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Multivariate Meixner polynomials related to holomorphic discrete series representations of SU$(1,d)$

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Dedicated to the memory of Gerrit van Dijk

Abstract

We show that Griffiths' multivariate Meixner polynomials occur as matrix coefficients of holomorphic discrete series representations of the group SU$(1,d)$. Using this interpretation we derive several fundamental properties of the multivariate Meixner polynomials, such as orthogonality relations and difference equations. Furthermore, we also show that matrix coefficients for specific group elements lead to degenerate versions of the multivariate Meixner polynomials and their properties.

Keywords: Multivariate Meixner polynomials; Holomorphic discrete series; Orthogonal polynomials

1. Introduction

In this paper we study Griffiths' [6] multivariate generalization of the Meixner polynomials. The (univariate) Meixner polynomials M_n are named after Jozef Meixner who studied the polynomials in [12], but the polynomials were introduced earlier by Ladislav Truksa in [13] who called them generalized Kummer polynomials. The Meixner polynomials can be defined through their generating function by

$$\left(1-\tfrac{t}{c}\right)^{x}(1-t)^{-x-\beta}=\sum_{n=0}^{\infty}\frac{(\beta)_n}{n!}M_n(x;\beta,c)t^n.$$

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These polynomials have many nice properties, e.g. they are of hypergeometric type,

$$M_n(x;\beta,c) = {}_2F_1\left(\begin{matrix}-n,-x\\ \beta\end{matrix};1-\frac{1}{c}\right),$$

and they are orthogonal with respect to the negative binomial distribution,

$$\sum_{x=0}^{\infty} M_n(x;\beta,c)M_{n'}(x;\beta,c)\frac{(\beta)_x}{x!}c^x(1-c)^\beta = 0, \qquad \text{if} \quad n \neq n',$$

where we assume $\beta > 0$ and $c \in (0,1)$. Here we use standard notation for shifted factorials and hypergeometric functions, see e.g. [1].

It is well known that the Meixner polynomials are related to the Lie group SU(1, 1), which is the group of complex 2 × 2-matrices $\left(\begin{smallmatrix} a & b \\ \overline{b} & \overline{a}\end{smallmatrix}\right)$ with $|a|^2 - |b|^2 = 1$. To be more precise, the Meixner polynomials occur as matrix coefficients of the holomorphic discrete series representations defined as follows. For a positive integer σ, the Bergman space $\mathcal{A}_\sigma$ is the Hilbert space of holomorphic functions on the complex unit disc $\mathbb{D}$ with inner product

$$\langle f_1, f_2\rangle = \int_{\mathbb{D}} f_1(z)\overline{f_2(z)}(1-|z|^2)^\sigma \mathrm{d}z.$$

The holomorphic discrete series are the irreducible unitary representations on $\mathcal{A}_\sigma$ given by

$$[\pi(g)f](z) = \frac{1}{(a+\overline{b}z)^\sigma} f\left(\frac{b+\overline{a}z}{a+\overline{b}z}\right), \qquad g = \begin{pmatrix} a & b \\ \overline{b} & \overline{a}\end{pmatrix} \in \mathrm{SU}(1,1).$$

The monomials z^n, $n = 0, 1, 2, \ldots$, form an orthogonal basis for $\mathcal{A}_\sigma$. Let $\pi_{m,n}(g)$ be the coefficients in the expansion of $\pi(g)z^n$ in terms of this basis, i.e.

$$(b+\overline{a}z)^n(a+\overline{b}z)^{-\sigma-n} = \sum_{m=0}^{\infty} \pi_{n,m}(g)z^m.$$

Comparing this to the generating function of the Meixner polynomials, it follows that the coefficients $\pi_{m,n}(g)$ are multiples of the Meixner polynomials $M_m(n;\sigma,|\frac{b}{a}|^2)$. From this representation theoretic interpretation of the Meixner polynomials several useful properties of the polynomials can easily be obtained: e.g. orthogonality relations, the three-term recurrence relation and the second order difference equation.

In [6], Griffiths introduced multivariate Meixner polynomials as orthogonal polynomials with respect to the negative multinomial distribution. As such, they are closely related to the multivariate Krawtchouk polynomials [5] which are orthogonal on a finite set with respect to the usual multinomial distribution. Iliev [7] showed that the multivariate Krawtchouk polynomials occur as matrix coefficients for finite dimensional representations of $\mathrm{SL}(d,\mathbb{C})$, and using this interpretation bispectrality of the Krawtchouk polynomials is shown. In a similar fashion Genest, Vinet and Zhedanov [4] used representation theory of $SO(d+1)$ to study the multivariate Krawtchouk polynomials. In the current paper we exploit a similar connection for the Meixner polynomials. In [8] Iliev showed, without references to representation theory, that the multivariate Meixner polynomials have many properties that resemble those of the Krawtchouk polynomials. A representation theoretic interpretation of Griffiths' bivariate Meixner polynomials is obtained by Genest, Miki, Vinet and Zhedanov [3] by showing they appear as matrix coefficients of SO(2, 1) representations on oscillator states. In the same paper, it is also indicated how the general multivariate Meixner polynomials arise similarly in the representation theory of SO(d, 1). Furthermore, in [2] it is shown that the bivariate Meixner polynomials occur as wave functions for a two-dimensional quantum oscillator, which is a result of the fact proved in [9] that orthogonal polynomials with respect to the negative multinomial distribution, such as Griffiths multivariate Meixner polynomials, are eigenfunctions of a second order partial difference operator.

In the present paper we provide an alternative way to study the multivariate Meixner polynomials by using holomorphic discrete series representations of $\mathrm{SU}(1,d)$, similar to the above described representation theoretic interpretation of the univariate Meixner polynomials. See also e.g. [10, Section 6.8] for the interpretation of the univariate Meixner polynomials in $\mathrm{SU}(1,1)$ representations. We expect that the results can be generalized to the more general case of holomorphic discrete series representations of $\mathrm{SU}(n,m)$ (for $n, m \geq 2$). This will be the topic of a future paper.

The organization of the paper is as follows. First in Section 2 we recall the holomorphic discrete series representation of $\mathrm{SU}(1,d)$. Then, in Section 3 we recall Griffiths' definition of the multivariate Meixner polynomials and show that the matrix coefficients of the holomorphic discrete series corresponding to generic $g \in \mathrm{SU}(1,d)$ can be expressed in terms of these Meixner polynomials. This immediately leads to several properties, such as orthogonality with respect to the negative multinomial distribution, of the Meixner polynomials. In Section 4, we consider the corresponding Lie algebra representations and derive difference equations for the multivariate Meixner polynomials. Finally, in Section 5 we consider degenerate versions of the multivariate Meixner polynomials, which correspond to the matrix coefficients for specific elements $g \in \mathrm{SU}(1,d)$.

1.1. Notations

For $\mathbf{x} = (x_1, \ldots, x_d) \in \mathbb{C}^d$, we define

$$\begin{aligned} \|\mathbf{x}\| &= \sqrt{|x_1|^2 + \cdots + |x_d|^2}, \\ |\mathbf{x}| &= x_1 + \cdots + x_d, \\ \overline{\mathbf{x}} &= (\overline{x_1}, \ldots, \overline{x_d}). \end{aligned}$$

We often consider elements in $\mathbb{C}^d$ as column vectors, which will be clear from the context. For $\mathbf{n} = (n_1, \ldots, n_d) \in \mathbb{N}_0^d$ and $\mathbf{x} = (x_1, \ldots, x_d) \in \mathbb{C}^d$, we set

$$\mathbf{n}! = n_1! \cdots n_d!, \quad \mathbf{x}^{\pm\mathbf{n}} = x_1^{\pm n_1} \cdots x_d^{\pm n_d}.$$

We denote by $\mathbb{M}_d$ the set of complex $d \times d$-matrices. For $A \in \mathbb{M}_d$, A^t is the transpose of A and $A^\dagger$ the conjugate transpose.

2. The holomorphic discrete series representations of SU(1, *d*)

$\mathrm{SL}(d+1;\mathbb{C})$ is the group of complex $(d+1) \times (d+1)$-matrices of determinant 1. $\mathrm{SU}(1,d)$ is the subgroup of $\mathrm{SL}(d+1;\mathbb{C})$ preserving the hermitian form associated with the matrix

$$J = \mathrm{diag}(1, -1, -1, \ldots, -1),$$

that is, a matrix $g \in \mathrm{SL}(d+1;\mathbb{C})$ is in $\mathrm{SU}(1,d)$ if and only if the following equation holds:

$$g^\dagger J g = J. \tag{2.1}$$

Throughout this paper it will be convenient to write $g \in \mathrm{SU}(1,d)$ in the form

$$g = \begin{pmatrix} a & \mathbf{b}^t \\ \mathbf{c} & D \end{pmatrix},$$

where $a \in \mathbb{C}$, $\mathbf{b}, \mathbf{c} \in \mathbb{C}^d$ and $D \in \mathbb{M}_d$. We will sometimes use notations such as $a = a(g)$ to stress the dependence on g. From the defining Eq. (2.1), it follows that the inverse of the matrix $g \in \mathrm{SU}(1,d)$ is given by

$$g^{-1} = J g^\dagger J = \begin{pmatrix} \overline{a} & -\mathbf{c}^\dagger \\ -\overline{\mathbf{b}} & D^\dagger \end{pmatrix}, \tag{2.2}$$

which implies identities such as

$$\begin{gathered} |a|^2 - \|\mathbf{b}\|^2 = |a|^2 - \|\mathbf{c}\|^2 = 1, \\ D^\dagger D = I_d + \overline{\mathbf{b}}\mathbf{b}^t, \quad DD^\dagger = I_d + \mathbf{c}\mathbf{c}^\dagger, \\ \overline{a}\mathbf{b}^t = \mathbf{c}^\dagger D, \qquad a\mathbf{c}^\dagger = \mathbf{b}^t D^\dagger, \end{gathered} \tag{2.3}$$

where I_d is the identity matrix in $\mathbb{M}_d$.

$\mathrm{SU}(1,d)$ has a family of representations called the holomorphic discrete series, on the weighted Bergman space $\mathcal{A}_\alpha$ that we now introduce, see e.g. [11, Chapter VI]. Let $\alpha > -1$. We define $\mathrm{d}v_\alpha$ to be the weighted Lebesgue measure on the open unit ball $\mathbb{B}_d = \{\mathbf{z} \in \mathbb{C}^d \mid \|\mathbf{z}\| < 1\}$ given by

$$\mathrm{d}v_\alpha = c_\alpha (1 - \|\mathbf{z}\|^2)^\alpha \mathrm{d}v, \tag{2.4}$$

with $\mathrm{d}v$ the standard volume measure on $\mathbb{B}_d$ and c_α is the normalizing constant so that $v_\alpha(\mathbb{B}_d) = 1$. A direct calculation shows that

$$c_\alpha = \frac{(\alpha+1)_d}{d!}.$$

The Bergman space $\mathcal{A}_\alpha$ is the space of holomorphic functions in $L^2(\mathbb{B}_d, \mathrm{d}v_\alpha)$. $\mathcal{A}_\alpha$ is a Hilbert space with inner product

$$\langle f, g \rangle = \int_{\mathbb{B}_d} f(\mathbf{z})\overline{g(\mathbf{z})}\, \mathrm{d}v_\alpha, \qquad f, g \in \mathcal{A}_\alpha.$$

An orthonormal basis for $\mathcal{A}_\alpha$ is given by the monomials

$$e_{\mathbf{m}}(\mathbf{z}) = \sqrt{\frac{(\alpha+d+1)_{|\mathbf{m}|}}{\mathbf{m}!}}\mathbf{z}^{\mathbf{m}}, \qquad \mathbf{m} \in \mathbb{N}_0^d,$$

see e.g. Lemma 1.11 and Proposition 2.6 in [14].

Now we are ready to define the representation of $\mathrm{SU}(1,d)$ we are interested in this paper. From here on, we assume $\alpha \in \mathbb{N}_0$ and we set

$$\sigma = \alpha + d + 1.$$

Then, π^σ given by

$$\pi^\sigma \begin{pmatrix} a & \mathbf{b}^t \\ \mathbf{c} & D \end{pmatrix} f(\mathbf{z}) = \left(a + \mathbf{c}^t\mathbf{z}\right)^{-\sigma} f\left(\frac{\mathbf{b} + D^t\mathbf{z}}{a + \mathbf{c}^t\mathbf{z}}\right), \tag{2.5}$$

defines a family of unitary representation of $\mathrm{SU}(1,d)$ on $\mathcal{A}_\alpha$ labeled by $\sigma \in \mathbb{N}_{\geq d+1}$.

3. Matrix coefficients and multivariate meixner polynomials

In [6,8], the multivariate Meixner polynomials are defined through their generating function. The generating function is given by

$$G(\mathbf{x},\mathbf{t},U,\beta) = \left(1-\sum_{j=1}^{d} t_j\right)^{-\beta-|\mathbf{x}|} \prod_{i=1}^{d}\left(1-\sum_{j=1}^{d} U_{i,j}t_j\right)^{x_i}, \qquad \mathbf{x},\mathbf{t}\in\mathbb{C}^d,$$

where $\beta \in \mathbb{C}\setminus(-\mathbb{N}_0)$, $U=(U_{i,j})\in\mathbb{M}_d$, and the principal branch of the power function is used. Then the polynomials $M_{\mathbf{n}}(\mathbf{x};U,\beta)$ are defined by

$$G(\mathbf{x},\mathbf{t},U,\beta) = \sum_{\mathbf{n}\in\mathbb{N}_0^d} \frac{(\beta)_{|\mathbf{n}|}}{\mathbf{n}!} M_{\mathbf{n}}(\mathbf{x};U,\beta)\mathbf{t}^{\mathbf{n}}, \tag{3.1}$$

for $\mathbf{t}$ sufficiently close to 0. They have an explicit expression as a Gelfand–Aomoto hypergeometric series given by

$$M_{\mathbf{n}}(\mathbf{x};U,\beta) = \sum_{(a_{i,j})\in\mathbb{M}_d(\mathbb{N}_0)} \frac{\prod_{j=1}^{d}(-n_j)_{\sum_{i=1}^{d}a_{i,j}} \prod_{i=1}^{d}(-x_i)_{\sum_{j=1}^{d}a_{i,j}}}{(\beta)_{\sum_{i,j=1}^{d}a_{i,j}}} \prod_{i,j=1}^{d} \frac{(1-U_{i,j})^{a_{i,j}}}{a_{i,j}!}, \tag{3.2}$$

where $\mathbb{M}_d(\mathbb{N}_0)$ denotes the subset of $\mathbb{M}_d$ consisting of matrices with entries in $\mathbb{N}_0$. The multivariate Meixner polynomials are the polynomials $M_{\mathbf{n}}(\mathbf{x};U,\beta)$ with conditions imposed on the matrix U to ensure orthogonality with respect to the negative multinomial distribution [6,8]. We will show that the multivariate Meixner polynomials occur as matrix coefficients for the holomorphic discrete series representation of $\mathrm{SU}(1,d)$. In this interpretation, the parameter matrix U depends on a $g\in\mathrm{SU}(1,d)$, which implies conditions for U that are closely related to the conditions imposed in [8], see the discussion at the end of this section.

Let $g\in\mathrm{SU}(1,d)$. The function $\pi^\sigma(g)e_{\mathbf{n}}$ is holomorphic on $\mathbb{B}_d$, hence it must equal its Taylor series which we can consider as the expansion in the basis $\{e_{\mathbf{m}} \mid \mathbf{m}\in\mathbb{N}_0^d\}$. We consider the corresponding matrix coefficients $\pi^\sigma_{\mathbf{m},\mathbf{n}}(g)$ which are determined by

$$\pi^\sigma(g)e_{\mathbf{n}}(\mathbf{z}) = \sum_{\mathbf{m}\in\mathbb{N}_0^d} \pi^\sigma_{\mathbf{m},\mathbf{n}}(g)e_{\mathbf{m}}(\mathbf{z}), \qquad \mathbf{z}\in\mathbb{B}_d, \tag{3.3}$$

or equivalently

$$\pi^\sigma_{\mathbf{m},\mathbf{n}}(g) = \langle \pi^\sigma(g)e_{\mathbf{n}}, e_{\mathbf{m}}\rangle. \tag{3.4}$$

Theorem 3.1. *Let* $g=\begin{pmatrix} a & \mathbf{b}^t \\ \mathbf{c} & D\end{pmatrix}\in\mathrm{SU}(1,d)$ *with* $a,b_i,c_i\neq 0$ *for* $i=1,\ldots,d$, *then*

$$\pi^\sigma_{\mathbf{m},\mathbf{n}}(g) = \sqrt{\frac{(\sigma)_{|\mathbf{m}|}(\sigma)_{|\mathbf{n}|}}{\mathbf{m}!\,\mathbf{n}!}}(-1)^{|\mathbf{m}|}a^{-\sigma}\tilde{\mathbf{p}}^{\mathbf{m}}\mathbf{p}^{\mathbf{n}}M_{\mathbf{m}}(\mathbf{n};U,\sigma)$$

with $\mathbf{p}=\mathbf{p}(g)=(p_1,\ldots,p_d)$, $\tilde{\mathbf{p}}=\tilde{\mathbf{p}}(g)=(\tilde p_1,\ldots,\tilde p_d)$ *and* $U=U(g)=(U_{i,j})$ *given by*

$$p_i=\frac{b_i}{a}, \quad \tilde p_i = \frac{c_i}{a}, \quad U_{i,j}=\frac{aD_{j,i}}{b_ic_j}.$$

Before we prove the theorem let us remark that the identities

$$|a|^2 - \|\mathbf{b}\|^2 = |a|^2 - \|\mathbf{c}\|^2 = 1,$$

imply

$$|a|^2 = \frac{1}{1-\sum_{i=1}^d |p_i|^2} = \frac{1}{1-\sum_{i=1}^d |\tilde{p}_i|^2}.$$

It is convenient to write $|a|^{-2} = |p_0|^2 = |\tilde{p}_0|^2$, so that

$$\sum_{i=0}^d |p_i|^2 = \sum_{i=0}^d |\tilde{p}_i|^2 = 1.$$

Proof. Assume g is as given in the theorem, then we can write out the left-hand side of (3.3) as follows:

$$\begin{aligned}\pi^\sigma(g)\mathbf{z}^\mathbf{n} &= \left(a + \sum_{j=1}^d c_j z_j\right)^{-\sigma} \prod_{i=1}^d \left(\frac{b_i + \sum_{j=1}^d D_{j,i} z_j}{a + \sum_{j=1}^d c_j z_j}\right)^{n_i} \\ &= a^{-\sigma-|\mathbf{n}|}\mathbf{b}^\mathbf{n} \left(1 - \sum_{i=1}^d \frac{-c_i z_i}{a}\right)^{-\sigma-|\mathbf{n}|} \prod_{i=1}^d \left(1 - \sum_{j=1}^d \frac{aD_{j,i}}{b_i c_j}\left(\frac{-c_j z_j}{a}\right)\right)^{n_i}.\end{aligned}$$

Comparing with the generating function for Meixner polynomials (3.1) we see that

$$\pi^\sigma(g)\mathbf{z}^\mathbf{n} = \sum_{\mathbf{m}\in\mathbb{N}_0^d} \frac{(\sigma)_{|\mathbf{m}|}}{\mathbf{m}!} a^{-\sigma-|\mathbf{m}|-|\mathbf{n}|}\mathbf{b}^\mathbf{n}(-\mathbf{c})^\mathbf{m} M_\mathbf{m}(\mathbf{n}; U, \sigma)\mathbf{z}^\mathbf{m},$$

where the parameter matrix U is given by $U_{ij} = \frac{aD_{ji}}{b_i c_j}$. From this the result follows.

From Theorem 3.1 we immediately obtain several properties of the multivariate Meixner polynomials:

Theorem 3.2. *The Meixner polynomials $M_\mathbf{m}(\mathbf{n}; U, \sigma)$ from Theorem 3.1 have the following properties.*

(i) Orthogonality relations:

$$\sum_{\mathbf{n}\in\mathbb{N}_0^d} \frac{(\sigma)_{|\mathbf{n}|}}{\mathbf{n}!} \mathbf{p}^\mathbf{n}\overline{\mathbf{p}}^\mathbf{n} M_\mathbf{m}(\mathbf{n}; U, \sigma)\overline{M_{\mathbf{m}'}(\mathbf{n}; U, \sigma)} = \delta_{\mathbf{m},\mathbf{m}'} \frac{\mathbf{m}!\tilde{\mathbf{p}}^{-\mathbf{m}}\overline{\tilde{\mathbf{p}}}^{-\mathbf{m}}}{(\sigma)_{|\mathbf{m}|}|p_0|^{2\sigma}},$$

$$\sum_{\mathbf{m}\in\mathbb{N}_0^d} \frac{(\sigma)_{|\mathbf{m}|}}{\mathbf{m}!} \tilde{\mathbf{p}}^\mathbf{m}\overline{\tilde{\mathbf{p}}}^\mathbf{m} M_\mathbf{m}(\mathbf{n}; U, \sigma)\overline{M_\mathbf{m}(\mathbf{n}'; U, \sigma)} = \delta_{\mathbf{n},\mathbf{n}'} \frac{\mathbf{n}!\mathbf{p}^{-\mathbf{n}}\overline{\mathbf{p}}^{-\mathbf{n}}}{(\sigma)_{|\mathbf{n}|}|p_0|^{2\sigma}}.$$

(ii) Integral representation:

$$(-1)^{|\mathbf{m}|}\tilde{\mathbf{p}}^\mathbf{m} M_\mathbf{m}(\mathbf{n}; U, \sigma) = \frac{(\sigma-d)_d}{d!}\int_{\mathbb{B}_d}\left(1 + \sum_{i=1}^d \tilde{p}_i z_i\right)^{-\sigma-|\mathbf{n}|} \prod_{i=1}^d \left(1 + \sum_{j=1}^d U_{i,j}\tilde{p}_j z_j\right)^{n_i} \overline{\mathbf{z}}^\mathbf{m}(1-\|\mathbf{z}\|^2)^{\sigma-d-1}\mathrm{d}v.$$

(iii) Duality: $M_\mathbf{m}(\mathbf{n}; U, \sigma) = M_\mathbf{n}(\mathbf{m}; U^t, \sigma)$.

(iv) Sum identity:

$$\left(\frac{a(g_1)a(g_2)}{a(g_1g_2)}\right)^{\sigma} \tilde{\mathbf{p}}(g_1g_2)^{\mathbf{m}}\tilde{\mathbf{p}}(g_1)^{-\mathbf{m}}\mathbf{p}(g_1g_2)^{\mathbf{n}}\mathbf{p}(g_2)^{-\mathbf{n}}M_{\mathbf{m}}(\mathbf{n};U(g_1g_2),\sigma) = \sum_{\mathbf{k}\in\mathbb{N}_0^d}(-1)^{|\mathbf{k}|}\mathbf{p}(g_1)^{\mathbf{k}}\tilde{\mathbf{p}}(g_2)^{\mathbf{k}}\frac{(\sigma)_{|\mathbf{k}|}}{\mathbf{k}!}M_{\mathbf{m}}(\mathbf{k};U(g_1),\sigma)M_{\mathbf{k}}(\mathbf{n};U(g_2),\sigma),$$

where $g_1, g_2 \in \mathrm{SU}(1,d)$ *such that* g_1, g_2, g_1g_2 *satisfy conditions as in Theorem 3.1.*

Proof. The integral representation follows directly from Theorem 3.1 and (3.4). The duality property follows from unitarity of π^σ, which implies that $\pi^\sigma_{\mathbf{m},\mathbf{n}}(g) = \overline{\pi^\sigma_{\mathbf{n},\mathbf{m}}(g^{-1})}$, and from $g^{-1} = \left(\begin{smallmatrix}\overline{a} & -\mathbf{c}^\dagger \\ -\overline{\mathbf{b}} & D^\dagger\end{smallmatrix}\right)$, see (2.2). The identity $\pi^\sigma(g_1)\pi^\sigma(g_2) = \pi^\sigma(g_1g_2)$ leads to

$$\pi^\sigma_{\mathbf{m},\mathbf{n}}(g_1g_2) = \sum_{\mathbf{k}\in\mathbb{N}_0^d}\pi^\sigma_{\mathbf{m},\mathbf{k}}(g_1)\pi^\sigma_{\mathbf{k},\mathbf{n}}(g_2).$$

Writing this in terms of the Meixner polynomials gives the sum identity. Taking $g_2^{-1} = g_1 = g$ and using $\pi^\sigma_{\mathbf{m},\mathbf{n}}(g) = \overline{\pi^\sigma_{\mathbf{n},\mathbf{m}}(g^{-1})}$ we obtain the orthogonality relation

$$\delta_{\mathbf{m},\mathbf{n}} = \sum_{\mathbf{k}\in\mathbb{N}_d}\pi^\sigma_{\mathbf{m},\mathbf{k}}(g)\overline{\pi^\sigma_{\mathbf{n},\mathbf{k}}(g)}.$$

The first orthogonality relation now follows from Theorem 3.1, and the second orthogonality relation follows from the first one and the duality property. □

Remark 3.3. Since the monomials $\{e_{\mathbf{n}}\}$ form an orthonormal basis for $\mathcal{A}_\alpha$, it follows from Parseval's identity that the set $\{\pi_{\mathbf{m},\cdot}(g) \mid \mathbf{m} \in \mathbb{N}_0^d\}$ is an orthonormal basis for $\ell^2(\mathbb{N}_0^d)$. As a consequence, the set of Meixner polynomials $\{M_{\mathbf{m}}(\,\cdot\,;U,\sigma) \mid \mathbf{m}\in\mathbb{N}_0^d\}$ is an orthogonal basis for the weighted L^2-space $\ell^2(\mathbb{N}_0^d; \frac{(\sigma)_{|\mathbf{k}|}}{\mathbf{k}!}\mathbf{p}^{\mathbf{k}}\overline{\mathbf{p}}^{\mathbf{k}})$. This gives another proof of Griffiths' [6] result on completeness of the Meixner polynomials.

Next, we obtain an identity for the Meixner polynomials which corresponds to the fact that the tensor product of several representations π^σ contains a specific $\pi^{\sigma'}$ as a subrepresentation. The case $N = 2$ in the following theorem corresponds to the Runge-type identity in [6].

Theorem 3.4. *Let* $N \in \mathbb{N}_{\geq 2}$, $\sigma_1, \ldots, \sigma_N \in \mathbb{N}_{\geq d+1}$ *and* $\sigma = \sum_{i=1}^N \sigma_i$. *Define the linear map* $\Lambda : \mathcal{A}_{\sigma_1-d-1}\otimes\cdots\otimes\mathcal{A}_{\sigma_N-d-1} \to \mathcal{A}_{\sigma-d-1}$ *on basis elements by*

$$\Lambda(e_{\mathbf{m}_1}\otimes\cdots\otimes e_{\mathbf{m}_N}) = C_{\mathbf{m}_1,\ldots,\mathbf{m}_N}e_{\mathbf{m}_1+\cdots+\mathbf{m}_N}, \qquad \mathbf{m}_1,\ldots,\mathbf{m}_N\in\mathbb{N}_0^d,$$

with

$$C_{\mathbf{m}_1,\ldots,\mathbf{m}_N} = \sqrt{\frac{(\mathbf{m}_1+\cdots+\mathbf{m}_N)!}{(\sigma)_{|\mathbf{m}_1+\cdots+\mathbf{m}_N|}}\prod_{i=1}^N\frac{(\sigma_i)_{|\mathbf{m}_i|}}{\mathbf{m}_i!}},$$

then Λ *intertwines* $\pi^{\sigma_1}\otimes\cdots\otimes\pi^{\sigma_N}$ *with* π^σ. *As a consequence, the Meixner polynomials satisfy*

$$\frac{(\sigma)_{|\mathbf{m}|}}{\mathbf{m}!}M_{\mathbf{m}}(\mathbf{n};U,\sigma) = \sum_{\substack{\mathbf{m}_1,\ldots,\mathbf{m}_N\in\mathbb{N}_0^d\\ \mathbf{m}_1+\cdots+\mathbf{m}_N=\mathbf{m}}}\prod_{i=1}^N\frac{(\sigma_i)_{|\mathbf{m}_i|}}{\mathbf{m}_i!}M_{\mathbf{m}_i}(\mathbf{n}_i;U,\sigma_i),$$

where $\mathbf{m}, \mathbf{n}_1, \ldots, \mathbf{n}_N \in \mathbb{N}_0^d$ *and* $\mathbf{n} = \sum_{i=1}^N \mathbf{n}_i$.

Proof. We write $e^{\sigma}_{\mathbf{m}}$ for a basis vector of $\mathcal{A}_{\sigma-d-1}$. Using $\mathbf{z}^{\mathbf{m}_1}\cdots\mathbf{z}^{\mathbf{m}_N} = \mathbf{z}^{\mathbf{m}}$, with $\mathbf{m} = \mathbf{m}_1+\cdots+\mathbf{m}_N$, we have

$$\prod_{i=1}^{N} e^{\sigma_i}_{\mathbf{m}_i}(\mathbf{z}) = C_{\mathbf{m}_1,\ldots,\mathbf{m}_N} e^{\sigma}_{\mathbf{m}}(\mathbf{z}).$$

Let $\mathbf{n}_1,\ldots,\mathbf{n}_N \in \mathbb{N}_0^d$ such that $\sum_{i=1}^N \mathbf{n}_i = \mathbf{n}$. Then, using the expansion $\pi^{\sigma_i}(g)e_{\mathbf{n}_i} = \sum_{\mathbf{m}_i} \pi^{\sigma_i}_{\mathbf{m}_i,\mathbf{n}_i}(g)e_{\mathbf{m}_i}$, we find

$$\begin{aligned}
\Lambda\Big(\pi^{\sigma_1}(g)e^{\sigma_1}_{\mathbf{n}_1}\otimes\cdots\otimes\pi^{\sigma_N}(g)e^{\sigma_N}_{\mathbf{n}_N}\Big)(\mathbf{z})&\\
&= \sum_{\mathbf{m}_1,\ldots,\mathbf{m}_N\in\mathbb{N}_0^d} \pi^{\sigma_1}_{\mathbf{m}_1,\mathbf{n}_1}(g)\cdots\pi^{\sigma_N}_{\mathbf{m}_N,\mathbf{n}_N}(g) C_{\mathbf{m}_1,\ldots,\mathbf{m}_N} e^{\sigma}_{\mathbf{m}_1+\cdots+\mathbf{m}_N}(\mathbf{z})\\
&= \prod_{i=1}^{N}\sum_{\mathbf{m}_i\in\mathbb{N}_0^d} \pi^{\sigma_i}_{\mathbf{m}_i,\mathbf{n}_i}(g)e^{\sigma_i}_{\mathbf{m}_i}(\mathbf{z})\\
&= \prod_{i=1}^{N}\pi^{\sigma_i}(g)e_{\mathbf{n}_i}(\mathbf{z})\\
&= C_{\mathbf{n}_1,\ldots,\mathbf{n}_N}\pi^{\sigma}(g)e^{\sigma}_{\mathbf{n}}(\mathbf{z}),
\end{aligned}$$

where the last step follows from the obvious identity

$$\prod_{i=1}^{N}(a+\sum_l c_l z_l)^{-\sigma_i-|\mathbf{n}_i|}\prod_k(b_k+\sum_l D_{l,k}z_l)^{n_{i,k}} = (a+\sum_l c_l z_l)^{-\sigma-|\mathbf{n}|}\prod_k(b_k+\sum_l D_{l,k}z_l)^{\mathbf{n}_k}.$$

This proves the intertwining property of Λ. Taking the inner product with $e_{\mathbf{m}}$ shows that

$$\sum_{\substack{\mathbf{m}_1,\ldots,\mathbf{m}_N\in\mathbb{N}_0^d\\ \mathbf{m}_1+\cdots+\mathbf{m}_N=\mathbf{m}}} C_{\mathbf{m}_1,\ldots,\mathbf{m}_N}\pi^{\sigma_1}_{\mathbf{m}_1,\mathbf{n}_1}(g)\cdots\pi^{\sigma_N}_{\mathbf{m}_N,\mathbf{n}_N}(g) = C_{\mathbf{n}_1,\ldots,\mathbf{n}_N}\pi^{\sigma}_{\mathbf{m},\mathbf{n}}(g),$$

and then the stated identity for Meixner polynomials follows from Theorem 3.1. □

We conclude this section by having a closer look at the parameter matrix U of the multivariate Meixner polynomials. The fact that U comes from a matrix $g \in \mathrm{SU}(1,d)$ imposes conditions on U that ensure orthogonality of the Meixner polynomials with respect to a positive weight function, namely the weight of the negative binomial distribution, see Theorem 3.2. Let us compare the conditions on U with the conditions on U given in [6,8]. The Meixner polynomials as defined by Iliev [8], and similarly the ones defined by Griffiths [6] after an appropriate change of parameters, depend on a parameter β and on the following parameters: $\mathbf{c} = (c_1,\ldots,c_d)$, $\tilde{\mathbf{c}} = (\tilde{c}_1,\ldots,\tilde{c}_d) \in \mathbb{C}^d$ and $U = (U_{ij}) \in \mathbb{M}_d$. These parameters relate to each other as follows: let $C = \mathrm{diag}(1,-\mathbf{c})$, $\widetilde{C} = \mathrm{diag}(1,-\tilde{\mathbf{c}})$, $\hat{U} = \left(\begin{smallmatrix}1 & \mathbf{1}^t\\ \mathbf{1} & U\end{smallmatrix}\right) \in \mathbb{M}_{d+1}$ with $\mathbf{1}$ the vector in $\mathbb{C}^d$ with every entry equal to 1, then

$$\hat{U}^t C \hat{U}\widetilde{C} = c_0 I_{d+1}, \tag{3.5}$$

with $c_0 = 1-|\mathbf{c}|$. In other words, given $\mathbf{c}$ and U, the vector $\tilde{\mathbf{c}}$ is determined by (3.5).

In our approach, the matrix U together with the matrices $C := \mathrm{diag}(1,-|p_1|^2,\ldots,-|p_d|^2)$ and $\widetilde{C} := \mathrm{diag}(1,-|\tilde{p}_1|^2,\ldots,-|\tilde{p}_d|^2)$ are obtained from a matrix $g = \left(\begin{smallmatrix}a & \mathbf{b}^t\\ \mathbf{c} & D\end{smallmatrix}\right) \in \mathrm{SU}(1,d)$ as follows: $g = a\widetilde{P}\hat{U}^t P$ with $P = \mathrm{diag}(1, \mathbf{p}) = \mathrm{diag}(1, \frac{\mathbf{b}}{a})$, $\widetilde{P} = \mathrm{diag}(1, \tilde{\mathbf{p}}) = \mathrm{diag}(1, \frac{\mathbf{c}}{a})$ and $\hat{U} = \left(\begin{smallmatrix} 1 & \mathbf{1}^t \\ \mathbf{1} & U \end{smallmatrix}\right)$ with $U_{i,j} = \frac{aD_{j,i}}{b_i c_j}$. Then $C = P^\dagger J P$ and $\widetilde{C} = \widetilde{P}^\dagger J \widetilde{P}$, and the condition $g^\dagger J g = J$ is equivalent to

$$\hat{U}^\dagger C \hat{U} \widetilde{C} = |p_0|^2 I_{d+1},$$

with $|p_0|^2 = |a|^{-2} = 1 - \sum_i |p_i|^2$. In particular, the only difference with condition (3.5) is the use of the complex transpose of $\hat{U}$ instead of just the transpose. This means that, given $\mathbf{c}$ and U, the vector $\tilde{\mathbf{c}}$ is defined in a slightly different way. Because of this small difference, the orthogonality relations obtained in this paper, and also the difference equations in the next section, look slightly different from the results from [8] even though they are the same results. Note that in [8] the parameters $\beta, c_1, \dots, c_d$ are allowed to be non-real, thus allowing a non-real-valued weight function, whereas these parameters are real-valued in our setting.

4. The Lie algebra $\mathfrak{su}(1, d)$ and multivariate meixner polynomials

The Lie algebra of SU$(1, d)$ consists of matrices $X \in \mathbb{M}_{d+1}$ with trace zero such that $X^\dagger J = -JX$, where (recall) $J = \mathrm{diag}(1, -1, \dots, -1)$. We denote by $\mathfrak{su}(1, d)$ the complexification of the Lie algebra of SU$(1, d)$, i.e. $\mathfrak{sl}(d + 1, \mathbb{C})$, equipped with the $*$-structure defined by

$$X^* = JX^\dagger J.$$

A basis of $\mathfrak{su}(1, d)$ is given by

$$\mathcal{B} = \Big\{E_{i,j} \mid i, j = 0, \dots, d,\ i \neq j\Big\} \cup \Big\{H_i = E_{i,i} - \frac{1}{d+1} I_{d+1} \mid i = 1, \dots, d\Big\}, \tag{4.1}$$

where $E_{i,j}$ denotes the matrix unit with (i, j)-entry 1 and all other entries 0. Note that

$$\begin{aligned} H_i^* &= H_i, && i = 1, \dots, d,\\ E_{i,0}^* &= -E_{0,i}, && i = 1, \dots, d,\\ E_{i,j}^* &= E_{j,i}, && 1 \leq i, j \leq d,\ i \neq j. \end{aligned} \tag{4.2}$$

The unitary representation π^σ of SU$(1, d)$ on $\mathcal{A}_\alpha$ gives rise to an unbounded $*$-representation of $\mathfrak{su}(1, d)$ on $\mathcal{A}_\alpha$ that we also denote by π^σ. As a dense domain we choose the set of polynomials on $\mathbb{B}_d$. On the basis of monomials $\{e_\mathbf{n} \mid \mathbf{n} \in \mathbb{N}_0^d\}$, the basis of $\mathfrak{su}(1, d)$ acts as follows.

Lemma 4.1. *For* $\mathbf{n} \in \mathbb{N}_0^d$,

$$\begin{aligned} \pi^\sigma(H_i)e_\mathbf{n} &= \left(\frac{\sigma}{d+1} + n_i\right) e_\mathbf{n}, && i = 1, \dots, d,\\ \pi^\sigma(E_{0,j})e_\mathbf{n} &= \sqrt{(\sigma + |\mathbf{n}| - 1)n_j}\, e_{\mathbf{n}-\mathbf{v}_j}, && j = 1, \dots, d,\\ \pi^\sigma(E_{i,0})e_\mathbf{n} &= -\sqrt{(n_i + 1)(\sigma + |\mathbf{n}|)}\, e_{\mathbf{n}+\mathbf{v}_i}, && i = 1, \dots, d,\\ \pi^\sigma(E_{i,j})e_\mathbf{n} &= \sqrt{(n_i + 1)n_j}\, e_{\mathbf{n}+\mathbf{v}_i-\mathbf{v}_j}, && 1 \leq i, j \leq d,\ i \neq j, \end{aligned}$$

where $\mathbf{v}_i$ *is the standard basis vector of* $\mathbb{C}^{d+1}$ *with* j^{th} *entry 1 and the other entries are 0, and we use the convention* $e_\mathbf{n} = 0$ *if* $n_i = -1$ *for some* $1 \leq i \leq d$.

Proof. This follows directly from computing

$$\frac{d}{dt}\Big|_{t=0} \pi^\sigma(\exp(tX))e_\mathbf{n}(\mathbf{z}), \qquad X \in \mathcal{B}. \quad \square$$

We fix a $g \in \mathrm{SU}(1,d)$ as in Theorem 3.1, i.e.

$$g = \begin{pmatrix} a & \mathbf{b}^t \\ \mathbf{c} & D \end{pmatrix}, \qquad \text{with } a, b_i, c_i \neq 0 \text{ for } i = 1, \dots, d.$$

Furthermore, let $U, \mathbf{p}, \tilde{\mathbf{p}}$ be as in Theorem 3.1. With this g, we define a new basis of $\mathcal{A}_\alpha$ by

$$\tilde{e}_{\mathbf{n}} = \pi^\sigma(g) e_{\mathbf{n}}, \qquad \mathbf{n} \in \mathbb{N}_0^d,$$

and define a corresponding basis of $\mathfrak{su}(1,d)$ by

$$\widetilde{\mathcal{B}} = \{\widetilde{X} = gXg^{-1} \mid X \in \mathcal{B}\},$$

where $\mathcal{B}$ is the basis given in (4.1). It immediately follows that the action of $\widetilde{\mathcal{B}}$ on $\{\tilde{e}_{\mathbf{n}}\}$ is given by

$$\begin{aligned} \pi^\sigma(\widetilde{H}_i)\tilde{e}_{\mathbf{n}} &= \left(\frac{\sigma}{d+1} + n_i\right)\tilde{e}_{\mathbf{n}}, && i = 1, \dots, d, \\ \pi^\sigma(\widetilde{E}_{0,j})\tilde{e}_{\mathbf{n}} &= \sqrt{(\sigma + |\mathbf{n}| - 1)n_j}\,\tilde{e}_{\mathbf{n}-\mathbf{v}_j}, && j = 1, \dots, d, \\ \pi^\sigma(\widetilde{E}_{i,0})\tilde{e}_{\mathbf{n}} &= -\sqrt{(n_i+1)(\sigma + |\mathbf{n}|)}\,\tilde{e}_{\mathbf{n}+\mathbf{v}_i}, && i = 1, \dots, d, \\ \pi^\sigma(\widetilde{E}_{i,j})\tilde{e}_{\mathbf{n}} &= \sqrt{(n_i+1)n_j}\,\tilde{e}_{\mathbf{n}+\mathbf{v}_i-\mathbf{v}_j}, && i, j = 1, \dots, d,\ i \neq j. \end{aligned} \tag{4.3}$$

We will use the representation of $\mathfrak{su}(1,d)$ to derive the difference equations for the Meixner polynomials from [8, Theorem 4.1]. First, we need a few preliminary results.

Lemma 4.2.

(i) For $X \in \mathcal{B}$, we have $(\widetilde{X})^* = \widetilde{X^*}$, i.e.

$$(\widetilde{H}_i)^* = \widetilde{H}_i, \quad (\widetilde{E}_{0,i})^* = -\widetilde{E}_{i,0}, \quad (\widetilde{E}_{i,j})^* = \widetilde{E}_{j,i}.$$

for $i, j = 1, \dots, d$, $i \neq j$.

(ii) For $k, l = 1, \dots, d$, $k \neq l$,

$$\begin{aligned} \widetilde{H}_k &= \sum_{i=1}^d (|D_{i,k}|^2 + |b_k|^2)H_i + \sum_{\substack{i,j=1 \\ i\neq j}}^d D_{i,k}\overline{D_{j,k}}E_{i,j} + b_k \sum_{j=1}^d \overline{D_{j,k}}E_{0,j} - \overline{b_k}\sum_{i=1}^d D_{i,k}E_{i,0}, \\ \widetilde{E}_{k,l} &= \sum_{i=1}^d (D_{i,k}\overline{D_{i,l}} + b_k\overline{b_l})H_i + \sum_{\substack{i,j=1 \\ i\neq j}}^d D_{i,k}\overline{D_{j,l}}E_{i,j} + b_k \sum_{j=1}^d \overline{D_{j,l}}E_{0,j} - \overline{b_l}\sum_{i=1}^d D_{i,k}E_{i,0}, \\ \widetilde{E}_{0,l} &= \sum_{i=1}^d (c_i\overline{D_{i,l}} + a\overline{b_l})H_i + \sum_{\substack{i,j=1 \\ i\neq j}}^d c_i\overline{D_{j,l}}E_{i,j} + a \sum_{j=1}^d \overline{D_{j,l}}E_{0,j} - \overline{b_l}\sum_{i=1}^d c_iE_{i,0}, \\ \widetilde{E}_{k,0} &= \sum_{i=1}^d (D_{i,k}\overline{c_i} + b_k\overline{a})H_i + \sum_{\substack{i,j=1 \\ i\neq j}}^d D_{i,k}\overline{c_j}E_{i,j} + b_k \sum_{j=1}^d \overline{c_j}E_{0,j} - \overline{a}\sum_{i=1}^d D_{i,k}E_{i,0}. \end{aligned}$$

Proof. For the first statement we use $\widetilde{X} = gXg^{-1}$, $J^\dagger = J$ and $I = J^2$, to obtain

$$(\widetilde{X})^* = J(\widetilde{X})^\dagger J = J(gXg^{-1})^\dagger J = (Jg^{-1}J)^\dagger(JX^\dagger J)(Jg^\dagger J).$$

Using $g^{-1} = Jg^\dagger J$ and $JX^\dagger J = X^*$, it follows that $(\widetilde{X})^* = gX^*g^{-1} = \widetilde{X^*}$.

The second statement follows from a direct calculation. For $E_{k,l}$ we have

$$\begin{aligned}\widetilde{E}_{k,l} &= \begin{pmatrix} -b_k\overline{b_l} & b_k\overline{D_{1,l}} & \cdots & b_k\overline{D_{d,l}} \\ -D_{1,k}\overline{b_l} & D_{1,k}\overline{D_{1,l}} & \cdots & D_{1,k}\overline{D_{d,l}} \\ \vdots & \vdots & \ddots & \vdots \\ -D_{d,k}\overline{b_l} & D_{d,k}\overline{D_{1,l}} & \cdots & D_{d,k}\overline{D_{d,l}} \end{pmatrix} \\ &= \sum_{i=1}^{d} x_i H_i + \sum_{\substack{i,j=1\\ i\neq j}}^{d} D_{i,k}\overline{D_{j,l}} E_{i,j} + b_k \sum_{j=1}^{d} \overline{D_{j,l}} E_{0,j} - \overline{b_l} \sum_{i=1}^{d} D_{i,k} E_{i,0},\end{aligned}$$

where the coefficients $x_i \in \mathbb{C}$ are determined by the equations

$$\begin{aligned} -\frac{1}{d+1}\sum_{j=1}^{d} x_j &= -b_k\overline{b_l}, \\ x_i - \frac{1}{d+1}\sum_{j=1}^{d} x_j &= D_{i,k}\overline{D_{i,l}}, \qquad i = 1, \dots, d. \end{aligned}$$

Note that consistency of these equations follows from the identity $b_k\overline{b_l} + \sum_{i=1}^{d} D_{i,k} D_{i,l} = 0$, see (2.3). It follows that $x_i = D_{i,k}\overline{D_{i,l}} + b_k\overline{b_l}$.

The results for $\widetilde{E}_{k,0}$ and $\widetilde{E}_{0,l}$ follow by interpreting $D_{i,0}$ as c_i, and b_0 as a. The calculation for $\widetilde{H}_k$ runs along the same lines. □

We are now in a position to derive difference equations for the Meixner polynomials from the action of the Cartan elements H_k.

Theorem 4.3. *For $k = 1, \dots, d$, the Meixner polynomials $M_{\mathbf{m}}(\mathbf{n}) = M_{\mathbf{m}}(\mathbf{n}; U, \sigma)$ satisfy*

$$\begin{aligned} \left|\frac{p_0}{p_k}\right|^2 n_k M_{\mathbf{m}}(\mathbf{n}) = \left(\sigma + |\mathbf{m}| + \sum_{i=1}^{d} |U_{k,i}\tilde{p}_i|^2 m_i\right) M_{\mathbf{m}}(\mathbf{n}) + \sum_{\substack{i,j=1\\ i\neq j}}^{d} \overline{U_{k,i}} U_{k,j} |\tilde{p}_i|^2 m_j M_{\mathbf{m}-\mathbf{v}_j+\mathbf{v}_i}(\mathbf{n}) \\ - \sum_{i=1}^{d} U_{k,i} m_i M_{\mathbf{m}-\mathbf{v}_i}(\mathbf{n}) - (\sigma + |\mathbf{m}|) \sum_{i=1}^{d} \overline{U_{k,i}} |\tilde{p}_i|^2 M_{\mathbf{m}+\mathbf{v}_i}(\mathbf{n}). \end{aligned}$$

Proof. The result follows from evaluating $\langle \pi^\sigma(\widetilde{H}_k)\tilde{e}_{\mathbf{n}}, e_{\mathbf{m}}\rangle$ in two ways.

First note that $\langle \tilde{e}_{\mathbf{n}}, e_{\mathbf{m}}\rangle = \pi_{\mathbf{m},\mathbf{n}}(g)$. From the action of $\widetilde{H}_k$ (4.3), it follows that

$$\langle \pi^\sigma(\widetilde{H}_k)\tilde{e}_{\mathbf{n}}, e_{\mathbf{m}}\rangle = \left(\frac{\sigma}{d+1} + n_k\right)\pi_{\mathbf{m},\mathbf{n}}(g).$$

On the other hand, using $\langle \pi^\sigma(X)\tilde{e}_{\mathbf{n}}, e_{\mathbf{m}}\rangle = \langle \tilde{e}_{\mathbf{n}}, \pi^\sigma(X^*)e_{\mathbf{m}}\rangle$ and Lemma 4.2, we obtain

$$\begin{aligned} \langle \pi^\sigma(\widetilde{H}_k)\tilde{e}_{\mathbf{n}}, e_{\mathbf{m}}\rangle &= \sum_{i=1}^{d} (|D_{i,k}|^2 + |b_k|^2)\Big(\frac{\sigma}{d+1} + m_i\Big)\pi_{\mathbf{m},\mathbf{n}}(g) \\ &\quad + \sum_{\substack{i,j=1\\ i\neq j}}^{d} \overline{D_{i,k}} D_{j,k} \sqrt{(m_i+1)m_j}\, \pi_{\mathbf{m}-\mathbf{v}_j+\mathbf{v}_i,\mathbf{n}}(g) \end{aligned}$$

$$+ \overline{b_k} \sum_{j=1}^{d} D_{j,k} \sqrt{(\sigma + |\mathbf{m}| - 1)m_j}\, \pi_{\mathbf{m}-\mathbf{v}_j,\mathbf{n}}(g)$$
$$+ b_k \sum_{i=1}^{d} \overline{D_{i,k}} \sqrt{(m_i + 1)(\sigma + |\mathbf{m}|)}\, \pi_{\mathbf{m}+\mathbf{v}_i,\mathbf{n}}(g).$$

So we have a difference equation for the matrix coefficients $\pi_{\mathbf{m},\mathbf{n}}(g)$. Expressing the matrix coefficients in terms of Meixner polynomials using Theorem 3.1, and simplifying the diagonal terms using the identity $\sum_{i=1}^{d} |D_{i,k}|^2 = |b_k|^2 + 1$, we obtain a difference equation for the Meixner polynomials. □

Remark 4.4. The construction of the difference equations for the multivariate Meixner polynomials in Theorem 4.3 is similar to Iliev's [7] construction of difference equations for multivariate Krawtchouk polynomials from Cartan elements of $\mathfrak{sl}(d+1,\mathbb{C})$. The difference lies in the use of the $\mathfrak{su}(1,d)$ $*$-structure, instead of the antiautomorphism $\mathfrak{a}$ from [7].

We note that we can rewrite the difference equations as

$$n_k M_{\mathbf{m}}(\mathbf{n}) = \left|\frac{p_k}{p_0}\right|^2 \sum_{\substack{i,j=1\\ i\neq j}}^{d} \overline{U_{k,i}} U_{k,j} |\tilde{p}_i|^2 m_j \Big[M_{\mathbf{m}-\mathbf{v}_j+\mathbf{v}_i}(\mathbf{n}) - M_{\mathbf{m}}(\mathbf{n}) \Big]$$
$$- \left|\frac{p_k}{p_0}\right|^2 \sum_{i=1}^{d} U_{k,i} m_i \Big[M_{\mathbf{m}-\mathbf{v}_i}(\mathbf{n}) - M_{\mathbf{m}}(\mathbf{n}) \Big]$$
$$- \left|\frac{p_k}{p_0}\right|^2 \sum_{i=1}^{d} \overline{U_{k,i}} |\tilde{p}_i|^2 (\sigma + |\mathbf{m}|) \Big[M_{\mathbf{m}+\mathbf{v}_i}(\mathbf{n}) - M_{\mathbf{m}}(\mathbf{n}) \Big].$$

Comparing this with the difference equations from [8, Theorem 4.1], we see that the result is again very similar; the difference is the occurrence of complex conjugates of appropriate parameters.

In the same way as in Theorem 4.3 we obtain 'lowering and raising' relations for the Meixner polynomials from the actions of $E_{k,l}$.

Theorem 4.5. *For $k, l = 0, \ldots, d$, $k \neq l$,*

$$\left|\tfrac{p_0}{p_l}\right|^2 n_l M_{\mathbf{m}}(\mathbf{n} + \mathbf{v}_k - \mathbf{v}_l) =$$
$$\left(\sigma + |\mathbf{m}| + \sum_{i=1}^{d} \overline{U_{l,i}} U_{k,i} |\tilde{p}_i|^2 m_i \right) M_{\mathbf{m}}(\mathbf{n}) + \sum_{\substack{i,j=1\\ i\neq j}}^{d} \overline{U_{l,i}} U_{k,j} |\tilde{p}_i|^2 m_j M_{\mathbf{m}+\mathbf{v}_i-\mathbf{v}_j}(\mathbf{n})$$
$$- \sum_{i=1}^{d} U_{k,i} m_i M_{\mathbf{m}-\mathbf{v}_i}(\mathbf{n}) - (\sigma + |\mathbf{m}|) \sum_{i=1}^{d} \overline{U_{l,i}} |\tilde{p}_i|^2 M_{\mathbf{m}+\mathbf{v}_i}(\mathbf{n}),$$

where we use the notations $U_{0,i} = 1$, $\mathbf{v}_0 = 0$ and $n_0 = -\sigma - |\mathbf{n}|$.

Using the duality property of the multivariate Meixner polynomials $M_{\mathbf{m}}(\mathbf{n})$, Theorem 3.2(iii), it follows that they also satisfy difference equations and lowering/raising relations in the variable $\mathbf{m}$.

5. Degenerate multivariate meixner polynomials

So far we considered matrix coefficients $\pi_{\mathbf{m},\mathbf{n}}(g)$ where $g = \begin{pmatrix} a & \mathbf{b}^t \\ \mathbf{c} & D \end{pmatrix} \in \mathrm{SU}(1,d)$ with $a, b_i, c_i \neq 0$. These matrix coefficients correspond to Griffith's multivariate Meixner polynomials which are associated to the matrix $\hat{U} = \begin{pmatrix} 1 & \mathbf{1}^t \\ \mathbf{1} & U \end{pmatrix} \in \mathbb{M}_{d+1}$, see also the discussion at the end of Section 3. In the present section, we consider the degenerate case in which the vectors $\mathbf{b}$ and $\mathbf{c}$ may contain elements equal to 0. For convenience we assume

$$b_{k+1} = \cdots = b_d = 0 \qquad \text{and} \qquad c_{l+1} = \cdots = c_d = 0$$

for some $k, l \in \{1, \ldots, d\}$ and the other elements of $\mathbf{b}$ and $\mathbf{c}$ are nonzero. This will correspond to multivariate Meixner polynomials associated to a matrix $\hat{U}$ with zero entries in the first row and first column. We briefly describe some properties of these polynomials, which are similar to properties of Griffiths' multivariate Meixner polynomials.

In the non-degenerate case we have an explicit expression for the matrix coefficients using the hypergeometric expression for the multivariate Meixner polynomials. By taking limits we obtain an explicit expression for $\pi^\sigma_{\mathbf{m},\mathbf{n}}(g)$.

Lemma 5.1. *The matrix coefficient $\pi^\sigma_{\mathbf{m},\mathbf{n}}(g)$ is given by*

$$\begin{aligned}
\pi^\sigma_{\mathbf{m},\mathbf{n}}(g) = {} & \sqrt{\frac{(\sigma)_{|\mathbf{m}|}(\sigma)_{|\mathbf{n}|}}{\mathbf{m}!\,\mathbf{n}!}}(-1)^{|\mathbf{m}|} a^{-\sigma-|\mathbf{m}|-|\mathbf{n}|} \prod_{i=1}^{k}\prod_{j=1}^{l} b_i^{n_i} c_j^{m_j} \\
& \times \sum_{(a_{i,j})\in\mathbb{M}_{k,l}(\mathbf{m},\mathbf{n})} \frac{\prod_{j=1}^d(-m_j)_{\sum_{i=1}^d a_{i,j}} \prod_{i=1}^d (-n_i)_{\sum_{j=1}^d a_{i,j}}}{(\sigma)_{\sum_{i,j=1}^d a_{i,j}}} \prod_{i,j=1}^d \frac{1}{a_{i,j}!} \\
& \times \prod_{j=1}^{l}\left(\prod_{i=1}^{k}\left(1-\tfrac{aD_{j,i}}{b_i c_j}\right)^{a_{i,j}} \prod_{i=k+1}^{d}\left(-\tfrac{aD_{j,i}}{c_j}\right)^{a_{i,j}}\right) \\
& \times \prod_{j=l+1}^{d}\left(\prod_{i=1}^{k}\left(-\tfrac{aD_{j,i}}{b_i}\right)^{a_{i,j}} \prod_{i=k+1}^{d}(-aD_{j,i})^{a_{i,j}}\right),
\end{aligned}$$

where

$$\mathbb{M}_{k,l}(\mathbf{m},\mathbf{n}) = \Bigg\{ (a_{i,j}) \in \mathbb{M}_d(\mathbb{N}_0) : \sum_{j=1}^d a_{i,j} = n_i \text{ for } i = k+1,\ldots,d \quad \text{and} \quad \sum_{i=1}^d a_{i,j} = m_j \text{ for } j = l+1,\ldots,d \Bigg\}.$$

Proof. First, we write out the Meixner polynomial $M_{\mathbf{m}}(\mathbf{n}; U, \sigma)$ from Theorem 3.1, i.e. in the non-degenerate case, as a hypergeometric series using (3.2), which is a sum labeled by $(a_{i,j}) \in \mathbb{M}_d(\mathbb{N}_0)$ of the form

$$\pi^\sigma_{\mathbf{m},\mathbf{n}}(g) = C\left(\prod_{i,j=1}^d b_i^{n_i} c_j^{m_j}\right) \sum_{(a_{i,j})} B_{a_{i,j}} \prod_{i,j=1}^d \left(1 - \frac{aD_{j,i}}{b_i c_j}\right)^{a_{i,j}},$$

where C and $B_{a_{i,j}}$ are independent of $\mathbf{b}$ and $\mathbf{c}$. The factor $B_{a_{i,j}}$ contains a term $(-n_i)_{\sum_{j=1}^d a_{i,j}}$, which equals 0 if $\sum_{j=1}^d a_{i,j} > n_i$, so that the sum is a finite sum labeled by $(a_{i,j}) \in \mathbb{M}_d(\mathbb{N}_0)$

with $\sum_{j=1}^{d} a_{i,j} \leq n_i$ for $1 \leq i \leq d$. For taking the limit $b_i \to 0$ we use

$$\lim_{b_i \to 0} b_i^{n_i} \prod_{j=1}^{d} \left(1 - \frac{aD_{j,i}}{b_i c_j}\right)^{a_{i,j}} = \begin{cases} \prod_{j=1}^{d} \left(-\frac{aD_{j,i}}{c_j}\right)^{a_{i,j}}, & \text{if } \sum_{j=1}^{d} a_{ij} = n_i, \\ 0, & \text{if } \sum_{j=1}^{d} a_{ij} < n_i. \end{cases}$$

Applying this for $i = k+1, \ldots, d$, shows that the sum over matrices $(a_{i,j}) \in \mathbb{M}_d(\mathbb{N}_0)$ reduces to a sum over matrices for which the elements of the i^{th} row sum to n_i, of the form

$$C' \left(\prod_{j=1}^{d} c_j^{m_j}\right) \sum_{(a_{i,j})} B_{a_{i,j}} \prod_{j=1}^{d} \prod_{i=1}^{k} \left(1 - \frac{aD_{j,i}}{b_i c_j}\right)^{a_{i,j}} \prod_{i=k+1}^{d} \left(-\frac{aD_{j,i}}{c_j}\right)^{a_{i,j}},$$

where C' is independent of $\mathbf{c}$. $B_{a_{i,j}}$ also contains a term $(-m_j)_{\sum_{i=1}^{d} a_{i,j}}$, which equals 0 for $\sum_{i=1}^{d} a_{i,j} > m_j$. Furthermore, we have

$$\begin{aligned} &\lim_{c_j \to 0} c_j^{m_j} \prod_{i=1}^{k} \left(1 - \frac{aD_{j,i}}{b_i c_j}\right)^{a_{i,j}} \prod_{i=k+1}^{d} \left(-\frac{aD_{j,i}}{c_j}\right)^{a_{i,j}} \\ &\qquad = \begin{cases} \prod_{i=1}^{k} \left(-\frac{aD_{j,i}}{b_i}\right)^{a_{i,j}} \prod_{i=k+1}^{d} (-aD_{j,i})^{a_{i,j}}, & \text{if } \sum_{i=1}^{d} a_{i,j} = m_j, \\ 0, & \text{if } \sum_{i=1}^{d} a_{i,j} < m_j. \end{cases} \end{aligned}$$

Applying this for $j = l+1, \ldots, d$, we are left with a sum over matrices $(a_{i,j})$ for which the elements of the jth column sum to m_j, of the form

$$\begin{aligned} C'' \sum_{(a_{i,j})} B_{a_{i,j}} \prod_{j=1}^{l} &\left(\prod_{i=1}^{k} \left(1 - \tfrac{aD_{j,i}}{b_i c_j}\right)^{a_{i,j}} \prod_{i=k+1}^{d} \left(-\tfrac{aD_{j,i}}{c_j}\right)^{a_{i,j}} \right) \\ &\times \prod_{j=l+1}^{d} \left(\prod_{i=1}^{k} \left(-\tfrac{aD_{j,i}}{b_i}\right)^{a_{i,j}} \prod_{i=k+1}^{d} (-aD_{j,i})^{a_{i,j}} \right). \end{aligned}$$

Writing out C'' and $B_{a_{i,j}}$ explicitly gives the result. □

For $\sigma > 0$ and $U \in \mathbb{M}_d$, we define the degenerate multivariate Meixner polynomials $\hat{M}_{\mathbf{m}}(\mathbf{n}; U, \sigma)$ by

$$\begin{aligned} \hat{M}_{\mathbf{m}}(\mathbf{n}; U, \sigma) = \sum_{(a_{i,j}) \in \mathbb{M}_{k,l}(\mathbf{m},\mathbf{n})} &\frac{\prod_{j=1}^{d} (-m_j)_{\sum_{i=1}^{d} a_{i,j}} \prod_{i=1}^{d} (-n_i)_{\sum_{j=1}^{d} a_{i,j}}}{(\sigma)_{\sum_{i,j=1}^{d} a_{i,j}}} \prod_{i,j=1}^{d} \frac{1}{a_{i,j}!} \\ &\times \prod_{\substack{1 \leq i \leq k \\ 1 \leq j \leq l}} \left(1 - U_{i,j}\right)^{a_{i,j}} \prod_{\substack{1 \leq i,j \leq d \\ i \geq k+1 \text{ or } j \geq l+1}} \left(-U_{i,j}\right)^{a_{i,j}}. \end{aligned}$$

Define, similar as in Theorem 3.1,

$$\begin{aligned} p_i &= \frac{b_i}{a}, \quad i = 1, \ldots, k, \\ \tilde{p}_j &= \frac{c_j}{a}, \quad j = 1, \ldots, l, \end{aligned}$$

and $U \in \mathbb{M}_d$ by

$$U_{i,j} = \begin{cases} \frac{aD_{j,i}}{b_i c_j} & 1 \le i \le k,\ 1 \le j \le l, \\ \frac{aD_{j,i}}{b_i} & 1 \le i \le k,\ l+1 \le j \le d, \\ \frac{aD_{j,i}}{c_j} & k+1 \le i \le d,\ 1 \le j \le l, \\ aD_{j,i} & k+1 \le i \le d,\ l+1 \le j \le d. \end{cases}$$

It is convenient to define $\mathbf{p}, \tilde{\mathbf{p}} \in \mathbb{C}^d$ by

$$\mathbf{p} = (p_1, \ldots, p_k, \tfrac{1}{a}, \ldots, \tfrac{1}{a}), \qquad \tilde{\mathbf{p}} = (\tilde{p}_1, \ldots, \tilde{p}_l, \tfrac{1}{a}, \ldots, \tfrac{1}{a}).$$

Note that we now have $D_{j,i} = ap_i \tilde{p}_j U_{i,j}$ for $1 \le i, j \le d$. Then, it follows from Lemma 5.1 that $\pi^\sigma_{\mathbf{m},\mathbf{n}}(g)$ is a multiple of $\hat{M}_\mathbf{m}(\mathbf{n}; U, \sigma)$,

$$\pi^\sigma_{\mathbf{m},\mathbf{n}}(g) = \sqrt{\frac{(\sigma)_{|\mathbf{m}|}(\sigma)_{|\mathbf{n}|}}{\mathbf{m}!\,\mathbf{n}!}}(-1)^{|\mathbf{m}|}a^{-\sigma}\tilde{\mathbf{p}}^\mathbf{m}\mathbf{p}^\mathbf{n}\hat{M}_\mathbf{m}(\mathbf{n}; U, \sigma).$$

Properties of the matrix coefficients can easily be translated to properties of the degenerate Meixner polynomials, similar as in the previous sections. We will state the orthogonality relations, the generating function, the duality property and difference equations, and leave the other properties to the interested reader. First we need to introduce some notations, similarly to the notations used at the end of Section 3. Given $U \in \mathbb{M}_d$ and $p_i, p_j \in \mathbb{C}$ for $i = 1, \ldots, k$, $j = 1, \ldots, l$, such that $1 - \sum_{i=1}^k |p_i|^2 = 1 - \sum_{i=1}^l |\tilde{p}_i|^2$, we denote

$$\hat{U} = \begin{pmatrix} 1 & \mathbf{1}_l^t \\ \mathbf{1}_k & U \end{pmatrix},$$

where $\mathbf{1}_k$ is the vector $\mathbf{u}$ with $u_i = 1$ for $i = 1, \ldots, k$ and $u_i = 0$ for $i = k+1, \ldots, d$, and

$$C = \operatorname{diag}(1, -|p_1|^2, \ldots, -|p_k|^2, -|p_0|^2, \ldots, -|p_0|^2),$$
$$\widetilde{C} = \operatorname{diag}(1, -|\tilde{p}_1|^2, \ldots, -|\tilde{p}_l|^2, -|p_0|^2, \ldots, -|p_0|^2),$$

where $|p_0|^2 = 1 - \sum_{i=1}^k |p_i|^2$.

Theorem 5.2. *Let $\sigma \in \mathbb{N}_{\ge d+1}$, $p_i, \tilde{p}_j \in \mathbb{C}$ for $i = 1, \ldots, k$, $j = 1, \ldots, l$ and $U \in \mathbb{M}_d$ such that $\hat{U}^\dagger C \hat{U} \widetilde{C} = |p_0|^2 I_{d+1}$. The polynomials $\hat{M}_\mathbf{m}(\mathbf{n}) = \hat{M}_\mathbf{m}(\mathbf{n}; U, \sigma)$ have the following properties:*

(i) Orthogonality relations:

$$\sum_{\mathbf{n} \in \mathbb{N}_0^d} \frac{(\sigma)_{|\mathbf{n}|}}{\mathbf{n}!}\mathbf{p}^\mathbf{n}\overline{\mathbf{p}}^\mathbf{n}\,\hat{M}_\mathbf{m}(\mathbf{n})\overline{\hat{M}_{\mathbf{m}'}(\mathbf{n})} = \delta_{\mathbf{m},\mathbf{m}'}\frac{\mathbf{m}!\tilde{\mathbf{p}}^{-\mathbf{m}}\overline{\tilde{\mathbf{p}}}^{-\mathbf{m}}}{(\sigma)_{|\mathbf{m}|}|p_0|^{2\sigma}},$$

$$\sum_{\mathbf{m} \in \mathbb{N}_0^d} \frac{(\sigma)_{|\mathbf{m}|}}{\mathbf{m}!}\tilde{\mathbf{p}}^\mathbf{m}\overline{\tilde{\mathbf{p}}}^\mathbf{m}\,\hat{M}_\mathbf{m}(\mathbf{n})\overline{\hat{M}_\mathbf{m}(\mathbf{n}')} = \delta_{\mathbf{n},\mathbf{n}'}\frac{\mathbf{n}!\tilde{\mathbf{p}}^{-\mathbf{n}}\overline{\tilde{\mathbf{p}}}^{-\mathbf{n}}}{(\sigma)_{|\mathbf{n}|}|p_0|^{2\sigma}}.$$

(ii) Generating function:

$$\sum_{\mathbf{m}\in\mathbb{N}_0^d} \frac{(\sigma)_{|\mathbf{m}|}}{\mathbf{m}!} \hat{M}_{\mathbf{m}}(\mathbf{n}; U, \sigma)\mathbf{t}^{\mathbf{m}}$$

$$= \left(1 - \sum_{j=1}^{l} t_j\right)^{-\sigma-|\mathbf{n}|} \prod_{i=1}^{k} \left(1 - \sum_{j=1}^{d} U_{i,j} t_j\right)^{n_i} \prod_{i=k+1}^{d} \left(-\sum_{j=1}^{d} U_{i,j} t_j\right)^{n_i}.$$

(iii) Duality: $\hat{M}_{\mathbf{m}}(\mathbf{n}; U, \sigma) = \hat{M}_{\mathbf{n}}(\mathbf{m}; U^t, \sigma)$.

(iv) Difference equations: for $s = 1, \dots, d$,

$$\left|\tfrac{p_0}{p_s}\right|^2 n_s \hat{M}_{\mathbf{m}}(\mathbf{n}) =$$

$$\left(\chi_k(s)(\sigma + |\mathbf{m}|) + \sum_{i=1}^{d} |U_{s,i}\tilde{p}_i|^2 m_i\right) \hat{M}_{\mathbf{m}}(\mathbf{n}) + \sum_{\substack{i,j=1\\ i\neq j}}^{d} \overline{U_{s,i}} U_{s,j} |\tilde{p}_i|^2 m_j M_{\mathbf{m}-\mathbf{v}_j+\mathbf{v}_i}(\mathbf{n})$$

$$- \chi_k(s) \left(\sum_{i=1}^{d} U_{s,i} m_i \hat{M}_{\mathbf{m}-\mathbf{v}_i}(\mathbf{n}) + (\sigma + |\mathbf{m}|) \sum_{i=1}^{d} \overline{U_{s,i}} |\tilde{p}_i|^2 \hat{M}_{\mathbf{m}+\mathbf{v}_i}(\mathbf{n})\right).$$

where

$$\chi_k(s) = \begin{cases} 1, & s \le k, \\ 0, & s \ge k+1. \end{cases}$$

Proof. The orthogonality relations are orthogonality relations for the matrix coefficients $\pi_{\mathbf{m},\mathbf{n}}(g)$. In this case $g = a\widetilde{P}\hat{U}^t P$ with $\widetilde{P} = \mathrm{diag}(1, \tilde{\mathbf{p}})$ and $P = \mathrm{diag}(1, \mathbf{p})$, where p_i, $\tilde{p}_i$ and $U_{i,j}$ are related to g as described above. Note that $\tilde{C} = \widetilde{P}^\dagger J \widetilde{P}$ and $C = P^\dagger J P$. The identity $g^\dagger J g = J$ then leads to the condition $\hat{U}^\dagger C \hat{U} \widetilde{C} = |p_0|^2 I_{d+1}$. The generating function follows from writing out

$$\pi^\sigma(g) e_{\mathbf{n}}(\mathbf{z}) = \sum_{\mathbf{m}} \pi^\sigma_{\mathbf{m},\mathbf{n}}(g) e_{\mathbf{m}}(\mathbf{z})$$

in terms of the degenerate Meixner polynomials and setting

$$t_i = \begin{cases} -\frac{c_i z_i}{a}, & \text{for } i = 1, \dots, l, \\ -\frac{z_i}{a}, & \text{for } i = l+1, \dots, d. \end{cases}$$

The duality property and the difference equations are obtained in the same way as in Theorems 3.2 and 4.3. □

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