A note on the stability of accelerated motions

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Summary

When the directional stability of a ship or any other sea going vessel is investigated, it is usually assumed that the vessel has an initial constant forward reference motion. This assumption is relaxed in the note, the vessel being considered to have instead an accelerating or decelerating motion straight ahead. To permit this it is necessary to adopt some simplifying assumptions and those adopted have been employed previously for ships even though they are known not to be strictly valid; in particular, the hydrodynamic actions are expressed in terms of slow motion derivatives which are assumed to be constant and independent of forward speed in their dimensionless form. The analysis suggests that criteria of directional stability which are relevant to a steady reference motion do not necessarily apply if the reference motion is not constant.

1. INTRODUCTION

In the dynamics of ships or any other sea going vessel in general it is usual to investigate directional stability under two assumptions:

a. Slow motion derivatives provide an adequate description of the relevant hydrodynamic actions.

b. The reference motion, whose stability is being assessed, is forward with a constant velocity.

The writers know of no investigations in which these assumptions are completely discarded in ship dynamics, though attempts have been made to cope with varying velocity.

If, at an instability boundary departures from the reference motion is both uni-directional (ie divergent) and infinitely slow, then slow motion derivatives should indeed provide an adequate specification of the hydrodynamic actions. But what if the vessel is accelerating or decelerating (ie has an unsteady reference motion)?

The purpose of this note is to theoretically that accelerated motions cannot just be ignored. It is possible for an illustrate vessel to be stable at some constant reference speed $\overline{U}$ but to be directionally unstable when passing through that same speed during accelerated motion.

The comparable aeronautical problems appear to have received surprisingly little attention. Mention should be made, however, of a paper by Collar (1957) who seems to have been the first to show that the stability of accelerated motions could be a problem; he showed, inter alia, that questions arise as to what is meant by 'stability' in this context.

The practical significance of all this in the context of general ship dynamics is not only that, under certain extreme conditions, directional stability may be lost. The additional point, to which the writers would draw attention, is that if large changes of stability occur during accelerated motion they could have profound influence on handling characteristics, possibly with serious implications in the maintenance of a correct course.

This note is admittedly somewhat mathematical and possibly, artificial. It is solely concerned, with the logic (or lack thereof) of a mathematical technique. In addition it relates only to "accelerated motion" in a straight line. But it does cause one to wonder about the stability of turning, for that is another form of accelerated motion.

In other words, this is primarily an exercise to illustrate some of the problems which can arise in solving equations with time dependent coefficients and of the possibilities for stability which ensue. It is realised that, in practice, there may be marked differences between the flows in steady and unsteady reference motions. In particular, in retardation the loss of directional stability is probably due largely to the marked changes in flow near the stern of the vessel when the screws are in reverse. Nevertheless the present examples perhaps give some insight into the difficulties of the problems.

There are at least two classes of time dependent coefficients which can be present in the equations of motion, when the reference motion is not steady. First the velocity of the reference motion will be a function of time, which we will denote by U(t). Secondly if we assume that the hydrodynamic actions produced by a disturbance then these derivative are also likely to be functions of time. Indeed the concept of a slow motion derivative is not clear for unsteady reference motions and requires much further thought which is beyond the scope of this paper. For the present we will assume that constant slow motion derivatives can be defined. In this way it is hoped to illustrate in a qualitative sense the effect of including a time dependent term U(t) in the equations of motion.

Some indication of the results due to variations in the slow motion derivatives with reference velocity can be made by a quasi-steady analysis. In this approach the stability of the vessel is assessed in a conventional manner with respect to a series of steady reference speeds within the range of speeds encompassed by U(t). The validity of this technique would seem to be open to question and is only plausible for very slowly changing reference motions. It is not possible at present, however, to specify how "small" the magnitude of U(t) must be, in order that the quasi-steady assumption should be tenable.

2. STEADY REFERENCE MOTION

Let us initially consider the horizontal directional stability of a surface ship having a constant reference forward velocity $\bar{U}$. The general linearised equations of motion of the ship are obtained by performing the following operations.

a. Setting up equations describing the reference motion.

b. Developing equations describing small perturbations motions from this initial reference motion.

c. Subtracting a. from b. and linearising the resultant equations.

This procedure is discussed in the literature (see for example Abkowitz (1964), Mandel (1967) and Bishop and Parkinson(1970)) and the linearised equations for unsteady motion in the horizontal plane are found to be

$$(m - Y_{\dot{v}})\,\dot{v} - Y_v v - Y_{\dot{r}}\dot{r} + \{m\bar{U} - Y_r\}\,r = 0$$

$$- N_{\dot{v}}\dot{v} - N_v v + (I_z - N_{\dot{r}})\,\dot{r} - N_r r = 0$$

in the usual notation.

Elimination of either v(t) or r(t) produces a single differential equation, the constants involved being the same for v(t) and r(t). That is,

$$a\ddot{v} + b\dot{v} + cv = 0$$

$$a\ddot{r} + b\dot{r} + cr = 0$$

where

$$a = (m - Y_{\dot{v}})\,(I_z - N_{\dot{r}}) - N_{\dot{v}}\,Y_{\dot{r}}$$

$$b = N_{\dot{v}}(m\bar{U} - Y_r) - N_v Y_{\dot{r}} - N_r(m - Y_{\dot{v}}) - Y_v(I_z - N_{\dot{r}})$$

$$c = N_v\,(m\bar{U} - Y_r) + Y_v N_r$$

If a > o, the reference motion is stable, when b > o and c > o. The important stability criterion turns out to be c > o, that is

$$N_v(m\bar{U} - Y_r) + Y_v N_r > 0$$

3. UNSTEADY REFERENCE MOTION

Now consider the directional stability of a surface ship which does not possess a constant forward velocity but has, instead, a reference velocity U(t) and acceleration $\dot{U}(t)$. A positive value of $\dot{U}(t)$ implies acceleration and a negative one deceleration, of course. By applying the procedure previously described, we obtain the linearised equations of motion

$$(m - Y_{\dot{v}})\,\dot{v} - Y_v v - Y_{\dot{r}}\dot{r} + \{mU(t) - Y_r\}r = 0$$

$$- N_{\dot{v}}\dot{v} - N_v v + (I_z - N_{\dot{r}})\dot{r} - N_r r = 0$$

where the slow motion derivatives are assumed to be constants.

The single equation of motion in the yaw variable r(t) is now

$$\{(m - Y_{\dot{v}})(I_z - N_{\dot{r}}) - N_{\dot{v}} Y_{\dot{r}}\} \ddot{r} + \Big[N_{\dot{v}} \{m U(t) - Y_r\}$$

$$- N_v Y_r - N_r (m - Y_{\dot{v}}) - Y_v (I_z - N_{\dot{r}}) \Big] \dot{r} + \Big[N_v \{m U(t) - Y_r\}$$

$$+ N_{\dot{v}} m \dot{U}(t) + Y_v N_r \Big] r = 0$$

The equation in the sway variable, v(t), is more complicated since it is difficult to eliminate the term U(t) r(t) from the coupled differential equations.

The previous equations for the steady forward reference motion are seen to be a special case of these more general equations. The form of the latter now depends in part on the nature of U(t) and $\dot{U}(t)$ even though it is still assumed that the fluid actions may be represented in a satisfactory manner by the coefficients of a Taylor's series.*

The solution of the uncoupled equation for r(t) poses many difficulties. By way of illustration, let us consider the effect on the stability of motion when U(t) and $\dot{U}(t)$ have certain simple forms.

4. ACCELERATING REFERENCE MOTION

A simple example of accelerating reference motion can be found by assuming that

$$U(t) = U_o(1 + \alpha t), \quad \dot{U}(t) = U_o \alpha,$$

where α and U_o are both positive constants. The uncoupled equation of free yawing motion reduces to:

$$a\ddot{r} + (b + dt)\dot{r} + (c + d + et)r = 0$$

where the forms of a, b and c have been given before, provided that U is replace by U_o, and

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*In fact, the fluid actions depend on the past history of the ship motions and so the functional approach as developed by Bishop, Burcher and Price (1974 a, b, c) would be a more appropriate formulation of the fluid actions for arbitrary motions U(t) and $\dot{U}(t)$. Presumably the concept of slow motion derivatives really needs further examination if they are associated with small departures from an unsteady reference motion; are they, for example, to be regarded as time dependent? This is not something that we shall follow up here however.

$$d = m N_v U_o \alpha$$

$$e = m N_v U_o \alpha$$

This equation can be rewritten in the form

$$\ddot{r} + (B + Dt)\dot{r} + (C + Et)r = 0$$

where

$$B = b/a, \; D = d/a, \; C = (c + d)/a, \; E = e/a$$

Towards solving this equation, let

$$r(t) = \eta(\xi) \exp(- Et/D)$$

where

$$\xi = \sqrt{(|D|)} \left(t + \frac{BD - 2E}{D^2}\right)$$

The equation describing the free motion then becomes

$$\eta'' + \xi\eta' + D^{-3} (E^2 + CD^2 - BED) \eta = 0 \qquad \text{if } D > 0$$

$$\eta'' - \xi\eta' - D^{-3} (E^2 + CD^2 - BED) \eta = 0 \qquad \text{if } D < 0$$

As shown by Kamke (1943) solutions of this equation depend on the sign of D (= d/a).

<u>Case I, D > 0</u>

Let

$$\gamma = D^{-3} (E^2 + CD^2 - BED)$$

Then the equation describing the free motion is

$$\eta'' + \xi\eta' + \gamma \eta = 0.$$

This equation has a solution in terms of a confluent hypergeometric function F_1 as follows

$$\eta(\xi) = \xi^{-\frac{1}{2}} e^{-\frac{1}{4}\xi^2} \begin{cases} F_1\left(\frac{\gamma}{2} - \frac{1}{4}, \frac{1}{4}, \frac{\xi^2}{2}\right) & \text{if } \gamma > 0 \\ F_1\left(-\frac{\gamma}{2} - \frac{1}{4}, \frac{1}{4}, \frac{\xi^2}{2}\right) & \text{if } \gamma < 0 \end{cases}$$

and for the solution in the time domain, the term

$$e^{-\frac{1}{4}\xi^2} = e^{-\frac{1}{4}|D| \left(t + \frac{BD - 2E}{D^2}\right)^2}$$

appears to be dominant as $t \to \infty$. Thus we find that

$$\lim_{t\to\infty} r(t) \to 0$$

indicating that the motion is stable provided that $N_{\dot{v}} > 0$, as we have followed convention and expressed the equations in the form which makes $a > o$.

Case II, $D < 0$

The relevant differential equation describing the free yaw motion may be written in the form

$$\eta''' - \xi\eta'' + \gamma\, \eta = 0$$

where

$$\gamma = - D^{-3} (E^2 + CD^2 - BED)$$

Now we must consider the implications of whether γ is positive or negative.

Case (a). $\gamma > 0$

$$\eta(\xi) = e^{\frac{1}{2}\xi^2} u(\xi)$$

so that

$$u'' + \xi u' + (1 + \gamma)u = 0$$

This equation has a solution in terms of a hypergeometric function F_1 as follows:

$$u(\xi) = \xi^{-1/2}\, e^{\xi^2/4} F_1 \left[(\frac{\gamma}{2} + \frac{1}{4}),\ \frac{1}{4},\ \frac{\xi^2}{2} \right]$$

For the solution in the time domain, the term

$$e^{\frac{\xi^2}{4}} = e^{\frac{1}{4}|D| \left(t + \frac{BD - 2E}{D^2}\right)^2}$$

appears to be dominant when $t \to \infty$. Thus we find that

$$\lim_{t\to\infty} r(t) \to \infty$$

indicating that the motion is unstable.

The condition $\gamma > 0$ and $D < 0$ therefore implies that

$$E^2 + CD^2 - BED > 0$$

or substituting for B, C, D and E we have,

$$N_v^{\,2} (m - Y_{\dot v}) (I_z - N_{\dot r}) + m\, U_o\, \alpha\, N_{\dot v}^{\,3} + N_{\dot v}^{\,2}\, Y_v\, N_r$$

$$+ N_v\, N_{\dot v}\ \{N_r\, (m - Y_{\dot v}) + Y_v (I_z - N_{\dot r})\} > 0$$

Case (b). $\gamma < 0$

Let $\gamma_1 = -\gamma > 0$ so that

$$\eta'' - \xi\eta' - \gamma_1\, \eta = 0$$

This equation has a solution of the form

$$\eta(\xi) = C_1\{1 + \sum_{n=1}^{\infty} \frac{\gamma_1(\gamma_1 + 2)\,(\gamma_1 + 4) \ldots (\gamma_1 + 2n - 2)\, \xi^{2n}}{(2n)!}\}$$

$$+ C_2\{\xi + \sum_{n=1}^{\infty} \frac{(\gamma_1 + 1)\,(\gamma_1 + 3)\, \ldots\, (\gamma_1 + 2n - 1)\, \xi^{2n+1}}{(2n + 1)!}\}$$

The variation of $\eta(\xi)$ with time can be investigated in general, but by way of illustration consider the special cases for $\gamma_1 = 0$, 1 and 2.

In particular, if $\gamma_1 = 2$ the solution reduces to

$$\eta(\xi) = C_1\{1 + \xi\, e^{\xi^2/2} \int e^{-\frac{1}{2}\xi^2}\, d\xi\} + C_2\, \xi\, e^{\xi^2/2}$$

and, in the time domain, this solution is dominated by the term $e^{\xi^2/2}$.

Similarly, if $\gamma_1 = 1$ the solution of the equation becomes

$$\eta(\xi) = C_1\, e^{\frac{1}{2}\xi^2} + C_2\, e^{\frac{1}{2}\xi^2} \int e^{-\frac{1}{2}\xi^2}\, d\xi$$

If $\gamma_1 = 0$ on the other hand we have

$$\eta(\xi) = C_1 + C_2 \int e^{\frac{1}{2}\xi^2}\, d\xi$$

In all these solutions, the term

$$e^{\frac{1}{2}\xi^2} = e^{\frac{1}{2}|D| \left(t + \frac{BD - 2E}{D^2}\right)^2}$$

is dominant. Therefore,

$$\lim_{t\to\infty} r(t) \to \infty$$

implying that the motion is again unstable.

Thus we have the situation that, for all values of γ, the motion is unstable when

$$D = mU_o \alpha N_{\dot{v}} < 0$$

That is, instability occurs when $N_{\dot{v}} < 0$ provided that $U_o > 0$ and $\alpha > 0$.

If at the time t_1, the instantaneous velocity of the ship is such that

$$U_o(1 + \alpha t_1) = \overline{U}$$

then there evidently exists the possibility that the ship which is stable with the steady reference motion, U, will be unstable while accelerating through this instantaneous velocity - at least if $N_{\dot{v}} < 0$.

To summarise the findings for the particular accelerating reference motion considered we find that

α	$N_{\dot{v}}$	$D = m U_o \alpha N_{\dot{v}}$	$\lim_{t\to\infty} r(t) \to$
> 0	> 0	> 0	0 - stable motion
> 0	< 0	< 0	∞ - unstable motion

5. <u>DECELERATING REFERENCE MOTION</u>

Suppose that,

$$U(t) = U_o (1 - \beta t)$$

such that $\beta > 0$ and $U(t) = 0$ when $t = {}^1/_\beta$. It is seen that this is the case in Section 4 in which $\beta = -\alpha$. Using the previous findings we may conclude that

α	$N_{\dot{v}}$	$D = m U_o \alpha N_{\dot{v}}$	$\lim r(t) \rightarrow$
< 0	< 0	> 0	0 – stable motion
< 0	> 0	< 0	∞ – unstable motion

Thus, for a given value of $N_{\dot{v}}$, the stability of the motion is dependent on the initial unsteady reference motion as shown in the following table.

$N_{\dot{v}}$	α	Unsteady Reference Motion	Disturbance
> 0	> 0	acceleration	stable
> 0	< 0	deceleration	unstable
< 0	> 0	acceleration	unstable
< 0	< 0	deceleration	stable

Let us again consider another unsteady decelerating reference motion. Suppose that

$$U(t) = U_o (1 + \alpha t)^{-1}$$

where $U_o > 0$, $\alpha > 0$ and so $\dot{U}(t) = - U_o \alpha (1 + \alpha t)^{-2}$

The uncoupled equation of free motion may now be written

$$A\ddot{r} + \{D + EU_o(1 + \alpha t)^{-1}\}\dot{r} + \{F + GU_o(1 + \alpha t)^{-1} - EU_o\alpha(1 + \alpha t)^{-2}\}r = 0$$

where

$A = (m - Y_{\dot{v}})(I_z - N_{\dot{r}}) - N_{\dot{v}} Y_{\dot{r}}$; $\quad F = Y_v N_r - N_v Y_r$;

$D = - Y_v(I_z - N_{\dot{r}}) - N_r(m - Y_{\dot{v}}) - N_v Y_{\dot{r}} - N_{\dot{v}} Y_r$; $\quad G = mN_v$;

$E = mN_{\dot{v}}$;

With the substitutions

$$r(t) = y(x) \text{ and } x = (1 + \alpha t)U_o^{-1}$$

the equation reduces to

$$x^2 y'' + (ax^2 + bx)y' + (\delta x^2 + \beta x - b)y = 0$$

where

$$a = \frac{U_o D}{\alpha A} \qquad b = \frac{U_o E}{\alpha A} \qquad \delta = \frac{F U_o^2}{\alpha A} \qquad \beta = \frac{G U_o^2}{\alpha^2 A}$$

This differential equation has the solution

$$y(x) = x^{(1-b)/2} e^{-a/2x} \begin{cases} C_1 J_p(\lambda x) + C_2 Y_p(\lambda x) & \text{if } d > 0 \\ C_1 I_p(\lambda x) + C_2 I_{-p}(\lambda x) & \text{if } d < 0 \end{cases}$$

where

$$d = \delta - \frac{a^2}{4} \qquad \lambda = \sqrt{\left|\delta - \frac{a^2}{4}\right|} \qquad p = \frac{(1+b)}{2}$$

where

$J_p(x)$ is a Bessel function of the first kind

$Y_p(x)$ is a Bessel function of the second kind

$I_p(x)$ is a modified Bessel function of the first kind

When investigating stability, we are interested in the behaviour of the motion as time increases that is, in the $\lim_{t \to \infty} r(t)$. In effect this requires us to consider the behaviour of these Bessel functions for large values of x. Whittaker and Watson (1950) show that for large values of x,

$$J_p(x) = \sqrt{\left(\frac{2}{\pi x}\right)} \{\zeta_p \cos\psi - \xi_p \sin\psi\}$$

$$Y_p(x) = \sqrt{\left(\frac{2}{\pi x}\right)} \{\zeta_p \sin\psi + \xi_p \cos\psi\}$$

where

$$\zeta_p(x) = 1 - \frac{(1^2-4p^2)(3^2-4p^2)}{2!\,(8x)^2} + \frac{(4p^2-1^2)(4p^2-3^2)(4p^2-5^2)(4p^2-7^2)}{4!\,(8x)^4} + \ldots\; R_t$$

$$\xi_p(x) = \frac{(4p^2-1^2)}{1!(8x)^2} - \frac{(4p^2-1^2)(4p^2-3^2)(4p^2-5^2)(4p^2-7^2)}{3!\,(8x)^3} + \ldots\; R_s$$

where R_t and R_s are remainders and

$$\zeta_p(x) = \zeta_{-p}(x) = \zeta_p(-x)$$

$$\xi_p(x) = \xi_{-p}(x) = -\xi_{-p}(-x)$$

$$\psi = x - \tfrac{1}{4}\pi - \tfrac{1}{2}p\pi$$

It follows

$$\lim_{x \to \infty} J_p(x) \to 0 \text{ and } \lim_{x \to \infty} Y_p(x) \to 0$$

It would thus appear that for $d > 0$ the solution tends to zero when x increases indefinitely. Hence

$$r(t) = \left(\frac{1 + \alpha t}{U_o}\right)^{(1 - b)/2} e^{-a/2 \left(\frac{1 + \alpha t}{U_o}\right) x}$$

$$\left[C_1 J_p \{\lambda U_o^{-1} (1 + \alpha t)\} + C_2 Y_p \{\lambda U_o^{-1} (1 + \alpha t)\} \right]$$

and

$$\lim_{t \to \infty} r(t) \to 0$$

provided that $a > o$.

When $d < 0$, so that $\delta < a^2/4$, and p is not an integer, the solution is

$$y(x) = x^{(1 - b)/2} e^{-ax/2} \{C_1 I_p(\lambda x) + C_2 I_{-p}(\lambda x)\}$$

When (λx) is large enough and $0 \leq \text{phase } (x) \leq \pi$, the modified function may be expressed as

$$I_p(\lambda x) \doteq \frac{e^{\lambda x}}{\sqrt{(2\pi\lambda x)}} \left\{1 - \frac{(4p^2 - 1^2)}{1!(8\lambda x)} + \frac{(4p^2 - 1^2)(4p^2 - 3^2)}{2!(8\lambda x)^3} + \ldots\right\}$$

$$+ e^{(p + \frac{1}{2})i\pi} \frac{e^{-\lambda x}}{\sqrt{(2\pi\lambda x)}} \left\{1 + \frac{(4p^2 - 1^2)}{1!(8\lambda x)} + \ldots\ldots\right\}$$

or, when $-\pi < \text{phase } (x) \leq \pi$,

$$I_p(\lambda x) \doteq \frac{e^{\lambda x}}{\sqrt{(2\pi\lambda x)}} \{1 - \frac{(4p^2 - 1^2)}{1!(8\lambda x)} + \frac{(4p^2 - 1^2)(4p^2 - 3^2)}{2!(8\lambda x)^2} + \ldots\}$$

$$+ e^{-(p + \frac{1}{2})i\pi} \frac{e^{-\lambda x}}{\sqrt{(2\pi\lambda x)}} \{1 + \frac{(4p^2 - 1^2)}{1!(8\lambda x)} + \ldots\}$$

The dominant contribution from the series is $e^{\lambda x} = e^{x\sqrt{|a^2/4 - \delta|}}$ and the asymptotic solution for $y(x)$ is of the form

$$y(x) \doteq x^{-b/2} e^{x\{\sqrt{1\, a/4^2 - \delta|} - a/2\}} \left[1 - O(1/x)\right]$$

Instability of the motion occurs when

$$\sqrt{\left|\frac{a^2}{4} - \delta\right|} - a/2 > 0$$

since the solution then increases for increasing x. This inequality is always true if

$$\delta = Y_v N_r - N_v Y_r \leq 0$$

provided A > 0.

Thus the motion is stable if

$$Y_v N_r - N_v Y_r > 0$$

However, we see that from the previous analysis there exist different conditions for stability of the motion depending on the unsteady decelerating reference motion considered.

It is interesting to note that this is, in fact, the criterion that is found for a steady reference motion, but with $\bar{U} = 0$. The physical significance of this is by no means clear however, although it may relate to the fact that in the limit as $t \to \infty$, which we consider in assessing stability, $U(t) \to 0$. This indeed is a matter which poses a philosophical difficulty in assessing stability, when the reference motion is not steady. In the conventional approach the effect of a small disturbance is examined as $t \to \infty$. In the examples considered in this note however,

$$U(t) \to \infty, -\infty \text{ or } 0 \text{ as } t \to \infty$$

Thus it is not clear in what sense the stability of a ship with unsteady reference speed U(t) can be determined, when U(t) is really changing through a finite range around a particular value of interest in a finite time. For example, what is meant by claiming that a ship is stable while accelerating from say 10 to 20 knots in a finite time!

6. CONCLUSIONS

The foregoing analysis suggests that stability criteria that are relevant for a steady reference motion do not necessarily apply if the motion is not constant. In this latter case, the equations describing the perturbed motion have some time dependent coefficients and, while the instantaneous values of those coefficients can equal those relating to the steady reference motion, the resulting stability criteria may be completely different. A body that is stable at constant speed could be unstable at the same speed during acceleration or deceleration.

Attempts have been made to examine the stability of accelerated motion by Wagner Smitt and Chislett (1972) and by Hooft (1969). These authors have adopted the quasi-steady approach outlined in the introduction to this note. In this way they investigate the directional stability of surface ships during deceleration by considering various combinations of steady reference and propeller speeds. In this note it is shown, in effect, that this type of approach is certainly open to question. It is only likely to be valid for very slowly changing reference motions.

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