_0) \right. \\ & \left. + 8(V_0 + \alpha \sin(\omega t)) \Omega_n^{(1)} \left(A_{n0}(t_1) \cos(\Omega_n^{(1)} t_0) - B_{n0}(t_1) \sin(\Omega_n^{(1)} t_0) \right) \right\} \\ & + \sum_{n=8}^{\infty*} \left\{ \frac{kn}{(n^2 - k^2)\pi} 4\omega\alpha \cos(\omega t) A_{n0}(t_1) \sin(\Omega_n^{(2)} t_0) + B_{n0}(t_1) \cos(\Omega_n^{(2)} t_0) \right. \\ & \left. + 8(V_0 + \alpha \sin(\omega t)) \Omega_n^{(2)} \left(A_{n0}(t_1) \cos(\Omega_n^{(2)} t_0) - B_{n0}(t_1) \sin(\Omega_n^{(2)} t_0) \right) \right\}. \end{aligned} \quad (4.22)$$

From the equations (4.21) and (4.22) it can readily be seen that there are infinitely many values of ω that can give rise to internal resonances. In fact these values are (in an $\mathcal{O}(\varepsilon)$ neighbourhood of):

$$\begin{aligned}
 (i) \quad & \omega \pm \Omega_n^{(1)} = \pm \Omega_k^{(1)}, \text{ for } n, k = 1, 2, \dots, 7, \\
 (ii) \quad & \omega \pm \Omega_n^{(2)} = \pm \Omega_k^{(1)}, \text{ for } k = 1, 2, \dots, 7, \text{ and } n = 8, 9, \dots, \\
 (iii) \quad & \omega \pm \Omega_n^{(1)} = \pm \Omega_k^{(2)}, \text{ for } n = 1, 2, \dots, 7, \text{ and } k = 8, 9, \dots, \\
 (iv) \quad & \omega \pm \Omega_n^{(2)} = \pm \Omega_k^{(2)}, \text{ for } n = 8, 9, \dots, \text{ and } k = 8, 9, \dots.
 \end{aligned} \tag{4.23}$$

For all resonant cases (i)-(iv) the additional condition that $k \pm n$ is an odd number, still holds due to summation in (4.15). By interchanging n and k , the resonant case (ii) (derived out of the $\mathcal{O}(\varepsilon)^{(1)}$ -problem) becomes the resonant case (iii) (derived out of the $\mathcal{O}(\varepsilon)^{(2)}$ -problem). The resonant case (i) is a resonance condition for the string equation, and has been investigated in [16]. The resonant case (iv) is a resonance condition for the beam with string effect equation. The solutions and stability conditions for this case can be found in [18]. Due to the interactions of these two models the model as proposed here has additional resonance conditions (ii) and (iii), where ω might be the sum or difference of one natural frequency of the string and one natural frequency of the beam with string effect. It is also necessary to investigate additionally if ω in the resonant case (i) also satisfies the cases (ii), (iii), and (iv), and vice versa. In the following sections μ and ε will be taken equal to 0.002 and 0.01 respectively, and the initial value problem for (4.15) will be studied for values of the relative error (in the frequencies) and for different values of ω .

4.4 Application of the method with a relative error of 5%

In this section the non-resonant and some resonant cases will be studied when the relative error in the frequencies is at most 5%.

4.4.1 The non-resonant case

In this case it is assumed that the frequency ω of the velocity-fluctuations of the axially moving continuum is not equal to any combination of the resonance frequencies as listed in (4.23). To eliminate the secular terms in the solution of the $\mathcal{O}(\varepsilon)^{(1)}$ -problem and the $\mathcal{O}(\varepsilon)^{(2)}$ -problem it follows that A_{k0} and B_{k0}

have to satisfy

$$\begin{cases} \frac{dA_{k0}}{dt_1} = -\frac{c_1 k^3}{2} B_{k0}, \\ \frac{dB_{k0}}{dt_1} = \frac{c_1 k^3}{2} A_{k0}, \end{cases} \quad (4.24)$$

for $1 \leq k \leq 7$, and where c_1 is given by $\mu = c_1 \varepsilon$, and

$$\begin{cases} \frac{dA_{k0}}{dt_1} = 0, \\ \frac{dB_{k0}}{dt_1} = 0, \end{cases} \iff \begin{cases} A_{k0}(t_1) = A_{k0}(0), \\ B_{k0}(t_1) = B_{k0}(0), \end{cases} \quad (4.25)$$

for $8 \leq k < \infty$. In this case system (4.24) can be seen as some sort of correction on the slow time (t_1) behaviour of the solution in the first seven vibration modes due to presence of the small bending stiffness term in the $\mathcal{O}(\varepsilon)^{(1)}$ -problem. In fact it can be seen as a correction on the frequencies of the oscillation modes for $1 \leq k \leq 7$ since the solution of (4.24) is given by

$$\begin{cases} A_{k0}(t_1) = K_{1k} \cos\left(\frac{c_1 k^3}{2} t_1\right) - K_{2k} \sin\left(\frac{c_1 k^3}{2} t_1\right), \\ B_{k0}(t_1) = K_{1k} \sin\left(\frac{c_1 k^3}{2} t_1\right) + K_{2k} \cos\left(\frac{c_1 k^3}{2} t_1\right), \end{cases} \quad (4.26)$$

for $1 \leq k \leq 7$, and where K_{1k} and K_{2k} are all constant of integration. From the initial conditions (4.7) it follows that

$$\begin{aligned} f(x) = \sum_{k=1}^{\infty} u_k(0; \varepsilon) \sin(kx) &\iff u_k(0; \varepsilon) = \frac{2}{\pi} \int_0^{\pi} f(x) \sin(kx) dx, \\ r(x) = \sum_{k=1}^{\infty} \dot{u}_k(0; \varepsilon) \sin(kx) &\iff \dot{u}_k(0; \varepsilon) = \frac{2}{\pi} \int_0^{\pi} r(x) \sin(kx) dx. \end{aligned}$$

Moreover, since $u_k(0; \varepsilon) = v_k(0, 0; \varepsilon) = v_{k0}(0, 0; \varepsilon) + \varepsilon v_{k1}(0, 0; \varepsilon) + \varepsilon^2 v_{k2}(0, 0; \varepsilon) + \dots$, and $\dot{u}_k(0; \varepsilon) = \dot{v}_k(0, 0; \varepsilon) = \dot{v}_{k0}(0, 0; \varepsilon) + \varepsilon \dot{v}_{k1}(0, 0; \varepsilon) + \varepsilon^2 \dot{v}_{k2}(0, 0; \varepsilon) + \dots$ it follows that

$$\begin{aligned} v_k(0; \varepsilon) &= \frac{2}{\pi} \int_0^{\pi} f(x) \sin(kx) dx \quad \text{and} \\ \dot{v}_k(0; \varepsilon) &= \frac{2}{\pi} \int_0^{\pi} r(x) \sin(kx) dx. \end{aligned} \quad (4.27)$$

From (4.19), (4.20) and (4.28) the following condition for $A_{k0}(0)$ and $B_{k0}(0)$ can be obtained:

$$\begin{aligned} A_{k0}(0) &= \frac{2k}{\pi} \int_0^\pi r(x) \sin(kx) dx, \\ B_{k0}(0) &= \frac{2}{\pi} \int_0^\pi f(x) \sin(kx) dx. \end{aligned} \quad (4.28)$$

All constants of integration in (4.26) can now be found using (4.28). The solution $v_{k0}^{(1)}$ can now be obtained for $1 \leq k \leq 7$, that is,

$$v_{k0}^{(1)} = A_{k0}(0) \sin(k(t_0 + 0.1k^3 t_1)) + B_{k0}(0) \cos(k(t_0 + 0.1k^3 t_1)). \quad (4.29)$$

The solution $v_{k0}^{(2)}$ for $k = 8, 9, \dots$ can now also be found, using (4.20) and (4.25):

$$v_{k0}^{(2)} = A_{k0}(0) \sin(k\sqrt{1 + 0.002k^2}t_0) + B_{k0}(0) \cos(k\sqrt{1 + 0.002k^2}t_0). \quad (4.30)$$

From (4.29) it follows that $k(t_0 + 0.1k^3 t_1) = (k + 0.001k^3)t$, so the frequencies for the first seven modes are approximated by $k + 0.001k^3$. Observing (4.30) the frequency of the higher modes is $k\sqrt{1 + 0.002k^2}$. The exact frequencies for $k = 1, \dots, 7$ are also given by $k\sqrt{1 + 0.002k^2}$. The difference between the exact and the approximated frequencies are $|k + 0.001k^3 - k\sqrt{1 + 0.002k^2}| \leq 0.008$ for $1 \leq k \leq 7$. This means that due to the application of the two time-scales perturbation method there is a slow time t_1 correction in the string model frequency which represents the effect of the small bending stiffness for the lower oscillation modes with $1 \leq k \leq 7$.

4.4.2 Some resonant cases

The following resonant cases will be investigated:

$$\begin{aligned} \omega &= m^*, \text{ where } m^* \text{ is equal to } 1, 3, 5 \text{ or } 7, \\ \omega &= \Omega_9^{(2)} - \Omega_8^{(2)}, \text{ a difference type of resonance,} \\ \omega &= \Omega_1^{(1)} - \Omega_8^{(2)}, \text{ an additional difference type of resonance,} \\ \omega &= \Omega_9^{(2)} + \Omega_8^{(2)}, \text{ a sum type of resonance,} \\ \omega &= \Omega_1^{(1)} + \Omega_8^{(2)}, \text{ an additional sum type of resonance,} \\ \omega &= \Omega_7^{(1)} - \Omega_8^{(2)}, \text{ an additional difference type of resonance,} \\ \omega &= \Omega_7^{(1)} + \Omega_8^{(2)}, \text{ an additional sum type of resonance, and} \\ \omega &= 2\Omega_8^{(2)}, \text{ usually referred to as a principal parametric resonance.} \end{aligned} \quad (4.31)$$

The resonant case $\omega = m^*$, where m^* is equal to 1, 3, 5 or 7

First, the case $\omega = m^* = 1$ (the first “string” resonance frequency) will be studied. It was shown in [16] that the case (i) in (4.23), i.e. $\omega \pm \Omega_n^{(1)} = \pm \Omega_k^{(1)}$, with $n, k = 1, 2, \dots, 7$, and $\omega = 1$ has solutions: $k = n + 1$ and $k = n - 1$. Additionally it has to be checked whether the resonance conditions (ii) – (iv) in (4.23) also have solutions if $\omega = 1$ or not. It can readily be verified that there are no such k and n . Therefore, in order to eliminate secular terms in the solution of the $\mathcal{O}(\varepsilon)^{(1)}$ -problem (see (4.21)) A_{k0} and B_{k0} for $1 \leq k \leq 7$ have to satisfy

$$\begin{cases} \frac{dA_{k0}}{dt_1} = -\frac{c_1 k^3}{2} B_{k0} - \frac{\alpha}{\pi} (k-1) B_{(k-1)0} - \frac{\alpha}{\pi} (k+1) B_{(k+1)0}, \\ \frac{dB_{k0}}{dt_1} = \frac{c_1 k^3}{2} A_{k0} + \frac{\alpha}{\pi} (k-1) A_{(k-1)0} + \frac{\alpha}{\pi} (k+1) A_{(k+1)0}, \end{cases} \quad (4.32)$$

and the other A_{k0} and B_{k0} functions with $8 \leq k < \infty$, derived from the $\mathcal{O}(\varepsilon)^{(2)}$ -problem (4.22), satisfy the same equations as for the non-resonant case (see (4.25)). It is clear from system (4.32), and from system (4.25), that there are interactions between the first eight modes. For the higher order vibration modes (i.e. for the 9th vibration mode and higher) there are no such interactions. Therefore there is no problem with applying the truncation method as it was in [16] for a string model (without bending stiffness). For the case $\omega = m^*$ it is assumed that m^* is equal to 3, 5, or 7 ($\omega = m^*$ is a resonance frequency from the string model). Following the same procedure as for the case $\omega = 1$, the following equations for A_{k0} and B_{k0} can be found:

$$\begin{cases} \frac{dA_{k0}}{dt_1} = -\frac{c_1 k^3}{2m^*} B_{k0} - \frac{\alpha (k-m^*)(2k-2m^*+1)}{\pi m^*(2k-m^*)} B_{(k-m^*)0} \\ \quad - \frac{\alpha (k+m^*)(2k+2m^*-1)}{\pi m^*(2k+m^*)} B_{(k+m^*)0}, \\ \frac{dB_{k0}}{dt_1} = \frac{c_1 k^3}{2m^*} A_{k0} + \frac{\alpha (k-m^*)(2k-2m^*+1)}{\pi m^*(2k-m^*)} A_{(k-m^*)0} \\ \quad + \frac{\alpha (k+m^*)(2k+2m^*-1)}{\pi m^*(2k+m^*)} A_{(k+m^*)0}, \end{cases} \quad (4.33)$$

for $1 \leq k \leq 7$ (where A_{k0} and B_{k0} are assumed to be zero for $k \leq 0$). For the higher order modes ($8 \leq k < \infty$) system (4.25) still holds. It can be seen from (4.33) and (4.25) that there are interactions between the first $(7 + m^*)$ modes. For the higher order modes there are no interactions.

To check the stability of the solution the following approach can be used. From (4.32) and (4.33) it follows that, in general, the equation $\dot{X} = MX$ has to be solved, where the vector X contains the unknown functions $A_{k0}(t_1)$ and

$B_{k0}(t_1)$, and where M is a corresponding square matrix of the size $(7+m^*)$ with constant elements. The eigenvalues of the matrix M define the stability properties of the system. It turns out that for all m^* (that is, for $m^* = 1, 3, 5$, and 7) all eigenvalues of matrix M are purely imaginary or zeros, and that all solutions are bounded and stable in this case.

The resonant case $\omega = \Omega_9^{(2)} - \Omega_8^{(2)}$: a difference type of resonance

One of the difference type of resonances of the stretched beam model (see case (iv) of (4.23)) will be studied in this subsection. The frequency of the belt velocity fluctuations ω is assumed to be equal to $\Omega_9^{(2)} - \Omega_8^{(2)}$. The equation $\omega \pm \Omega_n^{(2)} = \pm \Omega_k^{(2)}$ for $n = 8, 9, \dots$ and $k = 8, 9, \dots$ has only trivial solutions for $n = 8$ and $k = 9$ if $\omega + \Omega_n^{(2)} = \Omega_k^{(2)}$, and $n = 9$ and $k = 8$ if $\omega + \Omega_n^{(2)} = \Omega_k^{(2)}$ (detailed calculations to prove this can be found in [18]). Additionally it has to be checked whether the resonance conditions (i) – (iii) in (4.23) also have solutions if $\omega = \Omega_9^{(2)} - \Omega_8^{(2)}$ or not. It can be verified that there are no such k and n . Therefore, in order to eliminate secular terms in the solution of the $\mathcal{O}(\varepsilon)^{(2)}$ -problem (see (4.22)) A_{k0} and B_{k0} have to satisfy

$$\begin{aligned} & \left\{ \begin{aligned} \frac{dA_{8,0}}{dt_1} &= -\frac{72\alpha}{17\pi} \left(\frac{\Omega_9^{(2)} + \Omega_8^{(2)}}{\Omega_8^{(2)}} \right) B_{9,0}, \\ \frac{dB_{8,0}}{dt_1} &= \frac{72\alpha}{17\pi} \left(\frac{\Omega_9^{(2)} + \Omega_8^{(2)}}{\Omega_8^{(2)}} \right) A_{9,0}, \end{aligned} \right. , \text{ and} \\ & \left\{ \begin{aligned} \frac{dA_{9,0}}{dt_1} &= -\frac{72\alpha}{17\pi} \left(\frac{\Omega_9^{(2)} + \Omega_8^{(2)}}{\Omega_9^{(2)}} \right) B_{8,0}, \\ \frac{dB_{9,0}}{dt_1} &= \frac{72\alpha}{17\pi} \left(\frac{\Omega_9^{(2)} + \Omega_8^{(2)}}{\Omega_9^{(2)}} \right) A_{8,0}. \end{aligned} \right. \end{aligned} \quad (4.34)$$

It can be seen from (4.34) that there are interactions between the 8th and the 9th vibration modes. In order to avoid secular terms in the solutions of the $\mathcal{O}(\varepsilon)^{(1)}$ and $\mathcal{O}(\varepsilon)^{(2)}$ -problems if $\omega = \Omega_9^{(2)} - \Omega_8^{(2)}$ the following results were found. For vibration modes with $1 \leq k \leq 7$ A_{k0} and B_{k0} are still given by (4.24). This can be seen as an influence of the small bending stiffness term in the $\mathcal{O}(\varepsilon)^{(1)}$ -problem. For the 8th and the 9th vibration modes A_{k0} and B_{k0} have to satisfy (4.34). And for the higher order vibration modes with $10 \leq k < \infty$ A_{k0} and B_{k0} have to satisfy (4.25). To check the stability the eigenvalues of the corresponding matrix M have to be calculated. It turns out

that all eigenvalues are purely imaginary or zeros, and that all solutions in this case are bounded and stable.

The resonant case $\omega = \Omega_1^{(1)} - \Omega_8^{(2)}$: an additional difference type of resonance

One of the additional difference type of resonances of the beam-string model (see case (ii) of (4.23)) will be studied in this subsection. The frequency of the belt velocity fluctuations ω is assumed to be equal to $\Omega_1^{(1)} - \Omega_8^{(2)}$. The equation $\omega \pm \Omega_n^{(2)} = \pm \Omega_k^{(1)}$ for $n = 8, 9, \dots$ and $k = 1, 2, \dots, 7$ has only a trivial solution for $n = 8$ and $k = 1$. Additionally it has to be checked whether the resonance conditions (i), (iii) and (iv) in (4.23) also have solutions if $\omega = \Omega_1^{(1)} - \Omega_8^{(2)}$ or not. It turns out that there is one solution in case (iii) (see (4.23)): the equation $\omega \pm \Omega_n^{(1)} = \pm \Omega_k^{(2)}$ for $k = 8, 9, \dots$ and $n = 1, 2, \dots, 7$ has a solution $n = 1$ and $k = 8$ if $\omega - \Omega_n^{(1)} = -\Omega_k^{(2)}$. Therefore, in order to eliminate secular terms in the solution of the $\mathcal{O}(\varepsilon)^{(1)}$ and $\mathcal{O}(\varepsilon)^{(2)}$ -problems (see (4.21) and (4.22)) A_{k0} and B_{k0} have to satisfy

$$\begin{aligned} & \left\{ \begin{aligned} \frac{dA_{1,0}}{dt_1} &= -\frac{c_1}{2\Omega_1^{(1)}} B_{1,0} + \frac{8\alpha}{63\pi} \left(\frac{\Omega_1^{(1)} + \Omega_8^{(2)}}{\Omega_1^{(1)}} \right) B_{8,0}, \\ \frac{dB_{1,0}}{dt_1} &= \frac{c_1}{2\Omega_1^{(1)}} A_{1,0} - \frac{8\alpha}{63\pi} \left(\frac{\Omega_1^{(1)} + \Omega_8^{(2)}}{\Omega_1^{(1)}} \right) A_{8,0}, \end{aligned} \right. , \text{ and} \\ & \left\{ \begin{aligned} \frac{dA_{8,0}}{dt_1} &= \frac{8\alpha}{63\pi} \left(\frac{\Omega_1^{(1)} + \Omega_8^{(2)}}{\Omega_8^{(2)}} \right) B_{1,0}, \\ \frac{dB_{8,0}}{dt_1} &= -\frac{8\alpha}{63\pi} \left(\frac{\Omega_1^{(1)} + \Omega_8^{(2)}}{\Omega_8^{(2)}} \right) A_{1,0}. \end{aligned} \right. \end{aligned} \quad (4.35)$$

It can be seen from (4.35) that there are interactions between the 1st and the 8th vibration modes. In order to avoid secular terms in the solutions of the $\mathcal{O}(\varepsilon)^{(1)}$ and $\mathcal{O}(\varepsilon)^{(2)}$ -problems if $\omega = \Omega_1^{(1)} - \Omega_8^{(2)}$ the following results were found. For the 1st vibration mode (with $k = 1$) A_{k0} and B_{k0} are given by the first system of (4.35). For the vibration modes with $2 \leq k \leq 7$ A_{k0} and B_{k0} are still given by (4.24). For the 8th vibration mode (with $k = 8$) A_{k0} and B_{k0} are given by the second system of (4.35). And for the higher order vibration modes with $9 \leq k < \infty$ A_{k0} and B_{k0} have to satisfy (4.25).

To check the stability the eigenvalues of the corresponding matrix M have to be calculated. It turns out that all eigenvalues are purely imaginary or zeros, and that all solutions in this case are bounded and stable.

The resonant case $\omega = \Omega_9^{(2)} + \Omega_8^{(2)}$: a sum type of resonance

One of the sum type of resonances of the stretched beam model (see case (iv) of (4.23)) will be studied in this subsection. The frequency of the belt velocity fluctuations ω is equal to $\Omega_9^{(2)} + \Omega_8^{(2)}$. The equation $\omega \pm \Omega_n^{(2)} = \pm \Omega_k^{(2)}$ for $n = 8, 9, \dots$ and $k = 8, 9, \dots$ has only trivial solutions for $n = 8$ and $k = 9$, and symmetrically $n = 9$ and $k = 8$ if $\omega - \Omega_n^{(2)} = \Omega_k^{(2)}$ (detailed calculations to prove this can be found in [18]). Additionally it has to be checked whether the resonance conditions (i)–(iii) in (4.23) also have solutions or not if $\omega = \Omega_9^{(2)} + \Omega_8^{(2)}$. It can be verified that there are no such k and n . Therefore, in order to eliminate secular terms in the solution of the $\mathcal{O}(\varepsilon)^{(2)}$ -problem (see (4.22)) A_{k0} and B_{k0} have to satisfy

$$\begin{aligned} & \left\{ \begin{aligned} \frac{dA_{8,0}}{dt_1} &= \frac{72\alpha}{17\pi} \left(\frac{\Omega_8^{(2)} - \Omega_9^{(2)}}{\Omega_8^{(2)}} \right) B_{9,0}, \\ \frac{dB_{8,0}}{dt_1} &= \frac{72\alpha}{17\pi} \left(\frac{\Omega_8^{(2)} - \Omega_9^{(2)}}{\Omega_8^{(2)}} \right) A_{9,0}, \end{aligned} \right. , \text{ and} \\ & \left\{ \begin{aligned} \frac{dA_{9,0}}{dt_1} &= \frac{72\alpha}{17\pi} \left(\frac{\Omega_8^{(2)} - \Omega_9^{(2)}}{\Omega_9^{(2)}} \right) B_{8,0}, \\ \frac{dB_{9,0}}{dt_1} &= \frac{72\alpha}{17\pi} \left(\frac{\Omega_8^{(2)} - \Omega_9^{(2)}}{\Omega_9^{(2)}} \right) A_{8,0}. \end{aligned} \right. \end{aligned} \quad (4.36)$$

It can be seen from (4.36) that there are interactions between the 8th and the 9th vibration modes. In order to avoid secular terms in the solutions of the $\mathcal{O}(\varepsilon)^{(1)}$ and $\mathcal{O}(\varepsilon)^{(2)}$ -problems if $\omega = \Omega_9^{(2)} + \Omega_8^{(2)}$ the following results were found. For the vibration modes with $1 \leq k \leq 7$ A_{k0} and B_{k0} still have to satisfy (4.24). This can be seen as an influence of the small bending stiffness term in the $\mathcal{O}(\varepsilon)^{(1)}$ -problem. For the 8th and the 9th vibration modes A_{k0} and B_{k0} have to satisfy (4.36). And for the higher order vibration modes with $10 \leq k < \infty$ A_{k0} and B_{k0} have to satisfy (4.25).

To check the stability the eigenvalues of the corresponding matrix M have to be calculated. It turns out that some of the eigenvalues have positive real parts, and that some of the solutions in this case are unstable.

The resonant case $\omega = \Omega_1^{(1)} + \Omega_8^{(2)}$: an additional sum type of resonance

One of the additional sum type of resonances of the beam-string model (see case (ii) of (4.23)) will be studied in this subsection. The frequency of the belt

velocity fluctuations ω is assumed to be equal to $\Omega_1^{(1)} + \Omega_8^{(2)}$. The equation $\omega \pm \Omega_n^{(2)} = \pm \Omega_k^{(1)}$ for $n = 8, 9, \dots$ and $k = 1, 2, \dots, 7$ has only a trivial solution for $n = 8$ and $k = 1$ if $\omega - \Omega_n^{(2)} = \Omega_k^{(1)}$. Additionally it has to be checked whether the resonance conditions (i), (iii) and (iv) in (4.23) also have solutions if $\omega = \Omega_1^{(1)} + \Omega_8^{(2)}$ or not. It turns out that there is one solution in case (iii) case (see (4.23)) : the equation $\omega \pm \Omega_n^{(1)} = \pm \Omega_k^{(2)}$ for $k = 8, 9, \dots$ and $n = 1, 2, \dots, 7$ has a solution $n = 1$ and $k = 8$ if $\omega - \Omega_n^{(1)} = \Omega_k^{(2)}$. Therefore, in order to eliminate secular terms in the solution of the $\mathcal{O}(\varepsilon)^{(1)}$ and $\mathcal{O}(\varepsilon)^{(2)}$ -problems (see (4.21) and (4.22)) A_{k0} and B_{k0} have to satisfy

$$\begin{aligned} & \left\{ \begin{aligned} \frac{dA_{1,0}}{dt_1} &= -\frac{c_1}{2\Omega_1^{(1)}} B_{1,0} + \frac{8\alpha}{63\pi} \left(\frac{\Omega_1^{(1)} - \Omega_8^{(2)}}{\Omega_1^{(1)}} \right) B_{8,0}, \\ \frac{dB_{1,0}}{dt_1} &= \frac{c_1}{2\Omega_1^{(1)}} A_{1,0} + \frac{8\alpha}{63\pi} \left(\frac{\Omega_1^{(1)} - \Omega_8^{(2)}}{\Omega_1^{(1)}} \right) A_{8,0}, \end{aligned} \right. , \text{ and} \\ & \left\{ \begin{aligned} \frac{dA_{8,0}}{dt_1} &= \frac{8\alpha}{63\pi} \left(\frac{\Omega_1^{(1)} - \Omega_8^{(2)}}{\Omega_8^{(2)}} \right) B_{1,0}, \\ \frac{dB_{8,0}}{dt_1} &= \frac{8\alpha}{63\pi} \left(\frac{\Omega_1^{(1)} - \Omega_8^{(2)}}{\Omega_8^{(2)}} \right) A_{1,0}. \end{aligned} \right. \end{aligned} \quad (4.37)$$

It can be seen from the equations in (4.37) that there are interactions between the 1st and the 8th vibration modes. In order to avoid secular terms in the solutions of the $\mathcal{O}(\varepsilon)^{(1)}$ and $\mathcal{O}(\varepsilon)^{(2)}$ -problems if $\omega = \Omega_1^{(1)} + \Omega_8^{(2)}$ the following results were found. For the 1st vibration mode (with $k = 1$) A_{k0} and B_{k0} are given by the first system of (4.37). For the vibration modes with $2 \leq k \leq 7$ A_{k0} and B_{k0} still have to satisfy (4.24). For the 8th vibration mode (with $k = 8$) A_{k0} and B_{k0} have to satisfy the second system of (4.37). And for the higher order vibration modes with $9 \leq k < \infty$ A_{k0} and B_{k0} have to satisfy (4.25).

To check the stability the eigenvalues of the corresponding matrix M have to be calculated. It turns out that some of the eigenvalues have positive real parts, and that some of the solutions in this case are unstable.

The resonant case $\omega = \Omega_7^{(1)} - \Omega_8^{(2)}$: an additional difference type of resonance

One of the additional difference type of resonances of the beam-string model (see case (ii) of (4.23)) will be studied in this subsection. The frequency of the

belt velocity fluctuations ω is assumed to be equal to $\Omega_7^{(1)} - \Omega_8^{(2)}$. The equation $\omega \pm \Omega_n^{(2)} = \pm \Omega_k^{(1)}$ for $n = 8, 9, \dots$ and $k = 1, 2, \dots, 7$ has only a trivial solution $n = 8$ and $k = 7$ if $\omega + \Omega_n^{(2)} = \Omega_k^{(1)}$. Additionally it has to be checked whether the resonance conditions (i), (iii) and (iv) in (4.23) also have solutions or not if $\omega = \Omega_7^{(1)} - \Omega_8^{(2)}$. It turns out that there is one solution in case (iii) (see (4.23)) : the equation $\omega \pm \Omega_n^{(1)} = \pm \Omega_k^{(2)}$ for $k = 8, 9, \dots$ and $n = 1, 2, \dots, 7$ has a solution $n = 7$ and $k = 8$ if $\omega - \Omega_n^{(1)} = -\Omega_k^{(2)}$. Therefore, in order to eliminate secular terms in the solution of the $\mathcal{O}(\varepsilon)^{(1)}$ and $\mathcal{O}(\varepsilon)^{(2)}$ -problems (see (4.21) and (4.22)) A_{k0} and B_{k0} have to satisfy

$$\begin{cases} \frac{dA_{7,0}}{dt_1} = -\frac{c_1 2401}{2\Omega_7^{(1)}} B_{7,0} + \frac{56\alpha}{15\pi} \left(\frac{\Omega_7^{(1)} + \Omega_8^{(2)}}{\Omega_7^{(1)}} \right) B_{8,0}, \\ \frac{dB_{7,0}}{dt_1} = \frac{c_1 2401}{2\Omega_7^{(1)}} A_{7,0} - \frac{56\alpha}{15\pi} \left(\frac{\Omega_7^{(1)} + \Omega_8^{(2)}}{\Omega_7^{(1)}} \right) A_{8,0}, \end{cases}, \text{ and} \quad (4.38)$$

$$\begin{cases} \frac{dA_{8,0}}{dt_1} = \frac{56\alpha}{15\pi} \left(\frac{\Omega_7^{(1)} + \Omega_8^{(2)}}{\Omega_8^{(2)}} \right) B_{7,0}, \\ \frac{dB_{8,0}}{dt_1} = -\frac{56\alpha}{15\pi} \left(\frac{\Omega_7^{(1)} + \Omega_8^{(2)}}{\Omega_8^{(2)}} \right) A_{7,0}. \end{cases}$$

It can be seen from (4.38) that there are interactions between the 7th and the 8th vibration modes. In order to avoid secular terms in the solutions of the $\mathcal{O}(\varepsilon)^{(1)}$ and $\mathcal{O}(\varepsilon)^{(2)}$ -problems if $\omega = \Omega_7^{(1)} - \Omega_8^{(2)}$ the following results were found. For the vibration modes with $1 \leq k \leq 6$ A_{k0} and B_{k0} still have to satisfy (4.24). For the 7th and 8th vibration modes (with $k = 7$ and $k = 8$) A_{k0} and B_{k0} have to satisfy (4.38). And for the higher order vibration modes with $9 \leq k < \infty$ A_{k0} and B_{k0} have to satisfy (4.25).

To check the stability the eigenvalues of the corresponding matrix M have to be calculated. It turns out that all eigenvalues are purely imaginary or zeros, and that all solutions in this case are bounded and stable.

The resonant case $\omega = \Omega_7^{(1)} + \Omega_8^{(2)}$: an additional sum type of resonance

One of the additional sum type of resonances of the beam-string model (see case (ii) of (4.23)) will be studied in this subsection. The frequency of the belt velocity fluctuations ω is assumed to be equal to $\Omega_7^{(1)} + \Omega_8^{(2)}$. The equation $\omega \pm \Omega_n^{(2)} = \pm \Omega_k^{(1)}$ for $n = 8, 9, \dots$ and $k = 1, 2, \dots, 7$ has only a trivial solution $n = 8$ and $k = 7$ if $\omega - \Omega_n^{(2)} = \Omega_k^{(1)}$. Additionally it has to be checked whether

the resonance conditions (i), (iii) and (iv) in (4.23) also have any solutions if $\omega = \Omega_7^{(1)} + \Omega_8^{(2)}$ or not. It turns out that there is one solution in case (iii) (see (4.23)) : the equation $\omega \pm \Omega_n^{(1)} = \pm \Omega_k^{(2)}$ for $k = 8, 9, \dots$ and $n = 1, 2, \dots, 7$ has a solution $n = 7$ and $k = 8$ if $\omega - \Omega_7^{(1)} = \Omega_8^{(2)}$. It was also found for the case (iv) of (4.23) that $n = 20$ and $k = 27$ if $\omega + \Omega_n^{(2)} = \Omega_k^{(2)}$ and $n = 27$ and $k = 20$ if $\omega - \Omega_n^{(2)} = -\Omega_k^{(2)}$ then $|\omega \pm \Omega_n^{(2)} \pm \Omega_k^{(2)}| = 0.0012 < \varepsilon$. This can be seen as a detuning case if $\omega = \Omega_7^{(1)} + \Omega_8^{(2)}$. Therefore, in order to eliminate secular terms in the solution of the $\mathcal{O}(\varepsilon)^{(1)}$ and $\mathcal{O}(\varepsilon)^{(2)}$ -problems (see (4.21) and (4.22)) A_{k0} and B_{k0} have to satisfy

$$\begin{cases} \frac{dA_{7,0}}{dt_1} = -\frac{c_1 2401}{2\Omega_7^{(1)}} B_{7,0} + \frac{56\alpha}{15\pi} \left(\frac{\Omega_7^{(1)} - \Omega_8^{(2)}}{\Omega_7^{(1)}} \right) B_{8,0}, \\ \frac{dB_{7,0}}{dt_1} = \frac{c_1 2401}{2\Omega_7^{(1)}} A_{7,0} + \frac{56\alpha}{15\pi} \left(\frac{\Omega_7^{(1)} - \Omega_8^{(2)}}{\Omega_7^{(1)}} \right) A_{8,0}, \end{cases}, \text{ and} \quad (4.39)$$

$$\begin{cases} \frac{dA_{8,0}}{dt_1} = \frac{56\alpha}{15\pi} \left(\frac{\Omega_7^{(1)} - \Omega_8^{(2)}}{\Omega_8^{(2)}} \right) B_{7,0}, \\ \frac{dB_{8,0}}{dt_1} = \frac{56\alpha}{15\pi} \left(\frac{\Omega_7^{(1)} - \Omega_8^{(2)}}{\Omega_8^{(2)}} \right) A_{7,0}, \end{cases}$$

and, additionally,

$$\begin{cases} \frac{dA_{20,0}}{dt_1} = -\frac{540\alpha}{329\pi} \left(\frac{\Omega_{27}^{(2)} + \Omega_{20}^{(2)}}{\Omega_{20}^{(2)}} \right) B_{27,0}, \\ \frac{dB_{20,0}}{dt_1} = \frac{540\alpha}{329\pi} \left(\frac{\Omega_{27}^{(2)} + \Omega_{20}^{(2)}}{\Omega_{20}^{(2)}} \right) A_{27,0}, \end{cases}, \text{ and} \quad (4.40)$$

$$\begin{cases} \frac{dA_{27,0}}{dt_1} = -\frac{540\alpha}{329\pi} \left(\frac{\Omega_{27}^{(2)} + \Omega_{20}^{(2)}}{\Omega_{27}^{(2)}} \right) B_{20,0}, \\ \frac{dB_{27,0}}{dt_1} = \frac{540\alpha}{329\pi} \left(\frac{\Omega_{27}^{(2)} + \Omega_{20}^{(2)}}{\Omega_{27}^{(2)}} \right) A_{20,0}. \end{cases}$$

It can be seen from (4.39) and (4.40) that there are interactions between the 7th and the 8th, and the 20th and the 27th vibration modes, respectively. In order to avoid secular terms in the solutions of the $\mathcal{O}(\varepsilon)^{(1)}$ and $\mathcal{O}(\varepsilon)^{(2)}$ -problems if $\omega = \Omega_7^{(1)} + \Omega_8^{(2)}$ the following results were found. For the vibration

modes with $1 \leq k \leq 6$ A_{k0} and B_{k0} are still given by (4.24). For the 7th and 8th vibration modes (with $k = 7$ and $k = 8$) A_{k0} and B_{k0} are given by (4.39). For the vibration modes with $9 \leq k \leq 19$ and $21 \leq k \leq 26$ A_{k0} and B_{k0} still have to satisfy (4.25). For the 20th and 27th vibration modes (with $k = 20$ and $k = 27$) A_{k0} and B_{k0} have to satisfy (4.40). And for the higher order vibration modes with $28 \leq k < \infty$ A_{k0} and B_{k0} have to satisfy (4.25).

To check the stability the eigenvalues of the corresponding matrix M have to be calculated. It turns out that some of the eigenvalues have positive real parts, and that some of the solutions in this case are unstable.

The resonant case $\omega = 2\Omega_8^{(2)}$: a principal parametric resonance

One of the principal parametric resonances of the stretched beam model will be studied in this subsection. The frequency of the belt velocity fluctuations ω is assumed to be equal to $2\Omega_8^{(2)}$. The equation $\omega \pm \Omega_n^{(2)} = \pm \Omega_k^{(2)}$ for $n = 8, 9, \dots$ and $k = 8, 9, \dots$ has no solutions. Additionally it has to be checked whether the resonance conditions (i)–(iii) in (4.23) also have solutions or not if $\omega = 2\Omega_8^{(2)}$. It can be verified that there are no such k and n . Therefore, in order to eliminate secular terms in the solution of the $\mathcal{O}(\varepsilon)^{(1)}$ and $\mathcal{O}(\varepsilon)^{(1)}$ -problems (see (4.21) and (4.22)) A_{k0} and B_{k0} have to satisfy (the same as for the non-resonant case): (4.24) for $1 \leq k \leq 7$ and (4.25) for $8 \leq k < \infty$. It has already been shown in [18] that in case of principal parametric resonance the possibility to have solutions (interactions between vibration modes) for the case (iv) of (4.23) depends on the value of μ . It turns out here that in the particular case $\mu = 0.002$ (as it was fixed in the beginning) $\omega = 2\Omega_8^{(2)}$ is not a resonance frequency.

To check the stability the eigenvalues of the corresponding matrix M have to be calculated. It turns out that all eigenvalues are purely imaginary or zeros, and that all solutions in this case are bounded and stable.

4.5 Application of the method and with a relative error of 3%

In this section the non-resonant and some resonant cases will be studied when the relative error in the frequencies is at most 3%. From Table 4.1 it can be seen that the original problem (4.5) can now be split up into two models. The 'string' model (see (4.12)) for $1 \leq k \leq 5$ and the 'stretched beam' model (see (4.13)) for $6 \leq k < \infty$. From (4.21) and (4.22) it can readily be seen that there are infinitely many values of ω that can give rise to internal resonances. If the

relative error is 3% these values are (in an $\mathcal{O}(\varepsilon)$ neighbourhood of):

$$\begin{aligned}
 (i) \quad & \omega \pm \Omega_n^{(1)} = \pm \Omega_k^{(1)}, \text{ for } n, k = 1, 2, \dots, 5, \\
 (ii) \quad & \omega \pm \Omega_n^{(2)} = \pm \Omega_k^{(1)}, \text{ for } k = 1, 2, \dots, 5, \text{ and } n = 6, 7, \dots, \\
 (iii) \quad & \omega \pm \Omega_n^{(1)} = \pm \Omega_k^{(2)}, \text{ for } n = 1, 2, \dots, 5, \text{ and } k = 6, 7, \dots, \\
 (iv) \quad & \omega \pm \Omega_n^{(2)} = \pm \Omega_k^{(2)}, \text{ for } n = 6, 7, \dots, \text{ and } k = 6, 7, \dots.
 \end{aligned} \tag{4.41}$$

4.5.1 The non-resonant case

In this case it is assumed that the frequency ω of the velocity-fluctuations of the axially moving continuum is not equal to any combination of the resonance frequencies as listed in (4.41). To eliminate the secular terms in the solution of the $\mathcal{O}(\varepsilon)^{(1)}$ -problem and the $\mathcal{O}(\varepsilon)^{(2)}$ -problem it follows that A_{k0} and B_{k0} have to satisfy equations (4.24) for $1 \leq k \leq 5$ and equations (4.25) for $6 \leq k < \infty$. In this case system (4.24) can again be seen as some sort of slow time (t_1) correction in the string model frequency for the first five vibration modes due to presence of the small bending stiffness term in the $\mathcal{O}(\varepsilon)^{(1)}$ -problem.

4.5.2 Some resonant cases

The same resonant cases (see (4.31)) as in Section 4.4 will be investigated in this subsection.

The resonant case $\omega = m^*$, where m^* is equal to 1, 3, 5 or 7

First, the case $\omega = m^* = 1$ (the first “string” resonance frequency) will be studied. It was shown in [16] that the case (i) in (4.23), i.e. $\omega \pm \Omega_n^{(1)} = \pm \Omega_k^{(1)}$, with $n, k = 1, 2, \dots, 5$, and $\omega = 1$ has solution: $k = n + 1$ and $k = n - 1$. Additionally it has to be checked whether the resonance conditions (ii) – (iv) in (4.41) also have solutions if $\omega = 1$ or not. It can readily be verified that there are no such k and n . Therefore, in order to eliminate secular terms in the solution of the $\mathcal{O}(\varepsilon)^{(1)}$ -problem (see (4.21)) A_{k0} and B_{k0} have to satisfy (4.32) for $1 \leq k \leq 5$ and (4.25) for $6 \leq k < \infty$. It is clear from system (4.32) and from system (4.25) that there are interactions between the first six modes. For the higher order vibration modes (i.e. for the 7th vibration mode and higher) there are no such interactions. Therefore there is no problem with applying the truncation method as it was in [16]. For the case $\omega = m^*$ it is assumed that m^* is equal to 3, 5 or 7 ($\omega = m^*$ is a resonance frequency for the string model). Following the same procedure as for the case $\omega = 1$, it was found that A_{k0} and B_{k0} have to satisfy (4.33) for $1 \leq k \leq 5$ and (4.25) for the higher order modes ($6 \leq k < \infty$). It can be seen from (4.33) and (4.25) that there are

interactions between first $(5 + m^*)$ modes. For the higher order modes there are no such interactions.

To check the stability of the solution the following approach can be used. From (4.33) it follows that, in general, the equation $\dot{X} = MX$ has to be solved, where vector $\dot{X}$ contains unknown $\dot{A}_{k0}(t_1)$ and $\dot{B}_{k0}(t_1)$, vector X contains $A_{k0}(t_1)$ and $B_{k0}(t_1)$, and M is a corresponding square matrix of the size $(5 + m^*)$. The eigenvalues of the matrix M define the stability properties of the system. It was found that for all m^* (that is, for $m^* = 1, 3, 5$, and 7) all eigenvalues of the matrix M are purely imaginary or zeros, and that all solutions are bounded and stable in this case.

The resonant case $\omega = \Omega_9^{(2)} - \Omega_8^{(2)}$: a difference type of resonance

The frequency of the belt velocity fluctuations ω is assumed to be equal to $\Omega_9^{(2)} - \Omega_8^{(2)}$. The equation $\omega \pm \Omega_n^{(2)} = \pm \Omega_k^{(2)}$ for $n = 6, 7, \dots$ and $k = 6, 7, \dots$ has only trivial solutions $n = 8$ and $k = 9$ if $\omega + \Omega_n^{(2)} = \Omega_k^{(2)}$ and $n = 9$ and $k = 8$ if $\omega + \Omega_n^{(2)} = \Omega_k^{(2)}$. Therefore A_{k0} and B_{k0} for the 8th and the 9th vibration modes ($k = 8$ and $k = 9$) still have to satisfy (4.34). Additionally it has to be checked whether the resonance conditions (i) – (iii) in (4.41) also have solutions if $\omega = \Omega_9^{(2)} - \Omega_8^{(2)}$ or not. It was found for the cases (ii) and (iii) of (4.41) that if $n = 6$ and $k = 5$ if $\omega - \Omega_n^{(2)} = -\Omega_k^{(1)}$, and $n = 5$ and $k = 6$ if $\omega + \Omega_n^{(1)} = \Omega_k^{(2)}$, then $|\omega \pm \Omega_n^{(2)} \pm \Omega_k^{(1)}| = 0.0071 < \varepsilon$. This can be seen as a detuning case if $\omega = \Omega_9^{(2)} - \Omega_8^{(2)}$. To avoid secular terms in the solution of the $\mathcal{O}(\varepsilon)^{(1)}$ and $\mathcal{O}(\varepsilon)^{(2)}$ -problems A_{k0} and B_{k0} have to satisfy

$$\begin{cases} \frac{dA_{5,0}}{dt_1} = -\frac{c_1 5^4}{2\Omega_5^{(1)}} B_{5,0} - \frac{30\alpha}{11\pi} \left(\frac{\Omega_6^{(2)} + \Omega_5^{(1)}}{\Omega_5^{(1)}} \right) B_{6,0}, \\ \frac{dB_{5,0}}{dt_1} = \frac{c_1 5^4}{2\Omega_5^{(1)}} A_{5,0} + \frac{30\alpha}{11\pi} \left(\frac{\Omega_6^{(2)} + \Omega_5^{(1)}}{\Omega_5^{(1)}} \right) A_{6,0}, \end{cases} \quad \text{and} \quad (4.42)$$

$$\begin{cases} \frac{dA_{6,0}}{dt_1} = -\frac{30\alpha}{11\pi} \left(\frac{\Omega_6^{(2)} + \Omega_5^{(1)}}{\Omega_6^{(2)}} \right) B_{5,0}, \\ \frac{dB_{6,0}}{dt_1} = \frac{30\alpha}{11\pi} \left(\frac{\Omega_6^{(2)} + \Omega_5^{(1)}}{\Omega_6^{(2)}} \right) A_{5,0}. \end{cases}$$

It can be seen from (4.34) and (4.42) that there are interactions between the 8th and the 9th, and between the 5th and the 6th vibration modes. Finally, in order to avoid secular terms in the solutions of the $\mathcal{O}(\varepsilon)^{(1)}$ and $\mathcal{O}(\varepsilon)^{(2)}$ -problems if $\omega = \Omega_9^{(2)} - \Omega_8^{(2)}$ the following results were found. For vibration

modes with $1 \leq k \leq 4$ A_{k0} and B_{k0} are still given by (4.24). This can be seen as an influence of the small bending stiffness term in the $\mathcal{O}(\varepsilon)^{(1)}$ -problem. For the 5th and the 6th vibration modes A_{k0} and B_{k0} have to satisfy (4.42). For the 8th and the 9th vibration modes A_{k0} and B_{k0} have to satisfy (4.34). And for the vibration modes with $k = 7$ and $10 \leq k < \infty$ A_{k0} and B_{k0} have to satisfy (4.25).

To check the stability the eigenvalues of the corresponding matrix M have to be calculated. It turns out that all eigenvalues are purely imaginary or zeros, and that all solution in this case are bounded and stable.

The resonant case $\omega = \Omega_1^{(1)} - \Omega_8^{(2)}$: an additional difference type of resonance

The frequency of the belt velocity fluctuations ω is assumed to be equal to $\Omega_1^{(1)} - \Omega_8^{(2)}$. The equation $\omega \pm \Omega_n^{(2)} = \pm \Omega_k^{(1)}$ for $n = 6, 7, \dots$ and $k = 1, 2, \dots, 5$ has only a trivial solution $n = 8$ and $k = 1$ if $\omega + \Omega_n^{(2)} = \Omega_k^{(1)}$. Additionally it has to be checked whether the resonance conditions (i), (iii) and (iv) in (4.41) also have solutions if $\omega = \Omega_1^{(1)} - \Omega_8^{(2)}$ or not. It turns out that there is one solution in case (iii) (see (4.23)) : $n = 1$ and $k = 8$ if $\omega - \Omega_n^{(1)} = -\Omega_k^{(2)}$. Therefore, in order to avoid secular terms in the solutions of the $\mathcal{O}(\varepsilon)^{(1)}$ and $\mathcal{O}(\varepsilon)^{(2)}$ -problems if $\omega = \Omega_1^{(1)} - \Omega_8^{(2)}$ the following results were found. For the 1st vibration mode (with $k = 1$) A_{k0} and B_{k0} are given by the first system of (4.35). For the vibration modes with $2 \leq k \leq 5$ A_{k0} and B_{k0} are still given by (4.24). For the 8th vibration mode (with $k = 8$) A_{k0} and B_{k0} are given by (4.35). And, finally, for the vibration modes with $k = 6, 7$ and for the higher order vibration modes with $9 \leq k < \infty$ A_{k0} and B_{k0} have to satisfy (4.25). It can be seen from (4.35) that there are interactions between the 1st and the 8th vibration modes in this case.

To check the stability the eigenvalues of the corresponding matrix M have to be calculated. It turns out that all eigenvalues are purely imaginary or zeros, and that all solution in this case are bounded and stable.

The resonant case $\omega = \Omega_9^{(2)} + \Omega_8^{(2)}$: a sum type of resonance

The frequency of the belt velocity fluctuations ω is assumed to be equal to $\Omega_9^{(2)} + \Omega_8^{(2)}$. The equation $\omega \pm \Omega_n^{(2)} = \pm \Omega_k^{(2)}$ for $n = 6, 7, \dots$ and $k = 6, 7, \dots$ has only trivial solutions: $n = 8$ and $k = 9$, and symmetrically $n = 9$ and $k = 8$ if $\omega - \Omega_n^{(2)} = \Omega_k^{(2)}$. Additionally it has to be checked whether the resonance conditions (i) – (iii) in (4.41) also have any solutions if $\omega = \Omega_9^{(2)} + \Omega_8^{(2)}$ or not.

It can readily be verified that there are no such k and n . Therefore, in order to eliminate secular terms in the solution of the $\mathcal{O}(\varepsilon)^{(1)}$ and $\mathcal{O}(\varepsilon)^{(2)}$ -problems (see (4.21) and (4.22)) A_{k0} and B_{k0} have to satisfy the following. For the vibration modes with $1 \leq k \leq 5$ A_{k0} and B_{k0} are still given by (4.24). For the 8th and the 9th vibration modes (with $k = 8$ and $k = 9$) A_{k0} and B_{k0} are given by (4.36). And, finally, for the vibration modes with $k = 6, 7$ and for the higher order vibration modes with $10 \leq k < \infty$ A_{k0} and B_{k0} have to satisfy (4.25). It can be seen from (4.36) that there are interactions between the 8st and the 9th vibration modes.

To check the stability the eigenvalues of the corresponding matrix M have to be calculated. It turns out that some of the eigenvalues have positive real parts, and that some of the solutions in this case are unstable.

The resonant case $\omega = \Omega_1^{(1)} + \Omega_8^{(2)}$: an additional sum type of resonance

The frequency of the belt velocity fluctuations ω is assumed to be equal to $\Omega_1^{(1)} + \Omega_8^{(2)}$. The equation $\omega \pm \Omega_n^{(2)} = \pm \Omega_k^{(1)}$ for $n = 6, 7, \dots$ and $k = 1, 2, \dots, 5$ has only a trivial solution $n = 8$ and $k = 1$ if $\omega - \Omega_n^{(2)} = \Omega_k^{(1)}$. Additionally it has to be checked whether the resonance conditions (i), (iii) and (iv) in (4.41) also have any solutions if $\omega = \Omega_1^{(1)} + \Omega_8^{(2)}$ or not. It turns out that there is one solution in case (iii) (see (4.23)): the equation $\omega \pm \Omega_n^{(1)} = \pm \Omega_k^{(2)}$ for $k = 6, 7, \dots$ and $n = 1, 2, \dots, 5$ has a solution $n = 1$ and $k = 8$ if $\omega - \Omega_n^{(1)} = \Omega_k^{(2)}$. Therefore, in order to eliminate secular terms in the solution of the $\mathcal{O}(\varepsilon)^{(1)}$ and $\mathcal{O}(\varepsilon)^{(2)}$ -problems (see (4.21) and (4.22)) A_{k0} and B_{k0} have to satisfy the following. For the 1st vibration mode (with $k = 1$) A_{k0} and B_{k0} are given by the first system of (4.37). For the vibration modes with $2 \leq k \leq 5$ A_{k0} and B_{k0} are still given by (4.24). For the 8th vibration mode (with $k = 8$) A_{k0} and B_{k0} are given by the second system of (4.37). And, finally for the vibration modes with $k = 6, 7$ and for the higher order vibration modes with $9 \leq k < \infty$ A_{k0} and B_{k0} have to satisfy (4.25). It can be seen from equations (4.37) that there are interactions between the 1st and the 8th vibration modes.

To check the stability the eigenvalues of the corresponding matrix M have to be calculated. It turns out that some of the eigenvalues have positive real parts, and that some the solutions in this case are unstable.

The resonant case $\omega = \Omega_7^{(1)} - \Omega_8^{(2)}$: an additional difference type of resonance

The frequency of the belt velocity fluctuations ω is assumed to be equal to $\Omega_7^{(1)} - \Omega_8^{(2)}$. The equation $\omega \pm \Omega_n^{(2)} = \pm \Omega_k^{(1)}$ for $n = 6, 7, \dots$ and $k = 1, 2, \dots, 5$ has no solution for k and n if $\omega + \Omega_n^{(2)} = \Omega_k^{(1)}$. Additionally it has to be checked whether the resonance conditions (i), (iii) and (iv) in (4.41) also have solutions if $\omega = \Omega_7^{(1)} - \Omega_8^{(2)}$ or not. It turns out that there are no such k and n . Therefore, in order to eliminate secular terms in the solution of the $\mathcal{O}(\varepsilon)^{(1)}$ and $\mathcal{O}(\varepsilon)^{(2)}$ -problems (see (4.21) and (4.22)) A_{k0} and B_{k0} have to satisfy the following. For the vibration modes with $1 \leq k \leq 5$ A_{k0} and B_{k0} are still given by (4.24). And for the higher order vibration modes with $6 \leq k < \infty$ A_{k0} and B_{k0} have to satisfy (4.25).

To check the stability the eigenvalues of the corresponding matrix M have to be calculated. It turns out that all eigenvalues are purely imaginary or zeros, and that all solution in this case are bounded and stable.

The resonant case $\omega = \Omega_7^{(1)} + \Omega_8^{(2)}$: an additional sum type of resonance

The frequency of the belt velocity fluctuations ω is assumed to be equal to $\Omega_7^{(1)} + \Omega_8^{(2)}$. The equation $\omega \pm \Omega_n^{(2)} = \pm \Omega_k^{(1)}$ for $n = 6, 7, \dots$ and $k = 1, 2, \dots, 5$ has no solution for n and k . Additionally it has to be checked whether the resonance conditions (i), (iii) and (iv) in (4.41) also have solutions if $\omega = \Omega_7^{(1)} + \Omega_8^{(2)}$ or not. It turns out that there are solutions in case (iv) (see (4.41)): $n = 20$ and $k = 27$ if $\omega + \Omega_n^{(2)} = \Omega_k^{(2)}$ and $n = 27$ and $k = 20$ if $\omega - \Omega_n^{(2)} = -\Omega_k^{(2)}$ (then $|\omega \pm \Omega_n^{(2)} \pm \Omega_k^{(2)}| = 0.0012 < \varepsilon$). This can be seen as a detuning case if $\omega = \Omega_7^{(1)} + \Omega_8^{(2)}$. Therefore, in order to eliminate secular terms in the solution of the $\mathcal{O}(\varepsilon)^{(1)}$ and $\mathcal{O}(\varepsilon)^{(2)}$ -problems (see (4.21) and (4.22)) A_{k0} and B_{k0} have to satisfy the following. For the vibration modes with $1 \leq k \leq 5$ A_{k0} and B_{k0} are still given by (4.24). For the vibration modes with $6 \leq k \leq 19$ and $21 \leq k \leq 26$ A_{k0} and B_{k0} are still given by (4.25). For the 20th and the 27th vibration modes (with $k = 20$ and $k = 27$) A_{k0} and B_{k0} are given by (4.40). And for the higher order vibration modes with $28 \leq k < \infty$ A_{k0} and B_{k0} have to satisfy (4.25). It can be seen from equations (4.40) that there are interactions between the 20th and the 27th vibration modes.

To check the stability the eigenvalues of the corresponding matrix M have to be calculated. It turns out that all eigenvalues are purely imaginary or zeros, and that all solution in this case are bounded and stable.

The resonant case $\omega = 2\Omega_8^{(2)}$: a principal parametric resonance

The frequency of the belt velocity fluctuations ω is assumed to be equal to $2\Omega_8^{(2)}$. The equation $\omega \pm \Omega_n^{(2)} = \pm \Omega_k^{(2)}$ for $n = 6, 7, \dots$ and $k = 6, 7, \dots$ has no solutions. Additionally it has to be checked whether the resonance conditions (i) – (iii) in (4.41) also have solutions if $\omega = 2\Omega_8^{(2)}$ or not. It can be verified that there are no such k and n . Therefore, in order to eliminate secular terms in the solution of the $\mathcal{O}(\varepsilon)^{(1)}$ and $\mathcal{O}(\varepsilon)^{(1)}$ -problems (see (4.21) and (4.22)) A_{k0} and B_{k0} have to satisfy (the same as for the non-resonant case): (4.24) for $1 \leq k \leq 5$ and (4.25) for $6 \leq k < \infty$. It has been already shown in [18] that in case of principal parametric resonance the possibility to have solutions (interactions between vibration modes) for the case (iv) of (4.41) depends on value of μ . It was shown here that in particular case $\mu = 0.002$ and relative error of 3% (as it was fixed in the beginning) $\omega = 2\Omega_8^{(2)}$ is not a resonance frequency in that case as it also was not in Section 4.4.

To check the stability the eigenvalues of the corresponding matrix M have to be calculated. It turns out that all eigenvalues are purely imaginary or zeros, and that all solution in this case are bounded and stable.

4.6 Application of the method with a relative error of 1%

In this section the non-resonant and some resonant cases will be studied when the relative error in the frequencies is at most 1%. From Table 4.1 it can be seen that the original problem (4.5) can now be split up (with an error of 1%) into two models. The 'string' model (see (4.12)) for $1 \leq k \leq 3$ and the 'stretched beam' model (see (4.13)) for $4 \leq k < \infty$. From (4.21) and (4.22) it can readily be seen that there are infinitely many values of ω that can give rise to internal resonances. If the relative error is 1% these values are (in an $\mathcal{O}(\varepsilon)$ neighbourhood of):

$$\begin{aligned}
 (i) \quad & \omega \pm \Omega_n^{(1)} = \pm \Omega_k^{(1)}, \text{ for } n, k = 1, 2, 3, \\
 (ii) \quad & \omega \pm \Omega_n^{(2)} = \pm \Omega_k^{(1)}, \text{ for } k = 1, 2, 3, \text{ and } n = 4, 5, \dots, \\
 (iii) \quad & \omega \pm \Omega_n^{(1)} = \pm \Omega_k^{(2)}, \text{ for } n = 1, 2, 3, \text{ and } k = 4, 5, \dots, \\
 (iv) \quad & \omega \pm \Omega_n^{(2)} = \pm \Omega_k^{(2)}, \text{ for } n = 4, 5, \dots, \text{ and } k = 4, 5, \dots.
 \end{aligned} \tag{4.43}$$

4.6.1 The non-resonant case

In this case it is assumed that the frequency ω of the velocity-fluctuations of the axially moving continuum is not equal to any combination of the resonance

frequencies as listed in (4.43). To eliminate the secular terms in the solution of the $\mathcal{O}(\varepsilon)^{(1)}$ -problem and the $\mathcal{O}(\varepsilon)^{(2)}$ -problem it follows that A_{k0} and B_{k0} have to satisfy (4.24) for $1 \leq k \leq 3$ and (4.25) for $4 \leq k < \infty$.

In this case system (4.24) can again be seen as some sort of slow time (t_1) correction in the string model frequency for the first three vibration modes due to presence of the small bending stiffness term in the $\mathcal{O}(\varepsilon)^{(1)}$ -problem.

4.6.2 Some resonant cases

The same resonant cases (see (4.31)) as in Section 4.4 will be investigated in this subsection.

The resonant case $\omega = m^*$, where m^* is equal to 1, 3, 5 or 7

First, the case $\omega = m^* = 1$ (the first “string” resonance frequency) will be studied. It was shown in [16] that the case (i) in (4.23), i.e. $\omega \pm \Omega_n^{(1)} = \pm \Omega_k^{(1)}$, with $n, k = 1, 2, 3$ and $\omega = 1$ has solution: $k = n + 1$ and $k = n - 1$. Additionally it has to be checked whether the resonance conditions (ii) – (iv) in (4.43) also have solutions if $\omega = 1$ or not. It can readily be verified that there are no such k and n . Therefore, in order to eliminate secular terms in the solution of the $\mathcal{O}(\varepsilon)^{(1)}$ -problem (see (4.21)) A_{k0} and B_{k0} have to satisfy (4.32) for $1 \leq k \leq 3$ and (4.25) for $4 \leq k < \infty$. It is clear from system (4.32) and from system (4.25) that there are interactions between the first four modes. For the higher order vibration modes (i.e. for the 5th vibration mode and higher) there are no such interactions. Therefore there is no problem with applying the truncation method as it was in [16]. For the case $\omega = m^*$ it is assumed that m^* is equal to 3, 5 or 7 ($\omega = m^*$ is a resonance frequency for the string model). Following the same procedure as for the case $\omega = 1$, A_{k0} and B_{k0} for $m^* = 3, 5$ have to satisfy (4.33) for $1 \leq k \leq 3$ and (4.25) for the higher order modes ($4 \leq k < \infty$). It can be seen from (4.33) and (4.25) that there are interactions between first $(3 + m^*)$ modes. For the higher order modes there are no such interactions. For $m^* = 7$ $\omega = m^*$ is not a resonance frequency in this case.

To check the stability of the solution the following approach can be used. From (4.33) it follows that, in general, the equation $\dot{X} = MX$ has to be solved, where vector $\dot{X}$ contains unknown $\dot{A}_{k0}(t_1)$ and $\dot{B}_{k0}(t_1)$, vector X contains $A_{k0}(t_1)$ and $B_{k0}(t_1)$, and M is a corresponding square matrix of the size $(3 + m^*)$. The eigenvalues of the matrix M define the stability properties of the system. It was found that for all m^* (that is, for $m^* = 1, 3, 5$, and 7) all eigenvalues of the matrix M are purely imaginary or zeros. That means that the solution is bounded and uniformly stable.

The resonant case $\omega = \Omega_9^{(2)} - \Omega_8^{(2)}$: a difference type of resonance

The frequency of the belt velocity fluctuations ω is assumed to be equal to $\Omega_9^{(2)} - \Omega_8^{(2)}$. Equation $\omega \pm \Omega_n^{(2)} = \pm \Omega_k^{(2)}$ for $n = 4, 5, \dots$ and $k = 4, 5, \dots$ has only trivial solutions $n = 8$ and $k = 9$ if $\omega + \Omega_n^{(2)} = \Omega_k^{(2)}$ and $n = 9$ and $k = 8$ if $\omega + \Omega_n^{(2)} = \Omega_k^{(2)}$. Therefore A_{k0} and B_{k0} for the 8th and the 9th vibration modes ($k = 8$ and $k = 9$) still have to satisfy (4.34). Additionally it has to be checked whether the resonance conditions (i) – (iii) in (4.43) also have solutions if $\omega = \Omega_9^{(2)} - \Omega_8^{(2)}$ or not. It can be verified that there are no such k and n . Therefore, in order to avoid secular terms in the solutions of the $\mathcal{O}(\varepsilon)^{(1)}$ and $\mathcal{O}(\varepsilon)^{(2)}$ -problems if $\omega = \Omega_9^{(2)} - \Omega_8^{(2)}$ the following results were found. For vibration modes with $1 \leq k \leq 3$ A_{k0} and B_{k0} are still given by (4.24). This can be seen as an influence of the small bending stiffness term in the $\mathcal{O}(\varepsilon)^{(1)}$ -problem. For the 8th and the 9th vibration modes A_{k0} and B_{k0} have to satisfy (4.34). And for the vibration modes with $4 \leq k \leq 7$ and $10 \leq k < \infty$ A_{k0} and B_{k0} have to satisfy (4.25). It can be seen from (4.34) that there are interactions between the 8th and the 9th vibration modes.

To check the stability the eigenvalues of the corresponding matrix M have to be calculated. It turns out that all eigenvalues are purely imaginary or zeros, and that all solution in this case are bounded and stable.

The resonant case $\omega = \Omega_1^{(1)} - \Omega_8^{(2)}$: an additional difference type of resonance

The frequency of the belt velocity fluctuations ω is assumed to be equal to $\Omega_1^{(1)} - \Omega_8^{(2)}$. The equation $\omega \pm \Omega_n^{(2)} = \pm \Omega_k^{(1)}$ for $n = 4, 6, \dots$ and $k = 1, 2, 3$ has only a trivial solution for $n = 8$ and $k = 1$. Additionally it has to be checked whether the resonance conditions (i), (iii) and (iv) in (4.43) also have any solutions if $\omega = \Omega_1^{(1)} - \Omega_8^{(2)}$ or not. It turns out that there is one solution in case (iii) (see (4.23)) : $n = 1$ and $k = 8$ if $\omega - \Omega_n^{(1)} = -\Omega_k^{(2)}$. Therefore, in order to avoid secular terms in the solutions of the $\mathcal{O}(\varepsilon)^{(1)}$ and $\mathcal{O}(\varepsilon)^{(2)}$ -problems if $\omega = \Omega_1^{(1)} - \Omega_8^{(2)}$ the following results were found. For the 1st vibration mode (with $k = 1$) A_{k0} and B_{k0} are given by the first system of (4.35). For the vibration modes with $2 \leq k \leq 3$ A_{k0} and B_{k0} are still given by (4.24). For the 8th vibration mode (with $k = 8$) A_{k0} and B_{k0} are given by (4.35). And, finally, for $4 \leq k \leq 7$ and for the higher order vibration modes with $9 \leq k < \infty$ A_{k0} and B_{k0} have to satisfy (4.25). It can be seen from equations (4.35) that there are interactions between the 1st and the 8th vibration modes in this case.

To check the stability the eigenvalues of the corresponding matrix M have

to be calculated. It turns out that all eigenvalues are purely imaginary or zeros, and that all solution in this case are bounded and stable.

The resonant case $\omega = \Omega_9^{(2)} + \Omega_8^{(2)}$: a sum type of resonance

The frequency of the belt velocity fluctuations ω is assumed to be equal to $(\Omega_9^{(2)} + \Omega_8^{(2)})$. The equation $\omega \pm \Omega_n^{(2)} = \pm \Omega_k^{(2)}$ for $n = 4, 5, \dots$ and $k = 4, 5, \dots$ has only trivial solutions: $n = 8$ and $k = 9$, and symmetrically $n = 9$ and $k = 8$ if $\omega - \Omega_n^{(2)} = \Omega_k^{(2)}$. Additionally it has to be checked whether the resonance conditions (i) – (iii) in (4.43) also have solutions if $\omega = \Omega_9^{(2)} + \Omega_8^{(2)}$ or not. It can be verified that there are no such k and n . Therefore, in order to eliminate secular terms in the solution of the $\mathcal{O}(\varepsilon)^{(1)}$ and $\mathcal{O}(\varepsilon)^{(2)}$ -problems (see (4.21) and (4.22)) A_{k0} and B_{k0} have to satisfy the following. For the vibration modes with $1 \leq k \leq 3$ A_{k0} and B_{k0} are still given by (4.24). For the 8th and the 9th vibration mode (with $k = 8$ and $k = 9$) A_{k0} and B_{k0} are given by (4.36). And, finally, for $4 \leq k \leq 7$ and for the higher order vibration modes with $10 \leq k < \infty$ A_{k0} and B_{k0} have to satisfy (4.25). It can be seen from (4.36) that there are interactions between the 8st and the 9th vibration modes.

To check the stability the eigenvalues of the corresponding matrix M have to be calculated. It turns out that some of the eigenvalues have positive real parts, and that some of the solutions in this case are unstable.

The resonant case $\omega = \Omega_1^{(1)} + \Omega_8^{(2)}$: an additional sum type of resonance

The frequency of the belt velocity fluctuations ω is assumed to be equal to $\Omega_1^{(1)} + \Omega_8^{(2)}$. The equation $\omega \pm \Omega_n^{(2)} = \pm \Omega_k^{(1)}$ for $n = 4, 5, \dots$ and $k = 1, 2, 3$ has only a trivial solution $n = 8$ and $k = 1$ if $\omega - \Omega_n^{(2)} = \Omega_k^{(1)}$. Additionally it has to be checked whether the resonance conditions (i), (iii) and (iv) in (4.43) also have any solutions if $\omega = \Omega_1^{(1)} + \Omega_8^{(2)}$ or not. It can be verified that there is one solution for (iii) case : the equation $\omega \pm \Omega_n^{(1)} = \pm \Omega_k^{(2)}$ for $k = 4, 5, \dots$ and $n = 1, 2, 3$ has a solution $n = 1$ and $k = 8$ if $\omega - \Omega_n^{(1)} = \Omega_k^{(2)}$. Therefore, in order to eliminate secular terms in the solution of the $\mathcal{O}(\varepsilon)^{(1)}$ and $\mathcal{O}(\varepsilon)^{(2)}$ -problems (see (4.21) and (4.22)) A_{k0} and B_{k0} have to satisfy the following. For the 1st vibration mode (with $k = 1$) A_{k0} and B_{k0} are given by the first system of (4.37). For the vibration modes with $2 \leq k \leq 3$ A_{k0} and B_{k0} are still given by (4.24). For the 8th vibration mode (with $k = 8$) A_{k0} and B_{k0} are given by the second system of (4.37). And, finally for the vibration modes with $4 \leq k \leq 7$ and for the higher order vibration modes with $9 \leq k < \infty$

A_{k0} and B_{k0} have to satisfy (4.25). It can be seen from (4.37) that there are interactions between the 1st and the 8th vibration modes.

To check the stability the eigenvalues of the corresponding matrix M have to be calculated. It turns out that some of the eigenvalues have positive real parts, and that some of the solutions in this case are unstable.

The resonant case $\omega = \Omega_7^{(1)} - \Omega_8^{(2)}$: an additional difference type of resonance

The frequency of the belt velocity fluctuations ω is assumed to be equal to $\Omega_7^{(1)} - \Omega_8^{(2)}$. The equation $\omega \pm \Omega_n^{(2)} = \pm \Omega_k^{(1)}$ for $n = 4, 5, \dots$ and $k = 1, 2, 3$ has no solution for k and n . Additionally it has to be checked whether the resonance conditions (i), (iii) and (iv) in (4.43) also have solutions if $\omega = \Omega_7^{(1)} - \Omega_8^{(2)}$ or not. It can be verified that there is no such k and n . Therefore, in order to eliminate secular terms in the solution of the $\mathcal{O}(\varepsilon)^{(1)}$ and $\mathcal{O}(\varepsilon)^{(2)}$ -problems (see (4.21) and (4.22)) A_{k0} and B_{k0} have to satisfy the following. For the vibration modes with $1 \leq k \leq 3$ A_{k0} and B_{k0} are still given by (4.24). And for the higher order vibration modes with $4 \leq k < \infty$ A_{k0} and B_{k0} have to satisfy (4.25).

To check the stability the eigenvalues of the corresponding matrix M have to be calculated. It turns out that all eigenvalues are purely imaginary or zeros, and that all solution in this case are bounded and stable.

The resonant case $\omega = \Omega_7^{(1)} + \Omega_8^{(2)}$: an additional sum type of resonance

The frequency of the belt velocity fluctuations ω is assumed to be equal to $\Omega_7^{(1)} + \Omega_8^{(2)}$. The equation $\omega \pm \Omega_n^{(2)} = \pm \Omega_k^{(1)}$ for $n = 4, 5, \dots$ and $k = 1, 2, 3$ has no solution for n and k . Additionally it has to be checked whether the resonance conditions (i), (iii) and (iv) in (4.43) also have solutions if $\omega = \Omega_7^{(1)} + \Omega_8^{(2)}$ or not. It turns out that for the case (iv) in (4.43) there are solutions: $n = 20$ and $k = 27$ if $\omega + \Omega_n^{(2)} = \Omega_k^{(2)}$ and $n = 27$ and $k = 20$ if $\omega - \Omega_n^{(2)} = -\Omega_k^{(2)}$. In this case $|\omega \pm \Omega_n^{(2)} \pm \Omega_k^{(2)}| = 0.0012 < \varepsilon$. This can be seen as a detuning case if $\omega = \Omega_7^{(1)} + \Omega_8^{(2)}$. Therefore, in order to eliminate secular terms in the solution of the $\mathcal{O}(\varepsilon)^{(1)}$ and $\mathcal{O}(\varepsilon)^{(2)}$ -problems (see (4.21) and (4.22)) A_{k0} and B_{k0} have to satisfy the following. For the vibration modes with $1 \leq k \leq 3$ A_{k0} and B_{k0} are still given by (4.24). For the vibration modes with $4 \leq k \leq 19$ and $21 \leq k \leq 26$ A_{k0} and B_{k0} are still given by (4.25). For the 20th and the 27th vibration modes (with $k = 20$ and $k = 27$) A_{k0} and B_{k0} are given by (4.40). And for the higher order vibration modes with $28 \leq k < \infty$ A_{k0} and

B_{k0} have to satisfy (4.25). It can be seen from equations (4.40) that there are interactions between the 20th and the 27th vibration modes.

To check the stability the eigenvalues of the corresponding matrix M have to be calculated. It turns out that all eigenvalues are purely imaginary or zeros, and that all solution in this case are bounded and stable.

The resonant case $\omega = 2\Omega_8^{(2)}$: a principal parametric resonance

The frequency of the belt velocity fluctuations ω is assumed to be equal to $2\Omega_8^{(2)}$. The equation $\omega \pm \Omega_n^{(2)} = \pm \Omega_k^{(2)}$ for $n = 4, 5, \dots$ and $k = 4, 5, \dots$ has no solutions. Additionally it has to be checked whether the resonance conditions (i) – (iii) in (4.43) also have any solutions if $\omega = 2\Omega_8^{(2)}$ or not. It turns out that there are no such k and n . Therefore, in order to eliminate secular terms in the solution of the $\mathcal{O}(\varepsilon)^{(1)}$ and $\mathcal{O}(\varepsilon)^{(1)}$ -problems (see (4.21) and (4.22)) A_{k0} and B_{k0} have to satisfy (the same as for the non-resonant case): (4.24) for $1 \leq k \leq 3$ and (4.25) for $4 \leq k < \infty$. It has been already shown in [18] that in case of principal parametric resonance the possibility to have solutions (interactions between vibration modes) for the case (iv) of (4.41) depends on value of μ . It was shown here that in particular case $\mu = 0.002$ and relative error of 1% (as it was fixed in the beginning) $\omega = 2\Omega_8^{(2)}$ is not a resonance frequency in this case as it was also not in Sections 4.4 and 4.5.

To check the stability the eigenvalues of the corresponding matrix M have to be calculated. It turns out that all eigenvalues are purely imaginary or zeros, and that all solution in this case are bounded and stable.

4.7 Application of the method with a relative error of 0.1%

In this section the non-resonant and some resonant cases will be studied when the relative error in the frequencies is at most 0.1%. From Table 4.1 it can be seen that the original problem (4.5) can now be split up (with an error of 0.1%) into two models. The 'string' model (see (4.12)) for $k = 1$ and the 'stretched beam' model (see (4.13)) for $2 \leq k < \infty$. From (4.21) and (4.22) it can readily be seen that there are infinitely many values of ω that can give rise to internal resonances. If the relative error is 0.1% these values are (in an

$\mathcal{O}(\varepsilon)$ neighbourhood of):

$$\begin{aligned}
(i) \quad & \omega \pm \Omega_n^{(1)} = \pm \Omega_k^{(1)}, \text{ for } n, k = 1, \\
(ii) \quad & \omega \pm \Omega_n^{(2)} = \pm \Omega_k^{(1)}, \text{ for } k = 1, \text{ and } n = 2, 3, \dots, \\
(iii) \quad & \omega \pm \Omega_n^{(1)} = \pm \Omega_k^{(2)}, \text{ for } n = 1, \text{ and } k = 2, 3, \dots, \\
(iv) \quad & \omega \pm \Omega_n^{(2)} = \pm \Omega_k^{(2)}, \text{ for } n = 2, 3, \dots, \text{ and } k = 2, 3, \dots.
\end{aligned} \tag{4.44}$$

4.7.1 The non-resonant case

In this case it is assumed that the frequency ω of the velocity-fluctuations of the axially moving continuum is not equal to any combination of the resonance frequencies as listed in (4.44). To eliminate the secular terms in the solution of the $\mathcal{O}(\varepsilon)^{(1)}$ -problem and the $\mathcal{O}(\varepsilon)^{(2)}$ -problem it follows that A_{k0} and B_{k0} have to satisfy (4.24) for $k = 1$ and (4.25) for $2 \leq k < \infty$. In this case system (4.24) can again be seen as some sort of slow time (t_1) correction in the string model frequency for the first vibration mode due to presence of the small bending stiffness term in the $\mathcal{O}(\varepsilon)^{(1)}$ -problem.

4.7.2 Some resonant cases

The same resonant cases (see (4.31)) as in Section 4.4 will be investigated in this subsection.

The resonant case $\omega = m^*$, where m^* is equal to 1, 3, 5 or 7

First, the case $\omega = m^* = 1$ (the first “string” resonance frequency) has to be studied. Equation $\omega \pm \Omega_n^{(1)} = \pm \Omega_k^{(1)}$, with $n, k = 1$ and $\omega = 1$ has no solution for n and k . Additionally it has to be checked whether the resonance conditions (ii) – (iv) in (4.44) also have solutions if $\omega = 1$ or not. It can be verified that there are no such k and n . Therefore, in order to eliminate secular terms in the solution of the $\mathcal{O}(\varepsilon)^{(1)}$ -problem (see (4.21)) A_{k0} and B_{k0} have to satisfy (4.24) for $k = 1$ and (4.25) for $2 \leq k < \infty$. For the case $\omega = m^*$ it is assumed that m^* is equal to 3, 5 or 7 ($\omega = m^*$ is a resonance frequency from the string model). Following the same procedure as for the case $\omega = 1$, A_{k0} and B_{k0} have to satisfy (4.24) for $k = 1$ and (4.25) for the higher order modes ($2 \leq k < \infty$). There are no interactions in that case.

To check the stability the eigenvalues of the corresponding matrix M have to be calculated. It turns out that all eigenvalues are purely imaginary or zeros, and that all solution in this case are bounded and stable.

The resonant case $\omega = \Omega_9^{(2)} - \Omega_8^{(2)}$: a difference type of resonance

The frequency of the belt velocity fluctuations ω is assumed to be equal to $\Omega_9^{(2)} - \Omega_8^{(2)}$. Equation $\omega \pm \Omega_n^{(2)} = \pm \Omega_k^{(2)}$ for $n = 2, 3, \dots$ and $k = 2, 3, \dots$ has only trivial solutions $n = 8$ and $k = 9$ if $\omega + \Omega_n^{(2)} = \Omega_k^{(2)}$, and $n = 9$ and $k = 8$ if $\omega + \Omega_n^{(2)} = \Omega_k^{(2)}$. Therefore A_{k0} and B_{k0} for the 8th and the 9th vibration modes ($k = 8$ and $k = 9$) still have to satisfy equations (4.34). Additionally it has to be checked whether the resonance conditions (i) – (iii) in (4.44) also have any solutions if $\omega = \Omega_9^{(2)} - \Omega_8^{(2)}$ or not. It turns out that there are no such k and n . Therefore, in order to avoid secular terms in the solutions of the $\mathcal{O}(\varepsilon)^{(1)}$ and $\mathcal{O}(\varepsilon)^{(2)}$ -problems if $\omega = \Omega_9^{(2)} - \Omega_8^{(2)}$ the following results were found. For the first vibration mode with $k = 1$ A_{k0} and B_{k0} are still given by (4.24). This can be seen as an influence of the small bending stiffness term in the $\mathcal{O}(\varepsilon)^{(1)}$ -problem. For the 8th and the 9th vibration modes A_{k0} and B_{k0} have to satisfy (4.34). And for the vibration modes with $2 \leq k \leq 7$ and $10 \leq k < \infty$ A_{k0} and B_{k0} have to satisfy (4.25). It can be seen from equations (4.34) that there are interactions between the 8th and the 9th vibration modes.

To check the stability the eigenvalues of the corresponding matrix M have to be calculated. It turns out that all eigenvalues are purely imaginary or zeros, and that all solution in this case are bounded and stable.

The resonant case $\omega = \Omega_1^{(1)} - \Omega_8^{(2)}$: an additional difference type of resonance

The frequency of the belt velocity fluctuations ω is assumed to be equal to $\Omega_1^{(1)} - \Omega_8^{(2)}$. The equation $\omega \pm \Omega_n^{(2)} = \pm \Omega_k^{(1)}$ for $n = 2, 3, \dots$ and $k = 1$ has only a trivial solution $n = 8$ and $k = 1$ if $\omega + \Omega_n^{(2)} = \Omega_k^{(1)}$. Additionally it has to be checked whether the resonance conditions (i), (iii) and (iv) in (4.44) also have solutions if $\omega = \Omega_1^{(1)} - \Omega_8^{(2)}$ or not. It turns out that there is one solution in case (iii) (see (4.23)) : $n = 1$ and $k = 8$ if $\omega - \Omega_n^{(1)} = -\Omega_k^{(2)}$. Therefore, in order to avoid secular terms in the solutions of the $\mathcal{O}(\varepsilon)^{(1)}$ and $\mathcal{O}(\varepsilon)^{(2)}$ -problems if $\omega = \Omega_1^{(1)} - \Omega_8^{(2)}$ the following results were found. For the 1st vibration mode (with $k = 1$) A_{k0} and B_{k0} are given by the first system of (4.35). For the 8th vibration mode (with $k = 8$) A_{k0} and B_{k0} are given by (4.35). And, finally, for the vibration modes with $2 \leq k \leq 7$ and for the higher order vibration modes with $9 \leq k < \infty$ A_{k0} and B_{k0} have to satisfy (4.25). It can be seen from (4.35) that there are interactions between the 1st and the 8th vibration modes in this case.

To check the stability the eigenvalues of the corresponding matrix M have

to be calculated. It turns out that all eigenvalues are purely imaginary or zeros, and that all solution in this case are bounded and stable.

The resonant case $\omega = \Omega_9^{(2)} + \Omega_8^{(2)}$: a sum type of resonance

The frequency of the belt velocity fluctuations ω is assumed to be equal to $\Omega_9^{(2)} + \Omega_8^{(2)}$. The equation $\omega \pm \Omega_n^{(2)} = \pm \Omega_k^{(2)}$ for $n = 2, 3, \dots$ and $k = 2, 3, \dots$ has only trivial solutions: $n = 8$ and $k = 9$, and symmetrically $n = 9$ and $k = 8$ if $\omega - \Omega_n^{(2)} = \Omega_k^{(2)}$. Additionally it has to be checked whether the resonance conditions (i) – (iii) in (4.44) also have solutions if $\omega = \Omega_9^{(2)} + \Omega_8^{(2)}$ or not. It can be verified that there are no such k and n . Therefore, in order to eliminate secular terms in the solution of the $\mathcal{O}(\varepsilon)^{(1)}$ and $\mathcal{O}(\varepsilon)^{(2)}$ -problems (see (4.21) and (4.22)) A_{k0} and B_{k0} have to satisfy the following. For the first vibration mode with $k = 1$ A_{k0} and B_{k0} are still given by (4.24). For the 8th and the 9th vibration modes (with $k = 8$ and $k = 9$) A_{k0} and B_{k0} are given by (4.36). And, finally, for the vibration modes with $2 \leq k \leq 7$ and for the higher order vibration modes with $10 \leq k < \infty$ A_{k0} and B_{k0} have to satisfy (4.25). It can be seen from (4.36) that there are interactions between the 8st and the 9th vibration modes.

To check the stability the eigenvalues of the corresponding matrix M have to be calculated. It turns out that some of the eigenvalues have positive real parts, and that some of the solutions in this case are unstable.

The resonant case $\omega = \Omega_1^{(1)} + \Omega_8^{(2)}$: an additional sum type of resonance

The frequency of the belt velocity fluctuations ω is assumed to be equal to $\Omega_1^{(1)} + \Omega_8^{(2)}$. The equation $\omega \pm \Omega_n^{(2)} = \pm \Omega_k^{(1)}$ for $n = 2, 3, \dots$ and $k = 1$ has only a trivial solution $n = 8$ and $k = 1$ if $\omega - \Omega_n^{(2)} = \Omega_k^{(1)}$. Additionally it has to be checked whether the resonance conditions (i), (iii) and (iv) in (4.44) also have solutions if $\omega = \Omega_1^{(1)} + \Omega_8^{(2)}$ or not. It turns out that there is one solution in case (iii) (see (4.23)) : the equation $\omega \pm \Omega_n^{(1)} = \pm \Omega_k^{(2)}$ for $k = 2, 3, \dots$ and $n = 1$ has a solution $n = 1$ and $k = 8$ if $\omega - \Omega_n^{(1)} = \Omega_k^{(2)}$. Therefore, in order to eliminate secular terms in the solution of the $\mathcal{O}(\varepsilon)^{(1)}$ and $\mathcal{O}(\varepsilon)^{(2)}$ -problems (see (4.21) and (4.22)) A_{k0} and B_{k0} have to satisfy the following. For the 1st vibration mode (with $k = 1$) A_{k0} and B_{k0} are given by the first system of (4.37). For the 8th vibration mode (with $k = 8$) A_{k0} and B_{k0} are given by the second system of (4.37). And, finally for the vibration modes with $2 \leq k \leq 7$ and for the higher order vibration modes with $9 \leq k < \infty$ A_{k0} and B_{k0} have to satisfy (4.25). It can be seen from (4.37) that there are interactions between

the 1st and the 8th vibration modes.

To check the stability the eigenvalues of the corresponding matrix M have to be calculated. It turns out that some of the eigenvalues have positive real parts, and that some of the solutions in this case are unstable.

The resonant case $\omega = \Omega_7^{(1)} - \Omega_8^{(2)}$: an additional difference type of resonance

The frequency of the belt velocity fluctuations ω is assumed to be equal to $(\Omega_7^{(1)} - \Omega_8^{(2)})$. The equation $\omega \pm \Omega_n^{(2)} = \pm \Omega_k^{(1)}$ for $n = 2, 3, \dots$ and $k = 1$ has no solution for k and n . Additionally it has to be checked whether the resonance conditions (i), (iii) and (iv) in (4.44) also have solutions if $\omega = \Omega_7^{(1)} - \Omega_8^{(2)}$ or not. It can be verified that there is no such k and n . Therefore, in order to eliminate secular terms in the solution of the $\mathcal{O}(\varepsilon)^{(1)}$ and $\mathcal{O}(\varepsilon)^{(2)}$ -problems (see (4.21) and (4.22)) A_{k0} and B_{k0} have to satisfy the following. For the 1st vibration mode (with $k = 1$) A_{k0} and B_{k0} are still given by (4.24). And for the higher order vibration modes with $2 \leq k < \infty$ A_{k0} and B_{k0} have to satisfy (4.25).

To check the stability the eigenvalues of the corresponding matrix M have to be calculated. It turns out that all eigenvalues are purely imaginary or zeros, and that all solution in this case are bounded and stable.

The resonant case $\omega = \Omega_7^{(1)} + \Omega_8^{(2)}$: an additional sum type of resonance

The frequency of the belt velocity fluctuations ω is assumed to be equal to $\Omega_7^{(1)} + \Omega_8^{(2)}$. The equation $\omega \pm \Omega_n^{(2)} = \pm \Omega_k^{(1)}$ for $n = 2, 3, \dots$ and $k = 1$ has no solution for n and k . Additionally it has to be checked whether the resonance conditions (i), (iii) and (iv) in (4.43) also have solutions if $\omega = \Omega_7^{(1)} + \Omega_8^{(2)}$ or not. That was found that the case (iv) in (4.43) have two solutions: $n = 20$ and $k = 27$ if $\omega + \Omega_n^{(2)} = \Omega_k^{(1)}$, and $n = 27$ and $k = 20$ if $\omega - \Omega_n^{(2)} = -\Omega_k^{(1)}$. In this case $|\omega \pm \Omega_n^{(2)} \pm \Omega_k^{(1)}| = 0.0012 < \varepsilon$. This can be seen as a detuning case if $\omega = \Omega_7^{(1)} + \Omega_8^{(2)}$. Therefore, in order to eliminate secular terms in the solution of the $\mathcal{O}(\varepsilon)^{(1)}$ and $\mathcal{O}(\varepsilon)^{(2)}$ -problems (see (4.21) and (4.22)) A_{k0} and B_{k0} have to satisfy the following. For the 1st vibration mode (with $k = 1$) A_{k0} and B_{k0} are still given by (4.24). For the vibration modes with $2 \leq k \leq 19$ and $21 \leq k \leq 26$ A_{k0} and B_{k0} are still given by (4.25). For the 20th and the 27th vibration modes (with $k = 20$ and $k = 27$) A_{k0} and B_{k0} are given by (4.40). And for the higher order vibration modes with $28 \leq k < \infty$ A_{k0} and B_{k0} have to satisfy (4.25). It can be seen from (4.40) that there are interactions between

the 20th and the 27th vibration modes.

To check the stability the eigenvalues of the corresponding matrix M have to be calculated. It turns out that all eigenvalues are purely imaginary or zeros, and that all solution in this case are bounded and stable.

The resonant case $\omega = 2\Omega_8^{(2)}$: a principal parametric resonance

The frequency of the belt velocity fluctuations ω is assumed to be equal to $2\Omega_8^{(2)}$. The equation $\omega \pm \Omega_n^{(2)} = \pm \Omega_k^{(2)}$ for $n = 2, 3, \dots$ and $k = 2, 3, \dots$ has no solutions. Additionally it has to be checked whether the resonance conditions (i) – (iii) in (4.44) also have solutions if $\omega = 2\Omega_8^{(2)}$ or not. It turns out that there are no such k and n . Therefore, in order to eliminate secular terms in the solution of the $\mathcal{O}(\varepsilon)^{(1)}$ and $\mathcal{O}(\varepsilon)^{(1)}$ -problems (see (4.21) and (4.22)) A_{k0} and B_{k0} have to satisfy (the same as for the non-resonant case): (4.24) for $k = 1$ and (4.25) for $2 \leq k < \infty$. It has been already shown in [18] that in case of principal parametric resonance the possibility to have solutions (interactions between vibration modes) for the case (iv) of (4.44) depends on value of μ . It was shown here that in particular case $\mu = 0.002$ and relative error of 0.1% (as it was fixed in the beginning) $\omega = 2\Omega_8^{(2)}$ is not a resonance frequency in that case as it was not in sections 4.4, 4.5, and 4.6.

To check the stability the eigenvalues of the corresponding matrix M have to be calculated. It turns out that all eigenvalues are purely imaginary or zeros, and that all solution in this case are bounded and stable.

4.8 Stability properties of the system

To verify the method which was proposed in this chapter the following approach will be used. Stability properties of the solutions which were derived in Sections 4.4, 4.5, 4.6, and 4.7 will be compared with the stability properties of the exact solution of the problem (4.5) (the exact solution can be found in [18]). The exact solution can readily be found for the same resonance frequencies (4.31) and stability properties can also be checked. Table 4.2 represents stability properties for the cases with the relative errors in frequencies at most of 5%, 3%, 1%, 0.1%, and 0.0% (the exact solution of the problem (4.5)). It can be seen that the stability properties remain the same for almost all these cases and correspond to the stability properties of the exact solution of the problem. For the case $\omega = \Omega_7^{(1)} + \Omega_8^{(2)}$ (see Table 4.2) and relative error of at most 5% the solution is unstable. On the other hand, for the same frequency but with better accuracy (relative error of 3%, 1%, and 0.1%) the solution is stable. That corresponds also the exact solution of the problem. This happen because the

resonance frequency $\omega = \Omega_7^{(1)} + \Omega_8^{(2)}$ was derived from the combination of the string and the stretched beam models. For the case of the relative error of 5% (Section 4.4) value $\Omega_7^{(1)}$ is still in the region of the string model. For the other cases the region of the string model is smaller, so that this combination is not a resonance frequency anymore. It can also be seen from Table 4.2 that difference type of resonances are stable and sum type or resonances are unstable. That behaviour corresponds the stability properties of the exact solution which was found in [18].

relative error frequency ω	5%	3%	1%	0.1%	exact 0.0% (see[18])
$\omega = m^*$, $m^* = 1, 3, 5$ and 7	stable	stable	stable	stable	stable
$\omega = \Omega_9^{(2)} - \Omega_8^{(2)}$	stable	stable	stable	stable	stable
$\omega = \Omega_1^{(1)} - \Omega_8^{(2)}$	stable	stable	stable	stable	stable
$\omega = \Omega_9^{(2)} + \Omega_8^{(2)}$	unstable	unstable	unstable	unstable	unstable
$\omega = \Omega_1^{(1)} + \Omega_8^{(2)}$	unstable	unstable	unstable	unstable	unstable
$\omega = \Omega_7^{(1)} - \Omega_8^{(2)}$	stable	stable	stable	stable	stable
$\omega = \Omega_7^{(1)} + \Omega_8^{(2)}$	unstable	stable	stable	stable	stable
$\omega = 2\Omega_8^{(2)}$	stable	stable	stable	stable	stable

Table 4.2: Stability properties of the solutions.

4.9 Conclusions and remarks

In this chapter an initial-boundary value problem for a linear equation, describing an axially moving stretched beam has been studied. This equation can be used as a model for the transversal vibrations of a conveyor belt system. The axially moving belt is assumed to move in one direction with a non-constant speed $V(\tau)$, that is, $V(\tau) = \varepsilon(V_0 + \alpha \sin(\omega\tau))$, where $0 < \varepsilon \ll 1$,

and where V_0, α and ω are positive constants. For V_0 it is assumed that $V_0 > 0$ and $V_0 > |\alpha|$. A new model approach describing the transient “from string to beam” behaviour, based on the calculations of the natural frequencies has been proposed. The influence of the bending stiffness on the stability properties of the solution of the problem has been studied.

Depending on the natural frequencies the original problem is split up into two submodels: a string model for the lower frequencies and a stretched beam model for the higher frequencies. Each submodel has its own physical and mathematical properties. For the string model, for instance, the discussion on the applicability of the truncation method is not (mathematically) relevant anymore (see also [16]). In this combination model the submodels are interacting due to internal resonances, and the model equations depend on the frequencies and on the vibration mode numbers. The proposed model is a more realistic approach to describe the dynamical behaviour of a travelling continuum as the bending stiffness becomes more important for the higher order vibration modes. For the lower frequencies the bending stiffness can be neglected and the string equation can be used. The regions of applicability of the simplified models were found for different values of the bending stiffness parameter and for different values of the relative errors in the frequencies. It turns out that there are infinitely many values of ω that give rise to internal resonances in the axially moving belt system. In fact, that happens when ω is equal to any sum or difference combination of the natural frequencies of the string and (or) the stretched beam equations. In the non-resonant case it can clearly be seen that for the lower frequencies (when the string model is used) the perturbation approach leads to improvements in the frequencies taking into account the small bending stiffness. The formal approximations of the solution and the stability properties in some resonant cases have been determined for four different values of the relative error in the frequencies: 5%, 3%, 1% and 0.1%. It can be concluded that the properties of the studied problem remain the same and correspond to the properties of the exact solution of the problem (4.5) when the relative errors in the frequencies are less than 5 %. An important implication of the results as presented in Chapter 4 is that for these type of problems the use of only string-like models is not appropriate. To describe the dynamics of these type of conveyor belt problems correctly one has to include bending stiffness in the model (also when the bending stiffness is assumed to be small).

It has also to be noticed that the introduction of a damping term does not solve the truncation problem for the string-like equation (4.1), at least if the damping is assumed to be small and of order ε . The viscous damping or the structural damping can be taken into account for the partial differential equations (4.1) or (4.2), leading to the appearance of extra terms in these equations:

$u_t + Vu_x$, $u_{txx} + Vu_{xxx}$ or $u_{txxxx} + Vu_{xxxxx}$, respectively (for the moving frame of reference). If the damping in the problem is assumed to be small, that is of order ε , then after applying the two time-scales perturbation method to the equations, terms $u_t + Vu_x$, $u_{txx} + Vu_{xxx}$ or $u_{txxxx} + Vu_{xxxxx}$ will appear. At high frequencies the structural damping terms can not be considered to be small. The solutions of the $\mathcal{O}(1)$ -problems still can not be truncated in these cases and still has to be written in an infinite series representation from the mathematical point of view.

Regarding the future research, it will be interesting to study more complicated cases for the transversal and longitudinal motions of axially moving beams, including those cases for which the boundary conditions are such that energy inflow or outflow is possible through these boundaries. Numerical results (based on spectral element methods) for these type of problems can already be found in [44] and [45].

CHAPTER 5

On the transversal vibrations of a conveyor belt system induced by a stoppage of the system

Abstract: In this chapter the energy transfer between the longitudinal motion and the transversal motion of a conveyor belt system after stoppage will be studied. It will turn out that complicated internal resonances can occur leading to interesting energy transfers between longitudinal and transversal motions. For certain parameter values it will turn out that relatively high mode longitudinal vibrations can induce low mode transversal vibrations with relatively large amplitudes.

5.1 Introduction

Axially moving systems are present in a vast class of engineering problems which arise in industrial, civil, aerospace, mechanical, electronic, medical, and automotive applications. Serpentine belts, aerial cables, tram and train ways, oil pipelines, magnetic tapes, power transmission belts, band saw blades, chair lifts in skiing resorts, and even models of human DNA are examples of objects where axial transport of mass can be associated with transversal vibrations.

Investigating transverse vibrations of moving materials is a challenging subject which has been studied for many years by many researchers and still is of interest today. Transversal vibrations of a moving belt system were studied in [10] and [46] for the case with a constant belt velocity. It was found that

the natural frequencies depend on the axial velocity of the belt, and decrease with increasing belt velocity. It was also found that there are instabilities at certain critical belt velocities. In reality, however, the systems are exposed to accelerating and decelerating motions due to different sorts of imperfections of the moving belt systems such as roll eccentricities. This observation implies that the constant belt speed has approximately a harmonic perturbation. Transversal vibrations of a belt system moving with a time-varying velocity were studied first in [47] and later in [12]. A similar problem (with additional nonlinear terms) was considered in [14]. The weakly nonlinear transversal vibrations of a conveyor belt with a low and time-varying velocity were studied in [20], and [26]. Additionally, in [20] and [26] not only transversal vibrations but also longitudinal vibrations were taken into account. Even if the operating speed is normally maintained constant, the axial speed does vary considerably during the starting and stopping phases. In [48] the nonlinear vibrations of a traveling beam with a constant axial acceleration were considered. Lyapunov's method was used to determine a sufficient condition for asymptotic stability of a beam having a constant acceleration.

In Figure 5.1 a schematic model of a conveyor belt system that moves with a time-varying velocity is given. Also in Figure 5.1 the velocity components in transversal and longitudinal directions of a point P are shown. The equation of

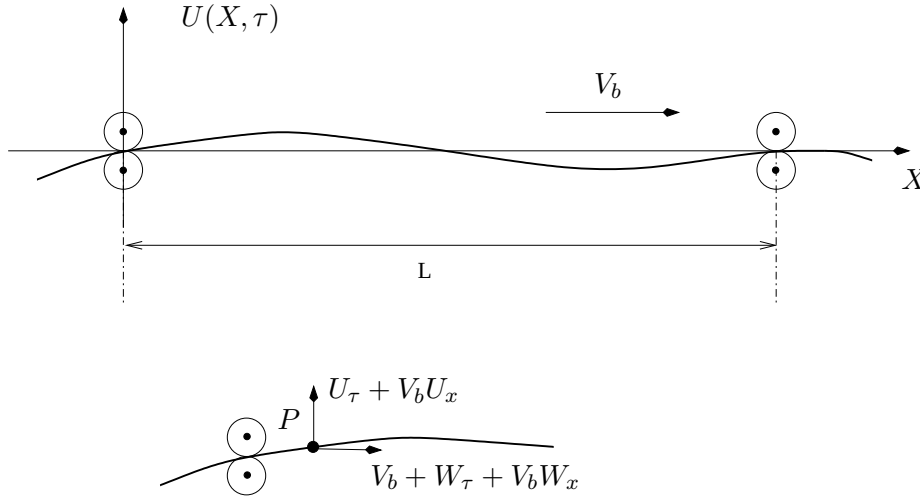


Figure 5.1: A traveling conveyor belt system with the velocity components at a point P.

motion of a conveyor belt system for the case of a constant belt velocity was for instance presented in [9] using Hamilton's principle. For the case of a low and time-varying velocity, namely, the following equations of motion describing the

transversal and longitudinal vibrations of a belt system were derived in [26] using Hamilton's principle:

$$\begin{aligned}
 \rho A W_{\tau\tau} + 2\rho A V_b W_{X\tau} + \rho A V_{b\tau}(1 + W_X) + (\rho A V_b^2 - EA)W_{XX} = \\
 (EA - R_0) \frac{(1 + W_X)U_X U_{XX} - U_X^2 W_{XX}}{[(1 + W_X)^2 + U_X^2]^{3/2}}, \\
 \rho A U_{\tau\tau} + 2\rho A V_b U_{X\tau} + \rho A V_{b\tau}U_X + (\rho A V_b^2 - EA)U_{XX} + EI U_{XXXX} = \\
 (R_0 - EA) \frac{(1 + W_X)^2 U_{XX} - (1 + W_X)U_X W_{XX}}{[(1 + W_X)^2 + U_X^2]^{3/2}},
 \end{aligned} \tag{5.1}$$

where

- $U(X, \tau)$: the transversal displacement of the belt,
- $W(X, \tau)$: the longitudinal displacement of the belt,
- $V_b(\tau)$: the time-varying belt speed,
- ρ : the mass density of the belt,
- A : the area of the cross section of the belt,
- E : the modulus of elasticity,
- I : the second moment of inertia with respect to the belt middle plane,
- X : the position along the horizontal axis,
- R_0 : the constant tension in a dynamic equilibrium,
- τ : the time, and
- L : the distance between the pulleys.

In some cases, when it can be assumed that the displacements in the longitudinal direction are much smaller than the displacements in the transversal directions, namely if $W = \mathcal{O}(U^2)$, the Kirchhoff approach can be applied, and the coupled system of nonlinear differential equations may be decoupled. This case has been studied in [20], assuming that the belt speed is low and time-varying, and is given by $V_b(\tau) = \varepsilon(V_0 + \alpha \sin(\omega\tau))$, where ε is a small parameter with $0 < \varepsilon \ll 1$, and where α, ω are some positive constants. The following ordering principle for the transversal and longitudinal displacements and for the belt velocity was assumed in that case : $U(X, \tau) = \mathcal{O}(\varepsilon)$, $W(X, \tau) = \mathcal{O}(\varepsilon^2)$, and $V_b(\tau) = \mathcal{O}(\varepsilon)$. Using these assumptions and Kirchhoff's approach the coupled system of partial differential equations (5.1) can be reduced to a single fourth order, weakly nonlinear beam equation for the transversal displacement $U(X, \tau)$. The dynamical behaviour of the conveyor belt system was studied in [20] for different values of the frequency ω in the harmonic perturbation of the belt velocity.

The main goal of this chapter is to study the dynamical behaviour of a conveyor belt system if the belt suddenly stops and (possible) "shock-waves"

occur, that is, an energy transfer from longitudinal motion to transversal motion occurs. A “shock-wave” is not a desirable state of a conveyor belt system after braking or stopping as it can lead to dropping off and (or) damaging the cargo a belt is transporting. To study and to predict these energy transfers from longitudinal to transversal motion is an important and challenging subject for engineers. This energy transfer will be studied in this chapter. Another interesting and challenging problem to study is the dynamic behaviour of the belt near the pulleys at τ_0 . This so-called boundary layer problem, however, will not be studied in this thesis. In Figure 5.2 the velocity of the conveyor belt system before and after the braking moment τ_0 is given. The belt speed

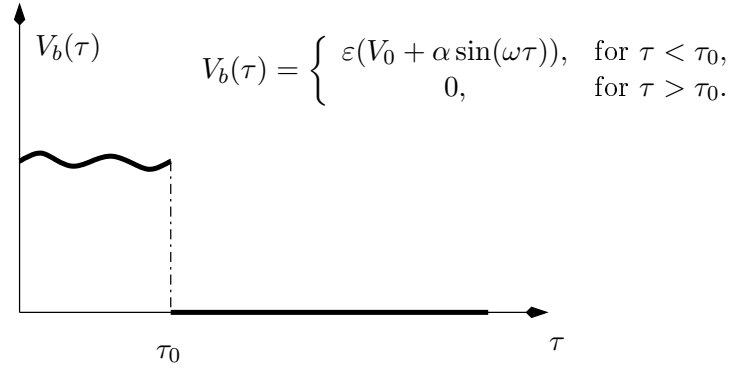


Figure 5.2: Velocity of a conveyor belt system before and after the braking moment τ_0 .

before the braking moment is assumed to be of order ε . After the moment when the belt stops, the velocity of the belt point P in longitudinal direction is now of order ε due to the inertia of the belt. The velocity of the belt point P in transversal direction is still of order ε as it was before the braking moment. In Figure 5.3 the velocity components of the belt point P in both directions are given. If the belt under consideration is long and rather elastic, the displacement of the belt in longitudinal direction is also of order ε for some time after time τ_0 .

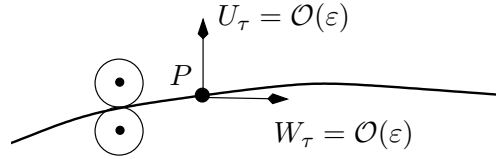


Figure 5.3: The velocity components of a conveyor belt point P after the braking moment τ_0 .

5.2 Formulation of the problem

To study the behaviour of the conveyer belt system after the braking moment and to study the (possible) occurrence of “shock wave(s)” the following assumptions are made:

1. Initially, before the braking moment, the conveyer belt system (see Figure 5.1) is moving with a low and time-varying velocity $V_b(\tau)$: $V_b(\tau) = \mathcal{O}(\varepsilon)$.
2. The belt stops simultaneously and instantly at both pulleys at the time τ_0 .
3. The belt is sufficiently long and elastic, so that after the braking moment τ_0 the velocity of most of the belt points in longitudinal direction, and the velocity of most of the belt points in transversal direction are of order ε , that is $W_\tau|_{\tau=\tau_0} = \mathcal{O}(\varepsilon)$, and $U_\tau = \mathcal{O}(\varepsilon)$.
4. Displacements of the belt in longitudinal and transversal directions are of the same order for some interval of time after the braking moment τ_0 , that is, $W = \mathcal{O}(U)$, and $U = \mathcal{O}(W)$.
5. The conveyer belt system is simply supported at the pulleys.

The 5th assumption was made only from the point of view to simplify the computations. If the conveyer belt system is simply supported at the edges, the boundary conditions allow to write the solution nicely in sine-series in x .

The equations of motion of the conveyer belt system after the braking moment τ_0 are given by

$$\begin{aligned} \rho A W_{\tau\tau} - E A W_{XX} &= (E A - R_0) \frac{(1 + W_X) U_X U_{XX} - U_X^2 W_{XX}}{[(1 + W_X)^2 + U_X^2]^{3/2}}, \\ \rho A U_{\tau\tau} - E A U_{XX} + E I U_{XXXX} &= (R_0 - E A) \frac{(1 + W_X)^2 U_{XX} - (1 + W_X) U_X W_{XX}}{[(1 + W_X)^2 + U_X^2]^{3/2}}. \end{aligned} \quad (5.2)$$

Using a Taylor series, the denominators in (5.2) can be approximated by

$$\begin{aligned} [(1 + W_X)^2 + U_X^2]^{-3/2} \\ = 1 - 3W_X + 6W_X^2 - \frac{3}{2}U_X^2 - 10W_X^3 + \frac{15}{2}W_X U_X^2 + \mathcal{O}(4). \end{aligned} \quad (5.3)$$

In approximation (5.3) $\mathcal{O}(4)$ stands for terms of degree 4 or higher. Assuming that the displacement in the transversal direction is of the same order as the displacement in the longitudinal direction it follows from (5.3) that $[(1 + W_X)^2 +$

$U_X^2]^{-3/2} \approx 1 - 3W_X + 6W_X^2 - \frac{3}{2}U_X^2$. Correspondingly, the equation of motions (5.2) can now be written approximately as

$$\begin{aligned} \rho A W_{\tau\tau} - E A W_{XX} &= (E A - R_0)(U_X U_{XX} - 2U_X U_{XX} W_X - U_X^2 W_{XX}), \\ \rho A U_{\tau\tau} - E A U_{XX} + E I U_{XXX} &= (R_0 - E A)(U_{XX} - U_{XX} W_X \\ &\quad + U_{XX} W_X^2 - \frac{3}{2} U_{XX} U_X^2 - U_X W_{XX} + 2U_X W_{XX} W_X). \end{aligned} \quad (5.4)$$

The following transformations are applied in order to rewrite the equations of motion (5.4) into a non-dimensional form: $w(x, t) = \frac{W(X, \tau)}{\varepsilon L}$, $u(x, t) = \frac{U(X, \tau)}{\varepsilon L}$, $x = \frac{X}{L}$, $c^2 = \frac{T_0}{\rho A}$, $t = \frac{c\tau}{L}$, $P_0^2 = \frac{EI}{T_0 L^2}$, $P_1^2 = \frac{EA}{T_0}$, $R_0 = T_0$, where ε is a small parameter with $0 < \varepsilon \ll 1$. The equations of motion in a non-dimensional form are given by

$$\begin{aligned} w_{tt} - P_1^2 w_{xx} &= (P_1^2 - 1)(\varepsilon u_x u_{xx} - 2\varepsilon^2 u_x u_{xx} w_x - \varepsilon^2 u_x^2 w_{xx}), \\ u_{tt} - u_{xx} + P_0^2 u_{xxx} &= (P_1^2 - 1)(\varepsilon u_{xx} w_x + \varepsilon u_x w_{xx} - \varepsilon^2 u_{xx} w_x^2 \\ &\quad + \frac{3}{2} \varepsilon^2 u_{xx} u_x^2 - 2\varepsilon^2 u_x w_{xx} w_x), \\ t &> 0, \quad 0 < x < 1. \end{aligned} \quad (5.5)$$

The boundary conditions are given by

$$w(0, t) = w(1, t) = 0, \quad \text{and} \quad u(x, t) = u_{xx}(x, t) = 0 \quad \text{for} \quad x = 0, 1. \quad (5.6)$$

The initial displacements and initial velocities satisfy the following:

$$\begin{aligned} w(x, 0) &= w_0(x), \quad w_t(x, 0) = w_1(x), \\ u(x, 0) &= u_0(x), \quad \text{and} \quad u_t(x, 0) = u_1(x). \end{aligned} \quad (5.7)$$

It has to be noticed that after the braking moment τ_0 almost all points of the belt still have velocities close to V_b in longitudinal direction w for some period of time, that effect is included in the initial conditions (5.7). Before studying the initial-boundary value problem (5.5)-(5.7) something has to be said about the orders of magnitude of the parameters P_0^2 and P_1^2 in (5.5). This will be done in the next subsection.

5.2.1 Three different ordering cases

To make the research more tractable the parameters ε , P_0^2 and P_1^2 will be estimated in orders of magnitude. Two different rubber sorts will be considered: EP 250/2 with $E = 340.914$ MPa, and EP 500/2 with $E = 250$ MPa.

The following characteristic values for the other parameters are chosen (other parameter values for belt systems can be found in [49]): $b = 0.6$ m, $h = 7.3$ mm, $\rho = 1197.2$ kg m⁻¹, and $T = 3500$ N. After calculating the values of P_0^2 and P_1^2 it will follow that $P_1^2 \gg P_0^2$. Taking this fact into account the order of all terms in equations (5.5) and the value of ε have to be defined now.

From the transformation for the transversal displacement of the belt $u(x, t) = \frac{U(X, \tau)}{\varepsilon L}$ it follows that there is an approximate relationship between ε and the belt length L . By assuming that $u(x, t) = \mathcal{O}(1)$, for instance, and assuming that the original displacement $U(X, \tau)$ is about 10 cm the following table can be derived:

L (in m)	1	10	30	50	80	100	300	500
ε	10^{-1}	10^{-2}	$3.3 \cdot 10^{-3}$	$2 \cdot 10^{-3}$	$1.3 \cdot 10^{-3}$	10^{-3}	$3.3 \cdot 10^{-4}$	$2 \cdot 10^{-4}$

Table 5.1: The approximate relationship between the belt length L and the small parameter ε (assuming that $u(x, t)$ is maximal 1 in absolute value).

The parameters P_0^2 and P_1^2 are given by $P_0^2 = \frac{EI}{T_0 L^2}$, $P_1^2 = \frac{EA}{T_0}$. It follows that P_0^2 depends also on the belt length L , and, accordingly, on the order of the small parameter ε (that is, the displacement in transversal direction, see Table 5.1). The values of the parameters P_0^2 and P_1^2 can now be calculated.

		P_1^2		P_0^2	
$\varepsilon \approx$	$L, (\text{in m})$	EP 250/2	EP 500/2	EP 250/2	EP 500/2
1	$L = 0.1$	$4.3 \cdot 10^2$	$3.1 \cdot 10^2$	$2 \cdot 10^{-1}$	$1.2 \cdot 10^{-1}$
10^{-1}	$L = 1$	$4.3 \cdot 10^2$	$3.1 \cdot 10^2$	$2 \cdot 10^{-3}$	$1.2 \cdot 10^{-3}$
10^{-2}	$L = 10$	$4.3 \cdot 10^2$	$3.1 \cdot 10^2$	$2 \cdot 10^{-5}$	$1.2 \cdot 10^{-5}$
10^{-3}	$L = 30$	$4.3 \cdot 10^2$	$3.1 \cdot 10^2$	$2.1 \cdot 10^{-6}$	$1.5 \cdot 10^{-6}$
10^{-3}	$L = 50$	$4.3 \cdot 10^2$	$3.1 \cdot 10^2$	$7.56 \cdot 10^{-7}$	$5.56 \cdot 10^{-7}$
10^{-3}	$L = 80$	$4.3 \cdot 10^2$	$3.1 \cdot 10^2$	$2.95 \cdot 10^{-7}$	$2.17 \cdot 10^{-7}$
10^{-3}	$L = 100$	$4.3 \cdot 10^2$	$3.1 \cdot 10^2$	$1.89 \cdot 10^{-7}$	$1.39 \cdot 10^{-7}$
10^{-4}	$L = 300$	$4.3 \cdot 10^2$	$3.1 \cdot 10^2$	$2.1 \cdot 10^{-8}$	$1.54 \cdot 10^{-8}$
10^{-4}	$L = 500$	$4.3 \cdot 10^2$	$3.1 \cdot 10^2$	$7.56 \cdot 10^{-9}$	$5.56 \cdot 10^{-9}$
10^{-4}	$L = 800$	$4.3 \cdot 10^2$	$3.1 \cdot 10^2$	$2.95 \cdot 10^{-9}$	$2.17 \cdot 10^{-9}$
10^{-4}	$L = 1000$	$4.3 \cdot 10^2$	$3.1 \cdot 10^2$	$1.89 \cdot 10^{-9}$	$1.39 \cdot 10^{-9}$

Table 5.2: The approximate relationship between the order of the small parameter ε and the parameters P_0^2 and P_1^2 for two different sorts of rubber.

In Table 5.2 the values of the parameters P_0^2 and P_1^2 are presented, depending on the order of the small parameter ε and the length of the belt for two different rubber sorts - EP 250/2 and EP 500/2. With the help of Table 5.2 the order of all terms in the equations (5.5) can now be found. If $\varepsilon \approx 10^{-2}$, $L \approx 10\text{m}$, $P_0^2 \approx 10^{-5}$, $P_1^2 \simeq 10^2 \left(\approx \mathcal{O}\left(\frac{1}{\varepsilon}\right) \right)$, it follows that the terms in the equations (5.5) satisfy the following:

$$\begin{aligned} \mathcal{O}\left(\frac{1}{\varepsilon}\right) \text{ are : } & w_{tt}, P_1^2 w_{xx} \\ \mathcal{O}(1) \text{ are : } & P_1^2 \varepsilon u_x u_{xx}, u_{tt}, u_{xx}, P_0^2 u_{xxxx}, P_1^2 \varepsilon u_{xx} w_x, P_1^2 \varepsilon u_x w_{xx} \\ \mathcal{O}(\varepsilon) \text{ are : } & \varepsilon u_x u_{xx}, P_1^2 2\varepsilon^2 u_x u_{xx} w_x, P_1^2 \varepsilon^2 u_x^2 w_{xx}; P_1^2 \varepsilon^2 u_{xx} w_x^2, \\ & P_1^2 \frac{3}{2} \varepsilon^2 u_{xx} u_x^2, P_1^2 2\varepsilon^2 u_x w_{xx} w_x \end{aligned} \quad (5.8)$$

From the transformation for the transversal displacement of the belt, $u(x, t) = \frac{U(X, \tau)}{\varepsilon L}$, it follows that there is an approximate relationship between ε and the belt length L . By assuming that $u(x, t) = \mathcal{O}(1)$ and the original displacement $U(X, \tau)$ is about 10 cm Table 5.1 and Table 5.2 were derived. But this assumption can be chosen differently, for instance the choice can be (according to certain belt vibrations) that $u(x, t) = \mathcal{O}(\varepsilon)$ or $u(x, t) = \mathcal{O}(\varepsilon^2)$, and the original displacement $U(X, \tau)$ is about 10 cm. The following relationship between the belt length L and the small parameter ε can be derived for these two cases. For the case that $u(x, t) = \mathcal{O}(\varepsilon)$ and the original displacement $U(X, \tau)$ is about 10 cm. the relationship is given in Table 5.3.

$L(\text{in m})$	1	10	30	50	80	100	300	500
ε	$3 \cdot 10^{-1}$	10^{-1}	$5.8 \cdot 10^{-2}$	$4.4 \cdot 10^{-2}$	$3.5 \cdot 10^{-2}$	$3.1 \cdot 10^{-2}$	$1.8 \cdot 10^{-2}$	$1.4 \cdot 10^{-2}$

Table 5.3: The approximate relationship between the belt length L and the small parameter ε for the case that $u(x, t)$ is assumed to be maximal ε in absolute value.

In Table 5.4 the values of the parameters P_0^2 and P_1^2 are presented for the case that $u(x, t) = \mathcal{O}(\varepsilon)$.

For the case $u(x, t) = \mathcal{O}(\varepsilon^2)$ and the original displacement $U(X, \tau)$ is about 10 cm the approximate relationship between the belt length L and ε is given in Table 5.5, and the corresponding values of P_0^2 and P_1^2 are given in Table 5.6.

		P_1^2		P_0^2	
$\varepsilon \approx$	L (in m)	EP 250/2	EP 500/2	EP 250/2	EP 500/2
1	$L = 0.1$	$4.3 \cdot 10^2$	$3.1 \cdot 10^2$	$2 \cdot 10^{-1}$	$1.2 \cdot 10^{-1}$
10^{-1}	$L = 1$	$4.3 \cdot 10^2$	$3.1 \cdot 10^2$	$2 \cdot 10^{-3}$	$1.2 \cdot 10^{-3}$
10^{-1}	$L = 10$	$4.3 \cdot 10^2$	$3.1 \cdot 10^2$	$2 \cdot 10^{-5}$	$1.2 \cdot 10^{-5}$
10^{-2}	$L = 30$	$4.3 \cdot 10^2$	$3.1 \cdot 10^2$	$2.1 \cdot 10^{-6}$	$1.5 \cdot 10^{-6}$
10^{-2}	$L = 50$	$4.3 \cdot 10^2$	$3.1 \cdot 10^2$	$7.56 \cdot 10^{-7}$	$5.56 \cdot 10^{-7}$
10^{-2}	$L = 80$	$4.3 \cdot 10^2$	$3.1 \cdot 10^2$	$2.95 \cdot 10^{-7}$	$2.17 \cdot 10^{-7}$
10^{-2}	$L = 100$	$4.3 \cdot 10^2$	$3.1 \cdot 10^2$	$1.89 \cdot 10^{-7}$	$1.39 \cdot 10^{-7}$
10^{-2}	$L = 300$	$4.3 \cdot 10^2$	$3.1 \cdot 10^2$	$2.1 \cdot 10^{-8}$	$1.54 \cdot 10^{-8}$
10^{-2}	$L = 500$	$4.3 \cdot 10^2$	$3.1 \cdot 10^2$	$7.56 \cdot 10^{-9}$	$5.56 \cdot 10^{-9}$
10^{-2}	$L = 800$	$4.3 \cdot 10^2$	$3.1 \cdot 10^2$	$2.95 \cdot 10^{-9}$	$2.17 \cdot 10^{-9}$
10^{-2}	$L = 1000$	$4.3 \cdot 10^2$	$3.1 \cdot 10^2$	$1.89 \cdot 10^{-9}$	$1.39 \cdot 10^{-9}$

Table 5.4: The relationship between the order of the small parameter ε and the parameters P_0^2 and P_1^2 for two different sorts of rubber for the case $u(x, t) = \mathcal{O}(\varepsilon)$.

L , (in m)	1	10	30	50	80	100	300	500
ε	$0.5 \cdot 10^{-1}$	$2.1 \cdot 10^{-1}$	$1.5 \cdot 10^{-1}$	$1.3 \cdot 10^{-1}$	$1.1 \cdot 10^{-1}$	10^{-1}	$7 \cdot 10^{-2}$	$5.9 \cdot 10^{-2}$

Table 5.5: The approximate relationship between the belt length L and the small parameter ε for the case that $u(x, t)$ is maximal ε^2 in absolute value.

It follows from Tables 5.1-5.6, that the order of the terms in the equations (5.5) and the order of the parameter ε can vary depending on the initial assumption on the order of the transversal displacement $u(x, t)$.

It also has to be noticed that the orders of the terms u_{tt} , u_{xx} , u_{xxxx} , and w_{xx} depend on the mode number, namely $u_{tt} = \mathcal{O}(P_0^2 k^4)$, $u_{xx} = \mathcal{O}(k^2)$, $u_{xxxx} = \mathcal{O}(k^4)$, and $w_{xx} = \mathcal{O}(n^2)$, $w_{tt} = \mathcal{O}(P_1^2 n^2)$, where k and n are the vibrations mode numbers in transversal and in longitudinal direction, respectively.

From the equations (5.5) and (5.8) it follows that if $u(x, t) = \mathcal{O}(1)$ and $w(x, t) = \mathcal{O}(1)$ the quadratic and the cubic terms in equation (5.5) should be studied simultaneously. If $u(x, t) = \mathcal{O}(\varepsilon)$, and $w(x, t) = \mathcal{O}(\varepsilon)$ then only quadratic terms can be considered because the cubic terms are much smaller. For simplicity this case with only quadratic nonlinearities will be studied further in this chapter.

		P_1^2		P_0^2	
$\varepsilon \approx$	L (in m) m	EP 250/2	EP 500/2	EP 250/2	EP 500/2
1	$L = 0.1$	$4.3 \cdot 10^2$	$3.1 \cdot 10^2$	$2 \cdot 10^{-1}$	$1.2 \cdot 10^{-1}$
10^{-1}	$L = 1$	$4.3 \cdot 10^2$	$3.1 \cdot 10^2$	$2 \cdot 10^{-3}$	$1.2 \cdot 10^{-3}$
10^{-1}	$L = 10$	$4.3 \cdot 10^2$	$3.1 \cdot 10^2$	$2 \cdot 10^{-5}$	$1.2 \cdot 10^{-5}$
10^{-1}	$L = 30$	$4.3 \cdot 10^2$	$3.1 \cdot 10^2$	$2.1 \cdot 10^{-6}$	$1.5 \cdot 10^{-6}$
10^{-1}	$L = 50$	$4.3 \cdot 10^2$	$3.1 \cdot 10^2$	$7.56 \cdot 10^{-7}$	$5.56 \cdot 10^{-7}$
10^{-1}	$L = 80$	$4.3 \cdot 10^2$	$3.1 \cdot 10^2$	$2.95 \cdot 10^{-7}$	$2.17 \cdot 10^{-7}$
10^{-1}	$L = 100$	$4.3 \cdot 10^2$	$3.1 \cdot 10^2$	$1.89 \cdot 10^{-7}$	$1.39 \cdot 10^{-7}$
10^{-2}	$L = 300$	$4.3 \cdot 10^2$	$3.1 \cdot 10^2$	$2.1 \cdot 10^{-8}$	$1.54 \cdot 10^{-8}$
10^{-2}	$L = 500$	$4.3 \cdot 10^2$	$3.1 \cdot 10^2$	$7.56 \cdot 10^{-9}$	$5.56 \cdot 10^{-9}$
10^{-2}	$L = 800$	$4.3 \cdot 10^2$	$3.1 \cdot 10^2$	$2.95 \cdot 10^{-9}$	$2.17 \cdot 10^{-9}$
10^{-2}	$L = 1000$	$4.3 \cdot 10^2$	$3.1 \cdot 10^2$	$1.89 \cdot 10^{-9}$	$1.39 \cdot 10^{-9}$

Table 5.6: The approximate relationship between the order of the small parameter ε and the parameters P_0^2 and P_1^2 for two different sorts of rubber for the case $u(x, t) = \mathcal{O}(\varepsilon^2)$.

The case $u(x, t) = \mathcal{O}(\varepsilon)$ and damping in the problem

It follows from Table 5.3 and Table 5.4 that if $\varepsilon \simeq 10^{-2}$ and $L \approx 30$ then $P_0^2 \approx 10^{-6} (\approx \mathcal{O}(\varepsilon^3))$, and $P_1^2 \simeq 10^2 \approx (\mathcal{O}(\varepsilon^{-1}))$. The terms in equation (5.5) have the following order now: $w_{tt} = \mathcal{O}(n^2)$, $P_1^2 w_{xx} = \mathcal{O}(n^2)$, $u_{tt} = \mathcal{O}(1)$, $u_{xx} = \mathcal{O}(1)$, and $P_0^2 u_{xxxx} = \mathcal{O}(\varepsilon^3 k^4)$. From the applicational point of view the lower transversal modes are the most important ones because these modes usually lead to large amplitudes. With these assumptions the cubic terms in equation (5.5) can be neglected. But it has to be noticed here that if there is an interaction with the higher order modes, the equations have to be reconsidered accordingly. It can be also assumed that $P_0^2 u_{xxxx} = \mathcal{O}(1)$ (that is, if $P_0^2 u_{xxxx} = \mathcal{O}(\varepsilon^3 k^4)$ then $P_0^2 u_{xxxx} = \mathcal{O}(1)$). This assumption will help us to truncate the equation for the transversal vibrations (the second equation in (5.5)). Accordingly the following equations can be derived (with all terms of order smaller than ε^2 being neglected):

$$\begin{aligned}
 w_{tt} - P_1^2 w_{xx} &= \varepsilon P_1^2 u_x u_{xx}, \\
 u_{tt} - u_{xx} + P_0^2 u_{xxxx} &= \varepsilon P_1^2 u_{xx} w_x + \varepsilon P_1^2 u_x w_{xx}, \\
 t > 0, \quad 0 < x < 1.
 \end{aligned} \tag{5.9}$$

Conservative systems like the equations in (5.9) represent mathematical idealizations, and in practice all systems have some degree of damping, so that free vibrations die out eventually. Nowadays two kinds of damping are nor-

mally used to describe the energy loss in distributed damped systems: viscous damping and structural damping. For a closer study of these types of damping the readers are referred to, for instance, [2] or [50].

To make the system (as it has been given by the equations (5.5)) nonconservative, small viscous damping is introduced in both directions: $\varepsilon\delta_1 u_t$ is the damping in transversal direction, and $\varepsilon\delta_2 w_t$ is the damping in longitudinal direction, where δ_1 is the coefficient of viscous damping in transversal direction, δ_2 is the coefficient of viscous damping in longitudinal direction, and ε is still a small parameter. The damping terms in both equations in (5.5) are assumed to be of order ε . Neglecting terms of order smaller than $\mathcal{O}(\varepsilon)$ in (5.5) leads to the following model for the problem:

$$\begin{aligned} w_{tt} - P_1^2 w_{xx} &= \varepsilon P_1^2 u_x u_{xx} - \varepsilon\delta_2 w_t, \\ u_{tt} - u_{xx} + P_0^2 u_{xxx} &= \varepsilon P_1^2 u_{xx} w_x + \varepsilon P_1^2 u_x w_{xx} - \varepsilon\delta_1 u_t, \\ t > 0, \quad 0 < x < 1. \end{aligned} \quad (5.10)$$

The boundary and the initial conditions for (5.10) are still given by (5.6) and (5.7), respectively. To satisfy the boundary conditions all functions in (5.10) should be expanded in Fourier-sine-series. Therefore, the solutions can be written as $u(x, t; \varepsilon) = \sum_{k=1}^{\infty} u_k(t; \varepsilon) \sin(k\pi x)$ and $w(x, t; \varepsilon) = \sum_{n=1}^{\infty} w_n(t; \varepsilon) \sin(n\pi x)$. As all terms in the equations (5.10) are already odd in x , $u(x, t; \varepsilon) = \sum_{k=1}^{\infty} u_k(t; \varepsilon) \sin(k\pi x)$ and $w(x, t; \varepsilon) = \sum_{n=1}^{\infty} w_n(t; \varepsilon) \sin(n\pi x)$ can be substituted into (5.10), yielding

$$\begin{aligned} & \sum_{n=1}^{\infty} (\ddot{w}_n + P_1^2 (n\pi)^2 w_n) \sin(n\pi x) \\ &= \varepsilon \left(P_1^2 \sum_{n=1}^{\infty} \sum_{j=1}^{\infty} -u_n u_j n \pi^3 j^2 \cos(n\pi x) \sin(j\pi x) - \delta_2 \sum_{n=1}^{\infty} \dot{w}_n \sin(n\pi x) \right), \\ & \sum_{k=1}^{\infty} (\ddot{u}_k + (1 + P_0^2 (k\pi)^2) (k\pi)^2 u_k) \sin(k\pi x) \\ &= \varepsilon \left(P_1^2 \sum_{j=1}^{\infty} \sum_{n=1}^{\infty} (-u_j w_n - u_n w_j) n \pi^3 j^2 \cos(n\pi x) \sin(j\pi x) \right. \\ & \quad \left. - \delta_1 \sum_{k=1}^{\infty} \dot{u}_k \sin(k\pi x) \right). \end{aligned} \quad (5.11)$$

The transformation of (5.11) into a system of ordinary differential equations in t can be done by taking into account the following identities:

$$\int_0^1 \sin(n\pi x) \sin(m\pi x) dx = \begin{cases} 0, & \text{for } m \neq n, \\ \frac{1}{2}, & \text{for } m = n, \end{cases} \quad \text{and}$$

$$\int_0^1 \cos(n\pi x) \sin(j\pi x) \sin(m\pi x) dx = \begin{cases} 0, & \text{for } j \neq \pm n \pm m, \quad \text{and} \\ \frac{1}{4}, & \text{for } j = m \pm n, \quad j \geq 1, \quad \text{and} \\ -\frac{1}{4}, & \text{for } j = n - m, \quad j \geq 1. \end{cases} \quad (5.12)$$

By multiplying (5.11) with $\sin(m\pi x)$, and by integrating the so-obtained equations with respect to x from $x = 0$ to $x = 1$ (see also (5.12)), and some the simplifications the following equations can be obtained:

$$\begin{aligned} & \ddot{w}_m + P_1^2(m\pi)^2 w_m \\ &= -\varepsilon \frac{P_1^2 \pi^3}{2} \left(\sum_{n=1}^{\infty} u_n u_{m+n} (m+n) mn + \sum_{n=1}^{m-1} u_n u_{m-n} (m-n)^2 n \right) \\ & \quad - \varepsilon \delta_2 \dot{w}_m, \end{aligned}$$

$$\begin{aligned} & \ddot{u}_m + ((m\pi)^2 + P_0^2(m\pi)^4) u_m \\ &= -\varepsilon \frac{P_1^2 \pi^3}{2} \left(\sum_{n=1}^{\infty} (u_n w_{n+m} + u_{n+m} w_n) (m+n) mn \right. \\ & \quad \left. + \sum_{n=1}^{m-1} u_n w_{m-n} (m-n) nm \right) - \varepsilon \delta_1 \dot{u}_m, \quad (5.13) \end{aligned}$$

where for $t = 0$, $w_m(t)$ satisfies: $w_m(0) = 2 \int_0^1 w_0(x) \sin(m\pi x) dx$,
 $\dot{w}_m(0) = 2 \int_0^1 w_1(x) \sin(m\pi x) dx$, and $u_m(t)$ satisfies for $t = 0$: $u_m(0) = 2 \int_0^1 u_0(x) \sin(m\pi x) dx$, $\dot{u}_m(0) = 2 \int_0^1 u_1(x) \sin(m\pi x) dx$.

5.3 Application of the two time-scales perturbation method

To avoid secular terms in the approximate solution of (5.13) a two time-scales perturbation method is used. The two new time scales are $t_0 = t$ and $t_1 = \varepsilon t$,

implying that $w_m(t; \varepsilon) = \mathbf{w}_m(t_0, t_1; \varepsilon)$ and $u_m(t; \varepsilon) = \mathbf{u}_m(t_0, t_1; \varepsilon)$. The following transformations are needed for the time derivatives for the longitudinal displacement:

$$\begin{aligned}\frac{dw_m}{dt} &= \frac{\partial \mathbf{w}_m}{\partial t_0} + \varepsilon \frac{\partial \mathbf{w}_m}{\partial t_1}, \\ \frac{d^2 w_m}{dt^2} &= \frac{\partial^2 \mathbf{w}_m}{\partial t_0^2} + 2\varepsilon \frac{\partial^2 \mathbf{w}_m}{\partial t_0 \partial t_1} + \varepsilon^2 \frac{\partial^2 \mathbf{w}_m}{\partial t_1^2},\end{aligned}\tag{5.14}$$

and for the transversal displacement:

$$\begin{aligned}\frac{du_m}{dt} &= \frac{\partial \mathbf{u}_m}{\partial t_0} + \varepsilon \frac{\partial \mathbf{u}_m}{\partial t_1}, \\ \frac{d^2 u_m}{dt^2} &= \frac{\partial^2 \mathbf{u}_m}{\partial t_0^2} + 2\varepsilon \frac{\partial^2 \mathbf{u}_m}{\partial t_0 \partial t_1} + \varepsilon^2 \frac{\partial^2 \mathbf{u}_m}{\partial t_1^2}.\end{aligned}\tag{5.15}$$

It is assumed that $\mathbf{w}_m = \mathbf{w}_{m,0} + \varepsilon \mathbf{w}_{m,1} + \dots$, and $\mathbf{u}_m(t_0, t_1; \varepsilon) = \mathbf{u}_{m,0} + \varepsilon \mathbf{u}_{m,1} + \dots$. By substituting (5.14) and (5.15) into (5.13), and by substituting the expansions for $\mathbf{w}_m$ and for $\mathbf{u}_m$ into (5.13), and then by taking together

terms of equal powers in ε in the so-obtained equations it follows that

$$\begin{aligned}
 \mathcal{O}(1) : \quad & \frac{\partial^2 \mathbf{w}_{m,0}}{\partial t_0^2} + P_1^2(m\pi)^2 \mathbf{w}_{m,0} = 0, \text{ and} \\
 & \frac{\partial^2 \mathbf{u}_{m,0}}{\partial t_0^2} + (m\pi)^2(1 + P_0^2(m\pi)^2) \mathbf{u}_{m,0} = 0, \\
 \mathcal{O}(\varepsilon) : \quad & \frac{\partial^2 \mathbf{w}_{m,1}}{\partial t_0^2} + P_1^2(m\pi)^2 \mathbf{w}_{m,1} = -2 \frac{\partial^2 \mathbf{w}_{m,0}}{\partial t_0 \partial t_1} \\
 & - \frac{P_1^2 \pi^3}{2} \left(\sum_{n=1}^{\infty} \mathbf{u}_{n,0} \mathbf{u}_{m+n,0} (m+n)nm + \sum_{n=1}^{m-1} \mathbf{u}_{n,0} \mathbf{u}_{m-n,0} (m-n)^2 n \right) \\
 & - \delta_2 \frac{\partial \mathbf{w}_{m,0}}{\partial t_0}, \text{ and} \\
 & \frac{\partial^2 \mathbf{u}_{m,1}}{\partial t_0^2} + (m\pi)^2(1 + P_0^2(m\pi)^2) \mathbf{u}_{m,1} = -2 \frac{\partial^2 \mathbf{u}_{m,0}}{\partial t_0 \partial t_1} \\
 & - \frac{P_1^2 \pi^3}{2} \left(\sum_{n=1}^{\infty} (\mathbf{u}_{m+n,0} \mathbf{w}_{n,0} + \mathbf{u}_{n,0} \mathbf{w}_{m+n,0}) (m+n)nm \right. \\
 & \left. + \sum_{n=1}^{m-1} \mathbf{u}_{n,0} \mathbf{w}_{m-n,0} (m-n)nm \right) \\
 & - \delta_1 \frac{\partial \mathbf{u}_{m,0}}{\partial t_0}.
 \end{aligned} \tag{5.16}$$

Solutions of the $\mathcal{O}(1)$ -problems are given by

$$\begin{aligned}
 \mathbf{w}_{m,0}(t_0, t_1) &= A_{m,0}(t_1) \cos(\omega_m^l t_0) + B_{m,0}(t_1) \sin(\omega_m^l t_0), \\
 \mathbf{u}_{m,0}(t_0, t_1) &= C_{m,0}(t_1) \cos(\omega_m^t t_0) + D_{m,0}(t_1) \sin(\omega_m^t t_0),
 \end{aligned} \tag{5.17}$$

where $A_{m,0}(t_1)$, $B_{m,0}(t_1)$, $C_{m,0}(t_1)$, and $D_{m,0}(t_1)$ are still arbitrary functions that can be used to eliminate secular terms in the solutions of the $\mathcal{O}(\varepsilon)$ -problem. In (5.17) ω_m^l and ω_m^t are given by

$$\begin{aligned}
 (\omega_m^l)^2 &= (P_1 m \pi)^2 & : \omega_m^l \text{ is the longitudinal frequency,} \\
 (\omega_m^t)^2 &= (m\pi)^2 + P_0^2(m\pi)^4 & : \omega_m^t \text{ is the transversal frequency,}
 \end{aligned} \tag{5.18}$$

and

$$\begin{aligned}
 A_{m,0}(0) &= 2 \int_0^1 w_0(x) \sin(m\pi x) dx, & B_{m,0}(0) &= \frac{2}{\omega_m^l} \int_0^1 w_1(x) \sin(m\pi x) dx, \\
 C_{m,0}(0) &= 2 \int_0^1 u_0(x) \sin(m\pi x) dx, & D_{m,0}(0) &= \frac{2}{\omega_m^t} \int_0^1 u_1(x) \sin(m\pi x) dx.
 \end{aligned} \tag{5.19}$$

From the equations (5.13) it can be seen that possible internal resonance frequencies are (in an $\mathcal{O}(\varepsilon)$ neighbourhood of):

$$\begin{aligned}
 (i) \quad & \pm\omega_m^l = \pm\omega_n^t \pm \omega_{n+m}^t, \quad \text{for } n = 1, 2, \dots \quad \text{and } m = 1, 2, \dots, \\
 (ii) \quad & \pm\omega_m^l = \pm\omega_n^t \pm \omega_{m-n}^t, \quad \text{for } n = 1, 2, \dots, m-1 \quad \text{and } m = 1, 2, \dots, \\
 (iii) \quad & \pm\omega_m^t = \pm\omega_n^l \pm \omega_{n+m}^t, \quad \text{for } n = 1, 2, \dots \quad \text{and } m = 1, 2, \dots, \\
 (iv) \quad & \pm\omega_m^t = \pm\omega_n^t \pm \omega_{n+m}^l, \quad \text{for } n = 1, 2, \dots \quad \text{and } m = 1, 2, \dots, \\
 (v) \quad & \pm\omega_m^t = \pm\omega_n^t \pm \omega_{m-n}^l, \quad \text{for } n = 1, 2, \dots, m-1 \quad \text{and } m = 1, 2, \dots
 \end{aligned} \tag{5.20}$$

It can be seen from (5.20) that the resonance condition (i) (by interchanging n and m) corresponds to the resonance condition (iii), so that there are actually four different resonance conditions for (5.13). From these resonance conditions (5.20) and the formulas (5.18) it can be seen that the possibility of having an internal resonance depends entirely on the values of P_1^2 and P_0^2 , that is, on the mechanical properties and configuration of the belt system (see Figure 5.1).

5.3.1 The non-resonant case

It was found in the previous section that possible internal resonances can occur if at least one of the following conditions is fulfilled:

$$\begin{aligned}
 (1) \quad & \pm\omega_m^l = \pm\omega_n^t \pm \omega_{n+m}^t, \quad \text{for } n = 1, 2, \dots \quad \text{and } m = 1, 2, \dots, \\
 (2) \quad & \pm\omega_m^l = \pm\omega_n^t \pm \omega_{m-n}^t, \quad \text{for } n = 1, 2, \dots, m-1 \quad \text{and } m = 1, 2, \dots, \\
 (3) \quad & \pm\omega_m^t = \pm\omega_n^t \pm \omega_{n+m}^l, \quad \text{for } n = 1, 2, \dots \quad \text{and } m = 1, 2, \dots, \\
 (4) \quad & \pm\omega_m^t = \pm\omega_n^t \pm \omega_{m-n}^l, \quad \text{for } n = 1, 2, \dots, m-1 \quad \text{and } m = 1, 2, \dots,
 \end{aligned} \tag{5.21}$$

where ω_m^l and ω_m^t are still given by (5.18). In the non-resonant case it is assumed that non of the resonance combinations of longitudinal and transversal frequencies as listed in (5.21) occur (also not in an order ε -neighbourhood). To eliminate the secular terms in the solution of the $\mathcal{O}(\varepsilon)$ -problems (see equations (5.16)) it follows that $A_{m,0}(t_1)$, $B_{m,0}(t_1)$, $C_{m,0}(t_1)$ and $D_{m,0}(t_1)$ have to satisfy

$$\begin{aligned}
 A_{m,0}(t_1) &= A_{m,0}(0) \exp\left(-\frac{1}{2}\delta_2 t_1\right), \quad B_{m,0}(t_1) = B_{m,0}(0) \exp\left(-\frac{1}{2}\delta_2 t_1\right), \\
 C_{m,0}(t_1) &= C_{m,0}(0) \exp\left(-\frac{1}{2}\delta_1 t_1\right), \quad D_{m,0}(t_1) = D_{m,0}(0) \exp\left(-\frac{1}{2}\delta_1 t_1\right),
 \end{aligned} \tag{5.22}$$

where $A_{m,0}(0)$, $B_{m,0}(0)$, $C_{m,0}(0)$, and $D_{m,0}(0)$ are given by (5.19).

5.3.2 Some resonant cases

In this section the resonance conditions (1) – (4) in the formulas (5.21) will be studied. The following values of parameters are chosen: $\varepsilon = 10^{-2}$ and $L = 30$ then $P_0^2 = 10^{-6} (\approx \mathcal{O}(\varepsilon^3))$, and $P_1^2 = 10^2 \approx (\mathcal{O}(\varepsilon^{-1}))$.

The resonance condition $\pm\omega_m^l = \pm\omega_n^t \pm \omega_{n+m}^t$, for $n = 1, 2, \dots$ and $m = 1, 2, \dots$

In this subsection the resonance case (1) in the formulas (5.21) is considered. The condition $\pm\omega_m^l = \pm\omega_n^t \pm \omega_{n+m}^t$, for $n = 1, 2, \dots$ and $m = 1, 2, \dots$, has the following sub-cases:

$$\begin{aligned}
 (a) \quad & \omega_m^l = \omega_n^t + \omega_{n+m}^t, \\
 (b) \quad & -\omega_m^l = \omega_n^t + \omega_{n+m}^t, \\
 (c) \quad & \omega_m^l = \omega_n^t - \omega_{n+m}^t, \\
 (d) \quad & -\omega_m^l = \omega_n^t - \omega_{n+m}^t, \\
 (e) \quad & \omega_m^l = -\omega_n^t + \omega_{n+m}^t, \\
 (f) \quad & -\omega_m^l = -\omega_n^t + \omega_{n+m}^t, \\
 (g) \quad & \omega_m^l = -\omega_n^t - \omega_{n+m}^t, \\
 (h) \quad & -\omega_m^l = -\omega_n^t - \omega_{n+m}^t.
 \end{aligned} \tag{5.23}$$

For the case (a) in (5.23) Figure 5.4 represents solutions for m and n . After

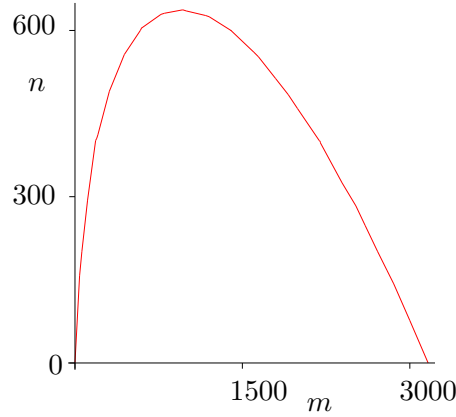


Figure 5.4: Solutions for m and n for the case (a) in (5.23).

calculations (case (a) was solved numerically in Maple), some of the possible solutions for $n = 1, 2, \dots$ and $m = 1, 2, \dots$ can found: $(m = 1, n = 5)$; $(m = 2, n = 9)$; $(m = 4, n = 18)$; $(m = 12, n = 53)$; $(m = 3137, n = 15)$; $(m = 3154, n = 7)$. It can be seen from these solutions, that there are (possible) interactions between: the 1st longitudinal and the 5th and the 6th transversal modes; the 2nd longitudinal and the 9th and the 11th transversal modes; the 4th longitudinal and the 18th and the 22nd transversal modes; the 12th longitudinal and the 53rd and the 65th transversal modes; the 3137th longitudinal and the 15th and the 3152nd transversal modes; and the 3154th longitudinal and the 7th and the 3161st transversal modes.

Case (b) in (5.23) has obviously no solutions for positive m and n .

Case (c) in (5.23) has no solutions for positive m and n as the right-hand side of (c) is always negative, whereas the left-hand side of (c) is always positive.

For the case (d) in (5.23) Figure 5.5 represents solutions for m and n . After calculations, some of the possible solutions for $n = 1, 2, \dots$ and $m =$

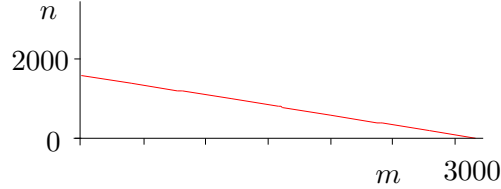


Figure 5.5: Solutions for m and n for the case (d) in (5.23).

$1, 2, \dots$ have been found: ($m = 3142, n = 14$); ($m = 3163, n = 3$). It can be seen from these solutions, that there are (possible) interactions between: the 3142nd longitudinal and the 14th and the 3156th transversal modes; the 3163rd longitudinal and the 3rd and the 3166th transversal modes.

Case (e) in (5.23) corresponds to the case (d) in (5.23) and, consequently has the same solutions as (d).

Case (f) in (5.23) has no solutions for positive m and n as the right-hand side is always positive, whereas the left-hand side of (f) is always negative.

Case (g) in (5.23) has obviously no solutions for positive m and n .

Case (h) in (5.23) corresponds to the case (a) in (5.23) and, consequently has the same solutions as (a).

The resonance condition $\pm\omega_m^l = \pm\omega_n^t \pm \omega_{m-n}^t$, for $n = 1, 2, \dots, m-1$ and $m = 1, 2, \dots$

In this subsection the resonance case (2) as given in (5.21) is considered. The condition $\pm\omega_m^l = \pm\omega_n^t \pm \omega_{m-n}^t$, for $n = 1, 2, \dots, m-1$ and $m = 1, 2, \dots$, has the following sub-cases:

$$\begin{aligned}
 (a) \quad & \omega_m^l = \omega_n^t + \omega_{m-n}^t, \\
 (b) \quad & -\omega_m^l = \omega_n^t + \omega_{m-n}^t, \\
 (c) \quad & \omega_m^l = \omega_n^t - \omega_{m-n}^t, \\
 (d) \quad & -\omega_m^l = \omega_n^t - \omega_{m-n}^t, \\
 (e) \quad & \omega_m^l = -\omega_n^t + \omega_{m-n}^t, \\
 (f) \quad & -\omega_m^l = -\omega_n^t + \omega_{m-n}^t, \\
 (g) \quad & \omega_m^l = -\omega_n^t - \omega_{m-n}^t, \\
 (h) \quad & -\omega_m^l = -\omega_n^t - \omega_{m-n}^t.
 \end{aligned} \tag{5.24}$$

For the case (a) in (5.24) Figure 5.6 represents solutions for m and n . After cal-

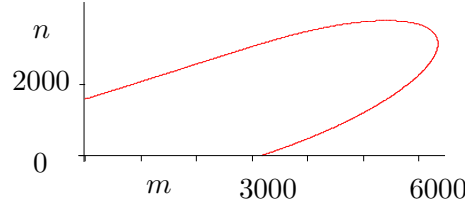


Figure 5.6: Solutions for m and n for the case (a) in (5.24).

culations, some of the possible solutions for $n = 1, 2, \dots, m-1$ and $m = 1, 2, \dots$ have been found: $(m = 3184, n = 8, n = 3176)$; $(m = 3203, n = 18, n = 3185)$. It can be seen from these solutions, that there are (possible) interactions between: the 3184th longitudinal and the 8th and the 3176th transversal modes; the 3203rd longitudinal and the 18th and the 3185th transversal modes.

Case (b) in (5.24) has obviously no solutions for positive m and n , and $n = 1, 2, \dots, m-1$.

For the case (c) in (5.24) Figure 5.7 represents possible solutions for m and n . It can be seen from this figure that there are infinitely many solutions for m and n but for only very large values of m and n ($m > 5000$).

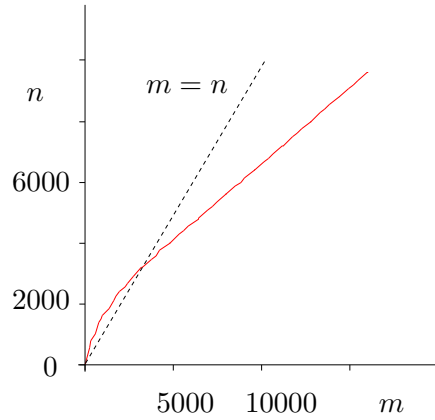


Figure 5.7: Solutions for m and n for the case (c) in (5.24).

For the case (d) in (5.24) Figure 5.8 represents possible solutions for m and n . After calculations, some of the possible solutions for $n = 1, 2, \dots, m-1$ and $m = 1, 2, \dots$ have been found: $(m = 3173, n = 2)$. It can be seen from these solutions, that there are (possible) interactions between: the 3173rd longitudinal and the 2nd and the 3171st transversal modes. It also can be

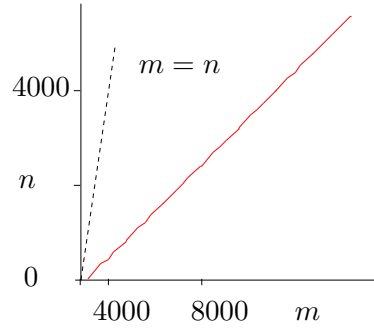


Figure 5.8: Solutions for m and n for the case (d) in (5.24).

seen from Figure 5.8 that there are infinitely many possible solutions for m and n , but only for very large values of m ($m > 3170$).

Case (e) in (5.24) corresponds to the case (d) in (5.24) and, consequently has the same solutions as (d).

Case (f) in (5.24) corresponds to the case (c) in (5.24) and, consequently has the same solutions as (c).

Case (g) in (5.24) has obviously no solutions for positive m and n , and $n = 1, 2, \dots, m - 1$.

Case (h) in (5.24) corresponds to the case (a) in (5.24) and, consequently has the same solutions as (a).

The resonance condition $\pm\omega_m^t = \pm\omega_n^t \pm \omega_{n+m}^l$, for $n = 1, 2, \dots$ and $m = 1, 2, \dots$

In this subsection the resonant case (3) as given in (5.21) is considered. The condition $\pm\omega_m^t = \pm\omega_n^t \pm \omega_{n+m}^l$, for $n = 1, 2, \dots$ and $m = 1, 2, \dots$, has the following sub-cases:

$$\begin{aligned}
 (a) \quad & \omega_m^t = \omega_n^t + \omega_{n+m}^l, \\
 (b) \quad & -\omega_m^t = \omega_n^t + \omega_{n+m}^l, \\
 (c) \quad & \omega_m^t = \omega_n^t - \omega_{n+m}^l, \\
 (d) \quad & -\omega_m^t = \omega_n^t - \omega_{n+m}^l, \\
 (e) \quad & \omega_m^t = -\omega_n^t + \omega_{n+m}^l, \\
 (f) \quad & -\omega_m^t = -\omega_n^t + \omega_{n+m}^l, \\
 (g) \quad & \omega_m^t = -\omega_n^t - \omega_{n+m}^l, \\
 (h) \quad & -\omega_m^t = -\omega_n^t - \omega_{n+m}^l.
 \end{aligned} \tag{5.25}$$

For the case (a) in (5.25) Figure 5.9 represents solutions for m and n . After calculations, some of the possible solutions for $n = 1, 2, \dots$ and $m = 1, 2, \dots$ have been found: $(m = 3171, n = 2)$; $(m = 3182, n = 12)$. It can be seen

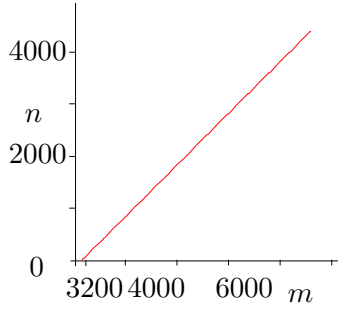


Figure 5.9: Solutions for m and n for the case (a) in (5.25).

from these solutions, that there are (possible) interactions between: the 3171st transversal and the 2nd transversal and the 3173rd longitudinal modes; the 3182nd transversal and the 12th transversal and the 3194th longitudinal modes. It can be seen from the Figure 5.9 that there are infinitely many solutions for m and n but for only very large values of m ($m > 3170$).

Case (b) in (5.25) has obviously no solutions for positive m and n .

Case (c) in (5.25) corresponds to the case (a), if m interchanges with n . Accordingly, the following possible solutions have been found: ($m = 2, n = 3171$); ($m = 12, n = 3182$). It can be seen from these solutions, that there are (possible) interactions between: the 2nd transversal and the 3171st transversal and the 3173rd longitudinal modes; the 12th transversal and the 3182nd transversal and the 3194th longitudinal modes. It can be seen from the Figure 5.10 that there are infinitely many solutions for m and n but for only very large values of n ($n > 3170$).

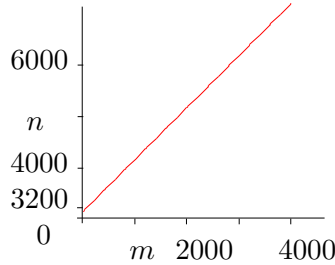


Figure 5.10: Solutions for m and n for the case (c) in (5.25).

For the case (d) in (5.25) Figure 5.11 represents solutions for m and n .

After calculations, some of the possible solutions for $n = 1, 2, \dots$ and $m = 1, 2, \dots$ have been found: ($m = 8, n = 3176$); ($m = 18, n = 3185$), and,

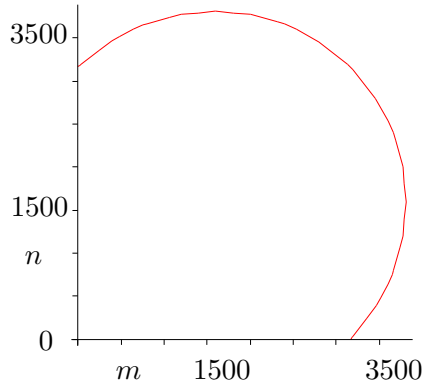


Figure 5.11: Solutions for m and n for the case (d) in (5.25).

symmetrically, $(m = 3176, n = 8)$; $(m = 3185, n = 18)$. It can be seen from these solutions, that there are (possible) interactions between: the 8th transversal and the 3176th transversal and the 3184th longitudinal modes; the 18th transversal and the 3185th transversal and the 3203rd longitudinal modes; and, symmetrically, the 3176th transversal and the 8th transversal and the 3184th longitudinal modes; the 3185th transversal and the 18th transversal and the 3203rd longitudinal modes.

Case (e) in (5.25) corresponds to the case (d) in (5.25) and, consequently has the same solutions as (d).

Case (f) in (5.25) corresponds to the case (c) in (5.25) and, consequently has the same solutions as (c).

Case (g) in (5.25) has obviously no solutions for positive m and n .

Case (h) in (5.25) corresponds to the case (a) in (5.25) and, consequently has the same solutions as (a).

The resonance condition $\pm\omega_m^t = \pm\omega_n^t \pm \omega_{m-n}^l$, for $n = 1, 2, \dots, m-1$ and $m = 1, 2, \dots$

In this subsection the resonant case (4) as given in (5.21) is considered. The condition $\pm\omega_m^t = \pm\omega_n^t \pm \omega_{m-n}^l$, for $n = 1, 2, \dots, m-1$ and $m = 1, 2, \dots$, has

the following sub-cases:

$$\begin{aligned}
 (a) \quad & \omega_m^t = \omega_n^t + \omega_{m-n}^l, \\
 (b) \quad & -\omega_m^t = \omega_n^t + \omega_{m-n}^l, \\
 (c) \quad & \omega_m^t = \omega_n^t - \omega_{m-n}^l, \\
 (d) \quad & -\omega_m^t = \omega_n^t - \omega_{m-n}^l, \\
 (e) \quad & \omega_m^t = -\omega_n^t + \omega_{m-n}^l, \\
 (f) \quad & -\omega_m^t = -\omega_n^t + \omega_{m-n}^l, \\
 (g) \quad & \omega_m^t = -\omega_n^t - \omega_{m-n}^l, \\
 (h) \quad & -\omega_m^t = -\omega_n^t - \omega_{m-n}^l.
 \end{aligned} \tag{5.26}$$

For the case (a) in (5.26) Figure 5.12 represents solutions for m and n . After calculations, some of the possible solutions for $n = 1, 2, \dots, m-1$ and

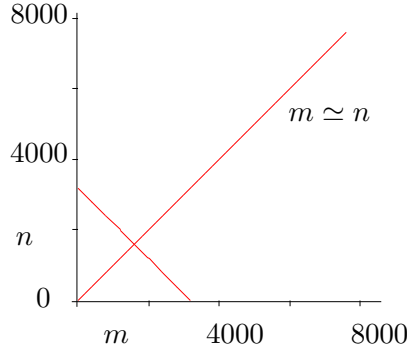


Figure 5.12: Solution for m and n for the case (a) in (5.26).

$m = 1, 2, \dots$ have been found: $(m = 3166, n = 3)$; $(m = 3167, n = 2)$. It can be seen from these solutions, that there are (possible) interactions between: the 3166th transversal and the 3rd transversal and the 3163th longitudinal modes; the 3167th transversal and the 2nd transversal and the 3165th longitudinal modes.

Case (b) in (5.26) has obviously no solutions for positive m and n , and for $n = 1, 2, \dots, m-1$.

For the case (c) in (5.26) Figure 5.13 represents solutions for m and n . It can be seen that for $n = 1, 2, \dots, m-1$ and $m = 1, 2, \dots$ there is no solutions for (c).

For the case (d) in (5.26) Figure 5.14 represents solutions for m and n .

After calculations, some of the possible solutions for $n = 1, 2, \dots, m-1$ and $m = 1, 2, \dots$ have been found: $(m = 6, n = 5)$; $(m = 11, n = 9)$; $(m = 17, n = 14)$; $(m = 22, n = 18)$; $(m = 3152, n = 15)$; $(m = 3161, n = 7)$.

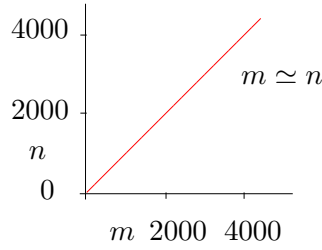


Figure 5.13: Solutions for m and n for the case (c) in (5.26).

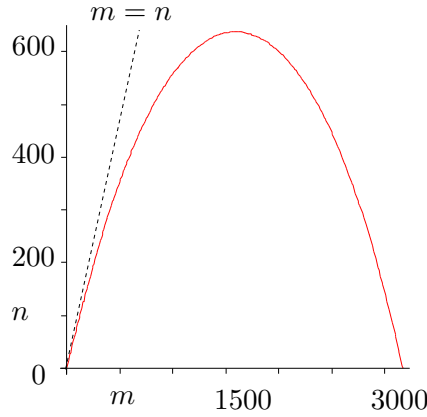


Figure 5.14: Solutions for m and n for the case (d) in (5.26).

It can be seen from these solutions, that there are (possible) interactions between: the 6th transversal and the 5th transversal and the 1st longitudinal modes; the 11th transversal and the 9th transversal and the 2nd longitudinal modes; the 17th transversal and the 14th transversal and the 3rd longitudinal modes; the 22nd transversal and the 18th transversal and the 4th longitudinal modes; the 3152nd transversal and the 15th transversal and the 3137th longitudinal modes; the 3161st transversal and the 7th transversal and the 3154th longitudinal modes.

Case (e) in (5.26) corresponds to the case (d) in (5.26) and, consequently has the same solutions as (d).

Case (f) in (5.26) corresponds to the case (c) in (5.26) and, consequently has the same solutions as (c).

Case (g) in (5.26) has obviously no solutions for positive m and n , and for $n = 1, 2, \dots, m - 1$.

Case (h) in (5.26) corresponds to the case (a) in (5.26) and, consequently has the same solutions as (a).

5.3.3 Examples of resonance

In this section four different kind of resonance will be studied: for a lower, odd longitudinal mode, for a lower, even longitudinal mode, for a lower, odd transversal mode, and for a lower, even transversal mode. Solutions for functions $A_{m,0}(t_1)$, $B_{m,0}(t_1)$, $C_{m,0}(t_1)$, and $D_{m,0}(t_1)$ will be found for each resonant case.

Example of resonance for a lower, odd longitudinal mode

Let us consider the resonance conditions (a) in (5.23) and (d) in (5.26). Assuming that $-\omega_6^t = \omega_5^t - \omega_1^l$, it follows that $\omega_1^l = \omega_5^t + \omega_6^t$, and $\omega_5^t = -\omega_6^t + \omega_1^l$. To eliminate the secular terms in the solutions of the $\mathcal{O}(\varepsilon)$ -problems in (5.16) (after the substitution into (5.16) of the solutions (5.17) of the $\mathcal{O}(1)$ -problems for $\mathbf{w}(t_0, t_1,)$ and for $\mathbf{u}(t_0, t_1)$) the following differential equations are obtained.

For $-\omega_6^t = \omega_5^t - \omega_1^l$:

$$\begin{aligned} \frac{dC_{6,0}}{dt_1} + \frac{\delta_1}{2}C_{6,0} + \frac{P_1^2\pi^3 30}{8\omega_6^t}(D_{5,0}A_{1,0} - C_{5,0}B_{1,0}) &= 0, \\ \frac{dD_{6,0}}{dt_1} + \frac{\delta_1}{2}D_{6,0} + \frac{P_1^2\pi^3 30}{8\omega_6^t}(C_{5,0}A_{1,0} + D_{5,0}B_{1,0}) &= 0. \end{aligned} \quad (5.27)$$

For $\omega_1^l = \omega_5^t + \omega_6^t$:

$$\begin{aligned} \frac{dA_{1,0}}{dt_1} + \frac{\delta_2}{2}A_{1,0} - \frac{P_1^2\pi^3 30}{8\omega_1^l}(D_{5,0}C_{6,0} + C_{5,0}D_{6,0}) &= 0, \\ \frac{dB_{1,0}}{dt_1} + \frac{\delta_2}{2}B_{1,0} + \frac{P_1^2\pi^3 30}{8\omega_1^l}(C_{5,0}C_{6,0} - D_{5,0}D_{6,0}) &= 0. \end{aligned} \quad (5.28)$$

For $\omega_5^t = \omega_1^l - \omega_6^t$:

$$\begin{aligned} \frac{dC_{5,0}}{dt_1} + \frac{\delta_1}{2}C_{5,0} - \frac{P_1^2\pi^3 30}{8\omega_5^t}(C_{6,0}B_{1,0} - D_{6,0}A_{1,0}) &= 0, \\ \frac{dD_{5,0}}{dt_1} + \frac{\delta_1}{2}D_{5,0} + \frac{P_1^2\pi^3 30}{8\omega_5^t}(C_{6,0}A_{1,0} + D_{6,0}B_{1,0}) &= 0. \end{aligned} \quad (5.29)$$

It follows from (5.16) that all other $A_{m,0}(t_1)$, $B_{m,0}(t_1)$, $C_{m,0}(t_1)$ and $D_{m,0}(t_1)$ have to satisfy the non-resonance solutions (5.22), where $A_{m,0}(0)$, $B_{m,0}(0)$, $C_{m,0}(0)$, and $D_{m,0}(0)$ are still given by (5.19). It can be seen from these differential equations that there is an energy transfer from the 5th and the 6th transversal modes to the 1st longitudinal mode and vice versa.

Example of resonance for a lower, even longitudinal and lower, odd transversal modes

Let us consider resonance conditions (a) in (5.23) and (d) in (5.26). Assuming, that $-\omega_{11}^t = \omega_9^t - \omega_2^l$, it follows that $\omega_2^l = \omega_9^t + \omega_{11}^t$, and $\omega_9^t = -\omega_{11}^t + \omega_2^l$. Where $\omega_2^l = \omega_9^t + \omega_{11}^t$ is the case (i) in (5.20) and $\omega_9^t = -\omega_{11}^t + \omega_2^l$ is the case (iii) in (5.20).

To eliminate the secular terms in the solutions of the $\mathcal{O}(\varepsilon)$ -problems in (5.16) the following differential equations are obtained.

For $-\omega_{11}^t = \omega_9^t - \omega_2^l$:

$$\begin{aligned} \frac{dC_{11,0}}{dt_1} + \frac{\delta_1}{2}C_{11,0} + \frac{P_1^2\pi^3 198}{8\omega_1^t}(D_{9,0}A_{2,0} - C_{9,0}B_{2,0}) &= 0, \\ \frac{dD_{11,0}}{dt_1} + \frac{\delta_1}{2}D_{11,0} + \frac{P_1^2\pi^3 198}{8\omega_6^t}(C_{9,0}A_{2,0} + D_{9,0}B_{2,0}) &= 0. \end{aligned} \quad (5.30)$$

For $\omega_2^l = \omega_9^t + \omega_{11}^t$:

$$\begin{aligned} \frac{dA_{2,0}}{dt_1} + \frac{\delta_2}{2}A_{2,0} - \frac{P_1^2\pi^3 198}{8\omega_2^l}(D_{9,0}C_{11,0} + C_{9,0}D_{11,0}) &= 0, \\ \frac{dB_{2,0}}{dt_1} + \frac{\delta_2}{2}B_{2,0} + \frac{P_1^2\pi^3 198}{8\omega_2^l}(C_{9,0}C_{11,0} - D_{9,0}D_{11,0}) &= 0. \end{aligned} \quad (5.31)$$

For $\omega_9^t = \omega_2^l - \omega_{11}^t$:

$$\begin{aligned} \frac{dC_{9,0}}{dt_1} + \frac{\delta_1}{2}C_{9,0} - \frac{P_1^2\pi^3 198}{8\omega_9^t}(C_{11,0}B_{2,0} - D_{11,0}A_{2,0}) &= 0, \\ \frac{dD_{9,0}}{dt_1} + \frac{\delta_1}{2}D_{9,0} + \frac{P_1^2\pi^3 198}{8\omega_1^t}(C_{11,0}A_{2,0} + D_{11,0}B_{2,0}) &= 0. \end{aligned} \quad (5.32)$$

It follows from (5.16) that all other $A_{m,0}(t_1)$, $B_{m,0}(t_1)$, $C_{m,0}(t_1)$ and $D_{m,0}(t_1)$ have to satisfy the non-resonance solutions (5.22), where $A_{m,0}(0)$, $B_{m,0}(0)$, $C_{m,0}(0)$, and $D_{m,0}(0)$ are still given by (5.19). It can be seen from these differential equations that there is an energy transfer from the 9th and the 11th transversal modes to the 2nd longitudinal mode and vice versa.

Example of resonance for a lower, even transversal mode

Let us consider the resonance conditions (d) in (5.24) (this resonance condition also corresponds to the resonance conditions (a) and (c) in (5.25)) and the

resonance condition (a) in (5.26). Assuming, that $-\omega_{3173}^l = \omega_2^t - \omega_{3171}^t$, it follows that $\omega_2^t = -\omega_{3173}^l + \omega_{3171}^t$, and $\omega_{3171}^t = \omega_{3173}^l + \omega_2^t$. From the resonant case (a) in (5.26) it follows that the 2nd transversal mode also interacts with the 3165th longitudinal and the 3167th transversal modes, namely, $\omega_{3167}^t = \omega_2^t + \omega_{3165}^l$. It follows that also the following resonance conditions are possible: $\omega_2^t = \omega_{3167}^t - \omega_{3165}^l$ and $\omega_{3165}^l = \omega_{3167}^t - \omega_2^t$. Consequently, for the second transversal mode there is the following resonance condition: $\omega_2^t = -\omega_{3173}^l + \omega_{3171}^t = \omega_{3167}^t - \omega_{3165}^l$. It has to be noticed that if $\omega_2^t = -\omega_{3173}^l + \omega_{3171}^t = \omega_{3167}^t - \omega_{3165}^l$, then $-\omega_{3173}^l + \omega_{3171}^t = \omega_{3167}^t - \omega_{3165}^l$ and consequently $\omega_8^l = -\omega_{3171}^t - \omega_{3167}^t$, but this is not a resonant case, as it does not fit to the resonance conditions (5.20). To eliminate the secular terms in the solutions of the $\mathcal{O}(\varepsilon)$ -problems in (5.16) (after the substitution into (5.16) of the solutions (5.17) of the $\mathcal{O}(1)$ -problems for $\mathbf{w}(t_0, t_1,)$ and for $\mathbf{u}(t_0, t_1)$) the following differential equations are obtained.

For $\omega_2^t = -\omega_{3173}^l + \omega_{3171}^t = \omega_{3167}^t - \omega_{3165}^l$:

$$\begin{aligned} \frac{dC_{2,0}}{dt_1} + \frac{\delta_1}{2}C_{2,0} - \frac{P_1^2\pi^3 2 \cdot 10^7}{8\omega_2^t}((D_{3171,0}A_{3173,0} - C_{3171,0}B_{3173,0} \\ + D_{3167,0}A_{3165,0} - C_{3167,0}B_{3165,0})) = 0, \\ \frac{dD_{2,0}}{dt_1} + \frac{\delta_1}{2}D_{2,0} + \frac{P_1^2\pi^3 2 \cdot 10^7}{8\omega_2^t}((C_{3171,0}A_{3173,0} + D_{3171,0}B_{3173,0} \\ + C_{3167,0}A_{3165,0} + D_{3167,0}B_{3165,0})) = 0. \end{aligned} \quad (5.33)$$

For $\omega_{3165}^l = \omega_{3167}^t - \omega_2^t$:

$$\begin{aligned} \frac{dA_{3165,0}}{dt_1} + \frac{\delta_2}{2}A_{3165,0} - \frac{P_1^2\pi^3 2 \cdot 10^7}{8\omega_{3165}^l}(C_{2,0}D_{3167,0} - D_{2,0}C_{3167,0}) = 0, \\ \frac{dB_{3165,0}}{dt_1} + \frac{\delta_2}{2}B_{3165,0} + \frac{P_1^2\pi^3 2 \cdot 10^7}{8\omega_{3165}^l}(C_{2,0}C_{3167,0} + D_{2,0}D_{3167,0}) = 0. \end{aligned} \quad (5.34)$$

For $\omega_{3167}^t = \omega_{3165}^l + \omega_2^t$:

$$\begin{aligned} \frac{dC_{3167,0}}{dt_1} + \frac{\delta_1}{2}C_{3167,0} - \frac{P_1^2\pi^3 2 \cdot 10^7}{8\omega_{3167}^t}(D_{2,0}A_{3165,0} + C_{2,0}B_{3165,0}) = 0, \\ \frac{dD_{3167,0}}{dt_1} + \frac{\delta_1}{2}D_{3167,0} + \frac{P_1^2\pi^3 2 \cdot 10^7}{8\omega_{3167}^t}(C_{2,0}A_{3165,0} - D_{2,0}B_{3165,0}) = 0. \end{aligned} \quad (5.35)$$

For $\omega_{3171}^t = \omega_{3173}^l + \omega_2^t$:

$$\begin{aligned} \frac{dC_{3171,0}}{dt_1} + \frac{\delta_1}{2}C_{3171,0} - \frac{P_1^2\pi^3 2 \cdot 10^7}{8\omega_{3171}^t}(D_{2,0}A_{3173,0} + C_{2,0}B_{3173,0}) &= 0, \\ \frac{dD_{3171,0}}{dt_1} + \frac{\delta_1}{2}D_{3171,0} + \frac{P_1^2\pi^3 2 \cdot 10^7}{8\omega_{3171}^t}(C_{2,0}A_{3173,0} - D_{2,0}B_{3173,0}) &= 0. \end{aligned} \quad (5.36)$$

For $\omega_{3173}^l = \omega_{3171}^t - \omega_2^t$:

$$\begin{aligned} \frac{dA_{3173,0}}{dt_1} + \frac{\delta_2}{2}A_{3173,0} - \frac{P_1^2\pi^3 2 \cdot 10^7}{8\omega_{3173}^l}(C_{2,0}D_{3171,0} - D_{2,0}C_{3171,0}) &= 0, \\ \frac{dB_{3173,0}}{dt_1} + \frac{\delta_2}{2}B_{3173,0} + \frac{P_1^2\pi^3 2 \cdot 10^7}{8\omega_{3173}^l}(C_{2,0}C_{3171,0} + D_{2,0}D_{3171,0}) &= 0. \end{aligned} \quad (5.37)$$

It follows from (5.16) that all other $A_{m,0}(t_1)$, $B_{m,0}(t_1)$, $C_{m,0}(t_1)$ and $D_{m,0}(t_1)$ have to satisfy the non-resonance solutions (5.22), where $A_{m,0}(0)$, $B_{m,0}(0)$, $C_{m,0}(0)$, and $D_{m,0}(0)$ are still given by (5.19). It can be seen from these differential equations that there is an energy transfer from the 2nd, the 3167th and the 3171st transversal modes to the 3165th and the 3173rd longitudinal mode and vice versa.

Example of resonance for a lower, odd transversal mode

Let us consider the resonance conditions (d) in (5.23) (this resonance condition also corresponds to the resonance conditions (e) in (5.23)) and the resonance condition (a) in (5.26). Assuming, that $-\omega_{3163}^l = \omega_3^t - \omega_{3166}^t$, it follows that $\omega_3^t = -\omega_{3163}^l + \omega_{3166}^t$, and $\omega_{3166}^t = \omega_{3163}^l + \omega_3^t$. To eliminate the secular terms in the solutions of the $\mathcal{O}(\varepsilon)$ -problems in (5.16) (after the substitution into (5.16) the solutions (5.17) of the $\mathcal{O}(1)$ -problems for $\mathbf{w}(t_0, t_1,)$ and for $\mathbf{u}(t_0, t_1)$) the following differential equations are obtained.

For $\omega_3^t = \omega_{3166}^t - \omega_{3163}^l$:

$$\begin{aligned} \frac{dC_{3,0}}{dt_1} + \frac{\delta_1}{2}C_{3,0} - \frac{P_1^2\pi^3 3 \cdot 10^7}{8\omega_3^t}(D_{3166,0}A_{3163,0} - C_{3166,0}B_{3163,0}) &= 0, \\ \frac{dD_{3,0}}{dt_1} + \frac{\delta_1}{2}D_{3,0} + \frac{P_1^2\pi^3 3 \cdot 10^7}{8\omega_3^t}(C_{3166,0}A_{3163,0} + D_{3166,0}B_{3163,0}) &= 0. \end{aligned} \quad (5.38)$$

For $\omega_{3166}^t = \omega_{3163}^l + \omega_3^t$:

$$\begin{aligned} \frac{dC_{3166,0}}{dt_1} + \frac{\delta_1}{2}C_{3166,0} - \frac{P_1^2\pi^3 3 \cdot 10^7}{8\omega_{3166}^t}(D_{3,0}A_{3163,0} + C_{3,0}B_{3163,0}) &= 0, \\ \frac{dD_{3163,0}}{dt_1} + \frac{\delta_1}{2}D_{3166,0} + \frac{P_1^2\pi^3 3 \cdot 10^7}{8\omega_{3166}^t}(C_{3,0}A_{3163,0} - D_{3,0}B_{3163,0}) &= 0. \end{aligned} \quad (5.39)$$

For $\omega_{3163}^l = \omega_{3166}^t - \omega_3^t$:

$$\begin{aligned} \frac{dA_{3163,0}}{dt_1} + \frac{\delta_2}{2}A_{3163,0} - \frac{P_1^2\pi^3 3 \cdot 10^7}{8\omega_{3163}^l}(C_{3,0}D_{3166,0} - D_{3,0}C_{3166,0}) &= 0, \\ \frac{dB_{3163,0}}{dt_1} + \frac{\delta_2}{2}B_{3163,0} + \frac{P_1^2\pi^3 3 \cdot 10^7}{8\omega_{3163}^l}(C_{3,0}C_{3166,0} + D_{3,0}D_{3166,0}) &= 0. \end{aligned} \quad (5.40)$$

It follows from (5.16) that all other $A_{m,0}(t_1)$, $B_{m,0}(t_1)$, $C_{m,0}(t_1)$ and $D_{m,0}(t_1)$ have to satisfy the non-resonance solutions (5.22), where $A_{m,0}(0)$, $B_{m,0}(0)$, $C_{m,0}(0)$, and $D_{m,0}(0)$ are still given by (5.19). It can be seen from these differential equations that there is an energy transfer from the 3rd and the 3166th transversal modes to the 3163th longitudinal mode and vice versa.

5.4 Discussion on the equations of motion (5.1)

The equations (5.1) were derived from Hamilton's principle, assuming that the torsional force and the bending moment are given by

$$F = EA[(1 + W_X)^2 + U_X^2]^{1/2} - 1,$$

and

$$M = EIU_{XX},$$

respectively, where for the bending moment it is assumed that $dX = ds$, where ds is the belt curvature. This assumption is valid only for sufficiently small deflections. With increasing amplitudes, the influence of nonlinear factors becomes more important (see also [51] and [7]), and nonlinearities have to be taken into account. For example, the bending moment has to be written (by taking into account the actual curvature of the belt) in the following form:

$$M = \frac{EIU_{XX}}{[(1 + W_X)^2 + U_X^2]^{1/2}}. \quad (5.41)$$

Using the same procedure as in [26] the following equations of motion can be obtained:

$$\begin{aligned}
& \rho A W_{\tau\tau} + 2\rho A V_b W_{X\tau} + \rho A V_{b\tau}(1 + W_X) + (\rho A V_b^2 - EA) W_{XX} \\
&= (EA - R_0) \frac{(1 + W_X) U_X U_{XX} - U_X^2 W_{XX}}{[(1 + W_X)^2 + U_X^2]^{3/2}} - EI \frac{(1 + W_X) U_X U_{XXX}}{[(1 + W_X)^2 + U_X^2]^{3/2}} \\
&+ EI \frac{U_{XX}^2 ((W_{XX}(2(W_X + 1)^2 - U_X^2)) + 3U_{XX} U_X (W_X + 1))}{2[(1 + W_X)^2 + U_X^2]^{5/2}}, \\
\\
& \rho A U_{\tau\tau} + 2\rho A V_b U_{X\tau} + \rho A V_{b\tau} U_X + (\rho A V_b^2 - EA) U_{XX} \\
&+ EI \frac{U_{XXXX}}{2[(1 + W_X)^2 + U_X^2]^{1/2}} \\
&= (R_0 - EA) \frac{(1 + W_X)^2 U_{XX} - (1 + W_X) U_X W_{XX}}{[(1 + W_X)^2 + U_X^2]^{3/2}} \\
&+ EI \frac{U_{XXX}(U_{XX} U_X + 2W_{XX}(W_X + 1)) + W_{XXX} U_{XX}(W_X + 1)}{2[(1 + W_X)^2 + U_X^2]^{3/2}} \\
&+ EI \frac{U_{XX} W_{XX}(W_{XX} ((2(W_X + 1)^2 - U_X^2)) + 3U_{XX} U_X (W_X + 1))}{2[(1 + W_X)^2 + U_X^2]^{5/2}}.
\end{aligned} \tag{5.42}$$

The equations of motion after the braking moment τ_0 are given by

$$\begin{aligned}
& \rho A W_{\tau\tau} - EA W_{XX} \\
&= (EA - R_0) \frac{(1 + W_X) U_X U_{XX} - U_X^2 W_{XX}}{[(1 + W_X)^2 + U_X^2]^{3/2}} - EI \frac{(1 + W_X) U_X U_{XXX}}{[(1 + W_X)^2 + U_X^2]^{3/2}} \\
&+ EI \frac{U_{XX}^2 ((W_{XX}(2(W_X + 1)^2 - U_X^2)) + 3U_{XX} U_X (W_X + 1))}{2[(1 + W_X)^2 + U_X^2]^{5/2}}, \\
\\
& \rho A U_{\tau\tau} - EA U_{XX} + EI \frac{U_{XXXX}}{2[(1 + W_X)^2 + U_X^2]^{1/2}} \\
&= (R_0 - EA) \frac{(1 + W_X)^2 U_{XX} - (1 + W_X) U_X W_{XX}}{[(1 + W_X)^2 + U_X^2]^{3/2}} \\
&+ EI \frac{U_{XXX}(U_{XX} U_X + 2W_{XX}(W_X + 1)) + W_{XXX} U_{XX}(W_X + 1)}{2[(1 + W_X)^2 + U_X^2]^{3/2}} \\
&+ EI \frac{U_{XX} W_{XX}(W_{XX} ((2(W_X + 1)^2 - U_X^2)) + 3U_{XX} U_X (W_X + 1))}{2[(1 + W_X)^2 + U_X^2]^{5/2}}.
\end{aligned} \tag{5.43}$$

Using a Taylor series, the denominators in (5.43) can be approximated by $[(1 + W_X)^2 + U_X^2]^{-1/2} = 1 - W_X + W_X^2 - \frac{1}{2}U_X^2 + \mathcal{O}(3)$, $[(1 + W_X)^2 + U_X^2]^{-3/2} = 1 - 3W_X + 6W_X^2 - \frac{3}{2}U_X^2 + \mathcal{O}(3)$, and $[(1 + W_X)^2 + U_X^2]^{-5/2} = 1 - 5W_X + 15W_X^2 - \frac{5}{2}U_X^2 + \mathcal{O}(3)$, where $\mathcal{O}(3)$ stands for terms of degree 3 or higher.

Assuming that the displacement in the transversal direction is of the same order as the displacement in the longitudinal direction it follows that the equations of motion (5.43) can now be written (approximately) as

$$\begin{aligned}
 \rho A W_{\tau\tau} - E A W_{XX} &= (E A - R_0)(U_X U_{XX} - 2 U_X U_{XX} W_X - U_X^2 W_{XX}) \\
 &\quad - E I (U_{XXX} U_{XX} - 2 U_{XXX} U_{XX} W_X - U_{XX}^2 W_{XX}), \\
 2 \rho A U_{\tau\tau} - 2 E A U_{XX} + E I U_{XXXX} &= 2(R_0 - E A)(U_{XX} - U_{XX} W_X \\
 &\quad + U_{XX} W_X^2 - \frac{3}{2} U_{XX} U_X^2 - U_X W_{XX} + 2 U_X W_{XX} W_X) \\
 &\quad + E I (U_{XXXX} W_X - U_{XXXX} W_X^2 + \frac{1}{2} U_{XXXX} U_X^2 + U_{XXX} U_{XX} U_X \\
 &\quad + 2 U_{XXX} W_{XX} - 4 U_{XXX} W_{XX} W_X + W_{XXX} U_{XX} \\
 &\quad - 2 W_{XXX} U_{XX} W_X - 2 U_{XX} W_{XX}^2).
 \end{aligned} \tag{5.44}$$

In order to get more accurate results the equations (5.44) have to be studied and cubic nonlinear terms (probably) have to be included in some cases.

5.5 Conclusions and remarks

The dynamical behaviour of a conveyor belt system after a sudden stoppage was studied in this chapter. The equations of motion for the belt system under consideration included longitudinal and transversal displacements. It was assumed that displacements of the belt in longitudinal and transversal directions are of the same order for some interval of time after the braking moment τ_0 , that is, $W = \mathcal{O}(U)$, and $U = \mathcal{O}(W)$. Depending on the order of the initial displacements $u(x, t)$ and the order of $w(x, t)$, (namely $u(x, t) = \mathcal{O}(1)$, $u(x, t) = \mathcal{O}(\varepsilon)$, or $u(x, t) = \mathcal{O}(\varepsilon^2)$) the orders of all terms in the equations of motion were found for different values of the parameters P_0^2 , P_1^2 , L , and the (small) parameter ε . For the case $u(x, t) = \mathcal{O}(\varepsilon)$ small damping terms (of order ε) in both directions were introduced into the system. To avoid the secular terms in the solution of the problem, a two time-scales perturbation method was used. Resonant longitudinal and transversal frequency-combinations were found. It turns out that there are infinitely many complicated, resonance frequency-combinations. A possible energy transfer from the longitudinal direction to the transversal direction after the braking moment was studied. Four different resonance cases were studied for the lowest longitudinal modes (odd end even kind of resonances) and for the lowest transversal modes (odd end even kind of resonances). It turns out that if there is a small amount of energy in the higher order longitudinal mode then large vibrations in the lower transversal vibration modes can be induced. Because of that energy transfer between the longitudinal and transversal directions it may cause problems for

the normal maintenance process of the conveyor belt system. In order to get more accurate results, nonlinear moments have to be studied, consequently the equations of motion become (5.44). The influence of the belt velocity V_b after the braking moment τ_0 also have to be studied more closely. These are the subjects for the future research.

CHAPTER 6

Conclusions

In this chapter some conclusions will be given and some possibilities for future research will be discussed. The goal of this thesis was the investigation of different aspects (such as applicability of methods, internal resonances, stability, energy transfer between modes, etc.) related to axially moving continua, which are mathematically modelled as moving string or beam equations.

In Chapter 2 an initial-boundary value problem for a linear, non-homogeneous axially moving string equation has been studied. The velocity of the string is assumed to be constant, and the non-homogeneous terms in the string equation are due to external forces acting on the string. To solve the initial-boundary value problem two methods are presented. One of these methods is the method of modal analysis based on eigenfunction expansions. To apply this method an operator notation has to be introduced, an inner product has to be defined, an eigenvalue problem has to be solved, and orthonormality relations have to be determined. Altogether this method is rather complicated to apply to these types of problems. For that reason in chapter 2 it was proposed to apply the Laplace transform method to these types of problems. When this method is applied poles (and the order of the poles) have to be determined, residues have to be calculated, and Cauchy's theorem has to be used (that is, integrals have to be evaluated by using the theory of complex variables). To construct the solution by using the Laplace transform method is rather straightforward and more easy than the use of the method of modal analysis based on eigenfunction expansions.

In Chapter 3 an initial-boundary value problem for a linear equation, describing an axially moving string has been studied. This equation can be used

as a model for the transversal vibrations of a conveyor belt system. The conveyor belt is assumed to move in one direction with non-constant speed $V(t)$, that is, $V(t) = V_0 + \varepsilon\alpha \sin(\omega t)$, where $0 < \varepsilon \ll 1$ and V_0, α, ω are positive constants. For V_0 it is assumed that $V_0 < c$, where c is the wave speed, and for the belt speed $V(t)$ it is assumed that its value is of the same order as the wave speed. Formal approximations of the solution of the initial-boundary value problem have been constructed. In this chapter it was proposed to apply the two time-scales perturbation method directly to the partial differential equation for the moving string. The solutions of the following $\mathcal{O}(\varepsilon^n)$ -problems were constructed with the help of the Laplace transform method, in order to find formal approximations of the solution. It turns out that there are infinitely many values of ω that cause internal resonances in the conveyor belt system. In fact, that happens when ω is equal to any natural frequency of the moving string, that is, $\Omega_n = \frac{\pi n(V_0^2 - c^2)}{lc}$, where $n = 1, 2, \dots$, or when ω is equal to a sum or difference of those natural frequencies. It also turned out that only a one-mode approximation (that was used in some other papers on this subject) is not accurate, as the explicit solution of the boundary-value problem consists of infinitely many modes. Four cases were studied in this chapter: ω is not in a neighbourhood of any Ω_n , $\omega = \Omega_1$, $\omega = \Omega_2$, and $\omega = \Omega_{m^*}$, where $m^* \in \mathbb{Z}^+$ and fixed. For the second case, that is, when ω is equal to the lowest natural frequency of the moving string, a first integral was found and it was proved, that the truncation method does not give accurate results on long time-scales. All approximations obtained by the method as introduced in this chapter are valid on long time-scales, that is, on time-scales of order ε^{-1} . Moreover, the results as obtained in [16] were proved again.

In Chapter 4 an initial-boundary value problem for a linear equation, describing an axially moving stretched beam has been studied. This equation can be used as a model for the transversal vibrations of a conveyor belt system. The axially moving belt is assumed to move in one direction with a non-constant speed $V(t)$, that is, $V(t) = \varepsilon(V_0 + \alpha \sin(\omega t))$, where $0 < \varepsilon \ll 1$, and where V_0, α and ω are positive constants. For V_0 it is assumed that $V_0 > 0$ and $V_0 > |\alpha|$. A new model approach describing the transient “from string to beam” behaviour, based on the calculations of the natural frequencies has been proposed. The influence of the bending stiffness on the stability properties of the solution of the problem has been studied.

Depending on the natural frequencies the original problem is split up into two submodels: a string model for the lower frequencies and a stretched beam model for the higher frequencies. Each submodel has its own physical and mathematical properties. For the string model, for instance, the discussion on the applicability of the truncation method is not (mathematically) relevant anymore (see also [16]). In this combination model the submodels are

interacting due to internal resonances, and the model equations depend on the frequencies and on the vibration mode numbers. The proposed model is a more realistic approach to describe the dynamical behaviour of a traveling continuum as the bending stiffness becomes more important for the higher order vibration modes. For the lower frequencies the bending stiffness can be neglected and the string equation can be used. The regions of applicability of the simplified models were found for different values of the bending stiffness parameter and for different values of the relative errors in the frequencies. It turns out that there are infinitely many values of ω that give rise to internal resonances in the axially moving belt system. In fact, that happens when ω is equal to any sum or difference combination of the natural frequencies of the string and (or) the stretched beam equations. In the non-resonant case it can clearly be seen that for the lower frequencies (when the string model is used) the perturbation approach leads to improvements in the frequencies taking into account the small bending stiffness. The formal approximations of the solution and the stability properties in some resonant cases have been determined for four different values of the relative error in the frequencies: 5%, 3%, 1% and 0.1%. It can be concluded that the properties of the studied problem remain the same and correspond to the properties of the exact solution of the problem when the relative errors in the frequencies are less than 5 %. An important implication of the results as presented in Chapter 4 is that for these type of problems the use of only string-like models is not appropriate. To describe the dynamics of these type of conveyor belt problems correctly one has to include bending stiffness in the model (also when the bending stiffness is assumed to be small).

It has also to be noticed that the introduction of a damping term does not solve the truncation problem for the string-like equation, at least if the damping is assumed to be small and of order ε . The viscous damping or the structural damping can be taken into account for the partial differential equations, leading to the appearance of extra terms in these equations: $u_t + Vu_x$, $u_{txx} + Vu_{xxx}$ or $u_{txxxx} + Vu_{xxxxx}$, respectively (for the moving frame of reference). If the damping in the problem is assumed to be small, that is of order ε , then after applying the two time-scales perturbation method to the equations, terms $u_t + Vu_x$, $u_{txx} + Vu_{xxx}$ or $u_{txxxx} + Vu_{xxxxx}$ will appear. At high frequencies the structural damping terms can not be considered to be small. The solutions of the $\mathcal{O}(1)$ -problems still can not be truncated in these cases and still has to be written in an infinite series representation from the mathematical point of view. For future research it will be interesting to study more complicated cases for the transversal and longitudinal motions of axially moving beams, including those cases for which the boundary conditions are such that energy inflow or outflow is possible through these boundaries. Numerical results (based on

spectral element methods) for these type of problems can already be found in [44], and [45].

The dynamical behaviour of a conveyor belt system after a sudden stoppage was studied in Chapter 5. The nonlinear equations of motion for the belt system under consideration included longitudinal and transversal displacements. It was assumed that displacements of the belt in longitudinal and transversal directions are of the same order for some interval of time after the braking moment τ_0 , that is, $W = \mathcal{O}(U)$, and $U = \mathcal{O}(W)$. Depending on the order of the initial displacements $u(x, t)$ and the order of $w(x, t)$, (namely $u(x, t) = \mathcal{O}(1)$, $u(x, t) = \mathcal{O}(\varepsilon)$, or $u(x, t) = \mathcal{O}(\varepsilon^2)$) the orders of all terms in the equations of motion were found for different values of the parameters P_0^2 , P_1^2 , L , and the (small) parameter ε . For the case $u(x, t) = \mathcal{O}(\varepsilon)$ small damping terms (of order ε) in both directions were introduced into the system. To avoid the secular terms in the solution of the problem, a two time-scales perturbation method was used. Resonant longitudinal and transversal frequency-combinations were found. It turns out that there are infinitely many complicated, resonant frequency-combinations. A possible energy transfer from the longitudinal direction to the transversal direction after the braking moment was studied. Four different resonance cases were studied for the lowest longitudinal modes (odd and even kind of resonances) and for the lowest transversal modes (odd and even kind of resonances). It turns out that if there is a small amount of energy in the higher order longitudinal mode then large vibrations in the lower transversal vibration modes can be induced. Because of that energy transfer between the longitudinal and transversal directions it may cause problems for the normal maintenance process of the conveyor belt system. For future research it will be interesting to study equations of motion (5.44). These equations were derived by taking into account the actual curvature of the belt for the bending moment. Another interesting and challenging problem to study is the dynamic behaviour of the belt near the pulleys at the braking moment τ_0 . This so-called boundary layer problem is also a subject for future research.

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In axially moving structures like conveyor belt systems, magnetic tapes, and so on, vibrations occur due to the presence of different kinds of imperfections in the systems. For these structures internal resonances can lead to severe vibrations. Resonance free conveyor belt systems can be constructed if the frequencies of the system are known.

In this thesis two models for an axially moving continuum have been studied: a string-like model (mathematically modelled by a wave equation) and a beam-string-like model (mathematically modelled by the Euler-Bernoulli beam equation with an additional tension term). The corresponding initial-boundary value problems have been solved approximately or exactly by using the multiple timescales perturbation method or the Laplace transform method. Exact solutions and formal approximations of the solutions for some string-like problems have been found in the form of Fourier series. It has been shown that the truncation method for the string-like problem can not be applied in order to obtain an accurate solution or approximation on a long time-scale.

A new model approach describing the transient “from string to beam” behaviour, based on the calculations of the natural frequencies of the system has been proposed. The influence of the bending stiffness on the stability properties of the solution of the problem has been studied. An important implication of the results as presented in this thesis is that for these types of axially moving continua problems the use of only string-like models is not appropriate. To describe the dynamics of these types of problems correctly one has to include (small) bending stiffness in the model. It has also been shown that the introduction of a damping term does not solve the truncation problem for the string-like equation, at least if the damping is assumed to be small.

A possible energy transfer between transversal and longitudinal vibration-

directions has been studied in case of a sudden stoppage of the conveyor belt system. It has been shown that this energy transfer can occur depending on the existing internal resonances in the system which are determined by the belt system parameters.

In axiaal bewegende systemen zoals transportbanden, magnetische tapes, enz. treden trillingen op die worden veroorzaakt door de aanwezigheid van verschillende imperfecties. Voor deze systemen kan interne resonantie leiden tot extreme trillingen. Indien de natuurlijke frequenties bekend zijn, kunnen resonantie "vrije" systemen worden ontworpen.

In dit proefschrift worden twee modellen voor een axiaal bewegend continuüm bestudeerd: een snaarmodel (wiskundig gemodelleerd d.m.v. de golfvergelijking) en een balkmodel (wiskundig gemodelleerd d.m.v. de Euler-Bernoulli vergelijking voor een balk met een additionele voorspanningsterm). Voor de corresponderende begin-randwaarde problemen zijn of exacte oplossingen bepaald met behulp van de Laplace transformatie methode of zijn benaderingen van deze oplossingen geconstrueerd met behulp van de meer-tijdschalen storingsmethode. Voor enkele snaarmodellen zijn exacte oplossingen en formele benaderingen gevonden in de vorm van Fourier reeksen. Ook is voor het snaarmodel bewezen dat de "truncation" methode niet kan worden gebruikt om een oplossing of benadering te vinden welke geldig is op een lange tijdschaal.

Gebaseerd op de berekeningen van de natuurlijke frequenties wordt een nieuw model aanbevolen om de overgang van snaar naar balk gedrag van het systeem te beschrijven. Hierbij is ook gekeken naar de invloed van de buigstijfheid op de stabiliteitseigenschappen van de oplossing. Een belangrijke gevolgtrekking is dat voor dit soort axiaal bewegende systemen het gebruik van een snaarmodel niet correct is. Om de dynamica van dit soort systemen te beschrijven, is het noodzakelijk om de (kleine) buigstijfheid van het systeem te implementeren. Ook is aangetoond dat de introductie van een extra kleine dempingsterm het "truncation" probleem voor een snaarmodel niet oplost.

Voor het geval van een discontinue stop van een transportbandsysteem is

een mogelijke energieoverdracht tussen de laterale en axiale trillingsrichtingen bestudeerd. Deze energieoverdracht kan voorkomen afhankelijk van de natuurlijke frequenties van het systeem, die worden bepaald door de systeemvariabelen.

Аннотация

**Некоторые вопросы колебаний продольно движущихся
континуальных систем**

Светлана Хагераатс-Пономарева

Разного рода погрешности в изготовлении продольно движущихся континуальных систем, таких как конвейерные ленты и т.п., могут приводить к возникновению внутренних резонансов, которые, в свою очередь, вызывают опасные колебания этих конструкций. Знание собственных частот позволяет избежать нежелательных режимов работы конвейерных лент уже на стадии конструирования.

В диссертации рассмотрены две модели, позволяющие качественно описать колебания движущихся конвейерных лент: колебания струны и балки Бернулли-Эйлера с учетом продольных усилий. Для решения соответствующих начально-краевых задач применялись методы Лапласа и теории возмущений, в частности метод многих масштабов. Для продольных колебаний движущейся струны были найдены решения в виде рядов Фурье. Доказано, что метод усечения рядов не может применяться для построения приближений решения, действительных на больших временах.

В диссертации предложен новый метод, позволяющий качественно описывать переход от уравнения поперечных колебаний движущейся струны к уравнению поперечных колебаний движущейся балки, основанный на вычислении собственных частот рассматриваемой модели.

Отдельно исследован вопрос о влиянии изгибной жесткости балки на устойчивость решения. Главный вывод представленного исследования заключается в том, что для качественно верного описания поперечных колебаний продольно движущихся континуальных систем необходимо дополнительно учитывать в уравнении изгибную жесткость. Было также доказано, что учет внутреннего демпфирования не влияет на применимость метода усечений рядов к волновому уравнению, по крайней мере в случае малого внутреннего демпфирования.

При исследовании колебаний, вызванных внезапной остановкой конвейерной ленты, особое внимание было уделено обмену энергией между продольными и поперечными модами колебаний. Была исследована зависимость энергетического обмена между продольными и поперечными модами колебаний ленты от ее механических и геометрических параметров.

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June 2009
Svetlana Hageraats-Ponomareva

Curriculum Vitae

Svetlana Hageraats-Ponomareva (Khageraats-Ponomareva, Ponomareva) was born in Leningrad, Soviet Union (nowadays Saint-Petersburg, Russia), on October 28, 1979. In 1996 she completed her secondary education at school N126 in Saint-Petersburg. In the same year she attended the course Applied Mechanics at Saint-Petersburg State Polytechnical University. In 2002 she defended her Master thesis entitled: “Finite element simulation and research of spatial deflected mode of lumbar part of spinal cord“ under supervision of Prof. A.I. Borovkov and obtained her Master of Science degree (with honors). After the graduation she joined research group of Prof. B.E. Mel’nikov at Saint-Petersburg State Polytechnical University, Strength of Materials Department. In September 2003 she came to the Netherlands and joined as a research staff member Delft Institute of Applied Mathematics at Delft University of Technology, where in April 2004 she started her Ph.D. research under supervision of Dr.ir. W.T. van Horssen.