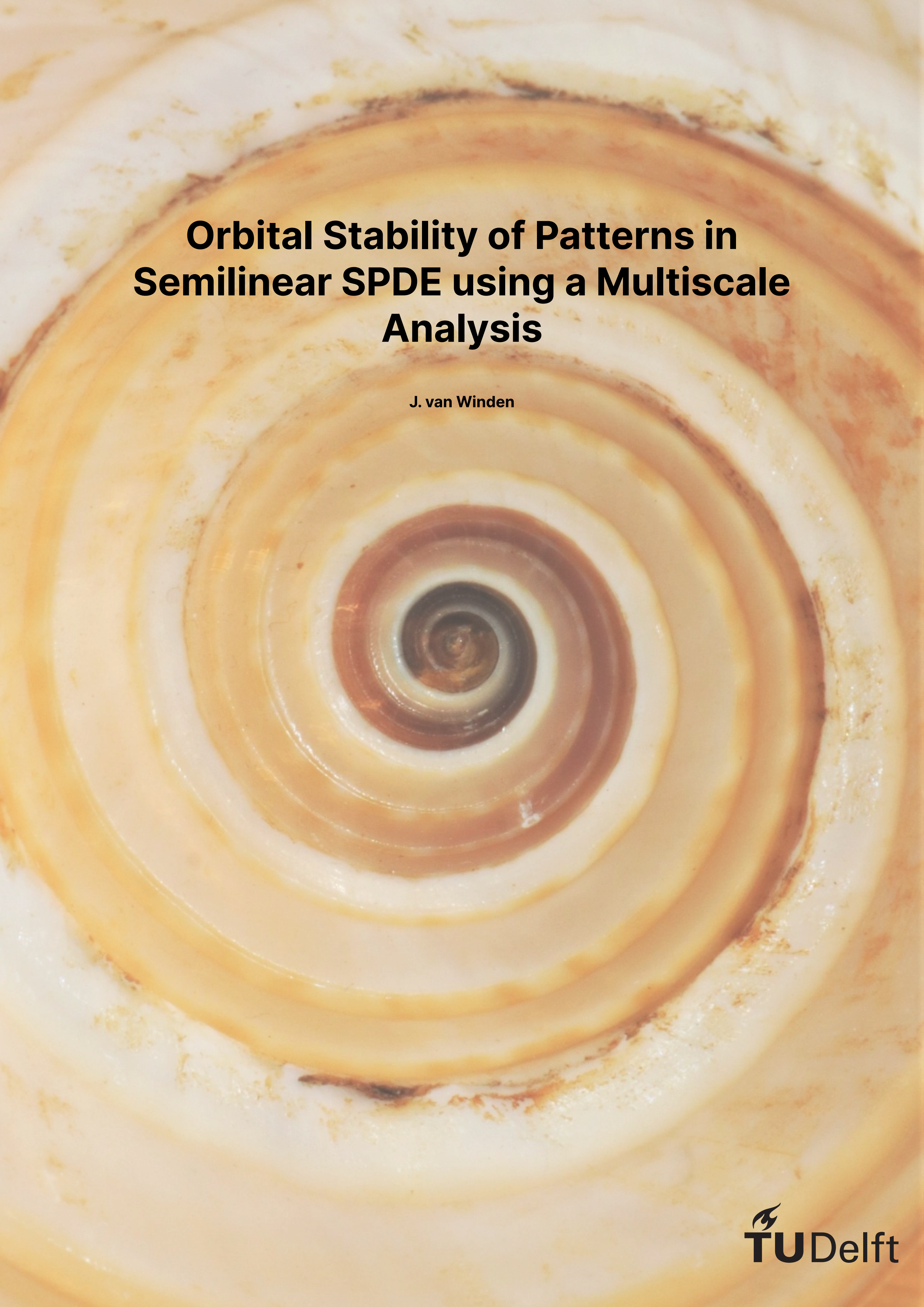
Orbital Stability of Patterns in Semilinear SPDE using a Multiscale Analysis

J. van Winden



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by

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Abstract

In this thesis we consider orbital stability of certain patterns in stochastic partial differential equations. We study two examples: a rotating wave in a two-dimensional reaction-diffusion equation and a soliton in a parametrically forced nonlinear Schrödinger equation. In both cases, we show that, for small noise, solutions to the stochastic equations remain close to a version of the pattern which is shifted according to some stochastic phase. We give explicit expressions for this phase, and show that it is optimal to first order in the strength of the noise.

To show stability, we construct a multiscale expansion of the solution around an arbitrarily shifted version of the pattern, and show that this expansion is accurate to second order. From this expansion an obvious candidate for the correct phase shift arises. For technical reasons we then construct a sequence of approximations to this phase shift, which is necessary to show the multiscale expansion around the correctly shifted pattern. We then combine this expansion with a deterministic stability result to get stochastic stability.

Finally, we take first steps towards formulating and proving the same results in a more general setting, where the pattern shift is represented by the action of a Lie group. We obtain some estimates necessary for the multiscale expansion, find the correct phase, and formulate necessary assumptions for the stability to hold.

Acknowledgments

I would like to thank dr. Manuel Gnann for his supervision. Our weekly meetings were always productive and enjoyable, and I looked forward to them every time. I also want to thank my friends and family for their support, and for always being willing to listen to me ramble on about mathematics.

Contents

Chapter 1

Introduction

In 1842, engineer John Scott Russell observed a large, solitary wave in a canal and followed it on horseback for two miles [46]. His observation was explained in 1895 by Korteweg and De Vries [30], who found solitary wave (soliton) solutions to the now celebrated KdV equation, a nonlinear PDE which models surface waves on shallow water. Since then, many more patterns in nonlinear PDE have been found. Notable examples are travelling pulses and wave trains in neural field equations [4, 9, 10, 13] and optics equations [2, 28, 40], rotating waves in reaction-diffusion equations [12, 22] and superconductivity equations [7], and surface waves on shallow water [3, 30]. Patterns have also been observed in chemical and biological systems [27].

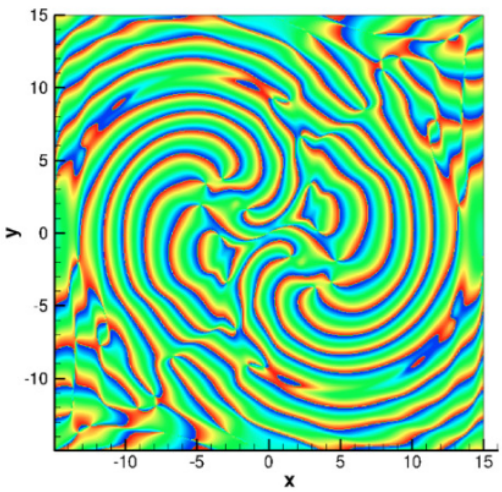


Figure 1.1: Phase plot of a spinning soliton in the quintic Ginzburg-Landau equation [5].

In physical, chemical or biological systems it is almost always impossible to measure or control the *exact* state of the system. Thus, to study these patterns it is important to ask:

How do the patterns occurring in PDE change when the system is slightly perturbed?

Much work has been done in a deterministic setting, where a small perturbation of the initial condition is introduced. Many of the aforementioned patterns exhibit *orbital stability* in this setting [6, 9, 11, 18, 28, 41, 47, 48], meaning the pattern persists but is shifted according to some phase.f

However, perturbing only the initial conditions of a PDE is not sufficient for modelling real noise, which is typically ever-present in any physical system due to random thermal fluctuations. Thus it is natural to introduce noise by turning the PDE into an SPDE and to consider orbital stability in the stochastic setting. This subject has received more attention in the last decade, and three distinct methods [23, 26, 50] have been developed to lift deterministic stability to the stochastic setting.

1.1 Stochastic stability of patterns

We consider a semilinear PDE taking the form

$$du(t) = [Au(t) + f(u(t))]dt, \tag{1.1}$$

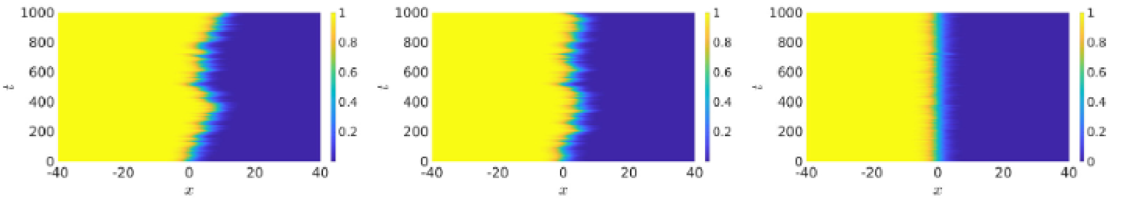
where u takes values in some Hilbert space H, and A is a linear operator generating a C_0-semigroup on H. We also assume (1.1) admits a pattern solution $u^*(t)$. This pattern can be any of the aforementioned ones, such as a travelling wave, wave train, rotating wave, or soliton. We now introduce noise to (1.1) to get the SPDE

$$\begin{aligned} du(t) &= [Au(t) + f(u(t))]dt + \sigma B(u(t))dW(t) \\ u(0) &= u^*(0) + \sigma v_0, \end{aligned} \tag{1.2}$$

where $W(t)$ is a Brownian motion taking values in H and σ is a parameter controlling the strength of the noise, which we assume to be small. Assuming equation (1.2) is well-posed, we aim to answer the following question:

Does the stochastic solution $u(t)$ remain close to (a shifted version of) $u^*(t)$ over long timescales?

Because the study of patterns originated with one-dimensional waves, the shift in u^* is typically called the *phase*. Keeping with the literature we will use the word phase even when more complicated patterns and transformations (such as rotations) are involved. The necessity of tracking the phase stochastically is illustrated by figure 1.2.



(a) $U(x + c_0 t, t)$ (b) $U(x + c_\sigma t, t)$ (c) $U(x + \Gamma(t), t)$

Figure 1.2: Three wave profiles of a reaction-diffusion equation. (a). Wave profile without phase correction. (b). Wave profile with corrected wave speed. (c). Wave profile with stochastic phase correction [25].

In order define the phase, it is intuitive to look at the function

$$E : \phi \to \|u(t) - \mathcal{T}_\phi u^*(t)\|_H^2, \tag{1.3}$$

where $\mathcal{T}_\phi$ is the shift corresponding to some phase ϕ. Because E measures the difference between the stochastic solution and the shifted deterministic solution, the value of ϕ which minimizes E is an obvious candidate for defining the phase. However, the dynamics of the global minimizer of (1.3) are highly irregular, since this minimizer may jump around between local minima. To remedy this, Stannat [50] considers a *phase-lag* method, where the phase is defined via the ODE

$$\dot{\phi}_m(t) = -m\frac{d}{d\phi_m}\|u(t) - \mathcal{T}_{\phi_m(t)}u^*(t)\|_H^2, \tag{1.4}$$

where $m > 0$ is some relaxation parameter. Equation (1.4) guarantees that ϕ_m continually changes to decrease E at a speed determined by m. As m goes to infinity, this ϕ_m converges to some ϕ_∞ which no longer lags behind the real phase. The advantage of this method is that explicit solutions for ϕ_m and ϕ_∞ are often available, giving insight into the phase dynamics. The method has primarily been used in stochastic neural field applications [15, 31, 32, 37].

Inglis and MacLaurin [26] took a more direct approach, deriving an SDE for ϕ which forces the equation

$$\frac{d}{d\phi}\|u^* - \mathcal{T}_{\phi_m}u^*\|^2 = 0$$

to hold. This enables tracking of a local minimum until the time where it becomes a saddle point. This method was later extended [38] and applied to more general patterns [39].

Finally, there is the most recent method by Hamster and Hupkes [23, 24, 25] which is less straightforward, defining the phase via a *stochastic freezing* condition, which guarantees that the shifted pattern only feels stochastic forcing. This method results in a significantly more complicated SDE for ϕ, but has been used to establish stability on a timescale of order $\mathcal{O}(\exp(C\sigma^{-1}))$ [25]. For more information on these techniques, we refer to the recent review by Kuehn [33].

In this thesis, we consider stochastic orbital stability of two particular patterns: rotating waves in two-dimensional reaction-diffusion equations two-dimensions, and a soliton in the parametrically forced nonlinear Schrödinger (PFNLS) equation. In both cases, we use an approach based on the *phase-lag* method introduced by Stannat [50] to obtain a first-order (in the noise strength) multi-scale expansion of the stochastic solution around the shifted pattern. However, our approach differs in a significant way. Instead of defining the lagging phase ϕ_m by equation (1.4) and linearizing (1.2) around $\mathcal{T}_{\phi_m}u^*$, we compute a linearization around $\mathcal{T}_\phi u^*$ for an *arbitrary* (but small) ϕ. This linear approximation is then split up using a Riesz spectral projection, which leads to a natural candidate for the phase. This natural candidate turns out to be equal to the ϕ_∞ obtained using the phase-lag method. For technical reasons, this ϕ_∞ then needs to be approximated by a sequence of (progressively measurable) differentiable processes, which are defined via the ODE

$$\dot{\phi}_m(t) = -m(\phi_m(t) - \phi_\infty(t)).$$

The resulting sequence also turns out to be equivalent (up to a difference in m) to the ϕ_m which solves (1.4). Thus, our approach gives the same phase correction as the phase-lag method without having to derive an SDE from (1.4). Although deriving this SDE is relatively straightforward for one-dimensional travelling waves, it is significantly more complicated in the case of rotating waves due to the noncommutativity of translations and rotations. More importantly, our approach still works when the derivative in equation (1.4) does not exist, which is generally the case in Banach spaces. Therefore, we believe this approach can be extended to work in general Banach spaces, since it does not utilize any tools specific to the Hilbert space setting.

1.2 Overview

This thesis contains three main chapters, in each of which a different pattern in an SPDE is examined. In each case, we show stochastic orbital stability by first constructing a multiscale expansion, using this to find the right phase correction, and subsequently combining this with a deterministic stability result.

Chapter 2 contains the mathematical preliminaries used throughout the thesis. In particular, we need theory concerning functional analysis, Sobolev spaces and elliptic PDE, (non)autonomous evolution equations, stochastic integration, and Lie groups and Lie algebras.

In Chapter 3, we treat rotating waves in a two-dimensional reaction-diffusion equation. Here, the main difficulty lies with the noncommutativity of the symmetry group $SE(2)$. In Chapter 4, the situation is slightly different. Here we treat solitons in a parametrically forced nonlinear Schrödinger equation, for which the symmetry is translational. However, in this case we have to deal with multiplicative noise, which presents an additional difficulty when constructing the multiscale expansion. Finally, in Chapter 5 we take first steps towards showing stochastic orbital stability for patterns in PDE with more general symmetries. We show some necessary estimates, formulate required assumptions assumptions and find an explicit expression for the (generalized) phase. Chapter 6 contains some auxiliary results, which are used throughout chapters 3, 4 and 5.

Chapter 2

Preliminary Theory

2.1 Functional analysis

We state some theorems and definitions from functional analysis for later use. For proofs of the statements, see [42] or [21].

Throughout this section, let H, H_1 and H_2 be seperable real or complex Hilbert spaces, and let X and Y be Banach spaces. We use the notation $(u, v)_H$ for the inner product of two elements $u, v \in H$. In case the Hilbert space is complex, we use the convention that the inner product is linear in the first variable and conjugate-linear in the second variable.

Definition 2.1.1. *A function $T : X \to Y$ is Fréchet differentiable at $x \in X$ if there exists a bounded linear operator from X to Y denoted by $f'(x)$ which satisfies*

$$\lim_{\|y\|_X \to 0} \frac{\|f(x+y) - f(x) - f'(x)y\|_Y}{\|y\|_X} = 0.$$

Theorem 2.1.2. *Let T be a linear operator defined and bounded on a dense subspace of X with values in Y. Then T extends uniquely to a bounded operator $\overline{T} : X \to Y$.*

Definition 2.1.3. *An operator $T : H \to H$ is positive if it satisfies*

$$(Th, h)_H \geq 0$$

for every $h \in H$.

Definition 2.1.4. *Let e_i be an orthonormal basis of H. The trace of an operator $T \in L(H)$ is defined as*

$$tr(T) := \sum_{i=1}^{\infty} (Te_i, e_i)_H.$$

Definition 2.1.5. *Let e_i be an orthonormal basis of H_1. The Hilbert-Schmidt norm of an operator $T \in L(H_1, H_2)$ is defined as*

$$\|T\|^2_{L_2(H_1,H_2)} := \Big(\sum_{i=1}^{\infty} \|Te_i\|^2_{H_2}\Big).$$

We denote the subspace of $L(H_1, H_2)$ consisting of operators with finite Hilbert-Schmidt norm by $L_2(H_1, H_2)$.

The trace and Hilbert-Schmidt norm are both independent of the choice of basis.

Proposition 2.1.6. *$L_2(H_1, H_2)$ is a Hilbert space with respect to the inner product*

$$(T, S)_{L_2(H_1,H_2)} = \sum_{i=1}^{\infty} (Te_i, Se_i)_{H_2}.$$

Theorem 2.1.7. *Let $T : H \to H$ be positive. Then there exists a unique operator $T^{1/2} \in L(H)$ which satisfies*

$$T^{1/2}T^{1/2} = T.$$

Furthermore there exists an orthonormal basis e_i of H consisting of eigenvectors of T. If T has finite trace, the sum of the eigenvalues (repeated according to multiplicity) is finite.

Theorem 2.1.8. *Let $T : D(T) \to H$ be a closed linear operator. Suppose the spectrum of T is the union of two disjoint compact sets K_1 and K_2. Define the linear operators Π_i by*

$$\Pi_i := \oint_{\gamma_i} (zI - T)^{-1} dz,$$

where γ_i is an admissible contour for K_i. Then each Π_i is a projection which commutes with T, and is called the spectral projection *onto K_i.*

2.2 Classical function spaces and derivatives

Throughout sections 2.2, 2.3, and 2.4, let $U \subset \mathbb{R}^n$ be open and let V be either $\mathbb{R}$ $\mathbb{R}^n$, $\mathbb{R}^{m \times n}$ or $\mathbb{C}$ with the obvious inner product and norm. Proofs of the statements can be found in [16, 21, 42].

The space $C^k(U, V)$ is the set of all k-times continuously differentiable functions on U taking values in V, where k may be infinity. By $C_c^\infty(U, V)$ we denote the subspace of functions $f \in C^\infty(U, V)$ which have their support compactly contained in U.

If $f \in C^1([a, b], V)$, we denote the derivative of f either by f' or $\dot{f}$. If $f \in C^1(U, V)$, we denote the partial derivative with respect to the i-th coordinate by $\partial_{x_i} f$. For repeated differentiation, we will sometimes write $\partial_{x_i x_j} f$ to mean $\partial_{x_i}\partial_{x_j} f$. To denote higher order derivatives we also use the concept of a multi-index, which is an element in $\mathbb{N}^n$. Given some multi-index $\alpha = (\alpha_1, ..., \alpha_n)$, the *order* of α is defined as

$$|\alpha| := \sum_{i=1}^{n} \alpha_i.$$

Given a function $f \in C^k(\mathbb{R}^n)$ and a multi-index α with order $|\alpha| \le k$, we define

$$\partial^\alpha f = \partial_{x_1}^{\alpha_1} ... \partial_{x_n}^{\alpha_n} f,$$

where $\partial_{x_i}^j f$ denotes the j-th order partial derivative in the i-th coordinate.

2.3 Sobolev spaces

In order to study elliptic PDE, we introduce the notion of weak differentiability and Sobolev spaces. We mainly follow the conventions of [16]. For a comprehensive treatment of Sobolev spaces, see [1].

Definition 2.3.1. *A measurable function $f : U \to V$ is locally integrable if it satisfies*

$$\int_K |f| dx < \infty,$$

for every compact set $K \subset U$.

Definition 2.3.2. *A locally integrable function $f : U \to V$ is k-times weakly differentiable if for every multi-index α with $|\alpha| \leq k$ there exists a locally integrable function, denoted by $\partial^\alpha f$, which satisfies*

$$\int_U f \partial^\alpha \phi dx = (-1)^{|\alpha|} \int_U \phi \partial^\alpha f dx$$

for every $\phi \in C_c^\infty(U)$ (here $\partial^\alpha \phi$ is the classical derivative).

Weak derivatives are unique (up to a set of measure zero), and satisfy many familiar properties of classical derivatives, such as linearity, the product rule, and the chain rule. If a function is weakly and classically differentiable, then the two derivatives coincide almost everywhere.

In order to be able to use the powerful tools from functional analysis, it is necessary to have a Banach space of weakly differentiable functions. The Sobolev spaces serve this purpose.

Definition 2.3.3. *The Sobolev space $W^{k,p}(U, V)$ consists of all k-times weakly differentiable functions $f : U \to V$ for which the weak derivatives of order $|\alpha| \leq k$ are in $L^p(U)$.*

Theorem 2.3.4. *The Sobolev space $W^{k,p}(U, V)$ is a Banach space when equipped with the norm*

$$\|f\|_{W^{k,p}(U,V)} = \Big(\sum_{|\alpha| \leq k} \|\partial^\alpha f\|^p_{L^p(U,V)} \Big)^{1/p}. \tag{2.1}$$

Theorem 2.3.5. *The Sobolev space $H^k(U, V) := W^{k,2}(U, V)$ is a Hilbert space with inner product*

$$(f, g)_{H^k(U,V)} = \sum_{|\alpha| \leq k} (\partial^\alpha f, \partial^\alpha g)_{L^2(U,V)} \tag{2.2}$$

It is easily verified that the norm induced by the inner product in equation (2.2) is consistent with the norm in equation (2.1). It turns out that L^p integrability of f and $\partial_{x_i} f$ sometimes implies L^q integrability of f for some $q \geq p$, where the relation between p and q is determined by the dimension n. Thus, it is possible to trade differentiability for integrability in some sense. This is made precise by the Sobolev embedding theorem.

Theorem 2.3.6. *(Gigliardo-Niremberg-Sobolev) Let $p \in (1, n)$, and let $p^* = \frac{np}{n-p}$. Then there exists a constant $C_{n,p}$ independent of $f : U \to V$ such that*

$$\|f\|_{L^{p^*}(U,V)} \leq C_{n,p} \|f\|_{W^{1,p}(U,V)},$$

The exponent p^* is called the *Sobolev conjugate* of p. From the computation

$$\frac{p^*}{p} = \frac{\frac{np}{n-p}}{p} = \frac{n}{n-p} > 1$$

we can see that $p^* > p$, which means we have gained integrability.

2.4 Weak solutions to PDE

Using Sobolev spaces, we define the notion of a *weak solution* to an elliptic PDE. Our treatment is similar to [16, Chapter 6], but similar statements are contained in [21, 42].

Definition 2.4.1. *A function $a : U \to \mathbb{R}^{n\times n}$ is uniformly elliptic if there exists a constant $c > 0$ independent of x, ζ, such that*

$$\sum_{i,j=1}^{n} a_{ij}(x)\zeta_i\zeta_j \geq c|\zeta|^2$$

for every $\zeta \in \mathbb{R}^n$ and almost every $x \in U$.

Consider the PDE

$$-\sum_{i,j=1}^{n} \partial_{x_i}(a_{ij}\partial_{x_j}u) + \sum_{i=1}^{n} b_i\partial_{x_i}u + cu = f, \tag{2.3}$$

where u is real-valued, $a : U \to \mathbb{R}^{n\times n}$, $b : U \to \mathbb{R}^n$ and $c : U \to \mathbb{R}$ are essentially bounded and measurable, and $f \in L^2(U, \mathbb{R})$, and additionally a is uniformly elliptic. If u is sufficiently smooth, we can multiply this PDE by a test function $\phi \in C_c^\infty(U, \mathbb{R})$, integrate over U and apply integration by parts to obtain the following equation.

$$\sum_{i,j=1}^{n} \int_U a_{ij}(\partial_{x_i}\phi)(\partial_{x_j}u)dx + \sum_{i=1}^{n} \int b_i(\partial_{x_i}u)\phi dx + \int_U cu\phi dx = \int_U f\phi dx. \tag{2.4}$$

Notice all the terms in equation (2.4) are well-defined even if u is only in $H^1(U, \mathbb{R})$. This motivates the notion of a *weak solution*.

Definition 2.4.2. *A function $u \in H^1(U, \mathbb{R})$ is a weak solution of (2.3) if it satisfies (2.4) for every $\phi \in C_c^\infty(U, \mathbb{R})$.*

The notion of a weak solution is a robust one: it is often possible to show existence and uniqueness of weak solutions using powerful tools from functional analysis, such as the Riesz-representation theorem or the Lax-Milgram theorem. If a, b, c and f are sufficiently smooth, it is even possible to show that this weak solution is also a classical solution to (2.3). This is usually done by showing that $u \in H^k$ for higher k and using Sobolev embeddings of $W^{k,p}$ into the Hölder spaces C^α. However, for our purposes the following theorem suffices.

Theorem 2.4.3 (Elliptic interior regularity)**.** *Suppose that a is uniformly elliptic, a, b and c are $k+1$-times classically differentiable, $f \in H^k(U, \mathbb{R})$, and $u \in H^1(U, \mathbb{R})$ is a weak solution to (2.3). Then u is $k+2$ times weakly differentiable.*

The additional weak derivatives of u do not necessarily lie in $L^2(U)$.

2.5 Semigroups and evolution families

The theory of strongly continuous evolution families can be used to study nonautonomous PDE in an abstract form. These definition and theorems, along with their proofs can be found in [44]. Throughout this section, let X be a banach space.

Definition 2.5.1. *A family of bounded linear operators $P_{t,t'} \in L(X)$, $t, t' \geq 0$ is a strongly continuous evolution family if it satisfies*

$$P_{t,t} = I, \quad \text{for all } t \geq 0$$
$$P_{t,t'}P_{t',t''} = P_{t,t''} \quad \text{for all } t, s \geq 0,$$

and the map $(t, t') \to P_{t,t'}x$ is continuous for every $x \in X$.

Note that, in general, $(t, t') \to P_{t,t'}$ is not continuous in the operator norm. If we can write $P_{t,t'} = S(t - t')$ for some set of operators $S(t)$, we say that $S(t)$ is a strongly continuous- or C_0-semigroup.

Definition 2.5.2. *The generator of $P_{t,t'}$ is a family of (possibly unbounded) linear operators $A_t : D(A_t) \to X$, $t \geq 0$, defined by*

$$A_t x = \lim_{s \to t} \frac{P_{s,t}x - x}{s - t}, \tag{2.5}$$

where $D(A_t)$ consists of all $x \in X$ for which this limit exists.

Equation (2.5) is reminiscent of the definition of the derivative. The following theorem states that A_t can indeed be interpreted as the derivative of $P_{t,t'}$.

Theorem 2.5.3. *Let $P_{t,t'}$ be a strongly continuous evolution family on X generated by A_t. If $x \in D(A_t)$, then*

$$\frac{d}{dt}P_{t,t'}x = A_t P_{t,t'}x$$

and

$$\frac{d}{dt'}P_{t,t'}x = -P_{t,t'}A_{t'}x \tag{2.6}$$

Theorem 2.5.3 shows that $P_{t,0}u_0$ is the solution to the nonautonomous abstract Cauchy problem

$$\frac{du}{dt} = A_t u(t) \tag{2.7}$$
$$u(0) = u_0.$$

A typical example is when A_t is the Laplacian, in which case (2.7) is the heat equation and $P_{t,t'} = S(t - t')$ is the heat semigroup.

Evolution families can also be used to solve nonlinear equations. Consider the nonlinear abstract Cauchy problem

$$\frac{du}{dt} = A_t u(t) + f(t, u(t)) \tag{2.8}$$
$$u(0) = u_0.$$

Motivated by the classical variation of parameters formula, we define the notion of a mild solution.

Definition 2.5.4. *A function $u : [0, T] \to X$ is a mild solution of (2.8) if $u(t)$ is continuous and satisfies*

$$u(t) = P_{t,0}u_0 + \int_0^t P_{t,t'} f(t', u(t'))dt'.$$

Existence of mild solutions can often be shown under certain conditions on f, such as Lipschitz continuity.

2.6 Stochastic Integration in Infinite Dimensions

Fix some stochastic basis $(\Omega, \mathcal{F}, \mathbb{P})$, along with a normal filtration $\mathcal{F}_t$, to be used throughout this section. When we speak of adapted, predictable or progressively measurable processes it is with respect to this filtration. We also fix some terminal time $T > 0$.

Throughout this section, let U and H be Hilbert spaces. First, we define the notion of a Wiener process in infinite dimensions.

Definition 2.6.1. *Let $Q \in L(U)$ be a positive with finite trace. Let e_i be an orthonormal basis of eigenvectors of Q with eigenvalues λ_i, and let β_i be a family of independent real-valued Brownian motions. A Q-Wiener process $W(t)$ is defined by*

$$W(t) := \sum_{i=1}^{\infty} \sqrt{\lambda_i}\beta_i(t)e_i$$

Theorem 2.1.7 guarantees that the sequence converges in $L^2(\Omega, U)$.

We now fix some Q-Wiener process $W(t)$ to be used throughout the rest of this section. For technical reasons we define $U_0 := Q^{1/2}U$, which is a Hilbert space with respect to the inner product

$$(Q^{1/2}u, Q^{1/2}v)_{U_0} = (u, v)_U.$$

In order to define the notion of a stochastic partial differential equation, we will need a stochastic integral to make sense of expressions such as

$$\int_0^t f(t)dW(t), \tag{2.9}$$

where f is a predictable process. The reason why this is not straightforward is that we cannot interpret equation (2.9) as a Stieltjes integral, since $W(t)$ is not sufficiently regular. Instead, (2.9) is defined for a class of simple processes, and extended by density to a larger space using the Itô isometry. The class of simple proceses is defined as follows.

Definition 2.6.2. *A stochastic process $\Phi(t)$ taking values in $L(U, H)$ is called* simple *if it takes the form*

$$\Phi(t) = \sum_{i=0}^{N-1} 1_{(t_i,t_{i+1}]}(t)\Phi_i,$$

with t_i being an increasing sequence of real numbers with $t_0 = 0$, $t_N = T$, and Φ_i being a sequence of $\mathcal{F}_{t_i}$-measurable random $L(U, H)$ variables with each Φ_i taking only finitely many values.

For a simple process Φ, the stochastic integral is naturally defined as

$$\int_0^t \Phi(t')dW(t') = \sum_{i=0}^{N-1} \Phi_i(W_{t_{i+1}\wedge t} - W_{t_i \wedge t}).$$

Using the basic properties of a Wiener-process, the following proposition can be shown.

Proposition 2.6.3. *For a simple process $\Phi(t)$, we have*

$$\mathbb{E}\Big[\|\int_0^T \Phi(t')dW(t')\|_H^2\Big] = \int_0^T \mathbb{E}\Big[\|\Phi(t')Q^{1/2}\|_{L_2(U,H)}^2\Big]dt'. \tag{2.10}$$

Using this isometry, it is possible to extend the definition of the stochastic integral to a large class of stochastic processes.

Definition 2.6.4. *Define $\mathcal{N}_W^2(0,T,H)$ as the space of all predictable $L(U,H)$-valued processes $\Phi(t)$ on $[0,T]$ for which the norm*

$$\|\Phi\|_{\mathcal{N}_W^2(0,T,H)} := \int_0^T \mathbb{E}\Big[\|\Phi(t')Q^{1/2}\|_{L_2(U,H)}\Big]^2 dt'$$

is finite.

Proposition 2.6.5. *The simple processes are dense in $\mathcal{N}_W^2(0,T,H)$.*

By density and equation (2.10), the stochastic integral now extends to $\mathcal{N}_W^2(0,T,H)$. We will frequently use the following two maximal estimate on the stochastic integral.

Theorem 2.6.6. *There exists a constant C independent of f such that*

$$\mathbb{E}\Big[\sup_{t\in[0,T]}\|\int_0^t f(t')dW(t')\|_H^2\Big] \leq C\cdot\int_0^T \mathbb{E}\Big[\|f(t')Q^{1/2}\|_H^2\Big]dt',$$

for all $f\in\mathcal{N}_W^2(0,T,H)$.

Theorem 2.6.7. *Let $S(t)$ be a C_0-semigroup on H. Then there exists a constant C, independent of f such that*

$$\mathbb{E}\Big[\sup_{t\in[0,T]}\|\int_0^t S(t-t')f(t')dW(t')\|_H^2\Big] \leq CM\cdot\int_0^T \mathbb{E}\Big[\|f(t')Q^{1/2}\|_H^2\Big]dt',$$

Now we formulate the notion of an SPDE and its solutions. Consider the SPDE

$$\begin{aligned} du(t) &= [A_t u(t) + f(t,u(t))]dt + B(t,u(t))dW(t) \\ u(0) &= \zeta \end{aligned} \tag{2.11}$$

where $f:\Omega\times[0,T]\times H\to H$ and $B:\Omega\times[0,T]\times H\to L_2(U_0,H)$ are measurable and A_t generates a strongly continuous evolution family on H. Motivated by the variation of constants formula, the notion of a mild solution to (2.11) is defined as follows.

Definition 2.6.8. *An H-valued predictable process $u(t)$ is a mild solution of (2.11) if it satisfies*

$$\mathbb{P}\Big[\int_0^T \|u(t)\|_H^2 < \infty\Big] = 1,$$

$$\mathbb{P}\Big[\int_0^T \|B(u(t))Q^{1/2}\|_{L_2(U,H)}^2 < \infty\Big] = 1$$

and

$$u(t) = P_{t,0}\zeta + \int_0^t P_{t,t'}f(t',u(t))dt' + \int_0^t P_{t,t'}B(t,u(t))dt', \quad \mathbb{P}-a.s.$$

Now we formulate the following existence and uniqueness results. Theorems 2.6.9 and 2.6.10 both follow from theorem 2.6.11.

Theorem 2.6.9. *(Linear SPDE with additive noise) Suppose that f is independent of u, and $B = B_0$ for some $B_0 \in L_2(U_0, H)$. Then there exists a unique mild solution of (2.11).*

Theorem 2.6.10. *(Linear SPDE with multiplicative noise) Suppose that f is independent of u, and $B(\omega, t, u) = B_0 u$ for some $B_0 \in L(H, L_2(U_0, H))$. Then there exists a unique mild solution of (2.11).*

Theorem 2.6.11. *(SPDE with Lipschitz nonlinearity) [49, Theorem 1.3] Suppose there exists a constant C such that*

$$\|f(\omega, t, u) - f(\omega, t, v)\|_H + \|B(\omega, t, u) - B(\omega, t, v)\|_{L_2(U_0,H)} \leq C\|u - v\|_H,$$

and

$$\|f(\omega, t, u)\|_H^2 + \|B(\omega, t, u)\|_{L_2(U_0,H)}^2 \leq C^2(1 + \|u\|_H)^2$$

hold uniformly in t, ω. Then there exists a unique mild solution to (2.11).

2.7 Lie groups and Lie algebras

To capture general continuous symmetries we will need the notion of a Lie group and a Lie algebra. We only require some definitions and basic theorems, which can be found for example in [29].

Definition 2.7.1. *A Lie group G is a group which is also a (finite-dimensional) smooth manifold, for which group multiplication and inversion are continuous.*

Definition 2.7.2. *A Lie algebra $\mathfrak{g}$ is a vector space which has an additional operation called the Lie bracket*

$$\begin{gathered} \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g} \\ (X, Y) \to [X, Y], \end{gathered}$$

which is symmetric, bilinear and satisfies the Jacobi identity

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0.$$

Definition 2.7.3. *A Lie group representation is a vector space V along with a smooth map*

$$\Pi : G \to GL(V),$$

which satisfies

$$\Pi(g)\Pi(h) = \Pi(gh),$$

for all $g, h \in G$.

Definition 2.7.4. *A Lie algebra representation is a vector space V along with a map*

$$\pi : \mathfrak{g} \to L(V),$$

which satisfies

$$\pi(X)\pi(Y) - \pi(Y)\pi(X) = \pi([X, Y]),$$

for all $X, Y \in \mathfrak{g}$.

There is a rich correspondence between Lie groups and Lie algebras, as illustrated by the next few theorems.

Theorem 2.7.5. *Let* $\mathfrak{g} := T_eG$ *be the tangent space of* G*. There exists a unique function* $\exp : \mathfrak{g} \to G$ *called the exponential map, which is continous and satisfies*

$$\begin{aligned} \exp(0) &= I, \\ \exp((t+s)X) &= \exp(tX)\exp(sX), \\ \frac{d}{dt}\exp(tX)\Big|_{t=0} &= X, \end{aligned}$$

for all $t, s \in \mathbb{R}$*,* $X \in \mathfrak{g}$*.*

Furthermore, $\mathfrak{g}$ *is a Lie algebra when equipped with the bracket*

$$[X,Y] := \frac{d}{dt}e^{tX}Ye^{-tX}\Big|_{t=0}.$$

Theorem 2.7.6. *Let* Π *be a Lie group representation of* G *on* V*. Then the map*

$$\begin{aligned} \pi &: \mathfrak{g} \to L(V) \\ \pi &: X \to \left(Y \to \frac{d}{dt}\Pi(\exp(tX)Y\Big|_{t=0}\right) \end{aligned}$$

is a Lie algebra representation of $\mathfrak{g}$ *on the* T_eV*. Furthermore, we have the equality*

$$\Pi(g)\pi(X)\Pi(g^{-1}) = \pi(gXg^{-1}) \tag{2.12}$$

The tangent space T_eG consists of equivalence classes of smooth curves γ which satisfy $\gamma(0) = e$. If $\gamma(t)$ is such a curve, then so is $g\gamma(t)g^{-1}$, and this operation is well-defined on the equivalence classes. This is the way in which we interpret $gXg^{-1} \in T_eG$ for $g \in G$, $X \in T_eG$.

Theorem 2.7.7. *The map*

$$\begin{aligned} Ad &: G \to GL(\mathfrak{g}) \\ & g \to (X \to gXg^{-1}) \end{aligned} \tag{2.13}$$

is a Lie group representation, and its corresponding Lie algebra representation is

$$ad : X \to (Y \to [X,Y]]). \tag{2.14}$$

We often write $Ad_g := Ad(g)$ *and* $ad_X := ad(X)$*.*
For $X, Y \in T_eG$ *we have the equality*

$$Ad_{\exp(X)}(Y) = e^{ad_X}(Y), \tag{2.15}$$

where the exponential on the right-hand side is an operator exponential.

Chapter 3

Rotating Waves

We consider a two-dimensional reaction-diffusion system, which satisfies the semilinear PDE

$$du(t,x) = D\Delta u(t,x)dt + f(u(t,x))dt, \tag{3.1}$$

where $u(t,x) \in \mathbb{R}^n$ and D is a diagonal matrix with strictly positive entries on the diagonal. Such systems are widespread throughout physics, chemistry [52] and biology [22, 36], and exhibit many interesting behaviours. A possible interpretation of (3.1) is as a chemical system. Here, $u(t,x)$ models the density at time t and position x of a compound made up of n different chemical species. The term $D\Delta u$ models the natural diffusion of chemicals, with D determining the diffusion speed of each species. The term $f(u)$ models a chemical reaction, turning some chemicals into others at a rate dependent on their densities.

Multi-component reaction-diffusion equations are well known for their propensity for pattern formation [14]. Typical patterns such as spots, stripes, and mazes have been numerically observed [20]. Although numerical simulations were not available at the time, Alan Turing knew about these patterns, and even postulated these reaction-diffusion equations to be the mechanism by which animals form patterns on their skin [51, 54].

In this thesis, we will treat one particular type of pattern: a rotating wave. This is a pattern having a fixed wave profile, which rotates along a fixed origin as time progresses. Rotating waves have been studied in a deterministic setting, where exponential decay [43] and stability [6] have been shown. Stability in a stochastic setting has previously been shown using the method of MacLaurin [34]. However, their SDE for the phase is extremely unwieldy compared to ours. Furthermore, our results about the multiscale expansion and approximate optimality of the phase are completely new.

3.1 Preliminaries

Throughout chapter 3 we will write

$$\begin{aligned} H^k &:= H^k(\mathbb{R}^2, \mathbb{R}^n), \\ L^2 &:= L^2(\mathbb{R}^2, \mathbb{R}^n), \\ L^\infty &:= L^\infty(\mathbb{R}^2, \mathbb{R}^n). \end{aligned}$$

We also fix some stochastic basis $(\Omega, \mathcal{F}, \mathbb{P})$ along with a normal filtration $\mathcal{F}_t$ to be used throughout these sections.

We will study a stochastic version of the reaction-diffusion equation (3.1). Let $W(t)$ be a Q-Wiener process. The stochastic version of (3.1) reads

$$du(t,x) = [D\Delta u(t,x) + f(u(t,x))]dt + \sigma dW(t), \tag{3.2}$$

where u takes values in H^2. We will assume f is such that the problem is well-posed.

Assumption 3.1.1. *The nonlinearity f is such that it satisfies the hypotheses of theorem 2.6.11. Thus, (3.2) has a unique mild solution.*

We will also need a notion of a small *phase-correction*. The phase-correction $\gamma(t)$ at time t will be described by a translation and a rotation, parameterized by

$$\gamma(t) = \begin{pmatrix} \theta(t) \\ b_1(t) \\ b_2(t) \end{pmatrix}.$$

We identify $\gamma(t) \in \mathbb{R}^3$ with the isometry of $\mathbb{R}^2$ given by

$$x \to R_\theta x + \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \tag{3.3}$$

which we also call $\gamma(t)$ in a slight abuse of notation. Here R_θ is a rotation matrix defined as

$$R_\theta := \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}. \tag{3.4}$$

We will also frequently write $b := \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$. The isometry $\gamma(t)$ further induces an isometry $\mathcal{T}_\gamma$ on H, defined by

$$\mathcal{T}_\gamma : \phi(x) \to \phi(\gamma^{-1}x) = \phi\Big(R_{-\theta}x - b\Big). \tag{3.5}$$

The reason we define $\mathcal{T}_\gamma$ with the inverse is so that we have $\mathcal{T}_{\gamma_1}\mathcal{T}_{\gamma_2} = \mathcal{T}_{\gamma_1\gamma_2}$.

Next we need some assumptions and preliminary results regarding the deterministic problem and rotating waves.

Assumption 3.1.2. *Equation (3.1) has a solution of the form*

$$\hat{u}(t,x) = \mathcal{T}_{R_{\omega t}} u^*(x) = u^*(R_{-\omega t}x) \tag{3.6}$$

This solution $\hat{u}$ is called the *rotating wave*, and u^* is called the *rotating wave profile* or *wave profile*. If u^* is sufficiently smooth, we can directly compute

$$\begin{aligned} \partial_t \hat{u}(t,x) &= \partial_t u^*(R_{-\omega t}x) \\ &= (\nabla u^*)(R_{-\omega t}x)\partial_t R_{-\omega t}x \\ &= -\omega(\nabla u^*)(R_{-\omega t}x)R_{\pi/2}R_{-\omega t}x \end{aligned}$$

Therefore if we change to co-rotating coordinates $y = R_{-\omega t}x$, equation (3.1) transforms into

$$-\omega \nabla u^*(y) R_{\pi/2} y = D\Delta u^*(y) + f(u^*(y)). \tag{3.7}$$

Introducing the differential operator

$$\partial_\psi \phi := x_1 \partial_{x_2}\phi - x_2 \partial_{x_1}\phi = \nabla\phi(x) R_{\pi/2} x, \tag{3.8}$$

equation (3.7) simplifies further to

$$D\Delta u^* + \omega \partial_\psi u^* + f(u^*) = 0. \tag{3.9}$$

In fact, any sufficiently smooth solution to (3.9) can be used as the wave profile for a rotating wave. Thus, existence of rotating waves is equivalent to existence of solutions to (3.9).

In equation 3.8 we chose the notation ∂_ψ because this operator is also the derivative in the radial direction in polar coordinates. This also means that ∂_ψ obeys the product rule.

3.1.1 Commutation relations

Later we will need various identities involving ∇, ∂_ψ and $\mathcal{T}_\gamma$, so we derive them all now. Throughout this section, let $\phi : \mathbb{R}^2 \to \mathbb{R}^n$ be a sufficiently smooth function.

We start by computing

$$\partial_t \mathcal{T}_{\gamma(t)}\phi = \partial_t \phi(\gamma^{-1}(t)x) = (\nabla\phi)(\gamma^{-1}(t)x)\partial_t\gamma^{-1}(t)x. \tag{3.10}$$

Using the product rule and the chain rule, we derive

$$\begin{aligned}
\partial_t\gamma^{-1}(t)x &= \partial_t\Big[R_{-\theta(t)}\big(x - b(t)\big)\Big] \\
&= -\dot\theta(t) R_{\pi/2} R_{-\theta(t)}\big(x - b(t)\big) - R_{-\theta(t)}\dot b(t) \\
&= -\dot\theta(t) R_{\pi/2}\gamma^{-1}(t)x - R_{-\theta(t)}\dot b(t).
\end{aligned}$$

Substitute this into equation (3.10), and use the definition of ∂_ψ (3.8) to get

$$\begin{aligned}
\partial_t \mathcal{T}_{\gamma(t)}\phi(x) &= -(\nabla\phi)(\gamma^{-1}(t)x)[(\dot\theta(t) R_{\pi/2}\gamma^{-1}(t)x + R_{-\theta(t)}\dot b(t))] \\
&= -\mathcal{T}_{\gamma(t)}\Big([\nabla\phi][(\dot\theta(t) R_{\pi/2} x + R_{-\theta(t)}\dot b(t))]\Big) \\
&= -\mathcal{T}_{\gamma(t)}\Big(\dot\theta(t)\partial_\psi\phi + \nabla\phi R_{-\theta(t)}\dot b(t))\Big).
\end{aligned}$$

Now introduce the three-dimensional matrix

$$R_\theta := \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) \\ 0 & \sin(\theta) & \cos(\theta) \end{pmatrix}. \tag{3.11}$$

to get

$$\partial_t \mathcal{T}_{\gamma(t)}\phi(x) = -\mathcal{T}_{\gamma(t)} \begin{pmatrix} \partial_\psi\phi & \partial_x\phi & \partial_y\phi \end{pmatrix} R_{-\theta(t)} \begin{pmatrix} \dot\theta \\ \dot b_1 \\ \dot b_2 \end{pmatrix}. \tag{3.12}$$

From now on, R_θ can either mean the two-dimensional matrix from (3.4) or the three-dimensional matrix from (3.11). It should always be clear from the context which one is meant. Note that the identity

$$R_a R_b = R_{a+b} = R_b R_a$$

holds in both cases. Next, we introduce the notational shorthand

$$\nabla_c \phi := \begin{pmatrix} \partial_\psi \phi & \partial_x \phi & \partial_y \phi \end{pmatrix} \tag{3.13}$$

to rewrite (3.12) to

$$\partial_t \mathcal{T}_{\gamma(t)} \phi(x) := -[\mathcal{T}_{\gamma(t)} \nabla_c \phi] R_{-\theta(t)} \dot{\gamma}(t). \tag{3.14}$$

Now we derive the commutation relation between ∂_ψ and ∇. First let $\xi \in C^2(\mathbb{R}^n, \mathbb{R})$. By the product rule, we have

$$\begin{aligned} \partial_\psi \nabla \xi &= (x_1 \partial_{x_2} - x_2 \partial_{x_1}) \begin{pmatrix} \partial_{x_1} \xi \\ \partial_{x_2} \xi \end{pmatrix}^T = \begin{pmatrix} x_1 \partial_{x_1 x_2} \xi - x_2 \partial_{x_1 x_1} \xi \\ x_1 \partial_{x_2 x_2} \xi - x_2 \partial_{x_1 x_2} \xi \end{pmatrix}^T \\ \nabla \partial_\psi \xi &= \begin{pmatrix} \partial_{x_1} (x_1 \partial_{x_2} \xi - x_2 \partial_{x_1} \xi) \\ \partial_{x_2} (x_1 \partial_{x_2} \xi - x_2 \partial_{x_1} \xi) \end{pmatrix}^T = \begin{pmatrix} x_1 \partial_{x_1 x_2} \xi - x_2 \partial_{x_1 x_1} \xi \\ x_1 \partial_{x_2 x_2} \xi - x_2 \partial_{x_1 x_2} \xi \end{pmatrix}^T + \begin{pmatrix} \partial_{x_2} \xi \\ -\partial_{x_1} \xi \end{pmatrix}^T \end{aligned}$$

which gives

$$\nabla \partial_\psi \xi = \partial_\psi \nabla \xi + [\nabla \xi] R_{\pi/2},$$

which immediately extends to

$$\nabla \partial_\psi \phi = \partial_\psi \nabla \phi + [\nabla \phi] R_{\pi/2}, \tag{3.15}$$

for vector-valued functions. Next is the relation between ∇ and $\mathcal{T}_\gamma$. Using the multivariable chain rule, we get

$$\nabla \mathcal{T}_\gamma \phi \overset{(3.5)}{=} \nabla \phi(\gamma^{-1} x) = [(\nabla \phi)(\gamma^{-1} x)] \nabla(\gamma^{-1} x). \tag{3.16}$$

Now observe that

$$\nabla \gamma^{-1} x \overset{(3.3)}{=} \nabla R_{-\theta}(x - b) = R_{-\theta}. \tag{3.17}$$

Thus, substituting (3.17) into (3.16) we get

$$\nabla \mathcal{T}_\gamma \phi = (\nabla \phi)(\gamma^{-1} x) R_\theta \overset{(3.5)}{=} \mathcal{T}_\gamma \nabla \phi R_{-\theta}. \tag{3.18}$$

Next we find

$$\begin{aligned} \partial_\psi \mathcal{T}_\gamma \phi &\overset{(3.8)}{=} [\nabla \mathcal{T}_\gamma \phi] R_{\pi/2} x \\ &\overset{(3.18)}{=} [\mathcal{T}_\gamma \nabla \phi R_{-\theta}] R_{\pi/2} x \\ &= \mathcal{T}_\gamma [\nabla \phi R_{\pi/2} R_{-\theta} \mathcal{T}_{\gamma^{-1}} x]. \end{aligned} \tag{3.19}$$

To simplify further, we compute

$$R_{-\theta}\mathcal{T}_{\gamma^{-1}}x \overset{(3.5)}{=} R_{-\theta}(R_\theta x + b) = x + R_{-\theta}b.$$

Substituting this back into (3.19) gives

$$\begin{aligned}\partial_\psi \mathcal{T}_\gamma \phi &= \mathcal{T}_\gamma[\nabla\phi R_{\pi/2}(x + R_{-\theta}b)] \\ &\overset{(3.8)}{=} \mathcal{T}_\gamma \partial_\psi \phi + \mathcal{T}_\gamma[\nabla\phi]R_{\pi/2}R_{-\theta}b).\end{aligned} \tag{3.20}$$

Using the ∇_c symbol introduced in (3.13), we may also combine (3.18) and (3.20) into

$$\nabla_c \mathcal{T}_\gamma \phi = [\mathcal{T}_\gamma \nabla_c \phi]R_{-\theta} + \mathcal{T}_\gamma \nabla\phi R_{\pi/2}R_{-\theta}be_\psi, \tag{3.21}$$

where $e_\psi := (1, 0, 0)$.

3.1.2 Differentiability of $\mathcal{T}_\gamma \hat{u}$

In this section we establish differentiability of $\mathcal{T}_\gamma \hat{u}$ with respect to γ in various function spaces. We introduce the following Hilbert spaces to be able to deal with rotational derivatives efficiently. Let V be either $\mathbb{R}$, $\mathbb{R}^n$ or $\mathbb{R}^{n\times n}$.

Definition 3.1.3. *For $l \le k$, the space $H^{k,l}(\mathbb{R}^n, V)$ is defined as*

$$H^{k,l}_{rot}(\mathbb{R}^n, V) := \{f \in H^k(\mathbb{R}^n, V) : \partial^j_\psi f \in H^{k-j}(\mathbb{R}^n, V) \text{ for all } j \le l\}$$

with the inner product

$$(f, g)_{H^{k,l}_{rot}(\mathbb{R}^n,V)} := \sum_{j=0}^{l} (\partial^j_\psi f, \partial^i_\psi g)_{H^{k-j}(\mathbb{R}^n,V)}. \tag{3.22}$$

Again, we will write $H^{k,l}_{rot} := H^{k,l}_{rot}(\mathbb{R}^2, V)$. The parameter l measures how many rotational derivatives we can 'safely' take. Notice that $H^{k,l}_{rot} \subset H^{k',l'}_{rot}$ and $\|\phi\|_{H^{k',l'}_{rot}} \le \|\phi\|_{H^{k,l}_{rot}}$ if $k' \ge k$ and $l' \ge l$. We also recover the regular Sobolev spaces if $l = 0$.

Proposition 3.1.4. *The space C^∞_c is dense in $H^{k,l}_{rot}$ for every k,l.*

Proposition 3.1.5. *The operator $\mathcal{T}_{R_\theta}$ is an isometry of $H^{k,l}_{rot}$ for every θ, k, l.*

Proof. We first observe that $\mathcal{T}_\gamma$ is an isometry of H^k for every $\gamma \in SE(2)$. Next, by (3.20) we see that ∂_ψ commutes with $\mathcal{T}_{R_\theta}$. Combining these facts we see that

$$\|\partial^j_\psi \mathcal{T}_{R_\theta}\phi\|_{H^{k-j}} = \|\mathcal{T}_{R_\theta}\partial^j_\psi \phi\|_{H^{k-j}} = \|\partial_\psi \phi\|_{H^{k-j}}$$

holds for every k, j with $j \le k$. Thus, the norm induced by (3.22) is invariant under $\mathcal{T}_{R_\theta}$. □

Proposition 3.1.6. *For each k, l there exists a constant C, independent of ϕ such that*

$$\|\mathcal{T}_\gamma \phi\|_{H^{k,l}_{rot}} \le C_{k,l}\|\phi\|_{H^{k,l}_{rot}} \sum_{j=0}^{l} |\gamma|^j. \tag{3.23}$$

Proof. We use induction on l. The case $l = 0$ follows from the fact that $\mathcal{T}_\gamma$ is an isometry of H^k. Now suppose equation (3.23) holds for some $l = l' \geq 0$. To show (3.23) for $l = l' + 1$, we see from (3.22) that we only need to estimate the additional term $\|\partial_\psi^{l'+1}\phi\|_{H^{k-l'-1}}$. By equation (3.20) we have

$$\begin{aligned}
\|\partial_\psi^{l'+1}\mathcal{T}_\gamma\psi\|_{H^{k-l'-1}} &\leq \|\mathcal{T}_\gamma\partial_\psi^{l'+1}\phi\|_{H^{k-l'-1}} + \|\mathcal{T}_\gamma[\nabla\partial_\psi^{l'}\phi]R_{\pi/2}R_{-\theta}b)\|_{H^{k-l'-1}} \\
&= \|\partial_\psi^{l'+1}\phi\|_{H^{k-l'-1}} + \|[\nabla\partial_\psi^{l'}\phi]R_{\pi/2}R_{-\theta}b)\|_{H^{k-l'-1}} \\
&\leq \|\partial_\psi^{l'+1}\phi\|_{H^{k-l'-1}} + |\gamma|\|\partial_\psi^{l'}\phi\|_{H^{k-l'}} \\
&\leq \|\phi\|_{H_{rot}^{k,l'+1}} + |\gamma|\|\phi\|_{H_{rot}^{k,l'}}.
\end{aligned}$$

The result follows from the induction hypothesis and the fact that $\|\phi\|_{H_{rot}^{k,l'}} \leq \|\phi\|_{H_{rot}^{k,l'+1}}$, which is trivial. ☐

Next we establish the following differentiability result.

Proposition 3.1.7. *Suppose $\phi \in H_{rot}^{k+2,l+2}$. Then the map*

$$\gamma \to \mathcal{T}_\gamma\phi$$

is differentiable as an $H_{rot}^{k,l}$-valued function, with Fréchet derivative

$$-\mathcal{T}_\gamma[\nabla_c\phi]R_{-\theta}. \tag{3.24}$$

Furthermore, we have the estimates

$$\|\mathcal{T}_\gamma\phi - \phi\|_{H_{rot}^{k,l}} \leq \|\phi\|_{H_{rot}^{k+1,l+1}} C_{k,l}\sum_{j=0}^{l}|\gamma|^{j+1}, \tag{3.25}$$

and

$$\|\mathcal{T}_\gamma\phi - \phi + [\nabla_c\phi]\gamma\|_{H_{rot}^{k,l}} \leq \|\phi\|_{H_{rot}^{k+2,l+2}} C_{k,l}\sum_{j=0}^{l}|\gamma|^{j+2}. \tag{3.26}$$

Proof. First assume $\phi \in C_c^\infty$. As a notational shorthand, we write

$$\begin{aligned}
\nabla_\gamma &:= \begin{pmatrix}\partial_\psi & \partial_{b_1} & \partial_{b_2}\end{pmatrix}, \\
H_\gamma &:= \nabla_\gamma[\nabla_\gamma]^T = \begin{pmatrix}\partial_{\psi\theta} & \partial_{\psi b_1} & \partial_{\psi b_2} \\ \partial_{\psi b_1} & \partial_{b_1 b_1} & \partial_{b_1 b_2} \\ \partial_{\psi b_2} & \partial_{b_1 b_2} & \partial_{b_2 b_2}\end{pmatrix}.
\end{aligned}$$

From equation (3.14) we see that

$$\gamma \to \mathcal{T}_\gamma\phi(x)$$

is differentiable for each x, with derivative

$$\nabla_\gamma\mathcal{T}_\gamma\phi(x) = -\mathcal{T}_\gamma[\nabla_c\phi(x)]R_{-\theta}.$$

Differentiating again, we find the Hessian matrix.

$$\begin{aligned}H_\gamma \mathcal{T}_\gamma \phi(x) &= \nabla_\gamma[\nabla_\gamma \mathcal{T}_\gamma \phi(x)]^T \\ &= -\nabla_\gamma \mathcal{T}_\gamma[\nabla_c \phi(x)]R_{-\theta}]^T \\ &= -R_\theta \nabla_\gamma[\mathcal{T}_\gamma[\nabla_c \phi(x)]^T \\ &= R_\theta \mathcal{T}_\gamma \nabla_c[\nabla_c \phi(x)]^T R_{-\theta}\end{aligned}$$

Thus, applying Taylor's theorem (in γ, with integral remainder) for a fixed x to $\mathcal{T}_\gamma\phi(x)$, we find that

$$\begin{aligned}\mathcal{T}_\gamma \phi(x) &= \mathcal{T}_\gamma \phi(x)\Big|_{\gamma=0} + \nabla_\gamma \mathcal{T}_\gamma \phi(x)\Big|_{\gamma=0}\gamma + \gamma^T \int_0^1 (1-t) H_\gamma \mathcal{T}_{t\gamma}\phi(x) dt\gamma \\ &= \phi(x) - \nabla_c \phi(x)\gamma + \gamma^T \int_0^1 (1-t) R_{t\theta} \mathcal{T}_{t\gamma} \nabla_c[\nabla_c \phi(x)]^T R_{-t\theta} dt\gamma \qquad (3.27)\end{aligned}$$

holds for each x. Now we estimate

$$\begin{aligned}\|R_{t\theta}\mathcal{T}_{t\gamma}\nabla_c[\nabla_c\phi(x)]^T R_{-t\theta}\|_{H^{k,l}_{rot}} &= \|\mathcal{T}_{t\gamma}\nabla_c[\nabla_c\phi(x)]^T\|_{H^{k,l}_{rot}} \\ &\overset{(3.23)}{\leq} C_{k,l}\|\nabla_c[\nabla_c\phi(x)]^T\|_{H^{k,l}_{rot}}\sum_{j=0}^{l}|\gamma|^j \\ &\leq C_{k,l}\|\phi\|_{H^{k+2,l+2}_{rot}}\sum_{j=0}^{l}|\gamma|^j\end{aligned}$$

Thus we may interpret (3.27) as an $H^{k,l}$-valued equality (replacing the integral by a Bochner integral). Rearranging (3.27) and taking the $H^{k,l}$ norm then also gives

$$\|\mathcal{T}_\gamma\phi - \phi + [\nabla_c\phi]\gamma\|_{H^{k,l}} \leq C_{k,l}\|\phi\|_{H^{k+2,l+2}}\sum_{j=0}^{l}|\gamma|^{j+2},$$

which shows (3.26). Dividing by $|\gamma|$ and letting $\gamma \to 0$ shows differentiability by definition of the Fréchet derivative.

Now let $\gamma(t) = t\gamma$. By the fundamental theorem of calculus (for Bochner integrals) we have

$$\mathcal{T}_\gamma\phi - \phi = \int_0^1 (\nabla_\gamma \mathcal{T}_{\gamma(t)}\phi)\gamma'(t)dt \overset{(3.24)}{=} -\int_0^1 \mathcal{T}_{t\gamma}[\nabla_c\phi]R_{-t\theta}dt\gamma.$$

Taking the $H^{k,l}_{rot}$-norm we find

$$\begin{aligned}\|\mathcal{T}_\gamma\phi - \phi\|_{H^{k,l}_{rot}} &\leq |\gamma|\int_0^1 \|\mathcal{T}_{t\gamma}[\nabla_c u]R_{-t\theta}\|_{H^{k,l}_{rot}}dt \\ &= |\gamma|\int_0^1 \|\mathcal{T}_{t\gamma}[\nabla_c u]\|_{H^{k,l}_{rot}}dt \\ &\overset{(3.23)}{\leq} C_{k,l}\|\nabla_c u\|_{H^{k,l}_{rot}}\sum_{j=0}^{l}|\gamma|^{j+1} \\ &\leq C_{k,l}\|u\|_{H^{k+1,l+1}_{rot}}\sum_{j=0}^{l}|\gamma|^{j+1},\end{aligned}$$

which shows (3.25). For general ϕ, the results follow by density using proposition 3.1.4. □

We would like to apply proposition 3.1.7 to $\hat{u}$. To do so, the following assumption is needed.

Assumption 3.1.8. $u^* \in H^{6,3}_{rot}$.

In section 6.3 we show using standard elliptic regularity methods that a weak solution to equation (3.9) automatically satisfies this assumption. Thus, assumption 3.1.8 is not restrictive. Hence we can specify to the case $\phi = \hat{u}(t)$, and formulate the estimates we will actually use.

Proposition 3.1.9. *For every t, the map*

$$\mathcal{T}_\gamma \hat{u}(t)$$

is differentiable with Fréchet derivative

$$-\mathcal{T}_\gamma [\nabla_c \hat{u}] R_{-\theta}.$$

Furthermore, we have the estimates

$$\|\mathcal{T}_\gamma \hat{u}(t) - \hat{u}(t)\|_{H^2} \leq \|u^*\|_{H^{3,1}_{rot}} C_{2,0} |\gamma|, \tag{3.28}$$

and

$$\|\mathcal{T}_\gamma \hat{u}(t) - \hat{u}(t) + [\nabla_c \hat{u}(t)]\gamma\|_{H^{4,1}_{rot}} \leq \|u^*\|_{H^{6,3}_{rot}} C_{4,1} \sum_{i=0}^{3} |\gamma|^{i+2}. \tag{3.29}$$

Proof. Since $\hat{u} = \mathcal{T}_{R_{\omega t}} u^*$, this follows directly from propositions 3.1.5 and 3.1.7, and assumption 3.1.8. □

3.1.3 Linearization operator

Now we show that the derivatives of u^* are eigenfunctions of some linearization operator. We first observe that the laplacian is invariant under rotations, so it commutes with ∂_ψ. Thus, assuming sufficient smoothness of u^* and f, we may apply ∂_ψ to (3.9) to find

$$D\Delta\partial_\psi u^* + \omega\partial_\psi\partial_\psi u^* + f'(u^*)\partial_\psi u^* = 0. \tag{3.30}$$

Similarly, applying the gradient operator (which also commutes with the laplacian) to (3.9) and using (3.15) gives

$$D\Delta\nabla u^* + \omega\partial_\psi \nabla u^* + f'(u^*)\nabla u^* = -\omega[\nabla u^*]R_{\pi/2}. \tag{3.31}$$

Introducing the "frozen-wave" operator

$$\mathcal{L}^{\#}\phi := D\Delta\phi + \omega\partial_\psi\phi + f'(u^*)\phi, \tag{3.32}$$

we may succinctly restate equations (3.30) and (3.31) as

$$\mathcal{L}^{\#}\partial_\psi u^* = 0, \qquad \mathcal{L}^{\#}\partial_{x_1} u^* = -\omega\partial_{x_2} u^*, \qquad \mathcal{L}^{\#}\partial_{x_2} u^* = \omega\partial_{x_1} u^* \tag{3.33}$$

or

$$\mathcal{L}^{\#}\partial_\psi u^* = 0 \tag{3.34a}$$
$$\mathcal{L}^{\#}(\partial_{x_1} + i\partial_{x_2})u^* = i\omega(\partial_{x_1} + i\partial_{x_2})u^* \tag{3.34b}$$
$$\mathcal{L}^{\#}(\partial_{x_1} - i\partial_{x_2})u^* = -i\omega(\partial_{x_1} - i\partial_{x_2})u^*. \tag{3.34c}$$

The following assumption on $\mathcal{L}^{\#}$ is critical for the deterministic stability of rotating waves.

Assumption 3.1.10. *$\mathcal{L}^{\#}$ generates a C_0-semigroup $P_t^{\#}$ on H.*

Later we will need to compute the action of $P_t^{\#}$ on linear combinations of $\partial_\psi u^*$, $\partial_{x_1} u^*$ and $\partial_{x_2} u^*$. Let V be the subspace of H spanned by these derivatives of u^*. We use the basis $\partial_\psi u^*$, $\partial_{x_1} u^*$ and $\partial_{x_2} u^*$ for V. In this basis, (3.33) becomes

$$\mathcal{L}^{\#}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ -c \\ b \end{pmatrix} = S\begin{pmatrix} a \\ b \\ c \end{pmatrix},$$

where

$$S := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\omega \\ 0 & \omega & 0 \end{pmatrix}.$$

Computing the matrix exponential

$$\exp(tS) = R_{\omega t},$$

we find that

$$P_t^{\#} v = R_{\omega t} v,$$

for any $v \in V$. In particular, this means that

$$P_t^{\#}[\nabla_c u^*]\gamma = [\nabla_c u^*] R_{\omega t}\gamma \tag{3.35}$$

for any $\gamma \in \mathbb{R}^3$. Note that equation (3.35) is just a different notation for taking linear combinations of $\partial_\psi u^*$, $\partial_{x_1} u^*$ and $\partial_{x_2} u^*$.

Next we define a spectral projection onto some eigenvalues of $\mathcal{L}^{\#}$. The following assumption guarantees that this is well-defined.

Assumption 3.1.11. *The point spectrum $\sigma_{pt}(\mathcal{L}^{\#})$ on H satisfies*

$$\sigma_{pt}(\mathcal{L}^{\#}) \subset \{\lambda \in \mathbb{C} : \Re(\lambda) \le -b\} \cup \{0, i\omega, -i\omega\}.$$

Furthermore, the eigenvalues $\{0, i\omega, -i\omega\}$ all have multiplicity one.

The second part of assumption 3.1.11 guarantees that $\mathcal{L}^{\#}$ has no eigenfunctions with eigenvalues $\{0, i\omega, -i\omega\}$ except for the ones found in equation (3.34c). From now on, we will call the span of these eigenvalues the *center space*. Next we define a projection onto the center space.

Definition 3.1.12. $\Pi^{\#,c}$ *is the spectral projection of $\mathcal{L}^{\#}$ onto $\{0, i\omega, -i\omega\}$ (see theorem 2.1.8). We further define*

$$\Pi^{\#} := I - \Pi^{\#,c} \tag{3.36}$$

$$\Pi^c_{R_{\omega t}} := \mathcal{T}_{R_{\omega t}} \Pi^{\#,c} \mathcal{T}_{R_{-\omega t}} \tag{3.37}$$

$$\Pi_{R_{\omega t}} := \mathcal{T}_{R_{\omega t}} \Pi^{\#} \mathcal{T}_{R_{-\omega t}} \tag{3.38}$$

Later we will need a more explicit form of $\Pi^{\#,c}$. Using assumption 3.1.11 and equation (3.34c), we find that $\Pi^{\#,c}$ has the form

$$\Pi^{\#,c}\phi = a\partial_\psi u^* + b\partial_{x_1}u^* + c\partial_{x_2}u^* \overset{(3.13)}{=} [\nabla_c u^*]\begin{pmatrix} a \\ b \\ c \end{pmatrix}, \tag{3.39}$$

where the coefficients a, b, c depend on ϕ. Taking inner products in L^2 from the right with $\partial_\psi u^*$, $\partial_{x_1}u^*$ and $\partial_{x_2}u^*$ results in the following system

$$\begin{pmatrix} (\Pi^{\#,c}\phi, \partial_\psi u^*) \\ (\Pi^{\#,c}\phi, \partial_x u^*) \\ (\Pi^{\#,c}\phi, \partial_y u^*) \end{pmatrix} = \begin{pmatrix} (\partial_\psi u^*, \partial_\psi u^*) & (\partial_{x_1}u^*, \partial_\psi u^*) & (\partial_{x_2}u^*, \partial_\psi u^*) \\ (\partial_\psi u^*, \partial_{x_1}u^*) & (\partial_{x_1}u^*, \partial_{x_1}u^*) & (\partial_{x_2}u^*, \partial_{x_1}u^*) \\ (\partial_\psi u^*, \partial_{x_2}u^*) & (\partial_{x_1}u^*, \partial_{x_2}u^*) & (\partial_{x_2}u^*, \partial_{x_2}u^*) \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix}. \tag{3.40}$$

where all inner products are taken in L^2. For notational purposes, we abbreviate the matrix on the left-hand side by defining

$$B := \begin{pmatrix} (\partial_\psi u^*, \partial_\psi u^*) & (\partial_{x_1}u^*, \partial_\psi u^*) & (\partial_{x_2}u^*, \partial_\psi u^*) \\ (\partial_\psi u^*, \partial_{x_1}u^*) & (\partial_{x_1}u^*, \partial_{x_1}u^*) & (\partial_{x_2}u^*, \partial_{x_1}u^*) \\ (\partial_\psi u^*, \partial_{x_2}u^*) & (\partial_{x_1}u^*, \partial_{x_2}u^*) & (\partial_{x_2}u^*, \partial_{x_2}u^*) \end{pmatrix}.$$

Since u^* and all its derivatives are real-valued, B is symmetric and strictly postive definite. Therefore we may invert B to solve equation (3.40) and substitute back into (3.39) to find

$$\Pi^{\#,c}\phi = [\nabla_c u^*]B^{-1}\begin{pmatrix} (\Pi^{\#,c}\phi, \partial_\psi u^*) \\ (\Pi^{\#,c}\phi, \partial_x u^*) \\ (\Pi^{\#,c}\phi, \partial_y u^*) \end{pmatrix}. \tag{3.41}$$

If we now introduce the bounded linear operator

$$\begin{aligned} \mathcal{P} : H^2(\mathbb{R}^2, \mathbb{R}^n) &\to \mathbb{R}^3 \\ \phi &\to \begin{pmatrix} (\Pi^{\#,c}\phi, \partial_\psi u^*) \\ (\Pi^{\#,c}\phi, \partial_x u^*) \\ (\Pi^{\#,c}\phi, \partial_y u^*) \end{pmatrix}, \end{aligned}$$

equation (3.41) further simplifies to

$$\Pi^{\#,c}\phi = [\nabla_c u^*]B^{-1}\mathcal{P}(\phi). \tag{3.42}$$

Now we formulate the final few assumptions required for deterministic stability.

Assumption 3.1.13. *There is a vector $u_\infty \in \mathbb{R}^n$ such that*

$$\sup_{|x|\geq R} |u^*(x) - u_\infty| \to 0 \text{ as } R \to \infty$$

Assumption 3.1.14. *$f \in C^4(\mathbb{R}^N; \mathbb{R}^N)$. Furthermore, f and its first, second and third derivatives are bounded.*

Assumption 3.1.15. *$f'(u_\infty)$ is a negative definite matrix.*

Finally we can formulate the linear stability result, due to Beyn and Lorenz [6].

Theorem 3.1.16. *Under assumptions 3.1.2, 3.1.8, 3.1.10, 3.1.11, 3.1.13, 3.1.14 and 3.1.15 we have*

$$\|P_t^{\#}\Pi^{\#}\|_{L(H^2)} \leq Ce^{-at} \tag{3.43}$$

for some constants C, $a > 0$.

3.1.4 Derivation of the SPDE

Let $\gamma(t) \in \mathbb{R}^3$ be a differentiable stochastic process which is adapted to $\mathcal{F}_t$ and satisfies $\gamma(0) = 0$. Now introduce the following three ways to write u:

$$u(t,x) =: \hat{u}(t,x) + v(t,x), \tag{3.44}$$
$$u(t,x) =: \mathcal{T}_{\sigma\gamma(t)}\hat{u}(t,x) + v_\gamma(t,x), \tag{3.45}$$
$$u(t,x) =: \mathcal{T}_{\sigma\gamma(t)}\hat{u}(t,x) + \sigma w_\gamma(t,x) + z_\gamma(t,x), \tag{3.46}$$

where w_γ will be specified later, after which the third equation serves as the definition for z_γ.

Combining the SPDE (3.2) for u, the definition (3.45) of v_γ, and (3.14) now gives the SPDE for v_γ.

$$\begin{aligned} dv_\gamma =& du(t,x) - d\mathcal{T}_{\sigma\gamma(t)}\hat{u}(t,x) \\ =& D\Delta u(t,x)dt + f(u(t,x))dt + \sigma dW(t,x) \\ &+ \sigma[\mathcal{T}_{\sigma\gamma(t)}\nabla_c\hat{u}(t,x)]R_{-\theta(t)}\dot{\gamma}(t)dt - \mathcal{T}_{\sigma\gamma(t)}d\hat{u}(t,x). \end{aligned} \tag{3.47}$$

We use (3.1) to find

$$\begin{aligned} \mathcal{T}_{\sigma\gamma(t)}d\hat{u}(t,x) &= \mathcal{T}_{\sigma\gamma(t)}[\Delta\hat{u}(t,x) + f(\hat{u}(t,x))]dt \\ &= \Delta\mathcal{T}_{\sigma\gamma(t)}\hat{u}(t,x)dt + \mathcal{T}_{\sigma\gamma(t)}f(\hat{u}(t,x))dt. \end{aligned}$$

Substituting this back into (3.47) gives

$$\begin{aligned} dv_\gamma =& D\Delta u(t,x)dt + f(u(t,x))dt + \sigma dW(t,x) \\ &+ \sigma[\mathcal{T}_{\sigma\gamma(t)}\nabla_c\hat{u}(t,x)]R_{-\theta(t)}\dot{\gamma}(t)dt \\ &- \Delta\mathcal{T}_{\sigma\gamma(t)}\hat{u}(t,x)dt - \mathcal{T}_{\sigma\gamma(t)}f(\hat{u}(t,x))dt, \end{aligned}$$

after which we rearrange some terms and use (3.45) to recombine the terms affected by Δ to find

$$\begin{aligned} dv_\gamma =& D\Delta v_\gamma(t,x)dt \\ &+ \sigma[\mathcal{T}_{\sigma\gamma(t)}\nabla_c\hat{u}(t,x)]R_{-\theta(t)}\dot{\gamma}(t)dt + \sigma dW(t,x) \\ &+ f(u(t,x))dt - \mathcal{T}_{\sigma\gamma(t)}f(\hat{u}(t,x))dt. \end{aligned}$$

Finally, add and subtract $[\mathcal{T}_{\sigma\gamma(t)}f'(\hat{u}(t,x))]v_\gamma(t,x)dt$ to get

$$\begin{aligned} dv_\gamma =& D\Delta v_\gamma(t,x)dt + [\mathcal{T}_{\sigma\gamma(t)}f'(\hat{u}(t,x))]v_\gamma(t,x)dt \\ &+ \sigma[\mathcal{T}_{\sigma\gamma(t)}\nabla_c\hat{u}(t,x)]R_{-\theta(t)}\dot{\gamma}(t)dt + \sigma dW(t,x) \\ &+ f(u(t,x))dt - \mathcal{T}_{\sigma\gamma(t)}f(\hat{u}(t,x))dt - [\mathcal{T}_{\sigma\gamma(t)}f'(\hat{u}(t,x))]v_\gamma(t,x)dt. \end{aligned} \tag{3.48}$$

Now define the linear operator

$$\mathcal{L}_{t,\gamma}\phi = D\Delta\phi(x) + [\mathcal{T}_{\gamma(t)}f'(\hat{u}(t,x))]\phi(x) \tag{3.49}$$

and the nonlinear term

$$\mathcal{R}_\gamma = f(u(t,x)) - \mathcal{T}_{\sigma\gamma(t)}f(\hat{u}(t,x)) - [\mathcal{T}_{\sigma\gamma}f'(\hat{u}(t,x))]v_\gamma(t,x). \tag{3.50}$$

We also introduce

$$\mathcal{L}_t\phi(x) := \Delta\phi(x) + f'(\hat{u}(t,x))\phi(x), \tag{3.51}$$

and observe that $\mathcal{L}_t = \mathcal{L}_{t,0}$. By a direct computation, we find the following relation between $\mathcal{L}_t$ and $\mathcal{L}^\#$.

$$(\partial_t - \mathcal{L}_t) \overset{(3.32)}{=} \mathcal{T}_{R_{\omega t}}(\partial_t - \mathcal{L}^\#)\mathcal{T}_{R_{-\omega t}}.$$

Using assumption 3.1.10 and the identity $\partial_t P^\#_{t-t'} = \mathcal{L}^\# P^\#_{t-t'}$, we also derive

$$\begin{aligned} (\partial_t - \mathcal{L}^\#)P^\#_{t-t'} &= 0 \\ \mathcal{T}_{R_{\omega t}}(\partial_t - \mathcal{L}^\#)\mathcal{T}_{R_{-\omega t}}\mathcal{T}_{R_{\omega t}}P^\#_{t-t'}\mathcal{T}_{R_{-\omega t'}} &= 0 \\ (\partial_t - \mathcal{L}_t)\mathcal{T}_{R_{\omega t}}P^\#_{t-t'}\mathcal{T}_{R_{-\omega t'}} &= 0. \end{aligned}$$

Now define

$$P_{t,t'} := \mathcal{T}_{R_{\omega t}}P^\#_{t-t'}\mathcal{T}_{R_{-\omega t'}}, \tag{3.52}$$

and observe that $P_{t,s}P_{s,r} = P_{t,r}$ to conclude that $P_{t,t'}$ is an evolution family on H generated by $\mathcal{L}_t$.

With this notation (3.48) simplifies to

$$\begin{aligned} dv_\gamma =& \mathcal{L}_{t,\sigma\gamma(t)}v_\gamma(t,x)dt \\ &+ \sigma[\mathcal{T}_{\sigma\gamma(t)}\nabla_c\hat{u}(t,x)]R_{-\theta(t)}\dot{\gamma}(t)]dt + \sigma dW(t,x) \\ &+ \mathcal{R}_\gamma dt. \end{aligned} \tag{3.53}$$

We now linearize (3.53), set γ (but not $\dot{\gamma}$) to zero and scale out σ to obtain the SPDE for the linearization w_γ.

$$dw_\gamma = \mathcal{L}_t w_\gamma(t,x)dt + [\nabla_c\hat{u}(t,x)]\dot{\gamma}(t)dt + dW(t,x). \tag{3.54}$$

In the next section we show that σw_γ is a good approximation to v_γ.

Finally we choose the following initial conditions for u and w_γ:

$$\begin{aligned} u(0,x) &= \hat{u}(0,x) + \sigma v_0(x) \\ w_\gamma(0,x) &= v_0(x) \end{aligned}$$

where v_0 is some fixed vector in H. The reason we choose these initial conditions is to make sure that

1. w_γ is independent of σ
2. the approximation $u(t,x) \approx \mathcal{T}_{\sigma\gamma(t)}\hat{u}(t,x) + \sigma w_\gamma(t,x)$ is exact at time 0.

The second property is easily verified using (3.45), remembering that we initially assumed $\gamma(0) = 0$.

3.2 Multiscale expansion

Now we formulate the first main result.

Theorem 3.2.1. *(a) Let*

$$\gamma(t) = \begin{pmatrix} \theta(t) \\ b_1(t) \\ b_2(t) \end{pmatrix}$$

be a progressively measurable stochastic process which is almost surely differentiable, satisfies $\gamma(0) = 0$ and

$$\mathbb{E}\Big[\sup_{t\in[0,T]} |\gamma(t)|^2\Big] = C_\gamma < \infty. \tag{3.55}$$

Let $u(t,x)$ be the solution to (3.2) with initial condition $u(0) = u^* + \sigma v_0$. Then equation (3.54) with initial condition $w_\gamma(0) = v_0$ has a unique mild solution in $C([0,T]; H^2)$, given by

$$w_\gamma(t,x) = P_{t,0}v_0 + \mathcal{T}_{R_{\omega t}}[\nabla_c u^*]R_{\omega t}\gamma(t) + \int_0^t P_{t,t'}dW(t'). \tag{3.56}$$

(b) Let $q \in (0, \frac{1}{2})$. Define the stopping times

$$\tau_{q,\sigma,\gamma} := \inf(\{t \in [0,T] : |\gamma(t)| \geq \sigma^{-q}\}) \wedge T \tag{3.57}$$

$$\tau_{q,\sigma,v} := \inf(\{t \in [0,T] : \|v(t,x)\|_{H^2} \geq \sigma^{1-q}\}) \wedge T$$

$$T_{q,\sigma,\gamma} := \tau_{q,\sigma,\gamma} \wedge \tau_{q,\sigma,v} \tag{3.58}$$

as well as

$$u(t,x) =: \mathcal{T}_{\sigma\gamma(t)}\hat{u}(t,x) + \sigma w_\gamma(t,x) + z_\gamma(t,x). \tag{3.59}$$

Then we have the estimate

$$\sup_{t\in[0,T_{q,\sigma,\gamma}]} \|z_\gamma(t,x)\|_{H^2} \leq C\sigma^{2-2q}, \tag{3.60}$$

for a constant C independent of γ, σ.

(c) For sufficiently small σ,

$$\mathbb{P}\Big[T_{q,\sigma,\gamma} = T\Big] \geq 1 - C(1 + C_\gamma)\sigma^{2q}, \tag{3.61}$$

with C_γ as in part (a), for a constant C independent of σ, γ.

3.2.1 Mild solution

We begin with the proof of (a). By theorem 2.6.9 we straightforwardly have existence and uniqueness of a mild solution to (3.54) given by

$$w_\gamma(t,x) = P_{t,0}v_0 + \int_0^t P_{t,t'}[\nabla_c \hat{u}(t',x)]\dot{\gamma}(t')dt' + \int_0^t P_{t,t'}dW(t',x). \tag{3.62}$$

It is possible to simplify the middle term. To do this we evaluate

$$\begin{aligned}
P_{t,t'}[\nabla_c \hat{u}(t',x)]\dot{\gamma}(t') &\overset{(3.52)}{=} \mathcal{T}_{R_{\omega t}} P^{\#}_{t-t'} \mathcal{T}_{R_{-\omega t'}} [\nabla_c \hat{u}(t',x)]\dot{\gamma}(t') \\
&\overset{(3.21)}{=} \mathcal{T}_{R_{\omega t}} P^{\#}_{t-t'} [\nabla_c \mathcal{T}_{R_{-\omega t'}} \hat{u}(t',x)] R_{\omega t'} \dot{\gamma}(t') \\
&\overset{(3.6)}{=} \mathcal{T}_{R_{\omega t}} P^{\#}_{t-t'} [\nabla_c u^*] R_{\omega t'} \dot{\gamma}(t') \\
&\overset{(3.35)}{=} \mathcal{T}_{R_{\omega t}} [\nabla_c u^*] R_{\omega(t-t')} R_{\omega t'} \dot{\gamma}(t') \\
&= \mathcal{T}_{R_{\omega t}} [\nabla_c u^*] R_{\omega t} \dot{\gamma}(t').
\end{aligned}$$

Substituting this into (3.62) gives

$$w_\gamma(t,x) = P_{t,0} v_0 + \int_0^t \mathcal{T}_{R_{\omega t}} [\nabla_c u^*] R_{\omega t} \dot{\gamma}(t') dt' + \int_0^t P_{t,t'} dW(t',x).$$

By linearity, we can now integrate the middle term (notice that only $\dot{\gamma}$ depends on the integration variable) and use $\gamma(0) = 0$ to find

$$w_\gamma(t,x) = P_{t,0} v_0 + \mathcal{T}_{R_{\omega t}} [\nabla_c u^*] R_{\omega t} \gamma(t) + \int_0^t P_{t,t'} dW(t',x),$$

which is equal to (3.56).

3.2.2 Estimate for z_γ

We now prove part (b). Throughout the remainder of this section, the symbol $A \lesssim B$ will mean $A \leq CB$ for some constant C depending only on f and u^*. By rewriting equations (3.45) and (3.46) we find that $z_\gamma(t,x) = v_\gamma(t,x) - \sigma w_\gamma(t,x)$. Therefore we can combine equations (3.53) and (3.54) to find the SPDE (which turns out to be a PDE) satisfied by z:

$$\begin{aligned}
dz_\gamma &= dv_\gamma - \sigma dw_\gamma \\
&= \mathcal{L}_{t,\sigma\gamma(t)} v_\gamma dt + \sigma \mathcal{T}_{\sigma\gamma(t)} [\nabla_c \hat{u}(t,x)] R_{-\sigma\theta(t)} \dot{\gamma}(t) dt + \sigma dW(t,x) + R_\gamma(t,x) dt \\
&\quad - \sigma \mathcal{L}_t w_\gamma(t,x) dt - \sigma [\nabla_c \hat{u}(t,x)] \dot{\gamma}(t) dt - \sigma dW(t,x).
\end{aligned}$$

Using the identity $\mathcal{L}_{t,\gamma}\phi = \mathcal{L}_t \phi + [\mathcal{T}_\gamma f'(\hat{u}(t,x)) - f'(\hat{u}(t,x))]\phi$ (which follows from (3.49) and (3.51)) and rearranging the terms gives

$$\begin{aligned}
dz_\gamma(t,x) =& \mathcal{L}_t z_\gamma(t,x) dt \\
&+ [\mathcal{T}_{\sigma\gamma(t)} f'(\hat{u}(t,x)) - f'(\hat{u}(t,x))] v_\gamma(t,x) dt \\
&+ \sigma \Big[\mathcal{T}_{\sigma\gamma(t)} [\nabla_c \hat{u}(t,x)] R_{-\sigma\theta(t)} \dot{\gamma}(t) - [\nabla_c \hat{u}(t,x)] \dot{\gamma}(t) \Big] dt \\
&+ \mathcal{R}_\gamma dt \\
=:& \mathcal{L}_t z_\gamma(t,x) + T_1 dt + T_2 dt + T_3 dt. \qquad (3.63)
\end{aligned}$$

We also establish the preliminary estimates

$$\sup_{t\in[0,T_{q,\sigma,\gamma}]} \| \int_0^t S(t-t') T(t') dt' \|_{H^2} \leq MT \sup_{t\in[0,T_{q,\sigma,\gamma}]} \|T(t)\|_{H^2}, \qquad (3.64)$$

and

$$\begin{aligned}\|v_\gamma(t,x)\|_{H^2} &\le \|v_\gamma(t,x) - v(t.x)\|_{H^2} + \|v(t.x)\|_{H^2}\\ &\overset{(3.44)(3.45)}{=} \|\mathcal{T}_{\sigma\gamma(t)}\hat{u}(t,x) - \hat{u}(t,x)\|_{H^2} + \|v(t.x)\|_{H^2}\\ &\overset{(3.28)}{\lesssim} \sigma|\gamma(t)| + \|v(t,x)\|_{H^2},\end{aligned}$$

which further implies

$$\sup_{t\in[0,T_{q,\sigma,\gamma}]} \|v_\gamma(t,x)\|_{H^2} \lesssim \sigma^{1-q} \tag{3.65}$$

by definition (3.58). By the same definition, it holds trivially that

$$\sup_{t\in[0,T_{q,\sigma,\gamma}]} |\gamma(t)| \le \sigma^{-q}. \tag{3.66}$$

Before we proceed, we remark that assumption 3.1.14 justifies all our uses of Taylor's theorem and Lipschitz continuity of f and its derivatives.

T1

First we estimate

$$T_1 = [\mathcal{T}_{\sigma\gamma(t)}f'(\hat{u}(t,x)) - f'(\hat{u}(t,x))]v_\gamma(t,x).$$

Using lemma 6.2.4 we immediately get

$$\|T_1\|_{H^2} \le \|\mathcal{T}_{\sigma\gamma(t)}f'(\hat{u}) - f'(\hat{u})\|_{H^2}\|v_\gamma\|_{H^2}, \tag{3.67}$$

where we have again suppressed the dependence on t and x. Thus it suffices to estimate

$$\begin{aligned}T_{1,1} := \mathcal{T}_{\sigma\gamma(t)}f'(\hat{u}) - f'(\hat{u}) &= f'(\mathcal{T}_{\sigma\gamma(t)}\hat{u}) - f'(\hat{u})\\ &= f'(\psi_2) - f'(\psi_1),\end{aligned} \tag{3.68}$$

where we have abbreviated $\psi_1(t,x) = \hat{u}$ and $\psi_2 = \mathcal{T}_{\sigma\gamma(t)}\hat{u}(t,x)$ to ease the notation. The reasoning behind equation (3.92) also result in

$$\|\psi_1(t,x)\|_{H^2} = \|\psi_2(t,x)\|_{H^2} = \|u^*(x)\|_{H^2}. \tag{3.69}$$

By Lipschitz continuity of f', we immediately get

$$|T_{1,1}| \lesssim |\psi_2 - \psi_1|$$

pointwise. Taking the L^2 norm gives

$$\|T_{1,1}\|_{L^2} \lesssim \|\psi_2 - \psi_1\|_{L^2} \le \|\psi_2 - \psi_1\|_{H^2} \tag{3.70}$$

Next we estimate the first derivative. Let $i \in \{1,2\}$. Taking the x_i-derivative of (3.68) gives

$$\partial_{x_i}T_{1,1} = f''(\psi_2)\partial_{x_i}\psi_2 - f''(\psi_1)\partial_{x_i}\psi_1. \tag{3.71}$$

Adding and subtracting $f''(\psi_2)\partial_{x_i}\psi_1$ and taking absolute values gives

$$\begin{aligned}|\partial_{x_i}T_{1,1}| \leq &|f''(\psi_2)\partial_{x_i}\psi_2 - f''(\psi_2)\partial_{x_i}\psi_1| \\ &+ |f''(\psi_2)\partial_{x_i}\psi_1 - f''(\psi_1)\partial_{x_i}\psi_1|\end{aligned}$$

pointwise. By boundedness and Lipschitz continuity of f'', we now have

$$|\partial_{x_i}T_{1,1}| \lesssim |\partial_{x_i}\psi_2 - \partial_{x_i}\psi_1| + |\psi_2 - \psi_1||\partial_{x_i}\psi_1|.$$

Next we take the L^2 norm and use lemma 6.2.2 to find

$$\begin{aligned}\|\partial_{x_i}T_{1,1}\|_{L^2} \lesssim &\|\partial_{x_i}\psi_2 - \partial_{x_i}\psi_1\|_{L^2} + \|\psi_2 - \psi_1\|_{L^\infty}\|\partial_{x_i}\psi_1\|_{L^2} \\ \lesssim &\|\psi_2 - \psi_1\|_{H^2} + \|\psi_2 - \psi_1\|_{H^2}\|\psi_1\|_{H^2} \\ \lesssim &\|\psi_2 - \psi_1\|_{H^2},\end{aligned} \tag{3.72}$$

where we have additionally used (3.69) for the final step.

Now we estimate the second derivative. Let $j \in \{1, 2\}$ and take the x_j-derivative of (3.71) to find

$$\begin{aligned}\partial_{x_ix_j}T_{1,1} = &f'''(\psi_2)\partial_{x_i}\psi_2\partial_{x_j}\psi_2 + f''(\psi_2)\partial_{x_ix_j}\psi_2 \\ &- f'''(\psi_1)\partial_{x_i}\psi_1\partial_{x_j}\psi_1 - f''(\psi_1)\partial_{x_ix_j}\psi_1.\end{aligned}$$

Rearrange the terms, and add and subtract

$$f'''(\psi_2)\partial_{x_i}\psi_2\partial_{x_j}\psi_1 + f'''(\psi_2)\partial_{x_i}\psi_1\partial_{x_j}\psi_1 + f''(\psi_2)\partial_{x_ix_j}\psi_1$$

to get

$$\begin{aligned}\partial_{x_ix_j}T_{1,1} = &f'''(\psi_2)\partial_{x_i}\psi_2\partial_{x_j}\psi_2 - f'''(\psi_2)\partial_{x_i}\psi_2\partial_{x_j}\psi_1 \\ &+ f'''(\psi_2)\partial_{x_i}\psi_2\partial_{x_j}\psi_1 - f'''(\psi_2)\partial_{x_i}\psi_1\partial_{x_j}\psi_1 \\ &+ f'''(\psi_2)\partial_{x_i}\psi_1\partial_{x_j}\psi_1 - f'''(\psi_1)\partial_{x_i}\psi_1\partial_{x_j}\psi_1 \\ &+ f''(\psi_2)\partial_{x_ix_j}\psi_2 - f''(\psi_2)\partial_{x_ix_j}\psi_1 \\ &+ f''(\psi_2)\partial_{x_ix_j}\psi_1 - f''(\psi_1)\partial_{x_ix_j}\psi_1.\end{aligned}$$

Taking absolute values and using boundedness of f''' and Lipschitz continuity of f'', we get

$$\begin{aligned}|\partial_{x_ix_j}T_{1,1}| \lesssim &|\partial_{x_i}\psi_2||\partial_{x_j}\psi_2 - \partial_{x_j}\psi_1| \\ &+ |\partial_{x_i}\psi_2 - \partial_{x_i}\psi_1||\partial_{x_j}\psi_1| \\ &+ |\psi_2 - \psi_1||\partial_{x_i}\psi_1\partial_{x_j}\psi_1| \\ &+ |\partial_{x_ix_j}\psi_2 - \partial_{x_ix_j}\psi_1| \\ &+ |\psi_2 - \psi_1||\partial_{x_ix_j}\psi_1|\end{aligned}$$

pointwise. Take the L^2 norm and use lemma 6.2.3 for the first and second line, and lemma 6.2.2 for the third and fifth line to find

$$\begin{aligned}\|\partial_{x_ix_j}T_{1,1}\|_{L^2} \lesssim &\|\psi_2\|_{H^2}\|\psi_2 - \psi_1\|_{H^2} \\ &+ \|\psi_2 - \psi_1\|_{H^2}\|\psi_1\|_{H^2} \\ &+ \|\psi_2 - \psi_1\|_{H^2}\|\psi_1\|_{H^2}^2 \\ &+ \|\psi_2 - \psi_1\|_{H^2} \\ &+ \|\psi_2 - \psi_1\|_{H^2}\|\psi_1\|_{H^2}.\end{aligned}$$

Using (3.69), this estimate immediately simplifies to

$$\|\partial_{x_i x_j} T_{1,1}\|_{L^2} \leq \|\psi_2 - \psi_1\|_{H^2}. \tag{3.73}$$

Combining (3.70), (3.72) and (3.73) now gives

$$\|T_{1,1}\|_{H^2} \lesssim \|\psi_2 - \psi_1\|_{H^2}.$$

Looking back at the definitions for ψ_1 and ψ_2, we see that $\psi_2 = \mathcal{T}_{\sigma\gamma}\psi_1$. Therefore we have

$$\|T_{1,1}\|_{H^2} \lesssim \|\psi_2 - \psi_1\|_{H^2} = \|\mathcal{T}_{\sigma\gamma}\psi_1 - \psi_1\|_{H^2} \overset{(3.28)}{\lesssim} \sigma|\gamma|.$$

Finally, substituting this back into (3.67) gives

$$\|T_1(t)\|_{H^2} \lesssim \sigma|\gamma(t)|\|v_\gamma\|_{H^2}.$$

Combining this with (3.64), (3.65) and (3.66), we immediately get

$$\sup_{t\in[0,T_{q,\sigma,\gamma}]} \|\int_0^t P_{t,t'} T_1(t',x)dt'\|_{H^2} \lesssim \sigma^{2-2q}. \tag{3.74}$$

T2

Now we estimate the convolution with

$$T_2 := \sigma\Big(\mathcal{T}_{\sigma\gamma(t)}[\nabla_c \hat{u}]R_{-\sigma\theta(t)}\dot{\gamma}(t) - [\nabla_c \hat{u}]\dot{\gamma}(t)\Big).$$

The main difficulty with this term is that T_2 depends linearly on $\dot{\gamma}$, but our assumptions only gives a bound on γ. To remedy this, we introduce the quantity

$$S_2 := \mathcal{T}_{\sigma\gamma(t)}\hat{u}(t,x) - \hat{u}(t,x) + [\nabla_c \hat{u}]\sigma\gamma(t). \tag{3.75}$$

Looking back at (3.14), we see that (3.75) is analogous to a first-order Taylor expansion of $\mathcal{T}_{\sigma\gamma}\hat{u}$ in γ.

We now apply $\partial_t + \omega\partial_\psi$ to S_2. To keep things readable, we examine the terms separately. For the middle term of (3.75), we immediately have

$$(\partial_t + \omega\partial_\psi)\hat{u}(t,x) = (\partial_t + \omega\partial_\psi)u^*(R_{-\omega t}x) = 0. \tag{3.76}$$

For the third term of (3.75), we use the product rule and (3.15) to find

$$\begin{aligned}(\partial_t + \omega\partial_\psi)[\nabla_c \hat{u}]\sigma\gamma(t) =& \sigma\gamma(t)\Big[\nabla_c\Big((\partial_t + \omega\partial_\psi)\hat{u}\Big)\Big] \\ &+ [\nabla_c \hat{u}]\sigma\dot{\gamma}(t) \\ &+ \omega[\nabla\hat{u}]R_{\pi/2}\sigma b(t) \\ \overset{(3.76)}{=}& [\nabla_c \hat{u}]\sigma\dot{\gamma}(t) \\ &- \omega\nabla\hat{u}\cdot R_{\pi/2}\sigma b(t).\end{aligned} \tag{3.77}$$

For the first term of (3.75), we use (3.14) and (3.20) to find

$$
\begin{aligned}
(\partial_t+\omega\partial_\psi)\mathcal{T}_{\sigma\gamma(t)}\hat{u} =&\mathcal{T}_{\sigma\gamma(t)}[(\partial_t+\omega\partial_\psi)\hat{u}] \\
&-\mathcal{T}_{\sigma\gamma(t)}[\nabla_c\hat{u}]R_{-\sigma\theta(t)}\sigma\dot{\gamma}(t) \\
&+\omega\mathcal{T}_{\sigma\gamma(t)}[\nabla\hat{u}]R_{\pi/2}R_{-\sigma\theta(t)}\sigma b(t) \\
\overset{(3.76)}{=}&-\mathcal{T}_{\sigma\gamma(t)}[\nabla_c\hat{u}]R_{-\sigma\theta(t)}\sigma\dot{\gamma}(t) \\
&+\omega\mathcal{T}_{\sigma\gamma(t)}[\nabla\hat{u}]R_{\pi/2}R_{-\sigma\theta(t)}\sigma b(t).
\end{aligned}
\tag{3.78}
$$

Combining (3.76), (3.78) and (3.77) we find

$$
\begin{aligned}
(\partial_t+\omega\partial_\psi)S_2 =&-\mathcal{T}_{\sigma\gamma(t)}[\nabla_c\hat{u}]R_{-\sigma\theta(t)}\sigma\dot{\gamma}(t) \\
&+[\nabla_c\hat{u}]\sigma\dot{\gamma}(t) \\
&+\omega\mathcal{T}_{\sigma\gamma(t)}[\nabla\hat{u}]R_{\pi/2}R_{-\sigma\theta(t)}\sigma b(t) \\
&-\omega[\nabla\hat{u}]^T R_{\pi/2}\sigma b(t) \\
=&-T_2+\omega\mathcal{T}_{\sigma\gamma(t)}[\nabla\hat{u}]R_{\pi/2}R_{-\sigma\theta(t)}\sigma b(t)-\omega[\nabla\hat{u}]R_{\pi/2}\sigma b(t) \\
=:&-T_2+T_{2,1},
\end{aligned}
$$

where the last line serves as the definition for $T_{2,1}$. Thus, if we define $T_{2,2}:=(\partial_t+\omega\partial_\psi)S_2$ we get

$$
T_2=T_{2,1}-T_{2,2}, \tag{3.79}
$$

and it suffices to estimate $T_{2,1}$ and $T_{2,2}$ separately. To estimate $T_{2,1}$, we add and subtract $\omega\mathcal{T}_{\sigma\gamma(t)}[\nabla\hat{u}]R_{\pi/2}\sigma b(t)$ to find

$$
\begin{aligned}
T_{2,1} =&\omega\mathcal{T}_{\sigma\gamma(t)}[\nabla\hat{u}]R_{\pi/2}R_{-\sigma\theta(t)}\sigma b(t)-\omega[\nabla\hat{u}]R_{\pi/2}\sigma b(t) \\
=&\omega\mathcal{T}_{\sigma\gamma(t)}[\nabla\hat{u}]R_{\pi/2}R_{-\sigma\theta(t)}\sigma b(t)-\omega\mathcal{T}_{\sigma\gamma(t)}[\nabla\hat{u}]R_{\pi/2}\sigma b(t) \\
&+\omega\mathcal{T}_{\sigma\gamma(t)}[\nabla\hat{u}]R_{\pi/2}\sigma b(t)-\omega[\nabla\hat{u}]R_{\pi/2}\sigma b(t).
\end{aligned}
$$

Taking the H^2-norm and using the triangle inequality we get

$$
\begin{aligned}
\|T_{2,1}\|_{H^2} \leq& \omega\sigma\|\mathcal{T}_{\sigma\gamma(t)}\nabla\hat{u}\|_{H^2}\|R_{\pi/2}\|_{L(\mathbb{R}^2)}\|R_{-\sigma\theta(t)}-I\|_{L(\mathbb{R}^2)}|b| \\
&+\omega\sigma\|(\mathcal{T}_{\sigma\gamma(t)}-I)\nabla\hat{u}\|_{H^2}\|R_{\pi/2}\|_{L(\mathbb{R}^2)}|b| \\
\lesssim& \sigma^2|\gamma|^2,
\end{aligned}
$$

because $\theta\to R_\theta$ is Lipschitz continuous and we have used equation (3.28). Combining this with (3.64) and (3.66), we immediately get

$$
\sup_{t\in[0,T_{q,\sigma,\gamma}]}\|\int_0^t P_{t,t'}T_{2,1}(t',x)dt'\|_H\lesssim\sigma^{2-2q}. \tag{3.80}
$$

For $T_{2,2}$, we apply the evolution family to get

$$
\begin{aligned}
-\int_0^t P_{t,t'}T_{2,2}(t')dt' =& \int_0^t P_{t,t'}(\partial_{t'}+\omega\partial_\psi)S_2(t')dt' \\
=&P_{t,t}S_2(t)-P_{t,0}S_2(0) \\
&+\int_0^t P_{t,t'}(\mathcal{L}_{t'}+\omega\partial_\psi)(S_2(t'))dt',
\end{aligned}
\tag{3.81}
$$

where we have used the identity

$$\partial_{t'}(P_{t,t'}f(t')) = P_{t,t'}\partial_{t'}f(t') + (\partial_{t'}P_{t,t'})f(t') \overset{(2.6)}{=} P_{t,t'}\partial_{t'}f(t') - (P_{t,t'}\mathcal{L}_{t'})f(t')$$

to integrate by parts. From (3.75) we see that $S_2(0) = 0$, since $\gamma(0) = 0$ by assumption. Furthermore, $P_{t,t}$ is the identity. Therefore, (3.81) simplifies to

$$-\int_0^t P_{t,t'}T_{2,2}(t')dt' = S_2(t) + \int_0^t P_{t,t'}(\mathcal{L}_{t'} + \omega\partial_\psi)(S_2(t'))dt'. \tag{3.82}$$

From equation (3.51) we now see that $\mathcal{L}_t$ is bounded (uniformly in t) when considered as an operator from the space $H^{4,1}$ (recall definition 3.1.3) to H^2. Therefore we may estimate

$$\|\mathcal{L}_{t'} + \omega\partial_\psi S_2(t')\|_{H^2} \lesssim \|S_2(t')\|_{H^{4,1}} \overset{(3.29)(3.75)}{\lesssim} \sum_{j=0}^{3} |\sigma\gamma|^{j+2}. \tag{3.83}$$

Trivially, we also have

$$\|S_2(t)\|_{H^2} \overset{(3.29)(3.75)}{\lesssim} \|S_2(t)\|_{H^{4,1}} \lesssim \sum_{j=0}^{3} |\sigma\gamma|^{j+2}. \tag{3.84}$$

Combining (3.82), (3.83) and (3.84) we find that

$$\|\int_0^t P_{t,t'}T_{2,2}(t')dt'\| \lesssim \sum_{j=0}^{3} |\sigma\gamma|^{j+2}.$$

Thus, by (3.66) we get

$$\sup_{t\in[0,T_{q,\sigma,\gamma}]} \|\int_0^t P_{t,t'}T_{2,2}(t')dt'\| \lesssim \sigma^{2-2q} \tag{3.85}$$

for $\sigma \le 1$.

T3

Recalling equation (3.50), we have

$$\begin{aligned} T_3(t,x) = \mathcal{R}_\gamma(t,x) &= f(u(t,x)) - \mathcal{T}_{\sigma\gamma(t)}f(\hat{u}(t,x)) - [\mathcal{T}_{\sigma\gamma(t)}f'(\hat{u}(t,x))]v_\gamma(t,x) \\ &= f(u(t,x)) - f(\mathcal{T}_{\sigma\gamma(t)}\hat{u}(t,x)) - f'(\mathcal{T}_{\sigma\gamma(t)}\hat{u}(t,x))v_\gamma(t,x). \end{aligned}$$

Now we fix some t, and write $\phi(x) = v_\gamma(t,x)$ and $\psi(x) = \mathcal{T}_{\sigma\gamma(t)}\hat{u}(t,x)$. Recalling (3.45) we see we can write T_3 as

$$T_3 = f(\psi + \phi) - f(\psi) - f'(\psi)\phi, \tag{3.86}$$

where we have suppressed the x and t dependence of T_3, ϕ and ψ. We will continue to do so whenever possible. We can apply Taylor's theorem so that

$$|T_3(x)| \lesssim |\phi(x)|^2.$$

pointwise. Taking the L^2 norm on both sides and using lemma 6.2.2 immediately gives

$$\|T_3\|_{L^2} \lesssim \|\phi^2\|_{L^2} \lesssim \|\phi^2\|_{H^2} \lesssim \|\phi\|_{H^2}^2. \tag{3.87}$$

Next find a bound on the first derivative. In the following derivation, let $i \in \{1,2\}$. Taking the x_i-derivative of (3.86) and using the chain rule gives

$$\begin{aligned}\partial_{x_i} T_3 &= f'(\psi+\phi)(\partial_{x_i}\psi + \partial_{x_i}\phi) - f'(\psi)(\partial_{x_i}\psi + \partial_{x_i}\phi) - f''(\psi)\phi\partial_{x_i}\psi \\ &= (f'(\psi+\phi) - f'(\psi) - f''(\psi)\phi)\partial_{x_i}\psi + (f'(\psi+\phi) - f'(\psi))\partial_{x_i}\phi.\end{aligned} \tag{3.88}$$

We use Taylor's theorem on the first term, and Lipschitz continuity of f' on the second term to get

$$|\partial_{x_i} T_3| \lesssim |\phi|^2 \partial_{x_i}\psi + |\phi||\partial_{x_i}\phi|.$$

pointwise. Now take the L^2 norm, and use lemma 6.2.2 on both terms to find

$$\begin{aligned}\|\partial_{x_i} T_3\|_{L^2} &\lesssim \|\phi\|_{L^\infty}^2 \|\partial_{x_i}\psi\|_{L^2} + \|\phi\|_{L^\infty}\|\partial_{x_i}\phi\|_{L^2} \\ &\lesssim \|\phi\|_{H^2}^2(1 + \|\psi\|_{H^2}).\end{aligned} \tag{3.89}$$

Next we estimate the second derivative. Let $i, j \in \{1,2,3\}$. Then taking the x_j-derivative of (3.88) gives

$$\begin{aligned}\partial_{x_i x_j} T_3 =& f''(\psi+\phi)(\partial_{x_i}\psi + \partial_{x_i}\phi)(\partial_{x_j}\psi + \partial_{x_j}\phi) \\ &+ f'(\psi+\phi)(\partial_{x_i x_j}\phi + \partial_{x_i x_j}\psi) \\ &- f''(\psi)(\partial_{x_i}\psi + \partial_{x_i}\phi)\partial_{x_j}\psi - f'(\psi)(\partial_{x_i x_j}\psi + \partial_{x_i x_j}\phi) \\ &- f'''(\psi)\phi\partial_{x_i}\psi\partial_{x_j}\psi - f''(\psi)\partial_{x_j}\phi\partial_{x_i}\psi - f''(\psi)\phi\partial_{x_i x_j}\psi.\end{aligned}$$

Now rearrange these terms to get

$$\begin{aligned}\partial_{x_i x_j} T_3 =& (f''(\psi+\phi) - f''(\psi) - f'''(\psi)\phi)\partial_{x_i}\psi\partial_{x_j}\psi \\ &+ (f'(\psi+\phi) - f'(\psi))\partial_{x_i x_j}\phi \\ &+ (f''(\psi+\phi) - f''(\psi))(\partial_{x_i}\phi\partial_{x_j}\psi + \partial_{x_j}\phi\partial_{x_i}\psi) \\ &+ f''(\psi+\phi)\partial_{x_i}\phi\partial_{x_j}\phi \\ &+ (f'(\psi+\phi) - f'(\psi) - f''(\psi)\phi)\partial_{x_i x_j}\psi.\end{aligned}$$

We estimate $\partial_{x_i x_j} T_3$ by applying Taylor's theorem to the first and fifth line, and Lipschitz continuity of f' and f'' to the second and third line, and boundedness of f'' to the fourth line. This gives

$$\begin{aligned}|\partial_{x_i x_j}| \lesssim& |\phi|^2|\partial_{x_i}\psi||\partial_{x_j}\psi| \\ &+ |\phi||\partial_{x_i x_j}\phi| \\ &+ |\phi|(|\partial_{x_i}\phi||\partial_{x_j}\psi| + |\partial_{x_j}\phi||\partial_{x_i}\psi|) \\ &+ |\partial_{x_i}\phi||\partial_{x_j}\phi| \\ &+ |\phi|^2|\partial_{x_i x_j}\psi|.\end{aligned}$$

Taking the L^2 norm, and using Hölder's inequality we get

$$\begin{aligned}\|\partial_{x_i x_j} T_3\|_{L^2} \lesssim& \|\phi\|_{L^\infty}^2 \|\partial_{x_i}\psi\partial_{x_j}\psi\|_{L^2} \\ &+ \|\phi\|_{L^\infty}\|\partial_{x_i x_j}\phi\|_{L^2} \\ &+ \|\phi\|_{L^\infty}(\|\partial_{x_i}\phi\partial_{x_j}\psi\|_{L^2} + \|\partial_{x_j}\phi\partial_{x_i}\psi\|_{L^2}) \\ &+ \|\partial_{x_i}\phi\partial_{x_j}\phi\|_{L^2} \\ &+ \|\phi\|_{L^\infty}^2\|\partial_{x_i x_j}\psi\|_{L^2}.\end{aligned}$$

Now applying lemmas 6.2.2 and 6.2.3 gives

$$\|\partial_{x_i x_j} T_3\|_{L^2} \lesssim \|\phi\|_{H^2}^2 (1 + \|\psi\|_{H^2} + \|\psi\|_{H^2}^2). \tag{3.90}$$

Combining (3.87), (3.89) and (3.90), now gives

$$\|T_3\|_{H^2} \lesssim \|\phi\|_{H^2}^2 (1 + \|\psi\|_{H^2} + \|\psi\|_{H^2}^2). \tag{3.91}$$

Recalling our definition for ψ as well as equation (3.6), we find

$$\|\psi\|_{H^2} = \|\mathcal{T}_{\sigma\gamma} \hat{u}(t,x)\|_{H^2} = \|\mathcal{T}_{(\sigma\gamma)R_{\omega t}} u^*\|_{H^2} = \|u^*\|_{H^2}, \tag{3.92}$$

because $\mathcal{T}_{(\sigma\gamma)R_{\omega t}}$ is an isometry. Substituting back our original definition $\phi = v_\gamma(t,x)$ into (3.91) and using (3.92) gives

$$\|T_3(t,x)\|_{H^2} \lesssim \|v_\gamma(t,x)\|_{H^2}^2.$$

Combining this with (3.64) and (3.65), we immediately get

$$\sup_{t \in [0, T_{q,\sigma,\gamma}]} \|\int_0^t P_{t,t'} T_3(t',x) dt'\|_H \lesssim \sigma^{2-2q}. \tag{3.93}$$

3.2.3 Combination of estimates

Now we combine all our estimates to estimate z_γ. Combining equation (3.63) with (3.79) we see that

$$dz_\gamma(t,x) = \mathcal{L}_t z_\gamma(t,x) + T_1 dt + T_{2,1} dt + T_{2,2} dt + T_3 dt.$$

Also note that by our choice of initial conditions for u and w_γ, we have $z_\gamma(0) = 0$. Thus z_γ has a mild solution given by

$$\begin{aligned} z_\gamma(t,x) = &\int_0^t P_{t,t'} T_1(t') dt' + \int_0^t P_{t,t'} T_{2,1}(t') dt' \\ &+ \int_0^t P_{t,t'} T_{2,2}(t') dt' + \int_0^t P_{t,t'} T_3(t') dt'. \end{aligned}$$

Applying the triangle inequality, taking the supremum and using our estimates (3.74), (3.80), (3.85), and (3.93), we find

$$\sup_{t \in [0, T_{q,\sigma,\gamma}]} \|z_\gamma(t)\|_{H^2} \lesssim \sigma^{2-2q}.$$

3.2.4 Convergence of stopping time

Now we show part (c) of theorem 3.2.1. We first split up the probability using (3.58) to get

$$\mathbb{P}\Big[T_{q,\sigma,\gamma} < T\Big] \leq \mathbb{P}\Big[\tau_{q,\sigma,\gamma} < T\Big] + \mathbb{P}\Big[\{\tau_{q,\sigma,\gamma} = T\} \cap \{\tau_{q,\sigma,v} < T\}\Big] \tag{3.94}$$

which is implied by the set-theoretic identity

$$(A \cap B)^c = (A^c \cap B) \cup B^c$$

combined with subadditivity of $\mathbb{P}$ and (3.58). To estimate the first probability we use Markov's inequality:

$$\begin{aligned}\mathbb{P}\Big[\tau_{q,\sigma,\gamma} < T\Big] &\leq \mathbb{P}\Big[\sup_{t\in[0,T]} |\gamma(t)| \geq \sigma^{-q}\Big] \\ &\leq \mathbb{P}\Big[\sup_{t\in[0,T]} |\gamma(t)|^2 \geq \sigma^{-2q}\Big] \\ &\leq \sigma^{2q}\mathbb{E}\Big[\sup_{t\in[0,T]} |\gamma(t)|^2\Big] \\ &\overset{(3.55)}{\leq} \sigma^{2q} C_\gamma, \end{aligned} \tag{3.95}$$

For the second probability we abbreviate $\{\tau_{q,\sigma,\gamma} = T\} \cap \{\tau_{q,\sigma,v} < T\} =: E$. By combining equations (3.44) and (3.46) and rearranging we get

$$v(s) = \mathcal{T}_{\sigma\gamma(t)}\hat{u}(s) - \hat{u}(s) + \sigma w_\gamma(s) + z_\gamma(s).$$

Taking the H^2-norm, using the triangle inequality and equation (3.28) then gives

$$\|v(s)\|_{H^2} \leq \sigma C'|\gamma(s)| + \sigma\|w_\gamma(s,x)\|_{H^2} + \|z_\gamma(s)\|_{H^2} \tag{3.96}$$

for some C' depending only on u^*. Notice that on E we have $\tau_{q,\sigma,v} = T_{q,\sigma,\gamma}$ by (3.58), which implies

$$\|z_\gamma(\tau_{q,\sigma,v})\|_{H^2} \leq \sup_{t\in[0,T_{q,\sigma,\gamma}]} \|z_\gamma(t)\|_{H^2} \overset{(3.60)}{\leq} C\sigma^{2-2q},$$

by part (b) of theorem 3.2.1. Since $\tau_{q,\sigma,v} < T$, we also have $\|v(\tau_{q,\sigma,v})\|_{H^2} = \sigma^{1-q}$. Substituting $\tau_{q,\sigma,v}$ for s into (3.96) gives that

$$\sigma^{1-q} \leq \sigma C'|\gamma(\tau_{q,\sigma,v})| + \sigma\|w_\gamma(\tau_{q,\sigma,v},x)\|_{H^2} + C\sigma^{2-2q}$$

holds on E. We rearrange this to

$$C'|\gamma(\tau_{q,\sigma,v})| + \|w_\gamma(\tau_{q,\sigma,v},x)\|_{H^2} \geq \sigma^{-q}(1 - C\sigma^{1-q}).$$

Then a fortiori, E implies

$$\sup_{t\in[0,T]} \Big(C'|\gamma(t)| + \|w_\gamma(t,x)\|_{H^2}\Big) \geq \sigma^{-q}(1 - C\sigma^{1-q})$$

which means that

$$\begin{aligned}\mathbb{P}[E] &\leq \mathbb{P}\Big[\sup_{t\in[0,T]} C'|\gamma(t)| + \|w_\gamma(t,x)\|_{H^2} \geq \sigma^{-q}(1 - C\sigma^{1-q})\Big] \\ &= \mathbb{P}\Big[\sup_{t\in[0,T]} (C'|\gamma(t)| + \|w_\gamma(t,x)\|_{H^2})^2 \geq \sigma^{-2q}(1 - C\sigma^{1-q})^2\Big] \\ &\leq \frac{\sigma^{2q}}{(1 - C\sigma^{1-q})^2}\mathbb{E}\Big[\sup_{t\in[0,T]} (C'|\gamma(t)| + \|w_\gamma(t,x)\|_{H^2})^2\Big] \\ &\lesssim \frac{\sigma^{2q}}{(1 - C\sigma^{1-q})^2}\Big(\mathbb{E}\Big[\sup_{t\in[0,T]} |\gamma(t)|^2\Big] + \mathbb{E}\Big[\sup_{t\in[0,T]} \|w_\gamma(t,x)\|_{H^2}^2\Big]\Big). \\ &\overset{(3.55)}{=} \frac{\sigma^{2q}}{(1 - C\sigma^{1-q})^2}\Big(C_\gamma + \mathbb{E}\Big[\sup_{t\in[0,T]} \|w_\gamma(t,x)\|_{H^2}^2\Big]\Big), \end{aligned} \tag{3.97}$$

where we have used Markov's inequality and Young's inequality for the intermediate steps. Applying theorem 2.6.6 to equation (3.56), we see that

$$\mathbb{E}\Big[\sup_{t\in[0,T]}\|w_\gamma(t,x)\|_{H^2}^2\Big] < \infty. \tag{3.98}$$

Substituting back our definition for E into (3.97) and using (3.98) we get

$$\mathbb{P}\Big[\{\tau_{q,\sigma,\gamma} = T\} \cap \{\tau_{q,\sigma,v} < T\}\Big] \lesssim (1 + C_\gamma)\sigma^{2q}, \tag{3.99}$$

for σ sufficiently small. Combining (3.94), (3.95) and (3.99) then gives

$$\mathbb{P}\Big[T_{q,\sigma,\gamma} < T\Big] \lesssim (1 + C_\gamma)\sigma^{2q}$$

which is the desired estimate (3.61).

3.3 Immediate relaxation

We take a closer look at the mild solution for w_γ. We repeat equation (3.56) here.

$$w_\gamma(t,x) = P_{t,0}v_0 + \mathcal{T}_{R_{\omega t}}[\nabla_c u^*]R_{\omega t}\gamma(t) + \int_0^t P_{t,t'}dW(t').$$

We will split w_γ up into two parts: one part which lies in the center space, and one part complementary to the center space. The part complementary to the center space will be estimated using theorem 3.1.16, and the part in the center space is minimized by choosing γ appropriately.

Using the identities

$$I \overset{(3.36)}{=} \Pi^{\#,c} + \Pi^{\#},$$
$$I \overset{(3.36),(3.37),(3.38)}{=} \Pi^c_{R_{\omega t}} + \Pi_{R_{\omega t}},$$

we get

$$\begin{aligned} w_\gamma = {} & P_{t,0}\Pi^{\#}v_0 + \int_0^t P_{t,t'}\Pi_{R_{\omega t'}}dW(t',x) \\ & + P_{t,0}\Pi^{\#,c}v_0 + \int_0^t P_{t,t'}\Pi^c_{R_{\omega t'}}dW(t',x) \\ & + \mathcal{T}_{R_{\omega t}}[\nabla_c u^*]R_{\omega t}\gamma(t). \end{aligned} \tag{3.100}$$

Now we rewrite the middle term of (3.100). We compute

$$
\begin{aligned}
&P_{t,0}\Pi^{\#,c}v_0 + \int_0^t P_{t,t'}\Pi^c_{R_{\omega t'}}dW(t',x) \\
&\overset{(3.37),(3.52)}{=} \mathcal{T}_{R_{\omega t}}P^{\#}_t\Pi^{\#,c}v_0 + \int_0^t \mathcal{T}_{R_{\omega t}}P^{\#}_{t-t'}\Pi^{\#,c}\mathcal{T}_{R_{-\omega t'}}dW(t',x) \\
&\overset{(3.42)}{=} \mathcal{T}_{R_{\omega t}}P^{\#}_t[\nabla_c u^*]B^{-1}\mathcal{P}v_0 + \int_0^t \mathcal{T}_{R_{\omega t}}P^{\#}_{t-t'}[\nabla_c u^*]B^{-1}\mathcal{P}\mathcal{T}_{R_{-\omega t'}}dW(t') \\
&\overset{(3.35)}{=} \mathcal{T}_{R_{\omega t}}[\nabla_c u^*]R_{\omega t}B^{-1}\mathcal{P}v_0 + \int_0^t \mathcal{T}_{R_{\omega t}}[\nabla_c u^*]R_{\omega(t-t')}B^{-1}\mathcal{P}\mathcal{T}_{R_{-\omega t'}}dW(t') \\
&= \mathcal{T}_{R_{\omega t}}[\nabla_c u^*]R_{\omega t}\Big[B^{-1}\mathcal{P}v_0 + \int_0^t R_{-\omega t'}B^{-1}\mathcal{P}\mathcal{T}_{R_{-\omega t'}}dW(t')\Big].
\end{aligned} \tag{3.101}
$$

We now define w_∞ and γ_∞ as follows:

$$w_\infty(t) := P_{t,0}\Pi^{\#}v_0 + \int_0^t P_{t,t'}\Pi_{R_{\omega t'}}dW(t'), \tag{3.102}$$

$$\gamma_\infty(t) := -B^{-1}\mathcal{P}v_0 - \int_0^t R_{-\omega t'}B^{-1}\mathcal{P}\mathcal{T}_{R_{-\omega t'}}dW(t'). \tag{3.103}$$

Notice that we have

$$
\begin{aligned}
\Pi^c_{R_{\omega t}}w_\infty(t) &\overset{(3.102)}{=} \Pi^c_{R_{\omega t}}P_{t,0}\Pi^{\#}v_0 + \int_0^t \Pi^c_{R_{\omega t}}P_{t,t'}\Pi_{R_{\omega t}}dW(t') \\
&\overset{(3.37),(3.81)}{=} \mathcal{T}_{R_{\omega t}}\Pi^{\#,c}P^{\#}_t\Pi^{\#}v_0 + \int_0^t \mathcal{T}_{R_{\omega t}}\Pi^{\#,c}P^{\#}_{t-t'}\Pi^{\#}\mathcal{T}_{R_{-\omega t'}}dW(t') \\
&=^{(3.37),(3.81)} \mathcal{T}_{R_{\omega t}}P^{\#}_t\Pi^{\#,c}\Pi^{\#}v_0 + \int_0^t \mathcal{T}_{R_{\omega t}}P^{\#}_{t-t'}\Pi^{\#,c}\Pi^{\#}\mathcal{T}_{R_{-\omega t'}}dW(t') \\
&= 0,
\end{aligned} \tag{3.104}
$$

since $\Pi^{\#,c}$ and $\Pi^{\#}$ are projections which satisfy (3.36). Using the substitutions (3.101), (3.102) and (3.103), we see (3.100) simplifies to

$$w_\gamma(t) = w_\infty(t) + \mathcal{T}_{R_{\omega t}}[\nabla_c u^*]R_{\omega t}(\gamma(t) - \gamma_\infty(t)). \tag{3.105}$$

If we can show that w_γ is small, we obtain stability of the rotating wave using theorem 3.2.1. With (3.104), we will be able to estimate $w_\infty(t)$ using theorem 3.1.16. This leaves only the term proportional to $\gamma(t)-\gamma_\infty(t)$. This term disappears if we choose $\gamma = \gamma_\infty$, but unfortunately this is not possible since γ_∞ does not satisfy the hypotheses of theorem 3.2.1, so the multiscale expansion does not apply. Hence we construct a sequence of approximations γ_m which do satisfy these hypotheses, and transfer the multiscale expansion to w_∞ using a limiting argument.

3.3.1 Phase-lag

We create the approximations to γ_∞ using the *phase-lag* method. For $m > 0$, define $\gamma_m(t)$ as the solution to the following first-order RODE:

$$\dot{\gamma}_m(t) = -m(\gamma_m(t) - \gamma_\infty(t)) \tag{3.106}$$

with initial conditions $\gamma_m(0) = 0$. By theorem 6.1.1 we have existence, uniqueness and progressive measurability of γ_m. Also γ_m is differentiable and starts at zero by definition. From equation (3.106), it is clear that γ_m will always 'chase after' γ_∞, at a rate depending on the parameter m. By theorem 6.1.1, we also have the solution representation:

$$
\begin{aligned}
\gamma_m(t) &= m\int_0^t e^{-m(t-t')}\gamma_\infty(t')dt' \\
&\overset{(3.103)}{=} -m\int_0^t e^{-m(t-t')}\Big[B^{-1}\mathcal{P}v_0 + \int_0^{t'} R_{-\omega t''}B^{-1}\mathcal{P}\mathcal{T}_{R_{-\omega t''}}dW(t'')\Big]dt' \\
&= -(1-e^{-mt})B^{-1}\mathcal{P}v_0 \\
&\quad - m\int_0^t\int_0^{t'} e^{-m(t-t')}R_{-\omega t''}B^{-1}\mathcal{P}\mathcal{T}_{R_{-\omega t''}}dW(t'')dt' \\
&= -(1-e^{-mt})B^{-1}\mathcal{P}v_0 \\
&\quad - m\int_0^t\int_{t''}^{t} e^{-m(t-t')}dt' R_{-\omega t''}B^{-1}\mathcal{P}\mathcal{T}_{R_{-\omega t''}}dW(t'') \\
&= -(1-e^{-mt})B^{-1}\mathcal{P}v_0 \\
&\quad - \int_0^t (1-e^{-m(t-t'')})R_{-\omega t''}B^{-1}\mathcal{P}\mathcal{T}_{R_{-\omega t''}}dW(t'').
\end{aligned} \tag{3.107}
$$

Using the sequence γ_m, we will prove the the multiscale expansion for w_∞.

3.3.2 Multiscale expansion for w_∞

Theorem 3.3.1. *(a) Let γ_m be defined as in equation (3.107). Then $\gamma_m(t)$ is progressively measurable, differentiable, $\gamma_m(0) = 0$ and*

$$\mathbb{E}\Big[\sup_{t\in[0,T]}|\gamma_m(t)|^2\Big] \le C_\infty, \tag{3.108}$$

for some constant C_∞ independent of m.

(b) Let $q \in (0, \frac{1}{2})$. Let w_∞ and γ_∞ be defined as in (3.102) (3.103). Let u and $\tau_{q,\sigma,v}$ be as in theorem 3.2.1, and define the additional stopping times

$$\tau_{q,\sigma,\infty} := \inf\{t \in [0,T] : |\gamma_\infty(t)| \ge \sigma^{-q}\} \wedge T, \tag{3.109}$$

$$T_{q,\sigma,\infty} := \tau_{q,\sigma,\infty} \wedge \tau_{q,\sigma,v}. \tag{3.110}$$

Then we have the following multiscale expansion for $u(t,x)$:

$$u(t,x) =: \mathcal{T}_{\sigma\gamma_\infty}\hat{u}(t,x) + \sigma w_\infty(t,x) + z_\infty(t,x), \tag{3.111}$$

where z_∞ satisfies

$$\sup_{t\in[0,T_{q,\sigma,\infty}]}\|z_\infty(t,x)\|_{H^2} \le C\sigma^{2-2q}, \tag{3.112}$$

where C is the same constant as in (3.60) (which is independent of σ).

(c) The stopping time $T_{q,\sigma.\infty}$ satisfies

$$\mathbb{P}\Big[T_{q,\sigma,\infty} = T\Big] \geq 1 - C\sigma^{2q},$$

for some constant C independent of σ.

Proof. We have already shown all of part (a) except (3.108). By theorem 6.1.1 we have

$$\mathbb{E}[\sup_{t\in[0,T]} |\gamma_m(t)|^2] \overset{(6.2)}{\leq} \mathbb{E}[\sup_{t\in[0,T]} |\gamma_\infty(t)|^2] = C_\infty, \tag{3.113}$$

where we know $C_\infty < \infty$ from equation (3.103) and theorem 2.6.6. To show part (b), we make three observations. Firstly, by theorem 6.1.1 we have

$$\lim_{m\to\infty} \sup_{t\in[\delta,T]} |\gamma_m(t) - \gamma_\infty(t)| \overset{(6.3)}{=} 0, \tag{3.114}$$

and also

$$\sup_{t\in[0,\tau_{q,\sigma,\infty}]} |\gamma_m(t)| \overset{(6.2)}{\leq} \sup_{t\in[0,\tau_{q,\sigma,\infty}]} |\gamma_\infty(t)| \leq \sigma^{-q}.$$

By definition of τ_{q,σ,γ_m} and $\tau_{q,\sigma,\infty}$, this implies

$$\tau_{q,\sigma,\infty} \overset{(3.57),(3.109)}{\leq} \tau_{q,\sigma,\gamma_m},$$

which further leads to

$$T_{q,\sigma,\infty} \overset{(3.58),(3.110)}{\leq} T_{q,\sigma,\gamma_m}. \tag{3.115}$$

Secondly, from equation (3.105) we see that

$$\|w_{\gamma_m}(t) - w_\infty(t)\| \leq C'|\gamma_m(t) - \gamma_\infty(t)|, \tag{3.116}$$

where $C' < \infty$ depends only on u^*. Thus, using equations (3.59) and (3.111) equal and using the triangle inequality we find

$$\begin{aligned} \|z_\infty(t)\| &\leq \|z_{\gamma_m}(t)\| + \sigma\|w_{\gamma_m}(t) - w_\infty(t)\| + \|\mathcal{T}_{\sigma\gamma_m(t)}\hat{u}(t) - \mathcal{T}_{\sigma\gamma_\infty(t)\hat{u}(t)}\| \\ &\overset{(3.28),(3.116)}{\leq} \|z_{\gamma_m}(t)\| + \sigma C''|\gamma_m(t) - \gamma_\infty(t)|, \end{aligned}$$

where $C'' < \infty$ depends only on u^*. Taking the supremum over $t \in [\delta, T_{q,\sigma,\infty}]$ we get

$$\begin{aligned} \sup_{t\in[\delta,T_{q,\sigma,\infty}]} \|z_\infty(t)\| &\leq \sup_{t\in[\delta,T_{q,\sigma,\infty}]} \Big(\|z_{\gamma_m}(t)\| + \sigma C''|\gamma_m(t) - \gamma_\infty(t)|\Big) \\ &\overset{(3.115)}{\leq} \sup_{t\in[\delta,T_{q,\sigma,\gamma_m}]} \Big(\|z_{\gamma_m}(t)\| + \sigma C''|\gamma_m(t) - \gamma_\infty(t)|\Big) \\ &\overset{(3.60)}{\leq} C\sigma^{2-2q} + \sigma C'' \sup_{t\in[\delta,T]} |\gamma_m(t) - \gamma_\infty(t)|, \end{aligned}$$

where C is the constant from (3.60). Note that our application of theorem 3.2.1 is justified by part (a) of theorem 3.3.1. Since neither C nor C'' depends on m, we can let m tend to infinity and use (3.114) to get

$$\sup_{t\in[\delta,T_{q,\sigma,\infty}]} \|z_\infty(t)\| \leq C\sigma^{2-2q}.$$

Choosing δ arbitrarily small we find

$$\sup_{t\in(0,T_{q,\sigma,\infty}]} \|z_\infty(t)\| \leq C\sigma^{2-2q}.$$

The estimate also holds at the left endpoint since z_∞ is continuous, as can be seen from (3.111).

Now we prove part (c). Applying part (c) of theorem 3.2.1 with $\gamma \equiv 0$ already gives

$$\mathbb{P}\Big[\tau_{q,\sigma,v} < T\Big] \leq C\sigma^{2q}, \tag{3.117}$$

for some $C < \infty$ independent of σ. Using Markov's inequality we also get

$$\begin{aligned} \mathbb{P}\Big[\tau_{q,\sigma,\infty} < T\Big] &\overset{(3.109)}{=} \mathbb{P}\Big[\sup_{t\in[0,T]} |\gamma_\infty(t)| \geq \sigma^{-q}\Big] \\ &\leq \sigma^{2q}\mathbb{E}\Big[\sup_{t\in[0,T]} |\gamma_\infty(t)|^2\Big] \\ &\overset{(3.113)}{=} C\sigma^{2q}, \end{aligned} \tag{3.118}$$

for some $C < \infty$ independent of m. After combining (3.110) with (3.117) and (3.118), part (c) follows. □

3.3.3 Stability and approximate minimization

Finally, we can prove orbital stability of the rotating wave by straightforwardly combining theorem 3.1.16 and theorem 3.3.1.

Proposition 3.3.2. *Let w_∞ be as in theorem 3.3.1. Then*

$$\mathbb{E}\Big[\|w_\infty(t,x)\|_{H^2}^2\Big] \leq 2C^2 e^{-2at}\|v_0\|_{H^2}^2 + \frac{C^2}{a}(1-e^{-2at})\|Q^{1/2}\|_{L_2}^2, \tag{3.119}$$

where C and a are the constants from theorem 3.1.16.

Proof. First we compute

$$\begin{aligned} \|w_\infty(t,x)\|_{H^2} &\overset{(3.102)}{\leq} \|P_{t,0}\Pi^\# v_0\|_{H^2} + \|\int_0^t P_{t,t'}\Pi_{R_{\omega t'}} dW(t',x)\|_{H^2} \\ &\overset{(3.38),(3.52)}{=} \|\mathcal{T}_{R_{\omega t}} P_t^\# \Pi^\# v_0\|_{H^2} + \|\int_0^t \mathcal{T}_{R_{\omega t}} P_{t-t'}^\# \Pi^\# \mathcal{T}_{R_{-\omega t'}} dW(t',x)\|_{H^2} \\ &= \|P_t^\# \Pi^\# v_0\|_{H^2} + \|\int_0^t P_{t-t'}^\# \Pi^\# \mathcal{T}_{R_{-\omega t'}} dW(t',x)\|_{H^2}, \end{aligned}$$

where the final step is justified since $\mathcal{T}_\gamma$ is an isometry of H^2. Taking the square, using $(a+b)^2 \le 2a^2 + 2b^2$ and taking the expectation gives

$$\mathbb{E}\Big[\|w_\infty(t,x)\|^2\Big] \le 2\|P_t^\# \Pi^\# v_0\|^2 + 2\mathbb{E}\Big[\|\int_0^t P_{t-t'}^\# \Pi^\# \mathcal{T}_{R_{-\omega t'}} dW(t',x)\|^2\Big]. \tag{3.120}$$

From theorem 3.1.16 we already have

$$\|P_t^\# \Pi^\# v_0\|^2 \overset{(3.43)}{\le} C^2 e^{-2at}\|v_0\|^2. \tag{3.121}$$

Furthermore, applying Itô's isometry to the second term of (3.120) gives

$$\begin{aligned}
\mathbb{E}\Big[\|\int_0^t P_{t-t'}^\# \Pi^\# \mathcal{T}_{R_{-\omega t'}} dW(t',x)\|^2\Big] &\overset{(2.10)}{=} \int_0^t \|P_{t-t'}^\# \Pi^\# \mathcal{T}_{R_{-\omega t'}} Q^{1/2}\|_{L_2}^2 dt' \\
&\le \|\mathcal{T}_{R_{-\omega t'}}\|_{L(H)}\|^2 Q^{1/2}\|_{L_2(H^2)}^2 \int_0^t \|P_{t-t'}^\# \Pi^\#\|_{L(H)}^2 dt' \\
&\overset{(3.43)}{\le} C^2 \|\|Q^{1/2}\|_{L_2(H^2)}^2 \int_0^t e^{-2a(t-t')} dt' \\
&= \frac{C^2}{2a}\|\|Q^{1/2}\|_{L_2(H^2)}^2 (1-e^{-2at}).
\end{aligned} \tag{3.122}$$

Substituting (3.121) and (3.122) into (3.122) gives the desired estimate (3.119). □

The multiscale expansion (3.111) combined with proposition 3.3.2 shows that the difference between $u(t)$ and $\mathcal{T}_{\gamma_\infty}(t)\hat{u}$ is $\mathcal{O}(\sigma)$, which shows orbital stability. Next we show that γ_∞ is a locally approximately optimal value for the phase, to first order in σ. This notion is made exact in the following proposition.

Proposition 3.3.3. *Define*

$$\begin{aligned}
E : \mathbb{R}^3 &\to \mathbb{R} \\
\gamma &\to \|\Pi_{R_{\omega t}}^c [u(t) - \mathcal{T}_\gamma \hat{u}(t)]\|_{H^2}^2.
\end{aligned}$$

Then $\sigma\gamma_\infty(t)$ is an approximate local minimizer of E, in the sense that

$$\nabla_\gamma E(\gamma)\Big|_{\gamma=\sigma\gamma_\infty(t)} = o(\sigma),$$

and the Hessian

$$\nabla_\gamma[\nabla_\gamma E(\gamma)]^T\Big|_{\gamma=\sigma\gamma_\infty}$$

is strictly positive to first order in σ.

Proof. We compute

$$\begin{aligned}
\nabla_\gamma \|\Pi_{R_{\omega t}}^c [u - \mathcal{T}_\gamma \hat{u}(t)]\|_H^2 &= \Big(\Pi_{R_{\omega t}}^c [u - \mathcal{T}_\gamma \hat{u}(t)], \nabla_\gamma \Pi_{R_{\omega t}}^c [u - \mathcal{T}_\gamma \hat{u}(t)]\Big) \\
&\overset{(3.24)}{=} -\Big(\Pi_{R_{\omega t}}^c [u - \mathcal{T}_\gamma \hat{u}(t)], \Pi_{R_{\omega t}}^c \mathcal{T}_\gamma [\nabla_c \hat{u}] R_{-\theta}]\Big).
\end{aligned} \tag{3.123}$$

Evaluating this expression at $\gamma = \sigma\gamma_\infty(t)$ and using theorem 3.3.1.

$$\begin{aligned}
\nabla_\gamma E(\gamma)\Big|_{\gamma=\sigma\gamma_\infty(t)} &\overset{(3.59)}{=} -\Big(\Pi^c_{R_{\omega t}}[\sigma w_\infty(t) + z_\infty(t)], \Pi^c_{R_{\omega t}} \mathcal{T}_{\sigma\gamma_\infty(t)}[\nabla_c \hat{u}]R_{-\sigma\theta_\infty(t)}]\Big) \\
&\overset{(3.104)}{=} -\Big(\Pi^c_{R_{\omega t}}[z_\infty(t)], \Pi^c_{R_{\omega t}} \mathcal{T}_{\sigma\gamma_\infty(t)}[\nabla_c \hat{u}]R_{-\sigma\theta_\infty(t)}]\Big) \\
&\overset{(3.112)}{=} O(\sigma^{2-2q}).
\end{aligned}$$

For the Hessian matrix, we transpose and differentiate (3.123) to find

$$\begin{aligned}
\nabla_\gamma[\nabla_\gamma \|\Pi^c_{R_{\omega t}}[u - \mathcal{T}_\gamma \hat{u}(t)]\|^2_H]^T \overset{(3.24)}{=} &\Big(\Pi^c_{R_{\omega t}} \mathcal{T}_\gamma[\nabla_c \hat{u}]R_{-\theta}, (\Pi^c_{R_{\omega t}} \mathcal{T}_\gamma[\nabla_c \hat{u}]R_{-\theta})^T\Big) \\
&+ \Big(\Pi^c_{R_{\omega t}}[u - \mathcal{T}_\gamma \hat{u}(t)], \Pi^c_{R_{\omega t}} \mathcal{T}_\gamma R_\theta \nabla_c[\nabla_c \hat{u}]^T R_{-\theta}]\Big).
\end{aligned} \tag{3.124}$$

Evaluating at $\gamma = \sigma\gamma_\infty(t)$ and using theorem 3.2.1 again we see that

$$\begin{aligned}
&\Big(\Pi^c_{R_{\omega t}}[u - \mathcal{T}_{\sigma\gamma_\infty(t)}\hat{u}(t)], \Pi^c_{R_{\omega t}} \mathcal{T}_{\sigma\gamma_\infty(t)} R_{\sigma\theta_\infty(t)} \nabla_c[\nabla_c \hat{u}]^T R_{-\sigma\theta_\infty(t)}]\Big) \\
&\quad \overset{(3.111)}{=} \Big(\Pi^c_{R_{\omega t}}[\sigma w_\infty(t) + z_\infty(t)], \Pi^c_{R_{\omega t}} \mathcal{T}_{\sigma\gamma_\infty(t)} R_{\sigma\theta_\infty(t)} \nabla_c[\nabla_c \hat{u}]^T R_{-\sigma\theta_\infty(t)}]\Big) \\
&\quad \overset{(3.104)}{=} \Big(\Pi^c_{R_{\omega t}} z_\infty(t), \Pi^c_{R_{\omega t}} \mathcal{T}_{\sigma\gamma_\infty(t)} R_{\sigma\theta_\infty(t)} \nabla_c[\nabla_c \hat{u}]^T R_{-\sigma\theta_\infty(t)}]\Big) \\
&\quad \overset{(3.112)}{=} \mathcal{O}(\sigma^{2-2q}).
\end{aligned}$$

Substituting this into (3.124) then gives

$$\nabla_\gamma[\nabla_\gamma E(\gamma)]^T\Big|_{\gamma=\sigma\gamma_\infty(t)} = \Big(\Pi^c_{R_{\omega t}} \mathcal{T}_\gamma[\nabla_c \hat{u}]R_{-\theta}, (\Pi^c_{R_{\omega t}} \mathcal{T}_\gamma[\nabla_c \hat{u}]R_{-\theta})^T\Big) + \mathcal{O}(\sigma^{2-2q}),$$

which shows the claim, since any matrix of the form vv^t is positive definite. □

Chapter 4

Solitons

The nonlinear schrodinger (NLS) equation (4.1) is a well-known PDE modelling propagation of waves in a nonlinear dispersive medium which has no dissipation.

$$du(t,x) = i\Delta u(t,x)dt + i\kappa|u(t,x)|^2 u(t,x)dt. \tag{4.1}$$

Equation (4.1) has many interpretations. For example, it can describe a classical field theory [45], a wave envelope in a fiber optics system [8] [17], or a wave profile of a (deep) water wave [53]. We will interpret (4.1) as a fiber optics equation. In this case, $z(t,x)$ describes the complex amplitude of a wave envelope of a pulse in a nonlinear dispersive medium, such as an optical fiber. In this context, x represents a temporal variable while t represents the position along the fiber. Even though this may seem unintuitive, we use this convention since it is convenient from a mathematical point of view to interpret (4.1) as an evolution equation. As such, when we refer to 'time' we mean the spatial variable t and not the physical time. For a derivation of the NLS equation for laser pulses in an isotropic medium, see [17].

An interesting feature of the NLS equation is that it supports solitary standing waves. This means there exists solution of (4.1), called *solitons*, which are independent of t. Since t models a spatial variable, this means the soliton has the same amplitude everywhere. Therefore it is possible for there to be an extremely bright pulse throughout the whole fiber, a fact which has interesting optical applications.

However, realistic physical models almost always have dissipative properties. To model this, a term proportional to u (with positive proportionality constant) may be added to (4.1). Unsurprisingly this term makes it impossible for solitons to exist. To compensate for the dissipation, a mechanism to induce a phase-sensitive amplification has been proposed [35]. This result in the parametrically forced nonlinear schrodinger (PFNLS) equation, which takes the form

$$du = [i\Delta u - i\delta u - \epsilon(\gamma u - \mu\overline{u})]dt + i\kappa|u|^2 u dt. \tag{4.2}$$

Here, the constant $\gamma > 0$ models the strength of the dissipation, and $v > 0$ and $\mu > 0$ model the amplification. Due to the amplification, the PFNLS equation supports soliton solutions depending on the parameters δ,γ and μ. In particular, the strength of the amplification μ must be greater than the dissipation γ [28].

4.1 Preliminaries

Throughout chapter 4, we will write

$$H^k := H^k(\mathbb{R}, \mathbb{C}),$$
$$L^2 := L^2(\mathbb{R}, \mathbb{C}),$$
$$L^\infty := L^\infty(\mathbb{R}, \mathbb{C}).$$

We also fix some stochastic basis $(\Omega, \mathcal{F}, \mathbb{P})$ along with a normal filtration $\mathcal{F}_t$ to be used throughout these sections.

Let W_H be a cylindrical Wiener process in L^2 which is adapted to $\mathcal{F}_t$ and let $\Phi \in L(L^2, H^1)$. We define the linear operator

$$A : \phi \to i\Delta\phi - i\delta\phi - \epsilon(\gamma u - \mu\overline{\phi}),$$

and consider the following stochastic version of the PFNLS equation.

$$du = Audt + 4i|u|^2udt - \frac{1}{2}uF_\Phi dt - iu\Phi dW_H, \tag{4.3}$$

where u takes values in H^1. The term $\frac{1}{2}uF_\Phi$ arises from the need to convert between the Stratonovich formulation of the SPDE (which is the correct one for the physical problem) and the Itô formulation which we use. Here F_Φ is defined as

$$F_\Phi(x) := \sum_{i=1}^{\infty}(\Phi e_i(x))^2, \tag{4.4}$$

where e_i is an orthonormal basis of L^2. This definition is independent of the choice of basis. We additionally require the following assumption on Φ.

Assumption 4.1.1. *Φ is Hilbert-Schmidt as an operator from L^2 to H^1. Furthermore, Φ is γ-radonifying from L^2 to $W^{1,2+\delta}$ for some $\delta > 0$.*

The property of being γ-radonifying is analogous to being Hilbert-Schmidt when the codomain is a Banach space.

With this assumption, it has been shown that (4.3) has a unique mild solution in the sense of definition 2.6.8 [19].

4.1.1 Solitons

By a direct computation it can be verified that the deterministic equation (4.2) supports soliton solutions given by

$$\hat{u}_{i,a}(x) := e^{i\theta_i}\sqrt{\frac{\omega + \epsilon\mu\sin(2\theta_i)}{2}}\operatorname{sech}(\sqrt{\omega + \epsilon\mu\sin(2\theta_i)}(x - a)),$$

where $\theta_i \in [0, 2\pi)$ is a solution to $\cos(2\theta_i) = \frac{\gamma}{\mu}$ [28]. The solitons are parametrized by $i \in \{1, 2\}$ and $a \in \mathbb{R}$. The parameter i determines which of the two (possibly non-distinct) solutions for θ_i is used, while a simply translates the soliton, and is present because (4.2) is translation invariant.

We now fix some soliton u^* to be used throughout the remainder of this thesis. Since u^* is time-independent and solves (4.2), it satisfies

$$Au^* = -4i|u^*|^2u^*. \tag{4.5}$$

We will show stability of this soliton in the stochastic equation (4.3). However, it is not reasonable to expect stability if the noise is large. Therefore, we introduce a parameter σ which controls the strength of the noise, and replace Φ by $\sigma\Phi$ in (4.3). Notice from (4.4) that F_Φ then needs to be replaced by $\sigma^2 F_\Phi$. We also define the Q-Wiener process $W(t) = \Phi W_H(t)$. Making these modifications to (4.3) we get the SPDE

$$du = Audt + 4i|u|^2udt - \frac{1}{2}\sigma^2 uF_\Phi dt + u\sigma dW. \tag{4.6}$$

4.1.2 Derivation of the SPDE

To show stability of the soliton u^* in the stochastic PFNLS equation (4.6), we take the same approach as in the case of the rotating wave. For an overview of the strategy, see section 1.1.

Let u be a solution to (4.6), and let $a(t)$ be a progressively measurable differentiable process starting at 0. Now introduce the following three ways to write u:

$$u(t,x) =: u^*(x) + v(t,x) \tag{4.7}$$
$$u(t,x) =: u^*(x+\sigma a(t)) + v_a(t,x) \tag{4.8}$$
$$u(t,x) =: u^*(x+\sigma a(t)) + \sigma w_a(t,x) + z_a(t,x) \tag{4.9}$$

with $w_a(t,x)$ being specified later, and the third line serving as a definition of $z_a(t,x)$.
We will determine the SPDE satisfied by v_a, linearize it and scale out σ to obtain an SPDE which will define w_a. Firstly, from equation (4.8) we have $v_a(t,x) = u(t,x) - u^*(x+\sigma a(t))$. Therefore, taking the differential and using (4.6) we get

$$\begin{aligned} dv_a(t,x) =& du - du^*(x+a(t)) \\ \overset{(4.6)}{=}& Audt - \frac{1}{2}\sigma^2 F_\Phi udt + 4i|u|^2udt + u\sigma dW - \sigma\dot{a}(t)u^*_x(x+a(t))dt \\ \overset{(4.8)}{=}& Av_adt - \frac{1}{2}\sigma^2 F_\Phi udt + 4i|u|^2udt + u\sigma dW - \sigma\dot{a}(t)u^*_x(x+a(t))dt \\ &+ Au^*(x+\sigma a(t))dt \\ \overset{(4.5)}{=}& Av_adt - \frac{1}{2}\sigma^2 F_\Phi udt + u\sigma dW - \sigma\dot{a}(t)u^*_x(x+a(t))dt \\ &+ 4i|u|^2udt - 4i|u^*(x+\sigma a(t))|^2u^*(x+\sigma a(t)). \end{aligned} \tag{4.10}$$

We want to linearize this equation around u^*, thus we compute (temporarily abbreviating $u^*(x+\sigma(t))$ by u^*)

$$\begin{aligned} |u|^2u - |u^*|^2u^* =& (u^*+v)\overline{(u^*+v)}(u^*+v) - |u^*|^2u^* \\ =& (2|u^*|^2v + (u^*)^2\overline{v}) \\ &+ 2u^*|v|^2 + \overline{u}^*v^2) \\ &+ |v|^2v, \end{aligned} \tag{4.11}$$

where we have seperated the terms which are, first, second or third order in v. Multiplying (4.11) by $4i$ and substituting into (4.10) we get

$$\begin{aligned} dv_a(t,x) =& Av_a dt - \frac{1}{2}\sigma^2 F_\Phi u dt + u\sigma dW - \sigma\dot{a}(t)u_x^*(x+a(t))dt \\ &+ 8i|u^*(x+\sigma a(t))|^2 v_a + 4iu^*(x+\sigma a(t))^2\overline{v}_a \\ &+ 8iu^*(x+\sigma a(t))|v_a|^2 + 4i\overline{u^*(x+\sigma a(t))}v_a^2 + 4i|v_a|^2 v_a. \end{aligned} \tag{4.12}$$

To ease the notation, we now define the family of linear operators

$$\mathcal{L}_a\phi(t,x) := A\phi(t,x) + 8i|u^*(x+a)|^2\phi(t,x) + 4i|u^*(x+a)\overline{\phi}(t,x) \tag{4.13}$$

as well as the nonlinear term

$$\mathcal{R}_a := 8iu^*(x+\sigma a(t))|v_a|^2 + 4i\overline{u^*(x+\sigma a(t))}v_a^2 + 4i|v_a|^2 v_a. \tag{4.14}$$

We also define

$$\mathcal{L}\phi(t,x) := \mathcal{L}_0\phi(t,x).$$

With this notation, equation (4.12) becomes

$$\begin{aligned} dv_a(t,x) =& \mathcal{L}_{\sigma a(t)}v_a dt - \frac{1}{2}\sigma^2 F_\Phi u dt + u\sigma dW - \sigma\dot{a}(t)u_x^*(x+\sigma a(t))dt \\ &+ \mathcal{R}_a dt. \end{aligned}$$

We next substitute $u = u^* + v$ in the $u\sigma dW$ term and change the order of the terms to get

$$\begin{aligned} dv_a(t,x) =& \mathcal{L}_{\sigma a(t)}v_a dt - \sigma\dot{a}(t)u_x^*(x+\sigma a(t))dt + u^*\sigma dW \\ &- \frac{1}{2}\sigma^2 F_\Phi u dt + \mathcal{R}_a dt + v\sigma dW. \end{aligned} \tag{4.15}$$

Finally, we set a to zero in (4.15), approximate to first order in σ and scale out σ to get an SPDE for the first-order linearization w_a.

$$dw_a(t,x) = \mathcal{L}w_a dt - \dot{a}(t)u_x^* dt + u^* dW. \tag{4.16}$$

4.1.3 Initial conditions

It remains to discuss initial conditions for w_a and u, which we have not specified until now. As we noted when introducing (4.8) and (4.9), it is the aim that $u^*(x+\sigma a(t))+\sigma w_a$ is a good approximation to $u(t,x)$. It is also desirable for w_a to be independent of σ, which means that the initial condition must be independent of σ as well. Therefore we use the initial conditions

$$\begin{aligned} u(0,x) &= u^*(x) + \sigma v_0 \\ w_a(0,x) &= v_0 \end{aligned}$$

for some fixed $v_0 \in H^1(\mathbb{R})$. Substituting $t = 0$ in (4.7) and (4.9) and using $a(0) = 0$ shows that our approximation $u^*(x+\sigma a(t)) + \sigma w_a$ exactly matches u at time zero. In the next chapters we will show that this approximation is still accurate after some finite time. To do this we first need to formulate some results about deterministic stability of the solitons.

4.1.4 Deterministic stability of solitons

It has been shown by Kapitula and Sandstede that soliton solutions to (4.2) are orbitally stable [28]. However, for our purposes we only require stability of the linearized solution around u^*. In their proof, Kapitula and Sandstede first establish the following facts, which we also need.

Theorem 4.1.2. *[28] $\mathcal{L}$ generates a C_0 semigroup $S(t)$ on $H^1(\mathbb{R})$.*

By differentiating (4.5) we can see that u_x^* (the subscript x denotes a spatial derivative) is an eigenvector of $\mathcal{L}$ with eigenvalue 0. This also means that

$$S(t)u_x^* = u_x^* \tag{4.17}$$

Kapitual and Sandstede also show that this eigenvalue is isolated. Thus it makes sense to define the following spectral projection.

Definition 4.1.3. *Π^0 is the spectral projection of $\mathcal{L}$ onto the eigenvalue 0. We also define $\Pi = I - \Pi^0$.*

With this definition, Π^0 and Π are both bounded linear operators on $H^1(\mathbb{R})$. Kapitual and Sandstede also show that u^* is the only eigenvector with this eigenvalue (up to scalar multiplication). Therefore, we have

$$\Pi^0\phi = a(\phi)u^* \tag{4.18}$$

for some scalar function a. Taking inner products in L^2 with u^* from the right we get

$$a(\phi)(u^*, u^*)_{L^2(\mathbb{R})} = (\Pi^0\phi, u^*)_{L^2(\mathbb{R})}.$$

Rewriting this and substituting back into (4.18) gives

$$\Pi^0\phi = \frac{(\Pi^0\phi, u_x^*)_{L^2(\mathbb{R})}}{(u_x^*, u_x^*)_{L^2(\mathbb{R})}} u_x^*.$$

For convenience, we define the (bounded) linear operator

$$\begin{aligned} \mathcal{P} : H^1(\mathbb{R}) &\to \mathbb{R} \\ \phi &\to \frac{(\Pi^0\phi, u_x^*)_{L^2(\mathbb{R})}}{(u_x^*, u_x^*)_{L^2(\mathbb{R})}}, \end{aligned}$$

to see that

$$\Pi^0\phi = u_x^*\mathcal{P}\phi. \tag{4.19}$$

We can now formulate the linear stability result which we need.

Theorem 4.1.4. *[28] There exist constants C, a (possibly depending on u^*), such that*

$$\|S(t)\Pi\|_{H^1(\mathbb{R})} \leq Ce^{-at}. \tag{4.20}$$

Finally we need some estimates involving translation of u^*.

Proposition 4.1.5. *There exists a constant $C < \infty$, depending only on u^* such that*

$$\|u^*(x+a) - u^*(x)\|_{H^1(\mathbb{R})} \leq Ca. \tag{4.21}$$

Proposition 4.1.6. *There exists a constant $C < \infty$, depending only on u^* such that*

$$\|u^*(x+a) - u^*(x) - au_x^*(x)\|_{H^3(\mathbb{R})} \leq Ca^2.$$

Note that the norm in proposition 4.1.6 is not the H^1 norm but the H^3 norm.

4.2 Multiscale expansion

As a first step towards stochastic stability, we formulate the following multiscale expansion of u around $u^*(x + \sigma a(t))$, where $a(t)$ is still arbitrary.

Theorem 4.2.1. *(a) Let $a(t)$ be a progressively measurable stochastic process which is almost surely differentiable and satisfies $a(0) = 0$ and*

$$\mathbb{E}\Big[\sup_{t\in[0,T]} |a(t)|^2\Big] = C_a < \infty. \tag{4.22}$$

Let $u(t,x)$ be the solution to (4.6) with initial condition $u(0) = u^* + \sigma v_0$ for some $v_0 \in H^1(\mathbb{R})$. Then equation (4.16) with initial condition $w_a(0) = v_0$ has a unique mild solution, given by

$$w_a(t,x) = S(t)v_0 - a(t)u^*_x + \int_0^t S(t-t')u^* dW(t'). \tag{4.23}$$

(b) Let $q \in (0, \frac{1}{2})$. Define the stopping times

$$\tau_{q,\sigma,a} := \inf(\{t \in [0,T] : |a(t)| \geq \sigma^{-q}\}) \wedge T$$

$$\tau_{q,\sigma,v} := \inf(\{t \in [0,T] : \|v(t,x)\|_{H^1} \geq \sigma^{1-q}\}) \wedge T$$

$$\tau_{q,\sigma,c}; = \inf(\{t \in [0,T] : \|\int_0^t S(t-t')v(t')dW(t')\|_{H^1} \geq \sigma^{1-2q}\}) \wedge T \tag{4.24}$$

$$T_{q,\sigma,a} := \tau_{q,\sigma,a} \wedge \tau_{q,\sigma,v} \wedge \tau_{q,\sigma,c} \tag{4.25}$$

as well as

$$u(t,x) =: u^*(x + \sigma a(t)) + \sigma w_a(t,x) + z_a(t,x). \tag{4.26}$$

Then we have the estimate

$$\sup_{t\in[0,T_{q,\sigma,a}]} \|z_a(t,x)\|_{H^1} \leq C\sigma^{2-2q} \tag{4.27}$$

for some $C < \infty$, independent of a, σ.

(c) For sufficiently small σ,

$$\mathbb{P}\Big[T_{q,\sigma,a} = T\Big] \geq 1 - C(1 + C_a)\sigma^{2q}$$

with C_a as in part (a) for some $C < \infty$, independent of a, σ.

4.2.1 Mild solution

We begin with the proof of part (a). Since (4.16) is a linear SPDE with additive noise, theorem 2.6.9 immediately gives existence and uniqueness of the following mild solution:

$$w_a(t,x) = S(t)v_0 - \int_0^t S(t-t')\dot{a}(t')u^*_x(x)dt' + \int_0^t S(t-t')u^*(t')dW(t').$$

Using (4.17), we can simplify the middle term to get

$$\begin{aligned} w_a(t,x) &= S(t)v_0 - \int_0^t \dot{a}(t')u_x^*(x)dt' + \int_0^t S(t-t')u^*(t')dW(t') \\ &= S(t)v_0 - a(t)u_x^*(x) + \int_0^t S(t-t')u^*(t')dW(t'), \end{aligned}$$

since $a(0) = 0$ by assumption.

4.2.2 Estimate for z_a

Now we prove part (b). By rewriting equations (4.8) and (4.9) we find that $z_a(t,x) = v_a(t,x) - \sigma w_a(t,x)$. Taking the differential and using equations (4.15) and (4.16) we find that z_a satisfies

$$\begin{aligned} dz_a =& dv_a - \sigma dw_a \\ =& \mathcal{L}_{\sigma a(t)} v_a dt - \sigma \dot{a}(t) u_x^*(x+\sigma a(t))dt + u^*\sigma dW - \frac{1}{2}\sigma^2 F_\Phi u dt + \mathcal{R}_a dt + v\sigma dW \\ & - \mathcal{L}\sigma w_a dt + \sigma \dot{a}(t) u_x^* dt - u^*\sigma dW \end{aligned}$$

Next we add and subtract $\mathcal{L}v_a$, use $v_a = \sigma w_a + z_a$ by (4.8) and (4.9) and rearrange the terms to find

$$\begin{aligned} dz_a =& \mathcal{L} z_a dt \\ & + [\mathcal{L}_{\sigma a(t)} v_a - \mathcal{L} v_a]dt \\ & - \sigma \dot{a}(t)(u_x^*(x+\sigma a(t)) - u_x^*(x))dt \\ & - \frac{1}{2}\sigma^2 F_\Phi u dt \\ & + \mathcal{R}_a dt \\ & + v\sigma dW \\ =:& \mathcal{L} z_a dt + T_1 dt + T_2 dt + T_3 dt + T_4 dt + T_5 dW. \end{aligned} \tag{4.28}$$

We will estimate these five terms separately. Before we do so, we record the elementary estimate

$$\sup_{t\in[0,T_{q,\sigma,a}]} \| \int_0^t S(t-t')T(t')dt' \|_{H^1} \le MT \sup_{t\in[0,T_{q,\sigma,a}]} \|T(t)\|_{H^1}, \tag{4.29}$$

as well as

$$\begin{aligned} \|v_a(t,x)\|_{H^1} &\le \|v_a(t,x) - v(t,x)\|_{H^1} + \|v(t,x)\|_{H^1} \\ &\overset{(4.7)(4.8)}{=} \|u^*(x+\sigma a(t)) - u^*(x)\|_{H^1} + \|v(t,x)\|_{H^1} \\ &\overset{(4.21)}{\lesssim} \sigma|a(t)| + \|v(t,x)\|_{H^1}, \end{aligned}$$

which implies

$$\sup_{t\in[0,T_{q,\sigma,a}]} \|v_a(t,x)\|_{H^1} \lesssim \sigma^{1-q} \tag{4.30}$$

by definition (4.25). From this same definition, the following estimates are also trivial:

$$\sup_{t\in[0,T_{q,\sigma,a}]} \|v(t,x)\|_{H^1} \lesssim \sigma^{1-q}, \tag{4.31}$$

$$\sup_{t\in[0,T_{q,\sigma,a}]} |a(t)| \lesssim \sigma^{-q} \tag{4.32}$$

T1

For the first term we estimate

$$\begin{aligned} T_1 :=& \mathcal{L}_{\sigma a(t)} v_a - \mathcal{L} v_a \\ \overset{(4.13)}{=}& 8i|u^*(x+\sigma a(t))|^2 v_a(t,x) - 8i|u^*(x)|^2 v_a(t,x) \\ &+ 4iu^*(x+\sigma a(t))^2 \overline{v_a}(t,x) - 4iu^*(x)^2 \overline{v_a}(t,x). \end{aligned}$$

Using the triangle inequality and lemma 6.2.6 we first derive

$$\begin{aligned} \|(ab - a'b')\|_{H^1} &\leq \|ab - ab'\|_{H^1} + \|ab' - a'b'\|_{H^1} \\ &\lesssim \|a\|_{H^1}\|b - b'\|_{H^1} + \|b'\|_{H^1}\|a - a'\|_{H^1} \end{aligned}$$

for $a, b, a', b' \in H^1(\mathbb{R})$. Substituting $a = u^*(x + \sigma a(t))$, $a' = u^*(x)$, $b = \overline{a}$ and $b' = \overline{a'}$ we find

$$\||u^*(x+\sigma a(t))|^2 - |u^*(x)|^2\|_{H^1} \lesssim \sigma|a(t)|$$

using proposition 4.1.5. Using lemma 6.2.6 again gives

$$\||u^*(x+\sigma a(t))|^2 v_a(t,x) - |u^*(x)|^2 v_a(t,x)\|_{H^1} \lesssim \sigma|a(t)|\|v_a\|_{H^1}. \tag{4.33}$$

If we instead substitute $b = u^*(x + \sigma a(t))$ and $b' = u^*(x)$, we find that

$$\begin{aligned} \|u^*(x+\sigma a(t))^2 \overline{v}_a(t,x) - u^*(x)^2 \overline{v}_a(t,x)\|_{H^1} &\lesssim \sigma|a(t)|\|\overline{v}_a\|_{H^1} \\ &= \sigma|a(t)|\|v_a\|_{H^1} \end{aligned} \tag{4.34}$$

Combining (4.33) and (4.34) now gives

$$\|T_1(t)\|_{H^1} \lesssim \sigma|a(t)|\|v_a\|_{H^1},$$

which we further combine with (4.29), (4.30) and (4.32) to find

$$\sup_{t\in[0,T_{q,\sigma,a}]} \|\int_0^t S(t-t')T_1(t')dt'\|_{H^1} \lesssim \sigma^{2-2q}. \tag{4.35}$$

T2

Now we estimate the term involving

$$T_2 := -\sigma\dot{a}(t)(u_x^*(x+\sigma a(t)) - u_x^*(x)).$$

The main difficulty with this term is that we have no prior control over $\dot{a}$. Therefore, we introduce

$$J_2(t,x) := u^*(x+\sigma a(t)) - u^*(x) - \sigma a(t)u_x^*(x),$$

which is a first-order Taylor expansion of $u^*(x + \sigma a(t))$. By proposition 4.1.6, we see that

$$\|J_2(t,x)\|_{H^3} \lesssim \sigma^2 a^2(t). \tag{4.36}$$

Differentiating S_2 with respect to t, we find

$$\partial_t J_2(t,x) = \sigma\dot{a}(t)u_x^*(x+\sigma a(t)) - \sigma\dot{a}(t)u_x^*(x) = -T_2(t,x).$$

Therefore, we can use integration by parts to compute

$$\begin{aligned}\int_0^t S(t-t')T_2(t')dt' &= -\int_0^t S(t-t')\partial_{t'}J_2(t')dt' \\ &= -S(0)J_2(t) + S(t)J_2(0) + \int_0^t (\partial_{t'}S(t-t'))J_2(t')dt' \\ &= -J_2(t) - \int_0^t S(t-t')\mathcal{L}J_2(t')dt'.\end{aligned}$$

From equation (4.13) it is clear that $\mathcal{L} = \mathcal{L}_0$ is a bounded operator from $H^3(\mathbb{R})$ to $H^1(\mathbb{R})$. Thus we can estimate

$$\begin{aligned}\|\int_0^t S(t-t')T_2(t')dt'\|_{H^1} &\le \|S_2(t)\|_{H^1} \\ &+ \int_0^t \|S(t-t')\|_{L(H^1)}\|\mathcal{L}\|_{L(H^3,H^1)}\|J_2(t)\|_{H^3}dt' \\ &\lesssim \sup_{t'\in[0,t]} \|J_2(t')\|_{H^1} \\ &\overset{(4.36)}{\lesssim} \sup_{t'\in[0,t]} \sigma^2 a^2(t').\end{aligned}$$

Taking the supremum over $t \in [0, T_{q,\sigma,a}]$ we find

$$\sup_{t\in[0,T_{q,\sigma,a}]} \|\int_0^t S(t-t')T_2(t')dt'\|_{H^1} \lesssim \sup_{t\in[0,T_{q,\sigma,a}]} \sigma^2 a^2(t) \overset{(4.25)}{\lesssim} \sigma^{2-2} \tag{4.37}$$

T3

Now we estimate

$$T_3(t) := -\frac{1}{2}\sigma^2 F_\Phi u(t)$$

We first derive

$$\begin{aligned}\|F_\Phi\|_{H^1} &\overset{(4.4)}{=} \|\sum_{i=1}^\infty (\Phi e_i)^2\|_{H^1} \\ &\le \sum_{i=1}^\infty \|(\Phi e_i)^2\|_{H^1} \\ &\overset{(6.17)}{\lesssim} \sum_{i=1}^\infty \|\Phi e_i\|_{H^1}^2 \\ &= \|\Phi\|_{L_2(L^2,H^1)}^2\end{aligned}$$

which is finite, since Φ is a Hilbert-Schmidt operator from L^2 to H^1 by assumption. Thus, by lemma 6.2.6 we get

$$\|T_3(t)\|_{H^1} \lesssim \sigma^2\|u(t)\| \overset{(4.7)}{\le} \sigma^2\Big(\|u^*\|_{H^1} + \|v(t)\|_{H^1}\Big).$$

Combining this with (4.29) and (4.31) gives

$$\sup_{t\in[0,T_{q,\sigma,a}]} \|\int_0^t S(t-t')T_3(t')dt'\|_{H^1(\mathbb{R})} \lesssim \sigma^2(1+\sigma^{1-q}) \lesssim \sigma^{2-2q} \tag{4.38}$$

for $\sigma \leq 1$.

T4

Now we estimate

$$T_4(t) := \mathcal{R}_a \overset{(4.14)}{=} 8iu^*(x+\sigma a(t))|v_a|^2 + 4i\overline{u^*(x+\sigma a(t))}v_a^2 + 4i|v_a|^2 v_a.$$

Applying the triangle inequality and repeatedly using lemma 6.2.6, we find

$$\|T_4(t)\| \lesssim \|v_a\|^2_{H^1(\mathbb{R})} + \|v_a\|^3_{H^1(\mathbb{R})}.$$

Combining this with (4.29) and (4.30) gives

$$\sup_{t\in[0,T_{q,\sigma,a}]} \|\int_0^t S(t-t')T_4(t')dt'\|_{H^1(\mathbb{R})} \lesssim \sigma^{2-2q} + \sigma^{3-3q} \lesssim \sigma^{2-2q} \tag{4.39}$$

for $\sigma \leq 1$.

T5

For $T_5(t) := \sigma v(t)$, it follows by definition of $T_{q,\sigma,a}$ (4.25) and $\tau_{q,\sigma,c}$ (4.24) that

$$\begin{aligned}
&\sup_{t\in[0,T_{q,\sigma,a}]} \|\int_0^t S(t-t')T_5(t')dW(t')\|_{H^1(\mathbb{R})} \\
&= \sigma \sup_{t\in[0,T_{q,\sigma,a}]} \|\int_0^t S(t-t')v(t')dW(t')\|_{H^1(\mathbb{R})} \\
&\leq \sigma \sup_{t\in[0,\tau_{q,\sigma,c}]} \|\int_0^t S(t-t')v(t')dW(t')\|_{H^1(\mathbb{R})} \leq \qquad \sigma^{2-2q}.
\end{aligned} \tag{4.40}$$

4.2.3 Combination of estimates

Recalling equation (4.28) as well as the fact that $z_a(0) = 0$ by our choice of initial conditions, we find that z_a takes the following form:

$$z_a(t) = \sum_{i=1}^{4} \int_0^t S(t-t')T_i(t')dt' + \int_0^t S(t-t')T_5(t')dW(t').$$

Thus, applying the triangle inequality, taking the supremum and using (4.35), (4.37), (4.38), (4.39) and (4.40), we get

$$\sup_{t\in[0,T_{q,\sigma,a}]} \|z_a(t)\|_{H^1(\mathbb{R})} \lesssim \sigma^{2-2q},$$

which concludes the proof of part (b).

4.2.4 Convergence of stopping time

Next we show part (c). We first split up the probability to get

$$\mathbb{P}[T_{q,\sigma,a} < T] \overset{(4.25)}{\leq} \mathbb{P}[\tau_{q,\sigma,a} < T] + \mathbb{P}[\{\tau_{q,\sigma,a} = T\} \cap \{\tau_{q,\sigma,v} \wedge \tau_{q,\sigma,c} < T\}]$$

which is implied by the set-theoretic identity

$$(A \cap B)^c = (A^c \cap B) \cup B^c$$

combined with subadditivity of $\mathbb{P}$. Next we split the event $\{\tau_{q,\sigma,v} \wedge \tau_{q,\sigma,c} < T\}$ up into the events $\{\tau_{q,\sigma,c} < T\} \cap \{\tau_{q,\sigma,c} \leq \tau_{q,\sigma,v}\}$ and $\{\tau_{q,\sigma,v} < T\} \cap \{\tau_{q,\sigma,v} \leq \tau_{q,\sigma,c}\}$. By subadditivity and monotonicity of $\mathbb{P}$ this gives the estimate

$$\begin{aligned}
\mathbb{P}[T_{q,\sigma,a} < T] \leq & \mathbb{P}[\tau_{q,\sigma,a} < T] \\
& + \mathbb{P}[\{\tau_{q,\sigma,a} = T\} \cap \{\tau_{q,\sigma,c} < T\} \cap \{\tau_{q,\sigma,c} \leq \tau_{q,\sigma,v}\}] \\
& + \mathbb{P}[\{\tau_{q,\sigma,a} = T\} \cap \{\tau_{q,\sigma,v} < T\} \cap \{\tau_{q,\sigma,v} \leq \tau_{q,\sigma,c}\}] \\
\leq & \mathbb{P}[\tau_{q,\sigma,a} < T] \\
& + \mathbb{P}[\tau_{q,\sigma,c} \leq \tau_{q,\sigma,v}] \\
& + \mathbb{P}[\{\tau_{q,\sigma,a} = T\} \cap \{\tau_{q,\sigma,v} < T\} \cap \{\tau_{q,\sigma,v} \leq \tau_{q,\sigma,c}\}]
\end{aligned} \tag{4.41}$$

and we shall estimate these three probabilities individually. Firstly, by Markov's inequality we have

$$\begin{aligned}
\mathbb{P}[\tau_{q,\sigma,a} < T] &\leq \mathbb{P}[\sup_{t\in[0,T]} |a(t)| \geq \sigma^{-q}] = \mathbb{P}[\sup_{t\in[0,T]} |a(t)|^2 \geq \sigma^{-2q}] \\
&\leq \sigma^{2q}\mathbb{E}\Big[\sup_{t\in[0,T]} |a(t)|^2\Big] \overset{(4.22)}{=} \sigma^{2q} C_a.
\end{aligned} \tag{4.42}$$

For the second probability we first estimate

$$\begin{aligned}
\mathbb{P}[\tau_{q,\sigma,c} \leq \tau_{q,\sigma,v}] &\leq \mathbb{P}\Big[\sup_{t\in[0,\tau_{q,\sigma,v}]} \|\int_0^t S(t-s)v(s)dW(s)\|_{H^1} \geq \sigma^{1-2q}\Big] \\
&= \mathbb{P}\Big[\sup_{t\in[0,\tau_{q,\sigma,v}]} \|\int_0^t S(t-s)v(s)dW(s)\|_{H^1}^2 \geq \sigma^{2-4q}\Big] \\
&\leq \sigma^{-2+4q}\mathbb{E}\Big[\sup_{t\in[0,\tau_{q,\sigma,v}]} \|\int_0^t S(t-s)v(s)dW(s)\|_{H^1}^2\Big],
\end{aligned} \tag{4.43}$$

where we have used Markov's inequality for the final step. Now for $s \leq t \leq \tau_{q,\sigma,v}$, we have $v(s) = v(s)1_{s\leq\tau_{q,\sigma,v}}$. Therefore, using theorem 2.6.7 with $M = \sup_{t\in[0,T]}\|S(t)\|_{L(H^1)}$, and $\|g(s)\| = \|v(s)1_{s\leq\tau_{q,\sigma,v}}\|_{H^1} \leq \sigma^{1-q}$ we get

$$\begin{aligned}
&\mathbb{E}\Big[\sup_{t\in[0,\tau_{q,\sigma,v}]} \|\int_0^t S(t-s)v(s)dW(s)\|_{H^1}^2\Big] \\
&= \mathbb{E}\Big[\sup_{t\in[0,\tau_{q,\sigma,v}]} \|\int_0^t S(t-s)v(s)1_{s\leq\tau_{q,\sigma,v}}dW(s)\|_{H^1}^2\Big] \\
&\leq \mathbb{E}\Big[\sup_{t\in[0,T]} \|\int_0^t S(t-s)v(s)1_{s\leq\tau_{q,\sigma,v}}dW(s)\|_{H^1}^2\Big] \\
&\lesssim \sigma^{2-2q}
\end{aligned} \tag{4.44}$$

Thus, substituting (4.44) into (4.43) gives

$$\mathbb{P}[\tau_{q,\sigma,c} \leq \tau_{q,\sigma,v}] \lesssim \sigma^{2q}. \tag{4.45}$$

Now we estimate the third probability, which we abbreviate be $E_3 := \{\tau_{q,\sigma,a} = T\} \cap \{\tau_{q,\sigma,v} < T\} \cap \{\tau_{q,\sigma,v} \leq \tau_{q,\sigma,c}\}$. By rearranging (4.7) and (4.9) and using proposition 4.1.5, we get

$$\|v(s)\|_{H^1} \leq C'\sigma|a(s)| + \sigma\|w_a(s,x)\|_{H^1} + \|z_a(s,x)\|_{H^1}, \tag{4.46}$$

where $C' < \infty$ only depends on u^*. Now observe that on the event E_3, we have $\tau_{q,\sigma,v} \leq \tau_{q,\sigma,a} \wedge \tau_{q,\sigma,c}$, and therefore $\tau_{q,\sigma,v} = T_{q,\sigma,a}$ by (4.25) which implies

$$\|z_a(\tau_{q,\sigma,v})\|_{H^1} \leq \sup_{t\in[0,T_{q,\sigma,a}]} \|z_a(t,x)\|_{H^1} \overset{(4.27)}{\leq} C\sigma^{2-2q}.$$

Since $\tau_{q,\sigma,v} < T$ we also have $\|v(\tau_{q,\sigma,v}, x)\|_{H^1} = \sigma^{1-q}$. Therefore, substituting $s = \tau_{q,\sigma,v}$ in (4.46) gives that

$$\sigma^{1-q} \leq C'\sigma|a(\tau_{q,\sigma,v})| + \sigma\|w_a(\tau_{q,\sigma,v}, x)\|_{H^1} + C\sigma^{2-2q}$$

holds on E_3. We rearrange this to

$$C'|a(\tau_{q,\sigma,v})| + \|w_a(\tau_{q,\sigma,v}, x)\|_{H^1} \geq \sigma^{-q}(1 - C\sigma^{1-q}).$$

Therefore, E_3 implies

$$\sup_{t\in[0,T]} C'|a(t)| + \|w_a(t,x)\|_{H^1} \geq \sigma^{-q}(1 - C\sigma^{1-q}),$$

meaning that

$$\begin{aligned}
\mathbb{P}[E_3] &\leq \mathbb{P}[\sup_{t\in[0,T]} C'|a(t)| + \|w_a(t,x)\|_{H^1} \geq \sigma^{-q}(1 - C\sigma^{1-q})] \\
&= \mathbb{P}[(\sup_{t\in[0,T]} C'|a(t)| + \|w_a(t,x)\|_{H^1})^2 \geq \sigma^{-2q}(1 - C\sigma^{1-q})^2] \\
&\leq \frac{\sigma^{2q}}{(1 - C\sigma^{1-q})^2} \mathbb{E}\Big[\big(\sup_{t\in[0,T]} C'|a(t)| + \|w_a(t,x)\|_{H^1}\big)^2\Big] \\
&\lesssim \frac{\sigma^{2q}}{(1 - C\sigma^{1-q})^2} \Big(\mathbb{E}\Big[\sup_{t\in[0,T]} |a(t)|^2\Big] + \mathbb{E}\Big[\sup_{t\in[0,T]} \|w_a(t,x)\|_{H^1}^2\Big]\Big)
\end{aligned} \tag{4.47}$$

where we have used Markov's and Young's inequalities for the final two steps. Using (4.23) and theorem 2.6.6, we find

$$\mathbb{E}\Big[\sup_{t\in[0,T]} \|w_a(t,x)\|_{H^1(\mathbb{R})}^2\Big] < \infty. \tag{4.48}$$

Substituting (4.22) and (4.48) into (4.47), and recalling the definition of E_3 then gives

$$\mathbb{P}[\tau_{q,\sigma,a} = T, \tau_{q,\sigma,v} \wedge \tau_{q,\sigma,c} < T] \lesssim (1 + C_a)\sigma^{2q}. \tag{4.49}$$

Combining (4.42), (4.45) and (4.49) with (4.41) shows part (c) of theorem 4.2.1.

4.3 Immediate relaxation

We take a closer look at the solution for w_a. We repeat equation (4.23) here:

$$w_a(t) = S(t)v_0 - a(t)u_x^* + \int_0^t S(t-t')u^* dW(t').$$

We split up w_a into two parts, just like in section 3.3. Using definition 4.1.3 we get

$$\begin{aligned} w_a(t) = {} & S(t)\Pi v_0 + \int_0^t S(t-t')\Pi u^* dW(t') \\ & + S(t)\Pi^0 v_0 + \int_0^t S(t-t')\Pi^0 u^* dW(t') \\ & - a(t)u_x^*. \end{aligned} \tag{4.50}$$

Now we rewrite the middle term of (4.50). We compute

$$\begin{aligned} & S(t)\Pi^0 v_0 + \int_0^t S(t-t')\Pi^0 u^* dW(t') \\ & \overset{(4.19)}{=} S(t)u_x^* \mathcal{P} v_0 + \int_0^t S(t-t')u_x^* \mathcal{P} u^* dW(t') \\ & \overset{(4.17)}{=} u_x^* \mathcal{P} v_0 + \int_0^t u_x^* \mathcal{P} u^* dW(t') \\ & = \left[\mathcal{P} v_0 + \int_0^t \mathcal{P} u^* dW(t')\right] u_x^*. \end{aligned} \tag{4.51}$$

We now define w_∞ and a_∞ as follows:

$$w_\infty(t) = S(t)\Pi v_0 + \int_0^t S(t-t')\Pi u^* dW(t') \tag{4.52}$$

$$a_\infty(t) = \mathcal{P} v_0 + \int_0^t \mathcal{P} u^* dW(t'). \tag{4.53}$$

Notice by construction that we already have

$$\Pi^0 w_\infty(t) = 0, \tag{4.54}$$

since $\Pi^0\Pi \overset{(4.18)}{=} \Pi^0(I - \Pi^0) = \Pi^0 - \Pi^0 = 0$. Using the substitutions (4.51), (4.52) and (4.53), we see (4.50) simplifies to

$$w_a(t) = w_\infty(t) + (a_\infty(t) - a(t))u_x^*. \tag{4.55}$$

Similarly to the rotating waves case, we will transfer the multiscale expansion from theorem 4.2.1 to w_∞ by defining a_m to be the solution to the random ODE

$$\dot{a}_m(t) = -m(a_m(t) - a_\infty(t)),$$

with the initial condition $a(0) = 0$. Again, by theorem 6.1.1 we have existence, uniqueness and progressive measurability of a_m. By construction we have that a_m is differentiable and starts at zero. Also, we have the explicit solution representation

$$\begin{aligned}
a_m(t) &= m \int_0^t e^{-m(t-t')} a_\infty(t') dt' \\
&\overset{(4.53)}{=} m \int_0^t e^{-m(t-t')} \Big(\mathcal{P}v_0 + \int_0^{t'} \mathcal{P}u^* dW(t'') \Big) dt' \\
&= (1 - e^{-mt}) \mathcal{P}v_0 + m \int_0^t \int_0^{t'} e^{-m(t-t')} \mathcal{P}u^* dW(t'') dt' \\
&= (1 - e^{-mt}) \mathcal{P}v_0 + m \int_0^t \int_{t''}^t e^{-m(t-t')} dt' \mathcal{P}u^* dW(t'') \\
&= (1 - e^{-mt}) \mathcal{P}v_0 + \int_0^t (1 - e^{-m(t-t'')}) \mathcal{P}u^* dW(t''). \qquad (4.56)
\end{aligned}$$

We now formulate the multiscale expansion for w_∞.

4.3.1 Multiscale expansion for w_∞

Theorem 4.3.1. *(a) Let a_m be defined as in equation (4.56). Then $a_m(t)$ is progressively measurable, almost surely differentiable, $a_m(0) = 0$ and*

$$\mathbb{E}\Big[\sup_{t \in [0,T]} |a_m(t)|^2 \Big] \le C_\infty \qquad (4.57)$$

for some constant $C_\infty < \infty$ independent of m.

(b) Let $q \in (0, \frac{1}{2})$. Let w_∞ and a_∞ be as in (4.52) and (4.53), let u, $\tau_{q,\sigma,v}$ and $\tau_{q,\sigma,c}$ be as in theorem 4.2.1, and define the new stopping times

$$\begin{aligned}
\tau_{q,\sigma,\infty} &:= \inf\{t \in [0,T] : |a_\infty(t)| \ge \sigma^{-q}\} \wedge T \\
T_{q,\sigma,\infty} &:= \tau_{q,\sigma,\infty} \wedge \tau_{q,\sigma,v} \wedge \tau_{q,\sigma,c}.
\end{aligned}$$

Then we have the following multiscale expansion for $u(t,x)$:

$$u(t,x) =: u^*(x + \sigma a_\infty(t)) + \sigma w_\infty(t) + z_\infty(t), \qquad (4.58)$$

where $z_\infty(t)$ satisfies

$$\sup_{t \in [0, T_{q,\sigma,\infty}]} \|z_\infty(t)\|_{H^1} \le C\sigma^{2-2q}, \qquad (4.59)$$

with C being the same constant as in (4.27) (which is independent of σ).

(c) The stopping time $T_{q,\sigma,\infty}$ satisfies

$$\mathbb{P}\Big[T_{q,\sigma,\infty} = T \Big] \ge 1 - C\sigma^{2q}$$

for some constant C independent of σ.

Proof. We have already shown part of part (a) except (4.57). By theorem 6.1.1 we have

$$\mathbb{E}[\sup_{t\in[0,T]} |a_m(t)|^2] \overset{(6.2)}{\leq} \mathbb{E}[\sup_{t\in[0,T]} |a_\infty(t)|^2] = C_\infty, \tag{4.60}$$

where we know $C_\infty < \infty$ from equation (4.53) and theorem 2.6.6. To show part (b), we make the following three observations. Firstly, by theorem 6.1.1 we have

$$\lim_{m\to\infty} \sup_{t\in[\delta,T]} |a_m(t) - a_\infty(t)| = 0. \tag{4.61}$$

Secondly, by the same theorem we have

$$\sup_{t\in[0,\tau_{q,\sigma,\infty}]} |a_m(t)| \leq \sup_{t\in[0,\tau_{q,\sigma,\infty}]} |a_\infty(t)| \leq \sigma^{-q}.$$

By definition of τ_{q,σ,a_m} this implies

$$\tau_{q,\sigma,\infty} \leq \tau_{q,\sigma,a_m},$$

which further leads to

$$T_{q,\sigma,\infty} \leq T_{q,\sigma,a_m}. \tag{4.62}$$

Thirdly, from equation (4.55) we see that

$$\|w_{a_m}(t) - w_\infty(t)\|_{H^1} \leq C'|a_m(t) - a_\infty(t)|, \tag{4.63}$$

where C' only depends on u^*.
Therefore, by combining equations (4.26) and (4.58) and using the triangle inequality, we find

$$\begin{aligned}\|z_\infty(t)\|_{H^1} \leq& \|z_{a_m}(t)\|_{H^1} + \sigma\|w_{a_m}(t) - w_\infty(t)\|_{H^1} \\ &+ \|u^*(x+\sigma a_m(t)) - u^*(x+\sigma a(t))\|_{H^1} \\ &\overset{(4.21),(4.63)}{\leq} \|z_{a_m}(t)\|_{H^1} + \sigma C''|a_m(t) - a_\infty(t)|,\end{aligned}$$

with $C'' < \infty$ depending only on u^*.
Taking the supremum over $t \in [\delta, T_{q,\sigma,\infty}]$ we get

$$\begin{aligned}\sup_{t\in[\delta,T_{q,\sigma,\infty}]} \|z_\infty(t)\|_{H^1} \leq& \sup_{t\in[\delta,T_{q,\sigma,\infty}]} \Big(\|z_{a_m}(t)\|_{H^1} + \sigma C''|a_m(t) - a_\infty(t)|\Big) \\ &\overset{(4.62)}{\leq} \sup_{t\in[\delta,T_{q,\sigma,a_m}]} \Big(\|z_{a_m}(t)\|_{H^1} + \sigma C''|a_m(t) - a_\infty(t)|\Big) \\ &\overset{(4.27)}{\leq} C\sigma^{2-2q} + \sigma C'' \sup_{t\in[\delta,T]} |a_m(t) - a_\infty(t)|,\end{aligned}$$

where C is the constant from equation (4.27). Note that our use of theorem 4.2.1 is justified by part (a). Since neither C nor C'' depends on m, we can let m tend to infinity and use (4.61) to find

$$\sup_{t\in[\delta,T_{q,\sigma,\infty}]} \|z_\infty(t)\|_{H^1} \leq C\sigma^{2-2q}.$$

Now C is also independent of δ, so we can choose δ arbitrarily small to get

$$\sup_{t\in(0,T_{q,\sigma,\infty}]} \|z_\infty(t)\|_{H^1} \leq C\sigma^{2-2q}.$$

The estimate also holds at the left endpoint since z_∞ is continuous (as can be seen from (4.58)).

Now we prove part (c). Applying part (c) of theorem 4.2.1 with $a \equiv 0$ already gives

$$\mathbb{P}\Big[\tau_{q,\sigma,v} \wedge \tau_{q,\sigma,c} < T\Big] \leq C\sigma^{2q}$$

for some $C < \infty$ independent of σ. Using Markov's inequality we also get

$$\begin{aligned}\mathbb{P}\Big[\tau_{q,\sigma,\infty} < T\Big] &= \mathbb{P}\Big[\sup_{t\in[0,T]} |a_\infty(t)| \geq \sigma^{-q}\Big] \\ &\leq \sigma^{2q}\mathbb{E}\Big[\sup_{t\in[0,T]} |a_\infty(t)|\Big] \\ &\overset{(4.60)}{=} C\sigma^{2q},\end{aligned}$$

for some $C < \infty$ independent of σ. The result now follows from the definition of $T_{q,\sigma,\infty}$. □

4.3.2 Stability and approximate minimization

By combining theorems 4.1.4 and 4.3.1, we can now show stability of the soliton by the following proposition.

Proposition 4.3.2. *Let w_∞ be as in theorem 4.3.1. Then*

$$\mathbb{E}\Big[\|w_\infty(t)\|^2_{H^1}\Big] \leq 2C^2e^{-at}\|v_0\|^2_{H_1} + \frac{C^2}{a}(1-e^{-2at})\|u^*\|^2_{L(H^1)}\|Q^{1/2}\|^2_{L_2(H^1)},$$

where C and a are the constants from theorem 4.1.4.

Proof. First we find

$$\|w_\infty(t)\|_{H^1} \overset{(4.52)}{\leq} \|S(t)\Pi v_0\|_{H^1} + \|\int_0^t S(t-t')\Pi u^* dW(t')\|_{H^1}.$$

Taking the square, using $(a+b)^2 \leq 2a^2 + 2b^2$ and taking the expectation gives

$$\mathbb{E}\Big[\|w_\infty(t)\|^2_{H^1}\Big] \leq 2\|S(t)\Pi v_0\|^2_{H^1} + 2\mathbb{E}\Big[\|\int_0^t S(t-t')\Pi u^* dW(t')\|^2_{H^1}\Big]. \tag{4.64}$$

From theorem 4.1.4 we already have

$$\|S(t)\Pi v_0\|^2_{H^1} \overset{(4.20)}{\leq} C^2e^{-2at}\|v_0\|^2_{H^1}. \tag{4.65}$$

Furthermore, applying Itô's isometry to the second term of (4.64) gives

$$\begin{aligned}\mathbb{E}\Big[\|\int_0^t S(t-t')\Pi u^* dW(t')\|_{H^1}^2\Big] &\overset{(2.10)}{=} \int_0^t \|S(t-t')\Pi u^* Q^{1/2}\|_{L_2}^2 dt' \\ &\leq \|u^*\|_{L(H^1)}^2 \|Q^{1/2}\|_{L_2(H^1)}^2 \int_0^t \|S(t-t')\Pi\|_{L(H^1)}^2 dt' \\ &\overset{(4.20)}{\leq} C^2 \|u^*\|_{L(H^1)}^2 \|Q^{1/2}\|_{L_2(H^1)}^2 \int_0^t e^{-2a(t-t')} dt' \\ &= \frac{C^2}{2a}\|u^*\|_{L(H^1)}^2 \|Q^{1/2}\|_{L_2(H^1)}^2 (1-e^{-2at}). \end{aligned} \tag{4.66}$$

Substituting (4.65) and (4.66) into (4.64) gives the desired estimate. □

Combining the multiscale expansion (4.58) with proposition 4.3.2 shows that $u(t,x)-u^*(x+\sigma a_\infty(t))$ is $\mathcal{O}(\sigma)$, which shows the orbital stability. Next we show that a_∞ is (to first order) the right phase correction, in the sense that it approximately locally minimizes the fluctiations around $u^*(x+\sigma a(t))$. This is made precise in the following proposition.

Proposition 4.3.3. *Define*

$$\begin{aligned} E : \mathbb{R} &\to \mathbb{R} \\ a &\to \|\Pi^0 u(t,x) - u^*(x+a)\|_H^2. \end{aligned}$$

Then $\sigma a(t)$ is an approximate local minimizer of E, in the sense that

$$\partial_a E\Big|_{a=\sigma a_\infty(t)} = o(\sigma),$$

and the second derivative

$$\partial_{aa} E\Big|_{a=\sigma a_\infty(t)}$$

is strictly positive to first order in σ.

Proof. We compute

$$\partial_a \|\Pi^0 u(t,x) - u^*(x+a)\|_H^2 = -2\Big(\Pi^0[u(t,x)-u^*(x+a)], \Pi^0 u_x^*(x+a)\Big). \tag{4.67}$$

Evaluating at $a=\sigma a_\infty(t)$ and using theorem 4.3.1 we get

$$\begin{aligned} \partial_a E\Big|_{a=\sigma a_\infty(t)} &\overset{(4.58)}{=} -2\Big(\Pi^0[\sigma w_\infty(t) + z_\infty(t)], \Pi^0 u_x^*(x+\sigma a_\infty(t))\Big) \\ &\overset{(4.54)}{=} -2\Big(\Pi^0 z_\infty(t), \Pi^0 u_x^*(x+\sigma a_\infty(t))\Big) \\ &\overset{(4.59)}{=} \mathcal{O}(\sigma^{2-2q}). \end{aligned}$$

Differentiating (4.67) again gives

$$\partial_{aa}E(a) = 2\Big(\Pi^0 u_x^*(x+a), \Pi^0 u_x^*(x+a)\Big) - 2\Big(\Pi^0[u(t)-u^*(x+a)], \Pi^0 u_{xx}^*(x+a)\Big). \tag{4.68}$$

Evaluating at $a = \sigma a_\infty(t)$ and using theorem 4.3.1 again we find

$$\begin{aligned}\Big(\Pi^0[u(t) - u^*(x + \sigma a_\infty(t))], \Pi^0 u^*_{xx}(x + \sigma a_\infty(t))\Big) &\overset{(4.58)}{=} \Big(\Pi^0[\sigma w_\infty(t) + z_\infty(t)], \Pi^0 u^*_{xx}(x + \sigma a(t))\Big) \\ &\overset{(4.54)}{=} \Big(\Pi^0 z_\infty(t), \Pi^0 u^*_{xx}(x + \sigma a(t))\Big) \\ &\overset{(4.59)}{=} \mathcal{O}(\sigma^{2-2q}).\end{aligned}$$

Substituting this into (4.68) then gives

$$\partial_{aa} E(a)\Big|_{a=\sigma a_\infty(t)} = 2\Big(\Pi^0 u^*_x(x + \sigma a_\infty(t)), \Pi^0 u^*_x(x + \sigma a_\infty(t))\Big) + \mathcal{O}(\sigma^{2-2q}),$$

which shows the claim. □

Chapter 5

An investigation of general symmetries

In this section, we take first steps towards showing stability of patterns which have more general symmetries. Consider the PDE

$$du(t) = f(u(t))dt, \tag{5.1}$$

as well as its corresponding stochastic counterpart

$$du(t) = f(u(t))dt + \sigma dW(t), \tag{5.2}$$

where f is a possible unbounded linear operator which maps its domain $D(f)$ into H, $W(t)$ is a Q-Wiener process taking values in H, and σ is a parameter controlling the strength of the noise. We now assume that the PDE has certain symmetries, captured by a Lie group G. Let $\mathfrak{g}$ be the corresponding Lie algebra, which we assume to be finite-dimensional. We also choose some norm to turn $\mathfrak{g}$ into a normed vector space. By symmetry, we mean there exists a Lie group representation Π of G on H such that

$$\Pi(g)f(h) = f(\Pi(g)h) \quad g \in G, h \in H. \tag{5.3}$$

In the case of rotating waves, the symmetry group was $SE(2)$, while in the case of the PFNLS soliton is was the group of translations. The symmetry group will determine which types of 'corrections' (such are rotations or translations) we can make to the pattern solution. By a pattern solution, we mean there exists a solution $\hat{u}$ to (5.1) and an element $X \in \mathfrak{g}$ such that

$$\hat{u}(t) = \Pi(e^{tX})u^* \tag{5.4}$$

for some $u^* \in H$. Comparing this to the rotating wave, we see that we had $X = \omega\partial_\psi$ and $\Pi(e^{tX}) = \mathcal{T}_{R_{\omega t}}$ there. In the remainder, X will always denote this particular element. We will use the letters Y and Z to denote arbitrary elements of $\mathfrak{g}$.

To get a Lie algebra representation, we define the linear operator:

$$\begin{aligned} \pi(Y) := D(\pi(Y)) &\to H \\ \phi &\to \frac{d}{dt}\Pi(e^{tY})\phi\Big|_{t=0}, \end{aligned} \tag{5.5}$$

where $D(\pi(X))$ consists of all $\phi \in H$ for which this limit exists.
The following assumptions are needed for the multiscale expansions.

Assumption 5.0.1. *The linear operator $\Pi(e^Y) \in L(H)$ is an isometry.*

Assumption 5.0.2. *$u^* \in D(\pi(Y))$ for every Y, and also $\pi(Y)u^* \in D(\pi(Z))$ for every Z.*

Assumption 5.0.3. *f is Fréchet differentiable at $\Pi(g)u^*$ for every $g \in G$. We denote this derivative by f'. Furthermore, there exists a constant C independent of g, v such that*

$$\|f(\Pi(g)u^* + v) - f(\Pi(g)u^*) - f'(\Pi(g)u^*)v\|_H \leq C\|v\|_H^2 \tag{5.6}$$

for every $g \in G$, $v \in H$.

Assumption 5.0.4. *There exists a constant C, independent of Y such that*

$$\|\Pi(\exp(Y)) - I\|_{L(H)} \leq C|Y| \tag{5.7}$$

for every $Y \in \mathfrak{g}$.

Assumption 5.0.5. *There exists a Hilbert space H_1 continuously embedded into H such that $\Pi(g)u^* \in H_1$ for every $g \in G$, and furthermore*

$$\|f'(\Pi(g)u^*) + \pi(X)\|_{L(H_1,H)} \leq C,$$

for some $C < \infty$ which is independent of g.

Assumption 5.0.6. *There exists a constant C independent of Y such that*

$$\|\pi(Y)\hat{u}(t)\|_H \leq C|Y|, \tag{5.8}$$

for all $Y \in \mathfrak{g}$, $t \geq 0$.

5.1 Linearized problem

To show stability of the stochastic pattern we will linearize the SPDE (5.2) around $\hat{u}$. To do this we first find some identities regarding u^*. Substituting (5.4) into (5.1) gives

$$\frac{d}{dt}\Pi(e^{tX})u^* = f(\Pi(e^{tX})u^*).$$

Evalutuating this equation at $t = 0$ and using assumption 5.0.2 gives our first identity:

$$\pi(X)u^* \overset{(5.5)}{=} f(u^*). \tag{5.9}$$

This equation is analogous to (3.9) and (4.5). Next apply $\Pi(e^{tY})$ from the left and use 5.3 to get

$$\begin{aligned}\Pi(e^{tY})\pi(X)u^* &= \Pi(e^{tY})f(u^*)\\ &= f(\Pi(e^{tY})u^*).\end{aligned}$$

Differentiating this with respect to t (which is justified by assumptions 5.0.2 and 5.0.3) and evaluating at $t = 0$ now gives

$$\begin{aligned}\pi(Y)\pi(X)u^* &= f'(u^*)\pi(Y)u^* \\ \pi(Y)\pi(X)u^* - \pi(X)\pi(Y)u^* &= f'(u^*)\pi(Y)u^* - \pi(X)\pi(Y)u^* \\ \pi([Y,X])u^* &= (f'(u^*) - \pi(X))\pi(Y)u^*. \end{aligned} \tag{5.10}$$

We now introduce the linear operators

$$\mathcal{L}^* := f'(u^*) - \pi(X), \tag{5.11}$$

$$\mathcal{L}_{t,Y} := f'(\Pi(e^Y)\hat{u}(t)), \tag{5.12}$$

$$\mathcal{L}_t := f'(\hat{u}(t)), \tag{5.13}$$

as well as the linear map

$$\begin{aligned} L : \mathfrak{g} &\to \mathfrak{g} \\ Y &\to [Y, X]. \end{aligned} \tag{5.14}$$

Notice that equations (5.11), (5.12), and (5.13) are directly analogous to (3.32), (3.49) and (3.51). We may also easily verify that

$$\partial_t - \mathcal{L}_t = \Pi(e^{tX})(\partial_t - \mathcal{L}^*)\Pi(e^{-tX}) \tag{5.15}$$

The following assumption is crucial.

Assumption 5.1.1. *$\mathcal{L}^*$ generates a C_0-semigroup P_t^* on H.*

From assumption 5.1.1 and equation (5.15) the following proposition immediately follows.

Proposition 5.1.2. *$\mathcal{L}_t$ generates a strongly continuous evolution family $P_{t,t'}$. We also have the relation*

$$P_{t,t'} = \Pi(e^{tX})P^*_{t-t'}\Pi(e^{-t'X}) \tag{5.16}$$

With equations (5.14) and (5.11), equation (5.10) can be restated as

$$\mathcal{L}^*\pi(Y)u^* = \pi(LY)u^*. \tag{5.17}$$

Now we introduce the concept of the *center space*

Definition 5.1.3. *Let Y_i be the eigenvectors of L with corresponding eigenvalues λ_i. The center space of $\mathcal{L}^*$ consists of the span of $\pi(Y_i)u^*$.*

From equation (5.17) we see that the λ_i are then also eigenvalues of $\mathcal{L}^*$.

By direct differentiation using equation (5.17) it can now be verified that

$$P_t^* u^* = \pi(e^{tL}Y)u^*. \tag{5.18}$$

This also results in

$$\begin{aligned}
P_{t,s}\pi(Y)\hat{u}(s) &\overset{(5.16)}{=} \Pi(e^{tX})P^*_{t-s}\Pi(e^{-sX})\pi(Y)\hat{u}(s) \\
&\overset{(5.4)}{=} \Pi(e^{tX})P^*_{t-s}\Pi(e^{-sX})\pi(Y)\Pi(e^{sX})u^* \\
&\overset{(2.12)}{=} \Pi(e^{tX})P^*_{t-s}\pi(e^{-sX}Ye^{sX})u^* \\
&\overset{(2.13)}{=} \Pi(e^{tX})P^*_{t-s}\pi(Ad_{-sX}(Y))u^* \\
&\overset{(2.15)}{=} \Pi(e^{tX})P^*_{t-s}\pi(e^{-s\cdot ad_X}(Y))u^* \\
&\overset{(2.14),(5.14)}{=} \Pi(e^{tX})P^*_{t-s}\pi(e^{sL}Y)u^* \\
&\overset{(5.18)}{=} \Pi(e^{tX})\pi(e^{(t-s)L}e^{sL}Y)u^* \\
&= \Pi(e^{tX})\pi(e^{tL}Y)u^* \qquad (5.19)
\end{aligned}$$

The most important feature of this identity is that the final expression is independent of s.

As a further assumption, we require a linear deterministic stability result similar to theorem 3.1.16 and 4.1.4. We first introduce the following spectral projections, justified by the next assumption.

Assumption 5.1.4. *The eigenvalues λ_i introduced in definition 5.1.3 are isolated eigenvalues of $\mathcal{L}^*$. Furthermore, the combined span of the eigenspaces of $\mathcal{L}^*$ with eigenvalues λ_i is the same as the center space.*

Definition 5.1.5. *Define $\mathcal{P}^{*,c}$ to be the spectral projection of $\mathcal{L}^*$ onto the eigenvalues λ_i. Additionally define*

$$\begin{aligned}
\mathcal{P}^* &:= I - \mathcal{P}^{*,c}, \\
\mathcal{P}^c_t &:= \Pi(e^{tX})\mathcal{P}^{*,c}\Pi(e^{-tX}), \qquad (5.20)\\
\mathcal{P}_t &:= \Pi(e^{tX})\mathcal{P}^*\Pi(e^{-tX}).
\end{aligned}$$

Now we can formulate the deterministic stability assumption.

Assumption 5.1.6. *There exist constants C, $a > 0$ such that*

$$\|P^*_t\mathcal{P}^*\|_{L(H)} \leq C^{-at}.$$

5.2 Derivation of the SPDE

Now we obtain a multiscale expansion around a shifted version of $\hat{u}(t)$ for some arbitrary phase correction γ. Let $\gamma : [0, T] \to \mathfrak{g}$ be differentiable, adapted to $\mathcal{F}_t$ and satisfy $\gamma(0) = 0$. We also introduce

$$g(t) := \exp(\sigma\gamma(t)), \qquad (5.21)$$

where exp is the exponential map from $\mathfrak{g}$ to G. Let u be a solution to (5.2). Now we introduce the following three ways to write $u(t)$:

$$\begin{aligned}
u(t) &=: \hat{u}(t) + v(t), \\
u(t) &=: \Pi(g(t))\hat{u}(t) + v_\gamma(t), \qquad (5.22)\\
u(t) &=: \Pi(g(t))\hat{u}(t) + \sigma w_\gamma(t) + z_\gamma(t), \qquad (5.23)
\end{aligned}$$

where w_γ will be introduced later, at which point the third line becomes the definition for z_γ. To linearize (5.2) around $\Pi(g(t))\hat{u}(t)$ we will find the SPDE satisfied by $v_\gamma(t)$. By rewriting (5.22) and taking the differential we find

$$\begin{aligned} dv_\gamma(t) &= du(t) - d\Pi(g(t))\hat{u}(t) \\ &\overset{(5.2)}{=} f(u(t))dt + \sigma dW(t) - [\partial_t \Pi(g(t))]\hat{u}(t)dt - \Pi(g(t))d\hat{u}(t) \\ &\overset{(5.1),(5.3)}{=} f(u(t))dt + \sigma dW(t) - [\partial_t \Pi(g(t))]\hat{u}(t)dt - f(\Pi(g(t))\hat{u}(t))dt \end{aligned}$$

Now we add and subtract $f'(\Pi(g(t))\hat{u}(t))v_\gamma(t)dt$ and rearrange the terms to get

$$\begin{aligned} dv_\gamma(t) = {} & f'(\Pi(g(t))\hat{u}(t))v_\gamma(t)dt + [\partial_t \Pi(g(t))]\hat{u}(t)dt + \sigma dW(t) \\ & + [f(u(t)) - f(\Pi(g(t))\hat{u}(t)) - f'(\Pi(g(t))\hat{u}(t))v_\gamma(t)]dt \end{aligned}$$

Defining

$$\mathcal{R}_\gamma(t) := f(\Pi(g(t))u(t)) - f(\Pi(g(t))\hat{u}(t)) - f'(\Pi(g(t)\hat{u}(t)))v_\gamma(t), \tag{5.24}$$

and recalling equations (5.12) and (5.21) we get

$$dv_\gamma(t) = \mathcal{L}_{t,\sigma\gamma(t)}v_\gamma(t)dt + [\partial_t \Pi(g(t))]\hat{u}(t)dt + \sigma dW(t) + \mathcal{R}_\gamma(t)dt \tag{5.25}$$

We now linearize this equation and scale out σ to get the definition of w_γ.

$$dw_\gamma(t) = \mathcal{L}_t w_\gamma(t)dt + \pi(\dot{\gamma})\hat{u}(t)dt + dW(t) \tag{5.26}$$

Initial conditions

So far we have only stated the SPDE satisfied by u and w_γ without specifying initial conditions. Similar to our treatment of the PFNLS equation and the rotating wave, we set

$$\begin{aligned} u(0) &= \hat{u}(0) + \sigma v_0 \\ w_\gamma(0) &= v_0 \end{aligned}$$

for some $v_0 \in H$. By equation (3.44) and (3.46) this guarantees that σw_γ exactly matches v_γ at the initial time. We will now show that this approximation is still good after a finite time.

5.3 Multiscale expansions

Now we compute some estimates which are necessary to formulate a multiscale expansion of the form

$$u(t) = \Pi(\exp(\sigma\gamma(t)))\hat{u}(t) + \sigma w_\gamma(t) + z_\gamma(t),$$

where $z_\gamma(t) = \mathcal{O}(\sigma^{2-2q})$.

The SPDE satisfied by the difference $v_\gamma - \sigma w_\gamma =: z_\gamma$ reads:

$$\begin{aligned}
dz_\gamma(t) &\overset{(5.22),(5.23)}{=} dv_\gamma(t) - \sigma dw_\gamma(t) \\
&\overset{(5.25),(5.26)}{=} [\mathcal{L}_{t,\sigma\gamma(t)} v_\gamma(t) - \mathcal{L}_t \sigma w_\gamma(t)]dt \\
&\qquad + [[\partial_t \Pi(g(t))]\hat{u}(t) - \pi(\dot{\gamma}(t))\hat{u}(t)]dt \\
&\qquad + \mathcal{R}_t dt \\
&\overset{(5.12)}{=} \mathcal{L}_t z_\gamma(t) dt \\
&\qquad + [\mathcal{L}_{t,\sigma\gamma(t)} - \mathcal{L}_t] v_\gamma(t) dt \\
&\qquad + [[\partial_t \Pi(g(t))]\hat{u}(t) - \pi(\dot{\gamma}(t))\hat{u}(t)]dt \\
&\qquad + \mathcal{R}_t dt \\
&=: \mathcal{L}_t z_\gamma dt + T_1 dt - T_2 dt + T_3 dt.
\end{aligned}$$

We estimate the three terms separately.

5.3.1 T1

For the first term, we immediately see

$$\begin{aligned}
T_1(t) :&= [\mathcal{L}_{t,\sigma\gamma(t)} - \mathcal{L}_t] v_\gamma \\
&\overset{(5.12),(5.13)}{=} [f'(\exp(\sigma\gamma(t))\hat{u}(t)) - f'(\hat{u}(t))] v_\gamma \\
&\overset{(5.3)}{=} [\Pi(\exp(\sigma\gamma(t))) f'(\hat{u}(t)) - f'(\hat{u}(t))] v_\gamma .
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
\|T_1(t)\|_H &\leq \|\Pi(\exp(\sigma\gamma(t))) - I\|_{L(H)} \|f'(\hat{u})\|_{L(H)} \|v_\gamma\|_H \\
&\overset{(5.7)}{\leq} \|f'(\hat{u}(t))\| |\sigma| |\gamma(t)| \|v_\gamma\|_H ,
\end{aligned}$$

so we see this term is of second order.

5.3.2 T2

Now we estimate the convolution with T_2.

$$T_2(t) := (\partial_t \Pi(g(t)))\hat{u}(t) - \pi(\sigma\dot{\gamma}(t))\hat{u}(t). \tag{5.27}$$

To do this, we formulate the following expression, which is analogous to a first-order Taylor expansion:

$$J(t) := \Pi(\exp(\sigma\gamma(t)))\hat{u}(t) - \hat{u}(t) - \pi(\sigma\gamma(t))\hat{u}(t)$$

We apply $\partial_t - \pi(X)$ to $J(t)$ to get

$$\begin{aligned}(\partial_t - \pi(X))J(t) =& [\partial_t \Pi(\exp(\sigma\gamma(t)))]\hat{u}(t) + \Pi(\exp(\sigma\gamma(t)))\partial_t \hat{u}(t) - \pi(X)\Pi(\exp(\sigma\gamma(t)))\hat{u}(t) \\ &- \partial_t \hat{u}(t) + \pi(X)\hat{u}(t) \\ &- \pi(\sigma\dot{\gamma}(t))\hat{u}(t) - \pi(\sigma\gamma(t))\partial_t \hat{u}(t) + \pi(X)\pi(\sigma\gamma(t))\hat{u}(t) \\ &\overset{(5.1)}{=} [\partial_t \Pi(\exp(\sigma\gamma(t)))]\hat{u}(t) + \Pi(\exp(\sigma\gamma(t)))\pi(X)\hat{u}(t) - \pi(X)\Pi(\exp(\sigma\gamma(t)))\hat{u} \\ &- \pi(X)\hat{u}(t) + \pi(X)\hat{u}(t) \\ &- \pi(\sigma\dot{\gamma}(t))\hat{u}(t) - \pi(\sigma\gamma(t))\pi(X)\hat{u}(t) + \pi(X)\pi(\sigma\gamma(t))\hat{u}(t).\end{aligned}$$

We now cancel the two terms on the second line and rearrange the other terms to get

$$\begin{aligned}(\partial_t - \pi(X))J(t) = & [\partial_t \Pi(\exp(\sigma\gamma(t))]\hat{u}(t) - \sigma\pi(\dot{\gamma}(t))\hat{u}(t) \\ &+ \sigma\pi([X,\gamma])\hat{u}(t) \\ &+ [\Pi(\exp(\sigma\gamma(t)))\pi(X) - \pi(X)\Pi(\exp(\sigma\gamma(t))) - \pi(\sigma\gamma(t))\pi(X)]\hat{u}(t).\end{aligned}$$

After introducing

$$\begin{aligned}T_{2,1}(t) &:= \sigma\pi([X,\gamma(t)])\hat{u}(t), \\ T_{2,2}(t) &:= [\Pi(\exp(\sigma\gamma(t)))\pi(X) - \pi(X)\Pi(\exp(\sigma\gamma(t))) - \pi(\sigma\gamma(t))\pi(X)]\hat{u}(t),\end{aligned}$$

and using (5.27), we get

$$(\partial_t - \pi(X))J(t) = T_2(t) + T_{2,1}(t) + T_{2,2}(t). \tag{5.28}$$

Notice that we immediately have

$$\|T_{2,1}(t)\|_H \overset{(5.9)}{\lesssim} \sigma|[X,\gamma(t)]| \lesssim \sigma|\gamma(t)|, \tag{5.29}$$

since the bracket is continuous. To estimate $T_{2,2}$, we temporarily suppress the dependence on t and compute

$$\begin{aligned}T_{2,2} =& \Pi(\exp(\sigma\gamma))\Big(\pi(X) - \Pi(\exp(-\sigma\gamma))\pi(X)\Pi(\exp(\sigma\gamma)) + \pi([X,\sigma\gamma])\Big)\hat{u} \\ &+ (I - \Pi(\exp(\sigma\gamma))\pi([X,\sigma\gamma])\hat{u}\end{aligned} \tag{5.30}$$

Using assumption 5.0.4 and 5.0.6 we see that

$$\|(I - \Pi(\exp(\sigma\gamma))\pi([X,\sigma\gamma])\hat{u}\|_H \overset{(5.7),(5.8)}{\lesssim} |\sigma\gamma||[X,\sigma\gamma]| \lesssim \sigma^2|\gamma|^2. \tag{5.31}$$

To estimate the remaining terms of (5.30) we first compute

$$\begin{aligned}K &:= \pi(X) - \Pi(\exp(-\sigma\gamma))\pi(X)\Pi(\exp(\sigma\gamma)) + \pi([X,\sigma\gamma]) \\ &\overset{(2.12)}{=} \pi(X) - \pi(\exp(-\sigma\gamma)X\exp(\sigma\gamma)) + \pi([X,\sigma\gamma]) \\ &\overset{(2.13)}{=} \pi(X) - \pi(Ad_{\exp(-\sigma\gamma)}X) + \pi([X,\sigma\gamma]) \\ &\overset{(2.15)}{=} \pi(X) - \pi(e^{-ad_{\sigma\gamma}}X) + \pi([X,\sigma\gamma]) \\ &= \pi\Big(X + [X,\sigma\gamma] - \sum_{i=0}^{\infty}\frac{1}{i!}(-ad_{\sigma\gamma})^i X\Big) \\ &= \pi\Big(\sum_{i=2}^{\infty}\frac{1}{i!}(-ad_{\sigma\gamma})^i\Big),\end{aligned}$$

where the first two terms of the sum cancel since $ad_Y(Z) = [Y, Z]$. We now apply K to $\hat{u}$ and take the norm to get

$$\begin{aligned}
\|K\hat{u}\|_H &= \|\pi\Big(\sum_{i=2}^{\infty}\frac{1}{i!}(-ad_{\sigma\gamma})^i X\Big)\hat{u}\| \\
&\overset{(5.8)}{\lesssim} |\sum_{i=2}^{\infty}\frac{1}{i!}(-ad_{\sigma\gamma})^i X| \\
&\lesssim \sum_{i=2}^{\infty}\frac{1}{i!}|\sigma\gamma|^i \\
&= |\sigma\gamma|^2\sum_{i=0}^{\infty}\frac{|\sigma\gamma|^i}{(i+2)!} \\
&\le |\sigma\gamma|^2 e^{\sigma\gamma}.
\end{aligned}$$

Combining this estimate with (5.31) and (5.30) we see that

$$\|T_{2,2}\| \lesssim |\sigma\gamma|^2. \tag{5.32}$$

for $|\sigma\gamma| \le 1$. Finally we need an estimate for $J(t)$. It is not possible to show this estimate without knowing G and Π. Thus, for our purposes, we formulate this estimate as an assumption.

Assumption 5.3.1. *There exists a constant C, independent of σ, γ such that*

$$\|J(t)\|_{H^1} \le C|\sigma\gamma|^2. \tag{5.33}$$

Using (5.28) (5.29), (5.32) and (5.33) and assumption 5.0.5, it is now possible to show

$$\|\int_0^t P_{t,t'}T_2(t')dt'\| \lesssim \sup_{t'\in[0,t]} |\sigma\gamma(t')|^2,$$

by the same derivation as in the end of section 3.2.2.

5.3.3 T3

Now we estimate T_3. Firstly we have

$$\begin{aligned}
T_3(t) := \mathcal{R}_\gamma(t) &\overset{(5.24)}{=} f(u(t)) - f(\Pi(g(t))\hat{u}(t)) - f'(\Pi(g(t))\hat{u}(t,x))v_\gamma(t) \\
&\overset{(5.22)}{=} f(\Pi(g(t))\hat{u}(t) + v_\gamma(t))f(\Pi(g(t))\hat{u}(t)) - f'(\Pi(g(t))\hat{u}(t,x))v_\gamma(t).
\end{aligned}$$

Therefore, by assumption 5.0.3 we immediately have

$$\|T_3(t)\|_H \overset{(5.6)}{\le} C\|v_\gamma\|_H^2,$$

so this term is also of second order.

5.4 Mild solution

By theorem 2.6.9 we immediately see that (5.26) has a unique mild solution given by

$$w_\gamma(t) = P_{t,0}v_0 - \int_0^t P_{t,t'}\pi(\dot\gamma(t'))\hat u(t')dt' + \int_0^t P_{t,t'}dW(t').$$

Using (5.19) we can simplify to

$$\begin{aligned} w_\gamma(t) &\overset{(5.19)}{=} P_{t,0}v_0 - \int_0^t \Pi(e^{tX})\pi(e^{tL}\dot\gamma(t'))u^* dt' + \int_0^t P_{t,t'}dW(t') \\ &= P_{t,0}v_0 - \Pi(e^{tX})\pi(e^{tL})\gamma(t)u^* + \int_0^t P_{t,t'}dW(t'), \end{aligned}$$

where the final step can be taken since the map $Y \to \Pi(e^{tX})\pi(e^{tL}Y)u^*$ is linear and independent of t'.

We now split up w_γ using $I = \mathcal{P}_t^c + \mathcal{P}_t = \mathcal{P}^{*,c} + \mathcal{P}^*$, and find

$$\begin{aligned} w_\gamma(t) &= P_{t,0}\mathcal{P}^* v_0 + \int_0^t P_{t,t'}\mathcal{P}_{t'}dW(t') \\ &+ P_{t,0}\mathcal{P}^{*,c}v_0 + \int_0^t P_{t,t'}\mathcal{P}_{t'}^c dW(t') \\ &- \Pi(e^{tX})\pi(e^{tL}\gamma(t))u^* \end{aligned} \tag{5.34}$$

Before we proceed, it is convenient to obtain a more explicit representation of $\mathcal{P}^{*,c}$. From assumption 5.1.4 we see that $\mathcal{P}^{*,c}$ projects onto the space $\pi(Y)u^*$, $Y \in \mathfrak{g}$. Since π is linear, there exists a continuous map $A : H \to \mathfrak{g}$ such that

$$\mathcal{P}^{*,c}v = \pi(Av)u^*. \tag{5.35}$$

Thus we have

$$\begin{aligned} P_{t,t'}\mathcal{P}_{t'}^c v &\overset{(5.16),(5.20)}{=} \Pi(\exp(tX))P_{t-t'}\mathcal{P}^{*,c}\Pi(\exp(-t'X))v \\ &\overset{(5.35)}{=} \Pi(\exp(tX))P_{t-t'}\pi(A\Pi(\exp(-t'X))v)u^* \\ &\overset{(5.19)}{=} \Pi(\exp(tX))\pi(e^{(t-t')L}A\Pi(\exp(-t'X))v)u^*. \end{aligned} \tag{5.36}$$

In case $t' = 0$ this reduces to

$$P_{t,0}\mathcal{P}^{*,c} = \Pi(\exp(tX))\pi(e^{tL}Av)u^*. \tag{5.37}$$

Combining (5.36) and (5.37) we find that

$$P_{t,0}\mathcal{P}^{*,c}v_0 + \int_0^t P_{t,t'}\mathcal{P}_{t'}^c dW(t') = \Pi(\exp(tX))\pi(e^{tL}\Big[Av_0 + \int_0^t e^{-t'L}A\Pi(\exp(-t'X))dW(t')\Big]u^*.$$

Substituting this into (5.34) gives

$$\begin{aligned} w_\gamma(t) =& P_{t,0}\Pi^* v_0 + \int_0^t P_{t,s}\Pi_s dW(s) \\ &+ \Pi(e^{tX})\pi(e^{tL}\Big(Av_0 + \int_0^t e^{-sL}A\Pi(e^{-sX})dW(s)\Big))u^* \\ &+ \Pi(e^{tX})\pi(e^{tL}\gamma(t))u^* \end{aligned} \tag{5.38}$$

Now finally we introduce w_∞ and γ_∞ as follows:

$$w_\infty := P_{t,0}\Pi^* v_0 + \int_0^t P_{t,s}\Pi_s dW(s),$$
$$\gamma_\infty := Av_0 + \int_0^t e^{-sL} A\Pi(e^{-sX}) dW(s).$$

which simplifies (5.38) further to

$$w_\gamma(t) - w_\infty(t) = \Pi(e^{tX})\pi(e^{tL}(\gamma_\infty(t) - \gamma(t)))u^*. \tag{5.39}$$

Just like in the case of the rotating wave and the PFNLS soliton, equation (5.39) suggests that γ_∞ is the correct phase correction.

Chapter 6

Auxiliary results

6.1 Phase tracking

Let $y(t)$ be a continuous progressively measurable stochastic process taking values in some vector space V. The aim of this section is to approximate $y(t)$ by a sequence of progressively measurable process $x_m(t)$ which are differentiable and start at zero.

Theorem 6.1.1. *Let $y(t)$ be a continuous progressively measurable stochastic process on $[0, T]$ taking values in V. For any m, the random ODE*

$$\dot{x}_m(t) = -m(x_m(t) - y(t)),$$

with initial condition $x_m(0) = 0$ has a unique solution, given by '

$$x_m(t) = m \int_0^t e^{-m(t-t')} y(t') dt'. \tag{6.1}$$

This solution is progressively measurable and satisfies

$$\sup_{t' \in [0,t]} \|x_m(t')\|_V \leq \sup_{t' \in [0,t]} \|y(t')\|_V \tag{6.2}$$

for any $t' \in [0, T]$. Furthermore, if $y(t)$ is α-Hölder continuous, then

$$\lim_{m \to \infty} \sup_{t \in [\delta, T]} \|x_m(t) - y(t)\|_V = 0 \tag{6.3}$$

Proof. First, existence and uniqueness of a global solution is guaranteed by the Picard-Lindelöf theorem since $y(t)$ is continuous. By differentiating (6.1) we immediately see that it is indeed the solution. Progressive measurability of $x_m(t)$ is obvious from equation (6.1).

To show (6.2), we simply compute

$$\begin{aligned}\|x(t)\| &= \|m\int_0^t e^{-m(t-t')}y(t')dt'\|_V \\ &\leq \sup_{t'\in[0,t]}\|y(t')\|_V \cdot m\int_0^t e^{-m(t-t')}dt' \\ &\overset{(6.4)}{=} \sup_{t'\in[0,t]}\|y(t')\|_V \cdot (1-e^{-mt}) \\ &\leq \sup_{t'\in[0,t]}\|y(t')\|_V,\end{aligned}$$

which gives the estimate.

Finally we show (6.3). We first state the elementary integral

$$1 = m\int_0^t e^{-m(t-t')}dt' + e^{-mt}. \tag{6.4}$$

With this expression, we can write

$$\begin{aligned}x_m(t) - y(t) &\overset{(6.1),(6.4)}{=} m\int_0^t e^{-m(t-t')}y(t')dt' - y(t)m\int_0^t e^{-m(t-t')}dt' - y(t)e^{-mt} \\ &= m\int_0^t e^{-m(t-t')}(y(t') - y(t))dt' - y(t)e^{-mt}.\end{aligned} \tag{6.5}$$

By α-Hölder continuity of $y(t)$, we have

$$\begin{aligned}\|m\int_0^t e^{-m(t-t')}(y(t') - y(t))dt'\|_V &\leq m\int_0^t e^{-m(t-t')}\|y(t') - y(t)\|_V dt' \\ &\leq m\int_0^t e^{-m(t-t')}|t-t'|^\alpha dt' \\ &= m\int_0^t e^{-mu}|u|^\alpha du \\ &\leq m\int_0^\infty e^{-mu}|u|^\alpha du \\ &= m^{-\alpha}\int_0^\infty e^{-z}|z|^\alpha dz \\ &= m^{-\alpha}\Gamma(1+\alpha).\end{aligned} \tag{6.6}$$

Now taking the supremum of the norm of (6.5) and substituting (6.6) we get

$$\begin{aligned}\sup_{t\in[\delta,T]}\|x_m(t) - y(t)\|_V &\leq m^{-\alpha}\Gamma(1+\alpha) + \sup_{t\in[\delta,T]} y(t)e^{-mt} \\ &\leq m^{-\alpha}\Gamma(1+\alpha) + e^{-m\delta}\sup_{t\in[0,T]}\|y(t)\|_V,\end{aligned}$$

which converges to zero as $m \to \infty$. □

6.2 Properties of H^1 and H^2

In this section we establish some properties of $H^1(\mathbb{R})$ and $H^2(\mathbb{R}^2)$ which we will frequently use in the proofs of theorems 3.2.1 and 4.2.1. Throughout this section, $A \lesssim B$ mean that there exists a constant C (possibly depending on q,γ, d in lemma 6.2.1) such that $A \le CB$. We begin with Sobolev embeddings in the critical case where $p = d$.

Lemma 6.2.1. *$W^{1,d}(\mathbb{R}^d)$ embeds continuously into $L^q(\mathbb{R}^d)$ for any $q \in [d, \infty)$.*

Proof. Let $u \in C_c^\infty(\mathbb{R}^d)$. Let $\gamma \in (1, d]$. By manipulating exponents we get the estimate

$$\||u|^\gamma\|_{L^{\frac{d}{\gamma}}} = \|u\|_{L^d}^\gamma \le \|u\|_{W^{1,d}}^\gamma. \tag{6.7}$$

Next observe that $|u|^\gamma$ is weakly differentiable with derivative $\gamma|u|^{\gamma-1}|u_x|$. Using Hölder's inequality with $\frac{\gamma}{d} = \frac{\gamma-1}{d} + \frac{1}{d}$ we get

$$\|\gamma|u|^{\gamma-1}|u_x|\|_{L^{\frac{d}{\gamma}}} \le \gamma\||u|^{\gamma-1}\|_{L^{\frac{d}{\gamma-1}}}\|u_x\|_{L^d} = \gamma\|u\|_{L^d}^{\gamma-1}\|u_x\|_{L^d} \le \gamma\|u\|_{W^{1,d}}^\gamma. \tag{6.8}$$

Combining (6.7) and (6.8) gives

$$\||u|^\gamma\|_{W^{1,\frac{d}{\gamma}}} \lesssim \|u\|_{W^{1,d}}^\gamma. \tag{6.9}$$

Next, note that the Sobolev conjugate of $\frac{d}{\gamma}$ is equal to

$$\frac{d\frac{d}{\gamma}}{d - \frac{d}{\gamma}} = \frac{d}{\gamma - 1}.$$

Therefore by theorem 2.3.6 we have

$$\||u|^\gamma\|_{L^q} \lesssim \||u|^\gamma\|_{W^{1,\frac{d}{\gamma}}}, \tag{6.10}$$

for any $q \in [d, \frac{d}{\gamma-1}]$. Taking (6.10) to the power γ^{-1} and combining with (6.9) we get

$$\|u\|_{L^{\gamma q}} = \||u|^\gamma\|_{L^q}^{\frac{1}{\gamma}} \lesssim \||u|^\gamma\|_{W^{1,\frac{d}{\gamma}}}^{\frac{1}{\gamma}} \lesssim \|u\|_{W^{1,d}}.$$

Since $q \in [d, \frac{d}{\gamma-1}]$, we get $\gamma q \in [\gamma d, \frac{\gamma d}{\gamma-1}]$. Thus by taking γ close enough to 1, we can get γq anywhere in the interval (d, ∞). Therefore, setting $r = \gamma q$ we get that $W^{1,d}(\mathbb{R}^d)$ embeds continuously in $L^r(\mathbb{R}^d)$ for any $r \in [d, \infty)$ (the case $r = d$ is trivial). □

We will use this lemma with $p = d = 2$, $q = 4$, which gives $\|u\|_{L^4} \lesssim \|u\|_{H^1}$.

Lemma 6.2.2. *If $v \in H^2(\mathbb{R}^2)$, then $v \in L^\infty(\mathbb{R}^2)$ and*

$$\|v\|_{L^\infty(\mathbb{R}^2)} \le \|v\|_{H^2(\mathbb{R}^2)}. \tag{6.11}$$

Proof. Let $v \in C_c^\infty(\mathbb{R}^2)$. By the fundamental theorem of calculus:

$$\begin{aligned} v(x_1, x_2)^2 &= \int_{-\infty}^{x_1} 2v(x_1', x_2)\partial_{x_1}v(x_1', x_2)dx_1' \\ &= 2\int_{-\infty}^{x_2}\int_{-\infty}^{x_1} \partial_{x_1}v(x')\partial_{x_2}v(x') + v(x')\partial_{x_1x_2}v(x')dx_1'dx_2' \\ &\le \int_{\mathbb{R}^2} (\partial_{x_1}v)^2 + (\partial_{x_2}v)^2 + v^2 + (\partial_{x_1x_2}v)^2 dx \\ &\le \|v\|_{H^2(\mathbb{R}^2)}^2, \end{aligned}$$

where we have used Young's inequality for the penultimate step. Taking the supremum over $(x_1, x_2) \in \mathbb{R}^2$ gives

$$\|v\|_{L^\infty(\mathbb{R}^2)} \leq \|v\|_{H^2(\mathbb{R}^2)}.$$

The result follows by density of C_c^∞ in $H^2(\mathbb{R}^2)$. □

Lemma 6.2.3. *Let $u, v \in H^2(\mathbb{R}^2)$ and $i, j \in \{1, 2\}$. Then $\partial_{x_i} u \partial_{x_j} v \in L^2(\mathbb{R}^2)$ and*

$$\|\partial_i u \partial_j v\|_{L^2(\mathbb{R}^2)} \lesssim \|u\|_{H^2(\mathbb{R}^2)} \|v\|_{H^2(\mathbb{R}^2)}. \tag{6.12}$$

Proof. By Hölder's inequality and lemma 6.2.1 we have

$$\begin{aligned}
\|\partial_i u \partial_j v\|_{L^2(\mathbb{R}^2)} &\leq \|\partial_i u\|_{L^4(\mathbb{R}^2)} \|\partial_j v\|_{L^4(\mathbb{R}^2)} \\
&\lesssim \|\partial_i u\|_{H^1\mathbb{R}^2)} \|\partial_j v\|_{H^1(\mathbb{R}^2)} \\
&\leq \|u\|_{H^2(\mathbb{R}^2)} \|v\|_{H^2(\mathbb{R}^2)}.
\end{aligned}$$

□

Lemma 6.2.4. *If $u, v \in H^2(\mathbb{R}^2)$, then $uv \in H^2(\mathbb{R}^2)$ and*

$$\|uv\|_{H^2(\mathbb{R}^2)} \lesssim \|u\|_{H^2(\mathbb{R}^2)} \|v\|_{H^2(\mathbb{R}^2)}.$$

Proof. First we estimate

$$\begin{aligned}
\|uv\|_{L^2(\mathbb{R}^2)} &\leq \|u\|_{L^\infty(\mathbb{R}^2)} \|v\|_{L^2(\mathbb{R}^2)} \\
&\overset{(6.11)}{\leq} \|u\|_{H^2(\mathbb{R}^2)} \|v\|_{H^2(\mathbb{R}^2)}.
\end{aligned} \tag{6.13}$$

Next let $i \in \{1, 2\}$. Then by the product rule and Hölder's inequality we have

$$\begin{aligned}
\|\partial_{x_i}(uv)\|_{L^2(\mathbb{R}^2)} &= \|u \partial_{x_i} v + v \partial_{x_i} u\|_{L^2(\mathbb{R}^2)} \\
&\leq \|u \partial_{x_i} v\|_{L^2(\mathbb{R}^2)} + \|v \partial_{x_i} u\|_{L^2(\mathbb{R}^2)} \\
&\leq \|u\|_{L^\infty(\mathbb{R}^2)} \|\partial_{x_i} v\|_{L^2(\mathbb{R}^2)} + \|v\|_{L^\infty(\mathbb{R}^2)} \|\partial_{x_i} u\|_{L^2(\mathbb{R}^2)} \\
&\overset{(6.11)}{\leq} 2\|u\|_{H^2(\mathbb{R}^2)} \|v\|_{H^2(\mathbb{R}^2)}.
\end{aligned} \tag{6.14}$$

Finally, let $i, j \in \{1, 2\}$. Then we have

$$\begin{aligned}
\|\partial_{x_i x_j}(uv)\|_{L^2(\mathbb{R}^2)} &= \|u \partial_{x_i x_j} v + v \partial_{x_i x_j} u + \partial_{x_i} u \partial_{x_j} v + \partial_{x_j} u \partial_{x_i} v\|_{L^2(\mathbb{R}^2)} \\
&\overset{(6.12)}{\lesssim} \|u\|_{L^\infty(\mathbb{R}^2)} \|v\|_{H^2(\mathbb{R}^2)} + \|v\|_{L^\infty(\mathbb{R}^2)} \|u\|_{H^2(\mathbb{R}^2)} + \|u\|_{H^2(\mathbb{R}^2)} \|v\|_{H^2(\mathbb{R}^2)} \\
&\overset{(6.11)}{\leq} 2\|u\|_{H^2(\mathbb{R})} \|v\|_{H^2(\mathbb{R})}.
\end{aligned} \tag{6.15}$$

The result follows by combining (6.13), (6.14) and (6.15). □

Next we establish similar properties for $H^1(\mathbb{R}^1)$, for use the proof of theorem 4.2.1 and 4.3.1.

Lemma 6.2.5. *If $v \in H^1(\mathbb{R})$, then $v \in L^\infty(\mathbb{R})$ and*

$$\|v\|_{L^\infty(\mathbb{R})} \leq \|v\|_{H^1(\mathbb{R})}. \tag{6.16}$$

Proof. Let $v \in C_c^\infty(\mathbb{R})$. By the fundamental theorem of calculus,

$$\begin{aligned} v(x)^2 &= \int_{-\infty}^{x} \frac{d}{dx'} v(x')^2 dx' \\ &= \int_{-\infty}^{x} 2v(x')v'(x')dx' \\ &\leq \int_{-\infty}^{x} v(x')^2 + v(x')^2 dx' \\ &\leq \int_{\mathbb{R}} v(x')^2 + v(x')^2 dx' \\ &= \|v\|_{H^1(\mathbb{R})}^2, \end{aligned}$$

where we have used Young's inequality. Therefore,

$$\|v\|_{L^\infty(\mathbb{R})}^2 \leq \|v\|_{H^1(\mathbb{R})}^2.$$

The general result follows by density of $C_c^\infty(\mathbb{R})$ in $H^1(\mathbb{R})$. □

Lemma 6.2.6. *If $u, v \in H^1(\mathbb{R})$, then $uv \in H^1(\mathbb{R})$ and*

$$\|uv\|_{H^1(\mathbb{R})} \leq C\|u\|_{H^1(\mathbb{R})}\|v\|_{H^1(\mathbb{R})} \tag{6.17}$$

for some $C < \infty$ which is independent of u and v.

Proof. Firstly, by lemma 6.2.5 we have

$$\|uv\|_{L^2(\mathbb{R})} \leq \|u\|_{L^\infty(\mathbb{R})}\|v\|_{H^1(\mathbb{R})} \leq \|u\|_{H^1(\mathbb{R})}\|v\|_{H^1(\mathbb{R})}. \tag{6.18}$$

Furthermore, by the product rule we have

$$\begin{aligned} \|(uv)'\|_{L^2(\mathbb{R})} &= \|uv' + u'v\|_{L^2(\mathbb{R})} \\ &\leq \|uv'\|_{L^2(\mathbb{R})} + \|u'v\|_{L^2(\mathbb{R})} \\ &\leq \|u\|_{L^\infty(\mathbb{R})}\|v'\|_{L^2(\mathbb{R})} + \|v\|_{L^\infty(\mathbb{R})}\|u'\|_{L^2(\mathbb{R})} \\ &\overset{(6.16)}{\leq} 2\|u\|_{H^1(\mathbb{R})}\|v\|_{H^1(\mathbb{R})}. \end{aligned} \tag{6.19}$$

The result follows by combining (6.18) and (6.19). □

6.3 Regularity of $\hat{u}$

Throughout this section, the expression $A \lesssim B$ indicates that $A \leq CB$ for some constant C, independent of R and u. C may depend on a, b and ω.

Proposition 6.3.1. *Let $f \in L^2(\mathbb{R}^2)$, and let a, b, ω be real constants. If $u \in H^1(\mathbb{R}^2)$ is a weak solution to*

$$\Delta u + \omega\partial_\psi u + a\partial_{x_1}u + b\partial_{x_2}u = f, \tag{6.20}$$

then $u \in H^2(\mathbb{R}^2)$, and $\partial_\psi u \in L^2(\mathbb{R}^2)$

Proof. Since u is a weak solution to (6.20) on $\mathbb{R}^2$, it also solves (6.20) on any ball $B(0,R)$ around the origin. Applying theorem 2.4.3, we see that u is weakly differentiable on $B(0,R)$ for any R, and thus on $\mathbb{R}^2$. This also means u solves (6.20) pointwise. The only thing left to show is that the (weak) second derivatives of u are square-integrable.

Let $g \in C_c([0,\infty)$ be a monotonically decreasing function such that the support of $g(x) = 1$ for $x \in [0,1]$. For any $R \geq 1$, define $g_R(x) = g(|x|/R)$. Then $g_R \in C_c(\mathbb{R}^2)$ and by direct computation we see that

$$\begin{aligned} |g_R| &\leq 1, \\ |\nabla g_R| &\lesssim \frac{1}{R}, \\ |\partial_\psi g_R| &= 0, \\ |\Delta g_R| &\lesssim \frac{1}{R^2}. \end{aligned}$$

By applying the product rule, this leads to the estimates

$$\|ug_R\|_{L^2} \leq \|u\|_{L^2}, \tag{6.21}$$
$$\|\nabla(ug_R)\|_{L^2} \lesssim \|u\|_{H^1}, \tag{6.22}$$
$$\|\partial_\psi(ug_R)\|_{L^2} \leq \|\partial_\psi u\|_{H^1},$$
$$\|\Delta(ug_R)\|_{L^2} \lesssim \|u\|_{H^2}. \tag{6.23}$$

Now multiply (6.20) by $g_R\Delta(ug_R)$ and integrate over $\mathbb{R}^2$. The compact support of g_R guarantees all the integrals are well-defined. This gives

$$\begin{aligned} \int \Delta u \Delta(ug_R) g_R dx = &-\omega \int \partial_\psi u \Delta(ug_R) g_R dx \\ &- \int (a\partial_x u + b\partial_y u)\Delta(ug_R) g_R dx \\ &+ \int f\Delta(ug_R) g_R dx. \end{aligned} \tag{6.24}$$

Using integration by parts, we find that

$$\begin{aligned} -\int \partial_\psi u \Delta(ug_R) g_R dx &= \int \nabla(g_R \partial_\psi u) \cdot \nabla(ug_R) dx \\ &= \int \nabla(\partial_\psi(ug_R)) \cdot \nabla(ug_R) dx \\ &= \int (\partial_\psi \nabla(ug_R)) \cdot \nabla(ug_R) dx + \int \nabla(ug_R) R_{\pi/2} \cdot \nabla(ug_R) \\ &= \int \frac{1}{2} \partial_\psi |\nabla(ug_R)|^2 dx + \int \nabla(ug_R) R_{\pi/2} \cdot \nabla(ug_R) \\ &= 0. \end{aligned}$$

Substituting this into (6.24), and using the Cauchy-Schwarz inquality on the remaining terms we find

$$\int \Delta u \Delta(ug_R) g_R dx \leq (|a| \|g_R \partial_{x_1} u\|_{L^2} + |b| \|g_R \partial_{x_2} u\|_{L^2}) \|g_R \Delta u\|_{L^2} + \|fg_R\|_{L^2} \|\Delta ug_R\|_{L^2}. \tag{6.25}$$

Using (6.21), (6.22), and (6.23), we simplify this to

$$\int \Delta u \Delta(ug_R) g_R dx \lesssim (\|u\|_{H^1} \|f\|_{L^2}) \|g_R \Delta u\|_{L^2}$$

Now we compute

$$\int g_R \Delta u \Delta(ug_R) dx = \int g_R^2 (\Delta u)^2 + 2 g_R \Delta u (\nabla g_R \cdot \nabla u) + u g_R \Delta u \Delta g_R) dx,$$

which we rearrange to

$$\int g_R^2 (\Delta u)^2 dx = \int g_R \Delta u \Delta(ug_R) dx - 2 \int g_R \Delta u (\nabla g_R \cdot \nabla u) dx - \int u g_R \Delta u \Delta g_R) dx,$$

which gives the estimate

$$\begin{aligned} \|g_R(\Delta u)\|_{L^2}^2 &\lesssim \int g_R \Delta u \Delta(ug_R) dx + \|g_R \Delta u\|_{L^2} \|\nabla u\|_{L^2} + \|g_R \Delta u\|_{L^2} \|u \Delta g_R\|_{L^2} \\ &\lesssim \int g_R \Delta u \Delta(ug_R) dx + \|g_R \Delta u\|_{L^2} \|u\|_{H^1} \end{aligned} \tag{6.26}$$

by Cauchy-Schwarz. Substituting (6.25) into (6.26), and letting C be the proportionality constant gives

$$\|g_R(\Delta u)\|_{L^2}^2 \leq C \|g_R \Delta u\|_{L^2} (\|u\|_{H^1} + \|f\|_{L^2}),$$

at which point we use Young's inequality to find

$$\|g_R(\Delta u)\|_{L^2}^2 \leq \frac{1}{2} \|g_R \Delta u\|_{L^2} + \frac{C^2}{2} (\|u\|_{H^1} + \|f\|_{L^2}).$$

Rearranging this inequality gives

$$\|g_R(\Delta u)\|_{L^2}^2 \leq C^2 (\|u\|_{H^1} + \|f\|_{L^2}).$$

Now let $R \to \infty$, note that C is independent of R and use monotone convergence to find

$$\|\Delta u\|_{L^2}^2 \leq C^2 (\|u\|_{H^1} + \|f\|_{L^2}) < \infty,$$

which is sufficient to show $u \in H^2(\mathbb{R}^2)$. By rewriting (6.20), we now see that $\partial_\psi u$ is a linear combination of square integrable functions, so $\partial_\psi u \in L^2(\mathbb{R}^2)$. □

If we apply ∂_{x_1}, ∂_{x_2} or ∂_ψ to (6.20) and use (3.15), we see that the derivatives of u also solve (3.9) (with different values of a, b). Thus, by assumption 3.1.14 we use extend proposition 6.3.2 to get the regularity for u^* we need.

Proposition 6.3.2. *Suppose $u^* \in H^1(\mathbb{R}^2, \mathbb{R}^n)$ is a weak solution to (3.9). Then assumption 3.1.8 is satisfied.*

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