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The Frobenius problem for homomorphic embeddings of languages into the integers

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Abstract Let S be a map from a language $\mathcal{L}$ to the integers satisfying $S(vw) = S(v) + S(w)$ for all $v, w \in \mathcal{L}$. The classical Frobenius problem asks whether the complement of $S(\mathcal{L})$ in the natural numbers will be infinite or finite, and in the latter case the value of the largest element in this complement. This is also known as the 'coin-problem', and $\mathcal{L}$ is the full language consisting of all words over a finite alphabet. We solve the Frobenius problem for the golden mean language, any Sturmian language and the Thue-Morse language. We also consider two-dimensional embeddings.

1 Introduction

The Frobenius problem is also known as the 'coin problem'. Since the value of a coin can only be positive, we will consider exclusively embeddings into the natural numbers $\mathbb{N} = \{1, 2, 3, \dots\}$. Let $\mathcal{L}$ be a language, i.e., a factor-closed subset of the free semigroup generated by a finite alphabet under the concatenation operation.

A homomorphism of $\mathcal{L}$ into the natural numbers is a map $S : \mathcal{L} \to \mathbb{N}$ satisfying

$$S(vw) = S(v) + S(w), \quad \text{for all } v, w \in \mathcal{L}.$$

The two main questions to be asked about the image set $S(\mathcal{L})$ are

(Q1) Is the complement $\mathbb{N} \setminus S(\mathcal{L})$ finite or infinite?

(Q2) If the complement of $S(\mathcal{L})$ is finite, then what is the largest element in this set?

These two questions are known as the Frobenius problem in the special case that $\mathcal{L}$ is the full language consisting of *all* words over a finite alphabet. In this case they have been posed as a problem (with solution) for an alphabet $\{a, b\}$ of cardinality 2 by James Joseph Sylvester in 1884 [15]: $\mathbb{N} \setminus S(\mathcal{L})$ is finite if and only if $S(a)$ and $S(b)$ are relatively prime, in which case its largest element is

$$S(a)S(b) - S(a) - S(b).$$

In this paper we will also restrict ourselves to the two symbol case: alphabet $\{a, b\}$.

In Section 2 we prove that for the golden mean language ("no bb") the set $\mathbb{N} \setminus S(\mathcal{L})$ is finite when $S(a)$ and $S(b)$ are relatively prime, with largest element

$$S(a)^2 + S(a)S(b) - 3S(a) - S(b).$$

Our main interest is however not in sofic languages[1], but in languages with low complexity (slow growth of the number of subwords of length n), where the complement of $S(\mathcal{L})$ can be infinite.

In Section 3 we analyse the case of Sturmian languages, and show that for the Fibonacci language a $0-\infty$ law holds: either the complement is empty or it has infinite cardinality.

In Section 4 we show that for any homomorphism S the image of the Thue-Morse language will consist of a union of 5 arithmetic sequences.

In Section 5 we consider two-dimensional embeddings, which behave quite differently.

We usually suppose that $\gcd(S(a), S(b)) = 1$. First of all this is not a big loss since automatically the complement will have infinite cardinality if this is not the case. Secondly, if r divides both $S_1(a)$ and $S_1(b)$ for some homomorphism S_1, then

$$S_1(\mathcal{L}^n) = r^n S_2(\mathcal{L}^n), \quad \text{for } n = 1, 2, \ldots, \text{ where } S_2(a) = \frac{S_1(a)}{r}, \; S_2(b) = \frac{S_1(b)}{r}.$$

Here, and throughout the paper, we write $\mathcal{L}^n$ for the set of words of length n in a language $\mathcal{L}$—not to be confused with the n-fold concatenation of $\mathcal{L}$.

Our work is related to the work on *abelian complexity*, see, e.g., [3], [13], [8]. See Lemma 3.1 for such a connection.

Our work is also related to the notion of *additive complexity*, see [14] and [2]. The *additive complexity* of an infinite word w over a finite set of integers (see [2]) is the function $n \to \phi^+(w, n)$ that counts the number of distinct sums obtained by summing n consecutive symbols of w. Let $\mathcal{L}_w$ be the language of all words occurring in the infinite word w. Then the additive complexity is $\phi^+(w, n) = \mathrm{Card}\{S(u) : u \in \mathcal{L}_w^n\}$, where S is the identity map on the alphabet of w. Here we do, and may, assume that the alphabet of w is a subset of $\mathbb{N}$.

We finally mention that homomorphisms S from a language to the natural numbers already occur in the 1972 paper [4, Section 6] in the context of the Fibonacci language, where they are called *weights*.

2 Homomorphic images of the golden-mean language

The golden mean language is the language $\mathcal{L}_{\mathrm{GM}}$ consisting of all words over $\{a, b\}$ in which bb does not occur as a subword. Now if S satisfies $S(a) = 1$ or $S(b) = 1$, then it is easily seen that $S(\mathcal{L}_{\mathrm{GM}}) = \mathbb{N}$, so for these homomorphisms the golden mean and the full language both map to $\mathbb{N}$. One could say they both have Frobenius number 0. In general however, the Frobenius number will increase substantially. If we take S defined by

$$S(a) = 100, \; S(b) = 3,$$

---

[1]Regular languages, or equivalently, languages defined by the labelling of paths of an automaton, see [10, Section 1.5]

2

then the Frobenius number of the full language under S is $300 - 100 - 3 = 197$, and the Frobenius number of $S(\mathcal{L}_{\text{GM}})$ is equal to 9997. For arbitrary homomorphisms the solution of the Frobenius problem for the golden mean language is given by the following, where we write $S_a := S(a)$, $S_b := S(b)$.

Theorem 2.1 *Let* $S : \mathcal{L}_{\text{GM}} \to \mathbb{N}$ *be a homomorphism. Suppose* $\gcd(S_a, S_b) = 1$, *and both* $S_a > 1$ *and* $S_b > 1$. *Then the Frobenius number of* $S(\mathcal{L}_{\text{GM}})$ *is equal to*

$$\max \mathbb{N} \setminus S(\mathcal{L}_{\text{GM}}) = S_a(S_a - 3) + S_b(S_a - 1).$$

Proof: Let an S_a-*point* be defined as a multiple nS_a, $n = 0, 1, \ldots$, and an S_a-*interval* as the set of numbers between two consecutive S_a-points. We also consider S_b-*chains*, defined for $n \geq 0$ by

$$C(n) = \{nS_a + S_b, \ nS_a + 2S_b, \ldots, nS_a + (n+1)S_b\}.$$

The union of the S_a-points and the S_b-chains will give $\mathcal{L}_{\text{GM}}$: the S_a-points are the images of the words a^n, and the S_b-chains are the images of the words in which n letters a occur, and k letters b for $k = 1, 2, \ldots, n+1$. Here $n+1$ is the maximal number of letters b that can occur because of the "no bb" constraint.

The key observation is that the S_b-chain $C(S_a - 2)$ has $S_a - 1$ elements, which are all distinct modulo S_a. This is a consequence of $\gcd(S_a, S_b) = 1$. It follows that the S_b-chains fill in more and more points of the S_a-intervals. The last point to be filled in is equal to $S_a - S_b$ modulo S_a, produced by the last element of the chain $C(S_a - 2)$. This is the number

$$P := (S_a - 2)S_a + (S_a - 1)S_b.$$

But then the largest number in the complement of $\mathcal{L}_{\text{GM}}$ is $P - S_a$, which is the number as claimed in the theorem. In this argument we used that if a point in an S_a-interval is filled in, then the corresponding points modulo S_a in all later intervals will also be filled in, simply because the later chains will be extensions of the earlier ones. $\qquad\square$

$S(a) = 7, S(b) = 3$: row n shows $C(n-1)$ in green, $n = 1, \ldots, 8$ (truncated at 56)
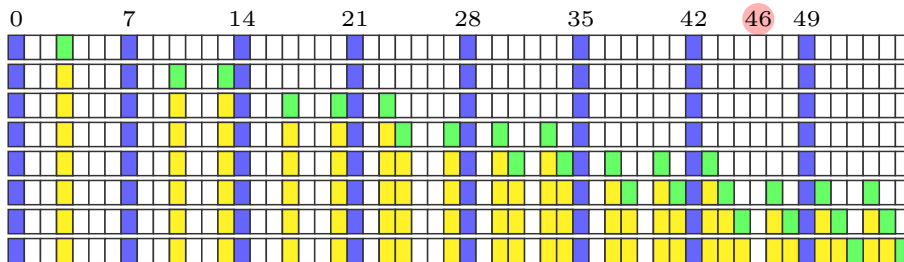
Figure 1: Example with $S(a) = 7, S(b) = 3$: row n shows the S_a-points in blue, and the S_b-chain $C(n-1)$ in yellow and green, for $n = 1, \ldots, 8$ (truncated at 56).

3   Sturmian languages

Sturmian words are infinite words over a two letter alphabet that have exactly $n + 1$ subwords for each $n = 1, 2, \ldots$. We call the collection of these subwords a

Sturmian language. There is a surprising characterization of Sturmian words: s is Sturmian if and only if s is irrational *mechanical*, which means that there exist an irrational number $\alpha \in (0,1)$ and a number ρ such that $s = s_{\alpha,\rho}$, or $s = s'_{\alpha,\rho}$, where

$$s_{\alpha,\rho} = \left([(n+1)\,\alpha + \rho] - [n\,\alpha + \rho]\right)_{n \geq 0}, \; s'_{\alpha,\rho} = \left(\lceil(n+1)\,\alpha + \rho\rceil - \lceil n\,\alpha + \rho\rceil\right)_{n \geq 0}.$$

See, e.g., [10, Prop. 2.1.13]. Because of this representation, we will use the alphabet $\{0,1\}$ instead of $\{a,b\}$ in this section.

Of special interest are the Sturmian words $s_\alpha := s_{\alpha,0}$ and $s'_\alpha := s'_{\alpha,0}$ of intercept 0. These have the property that they only differ in the first element:

$$s_\alpha = 0\,c_\alpha, \qquad s'_\alpha = 1\,c_\alpha.$$

Here $c_\alpha := s_{\alpha,\alpha}$ is called the *characteristic word* of α. For $n \geq 0$ we have

$$c_\alpha(n) = s_{\alpha,\alpha}(n) = [(n+1)\,\alpha + \alpha] - [n\,\alpha + \alpha] = [(n+2)\,\alpha] - [(n+1)\,\alpha].$$

The words s_α, s'_α and c_α generate the same language ([10, Prop. 2.1.18]), which we denote $\mathcal{L}_\alpha$. Recall that $\mathcal{L}_\alpha^n$ is the set of words of length n in $\mathcal{L}_\alpha$.

Lemma 3.1 *Let $\mathcal{L}_\alpha$ be a Sturmian language, and let S be a homomorphism with $\mathrm{S}(0) \neq \mathrm{S}(1)$. Then $\operatorname{Card}\mathrm{S}(\mathcal{L}_\alpha^n) = 2$ for all $n \geq 1$.*

Proof: This follows directly from the fact ([10, Th. 2.1.5]) that Sturmian words are *balanced*, i.e., any two words of the same length can differ by at most 1 in their number of ones. □

A sequence $([n\alpha])$, where $[.]$ denotes integer part, is called a *Beatty sequence* if $\alpha > 1$, and a *slow Beatty sequence* if $0 < \alpha < 1$ (terminology from [9]).

Theorem 3.1 *Let α be an irrational number from $(0,1)$. Let $\mathcal{L}_\alpha$ be the Sturmian language generated by α, and let $(q_n)_{n \geq 0}$ be the slow Beatty sequence defined by*

$$q_n = [(n+1)\alpha].$$

Let $\mathrm{S} : \mathcal{L}_\alpha \to \mathbb{N}$ be a homomorphism. Define $\mathrm{S}_0 = \mathrm{S}(0), \mathrm{S}_1 = \mathrm{S}(1)$. Then

$$\mathrm{S}(\mathcal{L}_\alpha) = \{(\mathrm{S}_1 - \mathrm{S}_0)q_n + n\mathrm{S}_0 + \mathrm{S}_0 : n = 0, \dots\} \cup \{(\mathrm{S}_1 - \mathrm{S}_0)q_n + n\mathrm{S}_0 + \mathrm{S}_1 : n = 0, \dots\}.$$

Proof: If $\mathrm{S}_0 = \mathrm{S}_1$ then this is certainly true, so suppose $\mathrm{S}_0 \neq \mathrm{S}_1$ in the sequel. We denote $c_\alpha[i,j] := c_\alpha(i)\dots c_\alpha(j)$ for integers i,j with $0 \leq i < j$. Let $N_\ell(w)$ denote the number of occurrences of the letter ℓ in a word w for $\ell = 0, 1$. Then

$$N_1(c_\alpha[0, n-1]) = \sum_{k=0}^{n-1} c_\alpha(k) = [(n+1)\alpha] - [\alpha] = q_n, \quad N_0(c_\alpha[0, n-1]) = n - q_n.$$

Of course all words $c_\alpha[0, n-1]$ are in the Sturmian language $\mathcal{L}_\alpha$, but $\mathcal{L}_\alpha$ also contains the words $0c_\alpha[0, n-1]$ and $1c_\alpha[0, n-1]$. It thus follows from Lemma 3.1 that $\mathrm{S}(\mathcal{L}_\alpha)$ is given by the union of all images $\mathrm{S}(0c_\alpha[0, n-1])$ and $\mathrm{S}(1c_\alpha[0, n-1])$. Since

$$\mathrm{S}(0c_\alpha[0, n-1]) = \mathrm{S}_0 + (n - q_n)\mathrm{S}_0 + q_n\mathrm{S}_1 = (\mathrm{S}_1 - \mathrm{S}_0)q_n + n\mathrm{S}_0 + \mathrm{S}_0,$$

the result follows. □

4

3.1 The Fibonacci language

Let $\Phi = (\sqrt{5}+1)/2 = 1.61803\ldots$ be the golden mean, and let $\alpha := 2 - \Phi = \Phi^{-2}$. We have (see, e.g., [10, Example 2.1.24])

$$c_\alpha = ([(n+1)\alpha] - [n\alpha])_{n\geq 1} = 0,1,0,0,1,0,1,0,0,1,0,0,1,0,1,0,0,\ldots,$$

the infinite Fibonacci word. We write $\mathcal{L}_F := \mathcal{L}_\alpha$.

Theorem 3.2 *Let* $S : \mathcal{L}_F \to \mathbb{N}$ *be a homomorphism. Then*

$$S(\mathcal{L}_F) = \big((S_0 - S_1)[n\Phi] + (2S_1 - S_0)n + S_0 - S_1\big)_{n\geq 1} \cup \big((S_0 - S_1)[n\Phi] + (2S_1 - S_0)n\big)_{n\geq 1}.$$

Proof: This is a corollary to Theorem 3.1, using $[-x] = -[x] - 1$ for non-integer x:

$$
\begin{aligned}
(S_1 - S_0)q_{n-1} + nS_0 &= (S_1 - S_0)[n\alpha] + nS_0 = (S_1 - S_0)[n(2-\Phi)] + nS_0 \\
&= 2(S_1 - S_0)n + (S_1 - S_0)[-n\Phi] + nS_0 \\
&= (2S_1 - S_0)n + (S_1 - S_0)(-[n\Phi] - 1) \\
&= (S_0 - S_1)[n\Phi] + (2S_1 - S_0)n + S_0 - S_1. \qquad \square
\end{aligned}
$$

Lemma 3.2 *For* $S(0) = 1$, $S(1) \leq 3$ *or* $S(0) = 2$, $S(1) = 1$ *one has* $S(\mathcal{L}_F) = \mathbb{N}$.

Proof: Take $(S_0, S_1) = (1,1)$. Then obviously $S(\mathcal{L}_F) = \mathbb{N}$.
Take $(S_0, S_1) = (2,1)$. Then $S(\mathcal{L}_F) = \mathbb{N}$, since by Theorem 3.2 $S(\mathcal{L}_F)$ is the union of $([n\Phi])$ and $([n\Phi] + 1)$, where the difference of two consecutive terms in $([n\Phi])$ is never more than 2.
Take $(S_0, S_1) = (1,2)$. Then $S(\mathcal{L}_F) = \mathbb{N}$, since $S(\mathcal{L}_F)$ is the union of $([n(3 - \Phi)])$ and $([n(3 - \Phi)]) + 1)$, where the difference of two consecutive terms in $([n(3 - \Phi)])$ is never more than 2.
Take $(S_0, S_1) = (1,3)$. This case is more complicated. Let $u := (-2[n\Phi] + 5n - 2)_{n\geq 1}$, and $v := u + 2$. Then according to Theorem 3.2, the union of the sets determined by u and v is $S(\mathcal{L}_F)$. Let Δu be the difference sequence defined by $\Delta u_n = u_{n+1} - u_n$ for $n \geq 0$. It is easy to see that the difference sequences Δv and Δu are both equal to the Fibonacci sequence $1,3,1,1,3,1,\ldots$ on the alphabet $\{1,3\}$ (cf. [1]). We claim that if two consecutive numbers $m, m+1$ are missing in u, then these two do appear in v, implying that $S(\mathcal{L}_F) = \mathbb{N}$. Indeed the two missing numbers are characterized by $u_{n+1} - u_n = 3$ for some n, and the missing numbers are $m = u_n + 1$ and $u_n + 2$. The second number appears in v, simply because $v = u + 2$. The first number appears because $u_{n+1} - u_n = 3$ implies $u_n - u_{n-1} = 1$ (no 33 in the 1-3-Fibonacci sequence), and so $v_{n-1} = v_n - 1 = u_n + 1$. $\qquad \square$

We define $\mathcal{E} := \{(1,1), (1,2), (1,3), (2,1)\}$.

Theorem 3.3 *Let* $S : \mathcal{L}_F \to \mathbb{N}$ *be a homomorphism. Then* $\mathbb{N} \setminus S(\mathcal{L}_F)$ *has infinite cardinality,* unless $(S(0), S(1)) \in \mathcal{E}$, *in which case the complement is empty.*

Proof: According to Lemma 3.2 the complement of $S(\mathcal{L}_F)$ is empty for $(S_0, S_1) \in \mathcal{E}$. The density of the set $S(\mathcal{L}_F)$ in the natural numbers exists, and equals

$$\delta := \frac{2}{(S_0 - S_1)\Phi + 2S_1 - S_0}.$$

The theorem will be proved if we show that $\delta < 1$ for (S_0, S_1) not in $\mathcal{E}$. First we note that the denominator of δ is positive:

$$(S_0 - S_1)(\Phi - 1) + S_1 > -S_1(\Phi - 1) + S_1 = S_1(2 - \Phi) > 0,$$

where we used that $1 < \Phi < 2$. We now have

$$\delta < 1 \iff (S_0 - S_1)\Phi + 2S_1 - S_0 > 2 \iff (S_0 - S_1)\Phi > S_0 - S_1 + 2 - S_1.$$

If $S_0 > S_1$, this is satisfied, since under this condition $(2 - S_1)/(S_0 - S_1) \leq 0$, unless $(S_0, S_1) = (2, 1) \in \mathcal{E}$. If $S_0 < S_1$, we have to see that $\Phi < 1 + (2 - S_1)/(S_0 - S_1)$. This holds for $S_0 \geq 2$, since then $(2 - S_1)/(S_0 - S_1) \geq 1$. If $S_0 = 1$, then this does not hold for $S_1 = 1, 2, 3$, i.e., for pairs from $\mathcal{E}$, but it will hold for all $S_1 \geq 4$. □

For particular values of $S(0)$ and $S(1)$ the complement of the embedding of the language has a nice structure, as it can be expressed in the classical Beatty sequences $A(n) = [n\Phi]$ for $n \geq 1$, and $B(n) = [n\Phi^2]$ for $n \geq 1$. The sequences A and B are called the *lower Wythoff sequence* and *upper Wythoff sequence*; they are extremely well-studied.

Example 1. Let S be given by $S(0) = 3$ and $S(1) = 2$. In the following we consider A and B as functions from $\mathbb{N}$ to $\mathbb{N}$, and define functions $pX + qY + r$ by

$$(pX + qY + r)(n) = pX(n) + qY(n) + r$$

for real numbers p, q, r and functions $X, Y : \mathbb{N} \to \mathbb{N}$. Then

$$S(\mathcal{L}_F) = B(\mathbb{N}) \cup (B+1)(\mathbb{N}), \quad \mathbb{N} \setminus S(\mathcal{L}_F) = \{1, 4, 9, 12, \dots\} = (2A + \text{Id} + 1)(\mathbb{N} \cup \{0\}).$$

The first statement follows from Theorem 3.2, and the relationship $B = A + \text{Id}$. The second statement follows in a number of steps from the fact that A and B form a Beatty pair: $A(\mathbb{N}) \cap B(\mathbb{N}) = \emptyset$, and $A(\mathbb{N}) \cup B(\mathbb{N}) = \mathbb{N}$. This implies that $A(A(\mathbb{N})) \cup A(B(\mathbb{N})) \cup B(\mathbb{N}) = \mathbb{N}$, where the three sets are disjoint. But $AA = B - 1$ (see, e.g., Formula (3.2) in [4]). Adding 1 to all three sequences it follows that

$$B(\mathbb{N}) \cup (B + 1)(\mathbb{N}) \cup (AB + 1)(\mathbb{N}) = \mathbb{N} \setminus \{1\}.$$

Moreover, according to [4, Formula (3.5)] one has $AB = A + B = 2A + \text{Id}$.
But then the three sequences $([n\Phi] + n)_{n \geq 1}$, $([n\Phi] + n + 1)_{n \geq 1}$, $(2[n\Phi] + n + 1)_{n \geq 0}$, form a complementary triple, i.e., as sets they are disjoint, and their union is $\mathbb{N}$.
A similar result holds for[2] $S(0) = 4$, $S(1) = 3$.

Example 2. Let S be given by $S(0) = 3$ and $S(1) = 1$, then by Theorem 3.2

$$S(\mathcal{L}_F) = (2A - \text{Id})(\mathbb{N}) \cup (2A - \text{Id} + 2)(\mathbb{N}).$$

It is proved in [1] that

$$\mathbb{N} \setminus S(\mathcal{L}_F) = \{2, 9, 20, 27, 38, 49, \dots\} = (4A + 3\text{Id} + 2)(\mathbb{N} \cup \{0\}),$$

and that the three sequences $(2[n\Phi] - n)_{n \geq 1}$, $(2[n\Phi] - n + 2)_{n \geq 1}$, $(4[n\Phi] + 3n + 2)_{n \geq 0}$, form a complementary triple.

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[2] In these two cases $\mathbb{N} \setminus S(\mathcal{L}_F)$ is given by sequences A276885, respectively A276886 in OEIS ([11]). It is easily seen that the definitions of these sequences in OEIS are equivalent to the way in which we obtain them.

4 The Thue-Morse language

Let θ given by $\theta(a) = ab$, $\theta(b) = ba$ be the Thue-Morse morphism. Let $\mathcal{L}_{\mathrm{TM}}$ be the language generated by this morphism.

Let $R_{r,s} = \{s, r+s, 2r+s, \dots\}$ be the set determined by the arithmetic sequence with terms $rn + s$ for $n = 0, 1 \dots$.

Theorem 4.1 *Let* $\mathrm{S} : \mathcal{L}_{\mathrm{TM}} \to \mathbb{N}$ *be a homomorphism. Define* $p = \mathrm{S}(a), q = \mathrm{S}(b)$. *Then*

$$\mathrm{S}(\mathcal{L}_{\mathrm{TM}}) = R_{p+q,0} \cup R_{p+q,p} \cup R_{p+q,q} \cup R_{p+q,2p} \cup R_{p+q,2q}.$$

Proof: Let $\mathcal{L}_{\mathrm{TM}}^n$ be the set of words of length n in the Thue-Morse language. Put $r = \mathrm{S}(ab) = p + q$. It is clear (and for $p = 0, q = 1$ also observed in [13]) that since the Thue-Morse word is a non-periodic concatenation of ab and ba that for $n = 1, 2, \dots$

$$\mathrm{S}(\mathcal{L}_{\mathrm{TM}}^{2n}) = \{rn, rn + q - p, rn + p - q\}, \quad \mathrm{S}(\mathcal{L}_{\mathrm{TM}}^{2n-1}) = \{rn + p, rn + q\}.$$

This implies the statement of the theorem. $\qquad\square$

Theorem 4.2 *Let* $\mathrm{S} : \mathcal{L}_{\mathrm{TM}} \to \mathbb{N}$ *be a homomorphism. Then* $\mathbb{N} \setminus \mathrm{S}(\mathcal{L}_{\mathrm{F}})$ *has infinite cardinality if and only if* $\mathrm{S}(a) + \mathrm{S}(b) \geq 6$. *For* $\mathrm{S}(a) + \mathrm{S}(b) < 6$, *the complement is either empty or a singleton.*

Proof: This follows directly from Theorem 4.1. If $\mathrm{S}(a) + \mathrm{S}(b) \geq 6$, then the density of $\mathbb{N} \setminus \mathrm{S}(\mathcal{L}_{\mathrm{TM}})$, is at least $1/6$, so the set has infinite cardinality.

When $\mathrm{S}(a) + \mathrm{S}(b) < 6$, then, because of symmetry, we only have to consider the four cases $(\mathrm{S}(a), \mathrm{S}(b)) = (1, 2)$, $(\mathrm{S}(a), \mathrm{S}(b)) = (1, 3)$, $(\mathrm{S}(a), \mathrm{S}(b)) = (1, 4)$, and $(\mathrm{S}(a), \mathrm{S}(b)) = (2, 3)$. In these cases one will find with Theorem 4.1 that the complement of $\mathrm{S}(\mathcal{L}_{\mathrm{F}})$ is empty in the first two cases, and equal to $\{3\}$, respectively $\{1\}$ in the last two. $\qquad\square$

Remark Let σ given by $\sigma(a) = ab, \sigma(b) = aa$ be the period-doubling or Toeplitz morphism. The difficulty—see [8, Lemma 6]—of determining the abelian complexity of the period-doubling morphism already indicates that solving the Frobenius problem for the period-doubling language will be much more involved than for the Thue-Morse language.

5 Two dimensional embeddings

Here we consider homomorphisms $\mathrm{S} : \mathcal{L} \to \mathbb{N} \times \mathbb{N}$ and $\mathrm{S} : \mathcal{L} \to \mathbb{Z} \times \mathbb{Z}$. The situation changes drastically for this 'double-coin' problem.

Proposition 5.1 *Let* $\mathcal{L}$ *be a language on the alphabet* $\{a, b\}$, *and let* $\mathrm{S} : \mathcal{L} \to \mathbb{N} \times \mathbb{N}$ *be a homomorphism. Then* $\mathbb{N} \times \mathbb{N} \setminus \mathrm{S}(\mathcal{L})$ *has infinite cardinality for all pairs* $\{\mathrm{S}(a), \mathrm{S}(b)\}$ *which are not equal to the pair* $\{(0, 1), (1, 0)\}$.

Proof: It suffices to prove this for the full language $\mathcal{L}_{\text{full}}$. The image under S is an integer lattice, with a complement of infinite cardinality, unless $S(a)$ and $S(b)$ are the unit vectors. $\qquad\square$

We learn from this that the alphabet is 'too small', and that we should rather consider embeddings in $\mathbb{Z} \times \mathbb{Z}$ instead of $\mathbb{N} \times \mathbb{N}$. We focus again on low complexity languages, in particular on those generated by a primitive morphism φ on an alphabet A. Such a morphism has a language $\mathcal{L}_\varphi$ associated to it, where each word $w \in \mathcal{L}_\phi$ has a measure $\mu_\varphi(w)$. For a given homomorphism $S : \mathcal{L}_\varphi \to \mathbb{Z} \times \mathbb{Z}$ we call the average

$$\Delta_\varphi(S) := \sum_{a \in A} \mu_\varphi(a) S(a)$$

the *drift* of S.

Proposition 5.2 *Let $\mathcal{L}_\varphi$ be a language generated by a primitive morphism on an alphabet A, and let $S : \mathcal{L}_\varphi \to \mathbb{Z} \times \mathbb{Z}$ be a homomorphism. Then $\mathbb{Z} \times \mathbb{Z} \setminus S(\mathcal{L})$ has infinite cardinality if $\Delta_\varphi(S) \neq (0,0)$.*

Proof: It is well-known that the measure μ_φ is strictly ergodic (see, e.g., [12]). Because of this, we have for words w from $\mathcal{L}_\varphi$, where $|w|$ denotes the length of w,

$$\frac{1}{|w|} S(w) = \frac{1}{|w|} \sum_{a \in A} N_a(w) S(a) \to \sum_{a \in A} \mu_\varphi(a) S(a) = \Delta_\varphi(S) \text{ as } |w| \to \infty.$$

Thus for long words w the images $S(w)$ will be concentrated around the line in the direction of the drift of S, and so the complement of $S(\mathcal{L}_\varphi)$ will have infinite cardinality if the drift is not $(0,0)$. $\qquad\square$

Can we say something about the Frobenius problem for homomorphic images of morphic languages of an embedding with drift $(0,0)$? We shall give an infinite family of morphic languages $\mathcal{L}_\theta$ on an alphabet $A = \{a,b,c,d\}$ of four letters where for the homomorphism $S^\oplus$ given by

$$S^\oplus(a) = (1,0),\ S^\oplus(b) = (0,1),\ S^\oplus(c) = (-1,0),\ S^\oplus(d) = (0,-1)$$

the homomorphic embedding is the whole $\mathbb{Z} \times \mathbb{Z}$—and thus the complement is empty. We shall make use of the paperfolding morphisms introduced in [6]. Let σ be the rotation morphism on the alphabet $\{a,b,c,d\}$ given by $\sigma(a) = b$, $\sigma(b) = c$, $\sigma(c) = d$, $\sigma(d) = a$, and let τ be the anti-morphism given by $\tau(w_1 \ldots w_n) = w_n \ldots w_1$.
A morphism θ on $\{a,b,c,d\}$ is called a *paperfolding* morphism if
  1) $\sigma\tau\theta = \theta\sigma\tau$,
  2) Letters from $\{a,c\}$ alternate[3] with letters from $\{b,d\}$ in $\theta(a)$.
A paperfolding morphism is called *symmetric* if $\sigma\theta = \theta\sigma$. It is clear that this happens if and only if the word $\theta(a)$ is a palindrome.

Let G be a (semi-) group with operation $+$ and unit e. In general an infinite word $x = (x_n)$ over an alphabet A and a homomorphism $S : A^* \to G$ generate a *walk* $Z = (Z_n)_{n \geq 0}$ by (cf. [5])

$$Z_0 = e, \qquad Z_{n+1} = Z_n + S(x_n) = S(x_0 \ldots x_n), \text{ for } n \geq 0.$$


[3]This corrects an omission in [6, Definition 1].

A paperfolding morphism θ with $\theta(a) = a...$ is called *perfect* if the four walks generated by the fixed point $x = \theta^\infty(a)$, and its three rotations over $\pi/2, \pi$ and $3\pi/2$ visit every integer point in the plane exactly twice (except the origin, which is visited 4 times).

In [6] it is—not explicitly—proved that for any odd integer N that is the sum of two squares there exists a perfect symmetric paperfolding morphism of length N. To make the proof explicit, one uses that, according to the paragraph at the end of Section 7 in [6], there exists a symmetric plane-filling and self-avoiding string for each such N, and then one observes that the construction of such a string in the proof of [6, Theorem 4] always satisfies the perfectness criterion given in [6, Theorem 5].

The smallest length is $N = 5$, with morphism θ given by

$$\theta(a) = abcba, \ \theta(b) = bcdcb, \ \theta(c) = cdadc, \ \theta(d) = dabad.$$
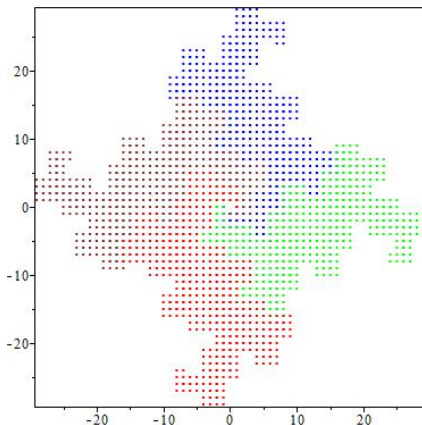
Figure 2: The four images of the words $\theta^4(a), \ldots, \theta^4(d)$ under $\mathrm{S}^\oplus$, where θ is the perfect symmetric 5-folding morphism. The origin is not covered, but it is the image of the word $abcd \in \mathcal{L}_\theta$.

Proposition 5.3 *Let $\mathcal{L}_\theta$ be the language generated by a perfect symmetric paperfolding morphism θ. Then $\mathrm{S}^\oplus(\mathcal{L}_\theta) = \mathbb{Z} \times \mathbb{Z}$.*

Proof: This follows directly from Theorem 5 in [6], using the observation above. □

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