



STATIC SHEAR BUCKLING OF CYLINDRICAL LIQUID TANKS:

An Analytical Nonlinear Elastic Approach

Thomas Karagiannis



MASTER THESIS REPORT

STATIC SHEAR BUCKLING OF CYLINDRICAL LIQUID TANKS: An Analytical Nonlinear Elastic Approach

by

Thomas P. Karagiannis

(4615719)

Master of Science
in Civil Engineering

Thesis committee: Chairman:	Prof. dr. ir. A. V. Metrikine	TU Delft, Faculty of Civil Engineering and Geosciences (CEG)
Committee Member:	Prof. ir. R. Nijse,	TU Delft, Faculty of Civil Engineering and Geosciences (CEG)
		&
		Faculty of Architecture and the Built Environment (Arch)
	Dr. ir. A. Tsouvalas,	TU Delft, Faculty of Civil Engineering and Geosciences (CEG)

"A man is like a fraction whose numerator is what he is and whose denominator is what he thinks of himself. The larger the denominator, the smaller the fraction."

Leo Tolstoy

Executive Summary

The usage of tanks has been quite common throughout the industry during the last decades. They fall into the category of shell structures and this is how they need to be studied. Shells, in general, due to their structural advantages and versatility can be seen in a large number of applications that extend from aerospace to architecture and civil engineering. Especially, tanks can take different geometrical shapes and often contain various liquids, such as water, oil, gas and other substances which in case of failure may result into environmental and financial disasters.

The focus of this study is concentrated on ground based thin cylindrical tanks filled with water made out of steel. Thin shells are susceptible to buckling, a phenomenon that takes several forms and calls for extra attention. The one which is investigated is shear buckling through the application of a static horizontal ground earthquake force on a clamped at the top tank. Such a problem is quite common nowadays to be tackled by engineers with the aid of *FEM* programs for practical purposes, something which on the other hand deprives the engineer of getting a deep understanding and insight of the phenomenon. For that reason, an analytic approach will be followed, in which shear buckling is studied within the boundaries of elasticity for a nonlinear shell theory in order, in that manner, to simulate reality in the best possible way.

The first chapter is devoted to help the reader understand the basic aspects of buckling found in the literature, which are vital for someone who desires to investigate buckling in general. At the very start, the terms of equilibrium, stability, instability and metastability are explained extensively. Then, buckling is defined as that phenomenon in which, the gradual increase of the loads acting on the system result after a certain point into a sudden change in geometry, as the structure transitions from one equilibrium state to another in an effort to withstand the forces with the expense a reduction in their stiffness. The load at that critical point is called buckling or bifurcation load. Also, the difference between elastic and plastic buckling are described as well as the inclusion of imperfections is justified in another section when dynamics are involved. Next, the most well-known buckling criteria, such as those of Neutral Equilibrium and Budiansky & Roth criterion are presented, and finally a synopsis of the modern structural codes and standards (American standard, Eurocode, Japanese code) is given regarding the matter, showing the lack of information on shear buckling for liquid filled tanks.

The next chapter starts by demonstrating the initial step towards formulating the problem which is no other than the choice of an appropriate displacement field along with a suitable middle surface strain theory. After the most common shell theories are discussed, it is concluded that a deep nonlinear thin shell theory would fit the case under study. For educational purposes, the applied theory is developed from scratch, despite the fact that popular shell theories, which meet the aforementioned criteria like those of Donnell, Flügge-Lur'e Byrne and Sanders-Koiter are thoroughly described. Moving on, the equations of motion are derived and a discussion is held on how a solution can be reached in order to acquire the critical-buckling load, revealing all the different methods found in the literature regardless if they were used or not, like the classical buckling theory, in an attempt to provide the potential reader with a universal understanding on the matter. Eventually, the initial thought on the approach that should be used is illustrated analytically through the example of a beam.

In the third chapter, the strong form of the initial problem is converted into a weak formulation due to the high complexity of the nonlinear partial differential equations of motion. For this reason, the perturbation method is utilized, an approximation technique which by expressing the displacements in terms of a very small perturbation parameter ϵ , it allows the breakdown of the nonlinear problem into an infinite number of linear

sub-problems. The simplest equations that could describe the problem are employed and the solution is divided into two different cases based on the order of ε ; the linear and the nonlinear one. At first, the solution of the 1st order in terms of ε linear problem is determined, in relation to the unknown buckling force, which serves as a tool for the 2nd order problem. This is realized by its substitution into certain nonlinear terms of the nonlinear problem, in a way making sure that the final product is linear. The resulted *EoMs*, despite the linearization process that has taken place, are still hard to tackle due to the existent variable coefficients. For that purpose, as a last resort calculation of the buckling load is searched numerically, a procedure which requires the final system of *EoMs* for the desired number of modes to be converted into a system of 1st order odes.

Due to the high computational cost, the numerical approach has been implemented solely for a beam clamped onto the ground under the application of a distributed load along its length and a concentrated force on the top, as an example to demonstrate the methodology. However, the process featured at the end of the second chapter, is modified since apart from the fact that it is not entirely applicable to the tank, now a solution is searched through the discretization of the space in an attempt to evaluate the variable coefficient of the *EoM* at certain points. By the aid of Matlab, different buckling loads are determined for various numerical values of the beam's stiffness. The steps of the numerical procedure that should be followed for the case of the cylindrical tank filled with water are then extensively described and illustrated.

In that sense, it is crystal clear of the method in which instability due to shear can be investigated in complex problems such that of a shell tank. After this thesis, the future researcher is provided with all those tools that will allow him to follow the correct path in order to extract some quite interesting results like; the effect of the tank's geometric characteristics, the various boundary conditions, the different levels of water as well as the effect of the miscellaneous existing non-linear shell theories on shear buckling. Such conclusions will definitely fill the gap in the current structural codes and standards and will contribute largely in the field of research regarding the cylindrical liquid tanks.

*To my father,
the best man I've ever known...*

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1 Literature Review

1.1 Introduction

Shell shaped structures have been an increasing choice among engineers for their designs during the last decades. The reason for this, is that compared to plates, which use their out of plane – bending stiffness, shells also take advantage of their in plane – membrane stiffness, something that offers the possibility for more versatile high strength lightweight structures [1].

Their application extends to several fields such as those of automobile, aerospace, civil and marine engineering. The initial trigger that led scientists to develop the first shell theories was the need to produce safe and reliable spacecrafts [2]. However, as the years were passing by, shells have also entered the building industry to facilitate the ambitions of architects who sought worldwide recognition through complex designs, known as *blobs*, that demonstrated a sense of maximalism and architectural boast. Nowadays, special interest has been gathered around maritime structures where the parameter of water plays an important role in the structural behaviour. Liquid pressure can be either external, such as in the case of submarines, or internal, for example onshore tanks, or both as it applies for floating storage tanks [3].

Ground based tanks, which after all will be the scope of this study, can take different geometries that include cylindrical, rectangular, ellipsoidal and spherical shapes [4],[5]. They can also be underground, semi-ground, over-ground or elevated, whereas most of them are usually covered by a roof and others are open with a stiffening ring on the top [6]. In industry, the most frequently used in practice, are the above ground steel cylindrical tanks, mainly due to the easiness of manufacturing and this is the reason why this kind will be the one elaborated in the present thesis. Shells express a rather sensitive category of structures, and thus should be designed with extra care against external loadings caused by natural phenomena. Wind pressures and earthquakes can lead to several types of failures such as tank sliding, base uplifting and damage from sloshing as shown in the pictures,

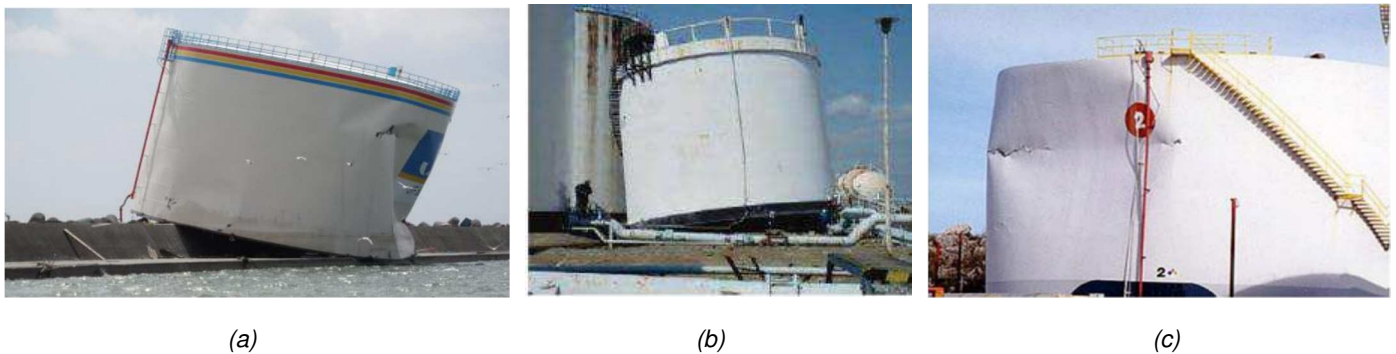


Figure 1 Types of tank failures: (a) base sliding, (b) base uplifting, (c) damage due to sloshing

Another very common failure mode that has attracted the interest of many researchers is buckling. As mentioned above, shells offer the opportunity of aesthetical designs, characterized in general by their low thickness compared to the other dimensions, something that accommodates the use of less fabrication material. However, these benefits do not come without a cost. Cylindrical tanks with a radius to thickness ratio between 500 and 2000, are considered *thin* and present a high vulnerability to buckling [4]. Occurrence of buckling may appear in diamond shaped patterns, as a result of axial compression, or in the shape of elephant foot caused by the compressive stresses at the bottom part due to the moments stemming from the hydrodynamic lateral pressures. Buckling deformations may also appear due to a considerable amount of external pressure, or finally in the form of shear buckling due to the development of excessive shear stresses [7].

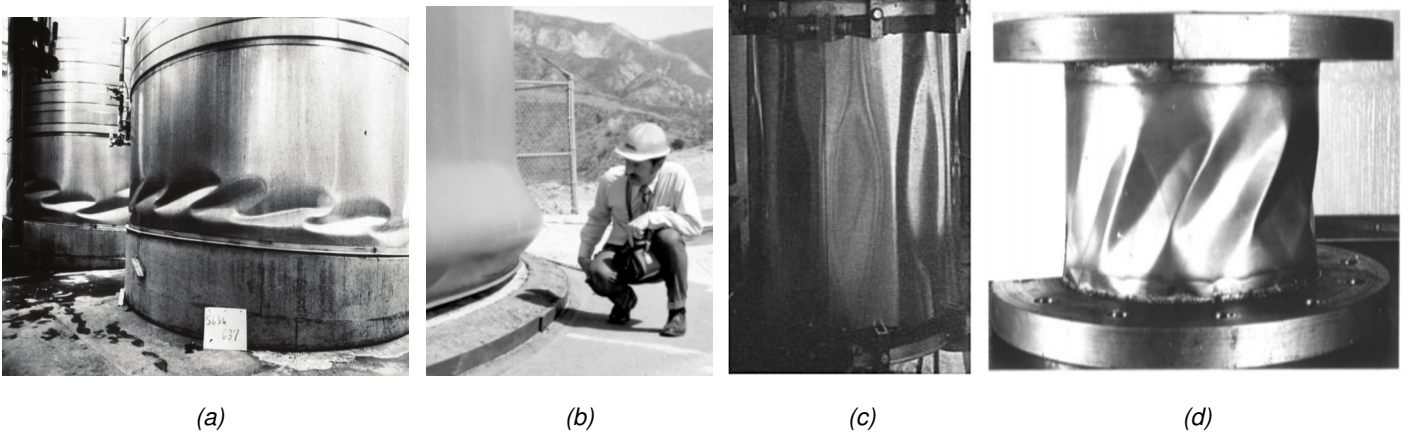


Figure 2 Types of buckling: (a) diamond shaped buckling, (b) elephant foot, (c) buckling due to external pressure, (d) shear buckling

Such types of failure cannot be always tolerated and must be avoided. Apart from water, most of the time, tanks are used as storage for fuel, oil, liquified gasses and hazardous chemicals. An improper design could lead not only to enormous economic losses but also to environmental disasters, such as the contamination of waters and forests, something which in turn could have a detrimental impact to public health [8].

The main interest of this thesis will be gathered around the topic of shear buckling. Our attention will be focused on how a filled or partially filled with water clamped on top steel cylindrical tank will behave under shearing forces developed by a horizontal seismic motion. An analytical approach will be followed and the critical shear buckling stress will be attempted to be determined. Before we delve more into this matter, it is vital the understanding of the basic aspects of buckling which I personally found valuable during the process of my study. The following paragraphs of this chapter are devoted to serving this purpose.

1.2 Basic Concepts

It is important that, from the very beginning the terms of stability, equilibrium, instability and metastability should be defined. In this way, it will be crystal clear to the reader how the buckling problem is approached and treated. For this reason, the principle of potential energy should be introduced. The potential energy, P , of a mechanical system can be described as a sum of two sub-energies,

$$P = U + V \quad (1.1)$$

where, U is the work produced by internal stresses and $V \equiv W_{in}$ the negative work by external forces. In order now, to accommodate a better understanding of the above terms, we represent, the energy of a mechanical system in relation to deformation with a graph (fig. 3) and its values with a ball.

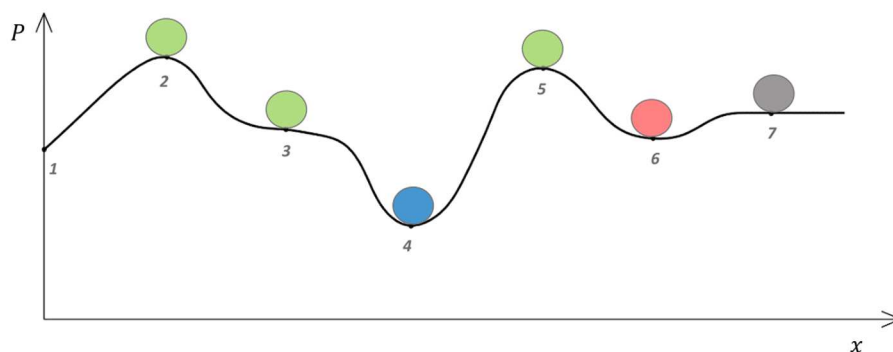


Figure 3 Potential energy of a single degree of freedom structural system

Equilibrium

A structural system is in equilibrium when its potential energy has stationary value and of course its kinetic energy is equal to zero. In other words, equilibrium is assured when the tangent of $P(x)$ is parallel to x axis,

$$\frac{dP}{dx} = 0 \quad (1.2)$$

** Often in literature, the requirement for equilibrium is expressed as $\delta P = 0$, i.e. a zero first variation of energy. This notion coincides with (1.2) as shown analytically in Appendix A.*

After establishing the equilibrium positions, applying now a perturbation, the potential energy is bound to change and so the mechanical system will no longer be in equilibrium. In order to provide a rational explanation of the movement, the nature of the initial equilibrium state regarding the degree of stability should be investigated.

Stability

A system lies in a stable equilibrium, if only after a perturbation always returns to its initial position. This can be easily depicted with a ball located at position 4 of (fig. 3). Any disturbance will cause the ball to return to the bottom of the valley. Mathematically speaking, the stability of an equilibrium state is provided by a local minimum of the potential energy,

$$\frac{d^2P}{dx^2} > 0 \quad (1.3)$$

** Alternatively, the sufficient condition for stability is expressed as a positive change in potential energy, $\Delta P > 0$ or a positive second variation of energy $\delta^2 P > 0$.*

Metastability

However, in case the ball, which as said before represents the value of potential energy, does not lie in a *deep valley* such as position 6 in (fig. 3), no one could tell with certainty whether the ball will return to its initial position after a *larger* disturbance. Such an equilibrium state which is highly dependable on the magnitude of the applied perturbation is often called metastable.

Instability

An equilibrium state which is bound to change irreversibly due to any kind of disturbance is called unstable. Let us take for example, a ball positioned at 3 or on a *hill*, 2 & 5 of (fig. 3). Even for the slightest perturbation, the ball will be dislocated and never return. An unstable equilibrium is defined as a local maximum of the potential energy i.e.

$$\frac{d^2P}{dx^2} < 0 \quad (1.4)$$

** In other words, instability is established either by a negative change of potential energy $\Delta P < 0$ or with its negative second variation $\delta^2 P < 0$.*

1.3 Buckling

Establishing the equilibrium of a mechanical system and the equations of motions that describe it, is the first step towards the awareness of the problem. However, it is not the last. As the loads acting on the system, gradually increase, a sudden change in the initial configuration takes place. This abrupt change in the geometry is called buckling. So, what is buckling exactly? It is the transition from one equilibrium state to another. That means, the deformation of the structure in its fundamental state after a certain point will stop and inevitably will continue deforming in a new stable equilibrium shape implying that the old one is now unstable [7].

The load at that critical point is called *buckling load*. In literature, buckling load is often referred also as *bifurcation load* due to the fact there is a bifurcation in the load - deformation curve [7]. Figure 4 shows the occurrence of bifurcations for three structural elements under compression,

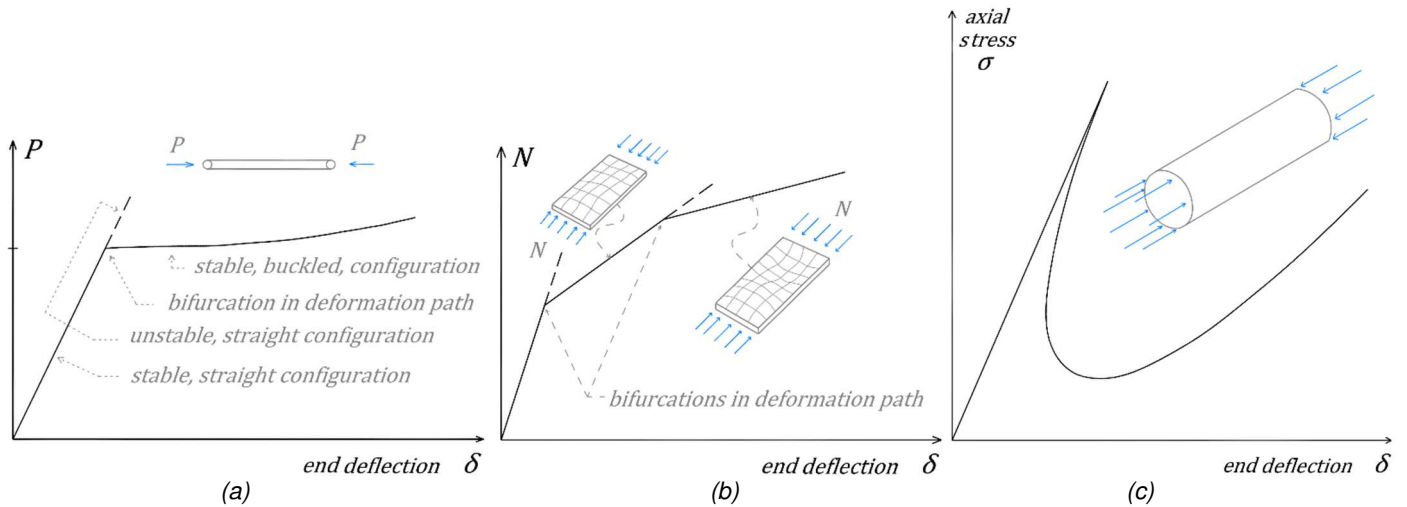


Figure 4 Bifurcations in load – deformation path of 3 structural elements under compression: (a) beam, (b) plate, (c) cylindrical shell.

Furthermore, a closer look at (fig. 4) indicates the existence of more than one bifurcation points, meaning that more than one buckling loads exist. In plates, for example, every buckling load signifies the stability limit of the corresponding equilibrium path. This answers the question of how a structure can sustain loads larger than the buckling load. Of course, the main scientific interest is focused on determining the lowest one, also known as *critical buckling load* [7].

Cylindrical shells are tackled by the current thesis. Parameters such as shell thickness, tank radius and internal pressure have a strong impact on buckling behaviour. Imperfections are also a major factor that can lead to a fraction of the theoretical buckling load. Koiter [8] delivers an extensive analysis of shells' buckling. In his work, pre-buckling and post-buckling behaviour, including the investigation of stable states adjacent to the critical load, are thoroughly discussed and presented in detail.

To sum up, buckling is a phenomenon whose physical meaning refers to the ability of structural elements to modify their shape with the cost of a reduction in their stiffness in order to withstand large forces. A sudden change in geometry, accompanied with a change in deflection configuration and the possibility of existence of more than one buckling mode, each described by a unique buckling load (or bifurcation load), are the three main characteristics that anyone who is eager to delve into the topic of buckling should be mindful of.

1.4 Elastic & Plastic Buckling

Buckling and yielding are meanings that can easily cause confusion as they may seem at first glance that are highly interconnected. The point in which the equilibrium path bifurcates, declaring its stability limit, does not and should not be interpreted as the yielding point of the material. A structure can buckle way before it enters the plasticity region, a phenomenon usually met in aircraft engineering [8]. In this case, also known as elastic buckling, there is the possibility of “travelling” up and down the equilibrium paths in the load – deformation diagrams of (fig. 4). In other words, a buckled structural element will be able to return to its initial configuration after a significant reduction of the applied load [7].

On the other hand, the fundamental geometrical shape cannot be retrieved when stresses have surpassed the yield limit. In general, plastic buckling requires a different approach. Reformulation of the equations of motion is necessary as well as the use of *tangent modulus* E_t . Tangent modulus is much lower than elastic – around 100 times for metals – and can be calculated for a specific value of strain, by the tangent line of stress – strain curve. Several examples can be found in the book of Lindberg & Florence [9], where this matter is discussed in detail. Inelastic problems will not be part of this thesis. Shear buckling of cylindrical tanks, will be studied within the boundaries of elasticity.

1.5 Dynamic Buckling

The study of buckling when dynamics are involved is considered to be a far more complicated topic than static buckling. During the last decades, the focus of many researchers has been concentrated on the matter, which without doubt has caused much trouble and conflicts among them [10].

The first studies by Yao and Bolotin give a comprehensive approach of the stability of empty cylindrical shells whereas a large number of authors whose work ranges around the subject can be found in [2]. However, the increasing needs in the fields of aerospace and rocket science regarding tanks that carry fuel, made it imperative for the scientists to study the dynamics of liquid filled tanks. This kind of studies were extended for similar problems such as the transport of liquid cargo, the stability of piping systems and finally cylindrical tanks placed onto the ground undergoing seismic motions [11]. In the literature two authors self-claim to be ones to have conducted the first analytic study concerning the dynamic stability of tanks filled with liquid. These papers come under the name of Rasim and Liu [12], [13] in which the dynamic stability of a cylindrical tank is investigated for various seismic rocking motions.

The scientific papers are divided into two groups depending on the nature of the load responsible for the excitation of the structure. These, in which dynamic buckling results from periodic loads and those from transient loads. The two types of dynamic buckling are also known as *vibration buckling* and *impulse buckling* accordingly [9].

Vibration buckling can be traced in many structural problems. Due to vertical seismic motion, axial stresses are being developed which may lead to buckling, whereas the lateral pressures on the top of the wall tank as the fluid moves by the horizontal rock motion, develop moments which may cause buckling by compression at the bottom [13]. Also, periodic shearing forces have been taken into consideration for a cylindrical tank in a few studies such as those of Yamaki [14] who examined the effect on stability for both static and dynamic forces and Michel [15] who conducted experiments on cylindrical tanks under shear. In vibration buckling, the buckling load can be far less than in static case, if the vibration frequency comes close to the natural frequency of the structure.

On the contrary, in impulse buckling, the transient load is always larger than the static buckling load. This is because of its small duration. So apart from the amplitude, the time of excitation is also a critical parameter that should be taken into account into the problem [9]. The simplest example that can be illustrated is the hammering of a nail or in a larger scale, a column used as a drop hammer in piling equipment, where a large force is not necessarily related to large deflections or surpassing the elastic limit, if the duration is short enough. Kubenko [11] and Shaw [2] examined dynamic buckling of cylindrical shells undergoing transient axial and torsional load, whereas Lindberg [9] in his book gives a plethora of examples for columns and tanks in which the main focus is not concentrated on the amplitude itself but on the timespan that a load of a certain amplitude can be applied without the loss of stability.

It should be noted that dynamic buckling differs from the known dynamic resonance, where the external force stands on its own at the right hand side of equations of motion. Here, it appears as a parameter multiplying a term of displacement and that is the reason often in literature, impulse and vibration buckling are referred to as buckling under time varying and oscillatory parametric loading accordingly.

Dynamic buckling is still up to this day a quite complicated and vague field in the scientific society. The lack of papers concerning liquid filled cylindrical tanks makes it hard for the young researcher to have a strong base of information. The above is corroborated by the fact that most of the existing publications concern the aerospace community, meaning that a high frequency range has been implemented (between 250 and 900 Hz), something which renders them not usable for the civil engineers whose main focus are the low frequency rocking motions.

1.6 Imperfections

The study of dynamic buckling does not make any sense without the inclusion of imperfections. As Shaw [2] mentions in the conclusions of his work, the calculated dynamic buckling load in the absence of imperfections is the same as the one in the static case. The impact of imperfections has come into consideration among scientists when experimental results deviated from theory, with Donnell being the pioneer in incorporating the effect of imperfections in the existing nonlinear shell equations. From that point on, many researchers focused on the matter, with Koiter especially presenting a rather comprehensive approach [8].

As said above, the buckling load of an imperfect shell comes at a value, way lower than its theoretical one. According to Kubenko [11], other parameters like the variability of homogeneity of the material, deviations in the definition of the boundary conditions etc. do not have such an impact in the critical load. This is mainly due to the high sensitivity of thin walled shells to small changes in geometry. More specifically, the abrupt change of membrane forces under a slight induced distortion, leads to a different critical equilibrium state than the one expected from the theory and hence the puzzling experimental results can be explained [8]. In general, the closer the load gets to the stability limit, the largest the effect of imperfections [11]. However, shells do not exhibit the same level of sensitivity for all kinds of loading; the influence of axial compression is far greater than in the case of pure torsion [2], [16]. In practice, throughout the earlier studies as it is described in detail in the book of Robert M. Jones [7], imperfections are presumed as the geometrical shape, the shell is expected to buckle. Donnell was the one who first implemented this approach for various loading cases, and final qualitative results are seen in the figures below.

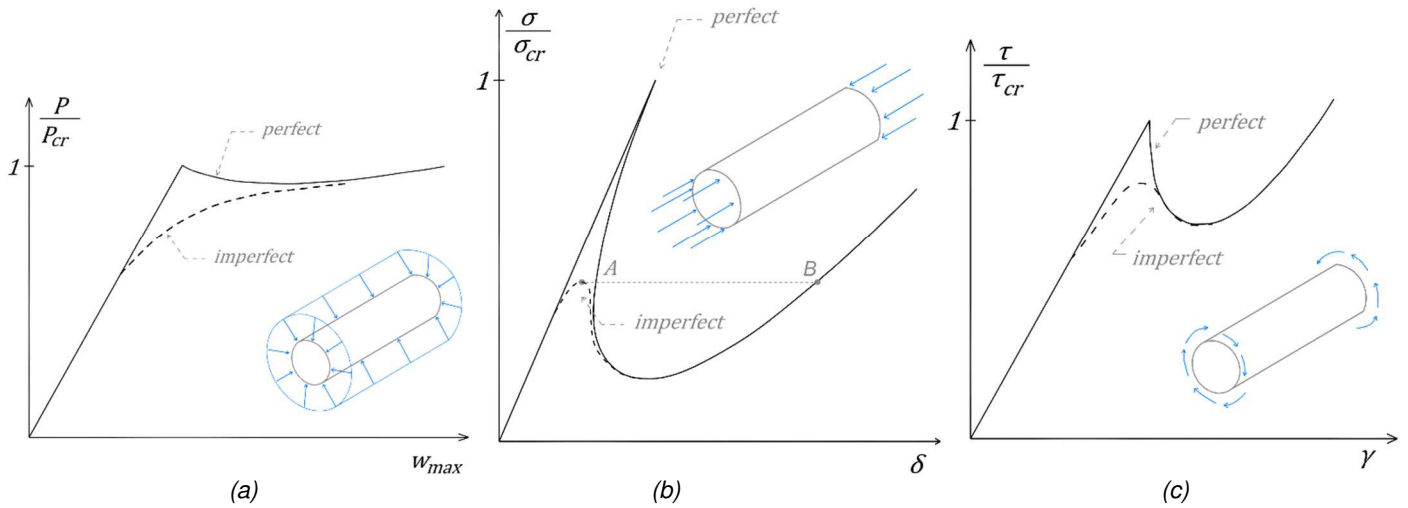


Figure 5 Equilibrium paths for perfect & imperfect cylindrical shell under various forcings : (a) external pressure, (b) axial compression, (c) torsion

Of course, this method does not correspond to reality and leads to an underestimation of the buckling load. This is why, imperfections are often introduced as pre-buckling initial strains which are expressed in equations of motion as an added term $w_0(x, y)$ to the existing displacement $w(x, y)$ and are expanded with the Fourier series. Many examples can be found in the book of Lindberg [9]. Here, before continuing, a closer look at (fig. 5) should be taken. It seems that buckling is accompanied either with large or small deformations. Pay attention, for instance, to the *leap* in deformation at the bifurcation point for a compressed shell. This phenomenon is called snap-through and expresses the initiation of buckling with the transition to an equal energy point in the load-deformation path.

The problem however closing up, comes down to that it is extremely difficult to accurately represent the imperfections, especially on a theoretical level. The only way is by collecting real data with the aid of electronic devices. So, given the fact that the final estimation of the buckling load, relies entirely on the initial assumption of the imperfection shape [11] and also, due the increased complexity of the problem, it has been decided that the effect of imperfections will not be included in the present thesis.

1.7 Buckling Criteria

When will the buckling occur? What is an accurate way to determine the critical buckling load? Many methods have been developed over the years by the scientists who attempted to give an answer to those questions. Understanding of buckling criteria is vital when tackling a buckling problem and for this reason some of them are briefly discussed below.

1.7.1 Neutral Equilibrium Criterion

It has been supported that in order for the buckling to take place, the initial equilibrium configuration should pass from a stable to an unstable state, so that a new adjacent or nonadjacent stable equilibrium could

be reached. Mathematically speaking, this means that the potential energy of the fundamental state should drift from a local minimum $\frac{d^2P}{dx^2} > 0$ to a local maximum $\frac{d^2P}{dx^2} < 0$. So, it is logical to assume that buckling load can be calculated at the transition point, *fig. 6* where,

$$\left. \frac{d^2P}{dx^2} \right|_{\frac{dP}{dx}} = 0 \quad (1.5)$$

Also expressed as a zero change in potential energy,

$$\Delta P = 0 \quad (1.6)$$

or after using its Taylor expansion,

$$\delta^2 P|_{\delta P=0} = 0 \quad (1.6)$$

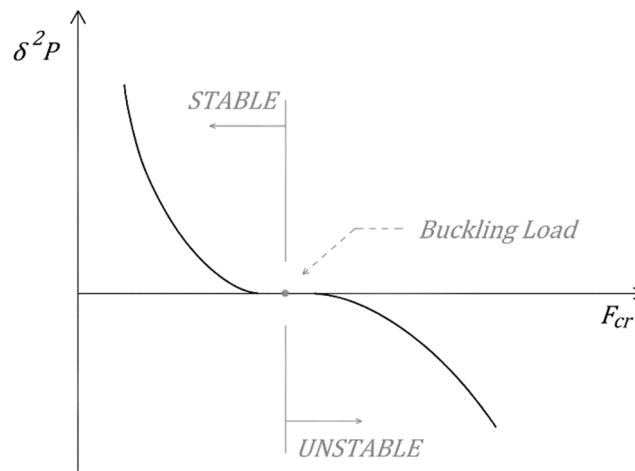


Figure 6 Second variation of potential energy in relation to the buckling load

This concept has been introduced by Simites [17] and has been used extensively in the work of Jones [7] for extracting the so called, *buckling equations*. Neutral equilibrium method “coincides mathematically” with Trefftz buckling criterion; $\delta(\delta^2 P) = 0$ which is derived by investigating two infinitesimal adjacent equilibrium states. The same philosophy is applied by Koiter [8] whereas, the analytical proof can be found in [7].

1.7.2 Budiansky & Roth Criterion

Unlike before, no energy is involved in this section. Their approach is based on the simple observation of the structural system’s response. A steep rise in the displacements’ magnitude at a certain value of the applied force, declares the occurrence of buckling [2]. This is the most frequently used buckling criterion in the literature, especially for dynamic buckling problems [10]. The only drawback is the high computational cost, since the displacement response should be calculated for each time step [2].

1.7.3 Other Criteria

Despite the fact, these are the most well-known buckling criteria, there are not the only ones. For example, according to Kubenko [18], a deflection with a magnitude as large as the shell’s thickness, is a sufficient condition for buckling. The same applies for some authors, in case a considerable level of stress (e.g. yield limit) is reached [9]. Other buckling criteria, such as the “Total Energy – Phase Plane Approach”, the “Criteria of Two Dynamic Curves” and the “Pseudo – Dynamic Curves” are described in detail in the work of Touati [10].

1.8 Structural Engineering Codes & Standards

A simple indicator of the complexities and difficulties introduced in the study of shells, is the shortage of information provided by the existing guidelines. Despite the fact that the usage of structural codes has been quite extensive among engineers throughout the last decades, instructions about cylindrical tanks is considered a rather newly introduced topic. The most frequently applied in practice codes are described below and briefly discussed.

1.8.1 API 650 (2007): “Welded Steel Tanks for Oil Storage” [19]

The design of empty or non-empty cylindrical tanks is carried out based on two empirical methods, known as “1-Foot Method” and “Variable Design Point Method”. They have been developed upon the concept of limiting the tensile stresses produced by the liquid and offer a way of calculating the thickness of each shell course. However, buckling seems to be almost a neglected area in the American standard. It is mainly accounted for in the case of wind action by recommending the stiffening of the shell by intermediate wind girders. The second and last reference about structural stability is found within the E section of Appendix, in which a limitation of the compressive force induced by the seismic motion is provided. Nonetheless, no analytical formulas of the buckling critical forces are existent. This fact renders the American standard not the best option for designing cylindrical tanks against buckling.

1.8.2 EN 1993-1-6: 2007 [21]

Eurocode offers a far more extensive analysis of shell's topic than the American Standard. Buckling is treated in a more scientific way as it provides the engineer numerous options of shells' analysis to base its design. In particular, the limit state of buckling suggests apart from various numerical methods, the “Stress Design” approach for calculating analytically critically buckling stresses. The choice of the appropriate method is defined by the consequence class of the tanks, a measure of structural reliability. Stress Design is the one most used in practice. Meridional (i.e. longitudinal) and shear buckling resistance stresses can be easily determined through analytical mathematical formulas derived by linear elastic analysis. Despite the fact that Eurocode may seem at first glance quite versatile in many aspects (such as the inclusions of imperfections for different quality classes), it still fails to tackle the problem of liquid filled tanks. Pressurized buckling resistance formula, is available only in the case of meridional buckling. Shear buckling resistance with regard to internal pressure is completely absent, meaning that its calculation will be performed with the formula referred to that of an empty tank, something which will obviously lead to a very conservative estimation of the final resistance.

1.8.3 Japanese Code of Storage Tanks (2010) [22]

Despite the fact that, the Japanese code does not treat cylindrical tank shells to the extent Eurocode does, such as the division in consequence and fabrication tolerance classes for choice of analysis or the inclusion of imperfections respectively, it attempts to provide instructions and cover all the possible cases of buckling. More specifically, meridional (i.e. compressive) and flexural buckling are described on the basis of two empirical formulas for empty and non-empty tanks. Shear buckling resistant stress stems from a 20% reduction in the Donnell's torsional buckling analytical expression. According to the Japanese standard, liquid filled tanks are not susceptible to shear buckling. For this reason, if the average circumferential stress caused by the internal pressure is larger than the 30% of yield material limit σ_y , buckling is not the predominant failure mode and shear yield-point stress τ_y is the critical one. For low liquid levels shear buckling stress is expressed by an empirical formula.

A review of the most widely known standards has shown that little has been done towards an appropriate and accurate guideline for designing shells against buckling without the aid of FEM programs or empirical formulas. Especially, shear buckling of liquid tanks still remains a vague and unexplored area. Further research needs to be conducted and that will be the goal of this thesis, in which any possible valuable information or conclusions will be attempted to be extracted towards this direction.

2 Problem Formulation

2.1 Introduction

The previous chapter ended with the impression that shear buckling in shells filled with liquid is a trivial matter. The Japanese Code [22] specified that failure due to shear stresses can take place, only for very low levels of water. So, why is it really worth the effort of studying shear buckling? Liquid filled tanks is a relatively new field of research and it should be kept in mind that codes & standards are not a panacea. To corroborate this, there are also other studies which contradict JPI. Chiba [2], for instance, in his work of cylindrical tanks under periodic shearing forces, addresses the occurrence of instability regions in the results with rising liquid. It has been also shown that application of torque has a negative impact on compressive buckling since it leads to lower critical loads [2]. Although not covered here, the interaction effect with other types of buckling is another excuse for studying and understanding the behavior of water tanks under shear.

One of the biggest struggles in shell problems filled with liquid is the simulation of fluid. Nowadays, FEM programs like Ansys and Abaqus are used for this purpose. However, due to the fact that validness of the results highly depends on the choice of the “correct” finite element, structural codes like Eurocode offer a simplified version 2D tank, also known as the Housner’s model in which a part of the water (convective mass) oscillates as a sdof connected with two horizontal springs and the rest is prescribed as rigid (impulsive) mass [6]. A similar model but a bit more detailed is the one developed by Haroun in 1983 where an additional division of water is included. Both are described extensively in [23]. A more in-depth mathematical approach for modeling fluid in a more realistic 3d dimensional way can be found in the book of Moiseyev and Rumyantsev [44]. Here, for the static case all the complexity of fluid simulation can be disregarded. All the above, should be taken into account when a dynamic analysis is performed and are referred just for informative reasons.

The problem that will be examined, involves a ground based cylindrical tank filled with water which is clamped on the top under a horizontal static seismic force:

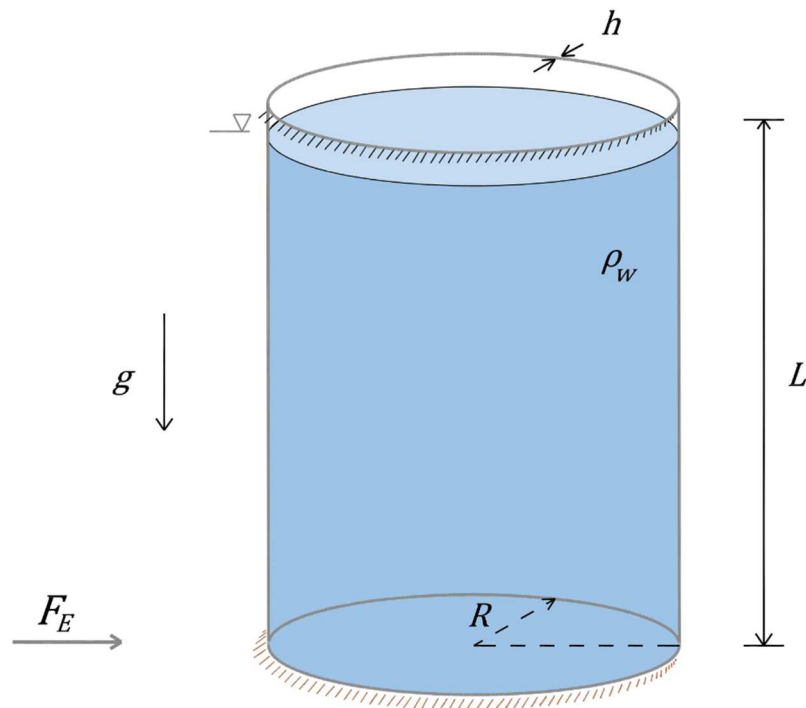


Figure 7 Fully filled cylindrical tank under a horizontal static earthquake force

In this chapter, the course of action that should be followed when shell buckling is tackled, is described. The “proper” shell theory needed for setting up the equations of motion will be explained as well as how a solution can be reached in order to acquire the critical – buckling load. As the procedure unfolds, at each stage all the available different options are discussed regardless of the fact if used or not. In this way, the potential reader will have a more universal understanding, since there is not a single approach and the literature is rather chaotic. Finally, a simple example of the chosen solution method will be demonstrated before we delve into more detail in the next chapter.

2.2 Strain - Displacement

The first step towards formulating the problem, should be the choice of an appropriate displacement field along with a suitable middle surface strain theory that would fit the case of the cylindrical tank under study. This can be achieved either by use of existent shell theories, such as those briefly featured in the book of Leissa [24] or by developing from scratch a general description of the middle surface strains and disregard at a later stage all those nonlinear terms considered irrelevant for the case at hand. In the following paragraphs, the most common shell theories, from the simplest to the most complicated, will be discussed and finally the method which was followed to the relations used in this study will be presented.

2.2.1 Shell Theories

Several questions should be answered at the beginning, when facing a shell problem. Are we going to study an open or a closed shell? On what scale does its thickness compare to its other dimension? What is the shell's deflection behavior prior to buckling? Could it be considered large or small? All those answers will determine the shell theory that should be used and will give a clearer picture about the general approach that will be followed in the future steps.

The first distinction among shells, is based on their shallowness. The term *shallow shell*, usually refers to open shell structures in which their height does not exceed the $1/5$ of its smallest dimension [1]. Often in literature, the shell's height is called *sagitta*, referring to that section where the rise of the arch is at its maximum. Equations of motion take a quite simple form in this case, which are not necessarily accompanied however with a lower order from that of the deep shell theories' odes. Cylindrical tanks do not fall in the group of open shells and for that reason, any further information can be searched and found in the work of Donnell and Mushtary [24]. Another, very important categorization, takes place according to the wall's thickness. Thin shell theories are the most common in practice. Such an approach is implemented only when the ratio of thickness to wavelength of deformation is lower than $1/20$. Rotatory inertia, and shear deformation can be neglected, and so a version of simplified equations is achievable. Love, Flügge, Lur'e-Byrne, Novozhilov and Sanders are among the most famous scientists who developed their theories based on that concept. However, in cases where the thickness to wavelength of deformation ranges between $1/10$ and $1/5$, rotatory inertia and shear deformation should be included in the analysis. Thick shell theories, also known as shear deformation theories [1], were first introduced by Reissner and Mindlin, and further developed by others like Liew and Lin [26]. Educational information on the topic is provided in the book of Soedel [25].

Last but not least, the study of shells falls either to the group of linear theories or to that of nonlinear ones. The criterion for this division is made upon the magnitude of deflection. In cases when the shell deforms to an extent greater than the wall's thickness a nonlinear theory must be used, otherwise huge errors will definitely appear eventually. The present discussion is being held for the phase before the shell buckles. The nonlinear terms in the strain displacement relations result into very complex pdes, which as it will be seen in the next chapter makes their solution quite a struggle. It should also be made clear that all those cases where the material's properties are inherently nonlinear, as it happens for plastics or at times the material itself has exceeded its yield limit, belong to the realm of nonlinear theories. For this reason, it is important for the reader to know that throughout this thesis, any reference to nonlinearity will be deformation-based.

In this context, it has been concluded that a deep nonlinear shell theory would be appropriate for the problem under study. The terms thin and deep, perfectly describe the geometry of water/oil tanks used in industry and by exploiting the benefits of nonlinearity, it provides the opportunity to cover a larger range of situations and make the most possible realistic approach. Popular theories that meet the aforementioned criteria, are presented and discussed below:

2.2.1.1 Donnell's Theory

The theory of Donnell was founded in 1933, and it appears to be the number one choice for scientists throughout the literature. The reason behind this, is the easiness and practicality it offers for studying buckling of cylindrical shells. It is by far, the simplest nonlinear theory as only the minimum nonlinear terms needed are maintained. Before we delve into any further, it is important to establish the coordinate system and the notation that will be used in the rest of this thesis.

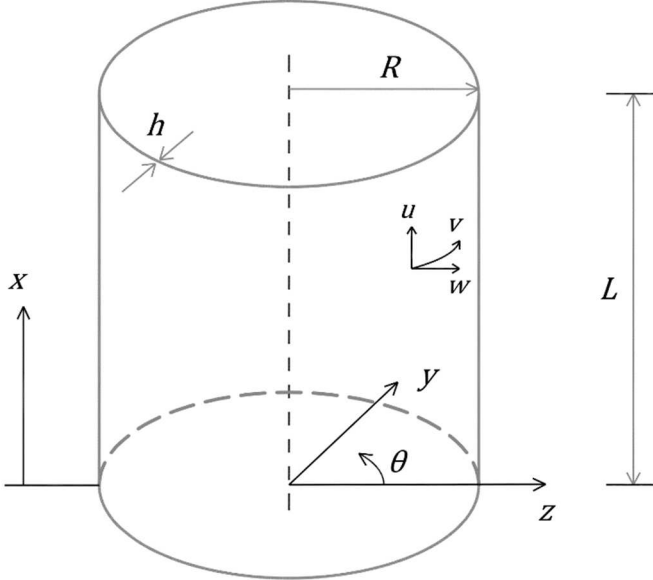


Figure 8 Cylindrical tank's coordinate system

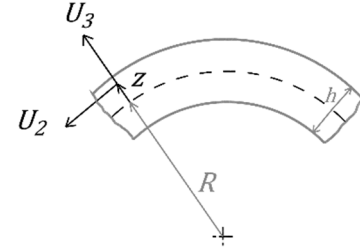


Figure 9 Displacements of a generic point at cylinder's cross section

The *assumptions* [27] upon the theory was developed are listed next,

- i) Shell's thinness is assured by: $h \ll R$ & $h \ll L$
- ii) The strains are very small. That means that the Hooke's law of linear elasticity is valid.
- iii) Straight lines normal to the mid-surface remain straight and normal to the mid-surface after the deformation.
- iv) Over the shell's thickness, strains develop in a linear distribution. Also, stresses normal to the surface are neglected even in the case where external loading normal to the surface is applied.
- v) The first derivatives of w , which represent the slope, are small: $\left| \frac{\partial w}{\partial x} \right| \ll 1$ & $\left| \frac{\partial w}{R \partial \theta} \right| \ll 1$
- vi) The deflection w is of the same magnitude as that of the shell's thickness, whereas the rest displacement u & v are considered very small. For this reason, only the nonlinear terms with respect to w are taken into account.

The 2nd and 3rd assumption stem from the plate theory and are the well-known Kirchhoff – Love hypothesis. Based on the latter and assuming also now that the mid-surface coincides with the outer side of the shell, i.e. $R + z \approx R$, the displacement field for a generic point is expressed by,

$$\begin{aligned} U_1(x, \theta) &= u(x, \theta) - z \frac{\partial w(x, \theta)}{\partial x} \\ U_2(x, \theta) &= v(x, \theta) - z \frac{\partial w(x, \theta)}{R \partial \theta} \\ U_3(x, \theta) &= w(x, \theta) \end{aligned} \quad (2.1)$$

in which $u(x, \theta)$, $v(x, \theta)$ and $w(x, \theta)$ represent the displacements for the mid-surface. Next, the *strain – displacement relations* at an arbitrary point are given by,

$$\begin{aligned} \varepsilon_{xx} &= \varepsilon_{xx,0} + z k_{xx} \\ \varepsilon_{\theta\theta} &= \varepsilon_{\theta\theta,0} + z k_{\theta\theta} \\ \gamma_{x\theta} &= \gamma_{x\theta,0} + z k_{x\theta} \end{aligned} \quad (2.2)$$

where $\varepsilon_{xx,0}$, $\varepsilon_{\theta\theta,0}$ and $\gamma_{x\theta,0}$ represent the mid-surface strains,

$$\begin{aligned}\varepsilon_{xx,0} &= \frac{\partial u}{\partial x} + \frac{1}{2} \left(\frac{\partial w}{\partial x} \right)^2 \\ \varepsilon_{\theta\theta,0} &= \frac{w}{R} + \frac{\partial v}{R \partial \theta} + \frac{1}{2R^2} \left(\frac{\partial w}{\partial \theta} \right)^2 \\ \gamma_{x\theta,0} &= \frac{\partial v}{\partial x} + \frac{1}{R} \left(\frac{\partial u}{\partial \theta} + \frac{\partial w}{\partial \theta} \frac{\partial w}{\partial x} \right)\end{aligned}\quad (2.3)$$

and k_{xx} , $k_{\theta\theta}$, $k_{x\theta}$ the curvatures,

$$\begin{aligned}k_{xx} &= -\frac{\partial^2 w}{\partial x^2} \\ k_{\theta\theta} &= -\frac{\partial^2 w}{R^2 \partial \theta^2} \\ k_{x\theta} &= -2 \frac{\partial^2 w}{R \partial x \partial \theta}\end{aligned}\quad (2.4)$$

As it can be seen, the outward displacement w plays a predominant role in the relations. To achieve a higher accuracy, more nonlinear terms should have been retained. The rest of the shear strains are neglected: $\gamma_{xz} = 0$ & $\gamma_{\theta z} = 0$. The above expressions can be also applied for thick shells, if the assumption of $R + z \approx R$ is dropped. The main downside of this theory is that it is mostly applicable for shallow shells. In order to avoid inaccurate results, the circumferential mode wavenumber k must be higher than 4 [9]. Still, it constitutes the most popular theory for studying shells' buckling.

2.2.1.2 Flügge-Lur'e-Byrne

In comparison to Donnell's theory where the displacement field is based on that of a plate, here the inclusion of v in the circumferential displacement, gives a more realistic representation of the shell's curvature [28].

$$\begin{aligned}U_1 &= u(x, \theta) - z \frac{\partial w}{\partial x} \\ U_2 &= v(x, \theta) - \frac{z}{R} \left(\frac{\partial w}{\partial \theta} - v \right) \\ U_3 &= w(x, \theta)\end{aligned}\quad (2.5)$$

The 6th assumption of Donnell mentioned before is completely disregarded. In plane displacement, u and v although small, they are no longer considered infinitesimal. For this reason, their contribution to the nonlinear terms will appear in the relations. Further, dropping now the thinness assumption of, $R \approx R + z$ along with the following approximations,

$$\frac{1}{R+z} = \frac{1}{R} \left[1 - \frac{z}{R} \right] \quad \& \quad \frac{1}{(R+z)^2} = \frac{1}{R^2} \left[1 - \frac{2z}{R} \right] \quad (2.6)$$

the strains of the middle are derived,

$$\begin{aligned}\varepsilon_{x,0} &= \frac{\partial u}{\partial x} + \frac{1}{2} \left[\left(\frac{\partial u}{\partial x} \right)^2 + \left(\frac{\partial v}{\partial x} \right)^2 + \left(\frac{\partial w}{\partial x} \right)^2 \right] \\ \varepsilon_{\theta,0} &= \frac{\partial v}{R \partial \theta} + \frac{w}{R} + \frac{1}{2R^2} \left[\left(\frac{\partial u}{\partial \theta} \right)^2 + \left(\frac{\partial v}{\partial \theta} + w \right)^2 + \left(\frac{\partial w}{\partial \theta} - v \right)^2 \right] \\ \gamma_{x\theta,0} &= \frac{\partial v}{\partial x} + \frac{\partial u}{R \partial \theta} + \frac{1}{R} \left[\frac{\partial u}{\partial x} \frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial x} \left(\frac{\partial v}{\partial \theta} + w \right) + \frac{\partial w}{\partial x} \left(\frac{\partial w}{\partial \theta} - v \right) \right]\end{aligned}\quad (2.7)$$

and also, the changes in curvature and torsion,

$$\begin{aligned}
k_x &= -\frac{\partial^2 w}{\partial x^2} - \frac{\partial u}{\partial x} \frac{\partial^2 w}{\partial x^2} - \frac{\partial v}{\partial x} \frac{\partial^2 w}{R \partial x \partial \theta} + \frac{1}{R} \left(\frac{\partial v}{\partial x} \right)^2 \\
k_\theta &= -\frac{\partial^2 w}{R^2 \partial \theta^2} - \frac{w}{R^2} - \frac{w}{R^3} \left(w + \frac{\partial^2 w}{\partial \theta^2} + \frac{\partial v}{\partial \theta} \right) - \frac{\partial u}{R^2 \partial \theta} \left(\frac{\partial u}{R \partial \theta} + \frac{\partial^2 w}{\partial x \partial \theta} \right) - \frac{\partial v}{R^3 \partial \theta} \frac{\partial^2 w}{\partial \theta^2} \\
k_{x\theta} &= -2 \frac{\partial^2 w}{R \partial x \partial \theta} + \frac{\partial v}{R \partial x} - \frac{\partial u}{R^2 \partial \theta} - \frac{\partial u}{R \partial x} \left(\frac{\partial u}{R \partial \theta} + \frac{\partial^2 w}{\partial x \partial \theta} \right) + \frac{\partial u}{R^2 \partial x} \left(\frac{\partial v}{\partial \theta} - \frac{\partial^2 w}{\partial x \partial \theta} \right) - \frac{\partial^2 w}{R^2 \partial x \partial \theta} \left(w + \frac{\partial v}{\partial \theta} \right) - \frac{\partial^2 w}{\partial x^2} \frac{\partial u}{R \partial \theta} - \frac{\partial v}{\partial x} \frac{\partial^2 w}{R^2 \partial \theta^2}
\end{aligned} \quad (2.8)$$

* Another difference between Donnell's and Flügge's theory concerns the imperfections, when these are included in the study. In all shell theories, in-plane imperfections are neglected, and only the outward initial deflection w_0 is taken into account [28]. As said in the first chapter, w_0 is added to w as an extra displacement. Imperfections are inserted in the analysis through the displacement field. For example, in Flügge's theory,

$$\begin{aligned}
U_1 &= u(x, \theta) - z \frac{\partial(w + w_0)}{\partial x} \\
U_2 &= v(x, \theta) - \frac{z}{R} \left(\frac{\partial(w + w_0)}{\partial \theta} - v \right) \\
U_3 &= w(x, \theta) + w_0
\end{aligned} \quad (2.9)$$

However, Donnell accounts w_0 , only for out of plane displacements,

$$U_3 = w(x, \theta) + w_0(x, \theta) \quad (2.10)$$

This fact, in comparison to Flügge's theory, makes the changes of curvature and torsion unaffected by imperfections.

2.2.1.2 Sanders – Koiter Theory

Without doubt, this constitutes the most famous and widely accepted theory among the family of shell theories. It was first established by Sanders in 1963 and due to fact three years later Koiter arrived at the same results, the equations employed in the theory come by the name of *Sanders – Koiter equations*. Similarly to Flügge, in plane displacements u & v are considerable components of the movement and again, transverse shear strains: γ_{xz} and $\gamma_{\theta z}$ are disregarded. Nevertheless, from a mathematical aspect, Sanders – Koiter theory lead to much simpler expressions, as the changes of curvature and torsion do not contain any nonlinear terms. So, strain – displacement relations are given by,

$$\begin{aligned}
\varepsilon_{x,0} &= \frac{\partial u}{\partial x} + \frac{1}{2} \left(\frac{\partial w}{\partial x} \right)^2 + \frac{1}{8} \left(\frac{\partial v}{\partial x} - \frac{\partial u}{R \partial \theta} \right)^2 \\
\varepsilon_{\theta,0} &= \frac{\partial v}{R \partial \theta} + \frac{w}{R} + \frac{1}{2} \left(\frac{\partial w}{R \partial \theta} - \frac{v}{R} \right)^2 + \frac{1}{8} \left(\frac{\partial u}{R \partial \theta} - \frac{\partial v}{\partial x} \right)^2 \\
\gamma_{x\theta,0} &= \frac{\partial u}{R \partial \theta} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x} \left(\frac{\partial w}{R \partial \theta} - \frac{v}{R} \right)
\end{aligned} \quad (2.11)$$

and

$$\begin{aligned}
k_x &= -\frac{\partial^2 w}{\partial x^2} \\
k_\theta &= \frac{\partial v}{R^2 \partial \theta} - \frac{\partial^2 w}{R^2 \partial \theta^2} \\
k_{x\theta} &= -2 \frac{\partial^2 w}{R \partial x \partial \theta} + \frac{1}{2R} \left(3 \frac{\partial v}{\partial x} - \frac{\partial u}{R \partial \theta} \right)
\end{aligned} \quad (2.12)$$

Based on the literature, Sanders – Koiter's approach gives good results for large deformation vibrations of cylindrical shells [28].

2.2.1.3 Other Theories

Apart from the above, there are actually more nonlinear theories in order to describe cylindrical shells. One very much alike Flügge's approach, is Novozhilov's theory [29], [28]. The main difference is pointed out in the 4th hypothesis where straight lines perpendicular to the mid-surface are not elongated and so a linear

distribution is no longer the case. This leads to different and rather complex expressions for changes in curvature and torsion, even though middle surface strains remain the same. Other shell theories can be found in the works of Libai, A., & Simmonds, J. G [30] and Naghdi, P. M., & Nordgren, R. P. [31].

2.2.2 Applied Theory

The development of this thesis, apart from its scientific purpose, also served as an educational medium for my personal interest and that could also be the case for the potential reader. For this reason, it was decided that the applied shell theory will be formed from scratch in the most simplified and understanding way possible. Before we get started the coordinate system upon which the problem will be described, and should be displayed.

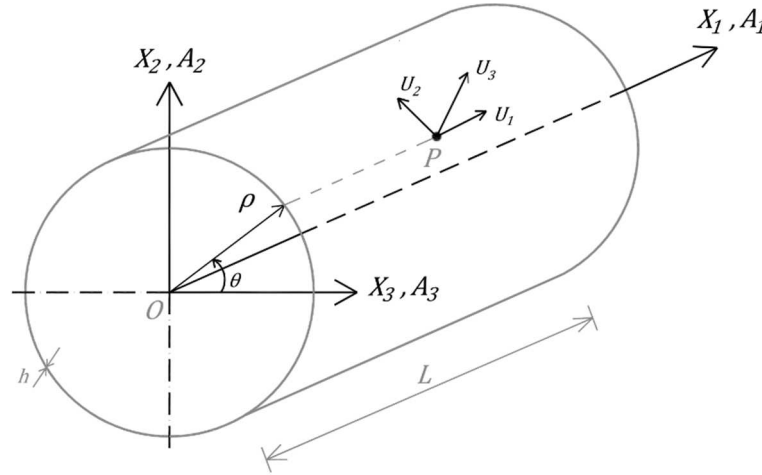


Figure 10 Global coordinate system for a cylindrical shell and displacements' notation of a generic point P

The $X_1X_2X_3$ cartesian system represents the original undeformed configuration, whereas the $A_1A_2A_3$ orthogonal system refers to the deformed final configuration. At each point P of the cylindrical shell, a local system $U_1U_2U_3$ is assigned in order to represent the displacements.

1st Step: The first crucial step is the ability of transitioning from one global system to the other. If (y_1, y_2, y_3) are the coordinates of a point in the new system and (x_1, x_2, x_3) the coordinates of the exact same point, without changing its position, in the old one, then the next transformation function can be defined,

$$y_i = y_i(x_1, x_2, x_3) \quad (2.13)$$

and inversely

$$x_i = x_i(y_1, y_2, y_3) \quad (2.14)$$

2nd Step: Now that the idea of transformation for points has been made clear, the same will be attempted for lines. If $P(x_1, x_2, x_3)$ and $P'(x_1 + dx_1, x_2 + dx_2, x_3 + dx_3)$ are two points infinitesimally close to each other of an undeformed cylindrical shell, then their distance is expressed by,

$$ds_0^2 = \sum_{i=1}^3 (dx_i)^2 = \sum_{i=1}^3 \sum_{j=1}^3 \delta_{ij} dx_i dx_j \quad (2.15)$$

Applying the chain rule for a multivariable function,

$$dx_i = \sum_{k=1}^3 \frac{\partial x_i}{\partial y_k} dy_k \quad \& \quad dx_j = \sum_{m=1}^3 \frac{\partial x_j}{\partial y_m} dy_m \quad (2.16)$$

the relation (2.15) can now be written in terms of the new coordinates,

$$ds_0^2 = \sum_{k=1}^3 \sum_{m=1}^3 \sum_{i=1}^3 \sum_{j=1}^3 \delta_{ij} \frac{\partial x_i}{\partial y_k} \frac{\partial x_j}{\partial y_m} dy_k dy_m \quad (2.17)$$

or by dropping the Kronecker δ_{ij} ,

$$ds_0^2 = \sum_{k=1}^3 \sum_{m=1}^3 \sum_{i=1}^3 \frac{\partial x_i}{\partial y_k} \frac{\partial x_j}{\partial y_m} dy_k dy_m \quad (2.18)$$

In order to get a more simplified version, the following functions are defined,

$$g_{km} = \sum_{i=1}^3 \frac{\partial x_i}{\partial y_k} \frac{\partial x_i}{\partial y_m} \quad (2.19)$$

and so finally,

$$ds_0^2 = \sum_{k=1}^3 \sum_{m=1}^3 g_{km} dy_k dy_m \quad (2.20)$$

3rd Step: Let us now assume that the initial points $P(x_1, x_2, x_3)$ and $P'(x_1 + dx_1, x_2 + dx_2, x_3 + dx_3)$ have undergone a displacement. Their new position is denoted by $P_{new}(a_1, a_2, a_3)$ and $P'_{new}(a_1 + da_1, a_2 + da_2, a_3 + da_3)$ accordingly. The coordinate system of the deformed configuration $A_1 A_2 A_3$ coincides with the initial one $X_1 X_2 X_3$ and so their distance is given in correspondence with (2.15) by,

$$ds^2 = \sum_{k=1}^3 \sum_{m=1}^3 \delta_{km} da_k da_m \quad (2.21)$$

Since it is presumed again,

$$a_i = a_i(x_1, x_2, x_3) \quad \& \quad y_i = y_i(a_1, a_2, a_3) \quad (2.22)$$

the concept of the chain rule is applied for the differentials da_k and da_m ,

$$da_k = \sum_{i=1}^3 \frac{\partial a_k}{\partial y_i} dy_i \quad \& \quad da_m = \sum_{j=1}^3 \frac{\partial a_m}{\partial y_j} dy_j \quad (2.23)$$

which means that (2.20) becomes,

$$ds^2 = \sum_{k=1}^3 \sum_{m=1}^3 \sum_{i=1}^3 \sum_{j=1}^3 \delta_{km} \frac{\partial a_k}{\partial y_i} \frac{\partial a_m}{\partial y_j} dy_i dy_j \quad (2.24)$$

4th Step: Derivation of strains is based on the difference of a line's length after some form of deformation. The same concept will be applied here using the commutative law of addition, so

$$ds^2 - ds_0^2 = \sum_{i=1}^3 \sum_{j=1}^3 \left(\sum_{k=1}^3 \sum_{m=1}^3 \delta_{km} \frac{\partial a_k}{\partial y_i} \frac{\partial a_m}{\partial y_j} - g_{ij} \right) dy_i dy_j \quad (2.25)$$

By definition Green's strains are given by,

$$ds^2 - ds_0^2 = 2 \sum_{i=1}^3 \sum_{j=1}^3 \varepsilon_{ij} \sqrt{g_{ii}} dy_i \sqrt{g_{jj}} dy_j \quad (2.26)$$

Comparing (2.25) and (2.26) it is obvious that,

$$\varepsilon_{ij} = \frac{1}{2} \left[\left(\sum_{k=1}^3 \sum_{m=1}^3 \delta_{km} \frac{\partial a_k}{\partial y_i} \frac{\partial a_m}{\partial y_j} \right) - g_{ij} \right] \frac{1}{\sqrt{g_{ii}}} \frac{1}{\sqrt{g_{jj}}} \quad (2.27)$$

5th Step: Usually for shell theories, it is convenient the usage of cylindrical coordinates. For this reason, it is considered that (y_1, y_2, y_3) correspond to (x, θ, ρ) where $\rho = R + z$ refers to a generic point of the shell's thickness. If the purpose was the transformation into spherical coordinates, (φ, θ, ρ) in place of (y_1, y_2, y_3) would be used instead. In our case, the cartesian coordinates of points $P(x_1, x_2, x_3)$ & $P_{new}(a_1, a_2, a_3)$ of the undeformed and deformed configuration are expressed based on the figure below,

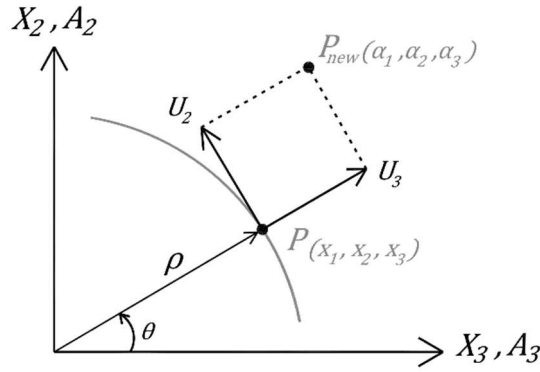


Figure 11 Final position of point P after displacement

$$x_1 = x, \quad x_2 = \rho \sin \theta, \quad x_3 = \rho \cos \theta \quad (2.28)$$

&

$$\alpha_1 = x + U_1, \quad \alpha_2 = \rho \sin \theta + U_3 \sin \theta + U_2 \cos \theta, \quad \alpha_3 = \rho \cos \theta + U_3 \cos \theta - U_2 \sin \theta \quad (2.29)$$

All the elements for calculating the strain – displacement relations have now been determined. For example, $\varepsilon_{11} \equiv \varepsilon_{xx}$ is computed as:

$$\varepsilon_{11} = \frac{1}{2} \left[\left(\sum_{k=1}^3 \sum_{m=1}^3 \delta_{km} \frac{\partial a_k}{\partial y_1} \frac{\partial a_m}{\partial y_1} \right) - g_{11} \right] \frac{1}{\sqrt{g_{11}}} \frac{1}{\sqrt{g_{11}}} \quad (2.30)$$

where

$$g_{11} = \left(\frac{\partial x_1}{\partial x} \right)^2 + \left(\frac{\partial x_2}{\partial x} \right)^2 + \left(\frac{\partial x_3}{\partial x} \right)^2 = 1^2 + 0^2 + 0^2 = 1 \quad (2.31a)$$

$$\left(\frac{\partial a_1}{\partial x} \right)^2 = \left(1 + \frac{\partial U_1}{\partial x} \right)^2 = 1 + 2 \frac{\partial U_1}{\partial x} + \left(\frac{\partial U_1}{\partial x} \right)^2 \quad (2.31b)$$

$$\left(\frac{\partial a_2}{\partial x} \right)^2 = \left(\frac{\partial U_3}{\partial x} \sin \theta + \frac{\partial U_2}{\partial x} \cos \theta \right)^2 = \left(\frac{\partial U_3}{\partial x} \right)^2 \sin^2 \theta + 2 \frac{\partial U_3}{\partial x} \frac{\partial U_2}{\partial x} \sin \theta \cos \theta + \left(\frac{\partial U_2}{\partial x} \right)^2 \cos^2 \theta \quad (2.31c)$$

$$\left(\frac{\partial a_3}{\partial x} \right)^2 = \left(\frac{\partial U_3}{\partial x} \cos \theta - \frac{\partial U_2}{\partial x} \sin \theta \right)^2 = \left(\frac{\partial U_3}{\partial x} \right)^2 \cos^2 \theta - 2 \frac{\partial U_3}{\partial x} \frac{\partial U_2}{\partial x} \sin \theta \cos \theta + \left(\frac{\partial U_2}{\partial x} \right)^2 \sin^2 \theta \quad (2.31d)$$

Substituting (2.31) into (2.30) the final expression is given by,

$$\varepsilon_{11} \equiv \varepsilon_{xx} = \frac{\partial U_1}{\partial x} + \frac{1}{2} \left[\left(\frac{\partial U_1}{\partial x} \right)^2 + \left(\frac{\partial U_2}{\partial x} \right)^2 + \left(\frac{\partial U_3}{\partial x} \right)^2 \right] \quad (2.32)$$

Following the same procedure for the rest cases by also introducing the thinness assumption $\rho = R + z \approx R$, with R and z as they are depicted in *fig. 9*, the final strain – displacement relations, including (2.32) are derived:

$$\varepsilon_{xx} = \frac{\partial U_1}{\partial x} + \frac{1}{2} \left[\left(\frac{\partial U_1}{\partial x} \right)^2 + \left(\frac{\partial U_2}{\partial x} \right)^2 + \left(\frac{\partial U_3}{\partial x} \right)^2 \right] \quad (2.33a)$$

$$\varepsilon_{\theta\theta} = \frac{1}{R} \frac{\partial U_2}{\partial \theta} + \frac{U_3}{R} + \frac{1}{2} \left[\left(\frac{1}{R} \frac{\partial U_1}{\partial \theta} \right)^2 + \left(\frac{1}{R} \frac{\partial U_2}{\partial \theta} + \frac{U_3}{R} \right)^2 + \left(\frac{1}{R} \frac{\partial U_3}{\partial \theta} - \frac{U_2}{R} \right)^2 \right] \quad (2.33b)$$

$$\varepsilon_{zz} = \frac{\partial U_3}{\partial z} + \frac{1}{2} \left[\left(\frac{\partial U_1}{\partial z} \right)^2 + \left(\frac{\partial U_2}{\partial z} \right)^2 + \left(\frac{\partial U_3}{\partial z} \right)^2 \right] \xrightarrow{\text{always taken as zero}} \varepsilon_{zz} = 0 \quad (2.33c)$$

$$\gamma_{x\theta} = \frac{\partial U_2}{\partial x} + \frac{1}{R} \frac{\partial U_1}{\partial \theta} + \frac{1}{R} \frac{\partial U_1}{\partial x} \frac{\partial U_1}{\partial \theta} + \left(\frac{1}{R} \frac{\partial U_2}{\partial \theta} + \frac{U_3}{R} \right) \frac{\partial U_2}{\partial x} + \frac{\partial U_3}{\partial x} \left(\frac{1}{R} \frac{\partial U_3}{\partial \theta} - \frac{U_2}{R} \right) \quad (2.33d)$$

$$\gamma_{xz} = \frac{\partial U_1}{\partial z} + \frac{\partial U_3}{\partial x} + \frac{\partial U_1}{\partial x} \frac{\partial U_1}{\partial z} + \frac{\partial U_2}{\partial x} \frac{\partial U_2}{\partial z} + \frac{\partial U_2}{\partial x} \frac{\partial U_2}{\partial z} + \frac{\partial U_3}{\partial x} \frac{\partial U_3}{\partial z} \quad (2.33e)$$

$$\gamma_{\theta z} = \frac{\partial U_2}{\partial z} + \frac{1}{1 + z/R} \left[\left(\frac{1}{R} \frac{\partial U_3}{\partial \theta} - \frac{U_2}{R} \right) + \left(\frac{1}{R} \frac{\partial U_2}{\partial \theta} + \frac{U_3}{R} \right) \frac{\partial U_2}{\partial z} + \frac{1}{R} \frac{\partial U_1}{\partial \theta} \frac{\partial U_1}{\partial z} + \left(\frac{1}{R} \frac{\partial U_3}{\partial \theta} - \frac{U_2}{R} \right) \frac{\partial U_3}{\partial z} \right] \quad (2.33f)$$

In case, the deflection is considered positive inwardly, as it is applied in many books, the sign of w in the relations should be the opposite. For the simplest version of the strains which may be used in a nonlinear elastic problem in which $z/R \approx 0$ is considered small, relations (2.33) can be reduced down to,

$$\varepsilon_{xx} = \frac{\partial U_1}{\partial x} + \frac{1}{2} \left(\frac{\partial U_3}{\partial x} \right)^2 \quad (2.34a)$$

$$\varepsilon_{\theta\theta} = \frac{1}{R} \frac{\partial U_2}{\partial \theta} + \frac{U_3}{R} + \frac{1}{2} \left(\frac{1}{R} \frac{\partial U_3}{\partial \theta} \right)^2 \quad (2.34b)$$

$$\varepsilon_{zz} = 0 \quad (2.34c)$$

$$\gamma_{x\theta} = \frac{\partial U_2}{\partial x} + \frac{1}{R} \frac{\partial U_1}{\partial \theta} + \frac{\partial U_3}{\partial x} \frac{1}{R} \frac{\partial U_3}{\partial \theta} \quad (2.34d)$$

$$\gamma_{xz} = \frac{\partial U_1}{\partial z} + \frac{\partial U_3}{\partial x} \quad (2.34e)$$

$$\gamma_{\theta z} = \frac{\partial U_2}{\partial z} + \frac{1}{R} \frac{\partial U_3}{\partial \theta} \quad (2.34f)$$

For in-plane strains γ_{xz} and $\gamma_{\theta z}$, since they are often neglected in most studies due to their relatively small value compared to the rest, their linear form is assumed to be adequate in calculations. The linear term U_2/R of $\gamma_{\theta z}$ has been disregarded for symmetry reasons.

Displacement Field

Of course, a shell theory cannot be considered complete without the appropriate displacement field. For the general case, we shall depend on Basset [32], who was the pioneer in developing the displacement concept, upon which many scientists have built their theories. Basset represented displacements as a series expansion in terms of z ,

$$\begin{aligned} U_1 &= u(x, \theta) + z \beta_x(x, \theta) + z^2 \varphi_x(x, \theta) + z^3 \psi_x(x, \theta) + z^4 \omega_x(x, \theta) \\ U_2 &= \left(1 + \frac{z}{R} \right) v(x, \theta) + z \beta_\theta(x, \theta) + z^2 \varphi_\theta(x, \theta) + z^3 \psi_\theta(x, \theta) + z^4 \omega_\theta(x, \theta) \\ U_3 &= w(x, \theta) \end{aligned} \quad (2.35)$$

The mid-surface strains are expressed again by u , v & w , while β_i are the rotations of transverse normals to the middle surface. The angles φ_i , ψ_i , ω_i are calculated in terms of u , v , w & β_i , by setting the shear stresses γ_{xz} and $\gamma_{\theta z}$ at the top and the bottom equal to zero [33].

$$\gamma_{xz} \Big|_{z=\pm \frac{h}{2}} = 0 \quad \& \quad \gamma_{\theta z} \Big|_{z=\pm \frac{h}{2}} = 0 \quad (2.36)$$

The number of z^n that are kept, determine the order of displacement theory. The higher the order the higher accuracy. In order to get a clearer picture, the three options which are mainly used, are depicted in the figure below [32].

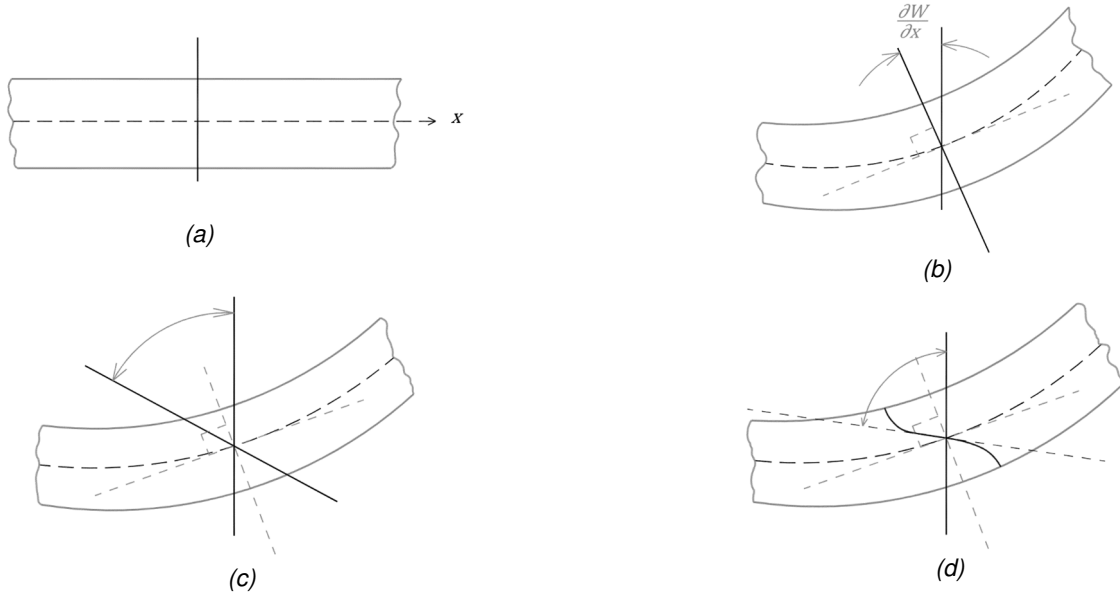


Figure 12 Displacement field of a circular cylindrical shell (a) undeformed, (b) deformed according to Kirchhoff, (c) deformed according to 1st order theory (transverse normal to the middle surface remain straight but not normal), (d) deformed according to 3rd order theory (transverse normal to the middle surface does not remain neither straight nor normal)

The two latter alternatives correspond to shear deformation theories, where the shell thickness plays an important role in the analysis. Since the shell is considered thin, Kirchhoff-Love approximations *fig. 12a*, also described in the previous section, along with $\frac{z}{R} \approx 0$ are regarded as an adequate choice for our case, meaning that,

$$\begin{aligned} U_1 &= u(x, \theta) + z \beta_x(x, \theta) \text{ with } \beta_x(x, \theta) = -\frac{\partial w}{\partial x} \\ U_2 &= v(x, \theta) + z \beta_\theta(x, \theta) \text{ with } \beta_\theta(x, \theta) = -\frac{\partial w}{R \partial \theta} \\ U_3 &= w(x, \theta) \end{aligned} \quad (2.37)$$

In conclusion, it should be noted that the initial intention was to also study dynamic buckling. In that case, the inclusion of imperfections would be a necessity, since according to Shaw* [2] the critical buckling load will result in the same value for both static and dynamic scenarios if the shell is *perfect*. Due to the parameter of imperfections, the probability for appearance of “large” pre-buckling deflections is much higher and thus a nonlinear shell theory should be chosen. The use of nonlinear equations gives the opportunity of analyzing shell’s vibrations that are accompanied by large deformations, and secondly, of studying the impact of geometric imperfections on the dynamic behavior. For instance, according to Kubenko [11] and Amabili [28], for larger amplitudes of response, imperfections of a shell may alter a hardening system (resonance frequency increases with amplitude) to a softening one (resonance frequency decreases with amplitude). Despite the fact, dynamic buckling will not be discussed in the following paragraphs eventually, the present thesis can act as an informative basis on how to approach nonlinear more complex problems in the future.

* Shaw [2] also studied the effect of imperfections on the critical buckling load for shells, under different types of forcing. It has been found that dynamic loads are not always lower than the static ones, as was expected for imperfect shells based on the rest of the literature. Imperfections tend to induce a reduction in the critical load only in the case of compression. It seemed that shear buckling load, due to torsional forcing, to be unaffected by the presence of imperfections. The investigation of such matter would be very interesting for liquid filled tanks under seismic motion but that will not be covered in the current thesis.

2.3 Equations of Motion

Although it may seem strange, more than a single way exists to formulate the equations of motion in order to study buckling. Lindberg [9] offers a practical and rather comprehensive approach by taking the equilibrium of stresses acting on an infinitesimal element of a plate. The adaptation to a shell is made through an adjustment in geometry in order to include the effect of curvature. Equilibrium method is not that versatile, and so in order to study the more general case, someone needs to resort to energy methods. The main ones include: 1) The principle of stationary value of the total potential energy, also mentioned in section 1.2 and applied by many, such as Shaw [2] and Robert Jones [7], 2) The Euler – Lagrange equations for functions of several variables. For an illustration, one may look into the work of Amabili [28] whereas a fairly extensive introduction of the theory can be found in [34], 3) Hamilton's principle. All the above methods produce an identical mathematical result, since they express the same concept with a different choice of words. Here, the Hamilton's principle will be applied and the procedure according to Soedel [25], will be followed.

Hamilton's principle:

"A dynamical system, for a specific time period, moves in space from one point to another in that particular path, which minimizes the time integral of the difference between kinetic and potential energy."

Its mathematical description is given by:

$$\delta \int_{t_0}^{t_1} (U - K - W_{in}) dt = 0 \quad (2.38)$$

In the static case, the kinetic energy K is zero and so (2.37) can be reformulated into,

$$\delta \int_{t_0}^{t_1} (U - W_{in}) dt = 0 \Rightarrow (U - W_{in}) = 0 \quad (2.39)$$

The mathematical symbol δ , declares the variation of the integral's energy quantities and is treated as a differential. The elastic strain U caused by the internal stresses σ_{ij} & τ_{ij} , in an infinitesimal volume of a cylindrical shell with length L and thickness h , is equal to,

$$U = \frac{1}{2} \int_0^L \int_0^{2\pi} \int_{-h/2}^{h/2} (\sigma_{xx}\varepsilon_{xx} + \sigma_{\theta\theta}\varepsilon_{\theta\theta} + \tau_{x\theta}\gamma_{x\theta} + \tau_{xz}\gamma_{xz} + \tau_{\theta z}\gamma_{\theta z}) dz R d\theta dx \quad (2.40)$$

$$\Rightarrow \delta U = \int_0^L \int_0^{2\pi} \int_{-h/2}^{h/2} (\sigma_{xx}\delta\varepsilon_{xx} + \sigma_{\theta\theta}\delta\varepsilon_{\theta\theta} + \tau_{x\theta}\delta\gamma_{x\theta} + \tau_{xz}\delta\gamma_{xz} + \tau_{\theta z}\delta\gamma_{\theta z}) dz R d\theta dx \quad (2.41)$$

As it can be observed, the normal stresses σ_{zz} , are neglected according to Kirchhoff's hypothesis. The internal stresses by definition, for an isotropic and homogeneous material, are given by

$$\sigma_{xx} = \frac{E}{1-\nu^2}(\varepsilon_{xx} + \nu\varepsilon_{\theta\theta}), \quad \sigma_{\theta\theta} = \frac{E}{1-\nu^2}(\varepsilon_{\theta\theta} + \nu\varepsilon_{xx}) \quad (2.42a, b)$$

$$\tau_{x\theta} = G\gamma_{x\theta} = \frac{E}{2(1+\nu)}\gamma_{x\theta}, \quad \tau_{xz} = G\gamma_{xz} = \frac{E}{2(1+\nu)}\gamma_{xz}, \quad \tau_{\theta z} = G\gamma_{\theta z} = \frac{E}{2(1+\nu)}\gamma_{\theta z} \quad (2.43a, b, c)$$

where ν is the Poisson's ratio and E the Young's modulus. Their existence, lead to internal forces and moments (fig. 13) per unit length,

$$N_{xx} = \int_{-h/2}^{h/2} \sigma_{xx} dz, \quad N_{\theta\theta} = \int_{-h/2}^{h/2} \sigma_{\theta\theta} dz, \quad N_{x\theta} = \int_{-h/2}^{h/2} \tau_{x\theta} dz \quad (2.44a, b, c)$$

$$M_{xx} = \int_{-h/2}^{h/2} \sigma_{xx} z dz, \quad M_{\theta\theta} = \int_{-h/2}^{h/2} \sigma_{\theta\theta} z dz, \quad M_{x\theta} = \int_{-h/2}^{h/2} \tau_{x\theta} z dz, \quad M_{\theta z} = \int_{-h/2}^{h/2} \tau_{\theta z} z dz \quad (2.45a, b, c, d)$$

$$Q_{xz} = \int_{-h/2}^{h/2} \tau_{xz} dz, \quad Q_{\theta z} = \int_{-h/2}^{h/2} \tau_{\theta z} dz \quad (2.46a, b)$$

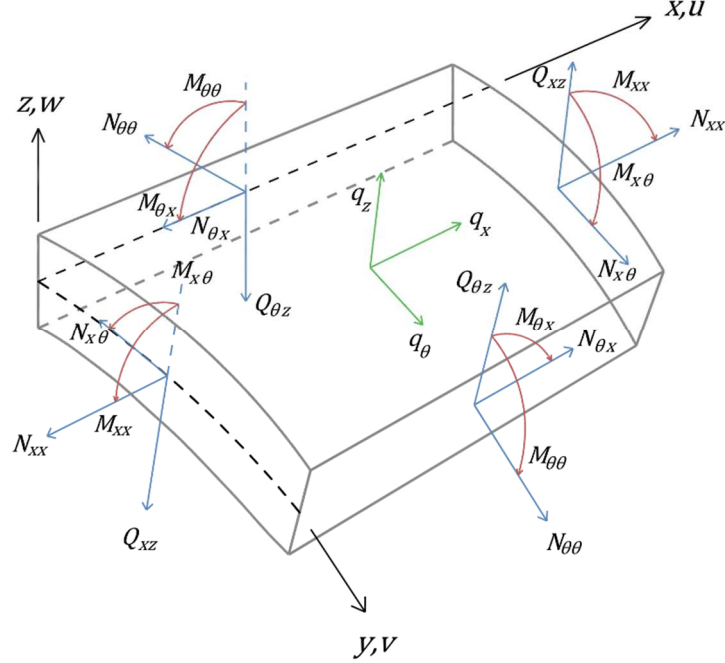


Figure 13 Internal forces and moments per unit length for an infinitesimal shell segment

The potential energy-work exerted by external forces, also regarded as an input energy to the system W_{in} , can be described as the sum of two sub-energies,

$$W_{in} = E_L + E_B \quad (2.47)$$

where E_L , is the energy induced by the distributed forces q_x, q_θ and q_z per unit area (fig. 13),

$$E_L = \int_0^L \int_0^{2\pi} (q_x u + q_\theta v + q_z w) R d\theta dx \Rightarrow \delta E_L = \int_0^L \int_0^{2\pi} (q_x \delta u + q_\theta \delta v + q_z \delta w) R d\theta dx \quad (2.48)$$

and E_B the energy due to the applied N_{ij}^*, Q_{ij}^* and M_{ij}^* at the boundary edges of the shell. In a similar way its variation is given by,

$$\begin{aligned} \delta E_B = & \int_0^L (N_{\theta x}^* \delta u + N_{\theta\theta}^* \delta v + Q_{\theta z}^* \delta w + M_{\theta x}^* \delta \beta_x + M_{\theta\theta}^* \delta \beta_\theta) dx \\ & + \int_0^{2\pi} (N_{xx}^* \delta u + N_{x\theta}^* \delta v + Q_{xz}^* \delta w + M_{xx}^* \delta \beta_x + M_{x\theta}^* \delta \beta_\theta) R d\theta \end{aligned} \quad (2.49)$$

It is assumed that all the aforementioned external forces act on the shell's mid-surface. Utilizing now the simplest version of the strain-displacement relations (2.34) along with the Hamilton's principle (2.38) by following the procedure shown in the book of Soedel [25], the equations of motion along with their boundary conditions are derived:

Equations of Motion

$$\frac{\partial N_{xx}}{\partial x} + \frac{\partial N_{x\theta}}{R\partial\theta} = 0 \quad (2.50a)$$

$$\frac{\partial N_{\theta\theta}}{R\partial\theta} + \frac{\partial N_{x\theta}}{\partial x} = 0 \quad (2.50b)$$

$$\begin{aligned} \frac{\partial N_{xx}}{\partial x} \frac{\partial w}{\partial x} + N_{xx} \frac{\partial^2 w}{\partial x^2} - \frac{N_{\theta\theta}}{R} + \frac{\partial N_{\theta\theta}}{R\partial\theta} \frac{\partial w}{\partial\theta} + N_{\theta\theta} \frac{\partial^2 w}{R^2 \partial\theta^2} + \frac{\partial N_{x\theta}}{R\partial\theta} \frac{\partial w}{\partial x} + N_{x\theta} \frac{\partial^2 w}{\partial x R \partial\theta} + \frac{\partial N_{x\theta}}{\partial x} \frac{\partial w}{R\partial\theta} \\ + N_{x\theta} \frac{\partial^2 w}{R\partial\theta \partial x} + \frac{\partial Q_{xz}}{\partial x} + \frac{\partial Q_{\theta z}}{R\partial\theta} = -q_z \end{aligned} \quad (2.50c)$$

where Q_{xz} and $Q_{\theta z}$, can be found through,

$$\frac{\partial M_{xx}}{\partial x} + \frac{\partial M_{x\theta}}{R\partial\theta} - Q_{xz} = 0 \quad (2.51a)$$

$$\frac{\partial M_{x\theta}}{\partial x} + \frac{\partial M_{\theta\theta}}{R\partial\theta} - Q_{\theta z} = 0 \quad (2.51b)$$

Boundary Conditions

$$\begin{aligned} N_{xx}^* &= N_{xx} & N_{\theta x}^* &= N_{x\theta} \\ N_{x\theta}^* &= N_{x\theta} & N_{\theta\theta}^* &= N_{\theta\theta} \\ Q_{xz}^* &= N_{xx} \frac{\partial w}{\partial x} + N_{x\theta} \frac{1}{R} \frac{\partial w}{\partial\theta} + Q_{xz} & Q_{\theta z}^* &= \frac{1}{R} N_{\theta\theta} \frac{\partial w}{\partial\theta} + N_{x\theta} \frac{\partial w}{\partial x} + Q_{\theta z} \\ M_{xx}^* &= M_{xx} & M_{\theta\theta}^* &= M_{\theta\theta} \end{aligned} \quad (2.52)$$

In the case of liquid filled tank, the distributed force due to water pressure is equal to $q_z = -f_w = -\gamma(L - x)$, with $\gamma = \rho_w \cdot g$ the water's weight density, whereas $q_x = q_\theta = 0$. In order to get a clearer picture of the nonlinearity involved from a mathematical point of view, equations (2.50) & (2.51) are rewritten in terms of displacements. It should be noted here, that in order to avoid any confusion between the symbols for circumferential displacement, v , and Poisson's ratio ν , the greek letter " ν " has been replaced by " n ".

Equations of Motion

$$\begin{aligned} DhR \frac{\partial^2 u(x, \theta)}{\partial x^2} + \frac{Gh}{R} \frac{\partial^2 u(x, \theta)}{\partial\theta^2} + (Gh + Dhn) \frac{\partial^2 v(x, \theta)}{\partial x \partial\theta} + Dhn \frac{\partial w(x, \theta)}{\partial x} + \frac{Gh}{R} \frac{\partial w(x, \theta)}{\partial x} \frac{\partial^2 w(x, \theta)}{\partial\theta^2} \\ + \left(\frac{Gh}{R} + \frac{Dhn}{R} \right) \frac{\partial w(x, \theta)}{\partial\theta} \frac{\partial^2 w(x, \theta)}{\partial x \partial\theta} + DhR \frac{\partial w(x, \theta)}{\partial x} \frac{\partial^2 w(x, \theta)}{\partial x^2} = 0 \end{aligned} \quad (2.53a)$$

&

$$\begin{aligned} (DhnR + GhR) \frac{\partial^2 u(x, \theta)}{\partial x \partial\theta} + GhR^2 \frac{\partial^2 v(x, \theta)}{\partial x^2} + Dh \frac{\partial^2 v(x, \theta)}{\partial\theta^2} + Dh \frac{\partial w(x, \theta)}{\partial\theta} + \frac{Dh}{R} \frac{\partial w(x, \theta)}{\partial\theta} \frac{\partial^2 w(x, \theta)}{\partial\theta^2} \\ + (GhR + DhnR) \frac{\partial w(x, \theta)}{\partial x} \frac{\partial^2 w(x, \theta)}{\partial x \partial\theta} + GhR \frac{\partial w(x, \theta)}{\partial\theta} \frac{\partial^2 w(x, \theta)}{\partial x^2} = 0 \end{aligned} \quad (2.53b)$$

&

$$\begin{aligned} \frac{Dhw(x, \theta)}{R} + \frac{Dh}{R} \frac{\partial v(x, \theta)}{\partial\theta} - \frac{Dh}{2R^2} \frac{\partial w(x, \theta)^2}{\partial\theta} - \frac{Dh}{R^2} \frac{\partial w(x, \theta)}{\partial\theta} \frac{\partial^2 v(x, \theta)}{\partial\theta^2} - \frac{Dh}{R^2} w(x, \theta) \frac{\partial^2 w(x, \theta)}{\partial\theta^2} - \frac{Dh}{R^2} \frac{\partial v(x, \theta)}{\partial\theta} \frac{\partial^2 w(x, \theta)}{\partial\theta^2} \\ - \frac{3Dh}{2R^3} \frac{\partial w(x, \theta)^2}{\partial\theta} \frac{\partial^2 w(x, \theta)}{\partial\theta^2} + \frac{Dh^3}{12R^3} \frac{\partial^4 w(x, \theta)}{\partial\theta^4} + Dhn \frac{\partial u(x, \theta)}{\partial x} - \frac{Dhn}{R} \frac{\partial^2 w(x, \theta)}{\partial\theta^2} \frac{\partial u(x, \theta)}{\partial x} - \frac{Gh}{R} \frac{\partial^2 u(x, \theta)}{\partial\theta^2} \frac{\partial w(x, \theta)}{\partial x} \\ - \frac{1}{2} Dhn \frac{\partial w(x, \theta)^2}{\partial x} - \left(\frac{Gh}{R} + \frac{Dhn}{2R} \right) \frac{\partial^2 w(x, \theta)}{\partial\theta^2} \frac{\partial w(x, \theta)^2}{\partial x} - \left(\frac{Gh}{R} + \frac{Dhn}{R} \right) \frac{\partial w(x, \theta)}{\partial\theta} \frac{\partial^2 u(x, \theta)}{\partial x \partial\theta} - (Gh + Dhn) \frac{\partial w(x, \theta)}{\partial x} \frac{\partial^2 v(x, \theta)}{\partial x \partial\theta} \\ - \frac{2Gh}{R} \frac{\partial u(x, \theta)}{\partial\theta} \frac{\partial^2 w(x, \theta)}{\partial x \partial\theta} - 2Gh \frac{\partial v(x, \theta)}{\partial x} \frac{\partial^2 w(x, \theta)}{\partial x \partial\theta} - \left(\frac{4Gh}{R} + \frac{2Dhn}{R} \right) \frac{\partial w(x, \theta)}{\partial\theta} \frac{\partial w(x, \theta)}{\partial x} \frac{\partial^2 w(x, \theta)}{\partial x \partial\theta} - DhR \frac{\partial w(x, \theta)}{\partial x} \frac{\partial^2 u(x, \theta)}{\partial x^2} \\ - Gh \frac{\partial w(x, \theta)}{\partial\theta} \frac{\partial^2 v(x, \theta)}{\partial x^2} - Dhnw(x, \theta) \frac{\partial^2 w(x, \theta)}{\partial x^2} - Dhn \frac{\partial v(x, \theta)}{\partial\theta} \frac{\partial^2 w(x, \theta)}{\partial x^2} - \left(\frac{Gh}{R} + \frac{Dhn}{2R} \right) \frac{\partial w(x, \theta)^2}{\partial\theta} \frac{\partial^2 w(x, \theta)}{\partial x^2} \end{aligned}$$

$$-DhR \frac{\partial u(x, \theta)}{\partial x} \frac{\partial^2 w(x, \theta)}{\partial x^2} - \frac{3}{2} DhR \frac{\partial w(x, \theta)}{\partial x} \frac{\partial^2 w(x, \theta)}{\partial x^2} + \left(\frac{Gh^3}{3R} + \frac{Dh^3 n}{6R} \right) \frac{\partial^4 w(x, \theta)}{\partial x^2 \partial \theta^2} + \frac{Dh^3}{12} R \frac{\partial^4 w(x, \theta)}{\partial x^4} = R\gamma(L - x) \quad (2.53c)$$

where $D = \frac{E}{1-\nu^2}$ and $G = \frac{E}{2(1+\nu)}$.

For the general case of strains – displacements (2.33a-d) where all the nonlinear terms are present, except of course again for the shear strains γ_{xz} and $\gamma_{\theta z}$, which remain as before, the resulted equations take a much more complex form,

Equations of Motion

$$-R \left[\frac{\partial N_{xx}}{\partial x} + \left(\frac{\partial N_{xx}}{\partial x} \frac{\partial u}{\partial x} + N_{xx} \frac{\partial^2 u}{\partial x^2} \right) + \left(\frac{\partial M_{xx}}{\partial x} \frac{\partial \beta_x}{\partial x} + M_{xx} \frac{\partial^2 \beta_x}{\partial x^2} \right) \right] - \frac{1}{R} \left(\frac{\partial N_{\theta\theta}}{\partial \theta} \frac{\partial u}{\partial \theta} + N_{\theta\theta} \frac{\partial^2 u}{\partial \theta^2} + \frac{\partial M_{\theta\theta}}{\partial \theta} \frac{\partial \beta_x}{\partial \theta} + M_{\theta\theta} \frac{\partial^2 \beta_x}{\partial \theta^2} \right) \quad (2.54a)$$

$$- \left[\frac{\partial N_{x\theta}}{\partial \theta} + \left(\frac{\partial N_{x\theta}}{\partial \theta} \frac{\partial u}{\partial x} + N_{x\theta} \frac{\partial^2 u}{\partial x \partial \theta} \right) + \left(\frac{\partial N_{x\theta}}{\partial x} \frac{\partial u}{\partial \theta} + N_{x\theta} \frac{\partial^2 u}{\partial \theta \partial x} \right) + \left(\frac{\partial M_{x\theta}}{\partial x} \frac{\partial \beta_x}{\partial \theta} + M_{x\theta} \frac{\partial^2 \beta_x}{\partial x \partial \theta} \right) + \left(\frac{\partial M_{x\theta}}{\partial \theta} \frac{\partial \beta_x}{\partial x} + M_{x\theta} \frac{\partial^2 \beta_x}{\partial \theta \partial x} \right) \right] = -\gamma h \ddot{u} R + q_x R$$

$$- \left(\frac{\partial N_{\theta\theta}}{\partial \theta} \right) - \frac{1}{R} \left(\frac{\partial N_{\theta\theta}}{\partial \theta} w + 2 N_{\theta\theta} \frac{\partial w}{\partial \theta} - N_{\theta\theta} v + \frac{\partial N_{\theta\theta}}{\partial \theta} \frac{\partial v}{\partial \theta} + N_{\theta\theta} \frac{\partial^2 v}{\partial \theta^2} - M_{\theta\theta} \beta_\theta + \frac{\partial M_{\theta\theta}}{\partial \theta} \frac{\partial \beta_\theta}{\partial \theta} + M_{\theta\theta} \frac{\partial^2 \beta_\theta}{\partial \theta^2} \right)$$

$$- \left[R \frac{\partial N_{x\theta}}{\partial x} + \left(\frac{\partial N_{x\theta}}{\partial x} w + N_{x\theta} \frac{\partial w}{\partial x} \right) + N_{x\theta} \frac{\partial w}{\partial x} + \left(\frac{\partial N_{x\theta}}{\partial \theta} \frac{\partial v}{\partial x} + N_{x\theta} \frac{\partial^2 v}{\partial x \partial \theta} \right) + \left(\frac{\partial N_{x\theta}}{\partial x} \frac{\partial v}{\partial \theta} + N_{x\theta} \frac{\partial^2 v}{\partial \theta \partial x} \right) + \left(\frac{\partial M_{x\theta}}{\partial x} \frac{\partial \beta_\theta}{\partial \theta} + M_{x\theta} \frac{\partial^2 \beta_\theta}{\partial x \partial \theta} \right) \right. \\ \left. + \left(\frac{\partial M_{x\theta}}{\partial \theta} \frac{\partial \beta_\theta}{\partial x} + M_{x\theta} \frac{\partial^2 \beta_\theta}{\partial \theta \partial x} \right) \right] - R \left[\left(\frac{\partial N_{xx}}{\partial x} \frac{\partial v}{\partial x} + N_{xx} \frac{\partial^2 v}{\partial x^2} \right) + \left(\frac{\partial M_{xx}}{\partial x} \frac{\partial \beta_\theta}{\partial x} + M_{xx} \frac{\partial^2 \beta_\theta}{\partial x^2} \right) \right] - Q_{\theta z} = -\gamma h \ddot{v} R + q_\theta R \quad (2.54b)$$

$$-R \left(\frac{\partial N_{xx}}{\partial x} \frac{\partial w}{\partial x} + N_{xx} \frac{\partial^2 w}{\partial x^2} \right) + N_{\theta\theta} + \frac{1}{R} \left(2 N_{\theta\theta} \frac{\partial v}{\partial \theta} + \frac{\partial N_{\theta\theta}}{\partial \theta} v + N_{\theta\theta} w - \frac{\partial N_{\theta\theta}}{\partial \theta} \frac{\partial w}{\partial \theta} - N_{\theta\theta} \frac{\partial^2 w}{\partial \theta^2} + \frac{\partial M_{\theta\theta}}{\partial \theta} \beta_\theta + 2 M_{\theta\theta} \frac{\partial \beta_\theta}{\partial \theta} \right) + N_{x\theta} \frac{\partial v}{\partial x} \\ + \left(\frac{\partial N_{x\theta}}{\partial x} v + N_{x\theta} \frac{\partial v}{\partial x} \right) - \left(\frac{\partial N_{x\theta}}{\partial \theta} \frac{\partial w}{\partial x} + N_{x\theta} \frac{\partial^2 w}{\partial x \partial \theta} \right) - \left(\frac{\partial N_{x\theta}}{\partial x} \frac{\partial w}{\partial \theta} + N_{x\theta} \frac{\partial^2 w}{\partial \theta \partial x} \right) + \left(\frac{\partial M_{x\theta}}{\partial x} \beta_\theta + M_{x\theta} \frac{\partial \beta_\theta}{\partial x} \right) + M_{x\theta} \frac{\partial \beta_\theta}{\partial x} \\ - R \frac{\partial Q_{xz}}{\partial x} - \frac{\partial Q_{\theta z}}{\partial \theta} = -\gamma h \ddot{w} R + q_z R \quad (2.54c)$$

where Q_{xz} and $Q_{\theta z}$, can be found through,

$$-R \left[\frac{\partial M_{xx}}{\partial x} + \left(\frac{\partial M_{xx}}{\partial x} \frac{\partial u}{\partial x} + M_{xx} \frac{\partial^2 u}{\partial x^2} \right) + \left(\frac{\partial \Lambda_{xx}}{\partial x} \frac{\partial \beta_x}{\partial x} + \Lambda_{xx} \frac{\partial^2 \beta_x}{\partial x^2} \right) \right] - \frac{1}{R} \left(\frac{\partial M_{\theta\theta}}{\partial \theta} \frac{\partial u}{\partial \theta} + M_{\theta\theta} \frac{\partial^2 u}{\partial \theta^2} \right) - R \left(\frac{\partial \Lambda_{\theta\theta}}{\partial \theta} \frac{\partial \beta_x}{\partial \theta} + \Lambda_{\theta\theta} \frac{\partial^2 \beta_x}{\partial \theta^2} \right) \quad (2.55a)$$

$$- \left[\left(\frac{\partial M_{x\theta}}{\partial \theta} \frac{\partial u}{\partial x} + M_{x\theta} \frac{\partial^2 u}{\partial x \partial \theta} \right) + \left(\frac{\partial M_{x\theta}}{\partial x} \frac{\partial u}{\partial \theta} + M_{x\theta} \frac{\partial^2 u}{\partial \theta \partial x} \right) + \frac{\partial M_{x\theta}}{\partial \theta} + \left(\frac{\partial \Lambda_{x\theta}}{\partial x} \frac{\partial \beta_x}{\partial \theta} + \Lambda_{x\theta} \frac{\partial^2 \beta_x}{\partial x \partial \theta} \right) + \left(\frac{\partial \Lambda_{x\theta}}{\partial \theta} \frac{\partial \beta_x}{\partial x} + \Lambda_{x\theta} \frac{\partial^2 \beta_x}{\partial \theta \partial x} \right) \right] + R Q_{xz} = 0$$

$$-R \left[\left(\frac{\partial M_{xx}}{\partial x} \frac{\partial v}{\partial x} + M_{xx} \frac{\partial^2 v}{\partial x^2} \right) + \left(\frac{\partial \Lambda_{xx}}{\partial x} \frac{\partial \beta_\theta}{\partial x} + \Lambda_{xx} \frac{\partial^2 \beta_\theta}{\partial x^2} \right) \right] - \frac{\partial M_{\theta\theta}}{\partial \theta} - \frac{1}{R} \left(-M_{\theta\theta} v + M_{\theta\theta} \frac{\partial w}{\partial \theta} + \frac{\partial M_{\theta\theta}}{\partial \theta} w + M_{\theta\theta} \frac{\partial w}{\partial \theta} + \frac{\partial M_{\theta\theta}}{\partial \theta} \frac{\partial v}{\partial \theta} + M_{\theta\theta} \frac{\partial^2 v}{\partial \theta^2} \right) \\ - \left[R \frac{\partial M_{x\theta}}{\partial x} + \left(\frac{\partial M_{x\theta}}{\partial \theta} \frac{\partial v}{\partial x} + M_{x\theta} \frac{\partial^2 v}{\partial x \partial \theta} \right) + \left(\frac{\partial M_{x\theta}}{\partial x} \frac{\partial v}{\partial \theta} + M_{x\theta} \frac{\partial^2 v}{\partial \theta \partial x} \right) + M_{x\theta} \frac{\partial w}{\partial x} + \left(\frac{\partial M_{x\theta}}{\partial x} w + M_{x\theta} \frac{\partial w}{\partial x} \right) + \left(\frac{\partial \Lambda_{x\theta}}{\partial x} \frac{\partial \beta_\theta}{\partial \theta} + \Lambda_{x\theta} \frac{\partial^2 \beta_\theta}{\partial x \partial \theta} \right) \right. \\ \left. + \left(\frac{\partial \Lambda_{x\theta}}{\partial \theta} \frac{\partial \beta_\theta}{\partial x} + \Lambda_{x\theta} \frac{\partial^2 \beta_\theta}{\partial \theta \partial x} \right) \right] - R \left(\frac{\partial \Lambda_{\theta\theta}}{\partial \theta} \frac{\partial \beta_\theta}{\partial \theta} + \Lambda_{\theta\theta} \frac{\partial^2 \beta_\theta}{\partial \theta^2} \right) + (R Q_{\theta z} - M_{\theta z}) = 0 \quad (2.55b)$$

with β_x & β_θ representing the angles (2.37) and additionally Λ_{xx} , $\Lambda_{\theta\theta}$ & $\Lambda_{x\theta}$,

$$\Lambda_{xx} = \int_{-h/2}^{h/2} \sigma_{xx} z^2 dz, \quad \Lambda_{\theta\theta} = \int_{-h/2}^{h/2} \sigma_{\theta\theta} z^2 dz, \quad \Lambda_{x\theta} = \int_{-h/2}^{h/2} \tau_{x\theta} z^2 dz \quad (2.56)$$

Boundary Conditions

$$N_{xx}^* = \left(N_{xx} + N_{xx} \frac{\partial u}{\partial x} + M_{xx} \frac{\partial \beta_x}{\partial x} \right) + N_{x\theta} \frac{1}{R} \frac{\partial u}{\partial \theta} + M_{x\theta} \frac{1}{R} \frac{\partial \beta_x}{\partial \theta} \quad \& \quad N_{\theta\theta}^* = (N_{\theta\theta}) + \frac{1}{R} \left(N_{\theta\theta} w + N_{\theta\theta} \frac{\partial v}{\partial \theta} + M_{\theta\theta} \frac{\partial \beta_\theta}{\partial \theta} \right) + N_{x\theta} \frac{\partial v}{\partial x} + M_{x\theta} \frac{\partial \beta_\theta}{\partial x}$$

$$\begin{aligned}
N_{x\theta}^* &= \left(N_{xx} \frac{\partial v}{\partial x} + M_{xx} \frac{\partial \beta_\theta}{\partial x} \right) + N_{x\theta} + N_{x\theta} \frac{1}{R} w + N_{x\theta} \frac{1}{R} \frac{\partial v}{\partial \theta} + M_{x\theta} \frac{1}{R} \frac{\partial \beta_\theta}{\partial \theta} & N_{\theta x}^* &= \frac{1}{R} \left(N_{\theta\theta} \frac{\partial u}{\partial \theta} + M_{\theta\theta} \frac{\partial \beta_x}{\partial \theta} \right) + N_{x\theta} + N_{x\theta} \frac{\partial u}{\partial x} + M_{x\theta} \frac{\partial \beta_x}{\partial x} \\
Q_{xz}^* &= \left(N_{xx} \frac{\partial w}{\partial x} \right) - N_{x\theta} \frac{1}{R} v + N_{x\theta} \frac{1}{R} \frac{\partial w}{\partial \theta} - M_{x\theta} \frac{1}{R} \beta_\theta + Q_{xz} & Q_{\theta z}^* &= \frac{1}{R} \left(-N_{\theta\theta} v + N_{\theta\theta} \frac{\partial w}{\partial \theta} - M_{\theta\theta} \beta_\theta \right) + N_{x\theta} \frac{\partial w}{\partial x} + Q_{\theta z} \\
M_{xx}^* &= \left(M_{xx} + M_{xx} \frac{\partial u}{\partial x} + \Lambda_{xx} \frac{\partial \beta_x}{\partial x} \right) + M_{x\theta} \frac{1}{R} \frac{\partial u}{\partial \theta} + \Lambda_{x\theta} \frac{1}{R} \frac{\partial \beta_x}{\partial \theta} & M_{\theta\theta}^* &= (M_{\theta\theta}) + \frac{1}{R} \left(M_{\theta\theta} w + M_{\theta\theta} \frac{\partial v}{\partial \theta} \right) + R \Lambda_{\theta\theta} \frac{\partial \beta_\theta}{\partial \theta} + M_{x\theta} \frac{\partial v}{\partial x} + \Lambda_{x\theta} \frac{\partial \beta_\theta}{\partial x}
\end{aligned} \quad (2.57)$$

Their version, in terms of displacements for space economy reasons is not presented here. Although not realized eventually, the initial purpose behind the usage of full-length strain relations for deriving equations of motion in their general form, was to investigate the effect of additional nonlinear terms on the final shear buckling stress. In that way, a comparison to the existing shell theories mentioned in the previous section would be accomplished as well.

At this point, a further step is taken by many researchers, and that's why it was considered worthwhile mentioning it here before the closure of this section. In some of the past studies, solely the equations of motion are not enough for studying the buckling of shells. The solution of the problem is searched over a new set of equations, also known as *buckling equations*. This concept is proposed by *classical buckling theory*. According to *classical buckling theory*, the problem is divided into the pre-buckling state and the buckling one which suggests the motion right after buckling has initiated. In the pre-buckled state, all the boundary conditions are disregarded, and the shell's behavior is that of a membrane. In other words, the shell is able, in the absence of any restrictions, to expand and contract like a membrane, without any signs of bending though. Due to the fact that, the outward radial deflection is assumed constant over the entire shell's surface, the equations (2.53) are simplified in a large degree and automatically become linear, as the derivatives of w , i.e. $\frac{\partial w}{\partial x}$ & $\frac{\partial w}{\partial \theta}$, and consequently $\frac{\partial^2 w}{\partial x^2}$, $\frac{\partial^2 w}{\partial \theta^2}$ & $\frac{\partial^2 w}{\partial x \partial \theta}$ vanish. In this sense, it is expected that no bending moments are present but only in-plane/membrane forces. On the contrary, when in buckled state, the shell gets restrained meaning that boundary conditions before and after are inconsistent. In order for the buckling equations to be derived, the total potential energy $P = U - E_L - E_B$ of the system, which undergoes a variation ΔP due to external forcing is utilized:

$$P + \Delta P \xrightarrow{\text{Taylor}} P + \delta P + \frac{1}{2} \delta^2 P + \dots \quad (2.58)$$

The first term refers to the pre-buckled configuration and setting it equal to zero, it coincides with the Hamilton's principle used above. As described in the previous chapter (fig 6), the transition to an unstable state takes place when the second term of the Taylor's expansion is zero, $\delta^2 P = 0$. So, for example the decoupled buckling equations for the simple case of Donnell's theory [7] are,

$$R \nabla^4 \bar{u} = -n \frac{\partial^3 \bar{w}}{\partial x^3} + \frac{\partial^3 \bar{w}}{\partial x R^2 \partial \theta^2} \quad (2.59a)$$

$$R \nabla^4 \bar{v} = -(2+n) \frac{\partial^3 \bar{w}}{\partial x^2 R \partial \theta} - \frac{\partial^3 \bar{w}}{R^3 \partial \theta^3} \quad (2.59b)$$

$$\frac{E h^3}{12(1-n^2)} \nabla^8 \bar{w} + \frac{E h}{R^2} \frac{\partial^4 \bar{w}}{\partial x^4} - \nabla^4 \left(N_x \frac{\partial^2 \bar{w}}{\partial x^2} + 2N_{xy} \frac{\partial^2 \bar{w}}{\partial x R \partial \theta} + N_y \frac{\partial^2 \bar{w}}{R^2 \partial \theta^2} \right) = 0 \quad (2.59c)$$

where the barred terms represent the buckling displacements. The force which induces buckling is inserted via the in-plane forces $N_{xx}, N_{x\theta}, N_{\theta\theta}$ calculated in the pre-buckled state. Further, the critical buckling load is searched in the simple manner by minimizing the coefficient of the desired load term, which form is determined in such a way, in order the assumed solution will satisfy the bcs and the 3rd buckling equation. For more details of the method, one may look into the book of Jones [7]. Despite the fact that *classical buckling theory* may seem at first glance attractive due to the simplifications it offers, it treats buckling in an unrealistic manner. In the past, the large deviations between experimental results and the theoretical ones were first attributed to the existence of imperfections, but many scientists proved that the inconsistencies of the classical theory were the real reason behind the inaccuracies, making it not an ideal option for our case.

2.4 Solution Methods

Definitely, one of the largest struggles of the thesis was by far, figuring out the proper way to reach a solution. A few approaches for studying buckling of shells have been detected in the literature with their pros and cons.

2.4.1 Mathieu's Equation for Stability

One method, especially used in the early studies is the investigation of stability through the readjustment of equations of motion into a Mathieu's formulation. The arrangement of Mathieu's equation is as simple as that of a 2nd order linear ordinary differential equation which includes a periodic force in the stiffness coefficient [35]. Once the formulation has been achieved, the following steps are well-known and so eventually, stability charts can be made in which the instability regions of the system are portrayed as shown in the work of Chiba [36]. However, Mathieu's equation concerns only dynamic problems under periodic forcing and thus will not be applied here.

2.4.2 Stein's and Almroth's Rigorous Approach for Stability

For the next approach, one should refer to the end of the previous section. A more rigorous way of solution exists which disregards all the simplifications and the inconsistencies of the classical buckling theory. As it has been mentioned, after the buckling equations are derived, in which not all the terms correspond to the buckled state, the pre-buckled terms are inserted. However, this time the latter ones are calculated taking into account bending deformations and also the boundary conditions of the initial configuration. This procedure was followed by Stein and Almroth [7] who examined the effect of various boundaries on the static buckling of cylindrical shells. In a later study, nevertheless, Touati [10] characterizes the principle of the zero second variation of the total potential energy, which after all is the basis of buckling equations, as not the ideal path for studying shell's buckling undergoing large deformations.

At this point, it should be made clear that choosing the right approach to investigate stability is only just the tip of the iceberg since at a second stage, one is called to resolve how the equations involved in the problem are going to be tackled. Unfortunately, exact solutions do not exist for every differential equation. Taking shells as an example, even the case of linear equations of motion is solvable only for simple theories and only for specific boundary conditions. For this reason, scientists have resorted to approximate analytical techniques, in order to reduce the complexity of the initial problem (strong form) in a weak formulation, so that a solution can be attained through a different path.

2.4.3 Rayleigh-Ritz Method

One of them is the Ritz method, a variational technique, also known as Rayleigh-Ritz, which utilizes the minimization of energy principle. It is used in "simple" boundary value problems whereas it is ineffective for complex cases of shell structures.

2.4.4 Galerkin Method

Another technique, which falls in the category of weighted residual methods, is the Galerkin method, which offers the possibility to discretize the initial continuous formulation of the equations. A solution that describes the expected deflection shape is assumed, and an effort to minimize its weighted average error takes place. Galerkin procedure has been employed in the majority of the literature.

In the book of Robert M. Jones [7] the application of Galerkin is explicitly explained through its application for various static stress states and boundary conditions for cylindrical shells. In a more recent study, Amabili [28] has also presented the same method to investigate nonlinear vibrations of circular cylindrical shells filled with water. Other works include those of Rasim [12], Shaw [2], Kubenko [11] and Goncalves [16]. A similar but more recent technique which has proved to be a rather efficient tool for solving problems in the field of shell's buckling is the Differential Quadrature Method (DQM), created by Bert [37]. During the beginning of the present study, many attempts have been made towards the implementation of Galerkin method. However, eventually it had to be rejected due to its main disadvantage. The solution, which has to be speculated beforehand, is written as a sum of functions – they will serve later as weighting functions – that should satisfy at least all the geometrical boundary conditions and if stress calculations is the main goal of the researcher such as in this study, an additional satisfaction of the shear and moment edge conditions is required [25].

This most of the times tends to be a very strenuous procedure and a preventing factor of using Galerkin, especially in cases of more complex theories.

2.4.5 Finite Elements & Finite Differences Methods

The difficulty increases significantly when the equations involved in the shell's motion are nonlinear. It is extremely rare for nonlinear differential equations to have an exact solution. Thus, an extra action needs to be undertaken, that of linearization. Conversion of a nonlinear problem into a linear set of equations and achievement of a solution at a second stage, can be easily done by the aid of modern programs which use finite elements (FEM) or the method of finite differences (FDM), a technique that estimates derivatives with finite differences [25]. However since, this is an analytically oriented thesis, an attempt towards this direction is not going to be made.

2.4.6 Applied Stability approach and Perturbation Method

Until now, two approaches have been described to study stability, and also some of the most common solution techniques. Here, the whole concept of the procedure that will be followed later, is going to be clarified through the simple example of a beam. A modification of the classic perturbation method is implemented that resembles the asymptotic approach introduced by Koiter [38]. Further details about the perturbation method will be given in the next chapter, where its application on the cylindrical shell tank under study unfolds extensively.

2.4.7 Beam Example

A beam (*fig. 14*) clamped onto the ground, under the application of a distributed load $q(x) = q$ along its length and a concentrated load P on the top, is taken as an example to demonstrate the methodology.

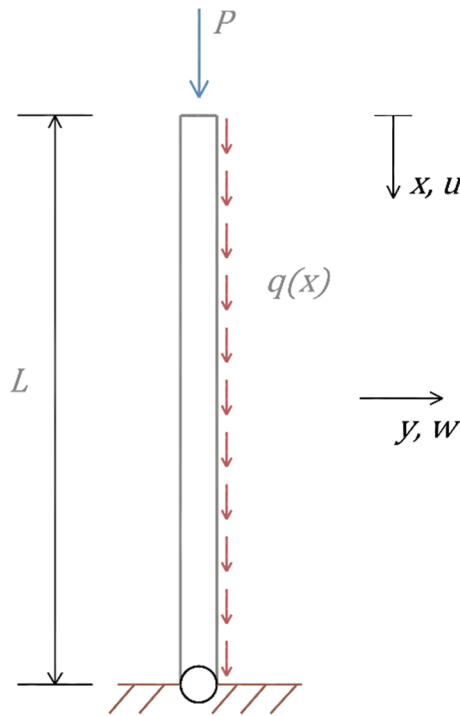


Figure 14 Cantilever beam under a concentrated force P and uniform distributed load $q(x)$

The equations that describe the motion in the two directions and the corresponding boundary conditions are given by,

Equations of Motion

$$EIw'''' - EA(u'w')' = 0 \quad (2.60a)$$

$$EAu' + EAu'w'' = q \quad (2.60b)$$

Boundary Conditions

At $x = L$,

$$u(L) = 0$$

$$w(L) = 0$$

$$w'(L) = 0$$

At $x = 0$,

$$EAu'(0) = -P \quad (2.61a, d)$$

$$w''(0) = 0 \quad (2.61b, e)$$

$$EIw'''(0) = 0 \quad (2.61c, f)$$

Step 1

i) The displacements are assumed in the form of:

$$u = u_0\varepsilon + u_1\varepsilon^2 \quad (2.62)$$

$$w = w_0\varepsilon + w_1\varepsilon^2 \quad (2.63)$$

ii) We now substitute (2.62), (2.63) into the equations of motion where only the terms in respect to ε are kept,

$$EI(w_0''''\varepsilon + w_1''''\varepsilon^2) - EA[(u_0'\varepsilon + u_1'\varepsilon^2)(w_0'\varepsilon + w_1'\varepsilon^2)]' = 0 \xrightarrow{\text{only the coefficient of } \varepsilon} \varepsilon EIw_0'''' = 0 \Rightarrow EIw_0'''' = 0 \quad (2.64)$$

$$EA(u_0''\varepsilon + u_1''\varepsilon^2) + EA[(w_0'\varepsilon + w_1'\varepsilon^2)(w_0''\varepsilon + w_1''\varepsilon^2)] = q \xrightarrow{\text{only the coefficient of } \varepsilon} \varepsilon EAu_0'' = q \quad (2.65)$$

Since ε has no physical meaning, it can be neglected and so (2.65) becomes,

$$EAu_0'' = q \quad (2.66)$$

iii) The perturbed displacements are also substituted into the boundary conditions and we apply the same notion as before ,

At $x = L$:

$$\varepsilon u_0(L) + \varepsilon^2 u_1(L) = 0 \xrightarrow{\text{only the coefficient of } \varepsilon} \varepsilon u_0(L) = 0 \Rightarrow u_0(L) = 0 \quad (2.67a)$$

$$\varepsilon w_0(L) + \varepsilon^2 w_1(L) = 0 \xrightarrow{\text{only the coefficient of } \varepsilon} \varepsilon w_0(L) = 0 \Rightarrow w_0(L) = 0 \quad (2.67b)$$

$$\varepsilon w_0'(L) + \varepsilon^2 w_1'(L) = 0 \xrightarrow{\text{only the coefficient of } \varepsilon} \varepsilon w_0'(L) = 0 \Rightarrow w_0'(L) = 0 \quad (2.67c)$$

At $x = 0$:

$$EA\varepsilon u_0'(0) + EA\varepsilon^2 u_1'(0) = -P \xrightarrow{\text{only the coefficient of } \varepsilon} EA\varepsilon u_0'(0) = -P \xrightarrow{\varepsilon \text{ is dropped}} EAu_0'(0) = -P \quad (2.67d)$$

$$\varepsilon w_0''(0) + \varepsilon^2 w_1''(0) = 0 \xrightarrow{\text{only the coefficient of } \varepsilon} \varepsilon w_0''(0) = 0 \Rightarrow w_0''(0) = 0 \quad (2.67e)$$

$$EI[\varepsilon w_0'''(0) + \varepsilon^2 w_1'''(0)] = 0 \xrightarrow{\text{only the coefficient of } \varepsilon} \varepsilon w_0'''(0) = 0 \Rightarrow w_0'''(0) = 0 \quad (2.67f)$$

Step 2

Equations of motion have now been linearized, meaning that either of their solutions can now be achieved. Currently, the 2nd equation is selected to be solved,

$$EAu_0'' = q \quad \text{with} \quad u_0(L) = 0 \quad \& \quad EAu_0'(0) = -P$$

i) A particular solution is searched in the form of,

$$u_0^p = Cx^2 \quad (2.68a)$$

which after substituting it in (2.66),

$$EA 2C = q \Rightarrow C = \frac{q}{2EA} \xrightarrow{\text{yields}} u_{p0} = \frac{q}{2EA} x^2 \quad (2.68b)$$

- ii) The solution of the homogeneous equation takes the form of,

$$EAu_0'' = 0 \Rightarrow u_0'' = 0 \Rightarrow u_{h0} = a + bx \quad (2.69)$$

- iii) The general solution $u_{g0} = u_{h0} + u_{p0} = a + bx + \frac{q}{2EA}x^2$ is now substituted in the two boundary conditions to determine the unknowns a, b :

$$\left. \begin{aligned} EAu_{g0}'(0) &= -P \Rightarrow EAb = -P \Rightarrow b = -\frac{P}{EA} \\ u_{g0}(L) &= 0 \Rightarrow a - \frac{P}{EA}L + \frac{q}{2EA}L^2 = 0 \Rightarrow a = \frac{P}{EA}L - \frac{q}{2EA}L^2 \end{aligned} \right\} \Rightarrow$$

$$u_{g0} = \frac{P}{EA}L - \frac{q}{2EA}L^2 - \frac{P}{EA}x + \frac{q}{2EA}x^2 \equiv u_0 \quad (2.70)$$

In this way, the force responsible for buckling has been inserted in the solution.

Step 3

Now we return to the initial problem, but this time we treat the equations differently. The perturbation method is applied again but except from the coefficient of ε , that of ε^2 is retained as well. The main focus turns in the 1st equation (2.60a), since the second one has been employed previously,

$$\begin{aligned} EIw_0'''' - EA(u'w')' &= 0 \Rightarrow EI(w_0''''\varepsilon + w_1''''\varepsilon^2) - EA[(u_0'\varepsilon + u_1'\varepsilon^2)(w_0'\varepsilon + w_1'\varepsilon^2)]' = 0 \Rightarrow \\ EI\varepsilon w_0'''' + EI\varepsilon^2 w_1'''' - EA[u_0'w_0'\varepsilon^2]' &= 0 \Rightarrow EIw_0'''' + EI\varepsilon w_1'''' - EA[u_0'w_0'\varepsilon]' = 0 \end{aligned} \quad (2.71a)$$

To overcome nonlinearity, u_0 is substituted by the solution (2.70) of the first order problem,

$$EIw_0'''' + \varepsilon EIw_1'''' - \varepsilon EA \left[\left(-\frac{P}{EA} + \frac{q}{EA}x \right) w_0' \right]' = 0 \quad (2.71b)$$

It is assumed that the unknown term w_1 of the w expansion can be neglected as well as the perturbation parameter ε due to its no physical meaning like before,

$$\begin{aligned} EIw_0'''' - EA \left[\left(-\frac{P}{EA} + \frac{q}{EA}x \right) w_0' \right]' &= 0 \\ EIw_0'''' - (-P + qx)w_0'' - qw_0' &= 0 \end{aligned} \quad (2.72)$$

Step 4

The second order of perturbation method is also applied in the rest of the boundary conditions that have not been utilized in the 1st order problem, i.e. (2.b,c,e), which after neglecting the unknown term w_1 become,

$$w_0(L) = w_0'(L) = w_0''(0) = 0 \quad (2.73a, b, c)$$

However, three bcs are not enough to solve a 4th order differential equation. The fourth relation rises by integrating at zero the 1st equation of motion, and working likewise in step 3.

$$\begin{aligned} EIw_0'''' - EA(u'w')' &= 0 \Rightarrow EIw_0'''' - EA(u'w') = 0 \Rightarrow EI(\varepsilon w_0 + \varepsilon^2 w_1)'''' - EA[(\varepsilon u_0 + \varepsilon^2 u_1)'(\varepsilon w_0 + \varepsilon^2 w_1)'] = 0 \\ \xrightarrow{u_1, w_1 \text{ are neglected}} EIw_0'''' - \varepsilon EAu_0'w_0' &= 0 \quad \xrightarrow{\varepsilon \text{ is dropped and } u_0 \text{ is substituted from the 1st order problem}} \\ EIw_0'''' - EA \left(-\frac{P}{EA} + \frac{q}{EA}x \right) w_0' &= 0 \xrightarrow{\text{at } x=0} EIw_0(0)'''' + Pw_0(0)' = 0 \end{aligned} \quad (2.74)$$

The problem that we are called to tackle is an ordinary, 4th order homogeneous ordinary differential equation with variable coefficients (2.72) with four boundary conditions (2.73) & (2.74).

Step 5

The solution is searched in the form of,

$$w_0(x) = \sum_{n=1}^{\infty} A_n \Psi_n(x) \quad (2.75)$$

and after its substitution into the equation of motion and the boundary conditions, the latter ones become

$$EI\Psi_n'''' - (-P + qx)\Psi_n'' - q\Psi_n' = 0 \quad (2.76)$$

with

$$\Psi_n(L) = \Psi_n'(L) = \Psi_n''(0) = 0 \quad (2.77a, b, c)$$

$$\& \\ EI\Psi_n(0)''' + P\Psi_n(0)' = 0 \quad (2.77d)$$

Step 6

Solving an ode with variable coefficients is not an easy task and since also stability is not affected by the uniform load the following two approximations are being made. Firstly, the system will be solved without the external load q , and so everything related to q in (2.76) is neglected:

$$EIw_0'''' - (-P + 0)w_0' - 0 = 0 \quad (2.78)$$

Secondly, an additional term is introduced, in which also a new, yet unknown parameter λ_n needs to be determined. The λ_n parameter has no relation at all with q .

$$EIw_0'''' + Pw_0' + \lambda_n^2 w_0 = 0 \xrightarrow{\text{yields}} EI\Psi_n'''' + P\Psi_n'' + \lambda_n^2 \Psi_n = 0 \quad (2.79)$$

All the boundary conditions remain the same. To put it as simply as possible, the main idea here is that the solution of the initial problem is searched through a similar problem with the same boundary conditions. The solution of (2.79) is assumed as,

$$\Psi_n(x) = \sum_{m=1}^4 z_m e^{\beta_m x} \quad (2.80)$$

i) Including λ_n , 5 unknowns but only four bcs exist. After substitution of (2.80) into the equation (2.79), the parameters β_m are calculated:

$$\beta_{1,2} = \pm \frac{\sqrt{-\frac{P}{EI} - \frac{\sqrt{P^2 - 4EI\lambda_n^2}}{EI}}}{\sqrt{2}} \quad \text{and} \quad \beta_{3,4} = \pm \frac{\sqrt{-\frac{P}{EI} + \frac{\sqrt{P^2 - 4EI\lambda_n^2}}{EI}}}{\sqrt{2}} \quad (2.81a - d)$$

The solution, now can be substituted into the bcs which in turn are written in matrix form as,

$$\begin{bmatrix} e^{\beta_1 L} & e^{-\beta_1 L} & e^{\beta_3 L} & e^{-\beta_3 L} \\ \beta_1 e^{\beta_1 L} & -\beta_1 e^{-\beta_1 L} & \beta_3 e^{\beta_3 L} & -\beta_3 e^{-\beta_3 L} \\ \beta_1^2 & \beta_1^2 & \beta_3^2 & \beta_3^2 \\ \beta_1^3 EI + \beta_1 P & -\beta_1^3 EI - \beta_1 P & \beta_3^3 EI + \beta_3 P & -\beta_3^3 EI - \beta_3 P \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad (2.82)$$

Taking the determinant of the coefficients of the z_m constants equal to zero, the parameter λ_n is able to be found in respect to P , i.e. $\lambda_n(P)$. It should be noted that, λ_n can be evaluated for numerical values only and for that reason during the next steps it will be considered as known.

ii) Utilizing (2.82), three out four equations are taken into account and the z_2, z_3, z_4 constants are expressed as a function of z_1 . In that way, solution (2.80) becomes,

$$\Psi_n(x) = \sum_{m=1}^4 z_m e^{\beta_m x} \Rightarrow \Psi_n(x) = z_1 e^{\beta_1 x} + z_1 g_2(P) e^{\beta_2 x} + z_1 g_3(P) e^{\beta_3 x} + z_1 g_4(P) e^{\beta_4 x}$$

or

$$\Psi_n(x) = z_1 \Psi_{0n}(x) \quad (2.81)$$

where

$$\Psi_{0n}(x) = e^{\beta_1 x} + g_2(P) e^{\beta_2 x} + g_3(P) e^{\beta_3 x} + g_4(P) e^{\beta_4 x}$$

The λ_n parameter is included in g_k and so it is clear that, g_k are functions of the load P .

Step 7

In order for the solution (2.81) to be of any importance, it should also be a solution of the initial problem. The boundary conditions are already satisfied, leaving us the ode (2.76) as our main concern. The two unknowns in (2.81) are the buckling load P and the constant z_1 , meaning that z_1 should be such that $\Psi_n(x)$ is able to satisfy (2.76). For this reason, substituting (2.81) into (2.75),

$$w_0(x) = \sum_{n=1}^{\infty} A_n \Psi_n(x) \Rightarrow w_0(x) = \sum_{n=1}^{\infty} A_n z_1 \Psi_{0n}(x) \quad (2.82)$$

and then in turn (2.82) into the original equation of motion, including also the terms which were neglected previously, we get:

$$(2.76) \Rightarrow \sum_n z_{1n} (EI \Psi_{0n}'''' + P \Psi_{0n}'') - qx \sum_n z_{1n} \Psi_{0n}'' - q \sum_n z_{1n} \Psi_{0n}' = 0 \quad (2.83)$$

By our definition, $\Psi_n(x)$ satisfies $EI \Psi_n'''' + P \Psi_n'' + \lambda_n^2 \Psi_n = 0$, so the above equation can be written as,

$$\sum_n z_{1n} (-\lambda_n^2 \Psi_{0n}) - qx \sum_n z_{1n} \Psi_{0n}'' - q \sum_n z_{1n} \Psi_{0n}' = 0 \quad (2.84)$$

Then if (2.84) is multiplied with Ψ_{0m} and integrated from 0 to L , it becomes:

$$z_{1m} \lambda_m^2 \int_0^L \Psi_{0m}^2 dx - q \sum_{n=1}^{N_{max}} z_{1n} \int_0^L (x \Psi_{0n}'' \Psi_{0m} + \Psi_{0n}' \Psi_{0m}) dx = 0 \quad (2.85)$$

The first summation disappears because the modes are orthogonal. (For $n \neq m$, $\Psi_{0m} * \Psi_{0n} = 0$). The critical buckling load will be determined upon a choice of a certain number for n and m . Both n and m run from 1 to ∞ and they must be equally valued. Suppose for example that $N_{max} = 2$ and $m = 2$, then 2 equations will occur:

$$\begin{aligned} \text{For } m = 1: \quad & z_{11} \lambda_1^2 \int_0^L \Psi_{01}^2 dx - q \sum_{n=1}^2 z_{1n} \int_0^L (x \Psi_{0n}'' \Psi_{01} + \Psi_{0n}' \Psi_{01}) dx = 0 \\ & \& \\ \text{For } m = 2: \quad & z_{12} \lambda_2^2 \int_0^L \Psi_{02}^2 dx - q \sum_{n=1}^2 z_{1n} \int_0^L (x \Psi_{0n}'' \Psi_{02} + \Psi_{0n}' \Psi_{02}) dx = 0 \end{aligned}$$

This system of two equations can now be written in the following matrix form,

$$\begin{bmatrix} \lambda_1^2 \int_0^L \Psi_{01}^2 dx - q \int_0^L (x \Psi_{01}'' \Psi_{01} + \Psi_{01}' \Psi_{01}) dx & -q \int_0^L (x \Psi_{02}'' \Psi_{01} + \Psi_{02}' \Psi_{01}) dx \\ \lambda_2^2 \int_0^L \Psi_{02}^2 dx - q \int_0^L (x \Psi_{01}'' \Psi_{02} + \Psi_{01}' \Psi_{02}) dx & -q \int_0^L (x \Psi_{02}'' \Psi_{02} + \Psi_{02}' \Psi_{02}) dx \end{bmatrix} \begin{bmatrix} z_{11} \\ z_{12} \end{bmatrix} = 0 \Rightarrow [A] \begin{bmatrix} z_{11} \\ z_{12} \end{bmatrix} = 0$$

By setting the determinant of the matrix A equal to zero, i.e. $|A| = 0$, the buckling load P is calculated. This is the method of the assumed modes and technically the problem of the beam is now solved.

3 Perturbation Method

3.1. Introduction

It is widely known that natural phenomena are inherently, on their whole, fully nonlinear. Nonlinearity exists everywhere and it has been a struggle for scientists through the years. Koiter, especially, mentions in his work that “the investigation of stability belongs to the domain of the non-linear theory of elasticity” [8]. Quite a few strategies have been utilized to deal with this topic, with numerical methods being the easiest ones to implement, such as the finite element or the method of finite differences [25]. But as Jones [7] underlines, engineers nowadays are so much dependable on computers that lose the deep understanding and the insight, analytical approaches have to offer for a structure’s behavior.

In addition to the previous chapter, for nonlinear problems, the most accurate methods available involve the Adomian’s decomposition method and Liao’s homotopy analysis method [39], [40]. Their main advantage is that they can lead to approximate analytical solutions of nonlinear partial differential equations without the need to subdivide the problem into linear cases. Since the first one exhibits the same disadvantage as the Galerkin technique, that of not satisfying all the boundary conditions, Liao’s homotopy analysis method gives the most accurate results. However, due to its mathematical complexity it was not applied in the present study. For this reason, the solution is approximated through a perturbation technique.

Traditional perturbation method [45],[43] bases its functionality in the existence of a very small parameter, also referred as perturbation parameter ε . This could be, in case of a circular cylindrical shell the ratio of radius and its thickness. Most likely, a large value of ε , $\varepsilon > 1$, will lead to fallacies. After the main displacement functions are expressed in an increasing order of ε , the initial nonlinear problem breaks down to an infinite number of linear problems which can be solved. The basic principle of perturbation method is that instead of trying to solve the set of non-linear *pdes*, the solution is approached through stages of increasing difficulty where the previous stage aids to the solution of the next. Its main advantage is that an accurate speculation of the solution beforehand is not needed like in Galerkin method for producing reliable results.

Of course, not all nonlinear phenomena include very small parameters which can act as perturbation parameters. In addition, its choice is generally vague. Since the accuracy of the results by the perturbation method depends mostly on the right choice of the perturbation parameter, in order to avoid this disadvantage of the traditional method, the concept of the two-step perturbation method [43], with a significant modification, is implemented, in which the constant ε has no physical meaning. This modification has been proposed by Prof. Andrei Metrikine and differentiates the method quite a lot than the one introduced by Shen and Zhang [43] in a way that will be explained in a later section (3.2.2.1).

In the current chapter the simplest equations that could describe the problem are employed. The solution is divided into two different cases; the linear and the nonlinear one, in which the perturbation method is used as the predominant tool to approach the problem at hand. The theoretical process is described at first, concluding with an example which can cover various examples of cylindrical shells.

3.2. Fully – Filled Liquid Tank

The main interest of this section will be focused around the case of a circular cylindrical shell which is filled with water until its highest point, undergoing also a horizontal static force at the bottom as it is illustrated in the following figure,

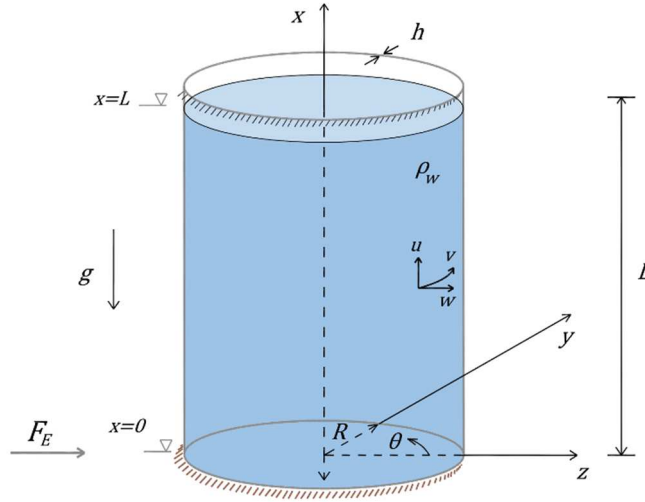


Figure 15 Fully filled cylindrical tank under horizontal force

The equations of motion have already been formulated in the previous chapters, but for reasons of simplicity will be repeated here in terms of forces and moments:

$$\frac{\partial N_{xx}}{\partial x} + \frac{\partial N_{x\theta}}{R \partial \theta} = 0 \quad (3.1a)$$

$$\frac{\partial N_{\theta\theta}}{R \partial \theta} + \frac{\partial N_{x\theta}}{\partial x} = 0 \quad (3.1b)$$

$$-\left(\frac{\partial N_{xx}}{\partial x} \frac{\partial w}{\partial x} + N_{xx} \frac{\partial^2 w}{\partial x^2}\right) + \frac{N_{\theta\theta}}{R} + \frac{1}{R^2} \left(-\frac{\partial N_{\theta\theta}}{\partial \theta} \frac{\partial w}{\partial \theta} - N_{\theta\theta} \frac{\partial^2 w}{\partial \theta^2}\right) - \frac{1}{R} \left(\frac{\partial N_{x\theta}}{\partial \theta} \frac{\partial w}{\partial x} + N_{x\theta} \frac{\partial^2 w}{\partial x \partial \theta}\right) - \frac{1}{R} \left(\frac{\partial N_{x\theta}}{\partial x} \frac{\partial w}{\partial \theta} + N_{x\theta} \frac{\partial^2 w}{\partial \theta \partial x}\right) - \frac{\partial Q_{xz}}{\partial x} - \frac{1}{R} \frac{\partial Q_{\theta z}}{\partial \theta} = \gamma (L - x) \quad (3.1c)$$

or in the form of displacements,

$$DhR \frac{\partial^2 u(x, \theta)}{\partial x^2} + \frac{Gh}{R} \frac{\partial^2 u(x, \theta)}{\partial \theta^2} + (Gh + Dhn) \frac{\partial^2 v(x, \theta)}{\partial x \partial \theta} + Dhn \frac{\partial w(x, \theta)}{\partial x} + \frac{Gh}{R} \frac{\partial w(x, \theta)}{\partial x} \frac{\partial^2 w(x, \theta)}{\partial \theta^2} + \left(\frac{Gh}{R} + \frac{Dhn}{R}\right) \frac{\partial w(x, \theta)}{\partial \theta} \frac{\partial^2 w(x, \theta)}{\partial x \partial \theta} + DhR \frac{\partial w(x, \theta)}{\partial x} \frac{\partial^2 w(x, \theta)}{\partial x^2} = 0 \quad (3.2a)$$

&

$$(DhnR + GhR) \frac{\partial^2 u(x, \theta)}{\partial x \partial \theta} + GhR^2 \frac{\partial^2 v(x, \theta)}{\partial x^2} + Dh \frac{\partial^2 v(x, \theta)}{\partial \theta^2} + Dh \frac{\partial w(x, \theta)}{\partial \theta} + \frac{Dh}{R} \frac{\partial w(x, \theta)}{\partial \theta} \frac{\partial^2 w(x, \theta)}{\partial \theta^2} + (GhR + DhnR) \frac{\partial w(x, \theta)}{\partial x} \frac{\partial^2 w(x, \theta)}{\partial x \partial \theta} + GhR \frac{\partial w(x, \theta)}{\partial \theta} \frac{\partial^2 w(x, \theta)}{\partial x^2} = 0 \quad (3.2b)$$

&

$$\frac{Dhw(x, \theta)}{R} + \frac{Dh}{R} \frac{\partial v(x, \theta)}{\partial \theta} - \frac{Dh}{2R^2} \frac{\partial w(x, \theta)^2}{\partial \theta} - \frac{Dh}{R^2} \frac{\partial w(x, \theta)}{\partial \theta} \frac{\partial^2 v(x, \theta)}{\partial \theta^2} - \frac{Dh}{R^2} w(x, \theta) \frac{\partial^2 w(x, \theta)}{\partial \theta^2} - \frac{Dh}{R^2} \frac{\partial v(x, \theta)}{\partial \theta} \frac{\partial^2 w(x, \theta)}{\partial \theta^2} - \frac{3Dh}{2R^3} \frac{\partial w(x, \theta)^2}{\partial \theta} \frac{\partial^2 w(x, \theta)}{\partial \theta^2} + \frac{Dh^3}{12R^3} \frac{\partial^4 w(x, \theta)}{\partial \theta^4} + Dhn \frac{\partial u(x, \theta)}{\partial x} - \frac{Dhn}{R} \frac{\partial^2 w(x, \theta)}{\partial \theta^2} \frac{\partial u(x, \theta)}{\partial x} - \frac{Gh}{R} \frac{\partial^2 u(x, \theta)}{\partial \theta^2} \frac{\partial w(x, \theta)}{\partial x}$$

$$\begin{aligned}
& -\frac{1}{2}Dhn \frac{\partial w(x, \theta)^2}{\partial x} - \left(\frac{Gh}{R} + \frac{Dhn}{2R}\right) \frac{\partial^2 w(x, \theta)}{\partial \theta^2} \frac{\partial w(x, \theta)}{\partial x} - \left(\frac{Gh}{R} + \frac{Dhn}{R}\right) \frac{\partial w(x, \theta)}{\partial \theta} \frac{\partial^2 u(x, \theta)}{\partial x \partial \theta} - (Gh + Dhn) \frac{\partial w(x, \theta)}{\partial x} \frac{\partial^2 v(x, \theta)}{\partial x \partial \theta} \\
& - \frac{2Gh}{R} \frac{\partial u(x, \theta)}{\partial \theta} \frac{\partial^2 w(x, \theta)}{\partial x \partial \theta} - 2Gh \frac{\partial v(x, \theta)}{\partial x} \frac{\partial^2 w(x, \theta)}{\partial x \partial \theta} - \left(\frac{4Gh}{R} + \frac{2Dhn}{R}\right) \frac{\partial w(x, \theta)}{\partial \theta} \frac{\partial w(x, \theta)}{\partial x} \frac{\partial^2 w(x, \theta)}{\partial x \partial \theta} - DhR \frac{\partial w(x, \theta)}{\partial x} \frac{\partial^2 u(x, \theta)}{\partial x^2} \\
& - Gh \frac{\partial w(x, \theta)}{\partial \theta} \frac{\partial^2 v(x, \theta)}{\partial x^2} - Dhnw(x, \theta) \frac{\partial^2 w(x, \theta)}{\partial x^2} - Dhn \frac{\partial v(x, \theta)}{\partial \theta} \frac{\partial^2 w(x, \theta)}{\partial x^2} - \left(\frac{Gh}{R} + \frac{Dhn}{2R}\right) \frac{\partial w(x, \theta)^2}{\partial \theta} \frac{\partial^2 w(x, \theta)}{\partial x^2} \\
& - DhR \frac{\partial u(x, \theta)}{\partial x} \frac{\partial^2 w(x, \theta)}{\partial x^2} - \frac{3}{2}DhR \frac{\partial w(x, \theta)^2}{\partial x} \frac{\partial^2 w(x, \theta)}{\partial x^2} + \left(\frac{Gh^3}{3R} + \frac{Dh^3n}{6R}\right) \frac{\partial^4 w(x, \theta)}{\partial x^2 \partial \theta^2} + \frac{Dh^3}{12}R \frac{\partial^4 w(x, \theta)}{\partial x^4} = R\gamma(L - x)
\end{aligned} \tag{3.2c}$$

In order to linearize the problem, the perturbation method is used in the way it is described below. Firstly, the displacements in three directions are assumed to be:

$$\begin{aligned}
u(x, \theta) &= \varepsilon u_0(x, \theta) + \varepsilon^2 u_1(x, \theta) \\
v(x, \theta) &= \varepsilon v_0(x, \theta) + \varepsilon^2 v_1(x, \theta) \\
w(x, \theta) &= \varepsilon w_0(x, \theta) + \varepsilon^2 w_1(x, \theta)
\end{aligned} \tag{3.3}$$

where ε is a small vague parameter.

The above assumptions are completely justified because, despite the fact that a non-linear theory of elasticity has been used, in order to take into account the effect of larger deflections than those provided by the linear theory prior to buckling, the resulting deflections are still small. At this point two very important remarks should be made. As mentioned previously, in contrast to the traditional perturbation method [45], [43], in which the perturbation parameter corresponds to a small geometrical quantity, here ε has no physical meaning, a “trick” that is also adopted by the two – step perturbation method [43]. In this way, the researcher is able to apply a linearization process without constraining himself on detecting small quantities. After all, not all problems in nature include geometrical parameters that can be considered small enough, meaning that their wrong selection can lead eventually to inaccurate results. Secondly, following strictly the principle of the traditional or the two – step perturbation method the form of displacement functions should be:

$$\begin{aligned}
u(x, \theta) &= u_0(x, \theta) + \varepsilon u_1(x, \theta) \\
v(x, \theta) &= v_0(x, \theta) + \varepsilon v_1(x, \theta) \\
w(x, \theta) &= w_0(x, \theta) + \varepsilon w_1(x, \theta)
\end{aligned} \tag{3.4}$$

This formulation is more logical in the sense that the original displacements of the system are followed by small perturbations. However, for the current case of circular cylindrical shell, the perturbation methods found in the literature could not lead to linear equations and for this reason the above assumption (3.3) was made. Substitution now of (3.3) into (3.2), the three equations of motion are expanded in terms of ε ,

Axial Motion

$$\begin{aligned}
& \varepsilon \frac{h}{R} \left[G \frac{\partial^2 u_0(x, \theta)}{\partial \theta^2} + DnR \frac{\partial w_0(x, \theta)}{\partial x} + (G + Dn)R \frac{\partial^2 v_0(x, \theta)}{\partial x \partial \theta} + DR^2 \frac{\partial^2 u_0(x, \theta)}{\partial x^2} \right] + \varepsilon^2 \frac{h}{R} \left[G \frac{\partial^2 u_{01}(x, \theta)}{\partial \theta^2} + G \frac{\partial^2 w_0(x, \theta)}{\partial \theta^2} \frac{\partial w_0(x, \theta)}{\partial x} \right. \\
& \left. + DnR \frac{\partial w_{01}(x, \theta)}{\partial x} + (G + Dn) \frac{\partial w_0(x, \theta)}{\partial \theta} \frac{\partial^2 w_0(x, \theta)}{\partial x \partial \theta} + DR^2 \frac{\partial^2 u_{01}(x, \theta)}{\partial x^2} + DR^2 \frac{\partial w_0(x, \theta)}{\partial x} \frac{\partial^2 w_0(x, \theta)}{\partial x^2} \right] + \varepsilon^3[\dots] + \varepsilon^4[\dots] = 0
\end{aligned} \tag{3.5a}$$

Circumferential motion

$$\begin{aligned}
& \varepsilon \frac{h}{R} \left[DR \frac{\partial w_0(x, \theta)}{\partial \theta} + DR \frac{\partial^2 v_0(x, \theta)}{\partial \theta^2} + (G + Dn)R^2 \frac{\partial^2 u_0(x, \theta)}{\partial x \partial \theta} + GR^3 \frac{\partial^2 v_0(x, \theta)}{\partial x^2} \right] + \frac{h}{R} \varepsilon^2 \left[DR \frac{\partial w_{01}(x, \theta)}{\partial \theta} + D \frac{\partial w_0(x, \theta)}{\partial \theta} \frac{\partial^2 w_0(x, \theta)}{\partial \theta^2} \right. \\
& \left. + (G + Dn)R^2 \frac{\partial^2 u_{01}(x, \theta)}{\partial x \partial \theta} + (G + Dn)R^2 \frac{\partial w_0(x, \theta)}{\partial x} \frac{\partial^2 w_0(x, \theta)}{\partial x \partial \theta} + GR^2 \frac{\partial w_0(x, \theta)}{\partial \theta} \frac{\partial^2 w_0(x, \theta)}{\partial x^2} \right] + \varepsilon^3[\dots] + \varepsilon^4[\dots] = 0
\end{aligned} \tag{3.5b}$$

Transverse motion

$$\varepsilon \left[\frac{Dhn}{R} \frac{\partial u_0(x, \theta)}{\partial x} + \frac{Dh}{R^2} \frac{\partial v_0(x, \theta)}{\partial \theta} + \frac{Dh}{R^2} w_0(x, \theta) + \frac{Dh^3}{12} \frac{\partial^4 w_0(x, \theta)}{\partial x^4} \right] + \frac{h^3}{6R^2} (2G + Dn) \frac{\partial^4 w_0(x, \theta)}{\partial x^2 \partial \theta^2} + \frac{Dh^3}{12R^4} \frac{\partial^4 w_0(x, \theta)}{\partial \theta^4}$$

$$\begin{aligned}
& + \varepsilon^2 \left[\frac{Dh}{R^2} w_{01}(x, \theta) - \frac{Dh}{2R^3} \left(\frac{\partial w_0(x, \theta)}{\partial \theta} \right)^2 - \frac{Dh}{R^3} \frac{\partial w_0(x, \theta)}{\partial \theta} \frac{\partial^2 v_0(x, \theta)}{\partial \theta^2} - \frac{Dh}{R^3} w_0(x, \theta) \frac{\partial^2 w_0(x, \theta)}{\partial \theta^2} - \frac{Dh}{R^3} \frac{\partial v_0(x, \theta)}{\partial \theta} \frac{\partial^2 w_0(x, \theta)}{\partial \theta^2} \right. \\
& + \frac{Dh^3}{12R^4} \frac{\partial^4 w_{01}(x, \theta)}{\partial \theta^4} - \frac{Dhn}{R^2} \frac{\partial^2 w_0(x, \theta)}{\partial \theta^2} \frac{\partial u_0(x, \theta)}{\partial x} + \frac{Dhn}{R} \frac{\partial u_{01}(x, \theta)}{\partial x} - \frac{Gh}{R^2} \frac{\partial^2 u_0(x, \theta)}{\partial \theta^2} \frac{\partial w_0(x, \theta)}{\partial x} - \frac{Dhn}{2R} \left(\frac{\partial w_0(x, \theta)}{\partial x} \right)^2 \\
& - \left(\frac{Gh}{R^2} + \frac{Dhn}{R^2} \right) \frac{\partial w_0(x, \theta)}{\partial \theta} \frac{\partial^2 u_0(x, \theta)}{\partial x \partial \theta} - \left(\frac{Gh}{R} + \frac{Dhn}{R} \right) \frac{\partial w_0(x, \theta)}{\partial x} \frac{\partial^2 v_0(x, \theta)}{\partial x \partial \theta} - \frac{2Gh}{R^2} \frac{\partial u_0(x, \theta)}{\partial \theta} \frac{\partial^2 w_0(x, \theta)}{\partial x \partial \theta} - \frac{2Gh}{R} \frac{\partial v_0(x, \theta)}{\partial x} \frac{\partial^2 w_0(x, \theta)}{\partial x \partial \theta} \\
& - Dh \frac{\partial w_0(x, \theta)}{\partial x} \frac{\partial^2 u_0(x, \theta)}{\partial x^2} - \frac{Gh}{R} \frac{\partial w_0(x, \theta)}{\partial x} \frac{\partial^2 v_0(x, \theta)}{\partial x^2} - \frac{Dhn}{R} w_0(x, \theta) \frac{\partial^2 w_0(x, \theta)}{\partial x^2} - \frac{Dhn}{R} \frac{\partial v_0(x, \theta)}{\partial \theta} \frac{\partial^2 w_0(x, \theta)}{\partial x^2} \\
& \left. - Dh \frac{\partial u_0(x, \theta)}{\partial x} \frac{\partial^2 w_0(x, \theta)}{\partial x^2} + \left(\frac{Gh^3}{3R^2} + \frac{Dh^3 n}{6R^2} \right) \frac{\partial^4 w_{01}(x, \theta)}{\partial x^2 \partial \theta^2} + \frac{1}{12} Dh^3 \frac{\partial^4 w_{01}(x, \theta)}{\partial x^4} \right] + \varepsilon^3[\dots] + \varepsilon^4[\dots] + \varepsilon^5[\dots] + \varepsilon^6[\dots] = (L - x)\gamma
\end{aligned} \tag{3.5c}$$

Since the coefficients of the higher order terms $\varepsilon^3, \varepsilon^4, \varepsilon^5, \varepsilon^6$ will not participate in the solution during the following stages and taking also into account their lengthy extent, it was considered pointless to address them here.

3.2.1 1st Order Problem

3.2.1.1 Linear EoMs

In perturbation method, the solution is approximated by breaking down the initial problem into sub problems according to the order of ε . Here, by keeping only the factor of ε^1 , the following linear set of 1st order $O(\varepsilon)$ equations is obtained,

$O(\varepsilon)$:

$$\begin{aligned}
\varepsilon \frac{1}{R} h \left[G \frac{\partial^2 u_0(x, \theta)}{\partial \theta^2} + DnR \frac{\partial w_0(x, \theta)}{\partial x} + (G + Dn)R \frac{\partial^2 v_0(x, \theta)}{\partial x \partial \theta} + DR^2 \frac{\partial^2 u_0(x, \theta)}{\partial x^2} \right] &= 0 \Rightarrow \\
G \frac{\partial^2 u_0(x, \theta)}{\partial \theta^2} + DR^2 \frac{\partial^2 u_0(x, \theta)}{\partial x^2} + (G + Dn)R \frac{\partial^2 v_0(x, \theta)}{\partial x \partial \theta} + DnR \frac{\partial w_0(x, \theta)}{\partial x} &= 0
\end{aligned} \tag{3.6a}$$

&

$$\begin{aligned}
\varepsilon \frac{1}{R} h \left[DR \frac{\partial w_0(x, \theta)}{\partial \theta} + DR \frac{\partial^2 v_0(x, \theta)}{\partial \theta^2} + (G + Dn)R^2 \frac{\partial^2 u_0(x, \theta)}{\partial x \partial \theta} + GR^3 \frac{\partial^2 v_0(x, \theta)}{\partial x^2} \right] &= 0 \Rightarrow \\
(G + Dn)R^2 \frac{\partial^2 u_0(x, \theta)}{\partial x \partial \theta} + GR^3 \frac{\partial^2 v_0(x, \theta)}{\partial x^2} + DR \frac{\partial^2 v_0(x, \theta)}{\partial \theta^2} + DR \frac{\partial w_0(x, \theta)}{\partial \theta} &= 0
\end{aligned} \tag{3.6b}$$

&

$$\begin{aligned}
\varepsilon \left[\frac{Dh}{R^2} w_0(x, \theta) + \frac{Dh}{R^2} \frac{\partial v_0(x, \theta)}{\partial \theta} + \frac{Dhn}{R} \frac{\partial u_0(x, \theta)}{\partial x} + \frac{h^3(2G + Dn)}{6R^2} \frac{\partial^4 w_0(x, \theta)}{\partial x^2 \partial \theta^2} + \frac{Dh^3}{12R^4} \frac{\partial^4 w_0(x, \theta)}{\partial \theta^4} + \frac{Dh^3}{12} \frac{\partial^4 w_0(x, \theta)}{\partial x^4} \right] &= \gamma(L - x) \\
\Rightarrow \frac{Dhn}{R} \frac{\partial u_0(x, \theta)}{\partial x} + \frac{Dh}{R^2} \frac{\partial v_0(x, \theta)}{\partial \theta} + \frac{Dhw_0(x, \theta)}{R^2} + \frac{Dh^3}{12} \frac{\partial^4 w_0(x, \theta)}{\partial x^4} + \frac{h^3(2G + Dn)}{6R^2} \frac{\partial^4 w_0(x, \theta)}{\partial x^2 \partial \theta^2} + \frac{Dh^3}{12R^4} \frac{\partial^4 w_0(x, \theta)}{\partial \theta^4} &= \gamma(L - x)
\end{aligned} \tag{3.6c}$$

In the 3rd equation, which describes the transverse motion, the existence of ε can be neglected as it is only a mathematical parameter. Next, the above set of equations can be written in matrix form,

$$\begin{bmatrix} L_{11} & L_{12} & L_{13} \\ L_{21} & L_{22} & L_{23} \\ L_{31} & L_{32} & L_{33} \end{bmatrix} \begin{bmatrix} u_0(x, \theta) \\ v_0(x, \theta) \\ w_0(x, \theta) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \gamma(L - x) \end{bmatrix} \tag{3.7}$$

where,

$$L_{11} = DR^2 \frac{\partial^2}{\partial x^2} + G \frac{\partial^2}{\partial \theta^2}, \quad L_{12} = R(G + Dn) \frac{\partial^2}{\partial x \partial \theta}, \quad L_{13} = RDn \frac{\partial}{\partial x}$$

$$\begin{aligned}
L_{21} &= R^2(G + Dn) \frac{\partial^2}{\partial x \partial \theta}, \quad L_{22} = GR^3 \frac{\partial^2}{\partial x^2} + DR \frac{\partial^2}{\partial \theta^2}, \quad L_{23} = DR \frac{\partial}{\partial \theta} \\
L_{31} &= \frac{Dhn}{R} \frac{\partial}{\partial x}, \quad L_{32} = \frac{Dh}{R^2} \frac{\partial}{\partial \theta}, \quad L_{33} = \frac{Dh}{R^2} + \frac{Dh^3}{12} \frac{\partial^4}{\partial x^4} + \frac{h^3}{6R^2} (2G + Dn) \frac{\partial^4}{\partial x^2 \partial \theta^2} + \frac{Dh^3}{12R^4} \frac{\partial^4}{\partial \theta^4}
\end{aligned} \quad (3.8a)$$

Since the two first equations are homogenous, the equation of axial motion can be multiplied by $\frac{h}{R^2}$ and the one of circumferential motion by $\frac{h}{R^3}$. In that case, the L_{ij} parameters become:

$$\begin{aligned}
L_{11} &= Dh \frac{\partial^2}{\partial x^2} + G \frac{h}{R^2} \frac{\partial^2}{\partial \theta^2}, \quad L_{12} = \frac{h}{R} (G + Dn) \frac{\partial^2}{\partial x \partial \theta} = L_{21}, \quad L_{13} = \frac{Dhn}{R} \frac{\partial}{\partial x} = L_{31} \\
L_{21} &= \frac{h}{R} (G + Dn) \frac{\partial^2}{\partial x \partial \theta} = L_{12}, \quad L_{22} = Gh \frac{\partial^2}{\partial x^2} + \frac{hD}{R^2} \frac{\partial^2}{\partial \theta^2}, \quad L_{23} = \frac{hD}{R^2} \frac{\partial}{\partial \theta} = L_{32} \\
L_{31} &= \frac{Dhn}{R} \frac{\partial}{\partial x} = L_{13}, \quad L_{32} = \frac{Dh}{R^2} \frac{\partial}{\partial \theta} = L_{23}, \quad L_{33} = \frac{Dh}{R^2} + \frac{Dh^3}{12} \frac{\partial^4}{\partial x^4} + \frac{h^3}{6R^2} (2G + Dn) \frac{\partial^4}{\partial x^2 \partial \theta^2} + \frac{Dh^3}{12R^4} \frac{\partial^4}{\partial \theta^4}
\end{aligned} \quad (3.8b)$$

As someone may observe, $L_{ij} = L_{ji}$ which is necessary according to some researchers for an accurate shell theory [25]. Relations (3.6) describe an δ^{th} order, in terms of x , nonhomogeneous system of three linear coupled partial differential equations. Separation of variables and decoupling of *EoMs*, are two essential actions that should be undertaken regardless of the order, so that a solution can be achieved. Starting out by simplifying (3.6) down to an ordinary system of differential equations, the displacements assumed in the form of,

$$\begin{aligned}
u_0(x, \theta) &= \sum_{k=0}^{\infty} U_0^{\bar{k}}(x) \cos \bar{k} \theta \\
v_0(x, \theta) &= \sum_{k=0}^{\infty} V_0^{\bar{k}}(x) \sin \bar{k} \theta \\
w_0(x, \theta) &= \sum_{k=0}^{\infty} W_0^{\bar{k}}(x) \cos \bar{k} \theta
\end{aligned} \quad (3.9)$$

in which $\bar{k}$, is the number of full waves in the circumferential direction. Substitution of (3.9) into (3.6) and utilizing the orthogonality of modes, results into the following set of ordinary linear equations:

$$\begin{aligned}
\sum_{k=0}^{\infty} \cos(\bar{k}\theta) \left[-\frac{Gh\bar{k}^2}{R^2} U_0(x) + DhU_0''(x) + \frac{h\bar{k}(G + Dn)}{R} V_0'(x) + \frac{Dhn}{R} W_0'(x) \right] &= 0 \\
\int_0^{2\pi} \cos m \theta (\dots) &\implies -\frac{Ghm^2}{R^2} U_0(x) + DhU_0''(x) + \frac{hm(G + Dn)}{R} V_0'(x) + \frac{Dhn}{R} W_0'(x) = 0
\end{aligned} \quad (3.10a)$$

&

$$\begin{aligned}
\sum_{k=0}^{\infty} \sin(\bar{k}\theta) \left[-\frac{h\bar{k}(G + Dn)}{R} U_0'(x) - \frac{Dh\bar{k}^2}{R^2} V_0(x) + GhV_0''(x) - \frac{Dh\bar{k}}{R^2} W_0(x) \right] &= 0 \\
\int_0^{2\pi} \sin m \theta (\dots) &\implies \begin{cases} 0 * [GhV_0''(x)] = 0, \text{ indeterminate for } m = 0 \\ -\frac{hm(G + Dn)}{R} U_0'(x) - \frac{Dhm^2}{R^2} V_0(x) + GhV_0''(x) - \frac{Dhm}{R^2} W_0(x) = 0, \quad m > 0 \end{cases}
\end{aligned} \quad (3.10b)$$

&

$$\sum_{k=0}^{\infty} \cos(\bar{k}\theta) \left[\frac{Dhn}{R} U_0'(x) + \frac{Dh\bar{k}}{R^2} V_0(x) + \left(\frac{Dh^3\bar{k}^4}{12R^4} + \frac{Dh}{R^2} \right) W_0(x) - \bar{k}^2 \left(\frac{Gh^3}{3R^2} + \frac{Dh^3n}{6R^2} \right) W_0''(x) + \frac{Dh^3}{12} W_0''''(x) \right] = \gamma(L - x)$$

$$\int_0^{2\pi} \cos m\theta (\dots = \dots) \rightarrow \begin{cases} \frac{Dhn}{R} U'_0(x) + \frac{Dhm}{R^2} V_0(x) + \left(\frac{Dh^3 m^4}{12R^4} + \frac{Dh}{R^2} \right) W_0(x) - m^2 \left(\frac{Gh^3}{3R^2} + \frac{Dh^3 n}{6R^2} \right) W_0''(x) + \frac{Dh^3}{12} W_0''''(x) = \gamma(L-x), m=0 \\ \frac{Dhn}{R} U'_0(x) + \frac{Dhm}{R^2} V_0(x) + \left(\frac{Dh^3 m^4}{12R^4} + \frac{Dh}{R^2} \right) W_0(x) - m^2 \left(\frac{Gh^3}{3R^2} + \frac{Dh^3 n}{6R^2} \right) W_0''(x) + \frac{Dh^3}{12} W_0''''(x) = 0, m>0 \end{cases} \quad (3.10c)$$

Two important remarks should be made based upon the previous relations. The effect of water pressure is present only for the symmetric mode $\bar{k} = 0$, in which as well, the circumferential motion is completely pointless and thus shall be disregarded from the analysis. It is obvious then, that different cases of modes call for different treatment.

Asymmetric Modes ($\bar{k} > 0$)

Deflections in terms of the independent variable x for the δ^{th} order system of ordinary differential equations, is searched in the form of,

$$\begin{aligned} U_0(x) &= \sum_{i=1}^8 \bar{U}_i^{\bar{k}} e^{\lambda_i x} \\ V_0(x) &= \sum_{i=1}^8 \bar{V}_i^{\bar{k}} e^{\lambda_i x} \\ W_0(x) &= \sum_{i=1}^8 \bar{W}_i^{\bar{k}} e^{\lambda_i x} \end{aligned} \quad (3.11)$$

where $\bar{U}_i^{\bar{k}}$, $\bar{V}_i^{\bar{k}}$ and $\bar{W}_i^{\bar{k}}$ represent unknown constants. Computation of the determinant in relation (3.7) and utilizing once again the orthogonality of modes, the system of equations is being decoupled and it leads to,

$$\begin{aligned} [L_{13}L_{22}L_{31} - L_{12}L_{23}L_{31} - L_{13}L_{21}L_{32} + L_{11}L_{23}L_{32} + L_{12}L_{21}L_{33} - L_{11}L_{22}L_{33}]f_0(x, \theta) &= (L_{22}L_{13} - L_{12}L_{23})\gamma(L-x) \Rightarrow \\ a_1 \frac{\partial^8 f_0(x, \theta)}{\partial \theta^8} + a_2 \frac{\partial^8 f_0(x, \theta)}{\partial x^2 \partial \theta^6} + a_3 \frac{\partial^4 f_0(x, \theta)}{\partial x^4} + a_4 \frac{\partial^8 f_0(x, \theta)}{\partial x^4 \partial \theta^4} + a_5 \frac{\partial^8 f_0(x, \theta)}{\partial x^6 \partial \theta^2} + a_6 \frac{\partial^8 f_0(x, \theta)}{\partial x^8} &= 0 \quad \xrightarrow{\text{orthogonality}} \\ \bar{k}^8 a_1 F_0(x) - \bar{k}^6 a_2 \frac{d^2 F_0(x)}{dx^2} + (a_3 + \bar{k}^4 a_4) \frac{d^4 F_0(x)}{dx^4} - \bar{k}^2 a_5 \frac{d^6 F_0(x)}{dx^6} + a_6 \frac{d^8 F_0(x)}{dx^8} &= 0 \end{aligned} \quad (3.12)$$

in which,

$$\begin{aligned} a_1 &= -\frac{D^2 Gh^5}{12R^8}, \quad a_2 = -\frac{D^3 h^5}{12R^6} - \frac{DG^2 h^5}{3R^6} + \frac{D^3 h^5 n^2}{12R^6}, \quad a_3 = -\frac{D^2 Gh^3}{R^2} + \frac{D^2 Gh^3 n^2}{R^2}, \\ a_4 &= -\frac{D^2 Gh^5}{2R^4} - \frac{D^3 h^5 n}{6R^4} + \frac{2DG^2 h^5 n}{3R^4} + \frac{2D^2 Gh^5 n^2}{3R^4} + \frac{D^3 h^5 n^3}{6R^4}, \quad a_5 = -\frac{D^3 h^5}{12R^2} - \frac{DG^2 h^5}{3R^2} + \frac{D^3 h^5 n^2}{12R^2}, \quad a_6 = -\frac{1}{12} D^2 Gh^5 \end{aligned}$$

Symbolic functions $f_0(x, \theta)$ and $F_0(x)$ refer to those of u_0, v_0, w_0 and $U_0^{\bar{k}}, V_0^{\bar{k}}, W_0^{\bar{k}}$ respectively, and have been used for space economy reasons. Substituting now into (3.12), $F_0(x)$ in the form dictated by relations (3.11), the following *characteristic equation* can be developed for evaluating the λ_i parameters.

$$\bar{k}^8 a_1 - \bar{k}^6 \lambda_i^2 a_2 + (a_3 + \bar{k}^4 a_4) \lambda_i^4 - \bar{k}^2 \lambda_i^6 a_5 + \lambda_i^8 a_6 = 0 \quad (3.13)$$

As it will be seen in the next section, only 8 boundary conditions are available, meaning that, at first glance, are not adequate for determining the 24 unknowns featured in the assumed solution (3.11). In order to overcome this obstacle, a relation is searched between $\bar{U}_i^{\bar{k}}, \bar{V}_i^{\bar{k}}$ and $\bar{W}_i^{\bar{k}}$. For this purpose, the formulas from appendix B referring to the decoupling procedure are applied onto the matrix form of equations (3.7),

$$(L_{32}L_{22}L_{11} - L_{32}L_{12}L_{21})u_0(x, \theta) = (L_{32}L_{12}L_{23} - L_{32}L_{22}L_{13})w_0(x, \theta) \Rightarrow$$

$$(L_{22}L_{11} - L_{12}L_{21})u_0(x, \theta) = (L_{12}L_{23} - L_{22}L_{13})w_0(x, \theta) \Rightarrow$$

$$\begin{aligned}
& \frac{D G h^2}{R^4} \frac{\partial^4 u_0(x, \theta)}{\partial \theta^4} + \left(\frac{D^2 h^2}{R^2} - \frac{2 D G h^2 n}{R^2} - \frac{D^2 h^2 n^2}{R^2} \right) \frac{\partial^4 u_0(x, \theta)}{\partial x^2 \partial \theta^2} + D G h^2 \frac{\partial^4 u_0(x, \theta)}{\partial x^4} = \frac{D G h^2}{R^3} \frac{\partial^3 w_0(x, \theta)}{\partial x \partial \theta^2} - \frac{D G h^2 n}{R} \frac{\partial^3 w_0(x, \theta)}{\partial x^3} \Rightarrow \\
& \sum_{k=0}^{\infty} \cos \bar{k} \theta e^{x \lambda_i} \bar{U}_i^{\bar{k}} \left[\frac{D G h^2 \bar{k}^4}{R^4} + \bar{k}^2 \left(-\frac{D^2 h^2}{R^2} + \frac{2 D G h^2 n}{R^2} + \frac{D^2 h^2 n^2}{R^2} \right) \lambda_i^2 + D G h^2 \lambda_i^4 \right] = \sum_{k=0}^{\infty} \cos \bar{k} \theta e^{x \lambda_i} \bar{W}_i^{\bar{k}} \left(-\frac{D G h^2 \bar{k}^2 \lambda_i}{R^3} - \frac{D G h^2 n \lambda_i^3}{R} \right) \\
& \xrightarrow{\int_0^{2\pi} \cos m \theta (\dots) d\theta} \bar{U}_i^m = \left[\frac{-\frac{D G h^2 m^2 \lambda_i}{R^3} - \frac{D G h^2 n \lambda_i^3}{R}}{\frac{D G h^2 m^4}{R^4} + m^2 \left(-\frac{D^2 h^2}{R^2} + \frac{2 D G h^2 n}{R^2} + \frac{D^2 h^2 n^2}{R^2} \right) \lambda_i^2 + D G h^2 \lambda_i^4} \right] \bar{W}_i^m \Rightarrow \bar{U}_i^m = \bar{A}_i^m \bar{W}_i^m \quad (3.14a)
\end{aligned}$$

&

$$(L_{31} L_{21} L_{12} - L_{31} L_{11} L_{22}) v_0(x, \theta) = (L_{31} L_{11} L_{23} - L_{31} L_{21} L_{13}) w_0(x, \theta) \Rightarrow$$

$$(L_{21} L_{12} - L_{11} L_{22}) v_0(x, \theta) = (L_{11} L_{23} - L_{21} L_{13}) w_0(x, \theta) \Rightarrow$$

$$\begin{aligned}
& -\frac{D G h^2}{R^4} \frac{\partial^4 v_0(x, \theta)}{\partial \theta^4} + \left(-\frac{D^2 h^2}{R^2} + \frac{2 D G h^2 n}{R^2} + \frac{D^2 h^2 n^2}{R^2} \right) \frac{\partial^4 v_0(x, \theta)}{\partial x^2 \partial \theta^2} - D G h^2 \frac{\partial^4 v_0(x, \theta)}{\partial x^4} = \\
& \frac{D G h^2}{R^4} \frac{\partial^3 w_0(x, \theta)}{\partial \theta^3} + \left(\frac{D^2 h^2}{R^2} - \frac{D G h^2 n}{R^2} - \frac{D^2 h^2 n^2}{R^2} \right) \frac{\partial^3 u_0(x, \theta)}{\partial x^2 \partial \theta} \Rightarrow \\
& \sum_{k=0}^{\infty} \sin \bar{k} \theta e^{x \lambda_i} \bar{V}_i^{\bar{k}} \left[-\frac{D G h^2 \bar{k}^4}{R^4} + \bar{k}^2 \left(\frac{D^2 h^2}{R^2} - \frac{2 D G h^2 n}{R^2} - \frac{D^2 h^2 n^2}{R^2} \right) \lambda_i^2 - D G h^2 \lambda_i^4 \right] = \\
& \sum_{k=0}^{\infty} \sin \bar{k} \theta e^{x \lambda_i} \bar{W}_i^{\bar{k}} \left[\frac{D G h^2 \bar{k}^3}{R^4} + \bar{k} \left(-\frac{D^2 h^2}{R^2} + \frac{D G h^2 n}{R^2} + \frac{D^2 h^2 n^2}{R^2} \right) \lambda_i^2 \right] \\
& \xrightarrow{\int_0^{2\pi} \sin m \theta (\dots) d\theta} \bar{V}_i^m = \left[\frac{\frac{D G h^2 m^3}{R^4} + m \left(-\frac{D^2 h^2}{R^2} + \frac{D G h^2 n}{R^2} + \frac{D^2 h^2 n^2}{R^2} \right) \lambda_i^2}{-\frac{D G h^2 m^4}{R^4} + m^2 \left(\frac{D^2 h^2}{R^2} - \frac{2 D G h^2 n}{R^2} - \frac{D^2 h^2 n^2}{R^2} \right) \lambda_i^2 - D G h^2 \lambda_i^4} \right] \bar{W}_i^m \Rightarrow \bar{V}_i^m = \bar{B}_i^m \bar{W}_i^m \quad (3.14b)
\end{aligned}$$

For each of the previously calculated values of λ_i parameters, $\bar{U}_i^{\bar{k}}$, $\bar{V}_i^{\bar{k}}$ and $\bar{W}_i^{\bar{k}}$ are now known.

Symmetric Mode ($\bar{k} = 0$)

Due to the fact that the shell expands and contracts in a uniform manner in the symmetric case of $\bar{k} = 0$, circumferential equation of motion has no meaning. Now, the angle θ does not varies along the cylinder's periphery and so all the derivatives in respect to θ are zero. This means that the coefficients $L_{12} = L_{21}$ & $L_{23} = L_{32}$ vanish and the set of equations results into a much simpler 6th order problem which nevertheless, accounts for the effect of water.

$$\begin{aligned}
& D h U_0''(x) + \frac{D h n}{R} W_0'(x) = 0 \\
& \frac{D h n}{R} U_0'(x) + \frac{D h}{R^2} W_0(x) + \frac{D h^3}{12} W_0''''(x) = \gamma(L - x)
\end{aligned} \quad (3.15)$$

All the responses as a function of x for the ordinary homogeneous equations take the form of,

$$\begin{aligned}
U_0(x) &= \sum_{i=1}^6 \bar{U}_i^0 e^{\lambda_i x} \\
W_0(x) &= \sum_{i=1}^6 \bar{W}_i^0 e^{\lambda_i x}
\end{aligned} \quad (3.16)$$

The values of λ_i are determined through the characteristic equation which is obtained in a similar way like before and thus there is no need to repeat it here. Also, the obstacle of 12 unknown constants against the 6 available boundary conditions for this situation, is handled once again by the following relation,

$$\begin{aligned}
L_{11} u_0(x, \theta) &= -L_{13} w_0(x, \theta) \Rightarrow Dh \frac{\partial^2 u_0(x, \theta)}{\partial x^2} = -\frac{Dhn}{R} \frac{\partial w_0(x, \theta)}{\partial x} \xrightarrow{(3.9) \text{ for } \bar{k}=0} Dh U_0''(x) = -\frac{Dhn}{R} W_0'(x) \\
&\xrightarrow{(3.16)} e^{x\lambda_i} \bar{U}_i^0 D h \lambda_i^2 = e^{x\lambda_i} \bar{W}_i^0 \left(-\frac{Dhn}{R} \lambda_i \right) \Rightarrow \bar{W}_i^0 = \left(-\frac{R\lambda_i}{n} \right) \bar{U}_i^0 \Rightarrow \bar{W}_i^0 = \bar{C}_i^0 \bar{U}_i^0
\end{aligned} \quad (3.17)$$

The general solution of the inhomogeneous problem, cannot be complete without a particular solution which takes into account the effect of water pressure. This can be easily calculated by assuming two 2nd order polynomials for functions $U_0(x)$ and $W_0(x)$ and substituting them in the initial formulation (3.15). The final result is given by,

$$\begin{bmatrix} U_0^p(x) \\ W_0^p(x) \end{bmatrix} = \begin{bmatrix} \frac{LR\gamma}{Dhn} x - \frac{nR\gamma}{2Dh(-1+n^2)} x^2 \\ \frac{R^2\gamma}{Dh(-1+n^2)} x \end{bmatrix} \quad (3.18)$$

3.2.1.2 Boundary Conditions

The cylindrical tank which rests on the ground undergoes a horizontal shear force due to earthquake that resolves into the two components as it is illustrated in the figure below.

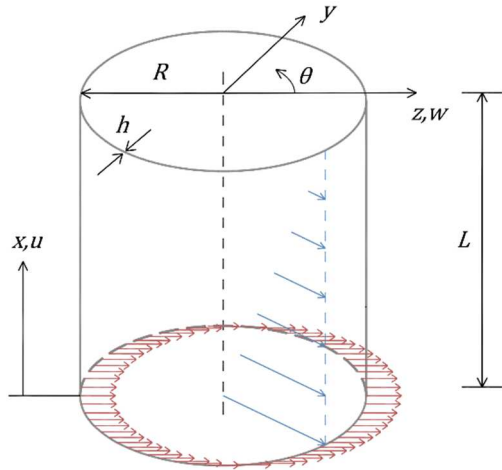


Figure 16 Cylindrical tank under horizontal shear force and water pressure

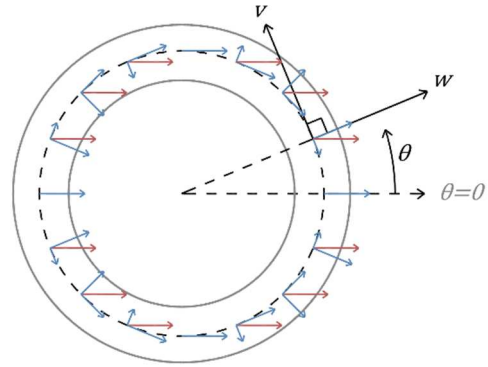


Figure 17 Components of uniform distributed shear at the bottom

Unlike dynamics, in order for buckling to take place in the case of a static load, the shell must be under some kind of restraint at the top. Of course, this is not a requirement for a time variant load where the mass of water contributes in the buckling phenomenon during the dynamic motion of the shell. Now taking all the aforementioned into consideration, the boundary conditions which will concern the case at hand are formulated as,

$$x = 0 : \quad u(0, \theta) = 0, \quad M_{xx}^*(0, \theta) = 0, \quad N_{x\theta}^*(0, \theta) = F_E \sin \theta, \quad Q_{xz}^*(0, \theta) = F_E \cos \theta \quad (3.19)$$

$$x = L :$$

$$\text{If clamped} \quad u(L, \theta) = 0, \quad v(L, \theta) = 0, \quad w(L, \theta) = 0, \quad \left| \frac{\partial w(x, \theta)}{\partial x} \right|_{x=L} = 0 \quad (3.20a)$$

or

$$\text{If hinged} \quad u(L, \theta) = 0, \quad v(L, \theta) = 0, \quad w(L, \theta) = 0, \quad M_{xx}^*(L, \theta) = 0 \quad (3.20b)$$

The assumed displacements prescribed by the perturbation method (3.3) should also be substituted in the relevant boundary conditions (2.52) of the previous chapter. Apart from the displacements' restraints, which are relatively obvious, the procedure for force and moment conditions will be displayed here only.

$$N_{x\theta}^* = N_{x\theta}, \quad M_{xx}^* = M_{xx}, \quad Q_{xz}^* = N_{xx} \frac{\partial w(x, \theta)}{\partial x} + N_{x\theta} \frac{\partial w(x, \theta)}{R \partial \theta} + Q_{xz} \quad (3.21)$$

Which may also be written, using (2.42-2.46), in the form of displacements,

$$\begin{aligned} N_{x\theta}^* &= \frac{Gh}{R} \frac{\partial u(x, \theta)}{\partial \theta} + Gh \frac{\partial v(x, \theta)}{\partial x} + \frac{Gh}{R} \frac{\partial w(x, \theta)}{\partial \theta} \frac{\partial w(x, \theta)}{\partial x} \\ M_{xx}^* &= -\frac{Dh^3 n}{12R^2} \frac{\partial^2 w(x, \theta)}{\partial \theta^2} - \frac{Dh^3}{12} \frac{\partial^2 w(x, \theta)}{\partial x^2} \\ Q_{xz}^* &= \frac{Gh}{R^2} \frac{\partial u(x, \theta)}{\partial \theta} \frac{\partial w(x, \theta)}{\partial \theta} + \frac{Gh}{R} \frac{\partial w(x, \theta)}{\partial \theta} \frac{\partial v(x, \theta)}{\partial x} + \frac{Dhn}{R} w(x, \theta) \frac{\partial w(x, \theta)}{\partial x} + \frac{Dhn}{R} \frac{\partial v(x, \theta)}{\partial \theta} \frac{\partial w(x, \theta)}{\partial x} + \\ &\quad \left(\frac{Gh}{R^2} + \frac{Dhn}{2R^2} \right) \left(\frac{\partial w(x, \theta)}{\partial \theta} \right)^2 \frac{\partial w(x, \theta)}{\partial x} + Dh \frac{\partial u(x, \theta)}{\partial x} \frac{\partial w(x, \theta)}{\partial x} + \frac{Dh}{2} \left(\frac{\partial w(x, \theta)}{\partial x} \right)^3 - \left(\frac{Gh^3}{6R^2} + \frac{Dh^3 n}{12R^2} \right) \frac{\partial^3 w(x, \theta)}{\partial x \partial \theta^2} - \frac{Dh^3}{12} \frac{\partial^3 w(x, \theta)}{\partial x^3} \end{aligned} \quad (3.22)$$

The assumed displacements (3.3) prescribed by the perturbation method are substituted in (3.22),

$$\begin{aligned} N_{x\theta}^* &= \varepsilon \left[\frac{Gh}{R} \frac{\partial u_0(x, \theta)}{\partial \theta} + Gh \frac{\partial v_0(x, \theta)}{\partial x} \right] + \varepsilon^2 \left[\frac{Gh}{R} \frac{\partial u_{01}(x, \theta)}{\partial \theta} + \frac{Gh}{R} \frac{\partial w_0(x, \theta)}{\partial x} \frac{\partial w_0(x, \theta)}{\partial \theta} \right] + \varepsilon^3 [\dots] + \varepsilon^4 [\dots] \\ M_{xx}^* &= M_{xx} = \varepsilon \left[-\frac{Dh^3 n}{12R^2} \frac{\partial^2 w_0(x, \theta)}{\partial \theta^2} - \frac{Dh^3}{12} \frac{\partial^2 w_0(x, \theta)}{\partial x^2} \right] + \varepsilon^2 \left[-\frac{Dh^3 n}{12R^2} \frac{\partial^2 w_{01}(x, \theta)}{\partial \theta^2} - \frac{Dh^3}{12} \frac{\partial^2 w_{01}(x, \theta)}{\partial x^2} \right] \\ Q_{xz}^* &= \varepsilon \left[\left(-\frac{Gh^3}{6R^2} - \frac{Dh^3 n}{12R^2} \right) \frac{\partial^3 w_0(x, \theta)}{\partial x \partial \theta^2} - \frac{Dh^3}{12} \frac{\partial^3 w_0(x, \theta)}{\partial x^3} \right] + \varepsilon^2 \left[\frac{Gh}{R^2} \frac{\partial u_0(x, \theta)}{\partial \theta} \frac{\partial w_0(x, \theta)}{\partial \theta} + \frac{Gh}{R} \frac{\partial w_0(x, \theta)}{\partial \theta} \frac{\partial v_0(x, \theta)}{\partial x} \right. \\ &\quad \left. + \frac{Dhn}{R} w_0(x, \theta) \frac{\partial w_0(x, \theta)}{\partial x} + \frac{Dhn}{R} \frac{\partial v_0(x, \theta)}{\partial \theta} \frac{\partial w_0(x, \theta)}{\partial x} + Dh \frac{\partial u_0(x, \theta)}{\partial x} \frac{\partial w_0(x, \theta)}{\partial x} - \left(\frac{Gh^3}{6R^2} + \frac{Dh^3 n}{12R^2} \right) \frac{\partial^3 w_{01}(x, \theta)}{\partial x \partial \theta^2} \right. \\ &\quad \left. - \frac{Dh^3}{12} \frac{\partial^3 w_{01}(x, \theta)}{\partial x^3} \right] + \varepsilon^3 [\dots] + \varepsilon^4 [\dots] + \varepsilon^5 [\dots] + \varepsilon^6 [\dots] \end{aligned} \quad (3.23)$$

Once again, there is no point of featuring the higher order terms as they will not be of use during the next steps. For the 1st order problem, (3.23) reduce to,

$O(\varepsilon^1)$:

$$\begin{aligned} N_{x\theta}^* &= \frac{Gh}{R} \frac{\partial u_0(x, \theta)}{\partial \theta} + Gh \frac{\partial v_0(x, \theta)}{\partial x} \\ M_{xx}^* &= -\frac{Dh^3 n}{12R^2} \frac{\partial^2 w_0(x, \theta)}{\partial \theta^2} - \frac{Dh^3}{12} \frac{\partial^2 w_0(x, \theta)}{\partial x^2} \\ Q_{xz}^* &= \left(-\frac{Gh^3}{6R^2} - \frac{Dh^3 n}{12R^2} \right) \frac{\partial^3 w_0(x, \theta)}{\partial x \partial \theta^2} - \frac{Dh^3}{12} \frac{\partial^3 w_0(x, \theta)}{\partial x^3} \end{aligned} \quad (3.24)$$

in which the perturbation quantity ε can be struck out because it is only a mathematical parameter. Taking also into account the assumed solutions for displacements' responses (3.9) and (3.11) as well as utilizing the property of orthogonality of modes, the boundary conditions (3.19 - 20) become:

$x = 0$:

$$1) \quad u_0(0, \theta) = 0 \xrightarrow{\int_0^{2\pi} \cos m \theta (\dots = \dots)} U_0(0) = 0 \quad (3.25a)$$

$$2) \quad M_{xx}^*(0, \theta) = 0 \xrightarrow{\int_0^{2\pi} \cos m \theta (\dots = \dots)} -n m^2 W_0(0) + R^2 \left| \frac{d^2 W_0(x)}{dx^2} \right|_{x=0} = 0 \quad (3.25b)$$

$$3) \quad N_{x\theta}^*(0, \theta) = F_E \sin \theta \xrightarrow{\int_0^{2\pi} \sin m \theta (\dots = \dots)} \begin{cases} 0 * (\dots) = 0, \text{ indeterminate case for } m = 0 \\ -\frac{Gh}{R} U_0(0) + Gh \left| \frac{dV_0(x)}{dx} \right|_{x=0} = F_E, m = 1 \\ -m \frac{Gh}{R} U_0(0) + Gh \left| \frac{dV_0(x)}{dx} \right|_{x=0} = 0, m > 1 \end{cases} \quad (3.25c)$$

$$4) \quad Q_{xz}^*(0, \theta) = F_E \cos \theta \xrightarrow{\int_0^{2\pi} \cos m \theta (\dots = \dots)} \begin{cases} \frac{h^3(2G + Dn)}{12R^2} \left| \frac{dW_0(x)}{dx} \right|_{x=0} - \frac{h^3 DR^2}{12R^2} \left| \frac{d^3 W_0(x)}{dx^3} \right|_{x=0} = F_E, m = 1 \\ \frac{h^3 m^2(2G + Dn)}{12R^2} \left| \frac{dW_0(x)}{dx} \right|_{x=0} - \frac{h^3 DR^2}{12R^2} \left| \frac{d^3 W_0(x)}{dx^3} \right|_{x=0} = 0, m \neq 1 \end{cases} \quad (3.25d)$$

$x = L :$

If clamped

$$5) \quad u_0(L, \theta) = 0 \xrightarrow{\int_0^{2\pi} \cos m \theta (\dots = \dots)} U_0(L) = 0 \quad (3.26a)$$

$$6) \quad v_0(L, \theta) = 0 \xrightarrow{\int_0^{2\pi} \sin m \theta (\dots = \dots)} \begin{cases} 0 * (\dots) = 0, \text{ indeterminate case for } m = 0 \\ V_0(L) = 0, m > 0 \end{cases} \quad (3.26b)$$

$$7) \quad w_0(L, \theta) = 0 \xrightarrow{\int_0^{2\pi} \cos m \theta (\dots = \dots)} W_0(L) = 0 \quad (3.26c)$$

$$8) \quad \left| \frac{\partial w_0(x, \theta)}{\partial x} \right|_{x=L} = 0 \xrightarrow{\int_0^{2\pi} \cos m \theta (\dots = \dots)} \left| \frac{dW_0(x)}{dx} \right|_{x=L} = 0 \quad (3.26d)$$

Or if hinged

$$8) \quad M_{xx}^*(L, \theta) = 0 \xrightarrow{\int_0^{2\pi} \cos m \theta (\dots = \dots)} -n m^2 W_0(L) + R^2 \left| \frac{d^2 W_0(x)}{dx^2} \right|_{x=L} = 0 \quad (3.26e)$$

3.2.1.3 1st Order Theoretical Solution

In the following stages, it was decided that the buckling of the circular cylindrical shell tank will be induced for the case of a clamped boundary at the top. The final “linear” solution comprises of the solutions for three sub-problems at first glance. The first one refers to the symmetric shell mode, in which from the previous sections it can be seen that only the effect of water is present. The anti-symmetric modes still need to be subdivided as well. For the majority of anti-symmetric modes, the problem remains unaltered, with the exception of the first anti-symmetric mode, in which the external force at the bottom responsible for buckling is present. With that being said, and in an effort to sum up the entirety of previous sections, the system of equations (3.10) are solved for each case of boundaries conditions substituting (3.11, 14) & (3.16-18) into (3.26):

Symmetric Problem ($\bar{k} = 0$) - Effect of the water pressure:

$$1) \quad u_0(0, \theta) = 0 \Rightarrow \sum_{i=1}^6 \bar{U}_i^0 = 0$$

$$2) \quad M_{xx}^*(0, \theta) = 0 \Rightarrow \sum_{i=1}^6 \lambda_i^2 \bar{C}_i \bar{U}_i^0 = 0$$

$$4) \quad u_0(L, \theta) = 0 \Rightarrow \sum_{i=1}^6 \bar{U}_i^0 e^{\lambda_i L} + \frac{L^2 R \gamma}{D h n} - \frac{L^2 n R \gamma}{2 D h (-1 + n^2)} = 0$$

$$5) \quad w_0(L, \theta) = 0 \Rightarrow \sum_{i=1}^6 \bar{C}_i \bar{U}_i^0 e^{\lambda_i L} + \frac{R^2 \gamma L}{D h (-1 + n^2)} = 0$$

$$3) Q_{xz}^*(0, \theta) = F \cos \theta \Rightarrow \sum_{i=1}^6 \lambda_i^3 \bar{C}_i \bar{U}_i^0 = 0 \quad 6) \left| \frac{\partial w_0(x, \theta)}{\partial x} \right|_{x=L} = 0 \Rightarrow \sum_{i=1}^6 \bar{C}_i \bar{U}_i^0 \lambda_i e^{\lambda_i L} + \frac{R^2 \gamma}{Dh(-1+n^2)} = 0 \quad (3.27)$$

Anti-symmetric Problem ($\bar{k} = 1$) - Effect of the buckling Force:

$$\begin{aligned} 1) u_0(0, \theta) = 0 &\Rightarrow \sum_{i=1}^8 \bar{A}_i \bar{W}_i^1 = 0 & 5) u_0(L, \theta) = 0 &\Rightarrow \sum_{i=1}^8 \bar{A}_i \bar{W}_i^1 e^{\lambda_i L} = 0 \\ 2) M_{xx}^*(0, \theta) = 0 &\Rightarrow \sum_{i=1}^8 (-n + R^2 \lambda_i^2) \bar{W}_i^1 = 0 & 6) v_0(L, \theta) = 0 &\Rightarrow \sum_{i=1}^8 \bar{B}_i \bar{W}_i^1 e^{\lambda_i L} = 0 \\ 3) N_{x\theta}^*(0, \theta) = F \sin \theta &\Rightarrow \sum_{i=1}^8 \left(-\frac{Gh}{R} \bar{A}_i + Gh \bar{B}_i \lambda_i \right) \bar{W}_i^1 = F_E & 7) w_0(L, \theta) = 0 &\Rightarrow \sum_{i=1}^8 \bar{W}_i^1 e^{\lambda_i L} = 0 \\ 4) Q_{xz}^*(0, \theta) = F \cos \theta &\Rightarrow \sum_{i=1}^8 \left[\frac{h^3 \lambda_i (2G + Dn - DR^2 \lambda_i^2)}{12R^2} \right] \bar{W}_i^1 = F_E & 8) \left| \frac{\partial w_0(x, \theta)}{\partial x} \right|_{x=L} = 0 &\Rightarrow \sum_{i=1}^8 \bar{W}_i^1 \lambda_i e^{\lambda_i L} = 0 \end{aligned} \quad (3.28)$$

Anti-symmetric Problem ($\bar{k} > 1$) :

$$\begin{aligned} 1) u_0(0, \theta) = 0 &\Rightarrow \sum_{i=1}^8 \bar{A}_i \bar{W}_i^{\bar{k}} = 0 & 5) u_0(h, \theta) = 0 &\Rightarrow \sum_{i=1}^8 \bar{A}_i \bar{W}_i^{\bar{k}} e^{\lambda_i L} = 0 \\ 2) M_{xx}^*(0, \theta) = 0 &\Rightarrow \sum_{i=1}^8 (-n m^2 + R^2 \lambda_i^2) \bar{W}_i^{\bar{k}} = 0 & 6) v_0(h, \theta) = 0 &\Rightarrow \sum_{i=1}^8 \bar{B}_i \bar{W}_i^{\bar{k}} e^{\lambda_i L} = 0 \\ 3) N_{x\theta}^*(0, \theta) = F \sin \theta &\Rightarrow \sum_{i=1}^8 \left(-m \frac{Gh}{R} \bar{A}_i + Gh \bar{B}_i \lambda_i \right) \bar{W}_i^{\bar{k}} = 0 & 7) w_0(h, \theta) = 0 &\Rightarrow \sum_{i=1}^8 \bar{W}_i^{\bar{k}} e^{\lambda_i L} = 0 \\ 4) Q_{xz}^*(0, \theta) = F \cos \theta &\Rightarrow \sum_{i=1}^8 \left[-\frac{h^3 \lambda_i (-m^2 (2G + Dn) + DR^2 \lambda_i^2)}{12R^2} \right] \bar{W}_i^{\bar{k}} = 0 & 8) \frac{\partial w_0(h, \theta)}{\partial x} = 0 &\Rightarrow \sum_{i=1}^8 \bar{W}_i^{\bar{k}} \lambda_i e^{\lambda_i L} = 0 \end{aligned} \quad (3.29)$$

The case of $\bar{k} > 1$ can be considered trivial since no motion takes place in this range of modes. Eventually, solving the two sets of *EoMs*, one is able to determine the 14 unknown constants $\bar{W}_i^0, \bar{W}_i^1$ (6 for the symmetric and 8 for each asymmetric case) which contain the unknown external shear force F_E , and so the general solutions of the linear problem are formulated as:

$$u_0^{1st\ order}(x, \theta) = \sum_{i=1}^6 \bar{U}_i e^{\lambda_i x} - \frac{nR\gamma}{2Dh(-1+n^2)} x^2 + \frac{LR\gamma}{Dhn} x + \sum_{i=1}^8 \bar{A}_i \bar{W}_i^1 e^{\lambda_i x} \cos \theta \quad (3.30a)$$

$$v_0^{1st\ order}(x, \theta) = \sum_{i=1}^8 \bar{B}_i \bar{W}_i^1 e^{\lambda_i x} \sin \theta \quad (3.30b)$$

$$w_0^{1st\ order}(x, \theta) = \sum_{i=1}^6 \bar{C}_i \bar{U}_i e^{\lambda_i x} + \frac{R^2 \gamma}{Dh(-1+n^2)} x + \sum_{i=1}^8 \bar{W}_i^1 e^{\lambda_i x} \cos \theta \quad (3.30c)$$

or using a more convenient notation,

$$u_0^{1st\ order}(x, \theta) = U_{symmetric}^{1st\ order}(x) + U_{asymmetric}^{1st\ order}(x) \cos \theta \quad (3.31a)$$

$$v_0^{1st\ order}(x, \theta) = V_{asymmetric}^{1st\ order}(x) \sin \theta \quad (3.31b)$$

$$w_0^{1st\ order}(x, \theta) = W_{symmetric}^{1st\ order}(x) + W_{asymmetric}^{1st\ order}(x) \cos \theta \quad (3.31c)$$

3.2.2 2nd order Problem

3.2.2.1 Non-Linear EoMs

Obtaining the solution for the linear case was an absolute prerequisite for solving the nonlinear equations of motion, which after all is the main purpose of this thesis. Here the set of equations (3.2) recalled, and after keeping only the terms with respect to ε & ε^2 we arrive to, the following non-linear set of 2nd order $O(\varepsilon^2)$ equations is obtained,

$O(\varepsilon^2)$:

$$\frac{h}{R} \left[G \frac{\partial^2 u_0(x, \theta)}{\partial \theta^2} + DnR \frac{\partial w_0(x, \theta)}{\partial x} + (G + Dn)R \frac{\partial^2 v_0(x, \theta)}{\partial x \partial \theta} + DR^2 \frac{\partial^2 u_0(x, \theta)}{\partial x^2} \right] + \frac{h}{R} \left[G \frac{\partial^2 u_{01}(x, \theta)}{\partial \theta^2} + G \frac{\partial^2 w_0(x, \theta)}{\partial \theta^2} \frac{\partial w_0(x, \theta)}{\partial x} + DnR \frac{\partial w_{01}(x, \theta)}{\partial x} + (G + Dn) \frac{\partial w_0(x, \theta)}{\partial \theta} \frac{\partial^2 w_0(x, \theta)}{\partial x \partial \theta} + DR^2 \frac{\partial^2 u_{01}(x, \theta)}{\partial x^2} + DR^2 \frac{\partial w_0(x, \theta)}{\partial x} \frac{\partial^2 w_0(x, \theta)}{\partial x^2} \right] = 0 \quad (3.32a)$$

&

$$\frac{h}{R} \left[DR \frac{\partial w_0(x, \theta)}{\partial \theta} + DR \frac{\partial^2 v_0(x, \theta)}{\partial \theta^2} + (G + Dn)R^2 \frac{\partial^2 u_0(x, \theta)}{\partial x \partial \theta} + GR^3 \frac{\partial^2 v_0(x, \theta)}{\partial x^2} \right] + \frac{h}{R} \left[DR \frac{\partial w_{01}(x, \theta)}{\partial \theta} + D \frac{\partial w_0(x, \theta)}{\partial \theta} \frac{\partial^2 w_0(x, \theta)}{\partial \theta^2} + (G + Dn)R^2 \frac{\partial^2 u_{01}(x, \theta)}{\partial x \partial \theta} + (G + Dn)R^2 \frac{\partial w_0(x, \theta)}{\partial x} \frac{\partial^2 w_0(x, \theta)}{\partial x \partial \theta} + GR^2 \frac{\partial w_0(x, \theta)}{\partial \theta} \frac{\partial^2 w_0(x, \theta)}{\partial x^2} \right] = 0 \quad (3.32b)$$

&

$$\begin{aligned} & \frac{Dhn}{R} \frac{\partial u_0(x, \theta)}{\partial x} + \frac{Dh}{R^2} \frac{\partial v_0(x, \theta)}{\partial \theta} + \frac{Dh}{R^2} w_0(x, \theta) + \frac{Dh^3}{12} \frac{\partial^4 w_0(x, \theta)}{\partial x^4} + \frac{h^3}{6R^2} (2G + Dn) \frac{\partial^4 w_0(x, \theta)}{\partial x^2 \partial \theta^2} + \frac{Dh^3}{12R^4} \frac{\partial^4 w_0(x, \theta)}{\partial \theta^4} \\ & + \frac{Dh}{R^2} w_{01}(x, \theta) - \frac{Dh}{2R^3} \left(\frac{\partial w_0(x, \theta)}{\partial \theta} \right)^2 - \frac{Dh}{R^3} \frac{\partial w_0(x, \theta)}{\partial \theta} \frac{\partial^2 v_0(x, \theta)}{\partial \theta^2} - \frac{Dh}{R^3} w_0(x, \theta) \frac{\partial^2 w_0(x, \theta)}{\partial \theta^2} - \frac{Dh}{R^3} \frac{\partial v_0(x, \theta)}{\partial \theta} \frac{\partial^2 w_0(x, \theta)}{\partial \theta^2} \\ & + \frac{Dh^3}{12R^4} \frac{\partial^4 w_{01}(x, \theta)}{\partial \theta^4} - \frac{Dhn}{R^2} \frac{\partial^2 w_0(x, \theta)}{\partial \theta^2} \frac{\partial u_0(x, \theta)}{\partial x} + \frac{Dhn}{R} \frac{\partial u_{01}(x, \theta)}{\partial x} - \frac{Gh}{R^2} \frac{\partial^2 u_0(x, \theta)}{\partial \theta^2} \frac{\partial w_0(x, \theta)}{\partial x} - \frac{Dhn}{2R} \left(\frac{\partial w_0(x, \theta)}{\partial x} \right)^2 \\ & - \left(\frac{Gh}{R^2} + \frac{Dhn}{R^2} \right) \frac{\partial w_0(x, \theta)}{\partial \theta} \frac{\partial^2 u_0(x, \theta)}{\partial x \partial \theta} - \left(\frac{Gh}{R} + \frac{Dhn}{R} \right) \frac{\partial w_0(x, \theta)}{\partial x} \frac{\partial^2 v_0(x, \theta)}{\partial x \partial \theta} - \frac{2Gh}{R^2} \frac{\partial u_0(x, \theta)}{\partial \theta} \frac{\partial^2 w_0(x, \theta)}{\partial x \partial \theta} - \\ & \frac{2Gh}{R} \frac{\partial v_0(x, \theta)}{\partial x} \frac{\partial^2 w_0(x, \theta)}{\partial x \partial \theta} - Dh \frac{\partial w_0(x, \theta)}{\partial x} \frac{\partial^2 u_0(x, \theta)}{\partial x^2} - \frac{Gh}{R} \frac{\partial w_0(x, \theta)}{\partial \theta} \frac{\partial^2 v_0(x, \theta)}{\partial x^2} - \frac{Dhn}{R} w_0(x, \theta) \frac{\partial^2 w_0(x, \theta)}{\partial x^2} - \\ & \frac{Dhn}{R} \frac{\partial v_0(x, \theta)}{\partial \theta} \frac{\partial^2 w_0(x, \theta)}{\partial x^2} - Dh \frac{\partial u_0(x, \theta)}{\partial x} \frac{\partial^2 w_0(x, \theta)}{\partial x^2} + \left(\frac{Gh^3}{3R^2} + \frac{Dh^3 n}{6R^2} \right) \frac{\partial^4 w_{01}(x, \theta)}{\partial x^2 \partial \theta^2} + \frac{1}{12} Dh^3 \frac{\partial^4 w_{01}(x, \theta)}{\partial x^4} = \gamma(L - x) \end{aligned} \quad (3.32c)$$

As argued before, ε has no physical meaning and so does ε^2 . For this reason, both of them can be disregarded from the equations. Furthermore, the functions $w_{01}(x, \theta)$ & $u_{01}(x, \theta)$ must also be disposed, as it is the only way of using this method, since they are and will remain unknown to us. In this way, (3.32) reduce to,

$O(\varepsilon^2)$:

$$G \frac{\partial^2 u_0(x, \theta)}{\partial \theta^2} + DnR \frac{\partial w_0(x, \theta)}{\partial x} + (G + Dn)R \frac{\partial^2 v_0(x, \theta)}{\partial x \partial \theta} + DR^2 \frac{\partial^2 u_0(x, \theta)}{\partial x^2} + \frac{\partial^2 w_0(x, \theta)}{\partial \theta^2} \frac{\partial w_0(x, \theta)}{\partial x} + (G + Dn) \frac{\partial w_0(x, \theta)}{\partial \theta} \frac{\partial^2 w_0(x, \theta)}{\partial x \partial \theta} + DR^2 \frac{\partial w_0(x, \theta)}{\partial x} \frac{\partial^2 w_0(x, \theta)}{\partial x^2} = 0 \quad (3.33a)$$

$$DR \frac{\partial w_0(x, \theta)}{\partial \theta} + DR \frac{\partial^2 v_0(x, \theta)}{\partial \theta^2} + (G + Dn)R^2 \frac{\partial^2 u_0(x, \theta)}{\partial x \partial \theta} + GR^3 \frac{\partial^2 v_0(x, \theta)}{\partial x^2} + D \frac{\partial w_0(x, \theta)}{\partial \theta} \frac{\partial^2 w_0(x, \theta)}{\partial \theta^2} + (G + Dn)R^2 \frac{\partial w_0(x, \theta)}{\partial x} \frac{\partial^2 w_0(x, \theta)}{\partial x \partial \theta} + GR^2 \frac{\partial w_0(x, \theta)}{\partial \theta} \frac{\partial^2 w_0(x, \theta)}{\partial x^2} = 0 \quad (3.33b)$$

$$\begin{aligned}
& \frac{Dh}{R^2} w_0(x, \theta) + \frac{Dh}{R^2} \frac{\partial v_0(x, \theta)}{\partial \theta} + \frac{Dhn}{R} \frac{\partial u_0(x, \theta)}{\partial x} + \frac{h^3}{6R^2} (2G + Dn) \frac{\partial^4 w_0(x, \theta)}{\partial x^2 \partial \theta^2} + \frac{Dh^3}{12} \frac{\partial^4 w_0(x, \theta)}{\partial x^4} + \frac{Dh^3}{12R^4} \frac{\partial^4 w_0(x, \theta)}{\partial \theta^4} \\
& - \frac{Dh}{2R^3} \left(\frac{\partial w_0(x, \theta)}{\partial \theta} \right)^2 - \frac{Dh}{R^3} \frac{\partial w_0(x, \theta)}{\partial \theta} \frac{\partial^2 v_0(x, \theta)}{\partial \theta^2} - \frac{Dh}{R^3} w_0(x, \theta) \frac{\partial^2 w_0(x, \theta)}{\partial \theta^2} - \frac{Dh}{R^3} \frac{\partial v_0(x, \theta)}{\partial \theta} \frac{\partial^2 w_0(x, \theta)}{\partial \theta^2} \\
& - \frac{Dhn}{R^2} \frac{\partial^2 w_0(x, \theta)}{\partial \theta^2} \frac{\partial u_0(x, \theta)}{\partial x} - \frac{Gh}{R^2} \frac{\partial^2 u_0(x, \theta)}{\partial \theta^2} \frac{\partial w_0(x, \theta)}{\partial x} - \frac{Dhn}{2R} \left(\frac{\partial w_0(x, \theta)}{\partial x} \right)^2 - \left(\frac{Gh}{R^2} + \frac{Dhn}{R^2} \right) \frac{\partial w_0(x, \theta)}{\partial \theta} \frac{\partial^2 u_0(x, \theta)}{\partial x \partial \theta} \\
& - \left(\frac{Gh}{R} + \frac{Dhn}{R} \right) \frac{\partial w_0(x, \theta)}{\partial x} \frac{\partial^2 v_0(x, \theta)}{\partial x \partial \theta} - \frac{2Gh}{R^2} \frac{\partial u_0(x, \theta)}{\partial \theta} \frac{\partial^2 w_0(x, \theta)}{\partial x \partial \theta} - \frac{2Gh}{R} \frac{\partial v_0(x, \theta)}{\partial \theta} \frac{\partial^2 w_0(x, \theta)}{\partial x \partial \theta} - Dh \frac{\partial w_0(x, \theta)}{\partial x} \frac{\partial^2 u_0(x, \theta)}{\partial x^2} - \\
& \frac{Gh}{R} \frac{\partial w_0(x, \theta)}{\partial \theta} \frac{\partial^2 v_0(x, \theta)}{\partial x^2} - \frac{Dhn}{R} w_0(x, \theta) \frac{\partial^2 w_0(x, \theta)}{\partial x^2} - \frac{Dhn}{R} \frac{\partial v_0(x, \theta)}{\partial \theta} \frac{\partial^2 w_0(x, \theta)}{\partial x^2} - Dh \frac{\partial u_0(x, \theta)}{\partial x} \frac{\partial^2 w_0(x, \theta)}{\partial x^2} = \gamma(L - x)
\end{aligned} \tag{3.33c}$$

In order to simplify the problem, it is necessary to move one step further by ignoring the self-multiplied nonlinear terms and only keep the mixed ones, making an effort in this way to include the interaction between the different displacements despite the simplification. In addition, the external forces in both the equations of motion and the boundary conditions are also neglected, since they have already been included in the previous section. Their influence will be existent in the nonlinear equations by inserting, for specific terms, the general solutions of the linear problem. Another very important remark is the fact that at this step the stability of the system is examined, which means that the right-hand side of the equations does not make any impact on the process. In other words, water pressure in the right-hand side does not affect the stability of the system, and in this sense, it can be disregarded. As said before, its influence will be introduced through the general solutions of the 1st order linear problem. Taking everything into account, we arrive at,

$O(\varepsilon^2)$:

$$G \frac{\partial^2 u_0(x, \theta)}{\partial \theta^2} + DnR \frac{\partial w_0(x, \theta)}{\partial x} + (G + Dn)R \frac{\partial^2 v_0(x, \theta)}{\partial x \partial \theta} + DR^2 \frac{\partial^2 u_0(x, \theta)}{\partial x^2} = 0 \tag{3.34a}$$

&

$$DR \frac{\partial w_0(x, \theta)}{\partial \theta} + DR \frac{\partial^2 v_0(x, \theta)}{\partial \theta^2} + (G + Dn)R^2 \frac{\partial^2 u_0(x, \theta)}{\partial x \partial \theta} + GR^3 \frac{\partial^2 v_0(x, \theta)}{\partial x^2} = 0 \tag{3.34b}$$

&

$$\begin{aligned}
& \frac{Dh}{R^2} w_0(x, \theta) + \frac{Dh}{R^2} \frac{\partial v_0(x, \theta)}{\partial \theta} + \frac{Dhn}{R} \frac{\partial u_0(x, \theta)}{\partial x} + \frac{h^3}{6R^2} (2G + Dn) \frac{\partial^4 w_0(x, \theta)}{\partial x^2 \partial \theta^2} + \frac{Dh^3}{12} \frac{\partial^4 w_0(x, \theta)}{\partial x^4} + \frac{Dh^3}{12R^4} \frac{\partial^4 w_0(x, \theta)}{\partial \theta^4} \\
& - \frac{Dh}{R^3} \frac{\partial w_0(x, \theta)}{\partial \theta} \frac{\partial^2 v_0(x, \theta)}{\partial \theta^2} - \frac{Dh}{R^3} \frac{\partial v_0(x, \theta)}{\partial \theta} \frac{\partial^2 w_0(x, \theta)}{\partial \theta^2} - \frac{Dhn}{R^2} \frac{\partial^2 w_0(x, \theta)}{\partial \theta^2} \frac{\partial u_0(x, \theta)}{\partial x} - \frac{Gh}{R^2} \frac{\partial^2 u_0(x, \theta)}{\partial \theta^2} \frac{\partial w_0(x, \theta)}{\partial x} \\
& - \left(\frac{Gh}{R^2} + \frac{Dhn}{R^2} \right) \frac{\partial w_0(x, \theta)}{\partial \theta} \frac{\partial^2 u_0(x, \theta)}{\partial x \partial \theta} - \left(\frac{Gh}{R} + \frac{Dhn}{R} \right) \frac{\partial w_0(x, \theta)}{\partial x} \frac{\partial^2 v_0(x, \theta)}{\partial x \partial \theta} - \frac{2Gh}{R^2} \frac{\partial u_0(x, \theta)}{\partial \theta} \frac{\partial^2 w_0(x, \theta)}{\partial x \partial \theta} - \\
& - \frac{2Gh}{R} \frac{\partial v_0(x, \theta)}{\partial x} \frac{\partial^2 w_0(x, \theta)}{\partial x \partial \theta} - Dh \frac{\partial w_0(x, \theta)}{\partial x} \frac{\partial^2 u_0(x, \theta)}{\partial x^2} - \frac{Gh}{R} \frac{\partial w_0(x, \theta)}{\partial \theta} \frac{\partial^2 v_0(x, \theta)}{\partial x^2} - \frac{Dhn}{R} w_0(x, \theta) \frac{\partial^2 w_0(x, \theta)}{\partial x^2} \\
& - Dh \frac{\partial u_0(x, \theta)}{\partial x} \frac{\partial^2 w_0(x, \theta)}{\partial x^2} = 0
\end{aligned} \tag{3.34c}$$

The process that is going to be followed, bases its foundation on the substitution of the previously found linear solutions (3.30) or (3.31) into the blue terms of the nonlinear products, which make so far the problem unsolvable. With this course of action, a nonlinear equation becomes linear. It can be observed that the first two equations after the simplifications include no more any nonlinear terms, so attention is being paid on the transverse equation of motion. At this point, it should be underlined that the perturbation method found in the literature is employed in a totally different way than the one described here. The main difference is traced to the solution of the higher order non – linear set of equations. As mentioned above, the original method described in [43] presumes the response functions in their perturbed form, $f_i(x, \theta) = f_{0,i}(x, \theta) + \varepsilon f_{1,i}(x, \theta)$ and starts by calculating the first order solutions $f_{0,i}(x, \theta)$, as done in the present thesis, which in turn however,

are substituted for all the zero subscripted terms in the 2^{nd} order system. This means that, the goal of $O(\varepsilon^2)$ problem focuses on the solution of non-homogenous *pdes*, where the right hand side encloses all the already found $f_{0,i}(x, \theta)$ terms and so, the only unknowns are the $f_{1,i}(x, \theta)$ functions. Here, on the other hand, the main idea unfolds around obtaining the best possible estimation for $f_{0,i}(x, \theta)$ by neglecting all the $f_{1,i}(x, \theta)$ functions. Both approaches represent similar approximate techniques and so no question is posed on whether which one is right or wrong. In case, nevertheless, the stability of a structural system is under study, since the buckling load is contained within the solution of the linear $O(\varepsilon^1)$ problem, the original perturbation method is not convenient. This is pretty simple for someone to realise, in the sense that the stability of a system depends on the coefficient of the response function and not on the non-homogenous term of the *pdes'* right hand side.

Returning now in the current case of liquid filled cylindrical tank, the same concept as in the orthogonality of modes is used again, (orthogonality has no practical meaning in the case of the nonlinear equation) by first assuming,

$$u_0(x, \theta) = \sum_{k=0}^{\infty} U_0^k(x) \cos k \theta \quad (3.35a)$$

$$v_0(x, \theta) = \sum_{k=0}^{\infty} V_0^k(x) \sin k \theta \quad (3.35b)$$

$$w_0(x, \theta) = \sum_{k=0}^{\infty} W_0^k(x) \cos k \theta \quad (3.35c)$$

which then after substituting them into (3.34) leads to,

$$\dots \xrightarrow{\int_0^{2\pi} \cos m \theta \dots} -m^2 G U_0^m(x) + DnR \frac{dW_0^m(x)}{dx} + m(G + Dn)R \frac{dV_0^m(x)}{dx} + DR^2 \frac{d^2 U_0^m(x)}{dx^2} = 0 \quad (3.36)$$

&

$$\dots \xrightarrow{\int_0^{2\pi} \sin m \theta \dots} \begin{cases} 0 * (\dots) = 0, & \text{indeterminate for } m = 0 \\ -mDRW_0^m(x) - m^2 DRV_0^m(x) - m(G + Dn)R^2 \frac{dU_0^m(x)}{dx} + GR^3 \frac{d^2 V_0^m(x)}{dx^2} = 0, & m > 0 \end{cases} \quad (3.37)$$

&

$$\begin{aligned} & \sum_{k=0}^{\infty} \cos k \theta \left[\frac{Dh}{R^2} W_0^k(x) + k \frac{Dh}{R^2} V_0^k(x) + \frac{Dhn}{R} \frac{dU_0^k(x)}{dx} - \frac{k^2 h^3}{6R^2} (2G + Dn) \frac{d^2 W_0^k(x)}{dx^2} + \frac{Dh^3}{12} \frac{d^4 W_0^k(x)}{dx^4} + \frac{k^4 Dh^3}{12R^4} W_0^k(x) \right] \\ & + \sum_{k=0}^{\infty} \sin k \theta \left(\frac{Dh}{R^3} \frac{\partial^2 v_0(x, \theta)}{\partial \theta^2} + \frac{Gh + Dhn}{R^2} \frac{\partial^2 u_0(x, \theta)}{\partial x \partial \theta} + \frac{Gh}{R} \frac{\partial^2 v_0(x, \theta)}{\partial x^2} \right) k W_0^k(x) \\ & - \sum_{k=0}^{\infty} \cos k \theta \left(\frac{Gh}{R^2} \frac{\partial^2 u_0(x, \theta)}{\partial \theta^2} + \frac{Gh + Dhn}{R} \frac{\partial^2 v_0(x, \theta)}{\partial x \partial \theta} + Dh \frac{\partial^2 u_0(x, \theta)}{\partial x^2} \right) \frac{dW_0^k(x)}{dx} \\ & - \sum_{k=0}^{\infty} \cos k \theta \left(\frac{Dh}{R^3} \frac{\partial^2 w_0(x, \theta)}{\partial \theta^2} + \frac{Dhn}{R} \frac{\partial^2 w_0(x, \theta)}{\partial x^2} \right) k V_0^k(x) - \sum_{k=0}^{\infty} \cos k \theta \left(\frac{Dhn}{R^2} \frac{\partial^2 w_0(x, \theta)}{\partial \theta^2} + Dh \frac{\partial^2 w_0(x, \theta)}{\partial x^2} \right) \frac{dU_0^k(x)}{dx} \\ & + \sum_{k=0}^{\infty} \sin k \theta \frac{2Gh}{R^2} \frac{\partial^2 w_0(x, \theta)}{\partial x \partial \theta} k U_0^k(x) - \sum_{k=0}^{\infty} \sin k \theta \frac{2Gh}{R} \frac{\partial^2 w_0(x, \theta)}{\partial x \partial \theta} \frac{dV_0^k(x)}{dx} = 0 \quad (3.38) \end{aligned}$$

The 1^{st} order solutions (3.30, 3.31) are utilized to replace the blue highlighted terms of the third equation,

$$\begin{aligned} & \sum_{k=0}^{\infty} \cos k \theta \left[\frac{Dh}{R^2} W_0^k(x) + k \frac{Dh}{R^2} V_0^k(x) + \frac{Dhn}{R} \frac{dU_0^k(x)}{dx} - \frac{k^2 h^3}{6R^2} (2G + Dn) \frac{d^2 W_0^k(x)}{dx^2} + \frac{Dh^3}{12} \frac{d^4 W_0^k(x)}{dx^4} + \frac{k^4 Dh^3}{12R^4} W_0^k(x) \right] \\ & + \sum_{k=0}^{\infty} W_0^k(x) k \sin k \theta \sin \theta \left[-\frac{Dh V_{\text{asymmetric}}^{1st \text{ order}}(x)}{R^3} + \frac{(-Gh - Dhn) U'_{\text{asymmetric}}(x)}{R^2} + \frac{Gh V''_{\text{asymmetric}}(x)}{R} \right] \end{aligned}$$

$$\begin{aligned}
& + \sum_{k=0}^{\infty} \frac{dW_0^k(x)}{dx} \cos k \theta \left[-DhU''_{\text{symmetric}}(x) + \cos \theta \left(\frac{GhU_{\text{asymmetric}}(x)}{R^2} + \frac{(-Gh - Dh)nV'_{\text{asymmetric}}(x)}{R} - DhU''_{\text{asymmetric}}(x) \right) \right] \\
& + \sum_{k=0}^{\infty} V_0^k(x) k \cos k \theta \left[-\frac{DhnW''_{\text{symmetric}}(x)}{R} + \cos \theta \left(\frac{DhW_{\text{asymmetric}}(x)}{R^3} - \frac{DhnW''_{\text{asymmetric}}(x)}{R} \right) \right] \\
& + \sum_{k=0}^{\infty} \frac{dU_0^k(x)}{dx} \cos k \theta \left[-DhW''_{\text{symmetric}}(x) + \cos \theta \left(\frac{DhnW_{\text{asymmetric}}(x)}{R^2} - DhW''_{\text{asymmetric}}(x) \right) \right] \\
& + \sum_{k=0}^{\infty} U_0^k(x) k \sin k \theta \sin \theta \left(-\frac{2GhW'_{\text{asymmetric}}(x)}{R^2} \right) + \sum_{k=0}^{\infty} \frac{dV_0^k(x)}{dx} \sin k \theta \sin \theta \frac{2GW'_{\text{asymmetric}}(x)}{R} = 0 \xrightarrow{\int_0^{2\pi} \cos m \theta \dots} \\
& \sum_{k=0}^{\infty} \int_0^{2\pi} \cos m \theta \cos k \theta \left[\frac{Dh}{R^2} W_0^k(x) + k \frac{Dh}{R^2} V_0^k(x) + \frac{Dhn}{R} \frac{dU_0^k(x)}{dx} - \frac{k^2 h^3}{6R^2} (2G + Dn) \frac{d^2 W_0^k(x)}{dx^2} + \frac{Dh^3}{12} \frac{d^4 W_0^k(x)}{dx^4} + \frac{k^4 Dh^3}{12R^4} W_0^k(x) \right] \\
& \sum_{k=0}^{\infty} W_0^k(x) \int_0^{2\pi} \cos m \theta k \sin k \theta \sin \theta \left[-\frac{DhV_{\text{asymmetric}}^{\text{1st order}}(x)}{R^3} + \frac{(-Gh - Dh)nU'_{\text{asymmetric}}(x)}{R^2} + \frac{GhV''_{\text{asymmetric}}(x)}{R} \right] \\
& + \sum_{k=0}^{\infty} \frac{dW_0^k(x)}{dx} \int_0^{2\pi} \cos m \theta \cos k \theta \left[-DhU''_{\text{symmetric}}(x) + \cos \theta \left(\frac{GhU_{\text{asymmetric}}(x)}{R^2} + \frac{(-Gh - Dh)nV'_{\text{asymmetric}}(x)}{R} - DhU''_{\text{asymmetric}}(x) \right) \right] \\
& + \sum_{k=0}^{\infty} V_0^k(x) \int_0^{2\pi} \cos m \theta k \cos k \theta \left[-\frac{DhnW''_{\text{symmetric}}(x)}{R} + \cos \theta \left(\frac{DhW_{\text{asymmetric}}(x)}{R^3} - \frac{DhnW''_{\text{asymmetric}}(x)}{R} \right) \right] \\
& + \sum_{k=0}^{\infty} \frac{dU_0^k(x)}{dx} \int_0^{2\pi} \cos m \theta \cos k \theta \left[-DhW''_{\text{symmetric}}(x) + \cos \theta \left(\frac{DhnW_{\text{asymmetric}}(x)}{R^2} - DhW''_{\text{asymmetric}}(x) \right) \right] \\
& + \sum_{k=0}^{\infty} U_0^k(x) \int_0^{2\pi} \cos m \theta k \sin k \theta \sin \theta \left(-\frac{2GhW'_{\text{asymmetric}}(x)}{R^2} \right) + \sum_{k=0}^{\infty} \frac{dV_0^k(x)}{dx} \int_0^{2\pi} \cos m \theta \sin k \theta \sin \theta \frac{2GW'_{\text{asymmetric}}(x)}{R} = 0
\end{aligned} \tag{3.39}$$

As it can be seen, in the initial non-linear part of the transverse equation, the indefinite sums do not disappear, since in this stage the orthogonality of the modes is pointless. However, we managed to linearize the equation, which can now be written in the following form,

$$\begin{aligned}
& \sum_{k=0}^{\infty} \int_0^{2\pi} \cos m \theta \cos k \theta \left[\frac{Dh}{R^2} W_0^k(x) + k \frac{Dh}{R^2} V_0^k(x) + \frac{Dhn}{R} \frac{dU_0^k(x)}{dx} - \frac{k^2 h^3}{6R^2} (2G + Dn) \frac{d^2 W_0^k(x)}{dx^2} + \frac{Dh^3}{12} \frac{d^4 W_0^k(x)}{dx^4} + \frac{k^4 Dh^3}{12R^4} W_0^k(x) \right] \\
& + \sum_{k=0}^{\infty} W_0^k(x) g_{W_0}^{mk}(x, \theta) + \sum_{k=0}^{\infty} \frac{dW_0^k(x)}{dx} g_{\frac{dW_0}{dx}}^{mk}(x, \theta) + \sum_{k=0}^{\infty} V_0^k(x) g_{V_0}^{mk}(x, \theta) + \sum_{k=0}^{\infty} \frac{dU_0^k(x)}{dx} g_{\frac{dU_0}{dx}}^{mk}(x, \theta) \\
& + \sum_{k=0}^{\infty} U_0^k(x) g_{U_0}^{mk}(x, \theta) + \sum_{k=0}^{\infty} \frac{dV_0^k(x)}{dx} g_{\frac{dV_0}{dx}}^{mk}(x, \theta) = 0
\end{aligned} \tag{3.40}$$

where,

$$\begin{aligned}
g_{W_0}^{mk}(x, \theta) &= \int_0^{2\pi} \cos m \theta k \sin k \theta \sin \theta \left[-\frac{DhV_{\text{asymmetric}}^{\text{1st order}}(x)}{R^3} + \frac{(-Gh - Dh)nU'_{\text{asymmetric}}(x)}{R^2} + \frac{GhV''_{\text{asymmetric}}(x)}{R} \right], \\
g_{\frac{dW_0}{dx}}^{mk}(x, \theta) &= \int_0^{2\pi} \cos m \theta \cos k \theta \left[-DhU''_{\text{symmetric}}(x) + \cos \theta \left(\frac{GhU_{\text{asymmetric}}(x)}{R^2} + \frac{(-Gh - Dh)nV'_{\text{asymmetric}}(x)}{R} - DhU''_{\text{asymmetric}}(x) \right) \right], \\
g_{V_0}^{mk}(x, \theta) &= \int_0^{2\pi} \cos m \theta k \cos k \theta \left[-\frac{DhnW''_{\text{symmetric}}(x)}{R} + \cos \theta \left(\frac{DhW_{\text{asymmetric}}(x)}{R^3} - \frac{DhnW''_{\text{asymmetric}}(x)}{R} \right) \right], \\
g_{\frac{dU_0}{dx}}^{mk}(x, \theta) &= \int_0^{2\pi} \cos m \theta \cos k \theta \left[-DhW''_{\text{symmetric}}(x) + \cos \theta \left(\frac{DhnW_{\text{asymmetric}}(x)}{R^2} - DhW''_{\text{asymmetric}}(x) \right) \right],
\end{aligned}$$

$$g_{w_0}^{mk}(x, \theta) = \int_0^{2\pi} \cos m \theta \, k \sin k \theta \sin \theta \left(-\frac{2GhW'_{\text{asymmetric}}(x)}{R^2} \right), \quad g_{\frac{dw_0}{dx}}^{mk}(x, \theta) = \int_0^{2\pi} \cos m \theta \sin k \theta \sin \theta \frac{2GW'_{\text{asymmetric}}(x)}{R} \quad (3.41a)$$

or by using the analytical solutions (3.30),

$$\begin{aligned} g_{w_0}^{mk}(x, \theta) &= \int_0^{2\pi} \cos m \theta \, k \sin k \theta \left[\sum_{i=1}^8 \left(-\frac{\bar{B}_i Dh}{R^3} - \frac{\bar{A}_i Gh \lambda_i}{R^2} - \frac{\bar{A}_i Dh n \lambda_i}{R^2} + \frac{\bar{B}_i Gh \lambda_i^2}{R} \right) \bar{W}_i^1 e^{x \lambda_i} \sin \theta \right], \\ g_{\frac{dw_0}{dx}}^{mk}(x, \theta) &= \int_0^{2\pi} \cos m \theta \cos k \theta \left[\sum_{i=1}^6 (-\bar{U}_i Dh \lambda_i^2) \bar{W}_i^0 e^{x \lambda_i} + \frac{n R \rho}{(-1+n^2)} + \sum_{i=1}^8 \left(\frac{\bar{A}_i Gh}{R^2} - \frac{\bar{B}_i Gh \lambda_i}{R} - \frac{\bar{B}_i Dh n \lambda_i}{R} - \bar{A}_i Dh \lambda_i^2 \right) \bar{W}_i^1 e^{x \lambda_i} \cos \theta \right], \\ g_{v_0}^{mk}(x, \theta) &= \int_0^{2\pi} \cos m \theta \, k \cos k \theta \left[\sum_{i=1}^6 \left(-\frac{Dh n \lambda_i^2}{R} \right) \bar{C}_i \bar{U}_i e^{x \lambda_i} + \sum_{i=1}^8 \left(\frac{Dh}{R^3} - \frac{Dh n \lambda_i^2}{R} \right) \bar{W}_i^1 e^{x \lambda_i} \cos \theta \right], \\ g_{\frac{dv_0}{dx}}^{mk}(x, \theta) &= \int_0^{2\pi} \cos m \theta \cos k \theta \left[\sum_{i=1}^6 (-Dh \lambda_i^2) \bar{C}_i \bar{U}_i e^{x \lambda_i} + \sum_{i=1}^8 \left(\frac{Dh n}{R^2} - Dh \lambda_i^2 \right) \bar{W}_i^1 e^{x \lambda_i} \cos \theta \right], \\ g_{u_0}^{mk}(x, \theta) &= \int_0^{2\pi} \cos m \theta \, k \sin k \theta \left[\sum_{i=1}^8 \left(-\frac{2Gh \lambda_i}{R^2} \right) \bar{W}_i^1 e^{x \lambda_i} \sin \theta \right], \quad g_{\frac{du_0}{dx}}^{mk}(x, \theta) = \int_0^{2\pi} \cos m \theta \sin k \theta \left[\sum_{i=1}^8 \left(\frac{2Gh \lambda_i}{R} \right) \bar{W}_i^1 e^{x \lambda_i} \sin \theta \right] \end{aligned} \quad (3.41b)$$

3.2.2.2 Boundary Conditions

The procedure regarding the equations of motions, should also be applied for the boundary conditions. This also includes all the assumptions which have been made above concerning the neglect of perturbation parameters ε & ε^2 , the unknown functions $w_{01}(x, \theta)$ & $u_{01}(x, \theta)$, the self-multiplied nonlinear terms and that of the external forces. Once again, the main interest will be focused on the moment and force boundaries, which after recalling (3.22) are written here in their final 2^{nd} order version.

$O(\varepsilon^2)$:

$$\begin{aligned} N_{x\theta}^* &= \varepsilon \left[\frac{Gh}{R} \frac{\partial u_0(x, \theta)}{\partial \theta} + Gh \frac{\partial v_0(x, \theta)}{\partial x} \right] + \varepsilon^2 \left[\frac{Gh}{R} \frac{\partial u_{01}(x, \theta)}{\partial \theta} + \frac{Gh}{R} \frac{\partial w_0(x, \theta)}{\partial x} \frac{\partial w_0(x, \theta)}{\partial \theta} \right] \Rightarrow \\ N_{x\theta}^* &= \frac{Gh}{R} \frac{\partial u_0(x, \theta)}{\partial \theta} + Gh \frac{\partial v_0(x, \theta)}{\partial x} \end{aligned} \quad (3.42a)$$

$$\begin{aligned} M_{xx}^*(x, \theta) &= \varepsilon \left(-\frac{Dh^3 n}{12R^2} \frac{\partial^2 w_0(x, \theta)}{\partial \theta^2} - \frac{Dh^3}{12} \frac{\partial^2 w_0(x, \theta)}{\partial x^2} \right) + \varepsilon^2 \left(-\frac{Dh^3 n}{12R^2} \frac{\partial^2 w_{01}(x, \theta)}{\partial \theta^2} - \frac{Dh^3}{12} \frac{\partial^2 w_{01}(x, \theta)}{\partial x^2} \right) \Rightarrow \\ M_{xx}^*(x, \theta) &= -\frac{Dh^3 n}{12R^2} \frac{\partial^2 w_0(x, \theta)}{\partial \theta^2} - \frac{Dh^3}{12} \frac{\partial^2 w_0(x, \theta)}{\partial x^2} \end{aligned} \quad (3.42b)$$

$$\begin{aligned} Q_{xz}^* &= \varepsilon \left[\left(-\frac{Gh^3}{6R^2} - \frac{Dh^3 n}{12R^2} \right) \frac{\partial^3 w_0(x, \theta)}{\partial x \partial \theta^2} - \frac{Dh^3}{12} \frac{\partial^3 w_0(x, \theta)}{\partial x^3} \right] + \varepsilon^2 \left[\frac{Gh}{R^2} \frac{\partial u_0(x, \theta)}{\partial \theta} \frac{\partial w_0(x, \theta)}{\partial \theta} + \frac{Gh}{R} \frac{\partial w_0(x, \theta)}{\partial \theta} \frac{\partial v_0(x, \theta)}{\partial x} \right. \\ &\quad + \frac{Dh n}{R} w_0(x, \theta) \frac{\partial w_0(x, \theta)}{\partial x} + \frac{Dh n}{R} \frac{\partial v_0(x, \theta)}{\partial \theta} \frac{\partial w_0(x, \theta)}{\partial x} + Dh \frac{\partial u_0(x, \theta)}{\partial x} \frac{\partial w_0(x, \theta)}{\partial x} - \left(\frac{Gh^3}{6R^2} + \frac{Dh^3 n}{12R^2} \right) \frac{\partial^3 w_{01}(x, \theta)}{\partial x \partial \theta^2} \\ &\quad \left. - \frac{Dh^3}{12} \frac{\partial^3 w_{01}(x, \theta)}{\partial x^3} \right] \Rightarrow Q_{xz}^* = \left(-\frac{Gh^3}{6R^2} - \frac{Dh^3 n}{12R^2} \right) \frac{\partial^3 w_0(x, \theta)}{\partial x \partial \theta^2} - \frac{Dh^3}{12} \frac{\partial^3 w_0(x, \theta)}{\partial x^3} + \frac{Gh}{R^2} \frac{\partial u_0(x, \theta)}{\partial \theta} \frac{\partial w_0(x, \theta)}{\partial \theta} \\ &\quad + \frac{Gh}{R} \frac{\partial w_0(x, \theta)}{\partial \theta} \frac{\partial v_0(x, \theta)}{\partial x} + \frac{Dh n}{R} \frac{\partial v_0(x, \theta)}{\partial \theta} \frac{\partial w_0(x, \theta)}{\partial x} + Dh \frac{\partial u_0(x, \theta)}{\partial x} \frac{\partial w_0(x, \theta)}{\partial x} \end{aligned} \quad (3.42c)$$

As it can be seen, the effect of non-linearity exists only for the out of plane shear force boundary condition Q_{xz}^* . For that particular case, the influence of the 1^{st} order problem should also be introduced by substituting the general linear solutions (3.30-31). In addition, the assumed solutions (3.35) should be substituted in (3.19 - 3.20), without of course the contribution of the external shear force. So, by taking also into account (3.42) and utilizing the orthogonality of modes again, the final boundary conditions become,

$x = 0 :$

$$1) \quad u(0, \theta) = 0 \xrightarrow{\int_0^{2\pi} \cos m \theta (... = ...)} U_0^m(0) = 0 \quad (3.43a)$$

$$2) \quad M_{xx}^*(0, \theta) = 0 \xrightarrow{\int_0^{2\pi} \cos m \theta (... = ...)} -n m^2 W_0^m(0) + R^2 \left| \frac{d^2 W_0^m(x)}{dx^2} \right|_{x=0} = 0 \quad (3.43b)$$

$$3) \quad N_{x\theta}^*(0, \theta) = 0 \xrightarrow{\int_0^{2\pi} \sin m \theta (... = ...)} \begin{cases} \text{for } m = 0 \rightarrow \text{indeterminate case} \\ -m U_0^m(0) + R \left| \frac{dV_0^m(x)}{dx} \right|_{x=0} = 0, m > 0 \end{cases} \quad (3.43c)$$

$$4) \quad Q_{xz}^*(0, \theta) = 0 \xrightarrow{\int_0^{2\pi} \cos m \theta (... = ...)} \left[\sum_{k=0}^{\infty} \int_0^{2\pi} \cos m \theta \cos k \theta \left[k^2 (2Gh^2 + Dh^2 n) \frac{dW_0^k(x)}{dx} - Dh^2 R^2 \frac{d^3 W_0^k(x)}{dx^3} \right] \right. \\ \left. - \sum_{k=0}^{\infty} W_0^k(x) \int_0^{2\pi} \cos m \theta k \sin k \theta \sin \theta [-12GU_{\text{asymmetric}}(x) + 12GRV'_{\text{asymmetric}}(x)] + \sum_{k=0}^{\infty} \frac{dW_0^k(x)}{dx} \int_0^{2\pi} \cos m \theta \right. \\ \left. \cos k \theta [12DR^2 U'_{\text{symmetric}}(x) + \cos \theta (12DR^2 U'_{\text{asymmetric}}(x) + 12DnRV_{\text{asymmetric}}(x))] \right] \Big|_{x=0} = 0 \\ \Rightarrow \left[\sum_{k=0}^{\infty} \int_0^{2\pi} \cos m \theta \cos k \theta \left[k^2 (2Gh^2 + Dh^2 n) \frac{dW_0^k(x)}{dx} - Dh^2 R^2 \frac{d^3 W_0^k(x)}{dx^3} \right] \right. \\ \left. - \sum_{k=0}^{\infty} W_0^k(x) \int_0^{2\pi} \cos m \theta k \sin k \theta \sin \theta \left[\sum_{i=1}^8 (-12G\bar{A}_i + 12GR\lambda_i \bar{B}_i) \bar{W}_i^1 e^{\lambda_i x} \right] \right. \\ \left. + \sum_{k=0}^{\infty} \frac{dW_0^k(x)}{dx} \int_0^{2\pi} \cos m \theta \cos k \theta \left[12DR^2 \left(\sum_{i=1}^6 \lambda_i \bar{U}_i e^{\lambda_i x} - \frac{nR\rho}{Dh(-1+n^2)} x + \frac{LR\rho}{Dhn} \right) \right] \right. \\ \left. + \sum_{k=0}^{\infty} \frac{dW_0^k(x)}{dx} \int_0^{2\pi} \cos m \theta \cos k \theta \cos \theta \left[\sum_{i=1}^8 (12DR^2 \lambda_i \bar{A}_i + 12DnR\bar{B}_i) \bar{W}_i^1 e^{\lambda_i x} \right] \right] \Big|_{x=0} = 0 \\ \Rightarrow \left[\sum_{k=0}^{\infty} \int_0^{2\pi} \cos m \theta \cos k \theta \left[k^2 (2Gh^2 + Dh^2 n) \frac{dW_0^k(x)}{dx} - Dh^2 R^2 \frac{d^3 W_0^k(x)}{dx^3} \right] \right. \\ \left. - \sum_{k=0}^{\infty} W_0^k(x) \int_0^{2\pi} \cos m \theta k \sin k \theta \sin \theta \left[\sum_{i=1}^8 (-12G\bar{A}_i + 12GR\lambda_i \bar{B}_i) \bar{W}_i^1 \right] \right. \\ \left. + \sum_{k=0}^{\infty} \frac{dW_0^k(x)}{dx} \int_0^{2\pi} \cos m \theta \cos k \theta \left[12DR^2 \left(\sum_{i=1}^6 \lambda_i \bar{U}_i + \frac{LR\rho}{Dhn} \right) \right] \right. \\ \left. + \sum_{k=0}^{\infty} \frac{dW_0^k(x)}{dx} \int_0^{2\pi} \cos m \theta \cos k \theta \cos \theta \left[\sum_{i=1}^8 (12DR^2 \lambda_i \bar{A}_i + 12DnR\bar{B}_i) \bar{W}_i^1 \right] \right] \Big|_{x=0} = 0 \quad (3.43d)$$

$x = L :$

If clamped

$$5) \quad u(L, \theta) = 0 \xrightarrow{\int_0^{2\pi} \cos m \theta (... = ...)} U_0^m(L) = 0 \quad (3.44a)$$

$$6) \quad v(L, \theta) = 0 \xrightarrow{\int_0^{2\pi} \sin m \theta (... = ...)} \begin{cases} 0 * (...) = 0, \text{indeterminate case for } m = 0 \\ V_0^m(L) = 0, m > 0 \end{cases} \quad (3.44b)$$

$$7) \quad w(L, \theta) = 0 \xrightarrow{\int_0^{2\pi} \cos m \theta (... = ...)} W_0^m(L) = 0 \quad (3.44c)$$

$$8) \quad \left| \frac{\partial w(x, \theta)}{\partial x} \right|_{x=L} = 0 \xrightarrow{\int_0^{2\pi} \cos m \theta (\dots) d\theta} \left| \frac{dW_0^m(x)}{dx} \right|_{x=L} = 0 \quad (3.44d)$$

Or if hinged

$$8) \quad M_{xx}^*(L, \theta) = 0 \xrightarrow{\int_0^{2\pi} \cos m \theta (\dots) d\theta} -n m^2 W_0^m(L) + R^2 \left| \frac{d^2 W_0^m(x)}{dx^2} \right|_{x=L} = 0 \quad (3.44e)$$

The above 8 relations include 9 unknown parameters. Among them are the 8 W_j constants and the external force F_E at the bottom, responsible for the shear buckling.

3.2.2.3 Solution Attempt

In the previous sections the problem of cylindrical tank has come down into a system of three linear homogeneous equations with variable coefficients. Ideally, in order for the buckling stress to be evaluated, certain steps need to be accomplished. To provide the reader with the bigger picture as far as the direction of the solution process is concerned, these are briefly mentioned next.

1st step: According to the total number $r > 1$ of the selected modes, some kind of general solution, approximate or not, needs to be determined for the second order response functions, $U_0^k(x)$, $V_0^k(x)$ and $W_0^k(x)$, in which $N = 6 + 8(r - 1)$ unknown constants C_i are enclosed.

2nd step: Substitution of the response's general solutions into the N second order boundary conditions leads to the following relation, expressed in matrix form: $\mathbf{A}_{N \times N} \mathbf{C}_{N \times 1} = \mathbf{0}$.

3rd step: Acknowledging that the horizontal external force F_E , has been inserted into the final relations $U_0^k(x)$, $V_0^k(x)$ and $W_0^k(x)$ through the **1st order** solutions (3.30 or 3.31), detected in the variable coefficients (3.41), and recalling also the beam example of the previous chapter, the critical value of the buckling load shall be calculated by setting the determinant of $\mathbf{A}_{N \times N}$ equal to zero; $\det(\mathbf{A}_{N \times N}) = 0$.

4th step: Buckling is expected to occur either due to compressive or shear stresses. A horizontal earthquake static force is likely to develop apart from the obvious shear lobes, a bulge due to compression in the region close to the fixed edge [46]. For this purpose, only the solutions from the second order problem will be utilized, since the effect of the first order responses have already been included in the latter ones.

Without doubt, the biggest obstacle to overcome, is not other but the determination of a general expression which describes the responses in the three directions. Linear odes with variable coefficients are a very difficult task to solve, as enclosed solutions exist only for certain problems. For this reason, approximate techniques, like those of power and Frobenius series [47] have been developed. However, even in this situation, only specific odes have been tackled successfully, with the most famous example that of Cauchy –Euler equations [47]. The general case still seems to be an incomplete area of research, although many mathematicians have attempted to provide an answer to the problem. According to Giannantoni [48], if the solving kernel of the N^{th} order ode is known, which is just a particular integral solution of the equation that satisfies its initial conditions, one is able to find a fundamental set, which represents the general solution of the ode consisted of N particular integrals with a non-zero Wronskian, designating their linear independency. Applying his “*fundamental theorem of the solving Kernel*”, and following the procedure proposed by [48] an analytical solution may be reached, by first adjusting the solution method for the existence of boundary conditions instead of initial ones. However, a way of determining the solving Kernel is yet unknown to me and neither found in the literature among the researchers and so the fundamental theorem of the solving Kernel, unfortunately cannot be applied, at least until now. Other remarkable attempts include the “*Chebyshev collocation method*” found in the works of Ramadan [49], [50] and [51].

Since it is important to begin from somewhere, our attempt starts by taking use of the fact that often in literature, to accommodate the solution process for linear odes of variable coefficients, the initial problem of N^{th} order, is transformed into a system of N **1st order** equations. To illustrate this, and also gain a better picture

through a practical example, the first three modes are selected. The relations (3.36 – 3.41) include two different parameters that represent the modes, namely m and k . These are both running numbers and the one should not exceed the other. In that sense, the problem of cylindrical tank becomes,

$$m = 0$$

$$k = 0$$

Axial Motion:

$$DnR \frac{dW_0^0(x)}{dx} + DR^2 \frac{d^2 U_0^0(x)}{dx^2} = 0$$

Circumferential motion:

no equation

Transverse motion:

$$2\pi \left[\frac{Dh}{R^2} W_0^0(x) + \frac{Dhn}{R} \frac{dU_0^0(x)}{dx} + \frac{Dh^3}{12} \frac{d^4 W_0^0(x)}{dx^4} \right] + W_0^0(x) g_{W_0^0}^{00}(x, \theta) + \frac{dW_0^0(x)}{dx} g_{\frac{dW_0^0(x)}{dx}}^{00}(x, \theta) + V_0^0(x) g_{V_0^0}^{00}(x, \theta) + \frac{dU_0^0(x)}{dx} g_{\frac{dU_0^0(x)}{dx}}^{00}(x, \theta) + U_0^0(x) g_{U_0^0}^{00}(x, \theta) + \frac{dV_0^0(x)}{dx} g_{\frac{dV_0^0(x)}{dx}}^{00}(x, \theta) = 0 \quad (3.45)$$

where after using the relations in appendix C for calculating the integrals that include the products among the cosine and sine functions for various values of k & m , the variable coefficients $g_i(x, \theta)$ are given by,

$$\begin{aligned} g_{\frac{dW_0^0}{dx}}^{00}(x, \theta) &= 2\pi [-DhU''_{\text{symmetric}}(x)], & g_{\frac{dU_0^0}{dx}}^{00}(x, \theta) &= 2\pi [-DhW''_{\text{symmetric}}(x)] \\ g_{W_0^0}^{00}(x, \theta) &= g_{V_0^0}^{00}(x, \theta) = g_{U_0^0}^{00}(x, \theta) = g_{\frac{dV_0^0}{dx}}^{00}(x, \theta) = 0 \end{aligned} \quad (3.46)$$

B.C.s :

$$\begin{aligned} U_0^0(0) &= 0 & U_0^0(L) &= 0 \\ \left| \frac{d^2 W_0^0(x)}{dx^2} \right|_{x=0} &= 0 & W_0^0(L) &= 0 \\ \left| \frac{d^3 W_0^0(x)}{dx^3} 2\pi(-Dh^2 R^2) + \frac{dW_0^0(x)}{dx} 2\pi[12DR^2 U'_{\text{symmetric}}(x)] \right|_{x=0} &= 0 & \left| \frac{dW_0^0(x)}{dx} \right|_{x=L} &= 0 \end{aligned} \quad (3.47)$$

&

$$m = 1$$

$$k = 0, 1$$

Axial Motion:

$$GU_0^1(x) + DnR \frac{dW_0^1(x)}{dx} + (G + Dn)R \frac{dV_0^1(x)}{dx} + DR^2 \frac{d^2 U_0^1(x)}{dx^2} = 0$$

Circumferential motion:

$$DRW_0^1(x) - DRV_0^1(x) - (G + Dn)R^2 \frac{dU_0^1(x)}{dx} + GR^3 \frac{d^2 V_0^1(x)}{dx^2} = 0$$

Transverse motion:

$$\begin{aligned} & 2\pi \left[\frac{Dh}{R^2} W_0^1(x) + \frac{Dh}{R^2} V_0^1(x) + \frac{Dhn}{R} \frac{dU_0^1(x)}{dx} - \frac{h^3}{6R^2} (2G + Dn) \frac{d^2 W_0^1(x)}{dx^2} + \frac{Dh^3}{12} \frac{d^4 W_0^1(x)}{dx^4} + \frac{Dh^3}{12R^4} W_0^1(x) \right] \\ & + \sum_{k=0}^1 W_0^k(x) g_{W_0^k}^{mk}(x, \theta) + \sum_{k=0}^1 \frac{dW_0^k(x)}{dx} g_{\frac{dW_0^k(x)}{dx}}^{mk}(x, \theta) + \sum_{k=0}^1 V_0^k(x) g_{V_0^k}^{mk}(x, \theta) + \sum_{k=0}^1 \frac{dU_0^k(x)}{dx} g_{\frac{dU_0^k(x)}{dx}}^{mk}(x, \theta) \\ & + \sum_{k=0}^1 U_0^k(x) g_{U_0^k}^{mk}(x, \theta) + \sum_{k=0}^1 \frac{dV_0^k(x)}{dx} g_{\frac{dV_0^k(x)}{dx}}^{mk}(x, \theta) = 0 \end{aligned} \quad (3.48)$$

where after utilizing appendix C,

$$\begin{aligned}
g_{\frac{dW_0}{dx}}^{10}(x, \theta) &= \pi \left(\frac{GhU_{\text{asymmetric}}(x)}{R^2} + \frac{(-Gh - Dhn)V'_{\text{asymmetric}}(x)}{R} - DhU''_{\text{asymmetric}}(x) \right), \\
g_{\frac{dU_0}{dx}}^{10}(x, \theta) &= \pi \left(\frac{DhnW_{\text{asymmetric}}(x)}{R^2} - DhW''_{\text{asymmetric}}(x) \right), \\
g_{W_0}^{10}(x, \theta) &= g_{V_0}^{10}(x, \theta) = g_{U_0}^{10}(x, \theta) = g_{\frac{dV_0}{dx}}^{10}(x, \theta) = 0
\end{aligned} \tag{3.49a}$$

and

$$\begin{aligned}
g_{\frac{dW_0}{dx}}^{11}(x, \theta) &= \pi [-DhU''_{\text{symmetric}}(x)], & g_{V_0}^{11}(x, \theta) &= \pi \left[-\frac{DhnW''_{\text{symmetric}}(x)}{R} \right], \\
g_{\frac{dU_0}{dx}}^{11}(x, \theta) &= \pi [-DhW''_{\text{symmetric}}(x)], & g_{W_0}^{11}(x, \theta) &= g_{U_0}^{11}(x, \theta) = g_{\frac{dV_0}{dx}}^{11}(x, \theta) = 0
\end{aligned} \tag{3.49b}$$

B.C.s :

$$\begin{aligned}
U_0^1(0) &= 0 & U_0^1(L) &= 0 \\
-nW_0^1(0) + R^2 \left| \frac{d^2W_0^1(x)}{dx^2} \right|_{x=0} &= 0 & W_0^1(L) &= 0 \\
-U_0^1(0) + R \left| \frac{dV_0^1(x)}{dx} \right|_{x=0} &= 0 & V_0^1(L) &= 0 \\
\left| \pi \left[(2Gh^2 + Dh^2n) \frac{dW_0^1(x)}{dx} - Dh^2R^2 \frac{d^3W_0^1(x)}{dx^3} \right] + \frac{dW_0^1(x)}{dx} \pi [12DR^2U'_{\text{symmetric}}(x)] \right. \\
&\quad \left. + \frac{dW_0^0(x)}{dx} \pi [12DR^2U'_{\text{asymmetric}}(x) + 12DnRV_{\text{asymmetric}}(x)] \right|_{x=0} &= 0 & \left| \frac{dW_0^1(x)}{dx} \right|_{x=L} = 0
\end{aligned} \tag{3.50}$$

&

$$m = 2$$

$$k = 0, 1, 2$$

Axial Motion:

$$-2^2GU_0^2(x) + DnR \frac{dW_0^2(x)}{dx} + 2(G + Dn)R \frac{dV_0^2(x)}{dx} + DR^2 \frac{d^2U_0^2(x)}{dx^2} = 0$$

Circumferential motion:

$$-2DRW_0^2(x) - 2^2DRV_0^2(x) - 2(G + Dn)R^2 \frac{dU_0^2(x)}{dx} + GR^3 \frac{d^2V_0^2(x)}{dx^2} = 0$$

Transverse motion:

$$\begin{aligned}
&\pi \left[\frac{Dh}{R^2} W_0^2(x) + 2 \frac{Dh}{R^2} V_0^2(x) + \frac{Dhn}{R} \frac{dU_0^2(x)}{dx} - \frac{2^2h^3}{6R^2} (2G + Dn) \frac{d^2W_0^2(x)}{dx^2} + \frac{Dh^3}{12} \frac{d^4W_0^2(x)}{dx^4} + \right. \\
&\quad \left. \frac{2^4Dh^3}{12R^4} W_0^2(x) \right] + \sum_{k=0}^2 W_0^k(x) g_{W_0}^{mk}(x, \theta) + \sum_{k=0}^2 \frac{dW_0^k(x)}{dx} g_{\frac{dW_0}{dx}}^{mk}(x, \theta) + \sum_{k=0}^2 V_0^k(x) g_{V_0}^{mk}(x, \theta) + \\
&\quad \sum_{k=0}^2 \frac{dU_0^k(x)}{dx} g_{\frac{dU_0}{dx}}^{mk}(x, \theta) + \sum_{k=0}^2 U_0^k(x) g_{U_0}^{mk}(x, \theta) + \sum_{k=0}^2 \frac{dV_0^k(x)}{dx} g_{\frac{dV_0}{dx}}^{mk}(x, \theta) = 0
\end{aligned} \tag{3.51}$$

where after utilizing again the information from appendix C,

$$g_{W_0}^{20}(x, \theta) = g_{\frac{dW_0}{dx}}^{20}(x, \theta) = g_{V_0}^{20}(x, \theta) = g_{\frac{dV_0}{dx}}^{20}(x, \theta) = g_{U_0}^{20}(x, \theta) = g_{\frac{dU_0}{dx}}^{20}(x, \theta) = 0 \tag{3.52a}$$

and

$$\begin{aligned}
g_{W_0}^{21}(x, \theta) &= -\frac{\pi}{2} \left[-\frac{DhV_{\text{asymmetric}}^{1\text{st order}}(x)}{R^3} + \frac{(-Gh - Dhn)U'_{\text{asymmetric}}(x)}{R^2} + \frac{GhV''_{\text{asymmetric}}(x)}{R} \right], \\
g_{\frac{dW_0}{dx}}^{21}(x, \theta) &= \frac{\pi}{2} \left(\frac{GhU_{\text{asymmetric}}(x)}{R^2} + \frac{(-Gh - Dhn)V'_{\text{asymmetric}}(x)}{R} - DhU''_{\text{asymmetric}}(x) \right) \\
g_{V_0}^{21}(x, \theta) &= \frac{\pi}{2} \left(\frac{DhW_{\text{asymmetric}}(x)}{R^3} - \frac{DhnW''_{\text{asymmetric}}(x)}{R} \right),
\end{aligned} \tag{3.52b}$$

$$g_{\frac{dU_0}{dx}}^{21}(x, \theta) = \frac{\pi}{2} \left(\frac{DhW_{\text{asymmetric}}(x)}{R^3} - \frac{DhnW''_{\text{asymmetric}}(x)}{R} \right)$$

$$g_{U_0}^{21}(x, \theta) = -\frac{\pi}{2} \left(-\frac{2GhW'_{\text{asymmetric}}(x)}{R^2} \right), \quad g_{\frac{dV_0}{dx}}^{21}(x, \theta) = -\frac{\pi}{2} \frac{2GW'_{\text{asymmetric}}(x)}{R}$$

and also finally,

$$g_{\frac{dW_0}{dx}}^{22}(x, \theta) = \pi [-DhU''_{\text{symmetric}}(x)], \quad g_{V_0}^{22}(x, \theta) = 2\pi \left[-\frac{DhnW''_{\text{symmetric}}(x)}{R} \right]$$

$$g_{\frac{dU_0}{dx}}^{22}(x, \theta) = \pi [-DhW''_{\text{symmetric}}(x)], \quad g_{W_0}^{22}(x, \theta) = g_{U_0}^{22}(x, \theta) = g_{\frac{dV_0}{dx}}^{22}(x, \theta) = 0 \quad (3.52c)$$

B.C.s :

$$U_0^2(0) = 0 \quad U_0^2(L) = 0$$

$$-n^2 W_0^2(0) + R^2 \left| \frac{d^2 W_0^2(x)}{dx^2} \right|_{x=0} = 0 \quad V_0^2(L) = 0$$

$$-2 U_0^2(0) + R \left| \frac{dV_0^2(x)}{dx} \right|_{x=0} = 0 \quad W_0^2(L) = 0$$

$$\left| \pi \left[2^2(2Gh^2 + Dh^2n) \frac{dW_0^2(x)}{dx} - Dh^2R^2 \frac{d^3 W_0^2(x)}{dx^3} \right] + \pi \frac{dW_0^2(x)}{dx} [12DR^2 U'_{\text{symmetric}}(x)] \right. \quad (3.53)$$

$$\left. - \frac{\pi}{2} W_0^1(x) [-12GU_{\text{asymmetric}}(x) + 12GRV'_{\text{asymmetric}}(x)] \right|_{x=L} = 0$$

$$+ \frac{dW_0^1(x)}{dx} \frac{\pi}{2} [12DR^2 U'_{\text{asymmetric}}(x) + 12DnRV_{\text{asymmetric}}(x)] \Big|_{x=0} = 0$$

Now the 22th order system can be transformed into a new system of 1st order differential equations, by reducing the order of response functions' derivatives through the following definitions and notation. Let,

$$f_1(x) = U_0^0(x), \quad f_2(x) = \frac{dU_0^0(x)}{dx},$$

$$f_3(x) = W_0^0(x), \quad f_4(x) = \frac{dW_0^0(x)}{dx}, \quad f_5(x) = \frac{d^2 W_0^0(x)}{dx^2}, \quad f_6(x) = \frac{d^3 W_0^0(x)}{dx^3}$$

$$\&$$

$$f_7(x) = U_0^1(x), \quad f_8(x) = \frac{dU_0^1(x)}{dx},$$

$$f_9(x) = V_0^1(x), \quad f_{10}(x) = \frac{dV_0^1(x)}{dx},$$

$$f_{11}(x) = W_0^1(x), \quad f_{12}(x) = \frac{dW_0^1(x)}{dx}, \quad f_{13}(x) = \frac{d^2 W_0^1(x)}{dx^2}, \quad f_{14}(x) = \frac{d^3 W_0^1(x)}{dx^3}$$

$$\&$$

$$f_{15}(x) = U_0^2(x), \quad f_{16}(x) = \frac{dU_0^2(x)}{dx},$$

$$f_{17}(x) = V_0^2(x), \quad f_{18}(x) = \frac{dV_0^2(x)}{dx},$$

$$f_{19}(x) = W_0^2(x), \quad f_{20}(x) = \frac{dW_0^2(x)}{dx}, \quad f_{21}(x) = \frac{d^2 W_0^2(x)}{dx^2}, \quad f_{22}(x) = \frac{d^3 W_0^2(x)}{dx^3} \quad (3.54)$$

So, the final problem can be written as,

$$\frac{df_1(x)}{dx} = f_2(x)$$

$$\frac{df_2(x)}{dx} = -\frac{n}{R} f_4(x)$$

$$\frac{df_3(x)}{dx} = f_4(x)$$

$$\frac{df_4(x)}{dx} = f_5(x)$$

$$\frac{df_5(x)}{dx} = f_6(x)$$

$$\begin{aligned}
\frac{df_6(x)}{dx} &= -\frac{6}{Dh^3\pi R^2} \left(2Dh\pi f_3(x) + 2Dhn\pi R f_2(x) + R^2 g_{dU}^{00}(x, \theta) f_2(x) + R^2 g_{dW}^{00}(x, \theta) f_4(x) \right) \\
\frac{df_7(x)}{dx} &= f_8(x) \\
\frac{df_8(x)}{dx} &= -\frac{1}{DR^2} [-Gf_7(x) + DnRf_{12}(x) + (G + Dn)Rf_{10}(x)] \\
\frac{df_9(x)}{dx} &= f_{10}(x) \\
\frac{df_{10}(x)}{dx} &= \frac{1}{GR^3} [DRf_{11}(x) + DRf_9(x) + (G + Dn)R^2 f_8(x)] \\
\frac{df_{11}(x)}{dx} &= f_{12}(x) \\
\frac{df_{12}(x)}{dx} &= f_{13}(x) \\
\frac{df_{13}(x)}{dx} &= f_{14}(x) \\
\frac{df_{14}(x)}{dx} &= -\frac{1}{\pi Dh^3} \left[\pi \left(\left(\frac{Dh}{R^2} + \frac{Dh^3}{12R^4} \right) f_{11}(x) + \frac{Dh}{R^2} f_9(x) + \frac{Dhn}{R} f_8(x) - \frac{h^3}{6R^2} (2G + Dn) f_{13}(x) \right) + f_4(x) g_{dW}^{10}(x, \theta) \right. \\
&\quad \left. + f_{12}(x) g_{dW}^{11}(x, \theta) + 0 * g_V^{10}(x, \theta) + f_9(x) * g_V^{11}(x, \theta) + f_2(x) g_{dU}^{10}(x, \theta) + f_8(x) g_{dU}^{11}(x, \theta) \right] \\
\frac{df_{15}(x)}{dx} &= f_{16}(x) \\
\frac{df_{16}(x)}{dx} &= -\frac{1}{DR^2} [-2^2 G f_{15}(x) + DnR f_{20}(x) + 2(G + Dn)R f_{18}(x)] \\
\frac{df_{17}(x)}{dx} &= f_{18}(x) \\
\frac{df_{18}(x)}{dx} &= \frac{1}{GR^3} [2DRf_{19}(x) + 2^2 DRf_{17}(x) + 2(G + Dn)R^2 f_{16}(x)] \\
\frac{df_{19}(x)}{dx} &= f_{20}(x) \\
\frac{df_{20}(x)}{dx} &= f_{21}(x) \\
\frac{df_{21}(x)}{dx} &= f_{22}(x) \\
\frac{df_{22}(x)}{dx} &= -\frac{1}{\pi Dh^3} \left[\pi \left(\left(\frac{Dh}{R^2} + \frac{2^4 Dh^3}{12R^4} \right) f_{19}(x) + 2 \frac{Dh}{R^2} f_{17}(x) + \frac{Dhn}{R} f_{16}(x) - \frac{2^2 h^3}{6R^2} (2G + Dn) f_{21}(x) \right) + f_{11}(x) g_W^{21}(x, \theta) \right. \\
&\quad + f_{12}(x) g_{dW}^{21}(x, \theta) + f_{20}(x) g_{dW}^{22}(x, \theta) + 0 * g_V^{20}(x, \theta) + f_9(x) * g_V^{21}(x, \theta) + f_{17}(x) * g_V^{22}(x, \theta) + f_8(x) g_{dU}^{21}(x, \theta) \\
&\quad \left. + f_{16}(x) g_{dU}^{22}(x, \theta) + f_7(x) g_U^{21}(x, \theta) + 0 * g_{dV}^{20}(x, \theta) + f_{10}(x) * g_{dV}^{21}(x, \theta) \right] \quad (3.55)
\end{aligned}$$

The new functions $f_N(x)$ should also be substituted into the boundary conditions, which is considered trivial and thus, it will not be presented here. Moving on, after the reduction of the response functions' order the final problem can be written in the following form,

$$\mathbf{F}' = \mathbf{A}\mathbf{F} \quad (3.56)$$

According to theory, if $\mathbf{A}$ is commutative i.e.

$$\mathbf{A}(x) \cdot \int_{x_1}^{x_2} \mathbf{A}(x) dx = \int_{x_1}^{x_2} \mathbf{A}(x) dx \cdot \mathbf{A}(x) \quad (3.57)$$

something which is valid in case of a symmetric or diagonal matrix, the obstacle encountered in the solving Kernel theorem can be overcome. In other words, a fundamental set of solutions can be calculated through

Liouville's formula without the need of knowing one particular solution. Unfortunately, this is not the case for the problem at hand and so the solution should be searched elsewhere.

Another thought would be to search for an answer by using the beam's example featured at the end of the previous chapter, as a guide for the solution. Let's now make a quick reminder of the process which was followed. After the perturbation method has been applied and the second order problem was now about to be confronted, the idea of finding a solution of a similar, more simple problem that would satisfy the boundary conditions of the original one has been implemented. This analogous problem has been formulated, firstly by disregarding the variable coefficient and secondly by introducing a new constant λ_n . Requiring now the calculated solution to also satisfy the initial equation of motion, the buckling load can be determined as shown through the rest steps. However, this philosophy cannot be applied in the case of the shell tank. The concept of neglecting all the variable coefficients in the expense of inserting new constants cannot be applicable firstly due to the complexity of the problem, and secondly because that would automatically mean the disregard of all the 1st order solutions which contain the buckling load. For this reason, the search of the load responsible for instability will take place numerically.

In order to get a clearer picture of the procedure that needs to be followed, once again our attention will be turned into the beam, taking of course a different route than before. The problem we are called to tackle after the perturbation method has been applied, is repeated here,

$$EIw_0''''(x) + (P - qx)w_0''(x) - qw_0'(x) = 0 \quad (3.58)$$

with

$$EIw_0'''(0) + Pw_0'(0) = 0 \quad \& \quad w_0''(0) = 0 \quad (3.59a, b)$$

$$w_0(L) = 0 \quad \& \quad w_0'(L) = 0 \quad (3.60a, b)$$

where EI is the stiffness, q the known uniform load and P the unknown force at the top responsible for buckling.

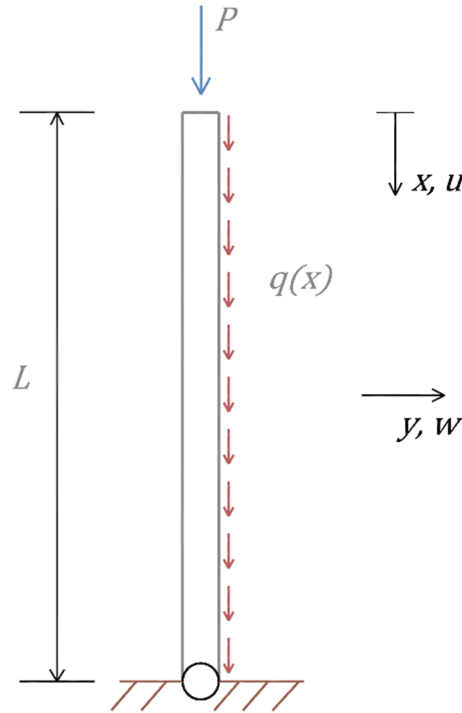


Figure 18 Cantilever beam under a concentrated force P and uniform distributed load $q(x)$

In case of a linear *eom*, buckling load can easily be found by substituting the solution into the boundary conditions and taking the coefficient matrix of unknown constants equal to zero. The same principle can be applied here by a numerical iterative approach which can be clear through the following steps:

Step 1

The length of the beam is broken into a total number of S segments (intervals).

Step 2

In each segment $j \in [1, S]$, equation (3.58) is written again as,

$$EIw_{0,j}''''(x) + \underbrace{(P - qx)}_{\substack{\text{variable} \\ \text{coefficient}}} w_{0,j}''(x) - qw_{0,j}'(x) = 0 \quad (3.61)$$

However, this time the variable coefficient $(P - qx)$ will be evaluated for $x \rightarrow x_{0,j}$, where $x_{0,j}$ is the middle of each interval j . This means that (3.61) should be rewritten as,

$$EIw_{0,j}''''(x) + (P - qx_{0,j})w_{0,j}''(x) - qw_{0,j}'(x) = 0 \quad (3.62)$$

with,

$$x_{0,j} = x_{j-1} + \frac{x_j - x_{j-1}}{2} \quad (3.63)$$

In this way, the equation of motion has been transformed into an *ode* with constant coefficients at each interval, by approximating the x coordinate of the variable coefficient with the middle numeric value of the corresponding segment.

Step 3

Working at each interval separately, the general solution of the related j^{th} *ode* is found. By assuming a solution in the form,

$$w_{0,j}(x) = Ce^{\lambda x} \quad (3.64)$$

and substitution into (3.62) leads to the following characteristic equation.

$$EI \lambda^4 + (P - qx_{0,j})\lambda^2 - q = 0 \quad (3.65)$$

which has 4 roots:

$$\lambda_{1,2} = \pm \frac{\sqrt{-\frac{P}{EI} + \frac{qx_{0,j}}{EI} - \frac{\sqrt{4EIq + (P - qx_{0,j})^2}}{EI}}}{\sqrt{2}} \quad \& \quad \lambda_{3,4} = \pm \frac{\sqrt{-\frac{P}{EI} + \frac{qx_{0,j}}{EI} + \frac{\sqrt{4EIq + (P - qx_{0,j})^2}}{EI}}}{\sqrt{2}} \quad (3.66)$$

The general solution for the j^{th} interval takes the following form,

$$w_{0,j}(x) = C_{4j-3}e^{\lambda_{4j-3}(P)x} + C_{4j-2}e^{\lambda_{4j-2}(P)x} + C_{4j-1}e^{\lambda_{4j-1}(P)x} + C_{4j}e^{\lambda_{4j}(P)x} \quad (3.67)$$

in which the eigenvalues are known functions of the unknown load P .

Step 4

Now the total number of unknowns is equal to $4 \cdot S$, where S is the number of segments (intervals). Apart from the boundary conditions above, the following 4 matching-interface (ICs) conditions should be introduced between the two consecutives j^{th} & $j + 1^{th}$ intervals in order to ensure continuity,

$$1) \quad w_{0,j}(x_j) = w_{0,j+1}(x_j) \quad (3.68a)$$

$$2) \quad w'_{0,j}(x_j) = w'_{0,j+1}(x_j) \quad (3.68b)$$

$$3) \quad M_j(x_j) = M_{j+1}(x_j) \Rightarrow EI w''_{0,j}(x_j) = EI w''_{0,j+1}(x_j) \Rightarrow w''_{0,j}(x_j) = w''_{0,j+1}(x_j) \quad (3.68c)$$

$$4) \quad V_j(x_j) = V_{j+1}(x_j) \Rightarrow EI w'''_{0,j}(x_j) - q \cdot x_{0,j} w'_{0,j}(x_j) = EI w'''_{0,j+1}(x_j) - q \cdot x_{0,j+1} w'_{0,j+1}(x_j) \quad (3.68d)$$

The first three interface conditions regarding the displacement, rotation and internal moment are quite clear. However, the fourth one about shear, requires a bit more explanation. This can be done by observing the beam in its deformed state and examining a small element of length dx :

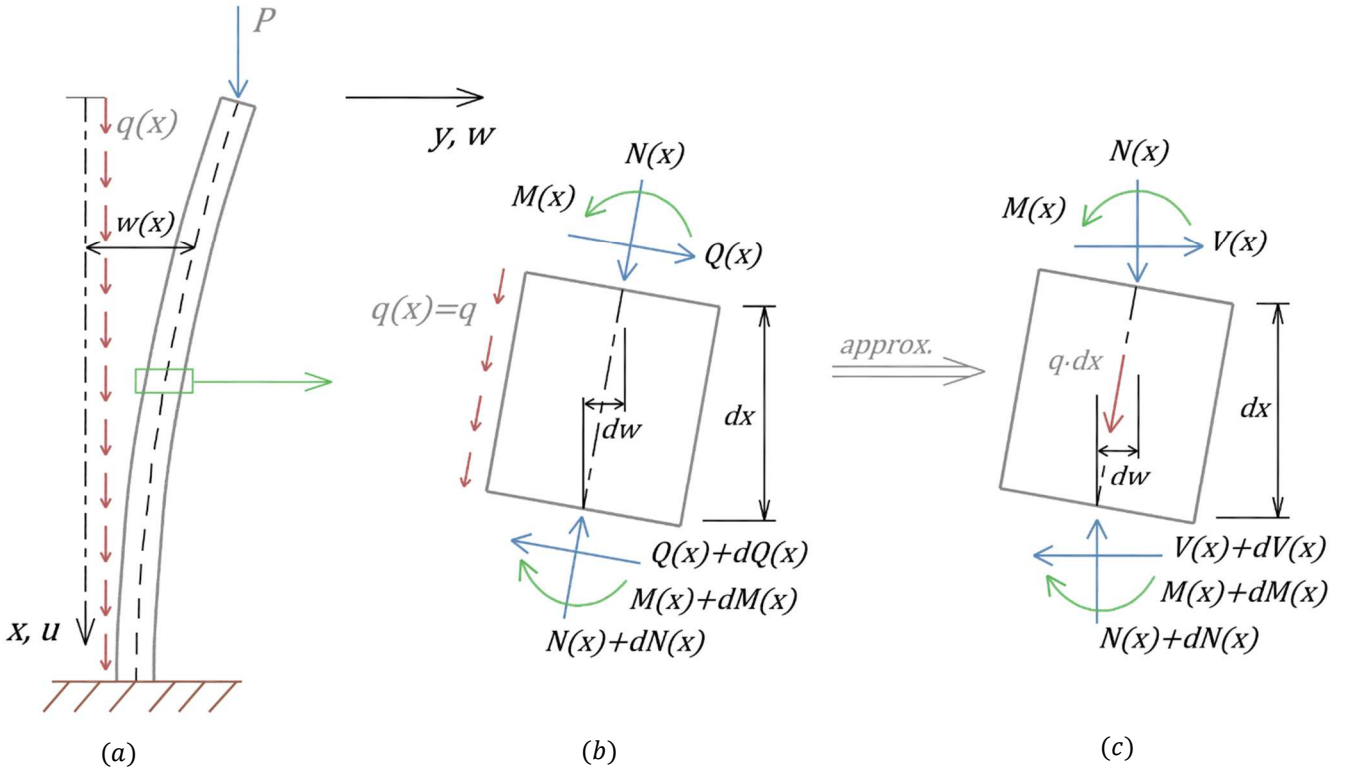


Figure 19 (a) Deformed shape of the beam, (b) a small element of length dx with shear force normal to the cross section, (c) a small element of length dx with shear force in the global direction and approximately normal force as well.

The only difference between the two last figures is the shear force $V(x)$ and the shear force normal to the cross section $Q(x)$. Based on fig. 19(b) and fig. 19(c), it can be easily observed that approximately,

$$V(x) = Q(x) - N(x) \frac{dw}{dx} \quad (3.69)$$

From simple statics, it is found that,

$$N(x) = P + q(x) \cdot x \Rightarrow N(x) = P + q \cdot x \quad (3.70)$$

Also from theory, it is known that in absolute value,

$$Q(x) = EI w'''(x) \quad (3.71)$$

Substituting (3.70) and (3.71) into (3.69), shear force becomes,

$$V(x) = EI w'''(x) - (P + q \cdot x)w'(x) \quad (3.72)$$

In this way, it can be seen that the shear force in the global direction contains apart from a bending related term, also a term that refers to compression. In our case that the beam is discretized, the uniform load is approximated as a piece-wise constant function of the axial coordinate, meaning that the latter part of the force is not continuous at the interfaces, i.e.

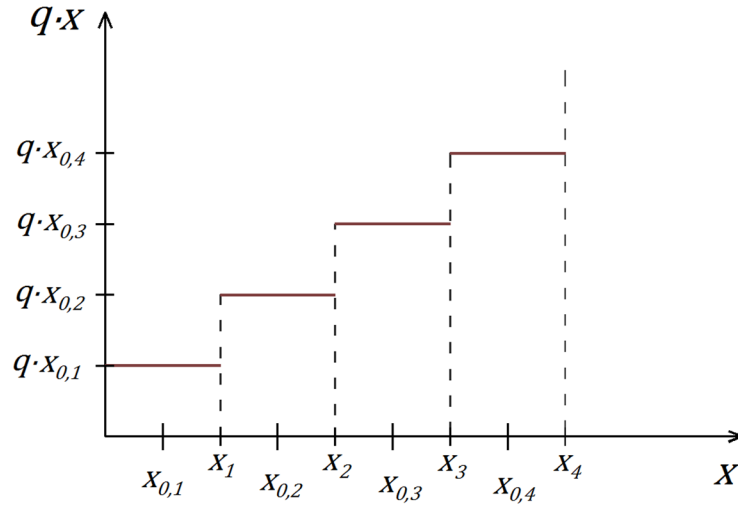


Figure 20 Approximation of the force as a piece-wise constant function of the axial coordinate.

It is reminded that the term $q \cdot x$ is evaluated for the numeric value at the middle of each interval, when the general solution for the corresponding segment j is determined.

$$q \cdot x \rightarrow q \cdot x_{0,j} \quad (3.73)$$

In order to have a correspondence with the general solutions $w_{0,j}(x)$, the same should be applied in relation (3.72). In that sense, the interface condition for the internal shear force $V(x)$ at x_j becomes:

$$V_j(x_j) = V_{j+1}(x_j) \xrightarrow{(3.72)} EI w'''_{0,j}(x_j) - (P + q \cdot x_{0,j})w'_{0,j}(x_j) = EI w'''_{0,j+1}(x_j) - (P + q \cdot x_{0,j+1})w'_{0,j+1}(x_j) \quad (3.74)$$

Making use of the 2nd interface condition (3.68b), the relation (3.74) is simplified into its final form,

$$EI w'''_{0,j}(x_j) - q \cdot x_{0,j} w'_{0,j}(x_j) = EI w'''_{0,j+1}(x_j) - q \cdot x_{0,j+1} w'_{0,j+1}(x_j) \quad (3.75)$$

Step 5

The general solutions should be substituted into the boundary and interface conditions depending on the corresponding segment (interval). For instance, at the 1st ($j = 1$) top segment:

BCs

$$EI w'''_{0,1}(0) + P w'_{0,1}(0) = 0 \Rightarrow$$

$$C_1(P \lambda_1 + EI \lambda_1^3) + C_2(P \lambda_2 + EI \lambda_2^3) + C_3(P \lambda_3 + EI \lambda_3^3) + C_4(P \lambda_4 + EI \lambda_4^3) = 0$$

&

$$w''_{0,1}(0) = 0 \Rightarrow C_1 \lambda_1^2 + C_2 \lambda_2^2 + C_3 \lambda_3^2 + C_4 \lambda_4^2 = 0$$

and also

ICs

$$w_{0,1}(x_1) = w_{0,2}(x_1) \Rightarrow$$

$$C_1 e^{\lambda_1(P)x_1} + C_2 e^{\lambda_2(P)x_1} + C_3 e^{\lambda_3(P)x_1} + C_4 e^{\lambda_4(P)x_1} = C_5 e^{\lambda_5(P)x_1} + C_6 e^{\lambda_6(P)x_1} + C_7 e^{\lambda_7(P)x_1} + C_8 e^{\lambda_8(P)x_1}$$

&

$$w'_{0,1}(x_1) = w'_{0,2}(x_1) \Rightarrow C_1 \lambda_1(P) e^{\lambda_1(P)x_1} + C_2 \lambda_2(P) e^{\lambda_2(P)x_1} + C_3 \lambda_3(P) e^{\lambda_3(P)x_1} + C_4 \lambda_4(P) e^{\lambda_4(P)x_1} \\ = C_5 \lambda_5(P) e^{\lambda_5(P)x_1} + C_6 \lambda_6(P) e^{\lambda_6(P)x_1} + C_7 \lambda_7(P) e^{\lambda_7(P)x_1} + C_8 \lambda_8(P) e^{\lambda_8(P)x_1}$$

&

$$w''_{0,1}(x_1) = w''_{0,2}(x_1) \Rightarrow C_1 \lambda_1^2(P) e^{\lambda_1(P)x_1} + C_2 \lambda_2^2(P) e^{\lambda_2(P)x_1} + C_3 \lambda_3^2(P) e^{\lambda_3(P)x_1} + C_4 \lambda_4^2(P) e^{\lambda_4(P)x_1} \\ = C_5 \lambda_5^2(P) e^{\lambda_5(P)x_1} + C_6 \lambda_6^2(P) e^{\lambda_6(P)x_1} + C_7 \lambda_7^2(P) e^{\lambda_7(P)x_1} + C_8 \lambda_8^2(P) e^{\lambda_8(P)x_1}$$

&

$$w'''_{0,1}(x_1) = w'''_{0,2}(x_1) \Rightarrow C_1 \lambda_1^3(P) e^{\lambda_1(P)x_1} + C_2 \lambda_2^3(P) e^{\lambda_2(P)x_1} + C_3 \lambda_3^3(P) e^{\lambda_3(P)x_1} + C_4 \lambda_4^3(P) e^{\lambda_4(P)x_1} \\ = C_5 \lambda_5^3(P) e^{\lambda_5(P)x_1} + C_6 \lambda_6^3(P) e^{\lambda_6(P)x_1} + C_7 \lambda_7^3(P) e^{\lambda_7(P)x_1} + C_8 \lambda_8^3(P) e^{\lambda_8(P)x_1}$$

The substitutions for the rest of the segments (intervals) is a rather standard and simple procedure and is conducted in the same manner as above. For this reason, the total picture of how the general solutions are substituted into the interface and boundary conditions for the whole beam, can be seen through the following figure.

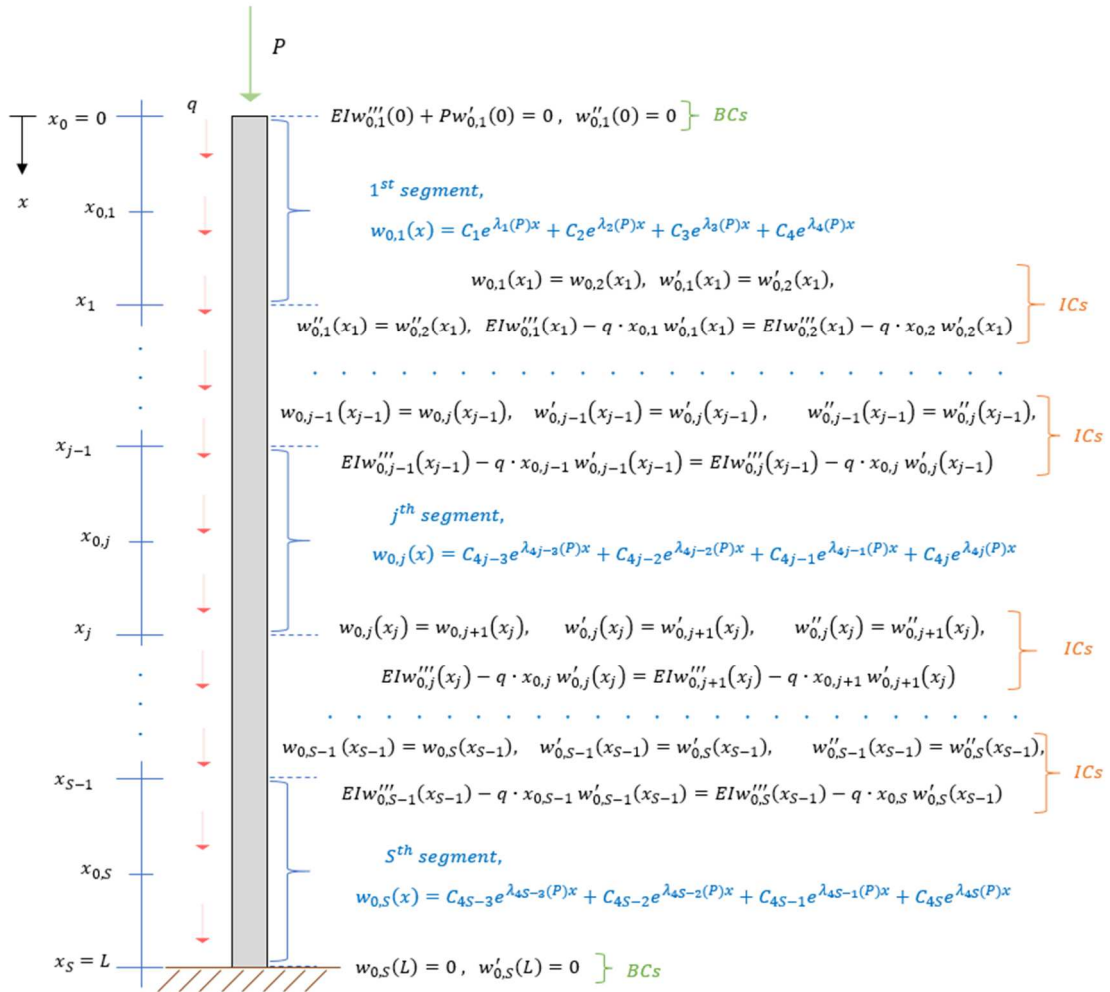


Figure 21 Discretization of the beam

Step 6

After all the substitutions of general solutions into the boundary and interface conditions has taken place, an analytical system of $4 \cdot S$ homogenous equations has been formed, which can be written in matrix form,

$$A_{4S \times 4S}^{final} C_{4S \times 1} = 0 \quad (3.76)$$

where $C_{4S \times 1}$ is the vector of the unknown constants and $A_{4S \times 4S}^{final}$ the square matrix coefficient where the load P is contained and which for example, in order to give a clear picture, in case $S = 4$ segments (intervals) are chosen, it takes the following form:

$$A_{final} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ a_{21} & a_{22} & a_{23} & a_{24} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & a_{36} & a_{37} & a_{38} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} & a_{46} & a_{47} & a_{48} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} & a_{56} & a_{57} & a_{58} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ a_{61} & a_{62} & a_{63} & a_{64} & a_{65} & a_{66} & a_{67} & a_{68} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & a_{75} & a_{76} & a_{77} & a_{78} & a_{79} & a_{710} & a_{711} & a_{712} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & a_{85} & a_{86} & a_{87} & a_{88} & a_{89} & a_{810} & a_{811} & a_{812} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & a_{95} & a_{96} & a_{97} & a_{98} & a_{99} & a_{910} & a_{911} & a_{912} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & a_{105} & a_{106} & a_{107} & a_{108} & a_{109} & a_{1010} & a_{1011} & a_{1012} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_{119} & a_{1110} & a_{1111} & a_{1112} & a_{1113} & a_{1114} & a_{1115} & a_{1116} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_{129} & a_{1210} & a_{1211} & a_{1212} & a_{1213} & a_{1214} & a_{1215} & a_{1216} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_{139} & a_{1310} & a_{1311} & a_{1312} & a_{1313} & a_{1314} & a_{1315} & a_{1316} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_{149} & a_{1410} & a_{1411} & a_{1412} & a_{1413} & a_{1414} & a_{1415} & a_{1416} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_{1513} & a_{1514} & a_{1515} & a_{1516} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_{1613} & a_{1614} & a_{1615} & a_{1616} \end{bmatrix}$$

Figure 22 Example matrix of unknown coefficients in case of a beam divided into 4 segments. Each one is presented with a different color.

Step 7

Finally, the value of load P is searched for which the determinant becomes zero.

$$|A_{4S \times 4S}^{final}| = 0 \quad (3.77)$$

Of course, this calculation cannot be performed by hand. The whole procedure will take place with the aid of *MATLAB*. The code in *appendix D* for the desired number of intervals, calculates the general solutions iteratively and constructs the final square matrix of coefficients. Finally, the buckling load is determined by setting the determinant equal to zero based on an initial approximation.

This can be also illustrated by an example beam of a length $L = 5m$, made of steel $E = 210 Gpa$, with a tube hollow section of diameter, on which apart from P , a uniform load $q = 1 kN/m$ is applied. The numeric values of the buckling load depending on the beam's stiffness EI after running the *MATLAB* code are given in the following graph.

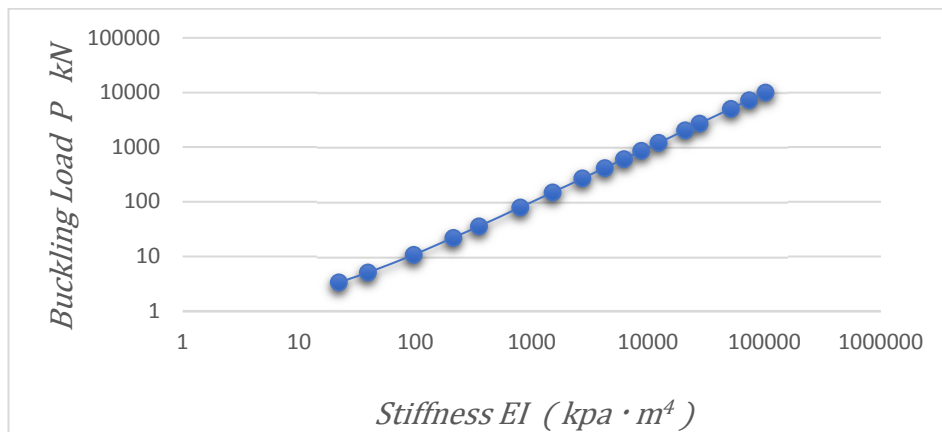


Figure 23 Buckling load in relation to beam's stiffness

As it was expected, the higher the stiffness the larger the value of the buckling load. For instance, a value of $I = 379 \text{ cm}^4$ for the 2nd moment of inertia results into a buckling load of $P = 80,685 \text{ kN}$. Now the same concept may also be applied in the case of the tank as well. Of course, the algorithm will be much larger in extent, but the main structure will remain the same. So, having the example of the beam as a standard, a numerical approximation of the buckling load may be achieved, through the following steps.

Step 1

Calculate symbolically the responses for $m = 0$ and $m = 1$ for the 1st order linear problem as it is indicated in section 3.2.1 .

Step 2

Formulate the N^{th} order nonlinear system, after it has been linearized, of the 1st order odes and bcs in terms of the x coordinate for the selected number r of modes, in the way it was conducted before. This step cannot be avoided, since in the initial form of the system (3.36, 3.37, 3.40), is nearly impossible for the computer engines to provide a solution.

Step 3

The tank, must now be divided into S segments. In each segment, the variable coefficients $g(x, F_E)$ of the corresponding system, which include the general solutions of the linear problem, will be evaluated at the middle numerical value of the corresponding interval,

$$x \rightarrow x_{0,j} = x_{j-1} + \frac{x_j - x_{j-1}}{2}$$

The only unknown inside g coefficients will now be the load F_E .

Step 4

Working at each part separately, now that the system has transformed into a system of odes with constant coefficients, an analytical solution for all the evolved responses $U_{0,j}^m(x)$, $V_{0,j}^m(x)$ & $W_{0,j}^m(x)$ must be calculated. The total number of unknowns at each interval j will be equal to the individual system's order N plus the load F_E .

Step 5

Apart from the $N/2$ boundary conditions at the top and the bottom, a number of matching - interface conditions equal to the total N^{th} order system of 1st order odes, must be introduced between two consecutive sections in order to ensure continuity.

$$1) \quad U_{0,j}^m(x_j) = U_{0,j+1}^m(x_j) \quad (3.78)$$

$$2) \quad V_{0,j}^m(x_j) = V_{0,j+1}^m(x_j) \quad (3.79)$$

$$3) \quad W_{0,j}^m(x_j) = W_{0,j+1}^m(x_j) \quad (3.80)$$

$$4) \quad W_{0,j}^m(x_j)' = W_{0,j+1}^m(x_j)' \quad (3.81)$$

$$5) \quad N_{xx,j}(x_j) = N_{xx,j+1}(x_j) \quad (3.82)$$

Perturbation method should also be applied in the normal force (fig.13) along with the corresponding previous assumptions, namely the neglect of the unknown functions of subscript 1, the self-multiplied functions as well as the mathematical vague epsilon parameters. So,

$$\begin{aligned}
(2.44a) \Rightarrow N_{xx} &= \varepsilon \left[\frac{Dhn}{R} w_0(x, \theta) + \frac{Dhn}{R} \frac{\partial v_0(x, \theta)}{\partial \theta} + Dh \frac{\partial u_0(x, \theta)}{\partial x} \right] \\
&+ \varepsilon^2 \left[\frac{1}{R} Dhn w_{01}(x, \theta) + \frac{Dhn}{2R^2} \frac{\partial w_0(x, \theta)^2}{\partial \theta} + Dh \frac{\partial u_{01}(x, \theta)}{\partial x} + \frac{Dh}{2} \frac{\partial w_0(x, \theta)^2}{\partial x} \right] \Rightarrow \\
N_{xx} &= \frac{Dhn}{R} w_0(x, \theta) + \frac{Dhn}{R} \frac{\partial v_0(x, \theta)}{\partial \theta} + Dh \frac{\partial u_0(x, \theta)}{\partial x}
\end{aligned} \quad (3.83)$$

This means that the interface condition after substituting the assumed solutions (3.35) and using then orthogonality, becomes,

$$\begin{aligned}
(3.82) \xrightarrow{(3.35)} \int_0^{2\pi} \cos m\theta (\dots) \frac{Dhn}{R} W_{0,j}^m(x_j) + m \frac{Dhn}{R} V_{0,j}^m(x_j) + Dh U_{0,j}^m(x_j)' &= \frac{Dhn}{R} W_{0,j+1}^m(x_j) + m \frac{Dhn}{R} V_{0,j+1}^m(x_j) + Dh U_{0,j+1}^m(x_j)' \\
&\xrightarrow{(3.79) \ \& \ (3.80)} Dh U_{0,j}^m(x_j)' = Dh U_{0,j+1}^m(x_j)' \Rightarrow U_{0,j}^m(x_j)' = U_{0,j+1}^m(x_j)'
\end{aligned} \quad (3.84)$$

$$6) \quad M_{xx,j}(x_j) = M_{xx,j+1}(x_j) \quad (3.85)$$

The same should be also applied for the moment (*fig. 13*).

$$\begin{aligned}
(2.45a) \Rightarrow M_{xx} &= -\varepsilon \left[\frac{nDh^3}{12R^2} \frac{\partial^2 w_0(x, \theta)}{\partial \theta^2} + \frac{Dh^3}{12} \frac{\partial^2 w_0(x, \theta)}{\partial x^2} \right] - \varepsilon^2 \left[\frac{nDh^3}{12R^2} \frac{\partial^2 w_{01}(x, \theta)}{\partial \theta^2} + \frac{Dh^3}{12} \frac{\partial^2 w_{01}(x, \theta)}{\partial x^2} \right] \\
&\Rightarrow M_{xx} = -\frac{nDh^3}{12R^2} \frac{\partial^2 w_0(x, \theta)}{\partial \theta^2} - \frac{Dh^3}{12} \frac{\partial^2 w_0(x, \theta)}{\partial x^2}
\end{aligned} \quad (3.86)$$

So finally,

$$\begin{aligned}
(3.85) \xrightarrow{(3.35)} \int_0^{2\pi} \cos m\theta (\dots) m^2 \frac{nDh^3}{12R^2} W_{0,j}^m(x_j) - \frac{Dh^3}{12} W_{0,j}^m(x_j)'' &= m^2 \frac{nDh^3}{12R^2} W_{0,j+1}^m(x_j) - \frac{Dh^3}{12} W_{0,j+1}^m(x_j)'' \\
&\xrightarrow{(3.80)} -\frac{Dh^3}{12} W_{0,j}^m(x_j)'' = -\frac{Dh^3}{12} W_{0,j+1}^m(x_j)'' \Rightarrow W_{0,j}^m(x_j)'' = W_{0,j+1}^m(x_j)''
\end{aligned} \quad (3.87)$$

$$7) \quad N_{x\theta,j}(x_j) = N_{x\theta,j+1}(x_j) \quad (3.88)$$

In-plane shear (*fig. 13*) is given by,

$$\begin{aligned}
(2.44c) \Rightarrow N_{x\theta} &= \varepsilon \left[\frac{Gh}{R} \frac{\partial u_0(x, \theta)}{\partial \theta} + Gh \frac{\partial v_0(x, \theta)}{\partial x} \right] + \varepsilon^2 \left[\frac{Gh}{R} \frac{\partial u_{01}(x, \theta)}{\partial \theta} + \frac{Gh}{R} \frac{\partial w_{01}(x, \theta)}{\partial \theta} \frac{\partial w_{01}(x, \theta)}{\partial x} \right] \\
&\Rightarrow N_{x\theta} = \frac{Gh}{R} \frac{\partial u_0(x, \theta)}{\partial \theta} + Gh \frac{\partial v_0(x, \theta)}{\partial x}
\end{aligned} \quad (3.89)$$

which means that,

$$\begin{aligned}
(3.88) \xrightarrow{(3.35)} \int_0^{2\pi} \sin m\theta (\dots) -m \frac{Gh}{R} U_{0,j}^m(x_j) + Gh V_{0,j}^m(x_j)' &= -m \frac{Gh}{R} U_{0,j+1}^m(x_j) + Gh V_{0,j+1}^m(x_j)' \\
&\xrightarrow{(3.78)} Gh V_{0,j}^m(x_j)' = Gh V_{0,j+1}^m(x_j)' \Rightarrow V_{0,j}^m(x_j)' = V_{0,j+1}^m(x_j)'
\end{aligned} \quad (3.90)$$

$$8) \quad Q_{xz,j}(x_j) = Q_{xz,j+1}(x_j) \quad (3.91)$$

Out-of-plane shear (*fig. 13*) is given by,

$$(2.46a) \Rightarrow Q_{xz} = \varepsilon \left[-\frac{2Gh^3 + Dh^3n}{12R^2} \frac{\partial^3 w_0(x, \theta)}{\partial x \partial \theta^2} - \frac{Dh^3}{12} \frac{\partial^3 w_0(x, \theta)}{\partial x^3} \right] + \varepsilon^2 \left[-\frac{2Gh^3 + Dh^3n}{12R^2} \frac{\partial^3 w_{01}(x, \theta)}{\partial x \partial \theta^2} - \frac{Dh^3}{12} \frac{\partial^3 w_{01}(x, \theta)}{\partial x^3} \right] \quad (3.92)$$

$$\Rightarrow Q_{xz} = -\frac{2Gh^3 + Dh^3n}{12R^2} \frac{\partial^3 w_0(x, \theta)}{\partial x \partial \theta^2} - \frac{Dh^3}{12} \frac{\partial^3 w_0(x, \theta)}{\partial x^3}$$

which means that,

$$(3.91) \xrightarrow{\int_0^{2\pi} \cos m\theta \, d\theta \dots} m^2 \frac{2Gh^3 + Dh^3n}{12R^2} W_{0,j}^m(x_j)' - \frac{Dh^3}{12} W_{0,j}^m(x_j)''' = m^2 \frac{2Gh^3 + Dh^3n}{12R^2} W_{0,j+1}^m(x_j)' - \frac{Dh^3}{12} W_{0,j+1}^m(x_j)''' \quad (3.35)$$

$$\xrightarrow{(3.81)} \frac{Dh^3}{12} W_{0,j}^m(x_j)''' = \frac{Dh^3}{12} W_{0,j+1}^m(x_j)''' \Rightarrow W_{0,j}^m(x_j)''' = W_{0,j+1}^m(x_j)''' \quad (3.93)$$

Step 6

Now, the general solutions which have been calculated at step 4 should be substituted into the N boundary (BCs) and $N \cdot S - N$ interface conditions (ICs). In order to get a clear picture, the following *fig. 24* has been used, which depicts the way the tank has been discretized as well as the substitution process that needs to take place for the interfaces and the boundaries.

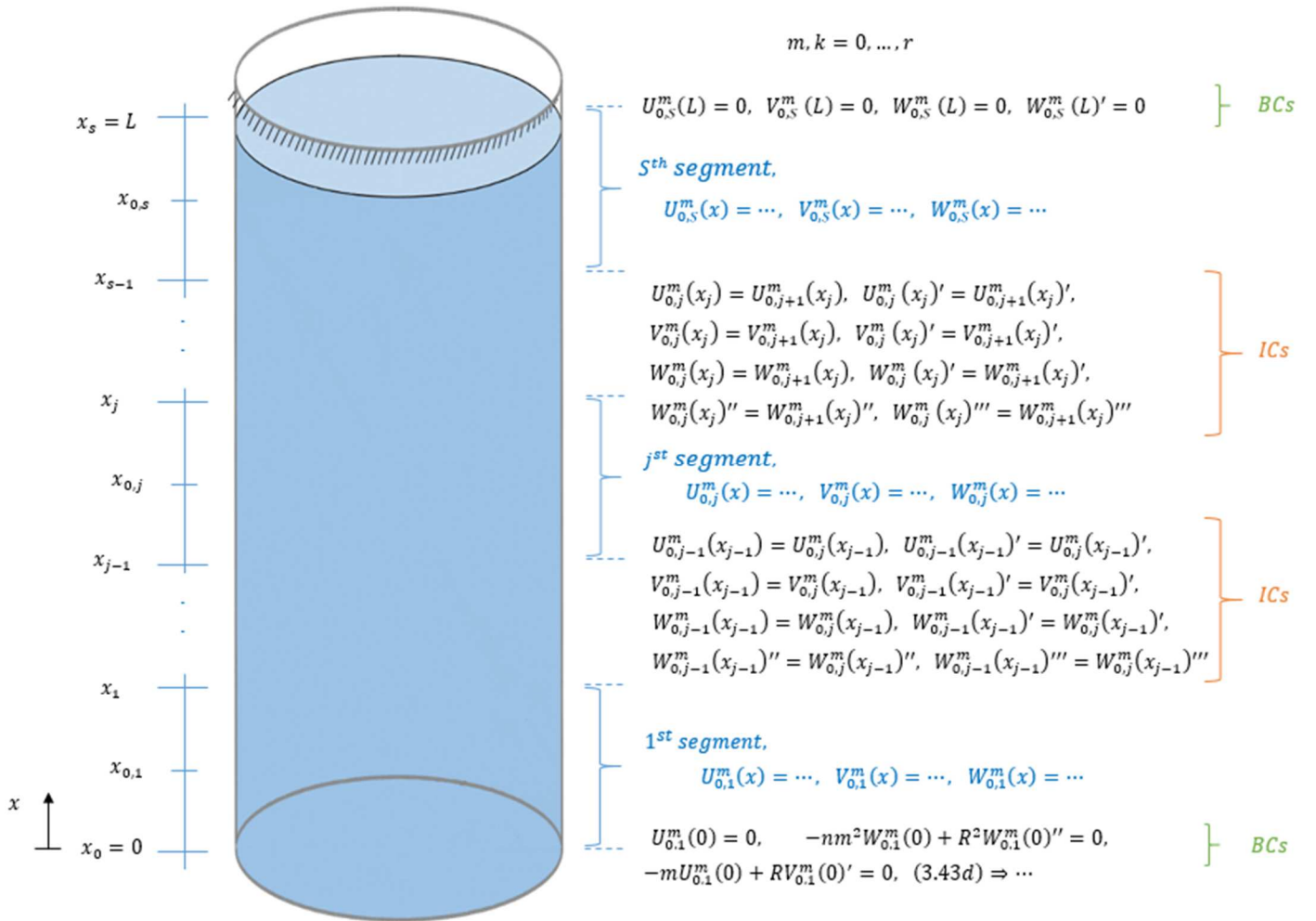


Figure 24 Discretization of the tank

Step 7

Now, an analytical system of $N \cdot S$ homogenous equations has been formed, which can be written in matrix form,

$$A_{N \cdot S \times N \cdot S}^{final} C_{N \cdot S \times 1} = 0 \quad (3.94)$$

where $C_{N \cdot S \times 1}$ is the vector of the unknown constants and $A_{N \cdot S \times N \cdot S}^{final}$ the square matrix coefficient where the load F_E is contained.

The final matrix A of the unknown coefficients once again can be built up as in the beam's example. An iterative procedure will be applied for the intermediate sections and a separate one for the top and the bottom. A typical form of the matrix $A_{N \cdot S \times N \cdot S}^{final}$ can be illustrated below.

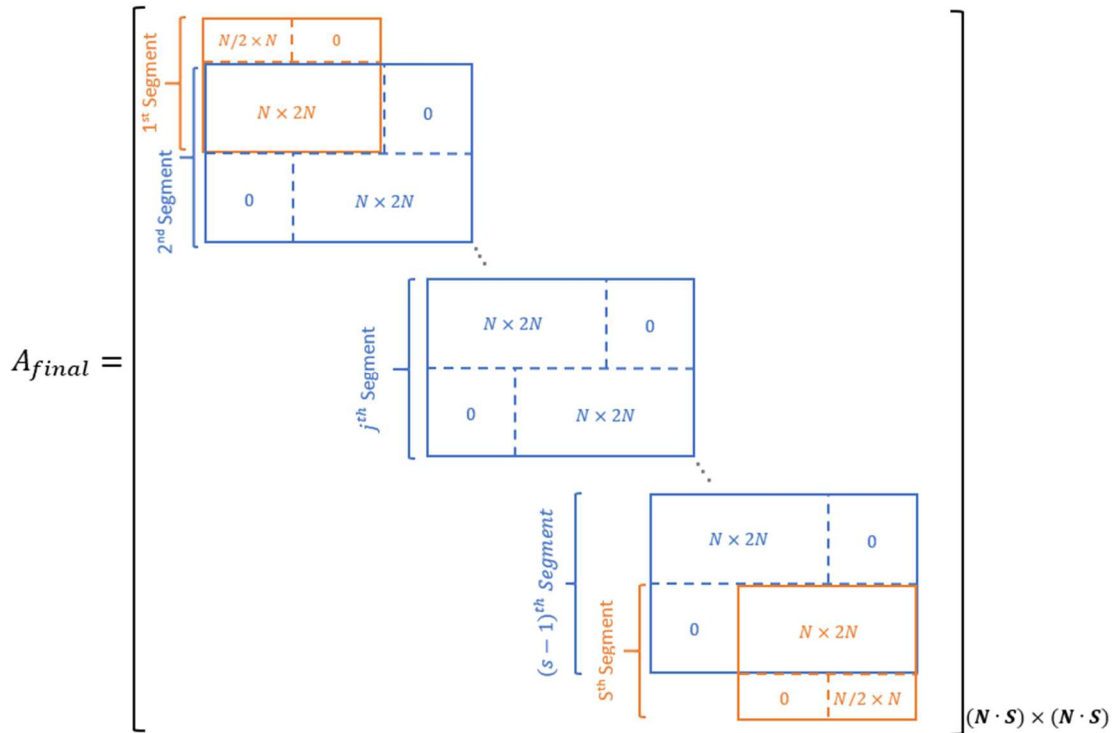


Figure 25 General form of the matrix of unknown coefficients in the case of a tank divided into S sections

For example, in the extremely simple case where firstly, $r = 3$ modes are utilized, something which is translated into a $N = 6 + 2 \cdot 8 = 22^{th}$ order of odes, and secondly just only $S = 3$ sections are used, A_{final} results into a 66×66 square matrix.

Step 8

Finally, to determine the critical shear buckling load F_E , the following equation

$$|A_{N \cdot S \times N \cdot S}^{final}| = 0 \quad (3.95)$$

needs to be formulated and *MATLAB's* routine *vpasolve* should be used to determine the value of F_E for which the above equation is valid. For that purpose, an initial approximation of the buckling load is highly recommended as it could help the process. That could be achieved by calculating first a value for the buckling load for the whole tank without any divisions and evaluating the variable coefficients at $x_0 = L/2$.

However, it is not certain that a result can be extracted. Solving a system even in the form of 1^{st} order odes with an unknown parameter or calculating the determinant of a symbolic matrix, or using *vpasolve* in a

large symbolic expression is not always feasible. In such an event in order to accommodate the process, the steps which were described above, should be performed repetitively for different consecutive values of load F_E . So now, at the 8th step, instead of using *vpasolve*, the buckling load will be approximated equally to that value for which the determinant is zero or close to zero.

At this point it must be underlined the following: both methods require that an analytical solution at *step 4* can be achieved. Even in this case, the computational cost is quite high. The higher the selected number r of modes, as well as the more divisions into segments, the better the approximation of the critical buckling load. In this way, it is now possible to determine the occurrence of instability for a cylindrical tank filled with water under shear.

4 Discussion & Conclusions

Delving into the topic of shell's shear buckling for a steel cylindrical tank filled with water, was personally without doubt quite a strenuous and demanding process. The literature is rather chaotic, as numerous approaches have been implemented by scientists to investigate this topic. The path that will be followed during the research, comes down to personal judgement, as one is called to decide, at every step of the study which is the most appropriate way to proceed. This is what also took place in the current thesis. As it was mentioned from the very beginning, the initial purpose was the study of both the static and dynamic case. The latter would be meaningless without the inclusion of imperfections, something which is inextricably associated with large deformations. This along with the fact that natural phenomena, and especially buckling, are inherently nonlinear, led us to select a large deformation nonlinear theory. This approach has been used by many, is more realistic and avoids inaccuracies. The choice among the various shell theories was based upon the characteristics of the shell tank under study and the complexity accompanied with each theory. After the formulation of equations and boundary conditions the question arose on how to handle the difficulty of nonlinearity. The answer to that, was the perturbation method, which allowed the transformation of the initial problem into a weaker, more manageable formulation. The equations of motion have become finally linear, after the problem's subdivision suggested by the perturbation method. The existence of variable coefficients called for the aid of *MATLAB*, when the stage of buckling investigation has arrived. The procedure which has been recommended is illustrated through the example of a beam, in order to get a clearer, more comprehensive picture of the path that needs to be followed.

Numerical values for the problem of tank has not been produced eventually. Although the approach is now known and applicable in simple problems, the computational cost is quite high for the complex case of the cylindrical shell tank. That maybe could be achieved by the next researcher with the aid of a powerful computer engine. Afterall, handling nonlinear equations of motion is not an easy task and it usually comes down to what technique will be used as well as what assumptions should be made in order to reach an easier, more simplified form of the system but at the same time not deviate from the initial problem. So, instead of the perturbation method alongside with its assumptions which I strongly believe are totally correct, maybe an answer with lower computational cost can be sought elsewhere. It could be the case that following the classical buckling-membrane theory despite its inconsistencies, might solve the problem by giving accurate results that would be verified with experiments in the future. Another idea, would be to first study and then try to implement the homotopy analysis of Liao, in order to tackle the nonlinear problem in a straightforward manner. Such an attempt would be a rather strenuous and time-consuming procedure, since homotopy analysis belongs in the field of advanced mathematics. Nevertheless, the easiest approach would be, without doubt, by the aid of *FEM* programs, something which is very common in the modern studies. Results produced by programs like *Ansys* and *Abaqus* could be verified by experiments for liquid filled steel tanks of a smaller scale.

In the event that one could reach a solution, either by the described numerical method of the latter chapter, or by one of the analytical alternatives or the use of *FEMs*, some quite interesting results could very easily be extracted, that would fill the gap of information in the existing structural codes and standards. The effect of tank's geometric characteristics, as well as the effect of the different levels of water on shear buckling would be some of the most meaningful conclusions to be drawn. The latter would also give an answer to what extent the recommendations provided by the Japanese code are valid or not. Another important aspect, that would be of interest, is how the different boundary conditions affect the buckling shear stress. Further, a more complete study on the topic of shear buckling for cylindrical water steel tanks, could be achieved by a comparison with other, more complex nonlinear shell theories.

As a final remark, it should be understood that the study of every natural phenomenon is not other but a simulation of the reality and thus, there always be deficiencies and difficulties. We are called to judge and draw conclusions based on the data that stem from the mathematical procedures which sometimes can be quite strenuous and challenging. Of course, nothing is predetermined or certain beforehand regarding the

final result, but this is also the joy and the challenge of research. After this thesis, the future researcher is provided with all those tools that will allow him to follow the correct path in which instability due to shear can be investigated in complex problems, such that of a liquid filled cylindrical tank.

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Appendix A

The potential energy P of a mechanical system after a finite perturbation is equal to,

$$P + \Delta P \quad (\text{A. 1})$$

The Taylor series of a function $P(x)$ is written as,

$$P(x) = P(x_0) + P'(x) (x - x_0) + \frac{P''(x)}{2!} (x - x_0)^2 + \frac{P'''(x)}{3!} (x - x_0)^3 + \dots \quad (\text{A. 2})$$

When x is very close to x_0 , then if the symbolism of $d \rightarrow \delta$ is made, it is obvious that

$$(x - x_0) = dx \equiv \delta x, \quad (x - x_0)^2 = dx^2 \equiv \delta x^2, \quad \dots \quad (\text{A. 3})$$

Also,

$$P'(x) = \frac{\delta P}{\delta x}, \quad P''(x) = \frac{\delta^2 P}{\delta x^2}, \quad \dots \quad (\text{A. 4})$$

Making use of (A.3) and (A.4) the relation (A.2) becomes,

$$\begin{aligned} P(x) &= P(x_0) + \frac{\delta P}{\delta x} \delta x + \frac{1}{2!} \frac{\delta^2 P}{\delta x^2} \delta x^2 + \frac{1}{3!} \frac{\delta^3 P}{\delta x^3} \delta x^3 + \dots \Rightarrow \\ P(x) - P(x_0) &= \delta P + \frac{1}{2!} \delta^2 P + \frac{1}{3!} \delta^3 P + \dots \Rightarrow \\ \Delta P &= \delta P + \frac{1}{2!} \delta^2 P + \frac{1}{3!} \delta^3 P + \dots \end{aligned} \quad (\text{A. 5})$$

In that sense, the new potential energy of the system is written as,

$$P + \Delta P = P + \delta P + \frac{1}{2!} \delta^2 P + \frac{1}{3!} \delta^3 P + \dots \quad (\text{A. 6})$$

In accordance to section 1.2 equilibrium is assured if,

$$\delta P = 0 \quad (\text{A. 7})$$

Now the prevailing term in ΔP is $\delta^2 P$, which for a stable equilibrium, where $\Delta P > 0$ *fig.3*, it means that,

$$\delta^2 P > 0 \quad (\text{A. 8})$$

If instability is expressed for $\delta^2 P < 0$, then it is logical that the buckling criterion for the transition from the stable into the unstable state is given by:

$$\delta^2 P = 0 \quad (\text{A. 9})$$

Appendix B

The system of the homogenous equations of motion are written in the following matrix form,

$$\begin{bmatrix} L_{11} & L_{12} & L_{13} \\ L_{21} & L_{22} & L_{23} \\ L_{31} & L_{32} & L_{33} \end{bmatrix} \begin{bmatrix} u_0(x, \theta) \\ v_0(x, \theta) \\ w_0(x, \theta) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad (\text{B. 1})$$

In order to find a relationship between $u_0(x, \theta)$ and $w_0(x, \theta)$, the displacement $v_0(x, \theta)$ must be eliminated. For this, the first equation is multiplied by $L_{32}L_{22}$, the second by $L_{12}L_{32}$ and the third one by $L_{12}L_{22}$.

$$\begin{bmatrix} L_{32}L_{22}L_{11} & L_{32}L_{22}L_{12} & L_{32}L_{22}L_{13} \\ L_{12}L_{32}L_{21} & L_{12}L_{32}L_{22} & L_{12}L_{32}L_{23} \\ L_{12}L_{22}L_{31} & L_{12}L_{22}L_{32} & L_{12}L_{22}L_{33} \end{bmatrix} \begin{bmatrix} u_0(x, \theta) \\ v_0(x, \theta) \\ w_0(x, \theta) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad (\text{B. 2})$$

Now by making use only two of the three equations, e.g. the 1st and the 2nd, if they are subtracted it leads to:

$$\begin{aligned} (L_{32}L_{22}L_{11} - L_{12}L_{32}L_{21})u_0(x, \theta) + (L_{32}L_{22}L_{13} - L_{32}L_{12}L_{23})w_0(x, \theta) &= 0 \Rightarrow \\ (L_{32}L_{22}L_{11} - L_{12}L_{32}L_{21})u_0(x, \theta) &= (L_{32}L_{12}L_{23} - L_{32}L_{22}L_{13})w_0(x, \theta) \Rightarrow \\ (L_{22}L_{11} - L_{12}L_{21})u_0(x, \theta) &= (L_{12}L_{23} - L_{22}L_{13})w_0(x, \theta) \end{aligned} \quad (\text{B. 3})$$

The same procedure should take place, when a relationship between $v_0(x, \theta)$ and $w_0(x, \theta)$ is pursued. However, this time the multiplications are performed as follows.

$$\begin{bmatrix} L_{21}L_{31}L_{11} & L_{21}L_{31}L_{12} & L_{21}L_{31}L_{13} \\ L_{11}L_{31}L_{21} & L_{11}L_{31}L_{22} & L_{11}L_{31}L_{23} \\ L_{11}L_{21}L_{31} & L_{11}L_{21}L_{32} & L_{11}L_{21}L_{33} \end{bmatrix} \begin{bmatrix} u_0(x, \theta) \\ v_0(x, \theta) \\ w_0(x, \theta) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad (\text{B. 4})$$

Subtracting the second equation from the first gives:

$$\begin{aligned} (L_{21}L_{31}L_{12} - L_{11}L_{31}L_{22})v_0(x, \theta) + (L_{21}L_{31}L_{13} - L_{11}L_{31}L_{23})w_0(x, \theta) &= 0 \Rightarrow \\ (L_{21}L_{31}L_{12} - L_{11}L_{31}L_{22})v_0(x, \theta) &= (L_{11}L_{31}L_{23} - L_{21}L_{31}L_{13})w_0(x, \theta) = 0 \Rightarrow \\ (L_{21}L_{12} - L_{11}L_{22})v_0(x, \theta) &= (L_{11}L_{23} - L_{21}L_{13})w_0(x, \theta) \end{aligned} \quad (\text{B. 5})$$

Appendix C

In order to have a better look of the multiplications among the cosines and sines it is better to sum them up in the following way,

Multiplication between cosines only

$\bar{k} = 0$:

$$\int_0^{2\pi} \cos(k * \theta) * \cos(m * \theta) d\theta = \frac{k \cos(2m\pi) \sin(2k\pi) - m \cos(2k\pi) \sin(2m\pi)}{k^2 - m^2} = \begin{cases} 0, k \neq m \\ \pi + \frac{\sin(4m\pi)}{4m}, & k = m \Rightarrow \begin{cases} 2\pi, & m = 0 \\ \pi, & m > 0 \end{cases} \end{cases}$$

$\bar{k} = 1$:

$$\int_0^{2\pi} \cos(k * \theta) * \cos(m * \theta) * \cos(\theta) d\theta = \frac{1}{2} \left(\frac{(k-m) \sin[2(k-m)\pi]}{-1 + (k-m)^2} + \frac{(k+m) \sin[2(k+m)\pi]}{(-1+k+m)(1+k+m)} \right) =$$

$$= \begin{cases} 0, k \neq -1+m \text{ \& } k \neq 1+m \text{ \& } k \neq 1-m \\ \frac{4(-1+m)m\pi + (-1+2m)\sin(4m\pi)}{8(-1+m)m}, & k = -1+m \Rightarrow \begin{cases} \pi, m = 0 \text{ not in our case } (k \geq 0) \\ \pi, m = 1 \\ \frac{\pi}{2}, m > 1 \end{cases} \\ \frac{4m(1+m)\pi + (1+2m)\sin(4m\pi)}{8m(1+m)}, & k = 1+m \Rightarrow \begin{cases} \pi, m = 0 \\ \frac{\pi}{2}, m > 0 \end{cases} \\ \frac{4(-1+m)m\pi + (-1+2m)\sin(4m\pi)}{8(-1+m)m}, & k = 1-m \Rightarrow \begin{cases} \pi, m = 0 \\ \pi, m = 1 \\ \frac{\pi}{2}, m > 1 \text{ not in our case } (k \geq 0) \end{cases} \end{cases}$$

$\bar{k} > 1$:

$$\int_0^{2\pi} \cos(k * \theta) * \cos(m * \theta) * \cos(\bar{k} * \theta) d\theta =$$

$$\frac{1}{4} \left(\frac{\sin[2(k-\bar{k}-m)\pi]}{k-\bar{k}-m} + \frac{\sin[2(k+\bar{k}-m)\pi]}{k+\bar{k}-m} + \frac{\sin[2(k-\bar{k}+m)\pi]}{k-\bar{k}+m} + \frac{\sin[2(k+\bar{k}+m)\pi]}{k+\bar{k}+m} \right) =$$

$$\frac{1}{4} \left(\frac{\sin[2(k-\bar{k}-m)\pi]}{k-\bar{k}-m} + \frac{\sin[2(k+\bar{k}-m)\pi]}{k+\bar{k}-m} + \frac{\sin[2(k-\bar{k}+m)\pi]}{k-\bar{k}+m} \right) =$$

$$0, k \neq -\bar{k}+m \text{ \& } k \neq \bar{k}+m \text{ \& } k \neq \bar{k}-m$$

$$= \begin{cases} \frac{1}{8} \left(4\pi + \frac{\sin(4\bar{k}\pi)}{\bar{k}} + \frac{\sin[4(\bar{k}-m)\pi]}{\bar{k}-m} + \frac{\sin(4m\pi)}{m} \right), & k = -\bar{k}+m \Rightarrow \begin{cases} \pi + \frac{\sin(4\bar{k}\pi)}{4\bar{k}} = \pi, m = 0 \\ \pi + \frac{\sin(4\bar{k}\pi)}{4\bar{k}} = \pi, m = \bar{k} \\ \frac{\pi}{2}, m \neq 0 \text{ and } m \neq \bar{k} \end{cases} \\ \frac{1}{8} \left(4\pi + \frac{\sin(4\bar{k}\pi)}{\bar{k}} + \frac{\sin(4m\pi)}{m} + \frac{\sin[4(\bar{k}+m)\pi]}{\bar{k}+m} \right), & k = \bar{k}+m \Rightarrow \begin{cases} \pi + \frac{\sin(4\bar{k}\pi)}{4\bar{k}} = \pi, m = 0 \\ \frac{\pi}{2}, m > 0 \end{cases} \\ \frac{1}{8} \left(4\pi + \frac{\sin(4\bar{k}\pi)}{\bar{k}} + \frac{\sin[4(\bar{k}-m)\pi]}{\bar{k}-m} + \frac{\sin(4m\pi)}{m} \right), & k = \bar{k}-m \Rightarrow \begin{cases} \pi + \frac{\sin(4\bar{k}\pi)}{4\bar{k}} = \pi, m = 0 \\ \pi + \frac{\sin(4\bar{k}\pi)}{4\bar{k}} = \pi, m = \bar{k} \\ \frac{\pi}{2}, m \neq 0 \text{ and } m \neq \bar{k} \end{cases} \end{cases}$$

Multiplication between Cosines and Sines

$\bar{k} = 1$:

$$\begin{aligned} & \int_0^{2*\pi} \sin(k * \theta) * \cos(m * \theta) * \sin(\theta) d\theta = \\ & \frac{1}{4} \left(-\frac{\sin[2(1+k-m)\pi]}{1+k-m} + \frac{\sin[2(1-k+m)\pi]}{1-k+m} + \frac{\sin[2(-1+k+m)\pi]}{-1+k+m} - \frac{\sin[2(1+k+m)\pi]}{1+k+m} \right) = \\ & \frac{1}{4} \left(-\frac{\sin[2(1+k-m)\pi]}{1+k-m} + \frac{\sin[2(1-k+m)\pi]}{1-k+m} + \frac{\sin[2(-1+k+m)\pi]}{-1+k+m} \right) = \\ & = \begin{cases} 0, & k \neq -1+m \text{ \& } k \neq 1+m \text{ \& } k \neq 1-m \text{ \& } k \neq -1-m \text{ (this is always the case) } (k \geq 0) \\ \frac{-4(-1+m)m\pi + \sin(4m\pi)}{8(-1+m)m}, & k = -1+m \Rightarrow \begin{cases} -\pi, m=0 \text{ not in our case } (k \geq 0) \\ 0, m=1 \\ -\frac{\pi}{2}, m>1 \end{cases} \\ \frac{4m(1+m)\pi + \sin(4m\pi)}{8m(1+m)}, & k = 1+m \Rightarrow \begin{cases} \pi, m=0 \\ \frac{\pi}{2}, m>0 \end{cases} \\ -\frac{-4(-1+m)m\pi + \sin(4m\pi)}{8(-1+m)m}, & k = 1-m \Rightarrow \begin{cases} \pi, m=0 \\ 0, m=1 \\ \frac{\pi}{2}, m>1 \text{ not in our case } (k \geq 0) \end{cases} \end{cases} \end{aligned}$$

$\bar{k} > 1$:

$$\begin{aligned} & \int_0^{2*\pi} \sin(k * \theta) * \cos(m * \theta) * \sin(\bar{k} * \theta) d\theta = \\ & \frac{1}{4} \left(\frac{\sin[2(k-\bar{k}-m)\pi]}{k-\bar{k}-m} - \frac{\sin[2(k+\bar{k}-m)\pi]}{k+\bar{k}-m} + \frac{\sin[2(k-\bar{k}+m)\pi]}{k-\bar{k}+m} - \frac{\sin[2(k+\bar{k}+m)\pi]}{k+\bar{k}+m} \right) = \\ & \frac{1}{4} \left(\frac{\sin[2(k-\bar{k}-m)\pi]}{k-\bar{k}-m} - \frac{\sin[2(k+\bar{k}-m)\pi]}{k+\bar{k}-m} + \frac{\sin[2(k-\bar{k}+m)\pi]}{k-\bar{k}+m} \right) = \\ & = \begin{cases} 0, & k \neq -\bar{k}+m \text{ \& } k \neq \bar{k}+m \text{ \& } k \neq \bar{k}-m \text{ \& } k \neq -1-m \text{ (this is always the case) } (k \geq 0) \\ \frac{1}{8} \left(-4\pi + \frac{\sin(4\bar{k}\pi)}{\bar{k}} + \frac{\sin[4(\bar{k}-m)\pi]}{\bar{k}-m} - \frac{\sin(4m\pi)}{m} \right), & k = -\bar{k}+m \Rightarrow \begin{cases} -\pi + \frac{\sin(4\bar{k}\pi)}{4\bar{k}} = -\pi, m=0 \\ 0, m=\bar{k} \\ -\frac{\pi}{2}, m \neq 0 \text{ and } m \neq \bar{k} \end{cases} \\ \frac{1}{8} \left(4\pi - \frac{\sin(4\bar{k}\pi)}{\bar{k}} + \frac{\sin(4m\pi)}{m} - \frac{\sin[4(\bar{k}+m)\pi]}{\bar{k}+m} \right), & k = \bar{k}+m \Rightarrow \begin{cases} \pi - \frac{\sin(4\bar{k}\pi)}{4\bar{k}} = \pi, m=0 \\ \frac{\pi}{2}, m>0 \end{cases} \\ \frac{1}{8} \left(4\pi - \frac{\sin(4\bar{k}\pi)}{\bar{k}} - \frac{\sin[4(\bar{k}-m)\pi]}{\bar{k}-m} + \frac{\sin(4m\pi)}{m} \right), & k = \bar{k}-m \Rightarrow \begin{cases} \pi - \frac{\sin(4\bar{k}\pi)}{4\bar{k}} = \pi, m=0 \\ 0, m=\bar{k} \\ \frac{\pi}{2}, m \neq 0 \text{ and } m \neq \bar{k} \end{cases} \end{cases} \end{aligned}$$

Appendix D

%% Data

EI = ...

q = ...

L = ...

segments = ...

dx = L / segments

% 2nd order Equation of Beam

%% 1st - Top Segment

*A_{matrix} = sym(zeros(4 * segments))*

syms w(x) P C1 C2 C3 C4

*ode1 = EI * diff(w(x), x, 4) + (P - q * (dx/2)) * diff(w(x), x, 2) - q * diff(w(x), x) == 0*

wSol_j1(x) = dsolve(ode1)

Constants1 = symvar(wSol_j1);

Constants1(Constants1 == P) = [];

Constants1(Constants1 == x) = [];

% Boundary Conditions

bc1 = subs(diff(wSol_j1, x, 2), 0) == 0;

*bc2 = EI * subs(diff(wSol_j1, x, 3), 0) + P * subs(diff(wSol_j1, x), 0) == 0;*

bcs = [bc1; bc2];

% Matrix Form

A = equationsToMatrix(bcs, Constants1);

Afinal(1:2, 1:4) = A;

%% Middle Segments

i = 0;

k = 0;

for x0 = dx + dx/2 : dx : L - dx/2

syms w(x) P C1 C2 C3 C4

*ode2 = EI * diff(w(x), x, 4) + (P - q * x0) * diff(w(x), x, 2) - q * diff(w(x), x) == 0*

wSol_j2(x) = dsolve(ode2);

i = i + 4;

*Cnew = [sym(sprintf('C%d', i + 1)), sym(sprintf('C%d', i + 2)), ...
sym(sym(sprintf('C%d', i + 3)), sprintf('C%d', i + 4))];*

wSol_j2(x) = subs(wSol_j2(x), [C1, C2, C3, C4], Cnew);

% Grab unknown constants C

Constants2 = symvar(wSol_j2);

Constants2(Constants2 == P) = [];

Constants2(Constants2 == x) = [];

ConstantsTotal = [Constants1 Constants2];

% Matching Boundary Conditions

matchingbc1 = subs(wSol_j1, x0 - dx/2) - subs(wSol_j2, x0 - dx/2) == 0;

matchingbc2 = subs(diff(wSol_j1, x), x0 - dx/2) - subs(diff(wSol_j2, x), x0 - dx/2) == 0;

matchingbc3 = subs(diff(wSol_j1, x, 2), x0 - dx/2) - subs(diff(wSol_j2, x, 2), x0 - dx/2) == 0;

*matchingbc4 = EI * subs(diff(wSol_j1, x, 3), x0 - dx/2) - q * (x0 - dx) * subs(diff(wSol_n1, x), x0 - dx/2) ...
- EI * subs(diff(wSol_j2, x, 3), x0 - dx/2) + q * x0 * subs(diff(wSol_n2, x), x0 - dx/2) == 0;*

matchingbcs = [matchingbc1; matchingbc2; matchingbc3; matchingbc4];

% Matrix Form

A = equationsToMatrix(matchingbcs, ConstantsTotal);

Afinal(3 + k: 6 + k, 1 + k: 8 + k) = A;

k = k + 4;

wSol_j1(x) = wSol_j2;

```

    Constants1 = Constants2;
end
%%% Last segment
bc3 = subs(wSol_j2,L) == 0;
bc4 = subs(diff(wSol_j2,x),L) == 0;
bcs = [bc3; bc4];
% Matrix Form
A = equationsToMatrix(bcs,Constants2);
Afinal(4 * segments - 1:4 * segments,4 * segments - 3:4 * segments) = A;
%%% Initial Approximation
x0 = L/2;
ode_whole = EI * diff(w,x,4) + (P - q * x0) * diff(w(x),x,2) - q * diff(w(x),x) == 0;
wSol_nosegments(x) = dsolve(ode_whole);
% Grab Unknown Constants
C = symvar(wSol_nosegments);
C(C == P) = [];
C(C == x) = [];
% Boundary Conditions
bc1 = subs(diff(wSol_nosegments,x,2),0) == 0;
bc2 = EI * subs(diff(wSol_nosegments,x,3),0) + P * subs(diff(wSol_nosegments,x),0) == 0;
bc3 = subs(wSol_nosegments,L) == 0;
bc4 = subs(diff(wSol_nosegments,x),L) == 0;
bcs_nosegments = [bc1; bc2; bc3; bc4];
% Matrix Form
A_nosegments = equationsToMatrix(bcs_nosegments,C);
Buckling_equation_approximation = det(A_nosegments) == 0;
Buckling_load_approximation = vpasolve(Buckling_equation_approximation,P)
% or sometimes it is needed to provide a certain range
%Buckling_load_approximation = vpasolve(Buckling_equation_approximation,P,[low limit top limit])
%%% Calculation of buckling load
Buckling_equation = det(Afinal) == 0;
Buckling_load = vpasolve(Buckling_equation,P,Buckling_load_approximation )

% or by

% DeterminantAfinal = det(Afinal);
% z = 0;
% for force = low limit : step : top limit % a low and a top limit must be provided based on the approximated load
%     z = z + 1;
%     determinant(z) = real(double(subs(DeterminantAfinal ,P,force)));
%     Load(z) = force;
% end
% plot(Load,determinant)

```



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