Christiaan Huygens' Planetarium

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WI3606, December 12, 2008

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Introduction

In this project you will read how Christiaan Huygens used continued fractions to build a Planetarium. Continued fractions are rational approximation for (ir)rational numbers. Continued fractions resembles the Pythagoras theorem, as neither the discoverer nor their origin is known. Examples of continued fractions can be found throughout mathematics over the last 2000 years. Nevertheless, we know that Christiaan was the first to demonstrate a practical application for them.

In Chapter 1 we will discuss the history of Continued fractions and their development over the years. You will also find that there were many notations used for continued fractions. Furthermore you will see that the biggest development took place in the last two centuries when numerous mathematicians such as Euler started working on the theory. All the information used in Chapter 1 can be found in [B]. If the reader is interested in the history of continued fractions, [B] is highly recommended.

In Chapter 2 we will work with continued fractions by introducing some examples. We will also explain what the maximum error is for continued fractions by initially working with the matrices. There are many theories involving continued fractions that will not be discussed such as complex and negative continued fractions. This is due to the fact that we will be solely using continued fractions for the Planetarium. Many of applications concerning continued fractions and the information used for Chapter 2 can be found in [L].

In Chapter 3 you will find a small biography of Chirstiaan Huygens - the full biography is available in book [A], where all the details are obtained from Chapter 3.

The last chapter, Chapter 4, describes the Planetarium of Christiaan Huygens. Bear in mind, that all the data for Chapter 4 is collected from the "Opuscula Postuma" script of 1703 - which can be found in [E]. Therefore much of the data used for the Planetarium by Christiaan Huygens is old and some of it is even wrong; for example the sizes of the planets. However, in Chapter 4 we will see that those factors will not play a significant role in the mechanics of the Planetarium. Also, most of the figures apart from Figure 7 in Chapter 4 come from the same script, hence the quality is not that good.

1 History of Continued Fractions

1.1 The beginning

It is not known who discovered continued fractions or where they originate from. A fact is that continued fractions are old. We can find examples of continued fractions throughout mathematics over the last 2000 years. Nevertheless the genuine discovery occurred between the 16th century and beginning of the 17th century. Many believe continued fractions started with the algorithm of Euclid. This algorithm determines the greatest common devisor of two integers. It is quite easy to show that the calculation of gcd of p and q is the same as the calculation of the continued fraction of $\frac{\mathrm{p}}{\mathrm{q}}$.

For more than 1000 years long continued fractions were only used as examples, because their effectiveness was much debated. The Indian mathematician, Aryabhata (500A.D), used continued fractions to solve linear Diophantine equations, named after the Greek mathematician Diophantus(250 A.D) Aryabhata wrote several books, the most recent interpretations show that he knew Euclid's gcd process and the recursive method for computing the convergents of the continued fraction. Say $\frac{\mathrm{p}_n}{\mathrm{q}_n}$ is the nth convergent of the continued fraction of x and $\frac{\mathrm{p}_{n-1}}{\mathrm{q}_{n-1}}$ the penultimate convergent, then the following holds

$$\mathrm{p}_{n-1}\mathrm{q}_n - \mathrm{p}_n\mathrm{q}_{n-1} = (-1)^n, \tag{1}$$

the proof for this will be given in Chapter 2. Aryabhata didn't generalise the method but only used it as a model for Diophantine equations

$$\mathrm{a}y + \mathrm{c} = \mathrm{b}z. \tag{2}$$

Many believe that not Aryabhata, but Bhascara II, who was born in Bijapur (in the present Mysore State) in 1115, was one of the most important Indian mathematicians. He wrote a book "Lilavati", named after one of his daughters, which means "beautiful", containing 13 chapters. In the book he treats Diophantine equations and proves that the solution can be obtained from the development of x into the continued fraction

$$x = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \frac{1}{a_n}}}}. \tag{3}$$

We are going to make some changes in (2). Take c $= x$, a $= \mathrm{p}_n$ and b $= \mathrm{q}_n$. Bhascara showed that the convergents of the continued fraction of $x = \frac{\mathsf{p}_n}{\mathsf{q}_n}$, with $\mathrm{p}_{-1} = 1,\ \mathrm{p}_0 = a_0,\ \mathrm{q}_{-1} = 0$ and $\mathrm{q}_0 = 1$ for $\mathrm{n} \geq 1$ equals

$$\mathsf{p}_n = \mathsf{q}_n\mathsf{p}_{n-1} + \mathsf{p}_{n-2},$$
$$\mathsf{q}_n = \mathsf{q}_n\mathsf{q}_{n-1} + \mathsf{q}_{n-2},$$
$$\mathsf{p}_n\mathsf{q}_{n-1} + \mathsf{p}_{n-1}\mathsf{q}_n = (-1)^{n-1},$$

and the solution is given by

$$y = \mp\, x\mathsf{p}_{n-1} + \mathsf{q}_n t,$$

$$z = \mp\, x\mathsf{p}_{n-1} + \mathsf{p}_n t,$$

eventually you get $\mathrm{p}_{n-1}\mathrm{q}_n - \mathrm{p}_n\mathrm{q}_{n-1} = \pm 1$ which is equal to (1).

Some say (see e.g.[B]) that the above mentioned recurrence relation of continued fractions was discovered in Europe by John Wallis, in 1655, which is 500 years later. The first English translation of Bhascara II's work appears in 1816 (by J.Taylor, from Bombay). Two men from Bologna (Italy), Rafeal Bombelli (1526-1572) and Pietro Cataldi (1548-1626), have given many examples of contiunued fractions. For instance, Bombelli expressed the root of 13 as a repeating continued fraction, whereas Cataldi did the same for the root of 18. However, neither progressed further with the generalization of continued fractions.

1.2 Notations

The preceding notation (6), which is the most commonly used today, was introduced in 1898 by Alfred Pringsheim, but before, many other notations were used.

Cataldi used in 1613 the notation

$$\sqrt{18} = 4.\&\frac{2}{8}.\&\frac{2}{8}.\&\frac{2}{8}.\&\frac{2}{8}.\&\frac{2}{8}.\&\frac{2}{8}.$$

John Wallis, continued with continued fractions. In his book "*Arithematica Infinitorium* (1655)", he displayed $\frac{\pi}{4}$ in this form:

$$x = \frac{\mathsf{a}}{\alpha}\frac{\mathsf{b}}{\beta}\frac{\mathsf{c}}{\gamma}\frac{\mathsf{d}}{\delta}\frac{\mathsf{e}}{\epsilon},$$

$$\frac{\pi}{4} = \frac{3}{2}\frac{3}{4}\frac{5}{4}\frac{5}{6}\frac{7}{6}.$$

Not did he only use this notation for his collected works of 1695 but also in a letter to Gottfried Wilhelm Leibniz (1646-1716), dated April 6, 1697.

Leibniz in a letter to Johan Bernoulli (1667-1748) dated December 28, 1696, used the following notation,

$$x = a_0 + \frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3}.$$

Christiaan Huygens (1629-1695) wrote in a posthumous text, published in 1698,

$$\frac{77708431}{2640858} = 29 + \frac{1}{2} + \frac{1}{2} + \frac{1}{1} + \frac{1}{5} + \frac{1}{1} + \frac{1}{4} \text{ etc.}$$

This notation of Leibniz and Huygens was adopted by many of the eighteenth century writers and nineteenth century authors, such as Joseph Alfred Serret (1819-1885).

In 1748, Leonhard Euler (1707-1783) used Leibniz and Huygens notation, but he found that it was space consuming and therefore, in 1762, he eventually represented the finite continued fractions of formula 3 by the notation

$$x = \frac{(a_0, a_1, a_2, \ldots, a_n)}{(a_1, a_2, \ldots, a_n)}.$$

The first president of the Royal Society, Lord Brouncker (1620-1684), showed that $\frac{4}{\pi}$ can be represented in the following form

$$\frac{4}{\pi} = 1 + \cfrac{1^2}{2 + \cfrac{3^2}{2 + \cfrac{5^2}{2 + \cfrac{7^2}{\ddots}}}}.$$

1.3 The Golden era

Brounker did not trouble himself to go further, whereas as Wallis did – his discovery prompted the first step in the generalization of continued fractions.

In his book "*Opera Mathematica* (1695)" he illustrated simple examples of continued fractions. In the book he explained how to compute the nth convergent, and with this, he discovered some well-known properties of the convergents of continued fractions that are still used today. Also, it was in his book that the name continued fractions was first used.

Whilst Wallis and Huygens studied the properties of continued fractions, Leonard Euler (1707-1783), Johan Heinrich Lambert (1728-1777) and Joseph Louis Lagrange (1736-1783) made a significant breakthrough in the generalization of continued fractions. Many of the modern theories concerning continued fractions are originated from the book "*De Fractionlous Continious* (1737)" written by Euler. He showed that a number is rational if and only if it can expressed as a simple finite continued fraction. He also expressed the base of the natural logarithm e ($\mathrm{e} \approx 2.718$) in the form continued fractions, which allowed him to prove that e^2 was in fact irrational. Euler also demonstrated how to go from a series to a continued fraction representation of the series, and conversely.

Lambert generalised Eulers work by showing that e^x and $\tan x$ are irrational if x is rational. He also proved that all the rational numbers have irrational natural logarithms, that the continued fraction for $\tan x$ converges and ended his work with the conjecture that "*no circular or logarithmic transcendental quantity into which no other transcendental quantity enters can be expressed by any irrational radical quantity.*" This conjecture was proved one century later in 1873 by Charles Hermite.

Lagrange also used continued fraction to calculate irrational roots. He used the resulting proof to show that the real root of a quadratic irrational is a periodic continued fraction, which we can refer to as Theorem 1.

Theorem 1 *Given an (eventually) periodic continued fraction,*

$$a_0 + [\mathbf{a}, \bar{\mathbf{b}}] = a_0 + [\mathbf{a,b,b,b,\ldots}] = [a_0; a_1, a_2, \ldots, b_1, \ldots, b_t, b_1, \ldots, b_t, \ldots],$$

the corresponding real number $\alpha = \alpha(\mathbf{a,b})$ *is a quadratic irrational* $\alpha = u + v\sqrt{w}$ *where u and v are rational and w is square-free and conversely.*

In his book, "*History of Continued Fractions and Padre Approximations*", Claude Brezinski states: "*the nineteenth century can be said to be popular period for continued fractions*", which describes the 19th century as the golden era for continued fractions. It was the time "*the subject was known to every mathematician.*" Due to this there was extensive growth in the use of continued fractions, especially the convergents. Continued fractions were also used for complex numbers. Many mathematicians worked in this field and produced results include Hermite, Gauss, Cauchy, Stieljes, Jacobi and Perron,.

At the start of the 20th century an old discovery of Gauss tied the theory of continued fractions to that of probability theory and ergodic theory. In 1802 and 1812 Guass found the invariant measure of the transformation underlying the regular continued fraction, and asked Lagrange in a letter in 1812 how fast

$$\lambda\left(T^{-n}\left([0,x]\right)\right) \tag{4}$$

converges to the invariant measure $\mu\left([0,x]\right|$. Here λ is the Lebesgue measure, and T is the map on [0,1) to [0,1) which will be introduced in (7). Gauss' question was answered independently in 1928 by Kuzmin, in 1929 by Paul Lévy. This "Guass-Kuzmin-Lévy Theorem" lead to many new results; see also [C] for more results and a more detailed description.

This was a brief introduction into the history and development of continued fractions, which have existed for centuries. Implementation of continued fractions faced some delay, but when the experts did use it many inventions followed by various mathematicians. The fact that continued fractions is still an integral part of today's math world shows that it still has a significant role to play in the future.

2 An introduction of Continued Fractions

2.1 The abstract behind Continued Fractions

As explained in Chapter 1 there is no genuine inventor of continued fractions. But what are continued fractions? What is their main purpose? Why do we use it for a mechanical Planetarium? All of this will be followed up in this chapter.

The theory of continued fractions arises from considerations of expressions given in the following form

$$x = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_n}}}}, \tag{5}$$

hence the name continued fractions, note that x is here taken to be a finite continued fraction.

The quantities $a_0, a_1, a_2, \dots, a_n$ are called the partial quotients and may be taken to be integers, real numbers, complex numbers, or functions of such a variables. Furthermore, complicated continued fractions may be considered in which the numerators are no longer positive.

The expression (5) is the commonly used form, you can also rewrite this in more compact form

$$x = [a_0; a_1, a_2, a_3, a_4, \cdots, a_n]. \tag{6}$$

Consider any (ir)rational number $x \in [0,1)$. We can now take the following transformation

$$T(x) = \frac{1}{x} - \left\lfloor \frac{1}{x} \right\rfloor \quad \text{with } x \neq 0;\ T(0) = 0. \tag{7}$$

The expression (5) can be changed into the form of a rational function of the partial quotients. If we begin by considering just the first few terms, we may write

$$a_1 = \left\lfloor \frac{1}{x} \right\rfloor,\ a_2 = \left\lfloor \frac{1}{T(x)} \right\rfloor,\ a_3 = \left\lfloor \frac{1}{T^2(x)} \right\rfloor = \left\lfloor \frac{1}{T(T(x))} \right\rfloor,\ \dots$$

and so on. Notice that if x is an irrational number so is $T(x)$, $T^2(x)$ since $T^2(x) = T(T(x))$ and hence $T^n(x)$ for any $n \in \mathbb{N}$. Due to Euclid's algorithm we have that if x is a rational number, eventually there is a n for which $T^n(x) = 0$.

We may rewrite $T(x)$ as follows

$$T(x) = \frac{1}{x} - a_1 \quad \Leftrightarrow \quad \frac{1}{x} = a_1 + T(x) \quad \Leftrightarrow \quad x = \frac{1}{a_1 + T(x)},$$

and for $T^2(x)$ we get,

$$T^2(x) = T(T(x)) = \frac{1}{T(x)} - \left\lfloor \frac{1}{T(x)} \right\rfloor = \frac{1}{T(x)} - a_2 \Leftrightarrow \frac{1}{T(x)} = a_2 + T^2(x) \Leftrightarrow T(x) = \frac{1}{a_2 + T^2(x)},$$

which leads to

$$x = \cfrac{1}{a_1 + \cfrac{1}{a_2 + T^2(x)}} = \cdots = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_n + T^n(x)}}}}.$$

2.2 Examples of Continued Fractions

Let us now consider some examples. Take for instance $x = \sqrt{2} - 1 \approx 0.4142$. We can now say

$$\frac{1}{x} = \frac{1}{\sqrt{2}-1} = \frac{1}{\sqrt{2}-1}\frac{\sqrt{2}+1}{\sqrt{2}+1} = \frac{\sqrt{2}+1}{2-1} = \sqrt{2}+1 \approx 2.4$$

now we can compute a_1

$$a_1 = \left\lfloor \frac{1}{x} \right\rfloor = 2 \implies T(x) = \frac{1}{x} - \left\lfloor \frac{1}{x} \right\rfloor = (\sqrt{2}+1) - 2 = \sqrt{2} - 1 = x.$$

We immediately get that $a_1 = a_2 = a_3 = \cdots = 2$, which will give us

$$x = \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{\ddots}}}}}. \tag{8}$$

We can use (6) and write $x = [0; 2, 2, 2, 2, \cdots]$. Because x is periodic we abbreviate this by $x = [0; \bar{2}]$, where the bar indicates the period.

In our second example we take $x = \mathrm{e} - 2 \approx 0.718281828$

$$\begin{aligned}
a_1 &= \left\lfloor \frac{1}{x} \right\rfloor = 1 \quad \implies \quad T(x) = \frac{1}{x} - \left\lfloor \frac{1}{x} \right\rfloor \approx 0.392211191 \\
a_2 &= \left\lfloor \frac{1}{T(x)} \right\rfloor = 2 \quad \implies \quad T^2(x) = \frac{1}{T(x)} - \left\lfloor \frac{1}{T(x)} \right\rfloor \approx 0.549646778 \\
a_3 &= \left\lfloor \frac{1}{T^2(x)} \right\rfloor = 1.
\end{aligned}$$

If you continue computing the continued fractions of x you will eventually get

$$x = \mathrm{e} - 2 = [0; 1, 2, 1, 1, 4, 1, 1, 6, 1, \cdots] = [0; \overline{1, 2\mathrm{k}, 1}]_{\mathrm{k} \geq 1}. \tag{9}$$

We now take the golden ratio as an example, $x = \varphi = \dfrac{1+\sqrt{5}}{2}$. We can use the following formula for the golden ratio,

$$\varphi = 1 + \frac{1}{\varphi}. \tag{10}$$

The continued fraction for φ will therefore be,

$$\varphi = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{\ddots}}}}}. \tag{11}$$

2.3 The margin of error

Let us define for $x \in [0,1)$ the matrices,

$$\mathbf{A_i} = \begin{pmatrix} 0 & 1 \\ 1 & a_i \end{pmatrix} \tag{12}$$

and

$$\mathbf{M_n} = \mathbf{A_1}\mathbf{A_2}\cdots\mathbf{A_n}, \text{ for n} = 1,2,3,\ldots \tag{13}$$

Furthermore, if a matrix $\mathbf{A} = \begin{pmatrix} \mathsf{a} & \mathsf{b} \\ \mathsf{c} & \mathsf{d} \end{pmatrix}$ with a,b,c,d $\in \mathbb{Z}$ and $\det(\mathbf{A}) = \pm 1 = \mathrm{ad}$ - bc, then define the Möbius transformation $\mathbf{A} : \mathbb{R} \cup \{\infty\} \longrightarrow \mathbb{R} \cup \{\infty\}$ by

$$\mathbf{A}(x) = \frac{\mathrm{a}x + \mathrm{b}}{\mathrm{c}x + \mathrm{d}}. \tag{14}$$

Notice that

$$\mathbf{A_n} = \begin{pmatrix} 0 & 1 \\ 1 & a_n \end{pmatrix}(0) = \frac{0 \cdot 0 + 1}{1 \cdot 0 + a_n} = \frac{1}{a_n}. \tag{15}$$

Now we compute $\mathbf{A_{n-1}}\mathbf{A_n}(0)$,

$$\mathbf{A_{n-1}}\mathbf{A_n}(0) = \begin{pmatrix} 0 & 1 \\ 1 & a_{n-1} \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & a_n \end{pmatrix}(0) = \begin{pmatrix} 0 & 1 \\ 1 & a_{n-1} \end{pmatrix}\left(\frac{1}{a_n}\right) = \frac{0 \cdot \frac{1}{a_n} + 1}{1 \cdot \frac{1}{a_n} + a_{n-1}} = \frac{1}{a_{n-1} + \cfrac{1}{a_n}}.$$

Repeating the same steps will lead to

$$\mathbf{M_n}(0) = \mathbf{A_1}\mathbf{A_2}\cdots\mathbf{A_{n-1}}\mathbf{A_n}(0) = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \frac{1}{a_n}}}}, \tag{16}$$

which is the continued fraction of x in (5).

Since $\mathbf{A_1}, \mathbf{A_2}, \cdots, \mathbf{A_{n-1}}, \mathbf{A_n}$ are matrices with only integers, hence $\mathbf{M_n}$ also contains only integers. We may write

$$\mathbf{M_n} = \begin{pmatrix} \mathrm{r}_n & \mathrm{p}_n \\ \mathrm{s}_n & \mathrm{q}_n \end{pmatrix} \tag{17}$$

therefore,

$$\mathbf{M_n}(0) = \begin{pmatrix} \mathrm{r}_n & \mathrm{p}_n \\ \mathrm{s}_n & \mathrm{q}_n \end{pmatrix}(0) = \frac{\mathrm{p}_n}{\mathrm{q}_n}.$$

Furthermore,

$$\mathbf{M_n} = \mathbf{A_1}\mathbf{A_2}\cdots\mathbf{A_{n-1}}\mathbf{A_n} = \mathbf{M_{n-1}}\mathbf{A_n} = \begin{pmatrix} \mathrm{r}_{n-1} & \mathrm{p}_{n-1} \\ \mathrm{s}_{n-1} & \mathrm{q}_{n-1} \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & a_n \end{pmatrix}.$$

When we work that out we get,

$$\mathbf{M_n} = \begin{pmatrix} \mathrm{r}_{n-1} & \mathrm{p}_{n-1} \\ \mathrm{s}_{n-1} & \mathrm{q}_{n-1} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & a_n \end{pmatrix} = \begin{pmatrix} \mathrm{p}_{n-1} & a_n\mathrm{p}_{n-1} + \mathrm{r}_{n-1} \\ \mathrm{q}_{n-1} & a_n\mathrm{q}_{n-1} + \mathrm{s}_{n-1} \end{pmatrix} = \begin{pmatrix} \mathrm{r}_n & \mathrm{p}_n \\ \mathrm{s}_n & \mathrm{q}_n \end{pmatrix}.$$

So eventually we have,

$$\mathbf{M_n} = \begin{pmatrix} \mathrm{r}_n & \mathrm{p}_n \\ \mathrm{s}_n & \mathrm{q}_n \end{pmatrix}, \qquad \text{with } \mathrm{r}_n = \mathrm{p}_{n-1} \text{ and } \mathrm{s}_n = \mathrm{q}_{n-1}. \tag{18}$$

Moreover, we find that

$$\mathrm{p}_{-1} = 1,\ \mathrm{p}_0 = 0,\ \mathrm{p}_n = \mathrm{a}_n\mathrm{p}_{n-1} + \mathrm{p}_{n-2} \quad \text{for n} \geq 1 \tag{19}$$

and

$$\mathrm{q}_{-1} = 0,\ \mathrm{q}_0 = 1,\ \mathrm{q}_n = \mathrm{a}_n\mathrm{q}_{n-1} + \mathrm{q}_{n-2} \quad \text{for n} \geq 1, \tag{20}$$

which are the recurrence relation we already mentioned in Chapter 1.

Lets go back to our example $x = \sqrt{2} - 1$, and put the data in a table

			$a_1 = 2$	$a_2 = 2$	$a_3 = 2$	$a_4 = 2$	$a_5 = 2$	$a_6 = 2$
p_n	1	0	1	2	5	12	29	70
q_n	0	1	2	5	12	29	70	169

The margin of error after six steps $\left|(\sqrt{2}-1) - \frac{70}{169}\right| = 0.0000123789$, as we take more steps the error decreases. We will now compute the maximum error for a given approximation $\frac{\mathrm{p}_n}{\mathrm{q}_n}$.

We know that the following holds for the determinant of a matrix. If $\mathbf{M_n} = \mathbf{A_1A_2}\cdots\mathbf{A_n}$ then $\det(\mathbf{M_n}) = \det(\mathbf{A_1A_2}\cdots\mathbf{A_{n-1}A_n}) = \det(\mathbf{A_1})\cdots\det(\mathbf{A_n})$.

We know that $\det(\mathbf{A}_i) = -1$ therefore we get,

$$\det(\mathbf{M_n}) = \det(\mathbf{A_1})\cdots\det(\mathbf{A_n}) = (-1)(-1)\cdots(-1) = (-1)^n. \tag{21}$$

We also know that

$$\mathbf{M_n} = \begin{pmatrix} \mathrm{p}_{n-1} & \mathrm{p}_n \\ \mathrm{q}_{n-1} & \mathrm{q}_n \end{pmatrix}, \tag{22}$$

which leads to

$$\det(\mathbf{M_n}) = \mathrm{p}_{n-1}\mathrm{q}_n - \mathrm{p}_n\mathrm{q}_{n-1} = (-1)^n. \tag{23}$$

A corollary is that gcd$(\mathrm{p}_n, \mathrm{q}_n) = 1$.

To compute the maximum error we replace $\mathbf{A_n}$ by $\tilde{\mathbf{A}}_\mathbf{n}$. Take

$$\tilde{\mathbf{A}}_\mathbf{n} = \begin{pmatrix} 0 & 1 \\ 1 & \mathrm{a}_n + T^n(x) \end{pmatrix} \tag{24}$$

and define a new matrix $\widetilde{\mathbf{M}}_\mathbf{n}$ by, $\widetilde{\mathbf{M}}_\mathbf{n} = \mathbf{A_1}\cdots\mathbf{A_{n-1}}\tilde{\mathbf{A}}_\mathbf{n} = \mathbf{M_{n-1}}\tilde{\mathbf{A}}_\mathbf{n}$. We will multiply out the matrices with our values in (19) and (20).

$$
\begin{aligned}
\widetilde{\mathbf{M}}_{\mathbf{n}} &= \mathbf{M}_{\mathbf{n-1}}\tilde{\mathbf{A}}_{\mathbf{n}} \\
&= \begin{pmatrix} \mathrm{p}_{n-2} & \mathrm{p}_{n-1} \\ \mathrm{q}_{n-2} & \mathrm{q}_{n-1} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & \mathrm{a}_n + T^2(x) \end{pmatrix} \\
&= \begin{pmatrix} \mathrm{p}_{n-1} & a_n \mathrm{p}_{n-1} + \mathrm{p}_{n-2} + \mathrm{p}_{n-1} T^n(x) \\ \mathrm{q}_{n-1} & a_n \mathrm{q}_{n-1} + \mathrm{q}_{n-2} + \mathrm{q}_{n-1} T^n(x) \end{pmatrix} \\
&= \begin{pmatrix} \mathrm{p}_{n-1} & \mathrm{p}_n + \mathrm{p}_{n-1} T^n(x) \\ \mathrm{q}_{n-1} & \mathrm{q}_n + \mathrm{q}_{n-1} T^n(x) \end{pmatrix}
\end{aligned}
$$

So,

$$
\widetilde{\mathbf{M}}_{\mathbf{n}}(0) = \frac{\mathrm{p}_n + \mathrm{p}_{n-1} T^n(x)}{\mathrm{q}_n + \mathrm{q}_{n-1} T^n(x)}.
$$

From (16) we then get,

$$
\begin{aligned}
\widetilde{\mathbf{M}}_{\mathbf{n}}(0) &= \mathbf{A_1} \cdots \mathbf{A_{n-1}} \tilde{\mathbf{A}}_{\mathbf{n}}(0) \\
&= \mathbf{A_1} \cdots \mathbf{A_{n-2}} \mathbf{A_{n-1}} \left(\frac{1}{a_n + T^n(x)} \right) \\
&= \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_n + T^n(x)}}}} \\
&= x,
\end{aligned}
$$

and we obtain that,

$$
x = \frac{\mathrm{p}_n + \mathrm{p}_{n-1} T^n(x)}{\mathrm{q}_n + \mathrm{q}_{n-1} T^n(x)}.
$$

The margin of error is given by,

$$
\begin{aligned}
\left| x - \frac{\mathrm{p}_n}{\mathrm{q}_n} \right| &= \left| \frac{\mathrm{p}_n + \mathrm{p}_{n-1} T^n(x)}{\mathrm{q}_n + \mathrm{q}_{n-1} T^n(x)} - \frac{\mathrm{p}_n}{\mathrm{q}_n} \right| \\
&= \left| \frac{\mathrm{p}_n \mathrm{q}_n + \mathrm{p}_{n-1} T^n(x) \mathrm{q}_n - \mathrm{p}_n \left(\mathrm{q}_n + \mathrm{q}_{n-1} T^n(x) \right)}{\mathrm{q}_n \left(\mathrm{q}_n + \mathrm{q}_{n-1} T^n(x) \right)} \right| \\
&= \frac{T^n(x) \left| \mathrm{p}_{n-1} \mathrm{q}_n - \mathrm{p}_n \mathrm{q}_{n-1} \right|}{\mathrm{q}_n^2 \left(1 + T^n(x) \frac{\mathrm{q}_{n-1}}{\mathrm{q}_n} \right)} \\
&< \frac{1}{\mathrm{q}_n^2}.
\end{aligned}
$$

3 Christiaan Huygens

Figure 1: Christiaan Huygens, portrait by C. Netscher, 1671; in the Collection Haags Gemeente museum, The Hague.

Christiaan Huygens was born on April 14, 1629. He was the son of Constantijn Huygens. Constantijn was known for his music and poetry. Christiaans mother, died when he was just 8 years of age.

A young 16 year old Christiaan gained admission to study Mathematics, Music Theory, Astronomy and Law at the University of Leiden. After two years he moved to finish his degree at an institution which was founded by his Father, the College of Oranje in Breda. Subsequently he embarked on a foreign diplomatic trip with Huygens senior and with the financial backing of his father, Christiaan decided to specialise in Mathematics and Physics.

It was during his college years that Chritiaan got in contact with the famous French scientist Mersenne. Christiaan and Mersenne were 'cyber' friends; they never physically meet. Communications happened through mail, and it often involved Christiaan having to solve mathematical problems. Mersenne died in the year 1648.

Christiaan is renowned as the Dutch physicist/astronomer who developed a wave theory of light and 31 tone to the octave keyboard instrument which made use of his discovery of 31 equal temperament. He also solved the majority of the mathematical problems of his days and started work on mathematical integrations and the fundamental mathematical basis for scalar diffraction theory.

Christiaan made many material inventions, such as pendulum clock, and also transformed telescopes which lead to the discovery of satellites and the ring of Saturn.

In the year 1666, France opened a scientific academy, l'Académie Royale des Sciences. Christiaan was invited to lead the organization by King Louis XIV. This enabled Huygens to spearhead Europe's science. Now many scientists appreciate him as one of the most prominent figures in the revolution of science.

In 1680, Christiaan began to develop a mechanical Planetarium for the Academie Royale des Sciences. He wrote a paper detailing the methods of using convergents of continued fractions to find the best rational approximations for gear ratios. These approximations allowed him to choose the gears with the correct number of teeth. During 1681 he fell seriously ill and returned home to Den Haag. It was in 1682 when he ultimately concluded this project. Eventually Johannes Ceulen received the honour of building the first mechanical Planetarium (see Figure 3), however because the equipment was a prototype technical issues soon surfaced.

Christiaan Huygens died on July 9, 1695. In 1792, Eise Eisinga built another mechanical Planetarium (see Figure 3), which is known as the oldest mechanical Planetarium in the world still functioning today.

Figure 2: The Planetarium of Eise Eisinga.

4 The description of the Planetarium

The description of the Planetarium by Christiaan Huygens is translated by Dr. J.A. Vollgraff and Dr. D.A.H. van Eck from the "Opuscula Postuma" script of 1703. In the time of Christiaan Huygens there were five planets known, the earth and not many moons (satellites).

4.1 The Planetarium

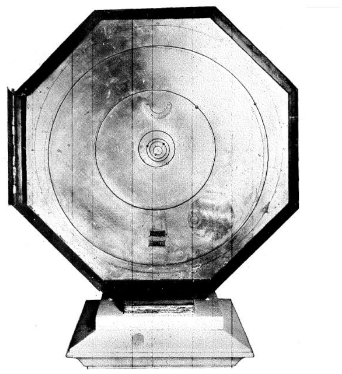

Figure 3: The Planetarium of Christiaan Huygens.

The Planetarium is an octagon made of wood with a diameter of two feet and a depth of six inches, which is hanging on a wall. On the left side you find a block, which is used to turn the apparatus and view the interior.

On the front you see a sheet of copper that forms the front of the octagon which is covered by a mirror glass. The large circles that you see represent the orbit of each respective planet, and the planets are represented by the small ball-like figures. The orbits of the planets on this sheet of copper are not according to the system of Copernicus but instead they follow the concept of Kepler. The copper sheet shows the course of the planet orbit, around that planet there is another orbit which indicates the course for the satellite. Hence, Saturn has five satellites, Jupiter has four, Earth one, which is our moon. There are also planets with no satellites such as Mars, but the illustration does show an orbits for those planets in order to improve their imagery. The Planetarium, along with movement of the planets around the sun, also takes into account the alternations in speed at which the planets complete their individual cycles. The Planetarium was build to accommodate the fact that it takes the moon 30 days to complete its cycle around the Earth. It was however not possible to include those respective details for Jupiter and Saturn, due to that fact that this Planetarium was to small and the manual labour to increase its size was simply too time consuming.

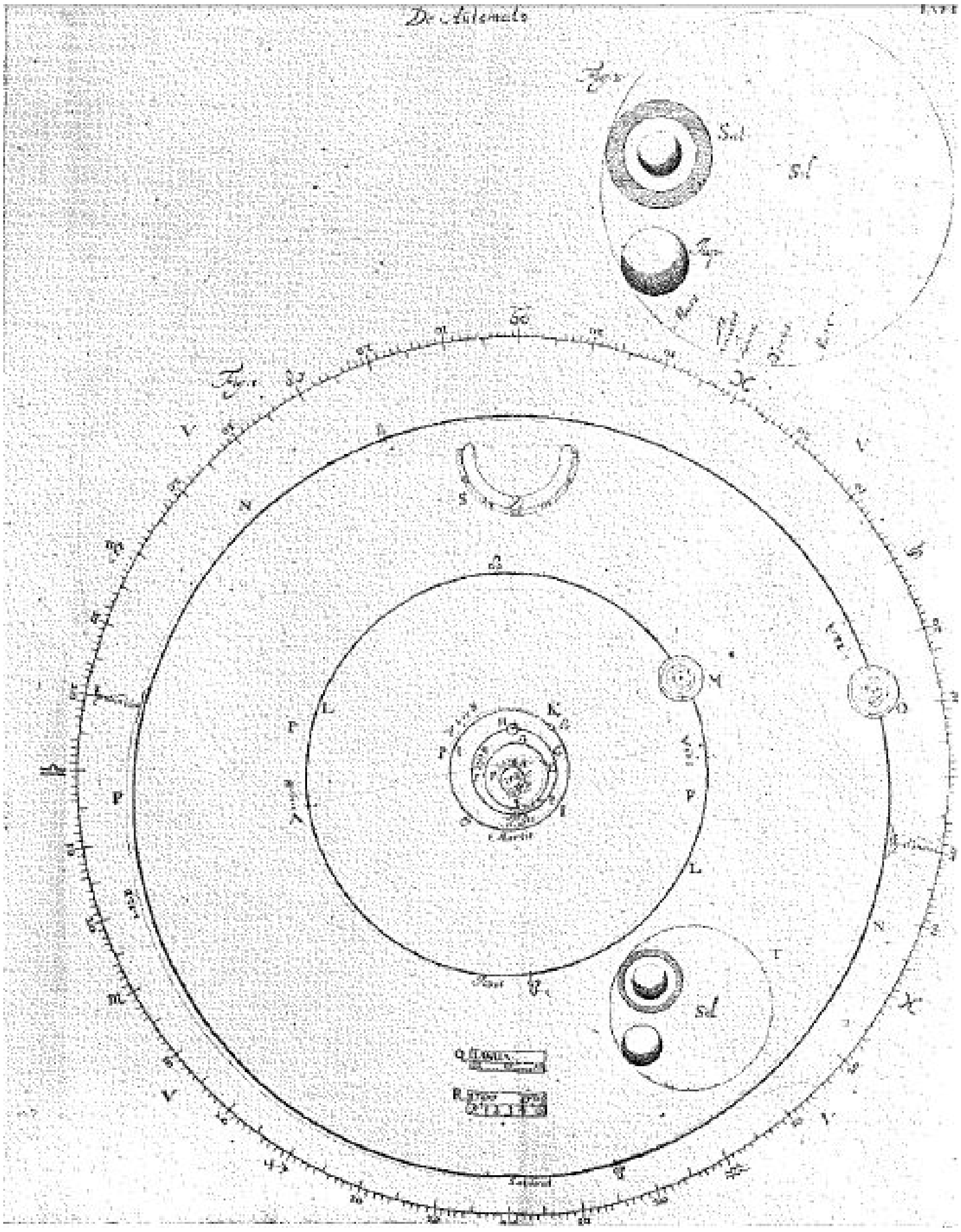

Figure 4: The Planetarium of Christiaan Huygens; front view with labels.

- VV the circle of the Ecliptic. This circle has been divided in 360 parts and in 12 parts to indicate the horoscopes.
- B the Sun.
- C the orbit of Mercury.
- D Mercury.
- E the orbit of Venus.
- F Venus.
- G the orbit of the Earth.
- H the Earth with the Moon, which orbits around her.
- I the orbit of Mars.
- K Mars.
- L the orbit of Jupiter.
- M Jupiter with his 4 satellites.
- N the orbit of Saturn.
- O Saturn with her 5 satellites.
- A.A. the positions of the Aphelia.
- $\wp$ ϑ This indicates the nodes for each planet.
- PP This indicates the cirkels of the planets.
- Q in this opening you can read the day and the month.
- R in this opening you can read the year.
- S in this opening you can read the time (in hour and minutes).
- T is a figure to show the sizes of the planets relative to each other and to the Sun.

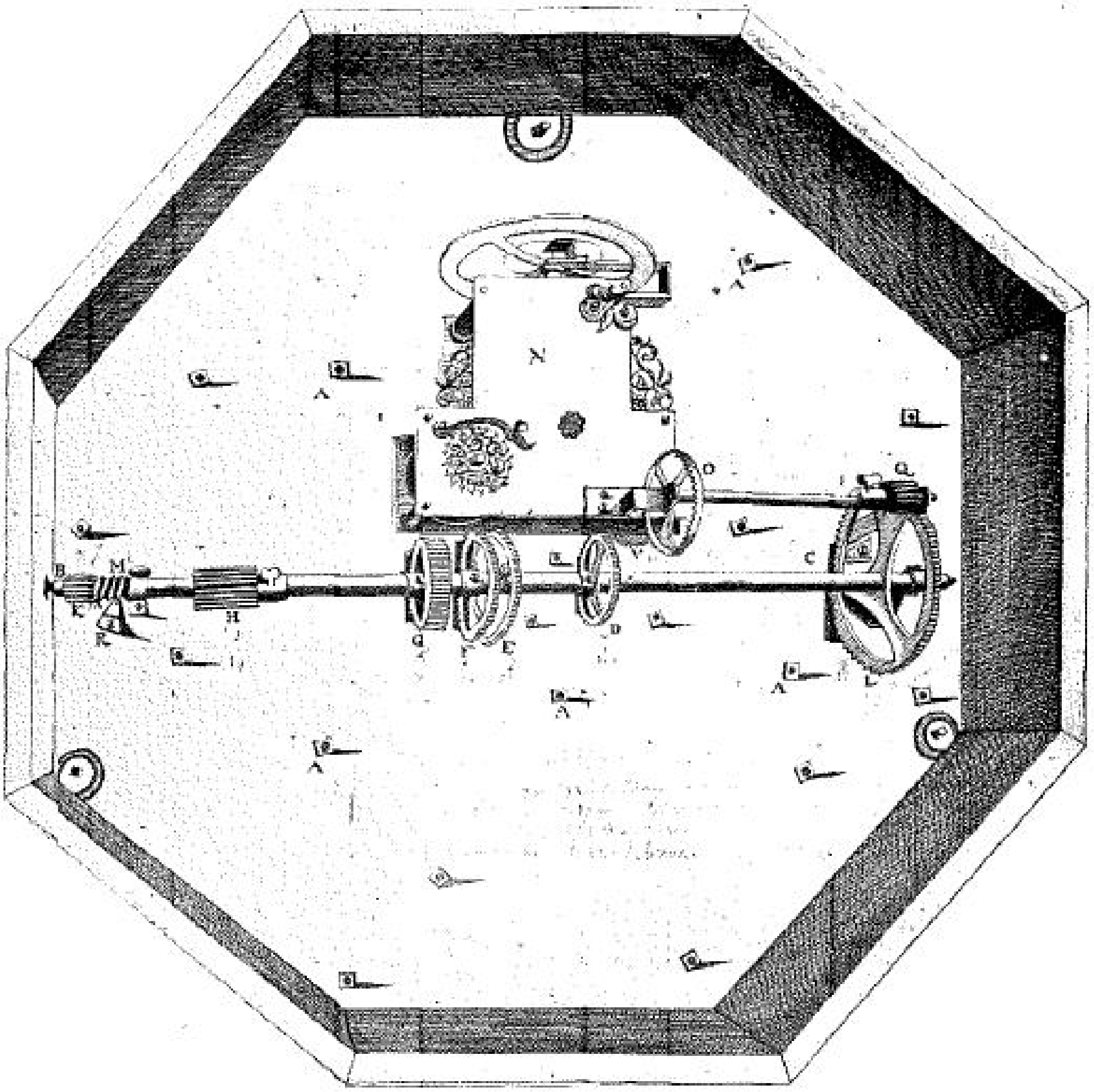

Figure 5: The Planetarium of Christiaan Huygens; inside view 1 with labels.

- A.A. are square bases, where the end of T.T (see Figure 6) are screwed into.
- CB is a two foot long iron rod.
- D is a gear with 121 teeth, that moves the gears of Mercury.
- E is a gear with 52 teeth, that moves the gears of Venus.
- F is a gear with 60 teeth, that moves the gears of Earth.
- G is a gear with 84 teeth, that moves the gears of Mars.
- H is a gear with 14 teeth, that moves the gears of Jupiter.
- K is a gear with 7 teeth, that moves the gears of Saturn.
- L is a gear with 73 teeth, that moves the circle that has the days and months printed on it.
- M is a piece of rigid metal spiral, that causes a cycle of 300 years with the aid of 2 gears (each with 6 teeth) attached to a common axis.
- N is the clock-mechanism.
- V is a gear that initiates the movement of the two foot long rod (CB). This gear turns once in every 96 hours.
- P are the 4 teeth of gear V.
- O is a gear with 45 teeth that is moved by teeth P.
- Q is gear attached to the axis of gear O with 9 teeth that moves gear L, which consequently turns CB.
- R is a copper plate, with a small opening that is intended for the gear in Figure 6.

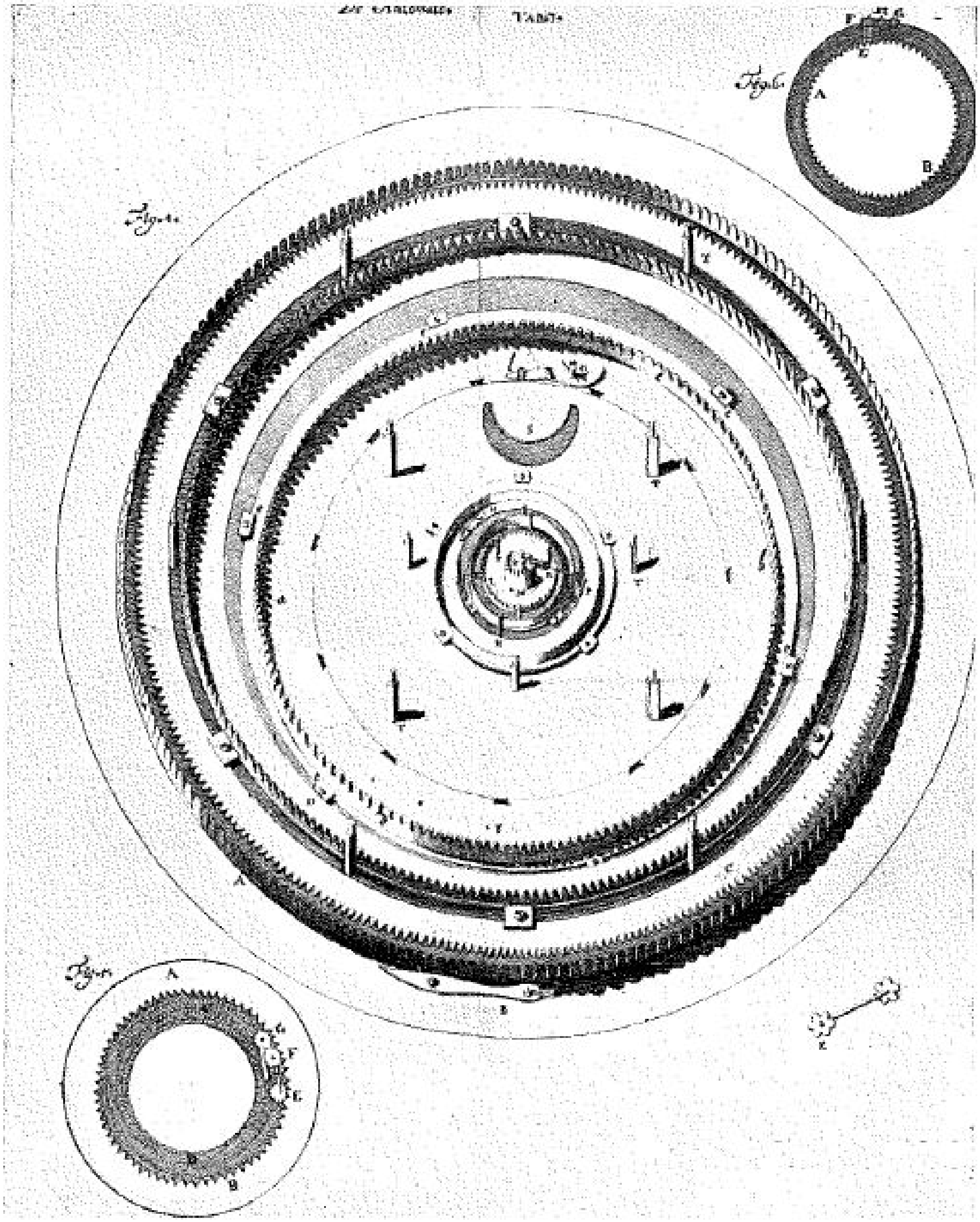

Figure 6: The Planetarium of Christiaan Huygens; inside view 2 with labels.

In Figure 6 you can see three different figures. All the distribution below refers to the middle figure.

- A is a gear of Saturn with 206 teeth.
- B is the "arm" where Saturn is attached to.
- C is the gear that has 300 years printed on it - consists 300 teeth. It turns with the help of the M, in Figure 5.
- D is the gear that has the months and days printed on it - consist of 219 teeth.
- F turns gear G.
- G is the "arm" that Jupiter is attached to.
- H is the gear of Mars, including the arm - consisting of 158 teeth.
- I is the gear of earth and the moon - consists of 60 teeth.
- K is a circle with 173 teeth, that is attached to the front plate of the Planetarium. Its task is to move the gears when the earth-gear turns.
- L is the gear of Venus with 32 teeth.
- M is the gear of Mercury with 17 teeth.
- N is a axis where Sun is attached to, which never moves.
- O is the axis of Mercury with two ends. At one end is where the Sun is positioned and at the other end you find P.
- P is a small pillar that is attached to the immovable plate of the earth.
- R is the gear of axis O with 7 teeth that moves mercury.
- Q is the other gear of axis O with 12 teeth.
- S is the opening where the plate that indicates time circulates.
- T.T.T.T are pillars that are securely screwed to the square bases AA (see Figure 5).
- a b – is a flat ring where a planet circulates on.
- c d – is a standing ring that consists of teeth.
- e e – are pins which control the flat ring during its course circulation.
- l m – is a arm.

On the Planetarium you also find the circle of Ecliptic. The Ecliptic represents our solar system, which has the Sun as the core, and inside the Ecliptic are all the planet orbits. In the lower half of the plate, between the orbits of Saturn and Jupiter there are two openings with a short distance within each other, each opening being two inches long and a half inch wide. In the upper opening you can read the month and in the other opening the year. The Earth has a cycle of 365 days and the Planetarium itself a cycle of 300 years. The Planetarium orbits per hour, the hours and minutes are given in the semi-circle shaped opening that is cut into the upper parts in between the orbits of Jupiter and Mars.

You can also operate the Planetarium manually, by twisting the knob at the right hand side. By twisting the knob clockwise the time on the Planetarium progresses one year due to the movement of the planets. As expected, twisting the knob anti-clockwise moves the time on the Planetarium a year backwards.

The Planetarium displays the different courses of orbits relative to each other. That was however not possible for the sizes of the planets and the Sun, because the Sun is several times larger than any of the planets. Nevertheless you can tell which planets are which due to their differences in size on the Planetarium; e.g. Jupiter is the biggest and Mercury is the smallest.

Christiaan Huygens deduced the sizes of the planets by analyzing them through a telescope. Uncertainty surrounded the size of the earth because he thought that it must have been bigger than Mars but smaller than Venus merely because the earth was positioned in between them. He couldn't confirm this because he had no means of looking at earth directly through a telescope as he was doing with the other planets. Ultimately he concluded that the Sun was 12000 earths and therefore 110 times the size of Earth. He then measured the lengths of the orbits using a more accurate method.

4.2 The mechanics of the Planetarium

If you flip the Planetarium and remove the lid you see Figure 6 and Figure 7. In Figure 6 you can see 6 rings, one assigned to each planet. These rings represent the orbits of each the planets. Each ring has a horizontal and a vertical side. We name the flat side the horizontal section, which has teeth attached to it. The other side, vertical section, has a pin attached to it with a small semi-ball (representing a planet), which you can see through the slit on the front side of the Planetarium.

The Planetarium is built in such a way that when the lid is closed the teeth of rings interlock with the gears of the lid (Figure 6 and Figure 7).

The plane of reference is the "ideal" orbit of a planet around the sun i.e. with no inclinations. To give an example - earth's equator. However, the planets do not orbit in such way. There is an inclination, which is the angle at which it does this relative to the plane of reference, given in Figure 8. Furthermore, we have two points where the orbit crosses the plane of reference which it is inclined to, we call them nodes. Making the planets circulate elliptically on the Planetarium was an easy thing for Christiaan Huygens to implement.

Now for the movement of the Moon, see the left bottom figure inside Figure 6. On the inside of the piece of the planet plate that lies in between the courses of Mars and Earth you find a ring attached that has 137 teeth in the inside of its edge AB. This ring has been secured by pins of CD. This ring carries a right angle axis that is secured by gear E at one end, and gear F at the other. The lowest gear has 12 teeth which are bonded to the teeth of ring AB. The highest gear has 13 teeth that are bonded to the 12 teeth of gear G. Gear G is also attached to CD and has a hole which has its opening at the front of the Planetarium where the Moon circle bar is attached to. You can see the small bar that connects gear G with CD and the other small bar that connects E and F. What this figure also does is to show the reader gears E, F and G relative to each other.

When the earth ring CD circulates in the direction indicated by letters AEB, then gears E and F will each turn in the opposite direction. Gear G will turn in the same direction as earth ring CD. In gear G you find a pin that has a ring attached to it which makes the Moon circulate its course and also spins the earth. This causes the correct circulation of the moon, which is 30 days.

Figure 7: The Planetarium of Christiaan Huygens; inside view.

1 Januari 1682

	Aphelia.	Klimmende Knoopen	Hellingen	Planeet-baan-stralen	Excentrici-teiten in dezelfde eenheden
van Mercurius	15° 11′ 19″ ♐	14° 29′ 47″ ♉	6° 54′ 0″	38806	8149
„ Venus	2° 59′ 44″ ♒	13° 54′ 52″ ♊	3° 22′ 0″	72400	500
„ Mars	0° 30′ 17″ ♍	17° 38′ 12″ ♉	1° 50′ 30″	152350	14115
„ de Aarde	7° 7′ 20″ ♑			100000	1800
„ Jupiter	7° 55′ 43″ ♎	5° 30′ 42″ ♋	1° 19′ 20″	519650	25058
„ Saturnus	27° 39′ 46″ ♐	21° 36′ 26″ ♋	2° 32′ 0″	951000	54207

Figure 8: The original table from the script of Christiaan Huygens

4.3 The theory behind the Planetarium

So far we have described the parts of the Planetarium and explained how Christiaan Huygens gathered the information for the Planetarium. Now we will describe what method he used to find the correct number of teeth for each gear to achieve the desired movement.

Christiaan decided to make the perpendicular bisector of the octagon shaped Planetarium 11.5 inches long. From the center he drew a circle with a radius of 10.5 inches. This circle has been divided in 360 parts and in 12 parts to indicate the horoscopes. Huygens then placed the sign Aries to the right, level with the middle point.

Furthermore the positions of Aphelia are shown in see Figure 4. Aphelion is the point on the orbit of the planet that circulates in an elliptical course of the sun and lies furthest away from the sun. The distances between each Aphelion on the Planetarium are drawn to scale in order to reflect their actual positions relative to each other. The Aphelion of the earth on the Planetarium is 1 inch.

To clarify this Huygens used Saturn as an example. The reader must first understand that the distance of Saturn from the center of the Sun is, a hundredth of the distance of the earth, multiplied 54, plus the distance of Earths itself. In the Planetarium the Earth's distance from the Sun is 1.00 inch, therefore Saturn's distance from the Sun is 1.54 inch.

To compute Saturn's precise position in the Ecliptic, in the year 1682 (that is the year the Planetarium was builts), take a line at 27 degrees from the center at the 40th minute. This then allows you to conclude Saturn's Aphelion.

The year orbit of Saturn around the Sun is given by $12^\circ\ 13'\ 34''\ 18'''$ and for the Earth we get a year orbit of $359^\circ\ 45'\ 40'\ 31'''$. When we convert everything to seconds we get $12^\circ\ 13'\ 34''\ 18''' =$ 2640858 and $359^\circ\ 45'\ 40'\ 31''' =$ 77708431. The ratio therefore is 2640858:77708431. For accuracy and smooth movements of the planets on the Planetarium we have to make sure the teeth gears of Saturn and the teeth gears of the Earth has the same ratio as 2640858:77708431. Since the numbers are very large Christiaan used the continued fractions to reduce this large ratio to a smaller ratio. He worked out the continued fraction of 2640858:77708431 to be,

$$\frac{2640858}{77708431} = 29 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{5 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \frac{1}{4}}}}}}}}. \tag{25}$$

Christiaan then decided to take the ratio 2640858:77708431 as the ratio 7:206, simply because

$$\frac{77708431}{2640858} \approx \frac{206}{7} = 29 + \cfrac{1}{2 + \cfrac{1}{2 + \frac{1}{1}}}. \tag{26}$$

If we know compute the margin of error we see that,

$$\left| \frac{77708431}{2640858} - \frac{206}{7} \right| = 0.00312. \tag{27}$$

That is the reason why Christiaan made the gear of Saturn contain 206 teeth and the gear, which is connected to the two feet long iron rod, that brings the gear of Saturn in movement contains 7 teeth. If you do the same for the other planets you get the following table,

The planet	number of teeth on the gear of the planet	number of teeth on the gear which is connected to the two feed long iron rod
Saturn	206	7
Jupiter	166	14
Mars	158	84
Venus	32	52
Mercury	17	7

Using the same method Christiaan worked out the gear ratio for Mercury and the Earth to be 847:204. Since 847 is a multiple of 121 and 7 and 204 a multiple of 12 and 17, Christiaan let the year gear contain (see Figure 5) 121 teeth and then used an axis with two ends. The axis containing on one end a gear with 7 teeth and on the other end a gear with 12 teeth. The gear with 12 teeth is connected to the year gear, which has 121 teeth. The other gear with 7 teeth is connected to the gear of Mercury, which has 17 teeth. This then gives him the ratio he required which is 847:204.

Conclusion

We have seen continued fractions yield good rational approximations for (ir)rational numbers. For centuries continued fractions were not used as a theorem, but only as examples. John Wallis was to first to use the name of continued fractions in his book "*Opera Mathematica*". Christiaan Huygens used the continued fractions for his Planetarium and was therefore the first one to demonstrate a practical application involving continued fractions, but it was two centuries after that when the continued fractions experienced a boost. Many mathematicians worked on continued fractions such as Euler, Lambert etc. Continued fractions are also used in computer algorithms to compute rational approximations for real numbers and for solving the Diophantine equations; see Knuth's famous book [K].

We have seen how Christiaan designed a Planetarium, what data he used and how accurate it was. The mechanical Planetarium of Christiaan is not working and therefore the oldest working Planetarium in the world today is Eise Eisinga's. In the 18th century many people could use the Planetarium for a better view of our solar system, but nowadays you have computers and the use of Planetarium is not common. For whom it is still an interested topic, you can best go and visit the Planetarium of Eise Eisinga in Franeker (Friesland).

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