Topology Optimization of Masonry Structures

P5 presentation

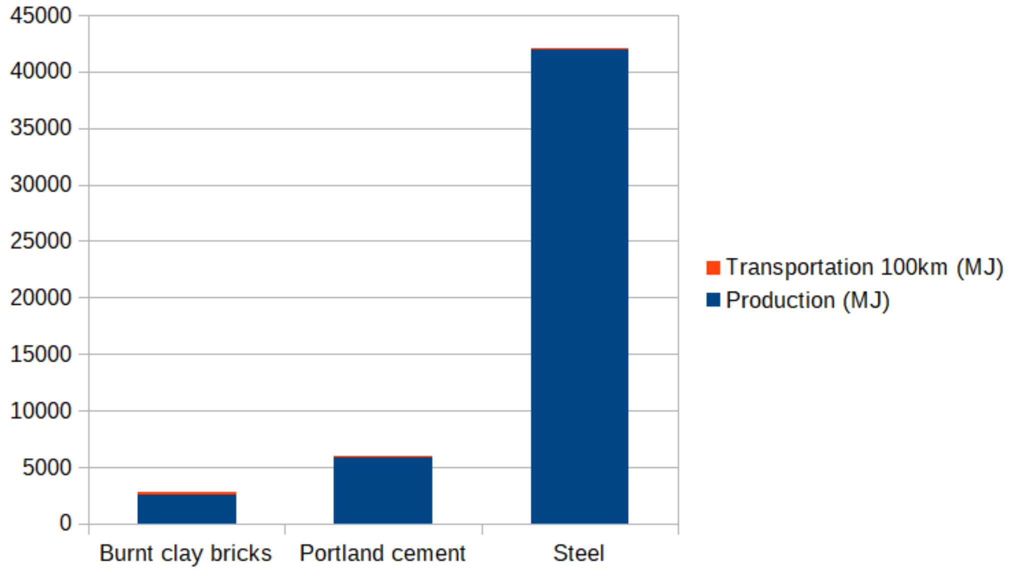
Kaan Akbaba

16/01/2023

Contents

Why masonry?

- To design sustainable structures
  - Masonry has relatively low embodied energy
  - Masonry is relatively cheap
  - Masonry is relatively durable



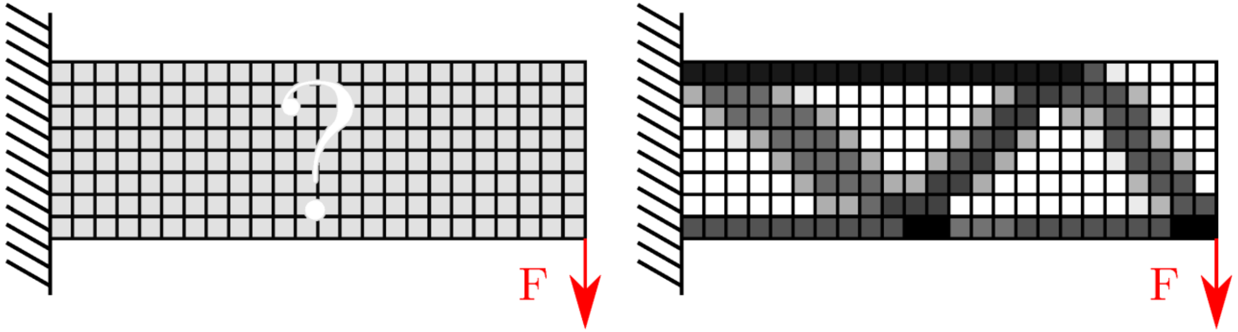
Embodied energy values of common construction materials

Historic bridges (18^{th}/19^{th} century Artvin, Turkey)

Habertürk. (n.d.). Artvin'deki çifte köprüler. https://www.haberturk.com/artvin-haberleri/73783783-artvindeki-cifte-kopruler-tarihe-meydan-okuyorarhavi-ilcesindeki-cifte-tas-kopruler-yeril

Topology optimization?

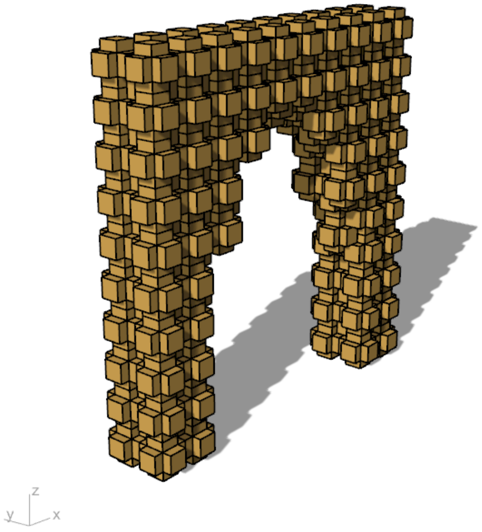
- Keep essential structural elements
- Remove unnecessary elements
- Different approach for masonry: discrete element analysis (DEA)



Sotola, M., Marsalek, P., Rybansky, D., Fusek, M., & Gabriel, D. (2021). Sensitivity Analysis of Key Formulations of Topology Optimization on an Example of Cantilever Bending Beam. *Symmetry*, *13*(4), 712. MDPI AG. http://dx.doi.org/10.3390/sym13040712

How can an arbitrary three-dimensional masonry structure be optimized using the discrete element method?

Relevant uses in design & construction



Generating new structures

Aiding in restoration or renovation projects

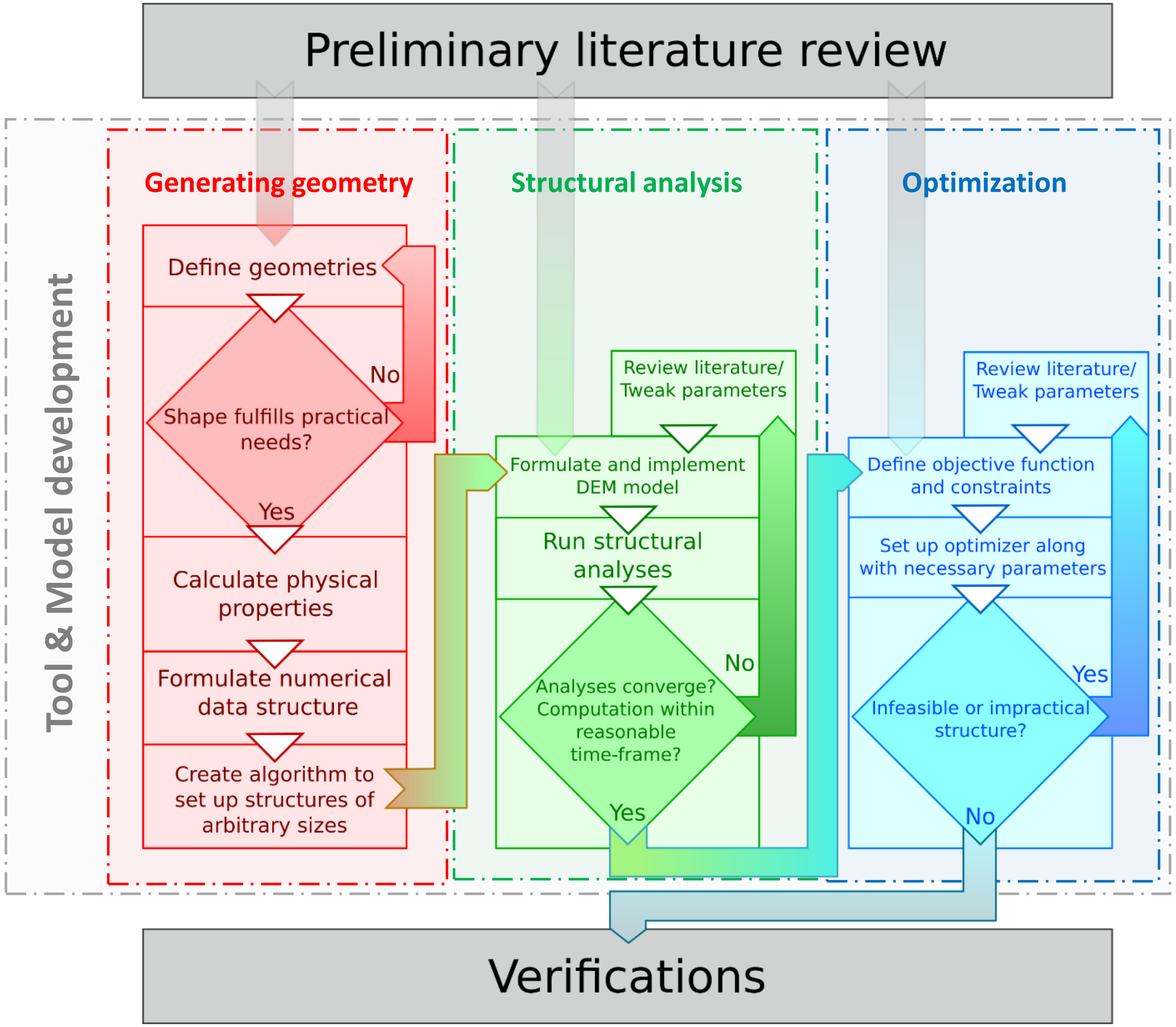
Decanter. (2019). Notre Dame cathderal, the day after the fire. https://www.decanter.com/wine-news/chateaux-owners-pledge-money-notre-dame-fire-412466/

Robotic construction

Craftsmac Labs. (n.d.). Masonry Robot. https://craftsmaclabs.com/about-us/

- Does not have to be limited to masonry
- All discrete structures can be optimized using this algorithm
  - Different material bricks/blocks
  - Modular buildings

Methodology



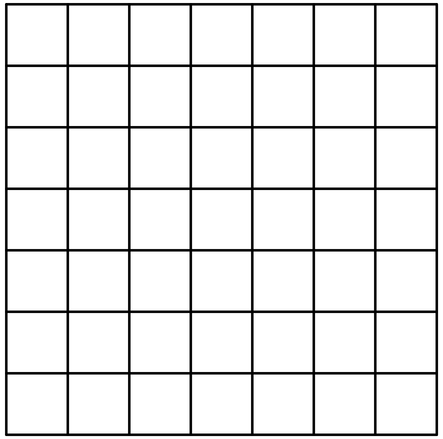
Implementation

Visualization

+

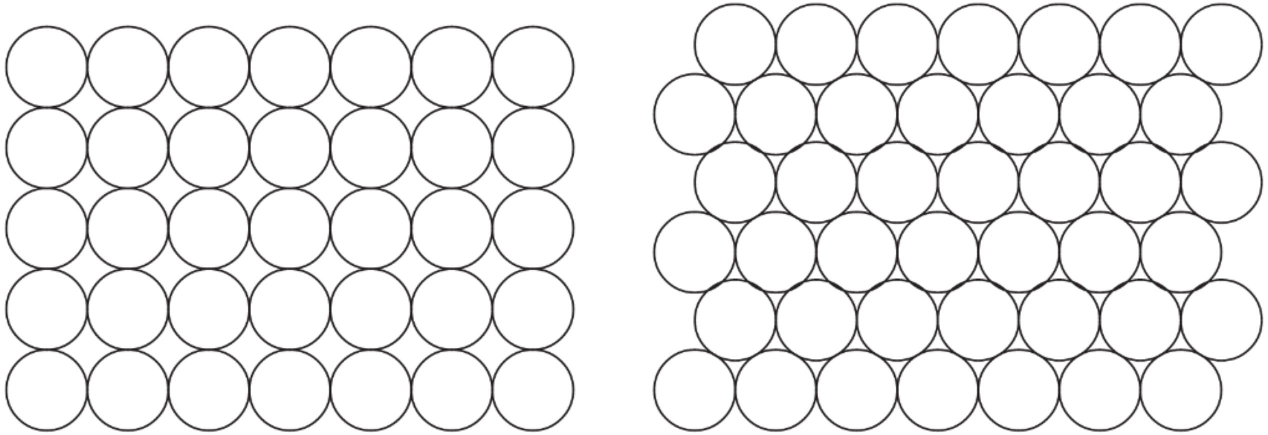
Generating geometry

- One type of element per structure
- Elements should be space-filling



Space-filling example

Wolfram MathWorld. (n.d.). Square Grid.
https://mathworld.wolfram.com/SquareGrid.html

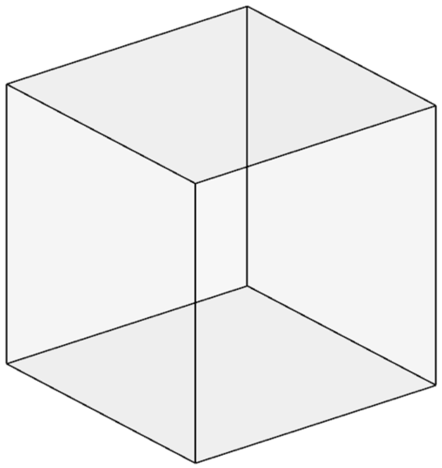


Non-space-filling example

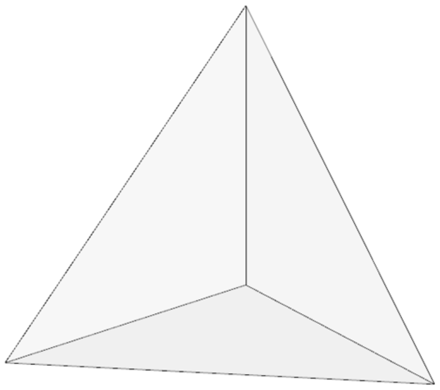
Wolfram MathWorld. (n.d.). Circle Packing.
https://mathworld.wolfram.com/CirclePacking.html

Generating geometry

- Elements consist of smaller shapes

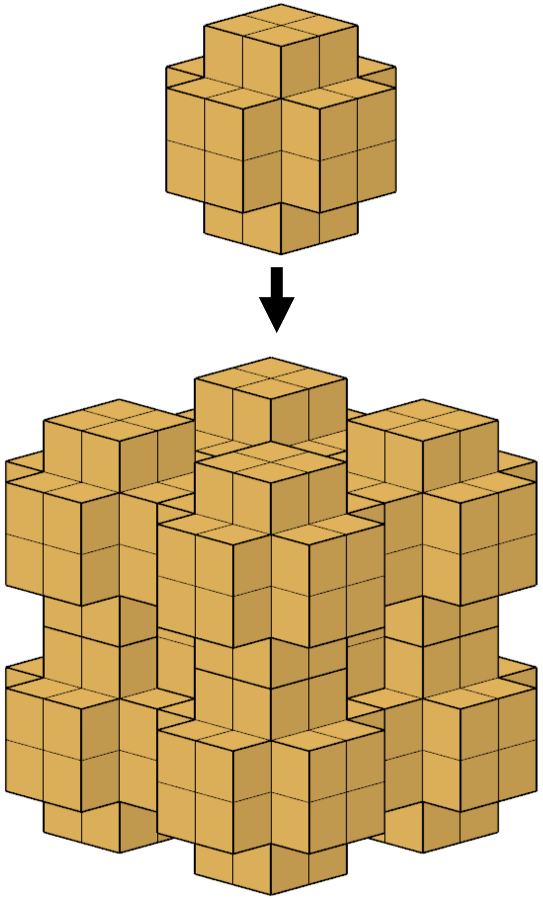


Cube

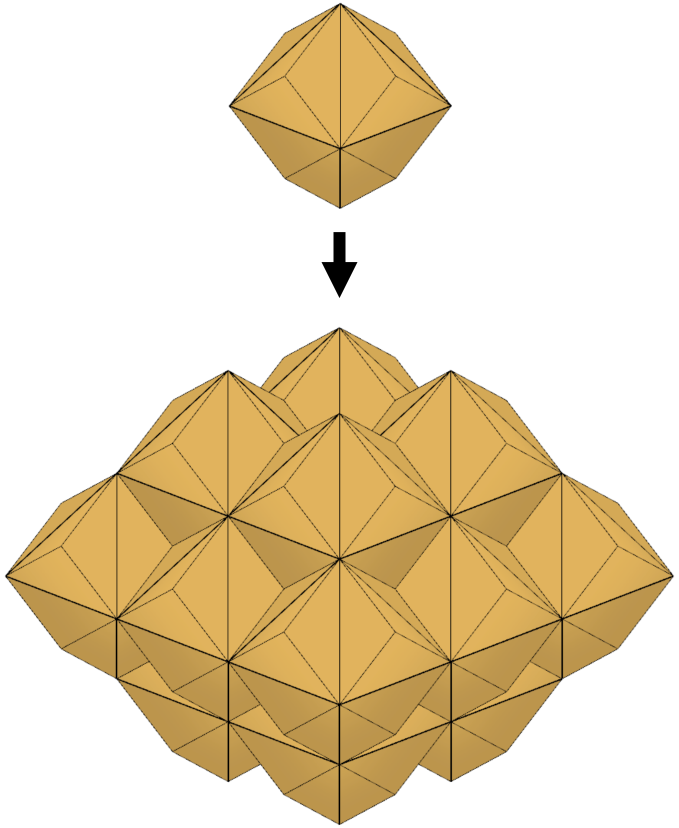


Tetrahedron

Generating geometry



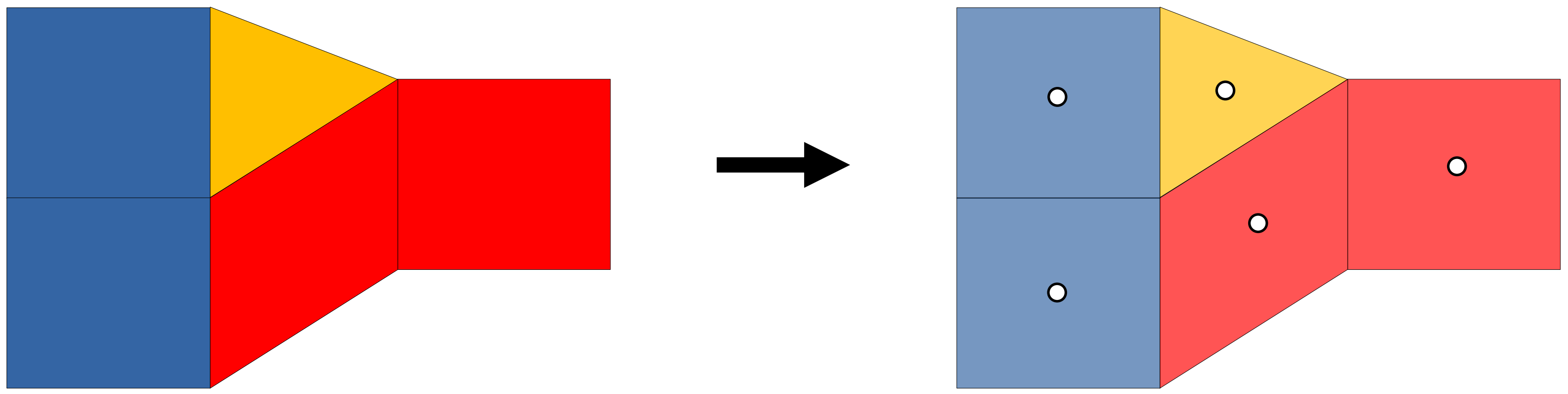
Cubic Sphere (CS)



Rhombic Dodecahedron (RD)

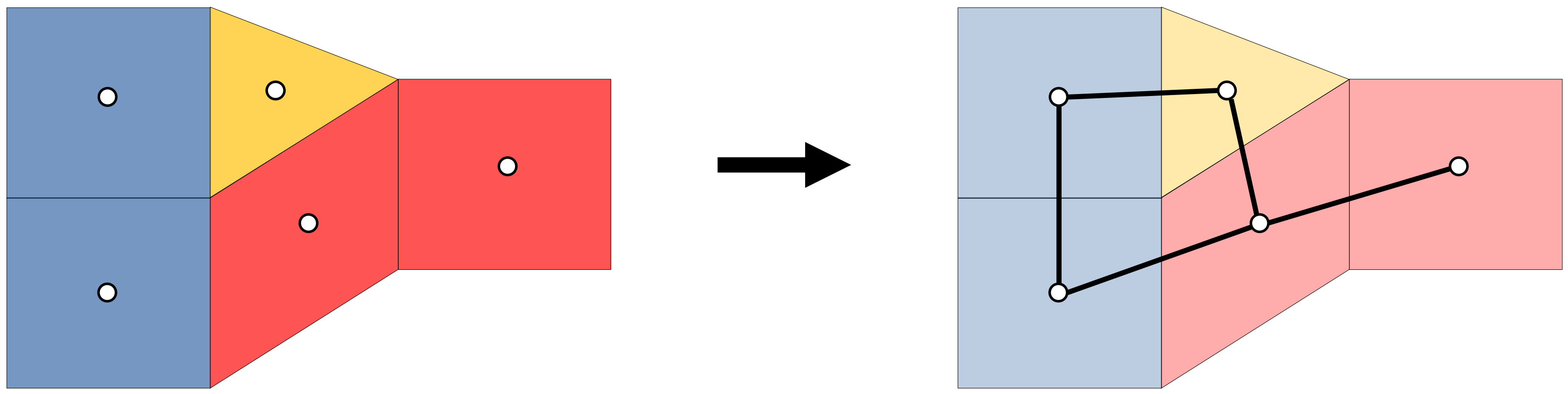
Generating geometry

- Geometry is simplified as a collection of nodes and edges (graph)



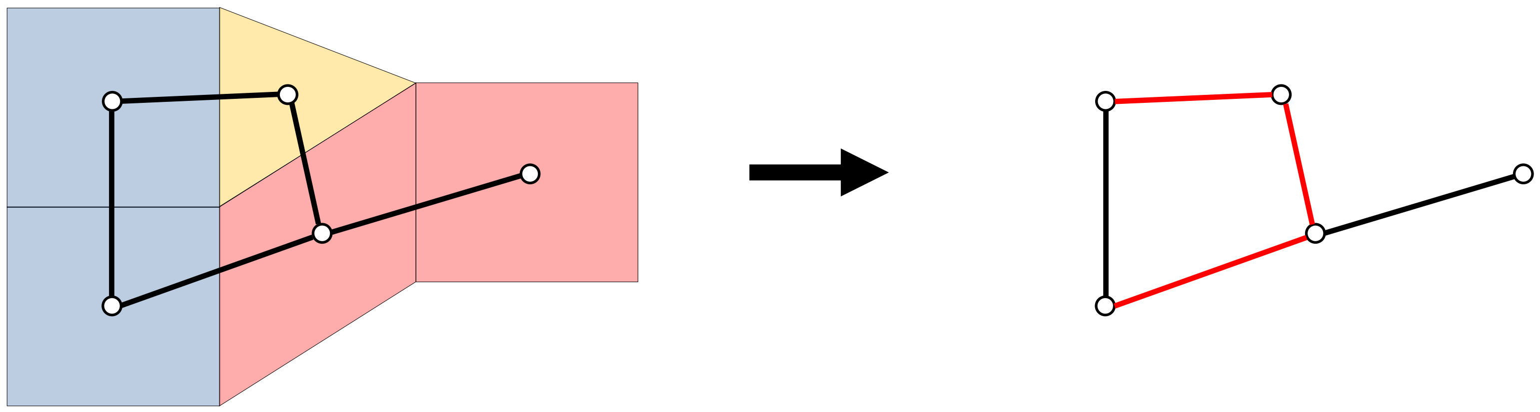
Centroids of shapes are represented by nodes

Generating geometry



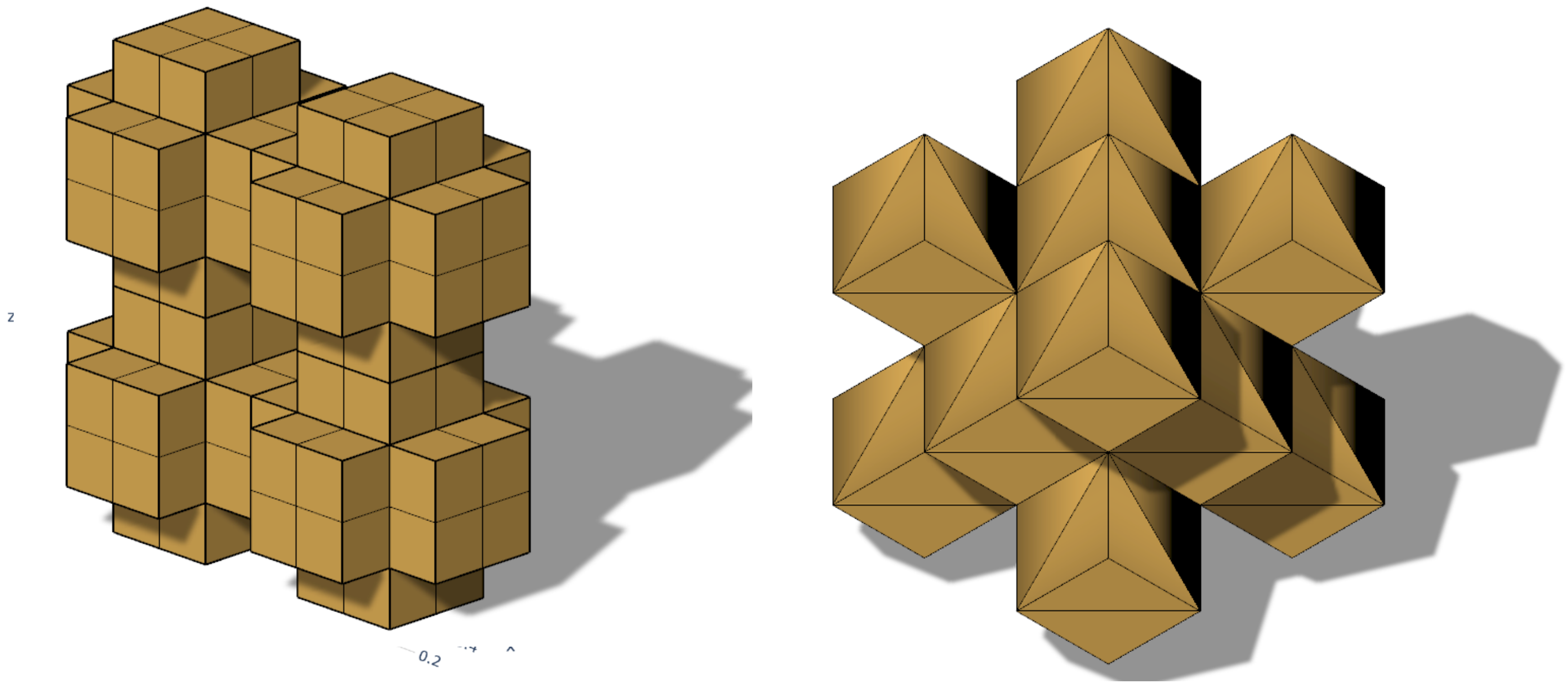
If shapes share a face (edge in 2D) their nodes are connected by an edge

Generating geometry



If edges connect two different elements, they represent an interface (red)

Generating geometry



Abstraction of CS and RD assemblies

Structural analysis

Time-explicit analysis

Globally damped

$$\hat{\boldsymbol{M}}^{(t)}\ddot{\hat{\boldsymbol{u}}}^{(t)} + C^{(t)}\dot{\hat{\boldsymbol{u}}}^{(t)} + \hat{\boldsymbol{K}}^{(t)}\hat{\boldsymbol{u}}^{(t)} = \hat{\boldsymbol{f}}^{(t)}$$

Locally damped

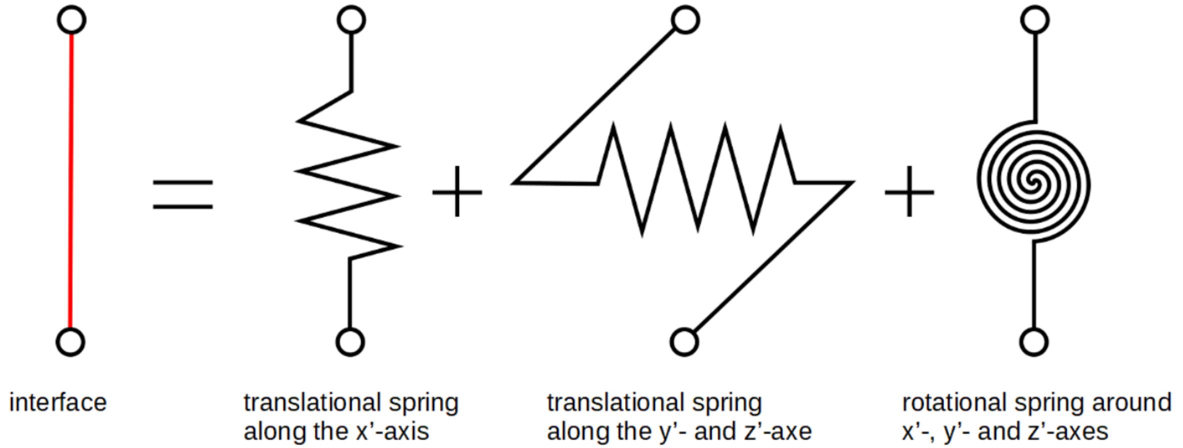
$$\dot{\boldsymbol{u}}^{(t+\frac{1}{2}\Delta t)} = \dot{\boldsymbol{u}}^{(t-\frac{1}{2}\Delta t)} + \Delta t\,\hat{\boldsymbol{M}}^{-1}\left(\hat{\boldsymbol{f}}_{\boldsymbol{int}}^{(t)} - \hat{\boldsymbol{f}}_{\boldsymbol{d}}^{(t)}\right)$$

Static analysis

$$\hat{\boldsymbol{K}}\hat{\boldsymbol{u}} = \hat{\boldsymbol{f}}$$

Structural analysis

- Interfaces represent multiple springs



$$G = \frac{E}{2(1+\nu)}$$

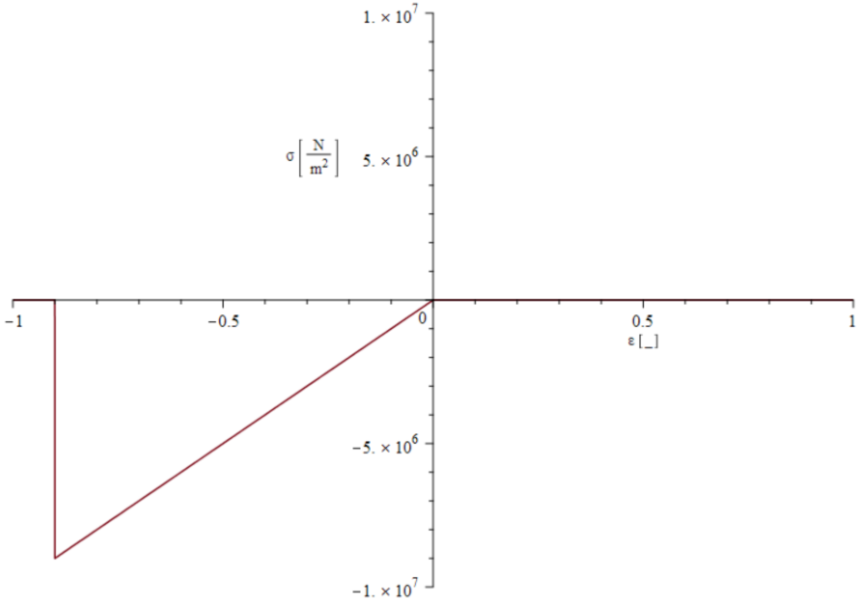
$$\frac{EA}{l_0} = k_n \qquad k_s = \frac{GA}{l_0}$$

E: the Young's Modulus of an element
A: the area that is represented by a spring
l_0: the original spring (or interface) length
ν: the Poisson's ratio of an element

Structural analysis

Time-explicit globally damped

- Time-explicit analyses model non-linear material behaviour



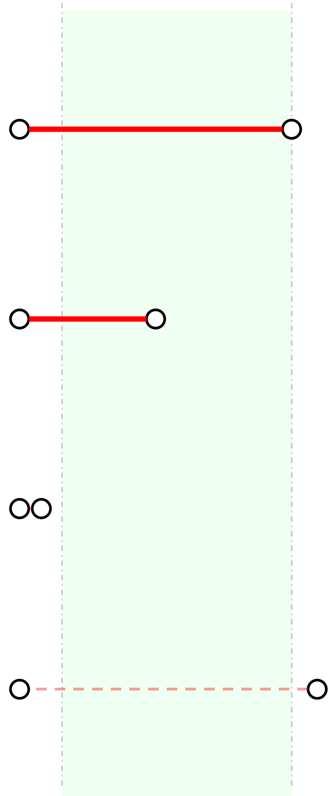
$$-90\% \leq \frac{\Delta l}{l_0} \leq 0\%$$

Original interface:

Compression 50%:

Compression 95%:
No stiffness modelled

Tension 10%:
No stiffness modelled

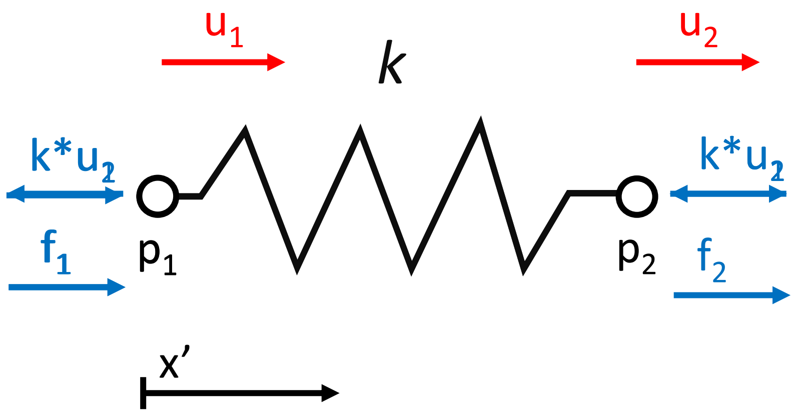


Structural analysis

Time-explicit globally damped

- Stiffness matrix relates the applied forces on an element to its degrees of freedom
- It is derived using the displacement method

$$\boldsymbol{K'u'} = \boldsymbol{f'}$$



$$= f_1$$

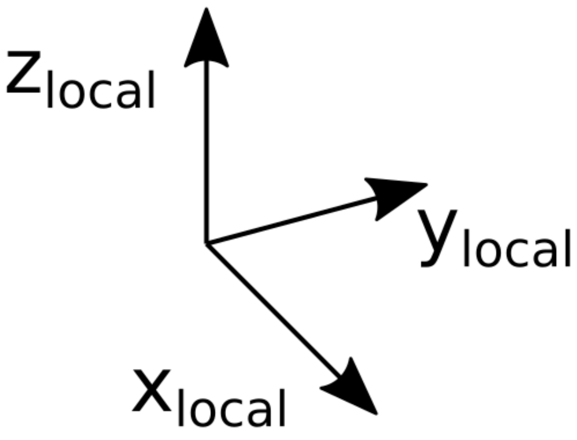
$$= f_2$$

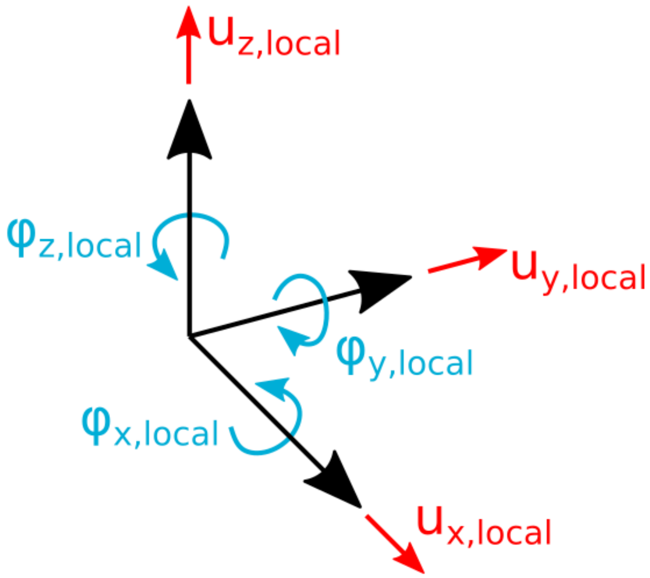
$$\begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$$

Structural analysis

Time-explicit globally damped

- Same procedure but 6 degrees of freedom per element instead of 1
- This results in a 12x12 local stiffness matrix instead of a 2x2 matrix

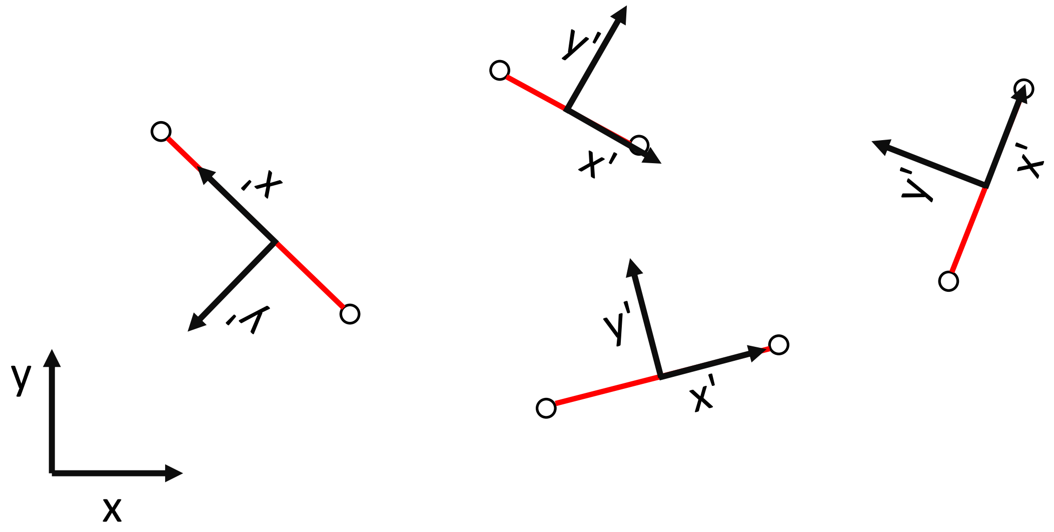




Structural analysis

Time-explicit globally damped

- Local coordinate systems need to be rotated to match the global coordinate system in order to do sensible calculations



$$T_{partial} = \begin{bmatrix} e'_{x[1]} & e'_{x[2]} & e'_{x[3]} \\ e'_{y[1]} & e'_{y[2]} & e'_{y[3]} \\ e'_{z[1]} & e'_{z[2]} & e'_{z[3]} \end{bmatrix}$$

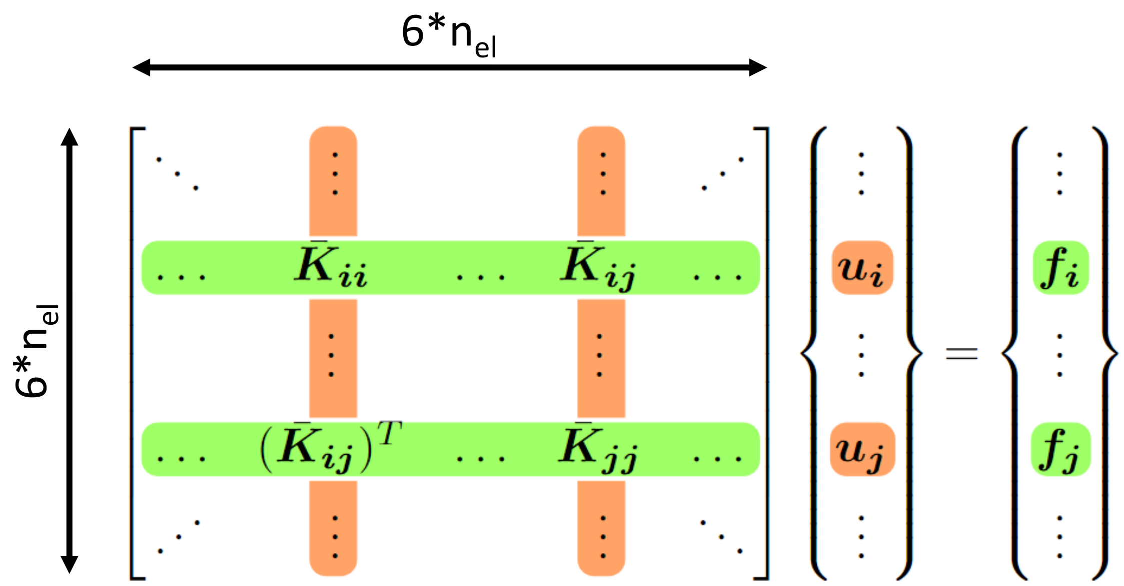
$$T = \begin{bmatrix} T_{partial} & 0 & 0 & 0 \\ 0 & T_{partial} & 0 & 0 \\ 0 & 0 & T_{partial} & 0 \\ 0 & 0 & 0 & T_{partial} \end{bmatrix}$$

$$a = T^T a' \qquad K = T^T K' T$$

Structural analysis

Time-explicit globally damped

- System stiffness matrix is assembled
- Assemble by summing values from all interface K-matrices



Structural analysis

Time-explicit globally damped

- Mass-matrix for each element is assembled by making use of the following expressions
- The inertia tensor is dependent on the geometry

$$\boldsymbol{f_i} = M_i \boldsymbol{a_i}$$

$$\boldsymbol{m_i} = \boldsymbol{I_i} \boldsymbol{\alpha_i}$$

$$\boldsymbol{I_i} = \begin{bmatrix} I_{xx,i} & -I_{xy,i} & -I_{xz,i} \\ -I_{yx,i} & I_{yy,i} & -I_{zz,i} \\ -I_{zx,i} & -I_{zy,i} & I_{zz,i} \end{bmatrix}$$

$$\begin{Bmatrix} \boldsymbol{f_i} \\ \boldsymbol{m_i} \end{Bmatrix} = \underbrace{\begin{bmatrix} M_i & 0 & 0 & 0 & 0 & 0 \\ 0 & M_i & 0 & 0 & 0 & 0 \\ 0 & 0 & M_i & 0 & 0 & 0 \\ 0 & 0 & 0 & I_{xx,i} & -I_{xy,i} & -I_{xz,i} \\ 0 & 0 & 0 & -I_{yx,i} & I_{yy,i} & -I_{zz,i} \\ 0 & 0 & 0 & -I_{zx,i} & -I_{zy,i} & I_{zz,i} \end{bmatrix}}_{\text{element mass matrix } \boldsymbol{M_i}} \begin{Bmatrix} \boldsymbol{a_i} \\ \boldsymbol{\alpha_i} \end{Bmatrix}$$

Structural analysis

Time-explicit globally damped

- Assembling system mass matrix:

$6*n_{el}$

$$\hat{M} = \begin{bmatrix} M_1 & 0 & \dots & 0 \\ 0 & M_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & M_n \end{bmatrix}$$

$6*n_{el}$

Structural analysis

Time-explicit globally damped

- Global damping coefficient is computed as

$$C = \beta \frac{\dot{\mathcal{E}}_k}{6 n_{el}}$$

$\dot{\mathcal{E}}_k$: change of kinetic energy of the system

β: parameter to scale the global damping

n_{el}: the number of elements in the structure

$$\mathcal{E}_k^{(t)} = \frac{1}{2} \left(\hat{\boldsymbol{u}}^{(t)} \right)^T \hat{\boldsymbol{M}}^{(t)} \hat{\boldsymbol{u}}^{(t)}$$

$$\dot{\mathcal{E}}_k^{(t)} \begin{cases} 0, & \text{if } t = 0 \\ \frac{\mathcal{E}_k^{(t)} - \mathcal{E}_k^{(t-\Delta t)}}{\Delta t}, & \text{otherwise} \end{cases}$$

Structural analysis

Time-explicit globally damped

- The equation of motion can finally be used to calculate accelerations at a time-step:

$$\hat{\ddot{\boldsymbol{u}}}^{(t)} = \hat{\boldsymbol{M}}^{(t)-1} \left(\hat{\boldsymbol{f}} - C^{(t)} \hat{\dot{\boldsymbol{u}}}^{(t)} - \hat{\boldsymbol{K}}^{(t)} \hat{\boldsymbol{u}}^{(t)} \right)$$

with

$$\hat{\boldsymbol{u}}_{[i]}^{(0)} = 0$$

$$\hat{\dot{\boldsymbol{u}}}_{[i]}^{(0)} = 0$$

Structural analysis

Time-explicit globally damped

- Displacements and velocities are calculated using midpoint numerical integration after the first time-step:

$$\dot{\hat{\boldsymbol{u}}}_{[i]}^{(\frac{1}{2}\Delta t)} = \dot{\hat{\boldsymbol{u}}}_{[i]}^{(0)} + \frac{1}{2}\Delta t\, \ddot{\hat{\boldsymbol{u}}}_{[i]}^{(0)}$$

$$\hat{\boldsymbol{u}}_{[i]}^{(\Delta t)} = \hat{\boldsymbol{u}}_{[i]}^{(0)} + \Delta t\, \dot{\hat{\boldsymbol{u}}}_{[i]}^{(\frac{1}{2}\Delta t)}$$

- And for subsequent time-steps:

$$\dot{\hat{\boldsymbol{u}}}_{[i]}^{(t+\frac{1}{2}\Delta t)} = \dot{\hat{\boldsymbol{u}}}_{[i]}^{(t-\frac{1}{2}\Delta t)} + \Delta t\, \ddot{\hat{\boldsymbol{u}}}_{[i]}^{(t)}$$

$$\dot{\hat{\boldsymbol{u}}}_{[i]}^{(t)} = \dot{\hat{\boldsymbol{u}}}_{[i]}^{(t+\frac{1}{2}\Delta t)} - \dot{\hat{\boldsymbol{u}}}_{[i]}^{(t-\frac{1}{2}\Delta t)}$$

$$\hat{\boldsymbol{u}}_{[i]}^{(t+\Delta t)} = \hat{\boldsymbol{u}}_{[i]}^{(t)} + \Delta t\, \dot{\hat{\boldsymbol{u}}}_{[i]}^{(t+\frac{1}{2}\Delta t)}$$

Structural analysis

Time-explicit globally damped

- The time-step size is related to the system's natural frequencies and the following value is taken to ensure stability:

$$\Delta t = 0.8\sqrt{\frac{diag\left(\hat{\boldsymbol{M}}^{(t)}\right)_{min}}{diag\left(\hat{\boldsymbol{K}}^{(t)}\right)_{max}}}$$

Structural analysis

Time-explicit globally damped

- The analyses converges when the out-of-balance forces at a time-step are smaller than 1% of the out-of-balance forces at the first time-step:

$$\hat{\Delta f}^{(t)} = \hat{f} - \hat{K}^{(t)}\hat{u}^{(t)} = \hat{f} - \hat{f}_{int}^{(t)}$$

$$\hat{\Delta f}_{[i]}^{(t)} = \begin{cases} 0, & \text{if } \hat{f}_{int[i]}^{(t)} = 0 \vee \hat{\dot{u}}_{[i]}^{(t)} > \dot{u}_{max} \\ \left(\hat{f} - \hat{f}_{int}^{(t)}\right)_{[i]}, & \text{otherwise} \end{cases}$$

$$\|\hat{\Delta f}^{(t)}\|_2 < 10^{-2}\,\|\hat{\Delta f}^{(\Delta t)}\|_2$$

- The second expression ensures that constrained DOFs or falling elements are not considered for convergence

Structural analysis

Time-explicit locally damped

- The following expressions are used in the time-explicit locally damped procedure:

$$\dot{\boldsymbol{u}}^{(t+\frac{1}{2}\Delta t)} = \dot{\boldsymbol{u}}^{(t-\frac{1}{2}\Delta t)} + \Delta t\, \hat{\boldsymbol{M}}^{-1} \left(\hat{\boldsymbol{f}}^{(t)}_{\boldsymbol{int}} - \hat{\boldsymbol{f}}^{(t)}_{\boldsymbol{d}} \right)$$

$$\hat{\boldsymbol{f}}^{(t)}_{\boldsymbol{d}} = \alpha |\vec{\hat{\boldsymbol{f}}}^{(t)}_{\boldsymbol{int}}| \odot sgn\left(\dot{\boldsymbol{u}}^{(t-\frac{1}{2}\Delta t)} \right)$$

where:

$|\vec{\hat{\boldsymbol{f}}}^{(t)}_{\boldsymbol{int}}|$: a vector containing the absolute values of the internal force vector

α: parameter to scale the local damping

$\odot$: Hadamard product (element-wise multiplication)

$$sgn\left(\dot{\boldsymbol{u}}^{(t-\frac{1}{2}\Delta t)}_{[i]} \right) = \begin{cases} 1, & \text{if } \dot{\boldsymbol{u}}^{(t-\frac{1}{2}\Delta t)}_{[i]} > 0 \\ -1, & \text{if } \dot{\boldsymbol{u}}^{(t-\frac{1}{2}\Delta t)}_{[i]} < 0 \\ 0 & \text{otherwise} \end{cases}$$

Structural analysis

Static

- The static solution procedure does not need time-steps to perform calculations.
- Calculates solutions considerably faster
- Does not consider non-linear material behaviour (tension is allowed)

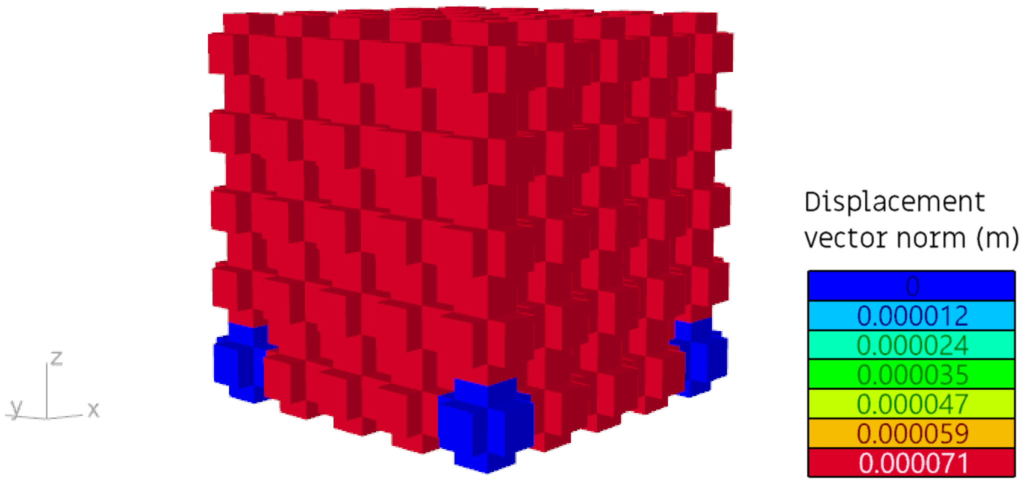
$$\hat{\boldsymbol{K}}\hat{\boldsymbol{u}} = \hat{\boldsymbol{f}}$$

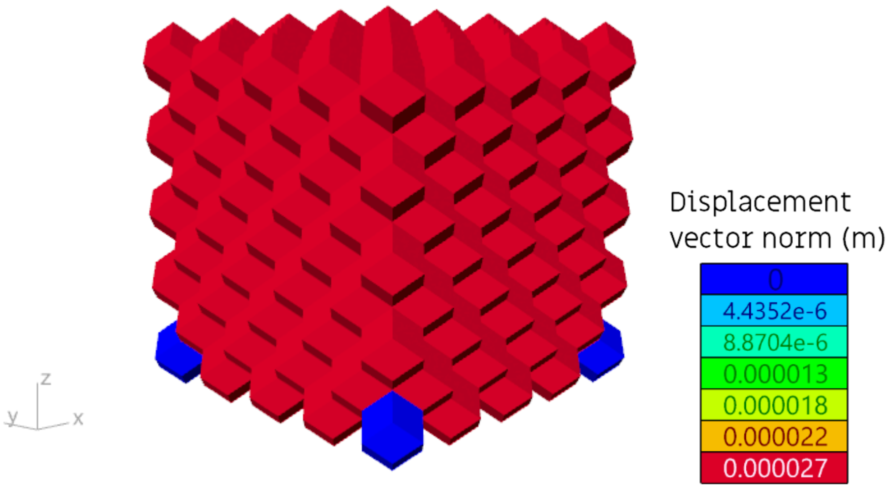
$$\hat{\boldsymbol{u}}_{\boldsymbol{f}} = \hat{\boldsymbol{K}}_{\boldsymbol{f}}^{-1}\hat{\boldsymbol{f}}_{\boldsymbol{f}}$$

Structural analysis

Results globally damped

- Unstable analysis
- Sudden extremely large displacements

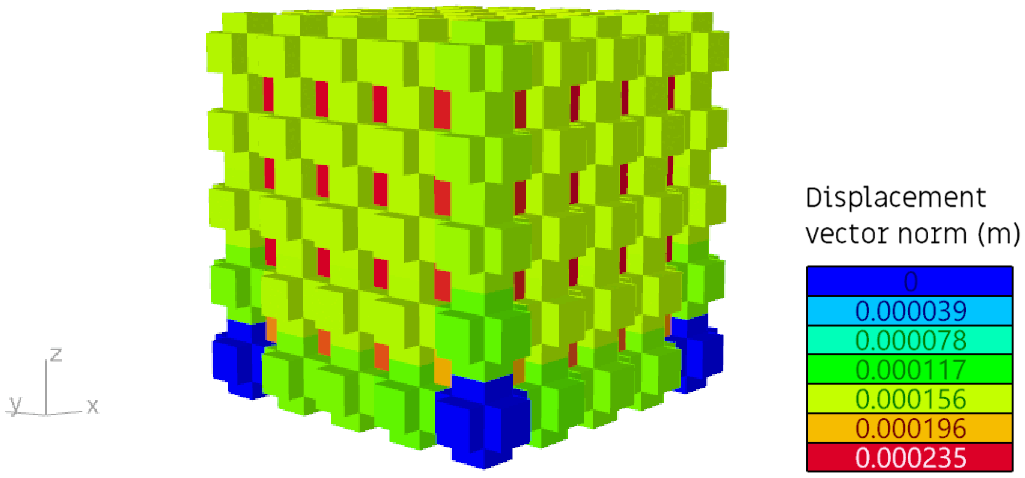


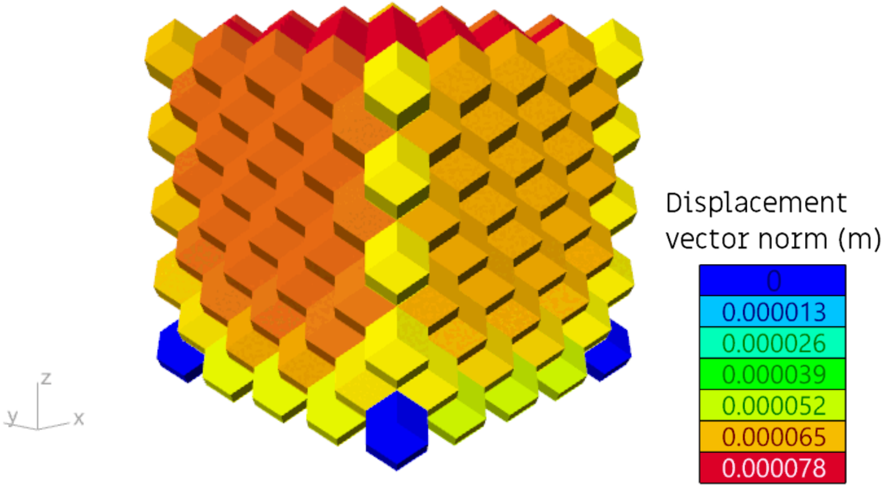


Structural analysis

Results locally damped

- Stable analysis

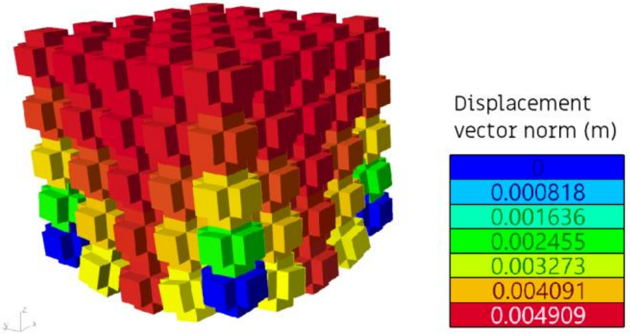


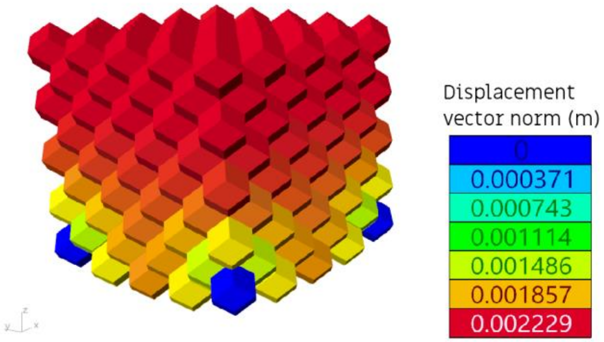


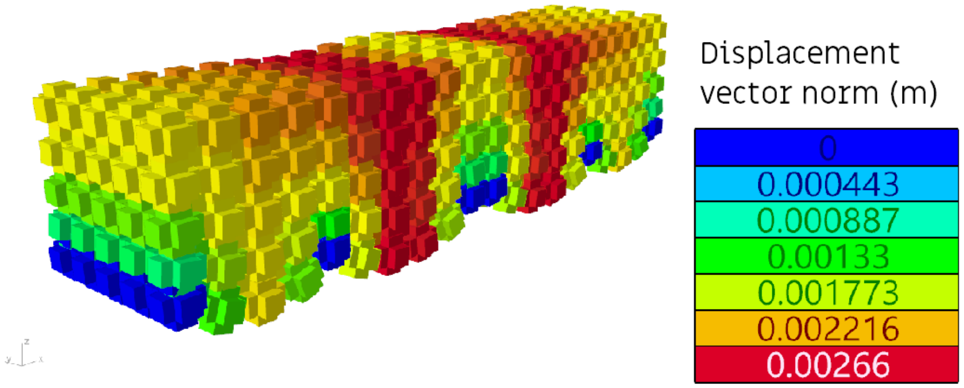
Structural analysis

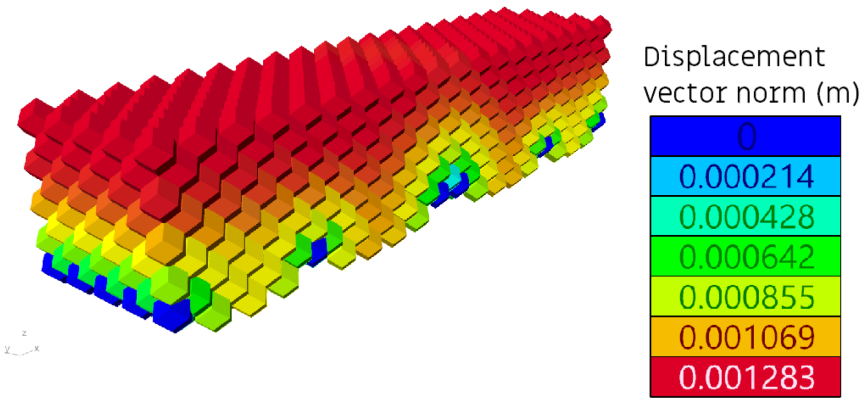
Results

- Larger structures
- Inclusion of tension









Optimization

- Optimization is carried out by using the Method of Moving Asymptotes (MMA)

  Svanberg, K. (1987). The method of moving asymptotes—a new method for structural optimization.International journal for numerical methods in engineering, 24(2), 359–373.

- Objective function should be minimized

$$min\left(c(\boldsymbol{\chi})\right) = \mu_0 \cdot \left(\hat{\boldsymbol{u}}^T \hat{\boldsymbol{f}} - \hat{\boldsymbol{u}}^T \hat{\boldsymbol{K}} \hat{\boldsymbol{u}}\right)$$

$\boldsymbol{\chi}$: vector containing design variables for the topology optimization, $0.001 \leq \boldsymbol{\chi}_{[i]} \leq 1$

μ_0: parameter to scale the objective function value

Optimization

- A volume constraint is implemented as follows:

$$\mu_1 \cdot \left(\frac{\sum_{i=1}^{n_{el}} \chi_{[i]}}{n_{el}} - f \right) = 0$$

V_0: the full 'volume' of the system if all design variables are set to 1, this is equal to the number of elements in the structure

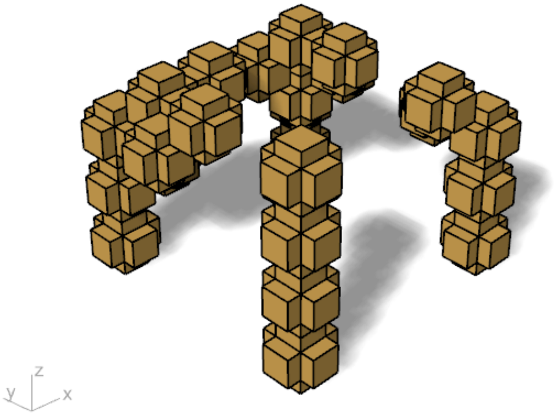
f: the target mean value of $\chi, 0 < f < 1$

μ_1: parameter to scale the volume constraint value

- This constraint ensures that the resulting structure has some set volume

Optimization

- A roof constraint should also be implemented:



$$\boldsymbol{g}(\chi) \begin{cases} \mu_2 \cdot \left(\left(1 - \frac{1}{q+1} \right) - \chi_{[i]} + \frac{1}{q+1} \chi_{[i]}^{q+1} + \sum_{K \neq i} \chi_{[i]} \chi_{[K]}^{q} \right), & \text{if } i \in K \\ 0, & \text{otherwise} \end{cases}$$

q: parameter to penalize intermediate values within the roof constraint $q = 3$ seemed to work well

μ_2: parameter to scale the roof constraint value

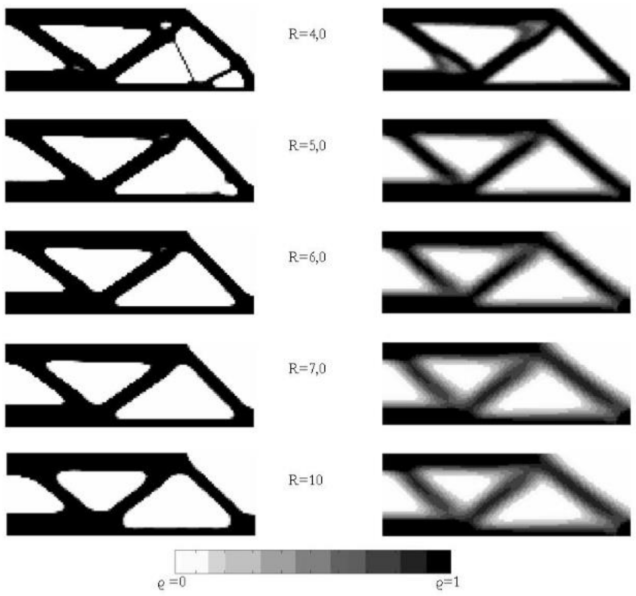
- This constraint ensures that the resulting structure has a spanning structure or roof

Optimization

- A sensitivity filter is used to ensure mesh-independence

Different results from different mesh sizes

Talischi, C., Paulino, G. H., Le, C. H., Paulino, G. H., Pindera, M.-J., Dodds, R. H., Rochinha, F. A., Dave, E., & Chen, L. (2008). Topology optimization using wachspress-type interpolation with hexagonal elements. AIP Conference Proceedings. https://doi.org/10.1063/1.2896796

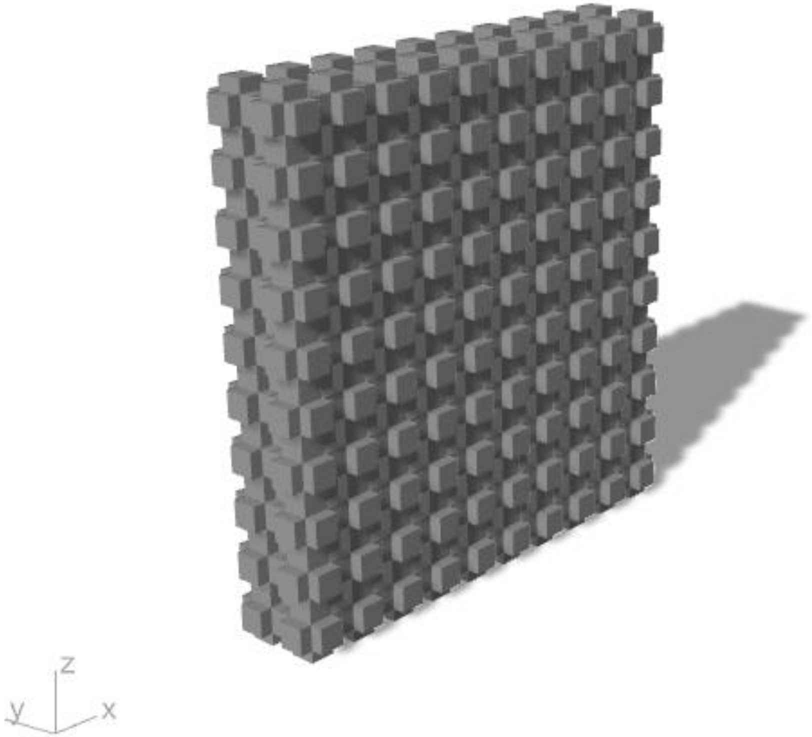


Effect of different filter sizes

Pedersen, C. G., Lund, J. J., Damkilde, L., & Kristensen, A. S. A. (2006). Topology optimization-improved checker-board filtering with sharp contours. Proceedings of the 19th Nordic Seminar on Computational Mechanics, 182–185.

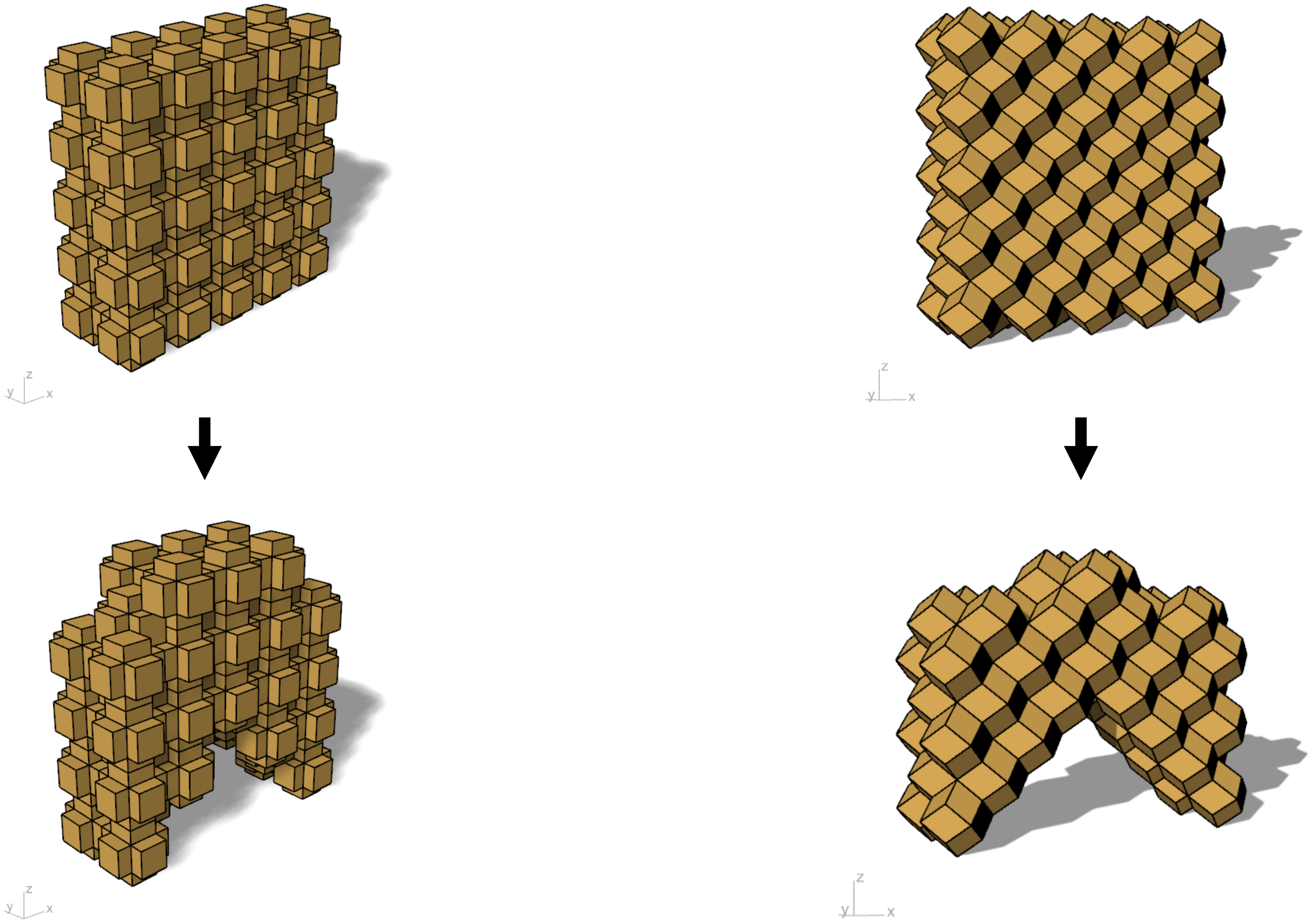
Optimization

Visualization of optimization process



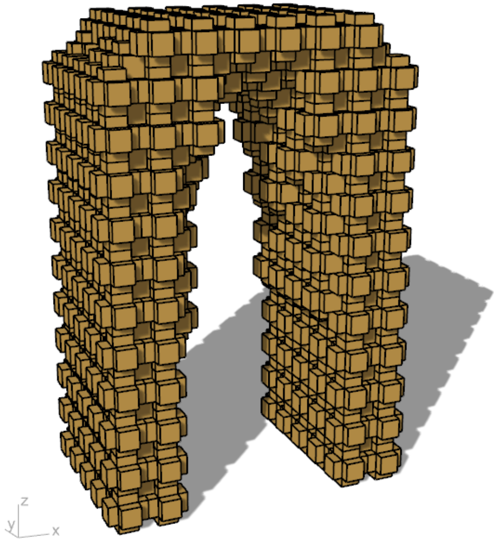
Optimization

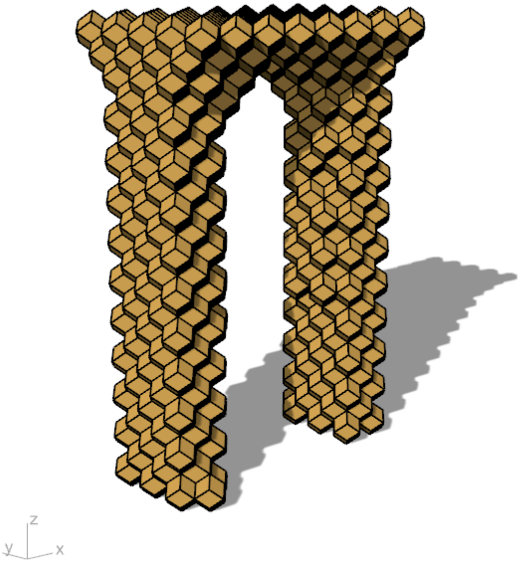
Results time-explicit

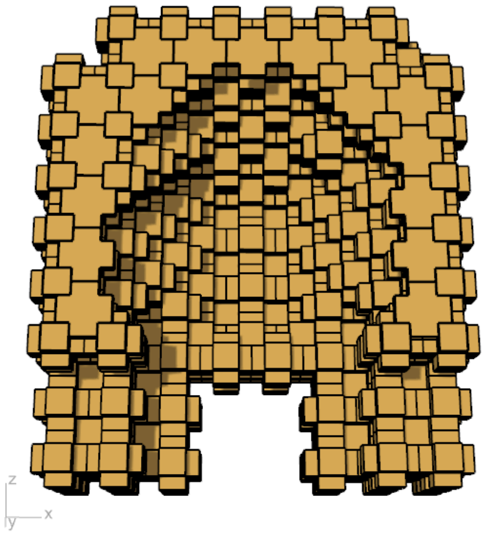


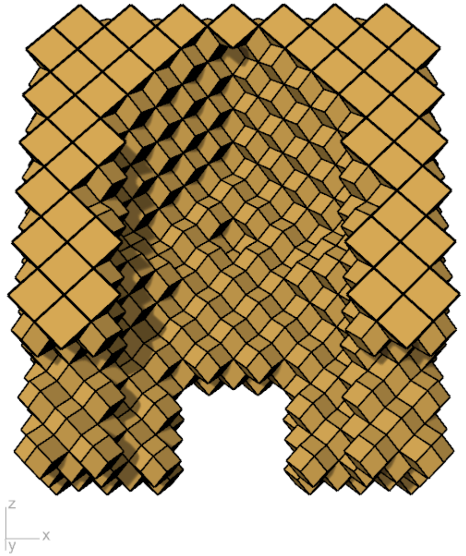
Optimization

Results static





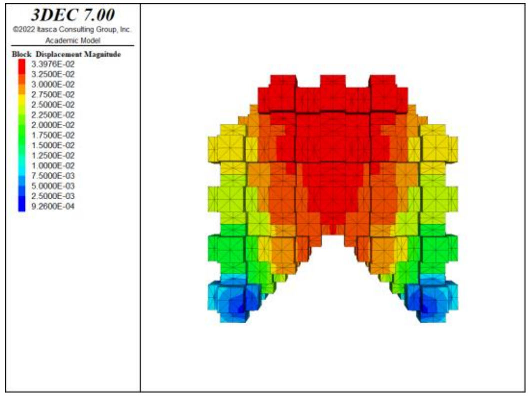


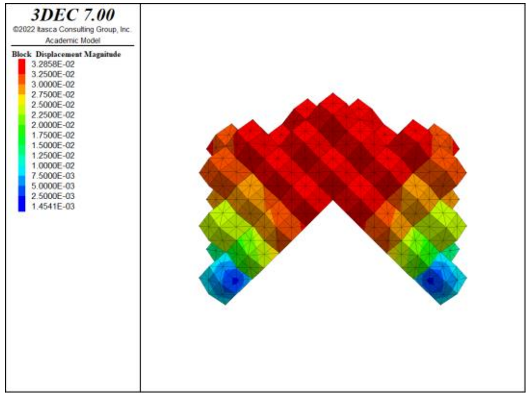


Verifications

3DEC™

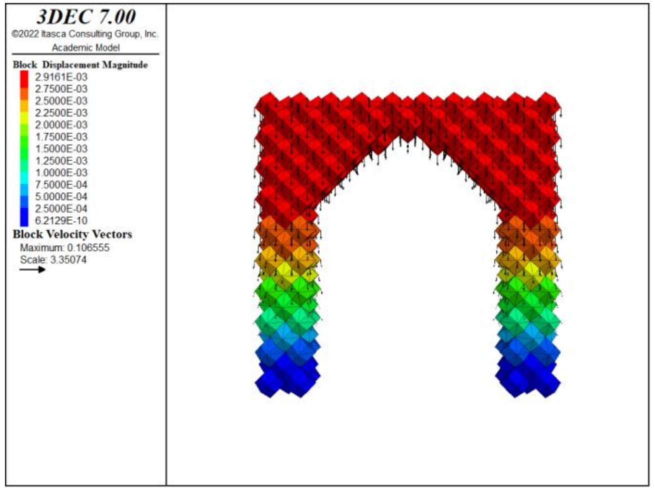
- Verifications using 3DEC
- 3DEC uses different stiffness definitions than the developed model (values need to be converted beforehand!)
- The analyses of time-explicitly optimized structures converged quickly and without failures

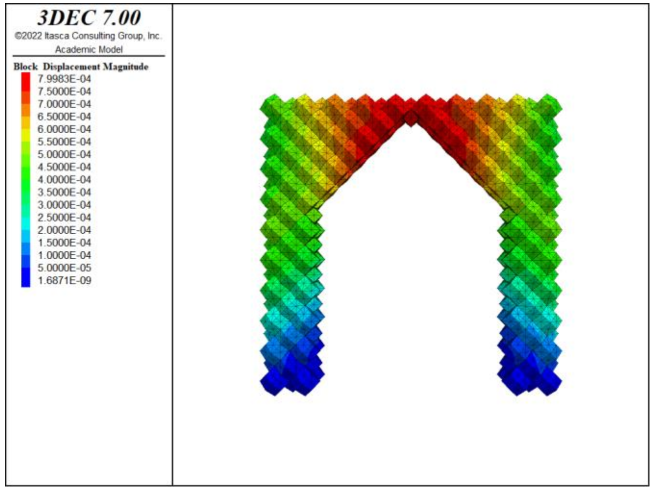




Verifications

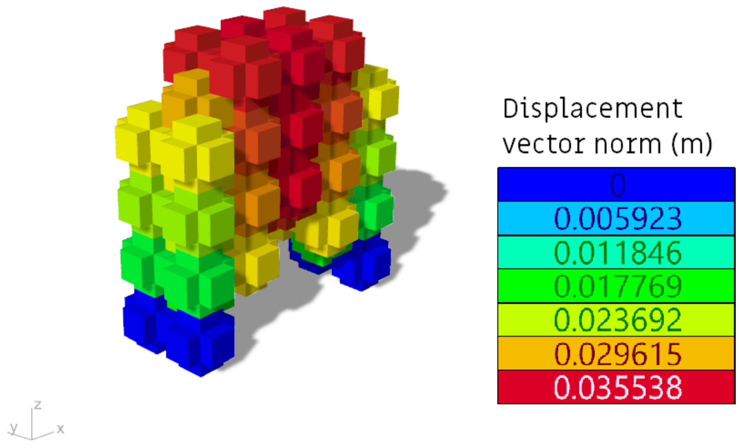
- In statically optimized structures some blocks may fall out
- After removing unsupported elements there are no further issues



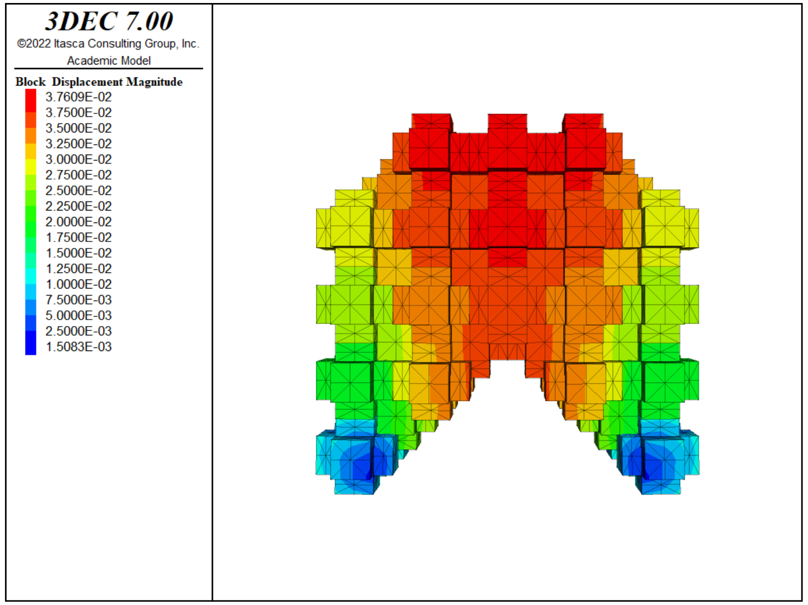


Verifications

- ~5% relative difference between the maximum displacements from 3DEC and implemented model



Max. displacement: $3.5538 \cdot 10^{-2}$m

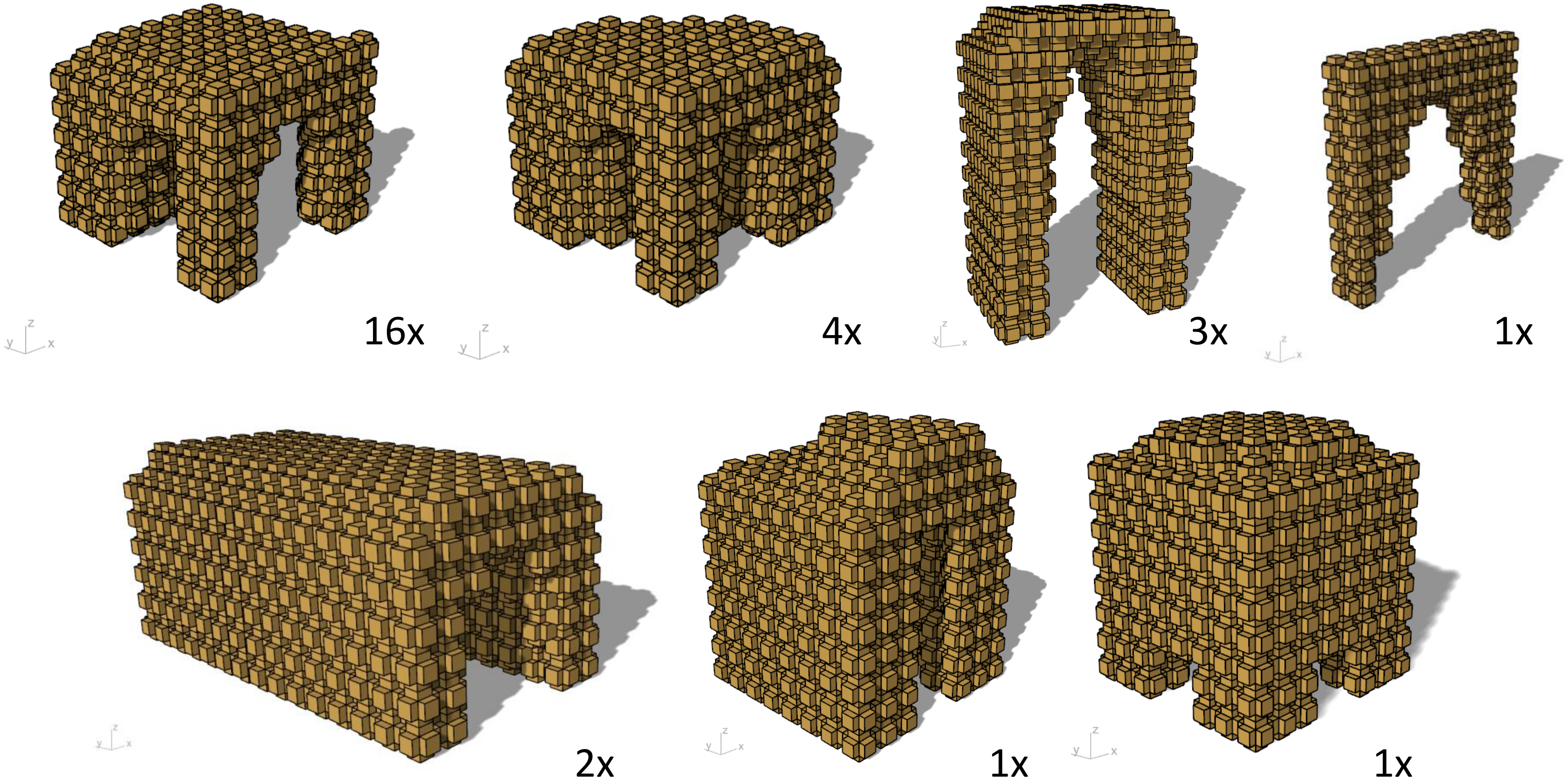


Max. displacement: $3.7609 \cdot 10^{-2}$m

Application

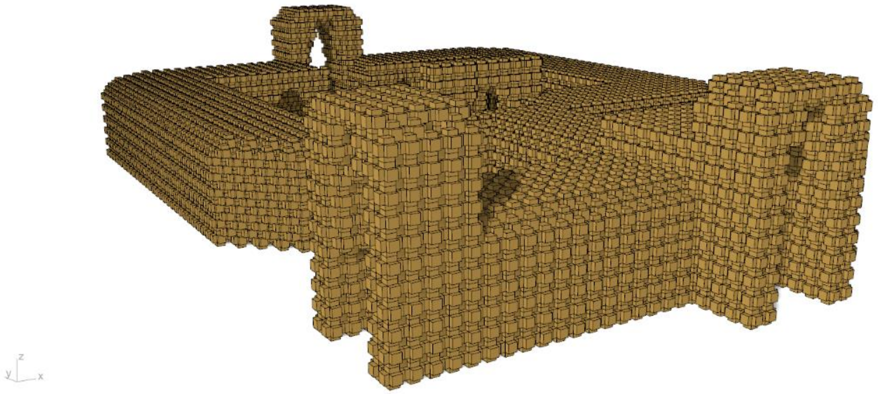
Current implementation

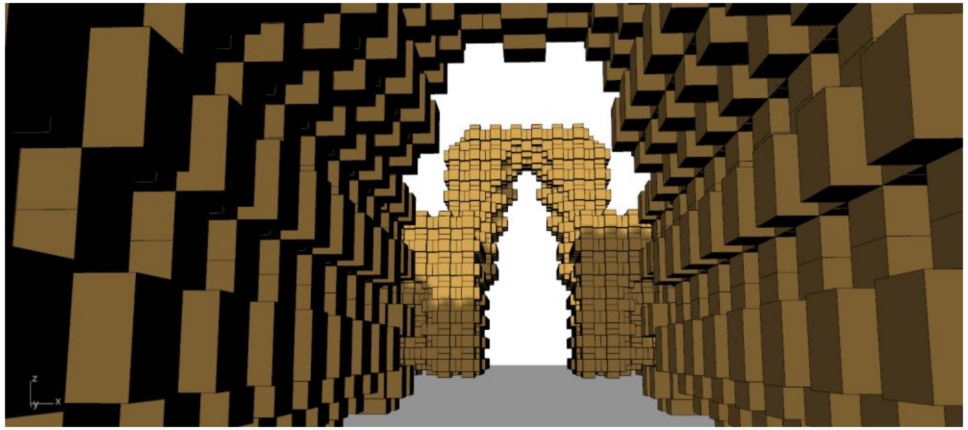
- Assemble a larger structure from smaller optimized parts

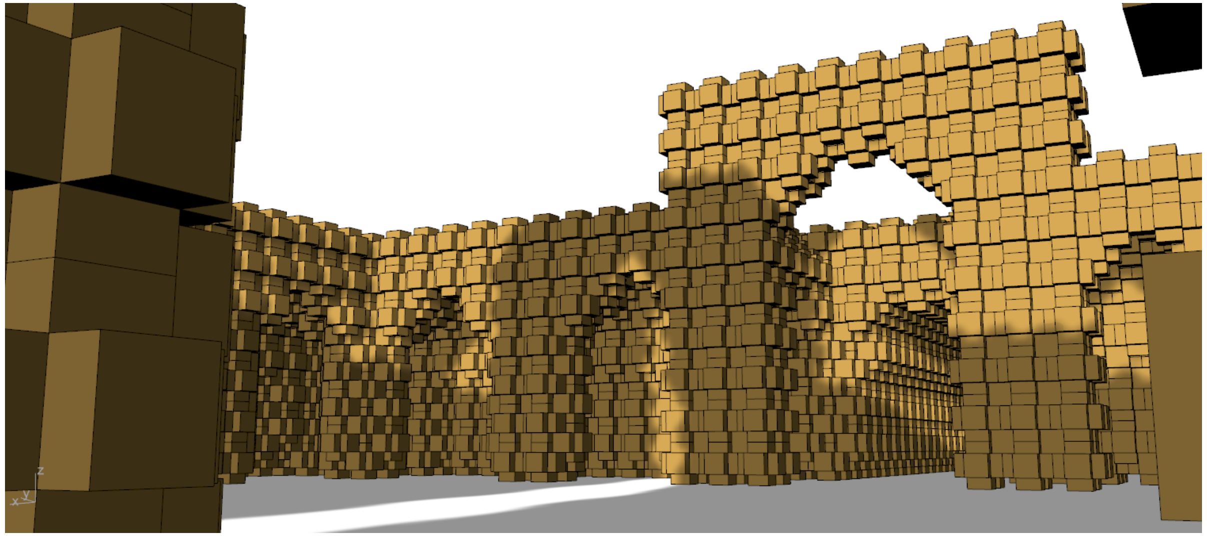


Application

Current implementation



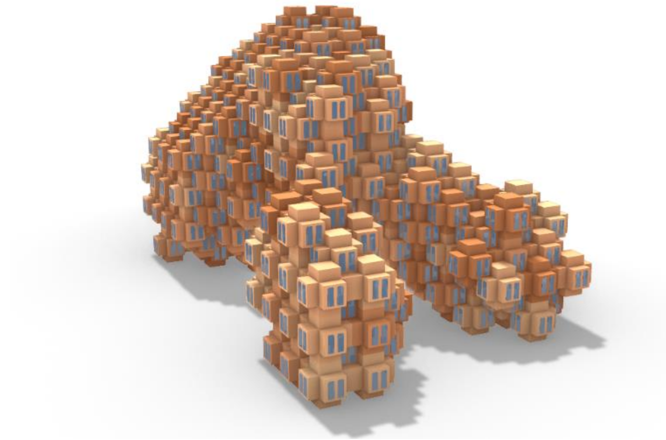


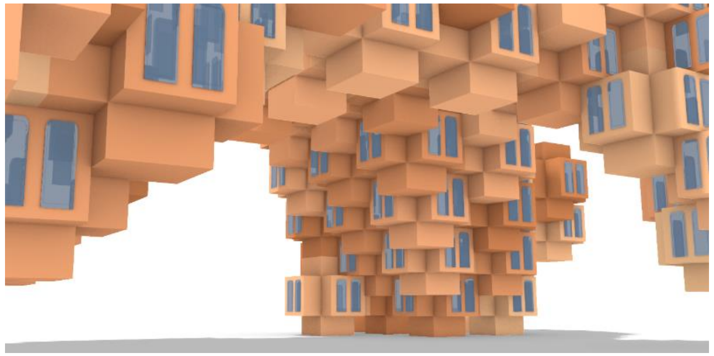


Application

Current implementation

- Optimizations considering elements not as bricks, but as building modules





5 Farming Bridges by Vincent Callebaut

Vincent Callebaut Architectures. (n.d.). The 5 farming bridges. https://vincent.callebaut .org/object/171023_mosul/mosul/projects

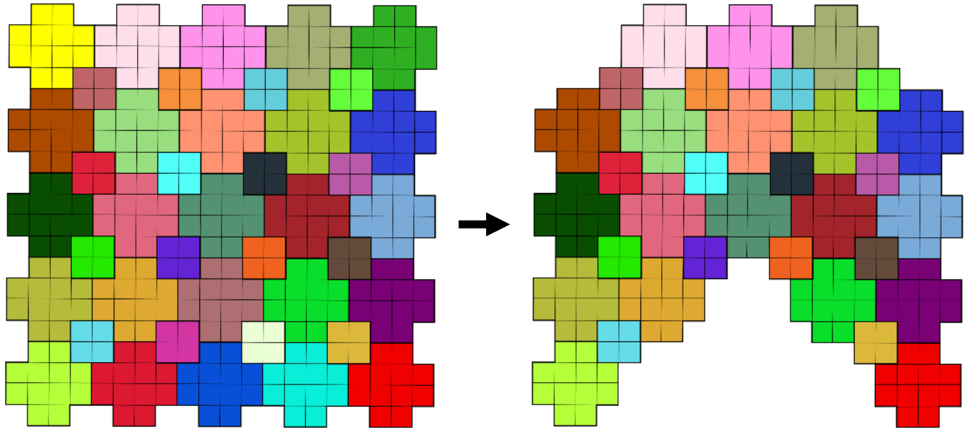
Habitat 67 by Moshe Safdie

E. Blue. (n.d.). Habitat 67. https://www.mtl.org/en/experience/revolutionary-montreal-icon-habitat-67

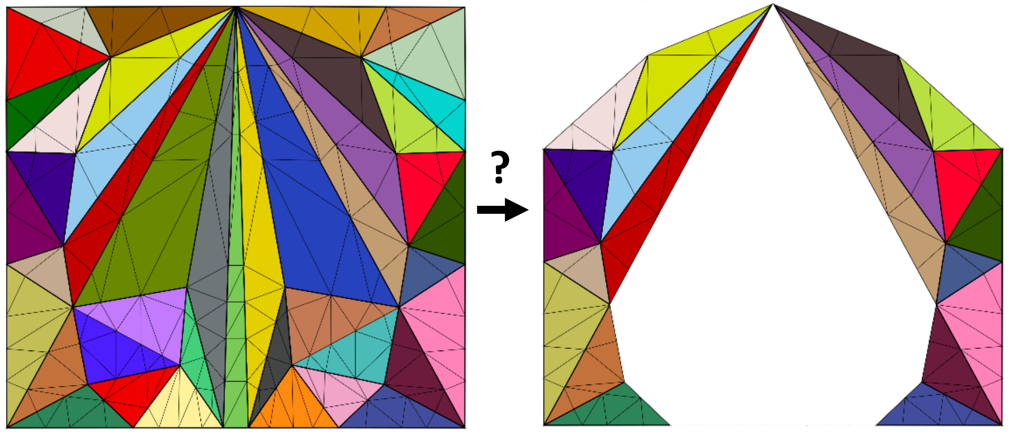
Application

Future developments

- Currently optimizations are limited to predefined element shapes/tessellations
- Implementing the option to import custom tessellations will provide more freedom in architectural designs of masonry structures



Optimization using CS elements

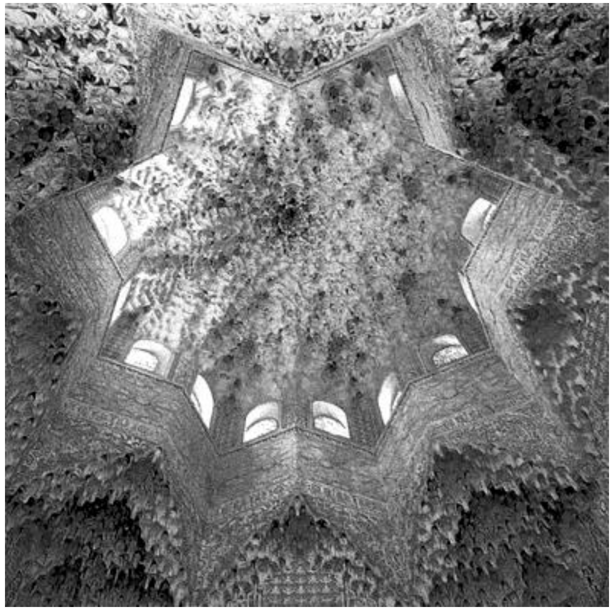


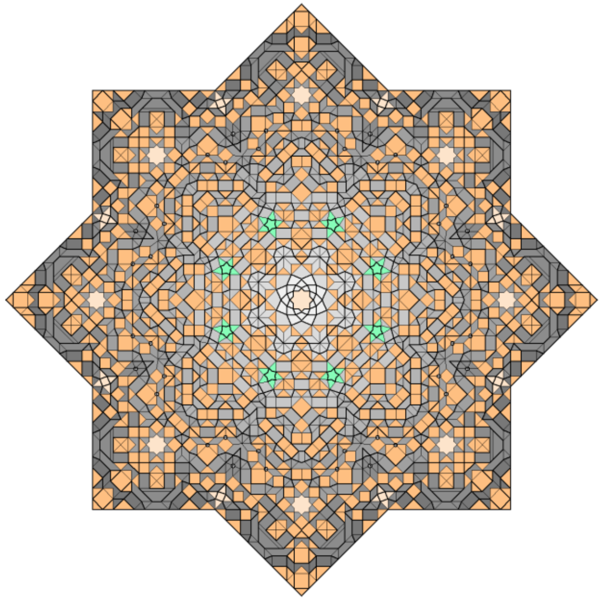
Possible optimization using custom elements

Application

Future visions

- Implementing custom tessellations may lead to the optimizer outputting muqarnas designs





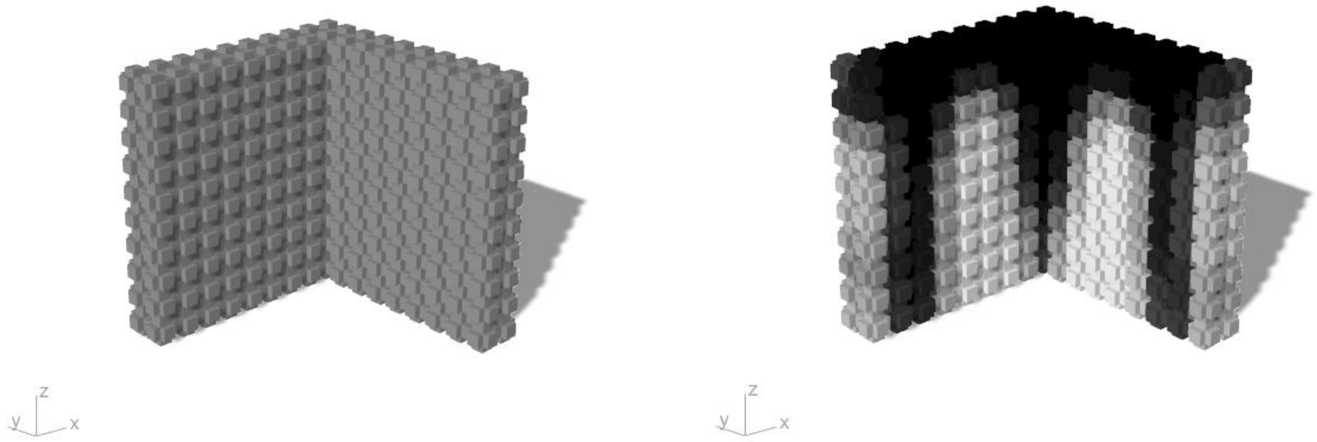
Photograph and drawing of a muqarnas in Alhambra (Granada, Spain)

S. Takahashi. (n.d.). Sala de los abencerrajes. alhambra palase, granada. spain. 1354-91. http://www.shiro1000.jp/muqarnas/data/034/34.htm

Application

Future visions

- Implementing custom tessellations may also prove to be useful in restoration projects of masonry structures
- The optimizer can give an overview of more or less essential structural elements in existing structures, which may aid in managing (re)construction projects



An overview of more and less essential structural elements of two CS walls

Conclusions

- Time-explicit optimizations are slower than static optimizations
- Static optimizations may not be as accurate as time-explicit optimizations, because it models tension between elements
- After removing any unsupported elements (static optimizations only), none of the optimized structures show signs of failure
- Large structures can be optimized by parts to speed up the optimization process using this algorithm
- Structural optimizations can also be carried out for modular buildings

Conclusions

- The current implementation of topology optimization of discrete element structures is limited to predefined elements, but implementing the functionality of importing custom tessellations may provide much more design freedom