RECENT RESULTS OBTAINED FROM A NUMERICAL WAVE THEORY FOR HIGHLY NONLINEAR SHALLOW WATER WAVES

Robert G. Dean

Department of Coastal and Oceanographic Engineering
University of Florida

Presented at the

SYMPOSIUM ON LONG WAVES

University of Delaware
Newark, Delaware

September 10-11, 1970

RECENT RESULTS OBTAINED FROM A NUMERICAL WAVE THEORY FOR HIGHLY NONLINEAR SHALLOW WATER WAVES

INTRODUCTION

Analytical representation of shallow water wave phenomena is complicated due, in part, to the fact that nonlinear features are important in a predominant number, if not all, problems of shallow water wave motion. It therefore may not even be approximately valid to utilize the Airy wave theory and to assume that various wave components behave independently of one another.

At present, even for periodic wave motion, predictions of shallow water wave phenomena based on various available theories differ by disturbing amounts. In this paper, several features of a numerical wave theory (Stream function) are reviewed and explored to demonstrate: (a) the agreement between theory and laboratory measurements, and (b) the differences between the numerical wave theory and the Airy wave theory. In particular, total wave energy, momentum and momentum flux, pressure response factors, maximum drag forces, shoaling coefficients, etc. are examined. The purpose of this paper is to direct attention to the very significant differences that exist between the two theories in shallow water and the need for additional research to resolve the differences noted.

STREAM FUNCTION WAVE THEORY

The Stream Function Wave Theory has been presented extensively elsewhere(1),(2),(3),(4) and will therefore only be described briefly in this paper. The theory is strictly applicable for a two dimensional wave propagating without change of form in water of uniform depth. Advantages of the theory include: (1) the form of the solution is inherently better suited (say than the Stokes' representation) for satisfying the nonlinear free surface boundary conditions, (2) the theory can be readily extended to reasonably high orders, (3) the theory has been shown to provide better fits to the boundary conditions than other theories established for shallow water conditions, and (4) for the limited data available, the theory provides a significantly better fit to horizontal water particle velocities measured in the laboratory.

Figure 1 is the result of a study to determine which theories provided best agreement to the two nonlinear free surface boundary conditions. The Stream function was calculated to the fifth order, and the study demonstrated that higher orders of the theory would have extended the "best fit" of the Stream function theory significantly into the shallow water region.

Figures 2, 3, and 4 represent comparisons of the maximum horizontal water particle velocities as calculated using a number of wave theories and as measured by Le Méhauté, et.al.(5); the wave conditions fall within the intermediate and shallow water ranges. Computation of the standard deviations between each of the theories and the measurements (8 different waves) indicated that the Stream function provided the least standard deviation, with the second lowest standard deviation approximately 40% higher than that provided by the Stream function theory. Figures 2, 3, and 4 span the range of relative agreement between the Stream function theory and the data (8 cases) with the best fit shown in Figure 2, an average fit in Figure 3, and the worst fit in Figure 4. Figures 5 and 6 are comparisons of the predictions of various wave theories with a measured maximum vertical water particle velocity distribution and with a measured wave profile, respectively. In both cases, the fit provided by the Stream function wave theory

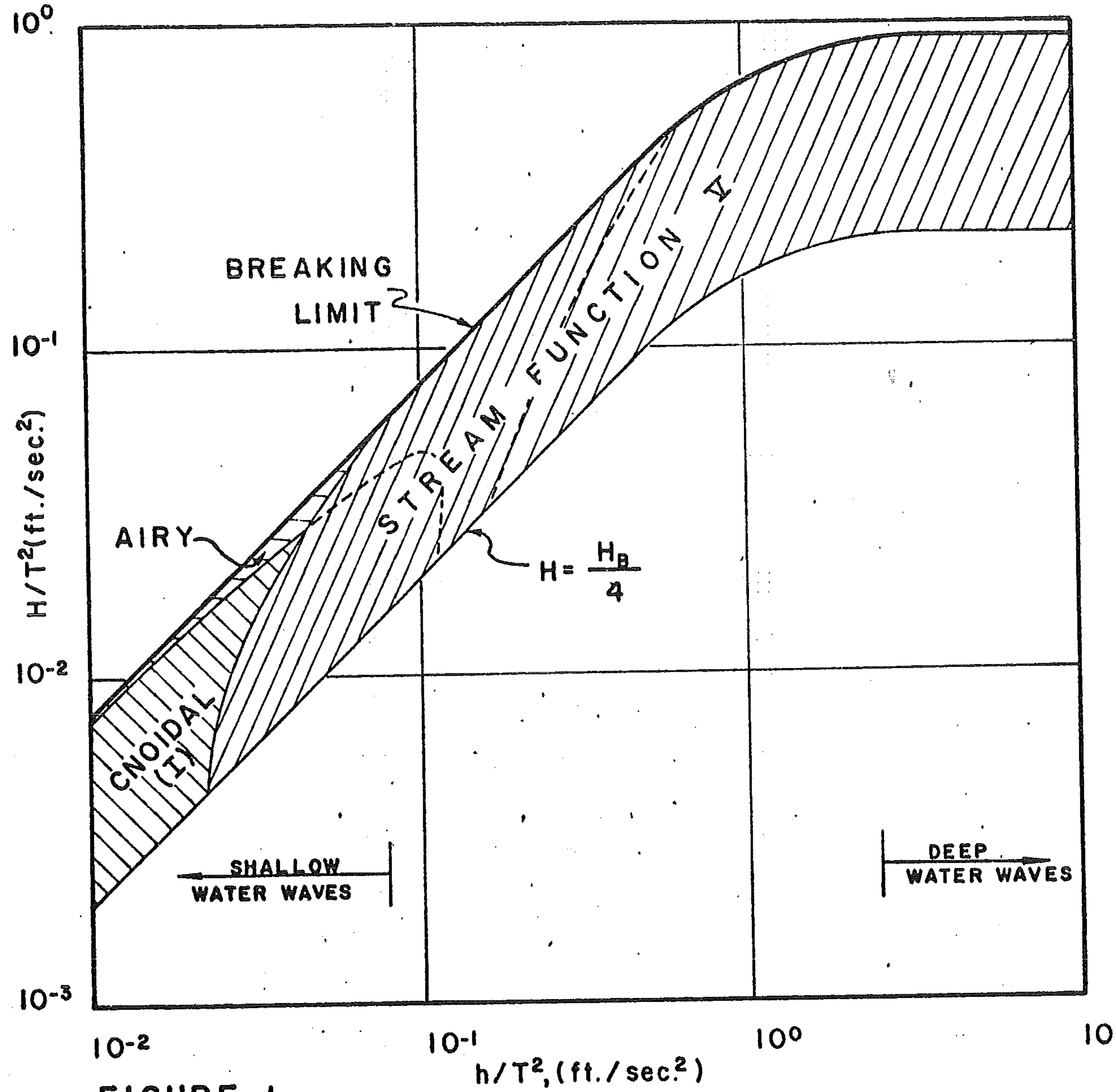


FIGURE I.
PERIODIC WAVE THEORIES PROVIDING BEST FIT TO DYNAMIC FREE SURFACE BOUNDARY CONDITION (ANALYTICAL AND STREAM FUNCTION V THEORIES)

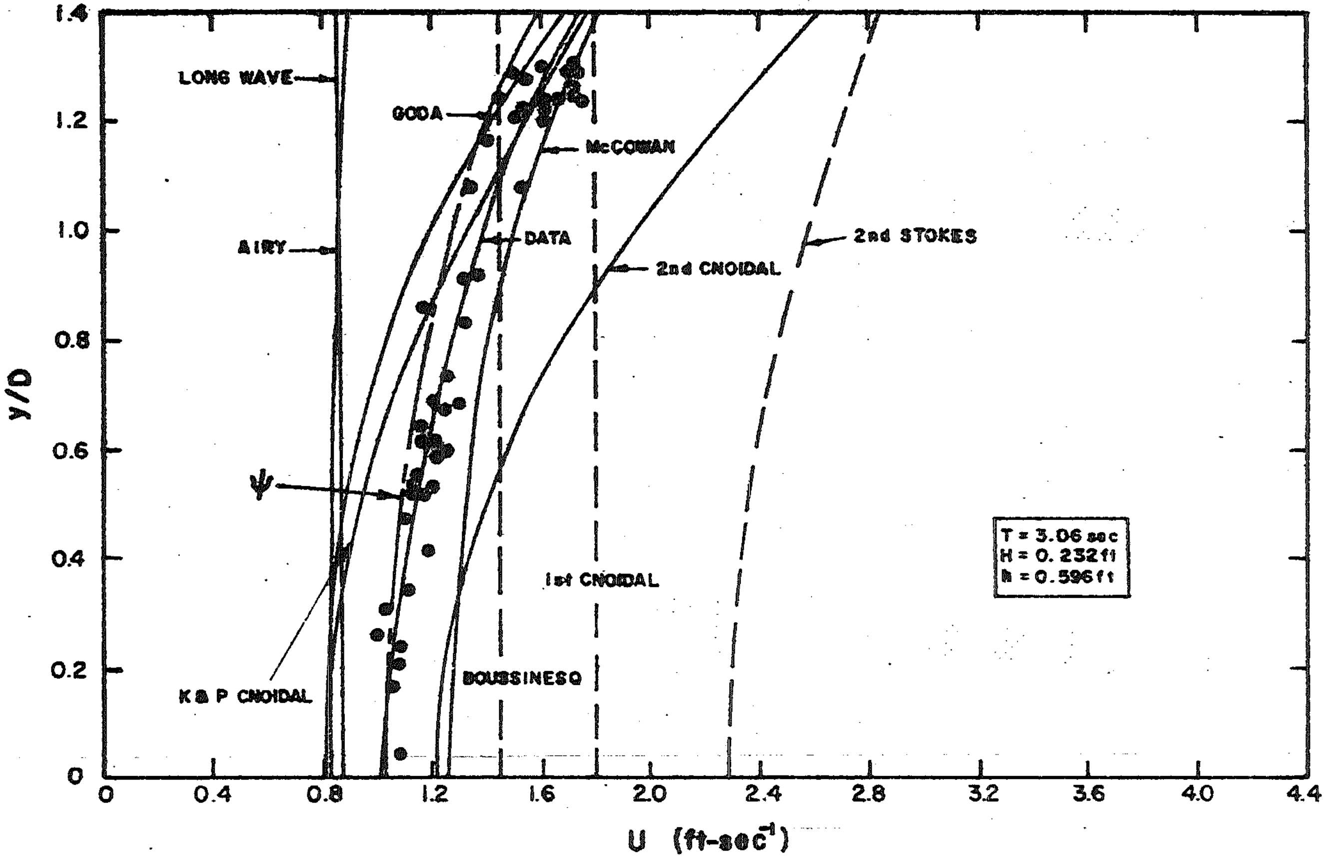


Figure 2. Horizontal Water Particle Velocity Under the Crest, Case 3.

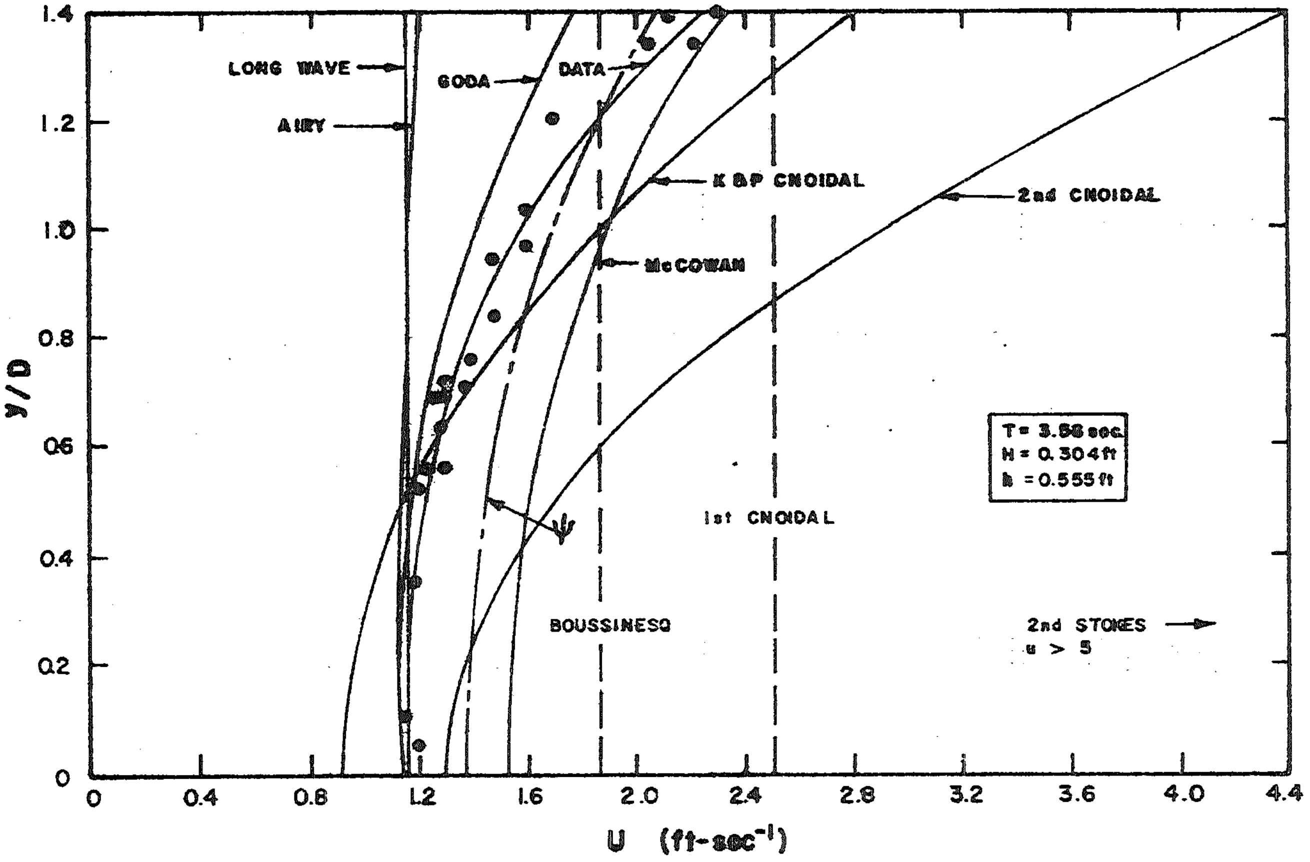


Figure 3. Horizontal Water Particle Velocity Under the Crest, Case 8.

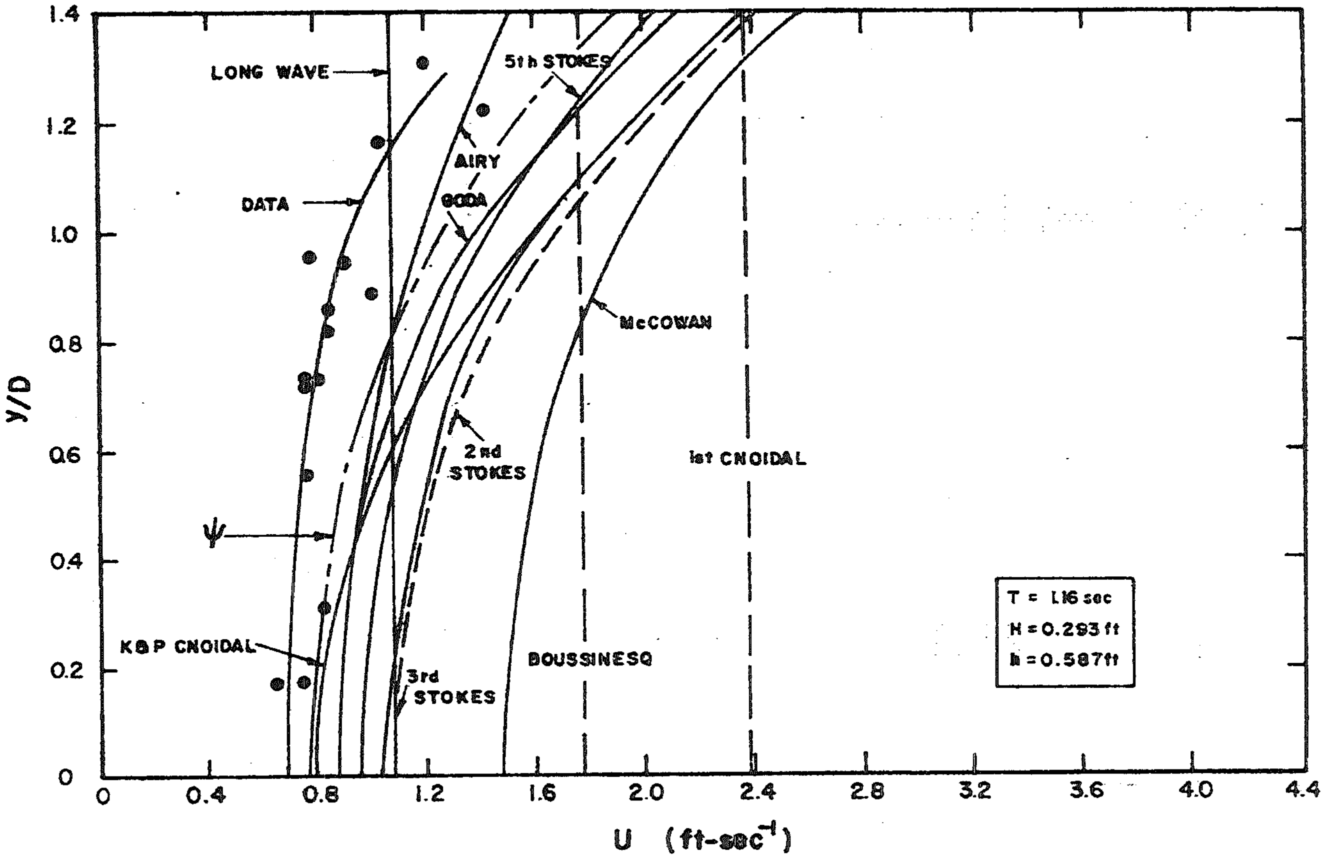


Figure 4. Horizontal Water Particle Velocity Under the Crest, Case 5.

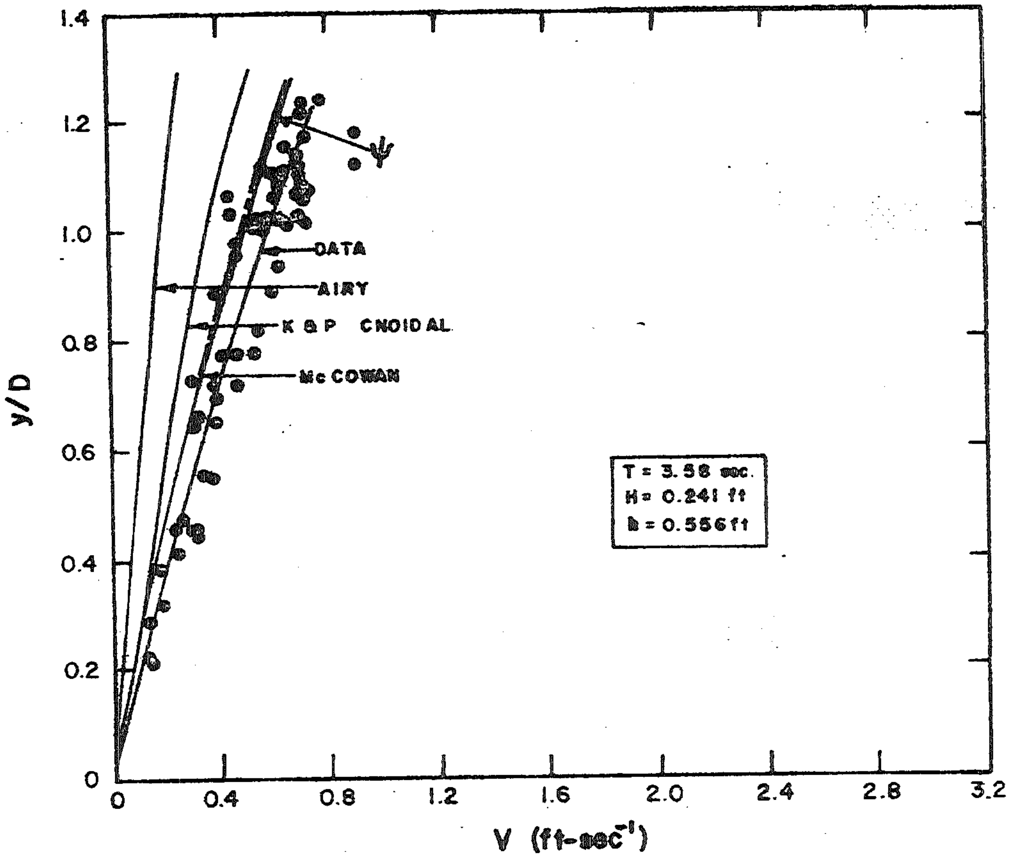


Figure 5. Vertical Water Particle Velocity, Case 9.

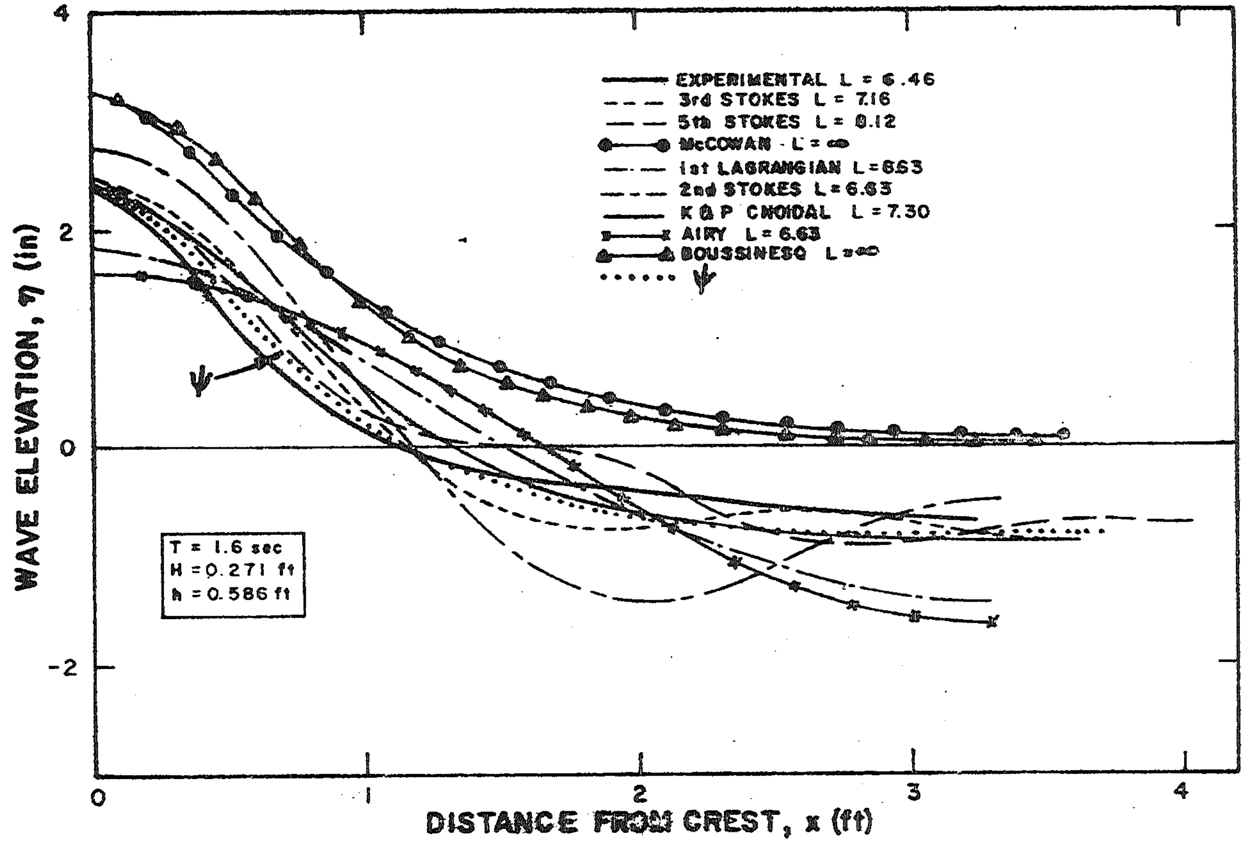


Figure 6. Free Surface Elevation, Case 10.

is considered as good or better than that provided by the other theories. Of particular interest, is the relatively poor fit associated with some of the theories developed particularly for shallow water conditions.

RESULTS OBTAINED FROM STREAM FUNCTION THEORY

If it is accepted that previous investigations indicate that the Stream function wave theory does provide some significant advantage over other available theories, then it is interesting to compare differences that would result by use of the Stream function wave theory and the Airy wave theory. The Airy wave theory was chosen for comparison, because it is the most widely employed theory for all relative depths, including the shallow water region. For example, important shallow water calculations employing the Airy wave theory include: shoaling, energy, momentum flux, refraction, etc. Examples of the differences in several variables follow.

Maximum Drag Forces and Moments

The maximum drag force is a variable of considerable engineering importance. Figure 7 presents the percentage differences in maximum drag forces that are obtained by use of the Stream function and Airy wave theorys. It is seen that in deep water, the maximum percentage difference is approximately 5% whereas in shallow water the corresponding difference is greater than 60%. It is reasonable to suppose that comparison with the Airy wave theory will unduly exaggerate the disagreement between theories in the shallow water region; this is not necessarily the case. For example, calculations presented in an earlier paper(3) have shown that the Cnoidal wave theory as presented by Laitone(6) would predict maximum drag forces that are 105% greater than the Stream function wave theory for a near-breaking wave in shallow water.

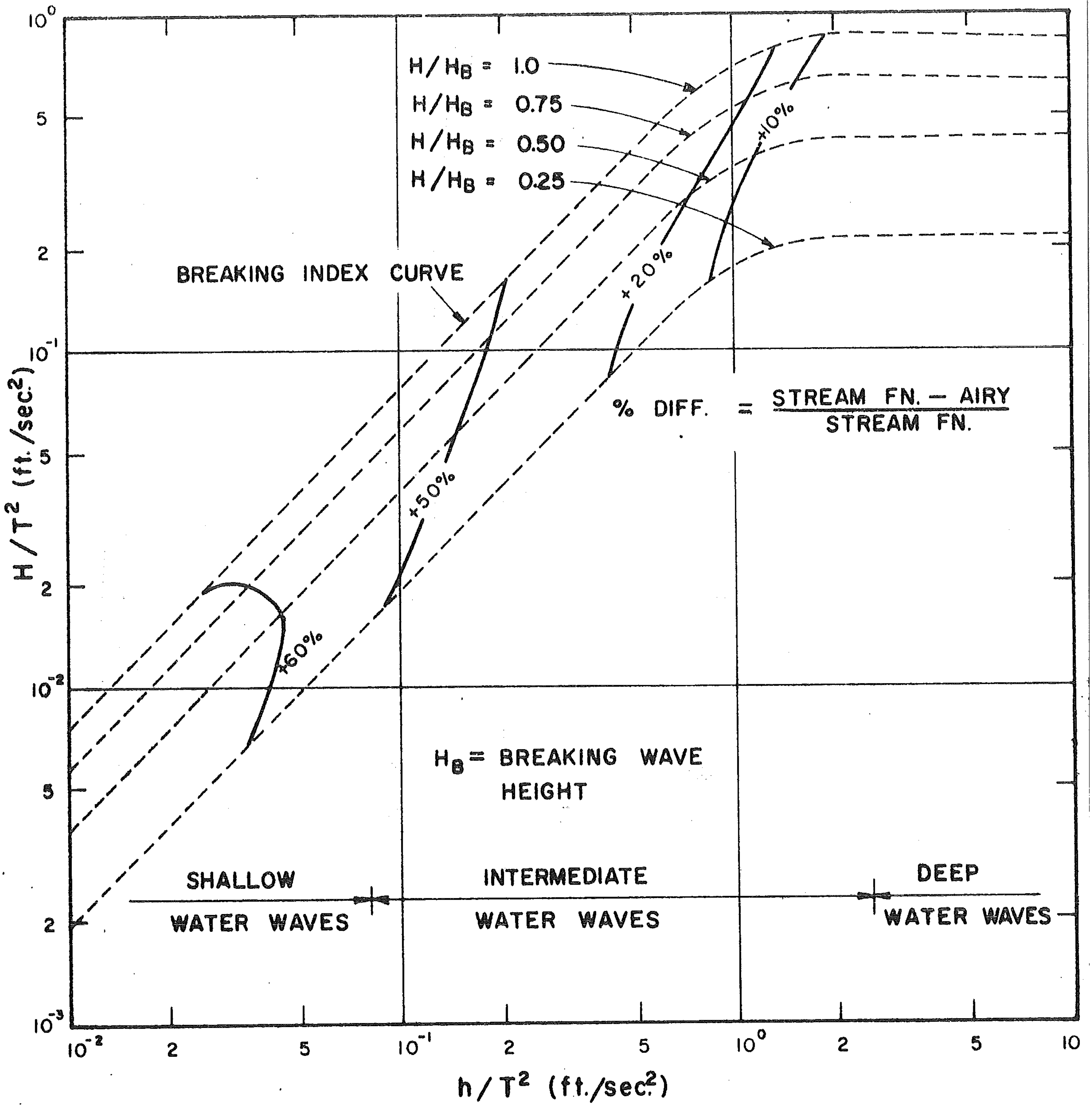


FIGURE 7. PERCENTAGE DIFFERENCES IN MAXIMUM DRAG FORCES; STREAM FUNCTION VS. LINEAR WAVE THEORY.

Figure 8 presents similar information for the maximum drag moment.

Pressure Response Factor

The pressure response factor, K_p, based on the wave height is defined as

$$K_p = \frac{P_D(0°) - P_D(180°)}{\gamma H}$$

where P_D denotes the dynamic pressure which the pressure sensor "sees" and γ is the specific weight of water. The pressure response factor is important, because the pressure sensor is still the most convenient type of wave gage to deploy, especially if there are no structures available as wave gage supports at the site of interest.

The percentage differences in pressure response factor for a pressure sensor located at mid-depth are shown in Figure 9. As an example, if a pressure sensor indicates a total pressure of 607 psf between crest and trough and h = 22.5 ft. and T = 15 sec., then the corresponding wave heights as calculated by the linear and Stream function wave theory would be:

$$H_{linear} = 9.9 \text{ ft.}$$

$$H_{\psi} = 13.3 \text{ ft.}$$

It is seen that the nonlinear effects can result in a considerable discrepancy in interpreted wave heights.

Total Average Wave Energy

Figure 10 presents the percentage differences between Stream function total wave energy and that obtained through use of the Airy theory. The negative signs indicate that, for a given wave height, the energies calculated by the Airy wave theory are the greater. This result may, at first, seem surprising, however, it is clear that the limiting potential energy would be zero as the wave troughs

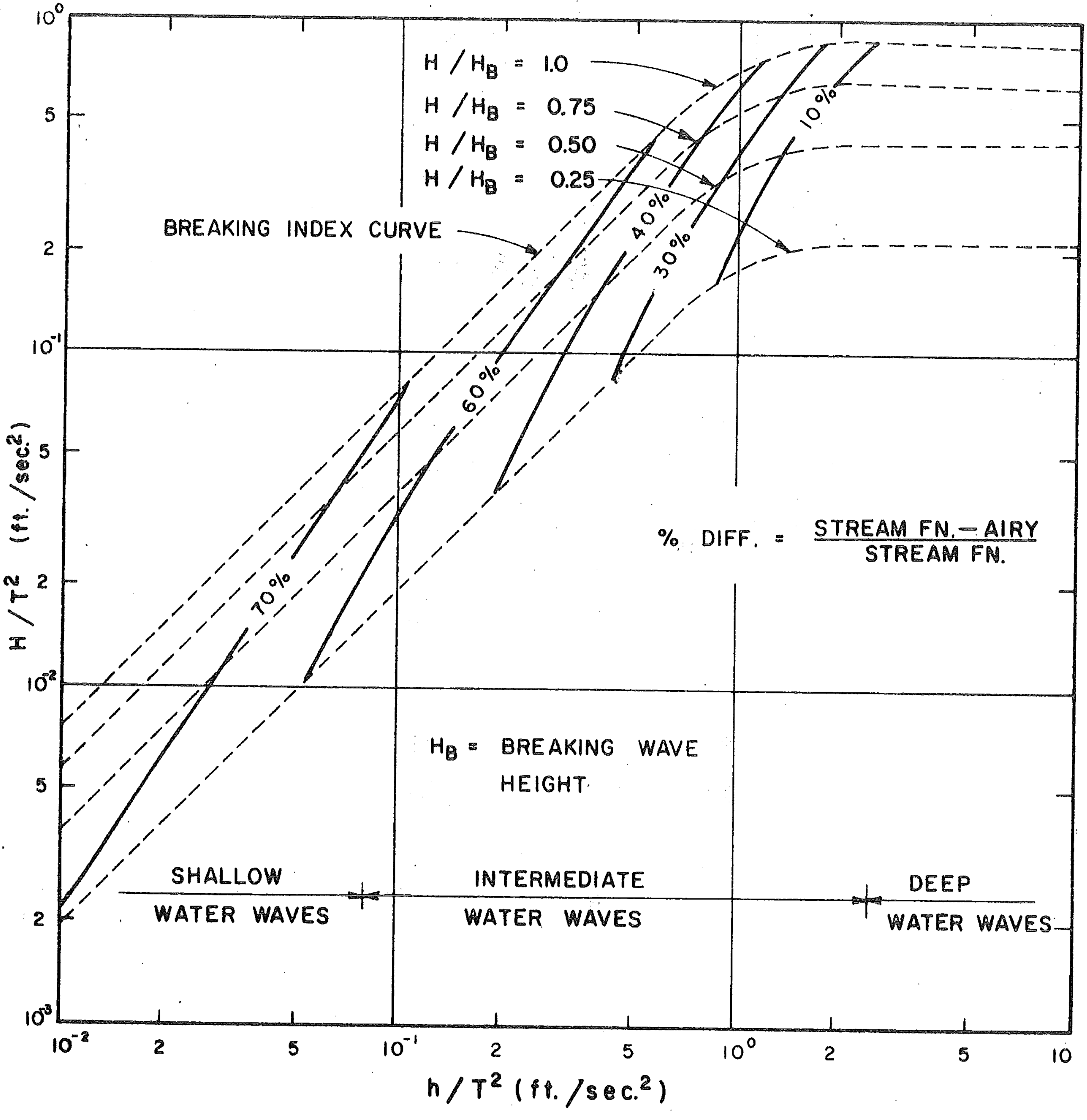


FIGURE 8. PERCENTAGE DIFFERENCES IN MAXIMUM DRAG MOMENTS; STREAM FUNCTION VS. LINEAR WAVE THEORY.

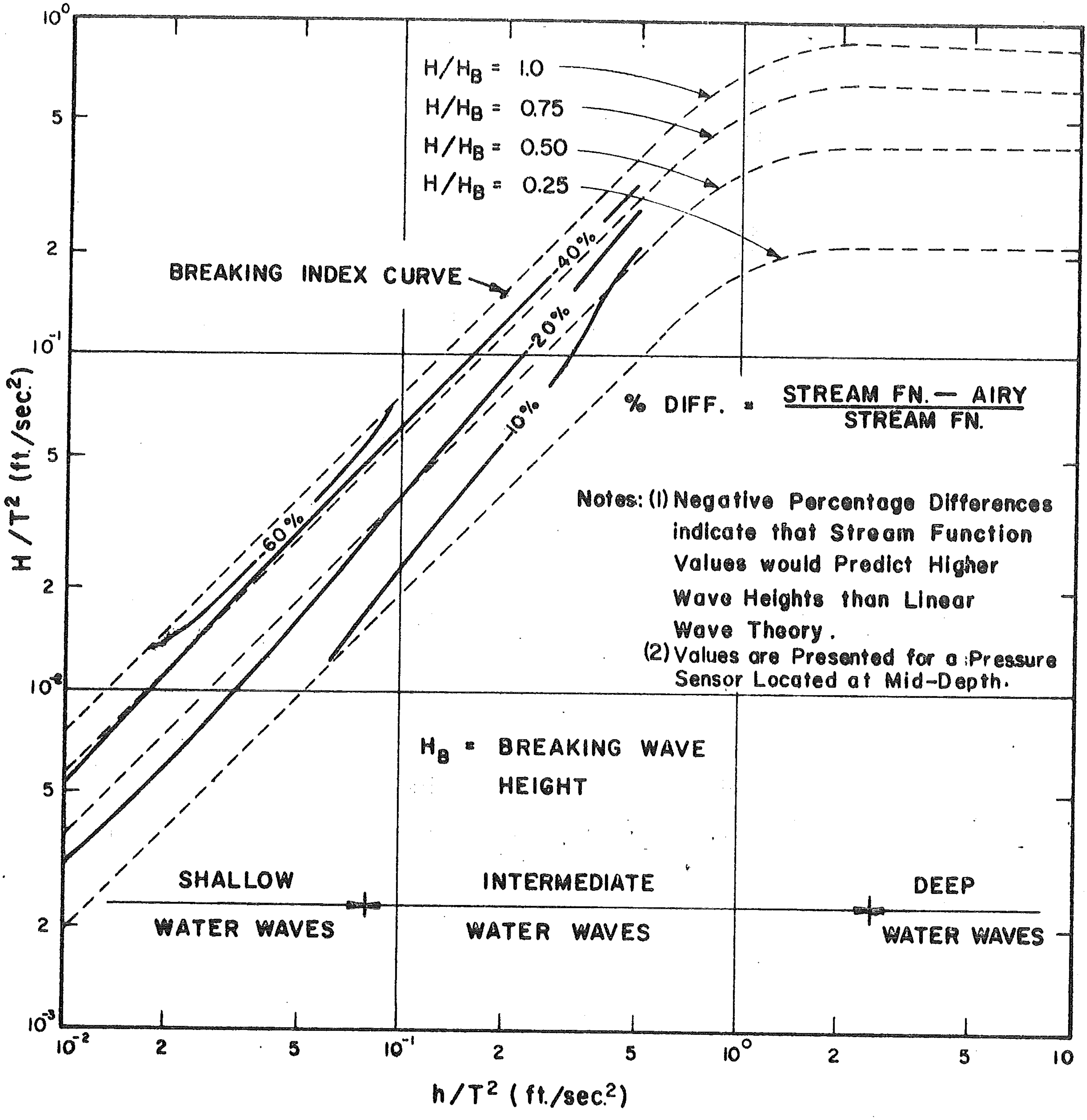


FIGURE 9. PERCENTAGE DIFFERENCES IN PRESSURE RESPONSE FACTOR; STREAM FUNCTION VS. LINEAR WAVE THEORY

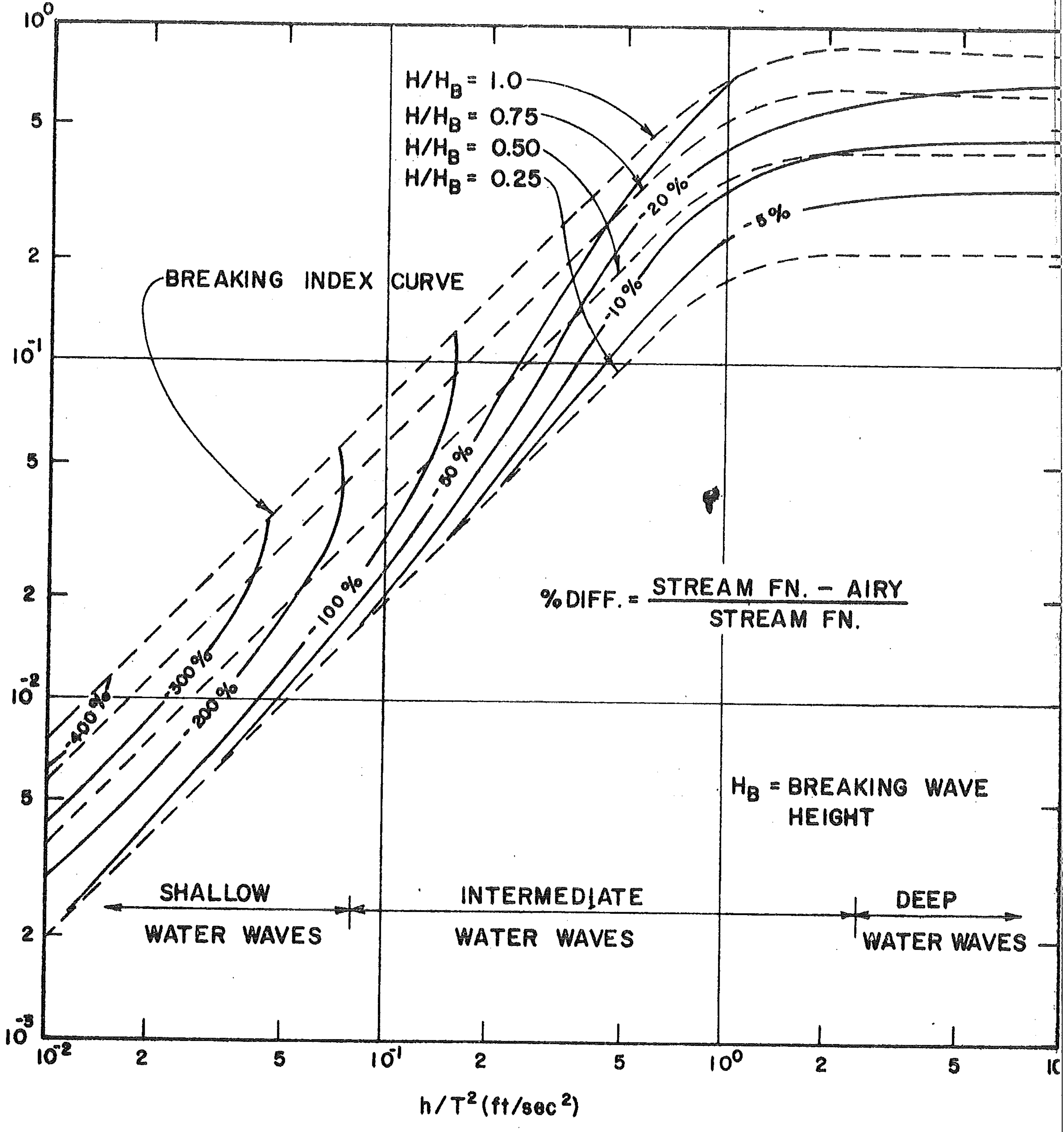


FIGURE 10. ISOLINES OF PERCENTAGE DIFFERENCES IN TOTAL ENERGY; AIRY VS. STREAM FUNCTION WAVE THEORIES.

become longer and shallower and the wave crest becomes more narrow. For example, the total average energy of the Stream function shallow water wave shown in Figure 11 is only 46% of that for the Airy theory shown.

Total Momentum

The differences for the total momentum are presented in Figure 12. The maximum percentage difference, as defined in the figure, is greater than 300%. The reason that the Stream function predicts such markedly smaller momentum values is due, in large part, to the peaked crests and long low troughs shown in Figure 11.

Total Momentum Flux

The momentum flux difference is presented in Figure 13. Again it is noted that the momentum flux calculated from the Stream function wave theory is generally less than calculated by the Airy wave theory. Since the momentum flux is an important agent in several surf zone mechanisms (e.g. set-up and longshore current) differences of the magnitude noted here could be of considerable importance.

Shoaling Coefficient

The percentage differences in shoaling coefficients for waves advancing normal to the shoreline are presented in Figure 14. As an example, assume the following deep water wave conditions

$$H_o = 4 \text{ ft.}$$
$$T = 10 \text{ sec.}$$
$$h = 1000 \text{ ft.}$$

According to linear wave theory, the wave height in 10 ft. of water depth would have increased to 4.9 ft. due to shoaling. The Stream function wave theory would predict a wave height of 5.9. As discussed previously, the primary reason for the difference is that, for a given wave height, the amount of wave

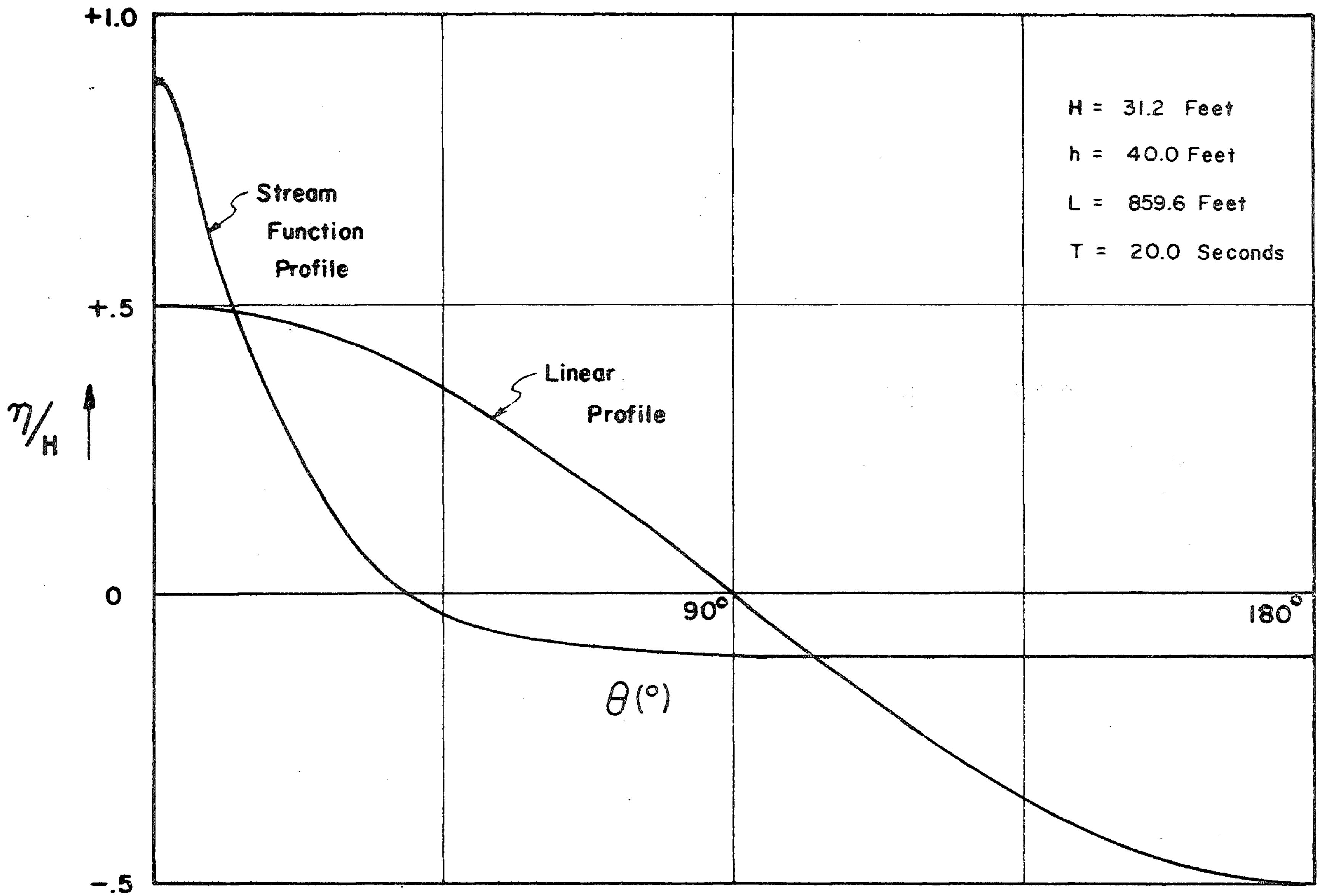


FIGURE 11. COMPARISON OF 11th ORDER STREAM FUNCTION AND LINEAR WAVE THEORY PROFILES: NEAR BREAKING SHALLOW WATER WAVE.

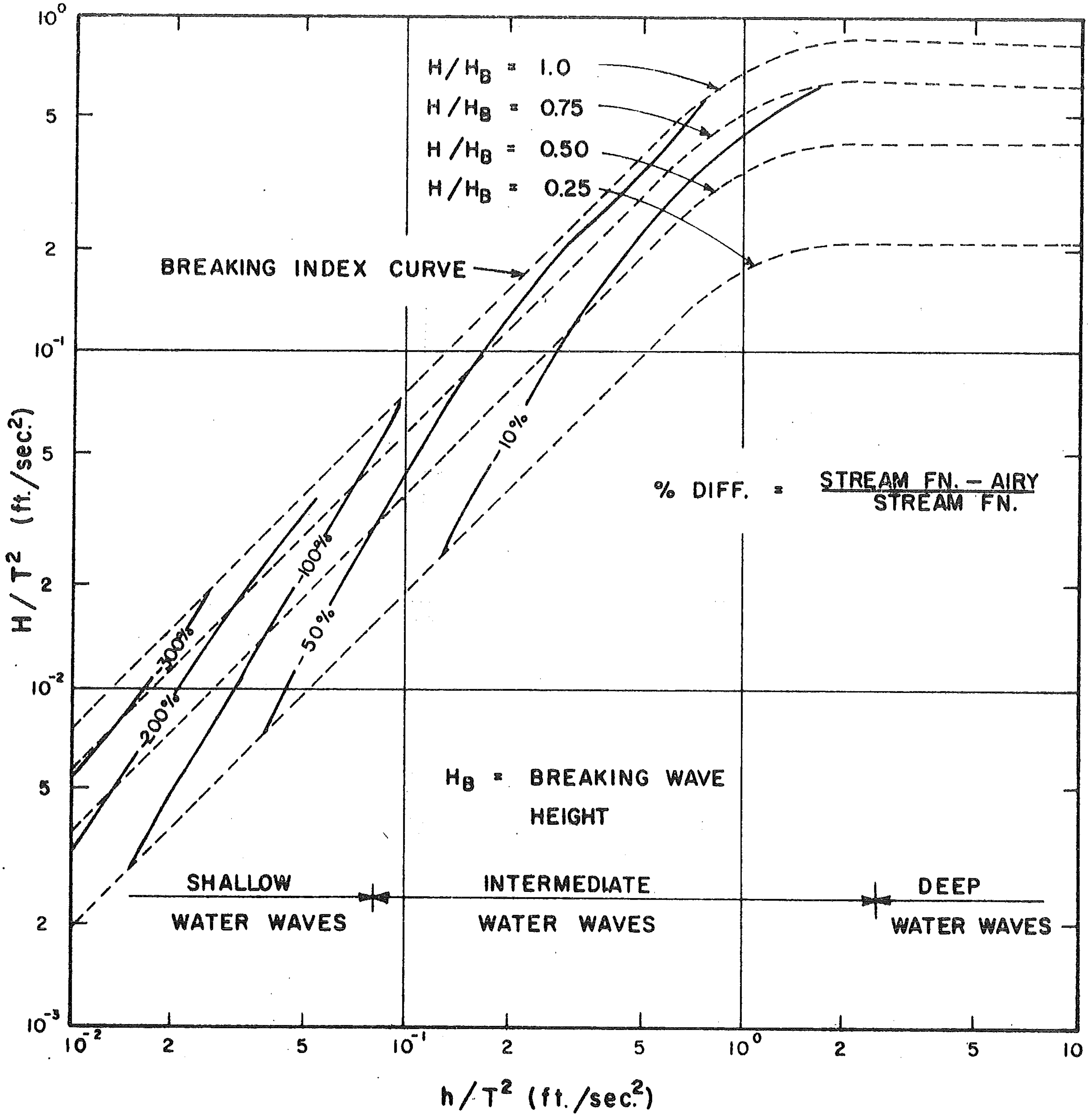


FIGURE 12. PERCENTAGE DIFFERENCES IN MOMENTUM; STREAM FUNCTION VS. LINEAR WAVE THEORY.

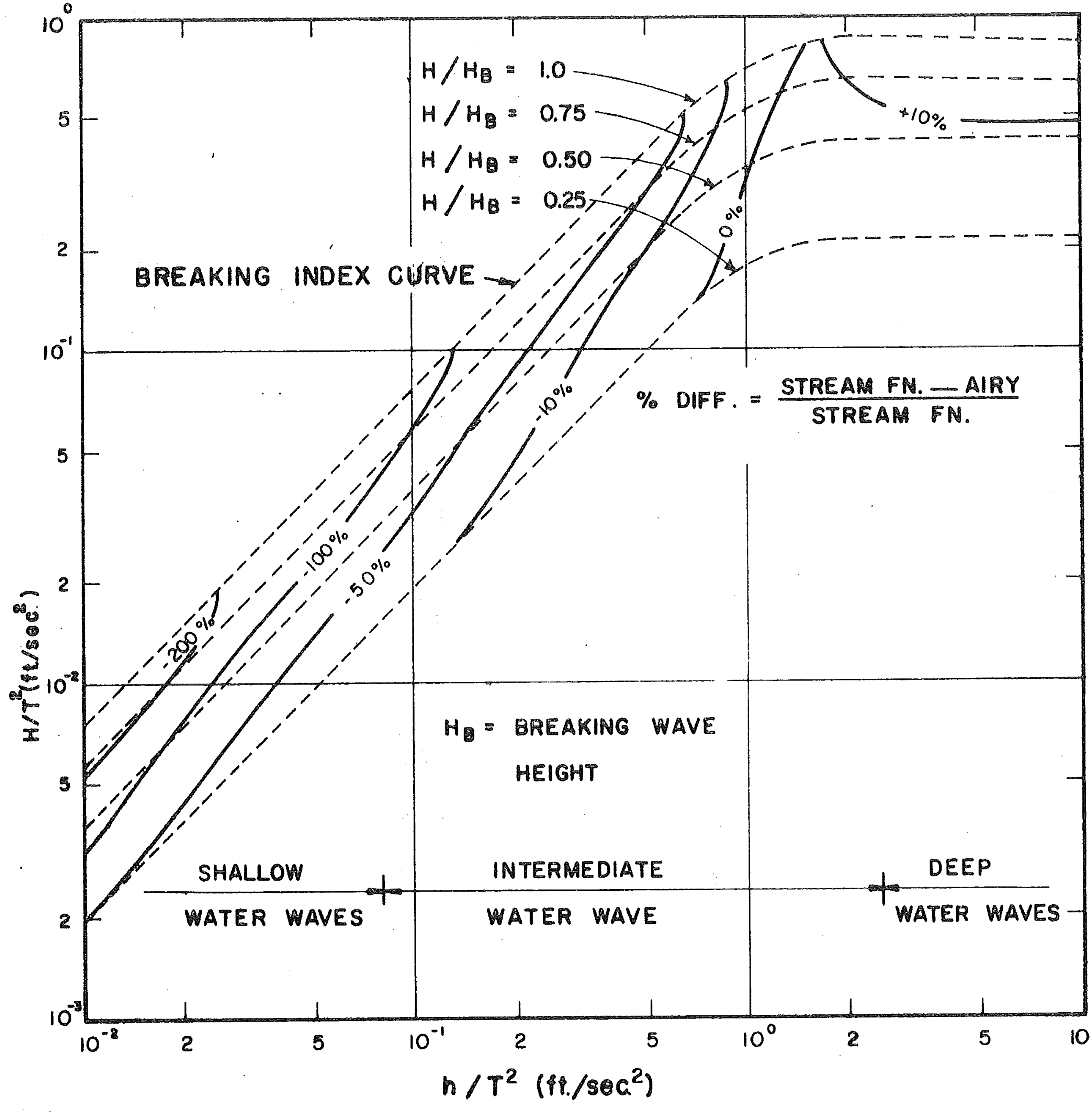


FIGURE 13. ISOLINES OF PERCENTAGE DIFFERENCES IN MOMENTUM FLUX; AIRY VS. STREAM FUNCTION WAVE THEORIES.

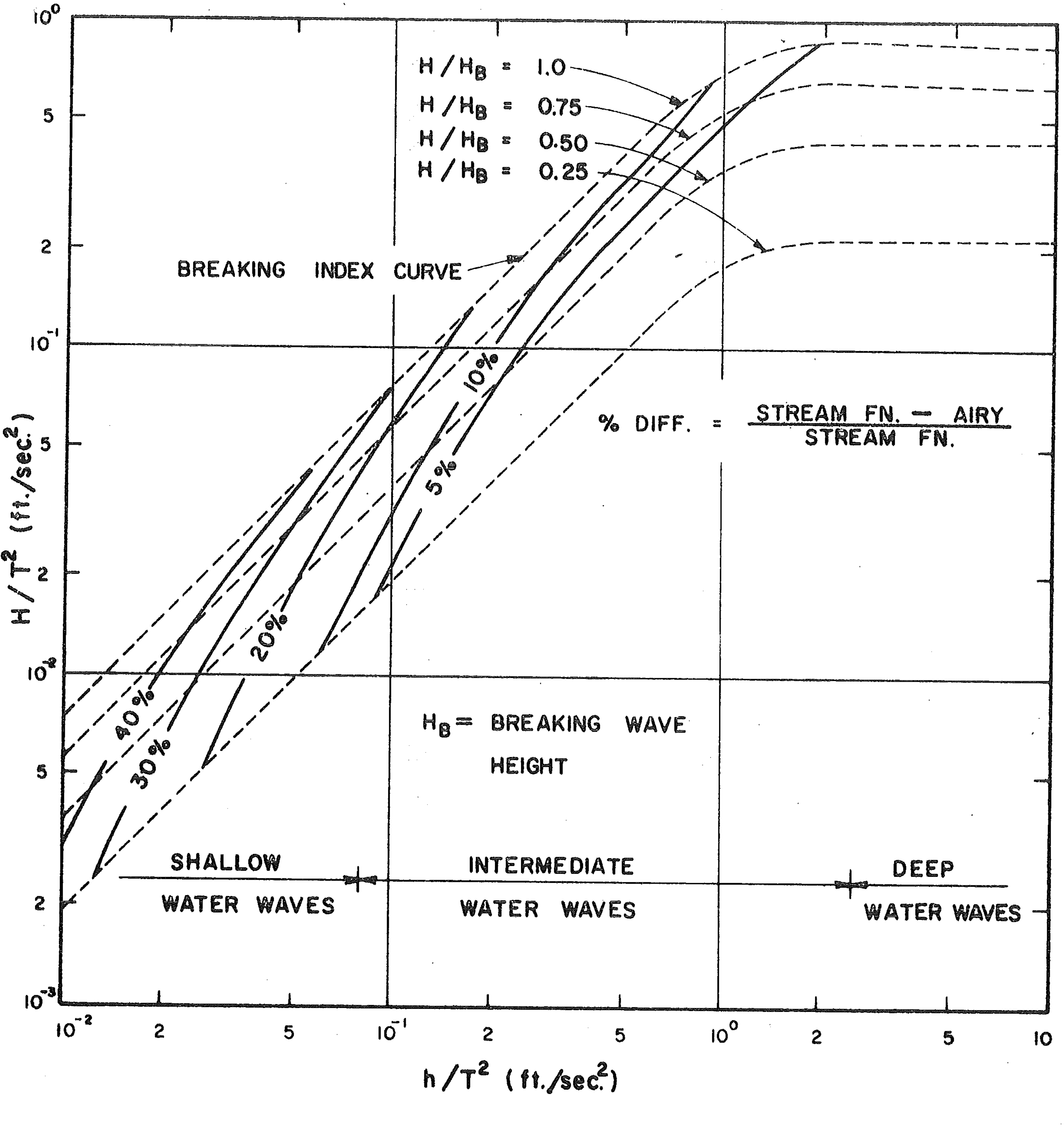


FIGURE 14. ISOLINES OF PERCENTAGE DIFFERENCES IN SHOALING COEFFICIENT; AIRY VS. STREAM FUNCTION WAVE THEORIES.

energy is less for a nonlinear wave than as calculated by the Airy wave theory.

For a comparison of shoaling effects as predicted by the Stream function and Airy wave theories and measurements, one set of laboratory data published by Bowen, et.al.(7) is used. Figure 15a illustrates the plane laboratory beach with a slope of 1:12. Figures 15b and 15c illustrate the wave shoaling and the relative crest and trough displacement, respectively.

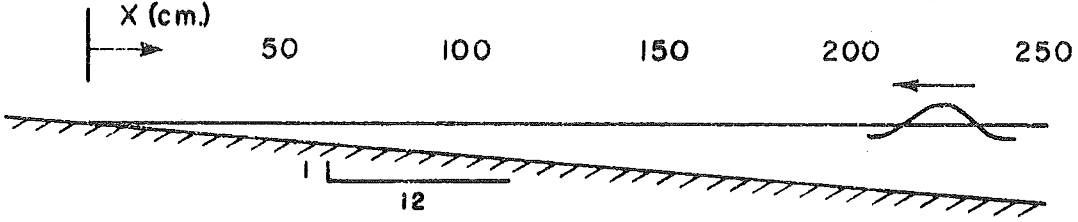
It is seen in Figure 15b that as the wave commences shoaling, the Airy predictions are in better agreement than the Stream function. As the wave approaches breaking, the measured wave height deviates sharply from the Airy theory and tends to be in somewhat better agreement with the Stream function predictions. It should be noted that a slope of 1:12 is fairly steep, and it is clear that there will be a time (or distance) "lag" between the time of passage of a wave over a particular depth and the response of the wave to that depth.

Figure 15c illustrates that the relative crest and trough elevations are in good agreement with the Stream function predictions, although there is some indication of a "lag" for near breaking conditions.

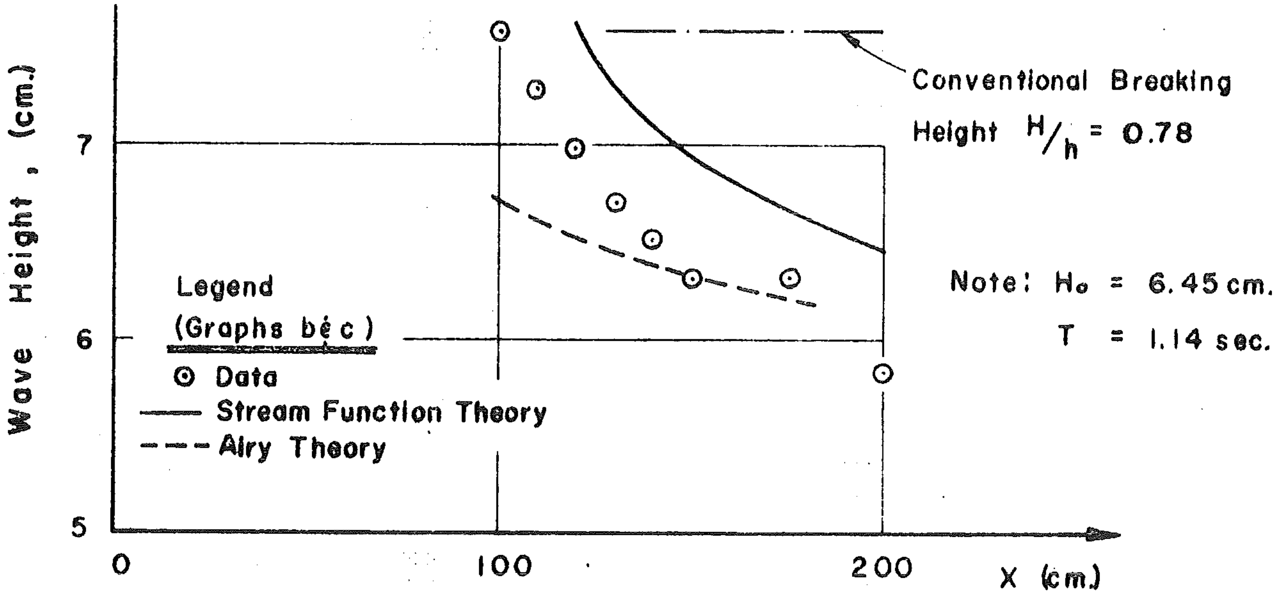
CONCLUSIONS

Comparisons have shown that the Stream function wave theory is in reasonable agreement with the equations describing water wave motion and with maximum horizontal water particle velocity data collected by Le Méhauté, et.al. for shallow water waves. A number of the theories derived for shallow water conditions do not agree satisfactorily with the measured data.

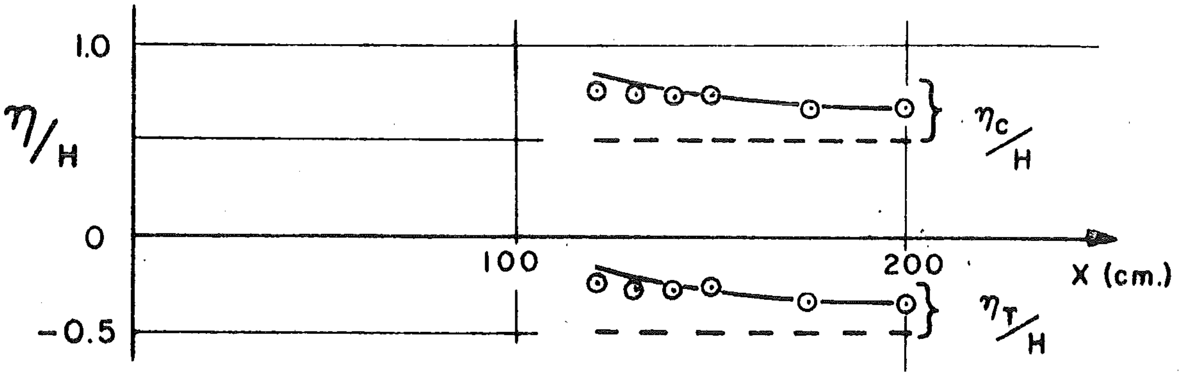
Although the linear wave theory is used in engineering practice for a number of important shallow water applications, including: shoaling, energy, pressure response factor, momentum flux, etc., it has been shown that significant differences



a) Experimental Beach



b) Wave Shoaling



c) Relative Crest and Trough Elevations

FIGURE 15. COMPARISON OF STREAM FUNCTION AND AIRY PREDICTIONS WITH MEASUREMENTS BY BOWEN ET. AL.

can exist between shallow water values as calculated by the linear and Stream function wave theories.

Much additional high quality laboratory and field shallow water data are required to further develop our understanding of the capabilities and limitations of the various available wave theories. The existing shallow water data are limited in number and in the variables and phase angles that they represent.

REFERENCES

1. Dean, R. G., "Stream Function Representation of Nonlinear Ocean Waves," Journal of Geophysical Research, 70(18), pp. 4561-4572, September, 1965.

2. Dean, R. G., "Stream Function Wave Theory; Validity and Application," Proceedings, ASCE Specialty Conference on Coastal Engineering, Chapter 12, pp. 269-300, 1965.

3. Dean, R. G., "Relative Validity of Water Wave Theories," Proceedings, ASCE Specialty Conference on Civil Engineering in the Oceans, San Francisco, pp. 1-30, 1968. (Also published in Waterways and Harbors Journal, American Society of Civil Engineers, p. 105-119, February, 1970)

4. Dean, R. G., and B. Le Méhauté, "Experimental Validity of Water Wave Theories," Paper presented at the 1970 ASCE Structural Engineering Conference, Portland, Oregon, April 8, 1970.

5. Le Méhauté, B., D. Divoky and A. Lin, "Shallow Water Waves: A Comparison of Theory and Experiment," Proceedings Eleventh International Conference on Coastal Engineering, Chap. 7, pp. 86-107, 1968.

6. Laitone, E. V., "The Second Approximation to Cnoidal and Solitary Waves," Journal of Fluid Mechanics, V. 9, Part 3, pp. 430-444, November, 1960.

7. Bowen, A. J., D. L. Inman and V. P. Simmons, "Wave 'Set-Down' and Set-Up," Journal of Geophysical Research, V. 73, No. 8, pp. 2569 - 2577, April, 1968.