# Subspace identification of continuous-time models using generalized orthonormal bases

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# Subspace identification of continuous-time models using generalized orthonormal bases

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*Abstract*— **The continuous-time subspace identification using state-variable filtering has been investigated for a long time. Due to the simple orthogonal basis functions that were adopted by the existing methods, the identification performance is quite sensitive to the selection of the system-dynamic parameter associated with an orthogonal basis. To cope with this problem, a subspace identification using generalized orthonormal (Takenaka-Malmquist) basis functions is developed, which has the potential to perform better than the existing methods since the adopted Takenaka-Malmquist basis has more degree of freedom in selecting the system-dynamic parameters. As a price for the flexibility of the generalized orthonormal bases, the transformed state-space model is time-varying or parameter-varying which cannot be identified using traditional subspace identification methods. To this end, a new subspace identification algorithm is developed by exploiting the structural properties of the time-variant system matrices, which is then validated by numerical simulations.**

## I. Introduction

The identification of continuous-time (CT) system models using sampled input and output (IO) data is investigated in this paper. So far, there have been a large number of identification algorithms for CT systems. See [1], [2] and the references therein. For numerical computational perspective, the identification methods can be classified into either prediction-error method (PEM) [3] or subspace identification method [4]. The PEM method is to estimate the system parameters using the gradient-typed algorithms so that the identification result is highly dependent on the initial parameter estimate. In contrast, the subspace identification method is a non-iterative identification method, which can produce accurate identification results based on reliable algebraic computations such as QR and SVD decompositions. Therefore, the subspace identification method is usually used in practical applications.

The CT subspace identification in the time domain has been studied for a long time [2], [5]–[10]. These algorithms are based on state-variable filtering technique, i.e., the system state sequence is filtered by a orthogonal basis filter chain, such as Poisson filter chain [5], [9], [11] and the Laguerre filter chain [6], [10], [12]. Either the Possion or Laguerre filter chain is completely determined by a system-dynamic (or pole) parameter. It has been shown in [5], [13] that the identification performance is sensitive to the selection of this system-dynamic parameter. More explicitly, if the adopted filter chain cannot accurately represent the inter sample behavior of the discrete-time (DT) IO data, the system transfer function cannot be correctly identified [13]. In the absence of some prior knowledge of the system dynamics to be identified, the proper selection of the system-dynamic parameter seems to be challenging.

In this paper, in order to alleviate the sensitivity of the identification performance with respect to the system-dynamic parameter selection, the CT subspace identification using general orthogonal basis filters will be investigated. More explicitly, the Takenaka-Malmquist bases are considered to be the generalized orthonormal bases in this paper and are used for the state-variable filtering. It is remarked that the Takenaka-Malmquist bases include the Poisson, Laguerre, Legendre and Hambo bases as special subclasses [14]. Moreover, the generalized orthonormal basis functions have more degree of freedom. Unlike the Possion or Laguerre basis that is determined by a single system-dynamic parameter, the Takenaka-Malmquist basese are determined by a sequence of system-dynamic (pole) parameters. The rich freedom of the generalized orthonormal basis enables us to design a system-dynamic sequence such that the inter sample behavior of the discrete-time IO data can be well represented and the concerned system model can be better identified. For example, reference [15] has shown that better identification results can be obtained by approximating the system transfer function using the Legendre basis, since its poles is nicely spread over the interested frequency band.

Using the generalized orthonormal bases (Takenaka-Malmquist bases), the transformed linear time-invariant (LTI) state-space model is not LTI any more, but linear time-variant (LTV) or linear parameter-varying (LPV), making the contribution of this paper significantly different from the existing CT subspace identification methods. It is noteworthy that Hambo basis was called "generalized orthonormal basis" in [8]; however, the transformed state-space model is still LTI. Due to the time-varying property of the transformed state-space model using the Takenaka-Malmquist basis, the classical subspace methods like N4SID and MOESP cannot be implemented here. To this end, we develop a new subspace method by exploiting the structural properties of the time-varying system matrices. Numerical simulation examples are provided to validate the new subspace identification method and demonstrate better identification performance against the Laguerre filtering method.

The rest of this paper is organized as follows. Section II

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gives the formulation of the considered CT system identification problem. Section III reviews the existing orthonormal basis functions that have been adopted in CT subspace identification methods, which motivates the usage of generalized orthonormal bases. Section IV presents the transformed LPV state-space model and the new developed subspace identification method. Section V shows the performance of the new developed identification method with relation to the Laguerre filtering method that is commonly used in the CT subspace identification algorithms. Finally, conclusions are made in Section VI.

## II. Problem Statement

We consider a continuous-time LTI state-space model as follows

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t), \end{aligned} \tag{1}$$

where $u(t) \in \mathbb{R}^m, x(t) \in \mathbb{R}^n$ and $y(t) \in \mathbb{R}^p$ are system input, state and output, respectively.

For the system model in (1), we assume the model is minimal and stable, and the input signal is persistently exciting of any order [3].

*The problem of interest* is stated as: given the sampled IO data pairs $\{u(t_k), y(t_k)\}$ for $k = 0, 1, \cdots, L-1$ that are generated by the model (1), the identification problem is to estimate the system matrices of (1) up to a similarity transformation, namely there exists a nonsingular matrix $Q$ such that the estimated system matrices $(\hat{A}, \hat{B}, \hat{C})$ satisfy

$$\hat{A} = Q^{-1}AQ,\ \hat{C} = CQ,\ \hat{B} = Q^{-1}B.$$

Note that the sampling-time sequence $\{t_k\}$ is known but might not be equidistant. In this case, the corresponding state-space model in the time domain is time-varying so that the traditional CT system identification routine [16], subspace identification of the DT system model using sampled IO data followed by the CT model identification using the discrete-to-continuous (D2C) conversion, may not be feasible here.

## III. Preliminaries of State-Filtering Methods

The main difficulty of the CT system identification is to deal with the higher-order derivatives associated with the input and output signal. Instead of the direct differentiation which may accentuate the noise effect, the state-filtering method is commonly used for the identification of state-space models. So far, the commonly used filters include Poisson filter chain [5], [9], [11] and the Laguerre filter chain [6], [10], [12]. The poisson filter chain has the form

$$\left[ \begin{array}{cccc} \frac{\sigma}{s+\sigma} & \left(\frac{\sigma}{s+\sigma}\right)^2 & \cdots & \left(\frac{\sigma}{s+\sigma}\right)^n \quad \cdots \end{array} \right], \quad s \in \mathbb{C}$$

while the Laguerre filter chain has the form

$$\left[ \begin{array}{cccc} \frac{\sqrt{2\sigma}}{s+\sigma} & \frac{\sqrt{2\sigma}(s-\sigma)}{(s+\sigma)^2} & \cdots & \frac{\sqrt{2\sigma}(s-\sigma)}{(s+\sigma)^2} \quad \cdots \end{array} \right], \quad s \in \mathbb{C},$$

where $\sigma$ is called system-dynamic parameter or pole parameter.

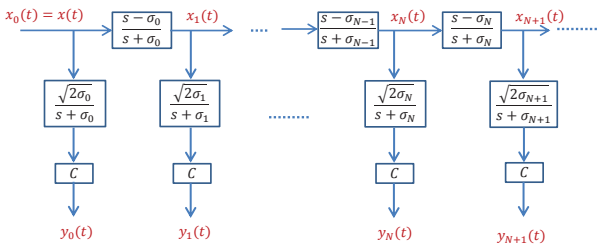


Fig. 1. The Takenaka-Malmquist network with a sequence of different poles. In the diagram, $x(t)$ is the continuous-time state of the original system model.

The Poisson or Laguerre bases are low-pass filters with their bandwidths being determined by the only parameter $\sigma$. When implementing the Poisson or Laguerre filters, the identification performance is sensitive to the selection of the parameter $\sigma$ [5], [13]. As long as the value of $\sigma$ increases, the bandwidth of the filter will becomes larger, so the inter sample behavior of sampled IO data can be better captured [13]. However, a large-value of $\sigma$ will decrease the smoothing capability of the Poisson or Laguerre filter, which may result in a large error when interpolating the continuous input/output signal from its sampled data. As a result, the value of $\sigma$ needs to be properly selected.

In order to reduce the sensitivity of the identification performance with respect to the selection of the sole system-dynamic parameter in Poisson or Laguerre basis, the generalized orthogonal basis functions (Takenaka-Malmquist bases) will be used in this paper (see Fig. 1) which have the following form

$$\left[ \begin{array}{cccc} \frac{\sqrt{2\sigma_0}}{s+\sigma_0} & \frac{\sqrt{2\sigma_1}}{s+\sigma_1}\frac{s-\sigma_0}{s+\sigma_0} & \cdots & \frac{\sqrt{2\sigma_m}}{s+\sigma_m}\prod_{l=0}^{m-1}\frac{s-\sigma_l}{s+\sigma_l} \quad \cdots \end{array} \right], \tag{2}$$

where $\sigma_m$ can be any positive real value or complex value with a positive real part.

The Legendre basis is a special kind of Takenaka-Malmquist basis for which the pole sequence is set to $\sigma_m = \sigma(m + \frac{1}{2})$. It has been shown in [15] that better identification results can be obtained by approximating the system transfer function using the Legendre basis, since the parameter $\sigma$ can be chosen such that the pole sequence $\sigma_m$ can be uniformly spread in the interested frequency band, even though the frequency response of the original may not be known as a priori knowledge. In practice, the interested frequency band is set to $[0, \frac{1}{T_s}]$ with $T_s$ being the sampling period.

## IV. New Subspace Identification Method

### A. Construction of data equation

First, we shall derive a transformed state-space model using the Takenaka-Malmquist basis given in (2). Define the transfer functions $W_k(s) = \frac{s-\alpha_k}{s+\alpha_k}$ and $M_k(s) = \frac{\sqrt{2\sigma_k}}{s+\sigma_k}$ with their time-domain impulse responses being $w_k(t)$ and $m_k(t)$, respectively.

*Lemma 1:* Using the Takenaka-Malmquist basis (2), the CT system model (1) can be transformed into the following

state-space model

$$\begin{aligned}[W_k x](t) &= A_k x(t) + B_k[M_k u](t) + E_k x_0 m_k(t)\\ [M_k y](t) &= C_k x(t) + D_k[M_k u](t) + F_k x_0 m_k(t),\end{aligned} \tag{3}$$

where $[W_k x](t)$ is a continuous-time convolution operator defined by $[W_k x](t) = \int_0^\infty w_k(\tau) x_k(t-\tau) d\tau$, $x_0$ denotes the initial value of $x(t)$, and the associated system matrices are

$$\begin{aligned}&A_k = (A+\sigma_k I)^{-1}(A-\sigma_k I), B_k = \sqrt{2\sigma_k}(A+\sigma_k I)^{-1}B\\ &C_k = \sqrt{2\sigma_k}C(A+\sigma_k I)^{-1}, D_k = -C(A+\sigma_k I)^{-1}B\\ &E_k = \sqrt{2\sigma_k}(A+\sigma_k I)^{-1}, F_k = -C(A+\sigma_k I)^{-1}.\end{aligned}$$

The above lemma can be proven following Lemma 3.4 in [7] by extending the constant $\sigma$ to variant $\sigma_k$. However, the following two aspects of the above state-space model needs to be noted:

1) Although the original system model does not have an input forward-feed term, the transformed model in (3) has one.
2) As the Takenaka-Malmquist basis functions may have different poles, the state-space model in (3) is time-varying (or linear-parameter varying).

Based on the state-space model (3), the continuous-time data equation shall be derived. Due to the heterogeneous property of the Takenaka-Malmquist chain network, the derived data equation is more involved than those in [7], [8], [10]. For the sake of the notational simplicity, the terms relevant to the initial state value will be omitted; however, the initial state value will be taken into account in the numerical simulation.

Define the matrix product

$$\Phi(j,i) = \begin{cases} A_j A_{j-1}\cdots A_i & j \geq i \\ I & j < i. \end{cases}$$

and the composite transformation operator

$$[\psi_{j,i}] = \begin{cases} [W_j \cdots W_i] & j \geq i \\ 1 & j < i. \end{cases}$$

It is remarked that the matrices $A_i$ and $A_j$ are commutative, namely $A_i A_j = A_j A_i$; therefore, the matrix $\Phi(j,i)$ can also be represented by the consecutive product of $\{A_k\}_{k=i}^{j}$ in an arbitrary ordering.

*Lemma 2:* According to the signal flow diagram in Fig. 1, the state signal $x_i(t)$ and output signal $y_i(t)$ can be represented as

$$\begin{aligned}x_i(t) &= \Phi(i-1,0)x(t) + \sum_{j=0}^{i-1}\Phi(j-1,0)B_j[\psi_{i-1,j+1}M_j u](t)\\ y_i(t) &= C_i\Phi(i-1,0)x(t) + \sum_{j=0}^{i-1}\frac{C_j\Phi(j-1,0)B}{\sqrt{2\sigma_i}}[\psi_{i-1,j+1}M_j u](t)\\ &\quad - \sum_{j=0}^{i}\frac{C_j\Phi(j-1,0)B}{\sqrt{2\sigma_i}}[\psi_{i,j+1}M_j u](t).\end{aligned} \tag{4}$$

*Proof:* The state signal $x_i(t)$ can be derived as follows:

$$\begin{aligned}x_1(t) &= [W_0 x](t) = A_0 x(t) + B_0[M_0 u](t),\\ x_2(t) &= [W_1 x_1](t) = A_0[W_1 x](t) + B_0[W_1 M_0 u](t)\\ &= A_0\left(A_1 x(t) + B_1[M_1 u](t)\right) + B_0[W_1 M_0 u](t)\\ &= A_0 A_1 x(t) + A_0 B_1[M_1 u](t) + B_0[W_1 M_0 u](t),\\ &\cdots\end{aligned} \tag{5}$$

Note that the above derivation is based on the expression of $[W_i x](t)$ in equation (3).

The expression of the output signal $y_i(t)$ is given by

$$\begin{aligned}y_i(t) &= C[M_i x_i](t) = \frac{C}{\sqrt{2\sigma_i}}\left(x_i(t) - [W_i x_i](t)\right)\\ &= \frac{C}{\sqrt{2\sigma_i}}\left(x_i(t) - x_{i+1}(t)\right).\end{aligned}$$

By using the expression of $x_i(t)$ in (5), the lemma is then proven. ■

By stacking the output signals $y_i(t)$ in (4), we can obtain the following continuous-time data equation

$$\mathbf{y}(t) = Ox(t) + T\mathbf{u}(t), \tag{6}$$

where

$$\mathbf{y}(t) = \begin{bmatrix} y_0(t)\\ y_1(t)\\ \vdots\\ y_{N-1}(t)\end{bmatrix}, \mathbf{u}(t) = \begin{bmatrix} \mathbf{u}_0(t)\\ \mathbf{u}_1(t)\\ \vdots\\ \mathbf{u}_{N-1}(t)\end{bmatrix},$$

$$\mathbf{u}_i(t) = \begin{bmatrix} [M_i u](t)\\ [\psi_{i+1,i+1}M_i u](t)\\ \vdots\\ [\psi_{N-1,i+1}M_i u](t)\end{bmatrix}, \quad O = \begin{bmatrix} C_0\\ C_1\Phi(0,0)\\ \vdots\\ C_{N-1}\Phi(N-2,0)\end{bmatrix}$$

and the convolution matrix is defined as

$$T = [T_0, T_1, \cdots, T_{N-1}]$$

with $T_i$ being a $N \times (N-i)$ block matrix defined by

$$T_0 = \begin{bmatrix} -D_0 & 0 & \cdots & 0\\ D_0 & -D_0 & & \vdots\\ \vdots & \ddots & \ddots & 0\\ 0 & \cdots & D_0 & -D_0\end{bmatrix}$$

$$D_0 = -C(A+\sigma_0 I)^{-1}B$$

$$T_i = \begin{bmatrix} 0 & 0 & \cdots & 0\\ -\Gamma_i & 0 & \cdots & 0\\ \Gamma_i & -\Gamma_i & & \vdots\\ \vdots & \ddots & \ddots & 0\\ 0 & \cdots & \Gamma_i & -\Gamma_i\end{bmatrix}$$

$$\Gamma_i = C_i\Phi(i-1,0)B \quad \text{for} \quad 1 \leq i \leq N-1.$$

From the definitions of the augmented input signal $\mathbf{u}(t)$ and the convolution matrix $T$, we can find that the data equation (6) is different from those of commonly confronted time-varying systems. More explicitly, the dimension of the stacked input signal $\mathbf{u}(t)$ is proportionally to the square of the dimension parameter $N$. As a result, the traditional subspace methods such as N4SID and MOESP cannot be implemented

here. To this end, we have to develop a new identification approach.

Using the sampled data at times $t_0, t_1, \cdots, t_{L-1}$, we can then obtain the following matrix-form data equation

$$Y = Ox + TU \tag{7}$$

where

$$\begin{aligned} Y &= [\mathbf{y}(t_0), \cdots, \mathbf{y}(t_{L-1})] \\ U &= [\mathbf{u}(t_0), \cdots, \mathbf{u}(t_{L-1})] \\ x &= [x(t_0), \cdots, x(t_{L-1})]. \end{aligned}$$

In the sequel, we assume that the value of $L$ is large enough such that the matrix $\begin{bmatrix} U \\ Y \end{bmatrix}$ is fat.

It can be observed from the data equation (7) that, compared with the uniform sampling strategy, the non-uniform sampling does not pose any difficulty to the subspace identification problem. The only difference is that the non-uniform sampling may cause larger approximation error when computing the filtered input/output samples, as will be shown in Subsection IV-C.

### *B. New subspace identification method*

According to the data equation (7), we shall develop a subspace identification method to identify the system matrices $(A, B, C)$ in (1) up to a similarity transformation. The subspace identification of time-varying systems or linear parameter-varying systems usually requires the system matrices to be periodically varying [17], [18]; however, the developed subspace identification method in this paper does not require this periodic property.

*Estimation of the matrices $C$ and $A$.* First, the column space of the extended observability matrix $O$ is to be extracted, as shown in the following lemma.

*Lemma 3:* Consider the data equation (7). Suppose that $\text{rank}\begin{bmatrix} U \\ x \end{bmatrix} = \text{rank}(U) + n$. Let the following LQ factorization be given

$$\begin{bmatrix} U \\ Y \end{bmatrix} = \begin{bmatrix} L_{11} & 0 \\ L_{21} & L_{22} \end{bmatrix} \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix}.$$

Then, we have that

$$\text{range}(O) = \text{range}(L_{22}).$$

The rank condition in the above lemma can be satisfied when the input sequence $u(t_k)$ is persistently exciting of any finite order [4].

Let the SVD decomposition of $L_{22}$ be given as

$$L_{22} = [\Theta_1 \;\; \Theta_2] \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V_1^T \\ V_2^T \end{bmatrix},$$

where $\Sigma \in \mathbb{R}^{n\times n}$ is a diagonal matrix with positive diagonal entries. Then, we have that

$$O = \Theta_1 Q, \tag{8}$$

where $Q \in \mathbb{R}^{n\times n}$ is a nonsingular matrix.

Next, the matrices $C$ and $A$ are to be estimated based on the obtained column subspace of the extended observability matrix $O$. Different from the traditional subspace identification for which there exists a shift property for the extended observability matrix, the extended observability matrix $O$ in (6) does not exist an explicit shift structure; however, we will show that the matrix $A$ can still be estimated (up to a similarity transformation) using the least-squares estimation method.

Let the matrices $O$ and $\Theta_1$ be partitioned as

$$O = \begin{bmatrix} O_0 \\ O_1 \\ \vdots \\ O_{N-1} \end{bmatrix}, \quad \Theta_1 = \begin{bmatrix} \Theta_{1,0} \\ \Theta_{1,1} \\ \vdots \\ \Theta_{1,N-1} \end{bmatrix}. \tag{9}$$

According to the definition of $O$ in (6), the block components of $O$ can be explicitly expressed as

$$O_i = \sqrt{2\sigma_i} C(A + \sigma_i I)^{-1} \left( \prod_{j=0}^{i-1} (A - \sigma_j I)(A + \sigma_j I)^{-1} \right) \tag{10}$$

for $i = 0, \cdots, N-1$. By exploiting the commutative property of the matrix product in the above equation, we can establish the following relationship:

$$\frac{O_i}{\sqrt{2\sigma_i}}(A + \sigma_i I) = \frac{O_{i-1}}{\sqrt{2\sigma_{i-1}}}(A - \sigma_{i-1} I) \tag{11}$$

for $i = 1, \cdots, N-1$. The matrix $O$ is unknown, but its estimate $\Theta_1$ up to a similarity transformation; therefore, the system matrices $(C, A)$ need to be computed based on the available matrix $\Theta_1$.

Define the matrices

$$M = \begin{bmatrix} \frac{\Theta_{1,0}}{\sqrt{2\sigma_0}} - \frac{\Theta_{1,1}}{\sqrt{2\sigma_1}} \\ \vdots \\ \frac{\Theta_{1,N-2}}{\sqrt{2\sigma_{N-2}}} - \frac{\Theta_{1,N-1}}{\sqrt{2\sigma_{N-1}}} \end{bmatrix}$$

$$R = \begin{bmatrix} \sqrt{\frac{\sigma_0}{2}}\Theta_{1,0} + \sqrt{\frac{\sigma_1}{2}}\Theta_{1,1} \\ \vdots \\ \sqrt{\frac{\sigma_{N-2}}{2}}\Theta_{1,N-2} + \sqrt{\frac{\sigma_{N-1}}{2}}\Theta_{1,N-1} \end{bmatrix}.$$

Then, the matrix $A$ (up to a similarity transform) satisfies the following linear equation:

$$MA = R, \tag{12}$$

where $M$ and $R$ are known. The estimate of $A$ can be obtained as:

$$\hat{A} = L_A^{\dagger} R_A, \tag{13}$$

where $^{\dagger}$ denotes the MoorePenrose pseudo inverse.

After obtaining the estimate $\hat{A}$, the matrix $C$ can be straightforwardly estimated according to the relation $O_0 = \sqrt{2\sigma_0} C(A + \sigma_0 I)^{-1}$, i.e.,

$$\hat{C} = \frac{\Theta_{1,0}}{2\sigma_0}(A + \sigma_0 I). \tag{14}$$

*Estimation of the matrix $B$.* The matrix $B$ shall be estimated based on the data equation (7) as well as the obtained

estimates $\hat{A}$ and $\hat{C}$. It can be observed from the structure of the convolution matrix $T$ in (6) that, given the estimates of $C$ and $A$, the matrix $T$ is linear with respect to the matrix variable $B$. As a result, the matrix $B$ can be estimated by solving the following least-squares estimation problem

$$[\hat{B}, \hat{x}] = \arg\min_{x,B} \|Y - \Theta_1 x - T(B)U\|_F^2. \tag{15}$$

### C. Calculation of filtered input and output signals

As a preliminary step for the new subspace identification method, the samples of the filtered IO signals in equation (6) need to computed. For the sake of brevity, only the output-signal filtering will be explained in details.

According to the signal flow graph in Fig. 1, the output signal $y_i(t)$ for $i = 0, \cdots, N-1$ can be written as

$$\begin{aligned} y_i(t) &= C[M_i x_i](t) \\ &= [M_i \phi_{i-1,0} y](t) \\ &= \frac{1}{\sqrt{2\sigma_i}} \left\{ [\phi_{i-1,0} y](t) - [\phi_{i,0} y](t) \right\}. \end{aligned} \tag{16}$$

Denote $z_i(t) = [\phi_{i,0} y](t)$. Then the signal $y_i(t)$ can be compactly represented as

$$y_i(t) = \frac{1}{\sqrt{2\sigma_i}} \left[ z_{i-1}(t) - z_i(t) \right]. \tag{17}$$

In particular, the signal $z_i(t)$ in the above equation can recursively computed as

$$\begin{aligned} z_0(t) &= [W_0 y](t) = \int_0^\infty w_0(\tau) y(t-\tau) d\tau \\ z_i(t) &= [W_i z_{i-1}](t) = \int_0^\infty w_i(\tau) z_{i-1}(t-\tau) d\tau. \end{aligned} \tag{18}$$

Again, for the sake of brevity, only the computation of the samples of $z_0(t)$ will be elaborated on.

The state-space form of $z_0(t) = [W_0 y](t)$ can be represented as

$$\begin{aligned} \dot{\xi}(t) &= -\sigma_0 \xi(t) + y(t) \\ z_0(t) &= -2\sigma_0 \xi(t) + y(t). \end{aligned} \tag{19}$$

The corresponding sampled output can be obtained as

$$\begin{aligned} z_0(t_k) &= y(t_k) - 2\sigma_0 e^{-\sigma_0 \Delta_k} \xi(t_{k-1}) \\ &\quad - 2\sigma_0 \int_0^{\Delta_k} e^{-\sigma_0 \tau} y(t_k - \tau) d\tau \\ &= y(t_k) + [z_0(t_{k-1}) - y(t_{k-1})] e^{-\sigma_i \Delta_k} \\ &\quad - 2\sigma_0 \int_0^{\Delta_k} e^{-\sigma_0 \tau} y(t_k - \tau) d\tau, \end{aligned} \tag{20}$$

where $\Delta_k = t_k - t_{k-1}$. The above equation shows that the sample data $z_0(t_k)$ can be computed based on its previous sample data $z_0(t_{k-1})$. However, the integration involved in the above equation can only be approximately computed since we do not know the continuous values of $y(t)$ for $t \in (t_{k-1}, t_k]$.

Here, based on the sample data $y(t_k)$ for $k = 0, \cdots, N-1$, the continuous segment of $y(t)$ at $t \in (t_{k-1}, t_k]$ is approximated by a $p$th-order polynomial [6]:

$$y(t_k - \tau) \approx g_0(k) + g_1(k)\tau + \cdots + g_p(k)\tau^p, \ \tau \in (0, \Delta_k], \tag{21}$$

where the coefficients $g_j(k)$ for $j = 0, \cdots, p$ are determined by interpolating the $p+1$ points $\{y(t_{k-j})\}$ for $j = \lfloor \frac{-p}{2} \rfloor, \cdots, \lfloor \frac{p}{2} \rfloor - 1$ with $y(t) = 0$ for $t < 0$ and $t > t_{N-1}$. It has been shown in [6] that, if the samples are uniformly sampled using the sampling interval $T_s$, a $p$-th order polynomial approximation gives an approximation error of $\mathcal{O}(T_s^{p+1})$. As a result, when $T_s \ll 1$, higher-order polynomial approximation will lead to higher accuracy.

Using the $p$th-order polynomial approximation, the integration in (20) can be represented as

$$\begin{aligned} \int_0^{\Delta_k} e^{-\sigma_i \tau} y(t_k - \tau) d\tau &\approx \sum_{j=0}^{2p} g_j(k) \int_{\tau=0}^{\Delta_k} e^{-\sigma_k \tau} \tau^j d\tau \\ &= e^{-\sigma_k \Delta_k} u_i(t_k) + \sum_{j=0}^{2p} g_j(k) \alpha_j, \end{aligned} \tag{22}$$

where

$$\begin{aligned} \alpha_0 &= -\frac{1}{\sigma_k} \left( e^{-\sigma_k \Delta_k} - 1 \right) \\ \alpha_j &= -\frac{1}{\sigma_k} \Delta_k^j e^{-\sigma_k \Delta_k} + \frac{j}{\sigma_k} \alpha_{j-1}, \ \text{ for } \ j \geq 1. \end{aligned}$$

To ease the reference, the sample data $y_i(t_k)$ for $k = 0, \cdots, N-1$ can be computed via carrying out the following four steps.

- S1. Interpolate the discrete-time samples using the $p$th-order polynomial fitting approach, as shown in (21).
- S2. Compute the sample data $z_0(t_k)$ recursively as shown in (20), where the involved integration is computed according to equation (22).
- S3. Similar to the computation of $z_0(t_k)$ as shown in S1-S2, compute the DT data $z_i(t_k)$ according to equation (18).
- S4. Compute the DT data $y_i(t_k)$ using the obtained quantities of $z_i(t_k)$, as shown in equation (17).

## V. Numerical simulations

To evaluate the performance of the proposed method in this paper, we shall test a fourth-order system model given as [13]:

$$G(s) = \frac{-0.8471s^3 + 0.9081s^2 - 5.972s + 5.519}{s^4 + 3.115s^3 + 5.036s^2 + 4.577s + 2.254},$$

which has two pairs of complex conjugated poles at $-0.4636 \pm 1.0972i$ and $-1.0939 \pm 0.6262i$. The bandwidth of the system is 1.3742 rad/s.

In the simulation, the continuous input signal is generated as a sequence of standard random Gaussian noise with the samples being interpolated by the first-order-hold operation. The sampling interval is set to $T_s = 0.005$s. The output samples are then acquired at the time points $t_k = kT_s$ for $k = 1, 2, \cdots$. When computing the filtered input and output data, the order of the local fitting polynomial (21) is set to $p = 3$. The dimension parameter $N$ in the data equation (6) is set to $N = 8$.

The sampled IO data used for the system identification are shown in Fig. 2. The data length is set to 400.

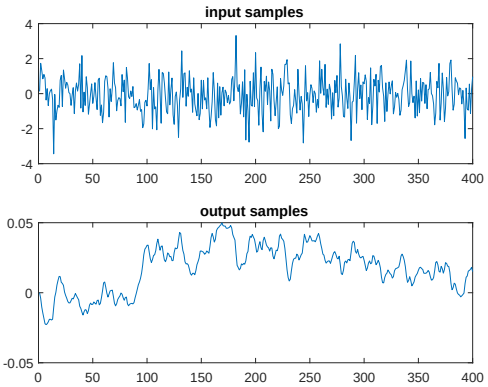


Fig. 2. Sequences of IO data.

For the ease of reference, the proposed identification method is called Takenaka-Malmquist method. The pole sequence of the Takenaka-Malmquist basis is set to $\{4, 12, 20 \cdots, 60\}$. For comparison purposes, the Laguerre filtering method is simulated as well. The Laguerre basis is determined by only one pole parameter. To have a comprehensive understanding of the Laguerre filtering method, we test the Laguerre filtering method using different poles at $\{4, 28, 60\}$, which are exactly the first, the middle and the last points of the pole sequence selected for the Takenaka-Malmquist basis.

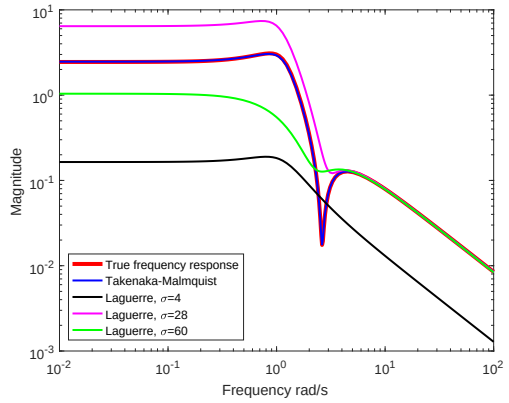


Fig. 3. Bode frequency responses of the original and identified systems. Note that the frequency-response curve of the Takenaka-Malmquist method overlaps with the true frequency-response curve.

Fig. 3 shows the identification performance of the Takenaka-Malmquist method and the Laguerre method with different poles, where we can observe that

1) The Takenaka-Malmquist method performs much better than the Laguerre filtering method, i.e., the identified system model using the Takenaka-Malmquist method is the most close to the true system model in terms of the frequency response.
2) The identification performance of the Laguerre method is sensitive to the selection of the pole value. Using a larger pole value, better frequency-response fitting can be obtained at the high-frequency band. However, by setting a small pole value, we cannot see the improvement of the frequency-response fitting in the low-frequency band. Therefore, the selection of the pole value seems to be a challenging task.

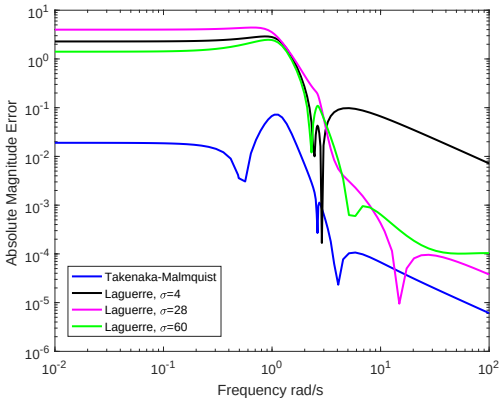


Fig. 4. Identification performance in terms of the frequency response error $|G(jw) - \hat{G}(jw)|$ with $\hat{G}(jw)$ being the estimated frequency response.

Fig. 4 shows the frequency-response errors of the identified system models with relation to that of the true system model. It is easy to see that the Takenaka-Malmquist method has better identification performance at both low-frequency bands and high-frequency bands. This can be explained that the poles of the Takenaka-Malmquist basis are uniformly spread in the frequency domain so that the system dynamics can be better captured using the Takenaka-Malquist basis than using the Laguerre basis.

## VI. Conclusion

In this paper, the continuous-time subspace identification problem has been investigated and a new identification method using the generalized basis functions (Takenaka-Malmquist bases) has been developed. The new subspace identification framework provides more freedom in selecting the system-dynamic parameters; hence, the sensitivity of the identification performance of existing methods with respect to the selection of a single system-dynamic parameter can be alleviated. However, the transformed state-space model using the generalized orthogonal basis is time-varying or parameter-varying, which cannot be identified using traditional subspace identification methods. To cope with this problem, a new subspace identification method has been developed by exploiting the hidden shift properties of the extended observability matrix corresponding to the parameter-varying system model.

This paper provides a general subspace identification framework for CT system in the sense that most of the existing orthogonal basis filters can be incorporated into this framework. In our future work, the process and measurement noise of the CT system model will be taken into account, and how to adaptively tuning the system-dynamic parameters of the generalized orthonormal bases will be investigated.

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