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Multi-vehicle automated driving as a generalized mixed-integer potential game

Filippo Fabiani and Sergio Grammatico

***Abstract*—This paper considers the multi-vehicle automated driving coordination problem. We develop a distributed, hybrid decision-making framework for safe and efficient autonomous driving of selfish vehicles on multi-lane highways, where each dynamics is modeled as a mixed-logical-dynamical system. We formalize the coordination problem as a generalized mixed-integer potential game, seeking an equilibrium solution that generates almost individually-optimal mixed-integer decisions, given the safety constraints. Finally, we embed the proposed best-response-based algorithms within distributed open- and closed-loop control policies.**

I. Introduction

The imminent revolution in road traffic, due to the envisioned uptake of autonomous vehicles, holds the promise to enhance traffic security, comfort and efficiency. At the same time, it poses severe engineering challenges. A fundamental milestone towards Automated Driving (AD) will be providing a high degree of decision autonomy to each vehicle. In this context, motion planning for Multi-Vehicle Automated Driving (MVAD) has been addressed via optimal control and Model Predictive Control (MPC) algorithms [2], [3], [4], multi-layer and probabilistic decision-making frameworks [5], [6], [7], [8].

Such a literature, however, does not entirely catch a key feature of the road traffic environment: human drivers are *selfish* decision makers. In fact, typically, each driver behaves according to its own individual interests, while sharing the road space-time with the other drivers. Thus, *game theory* has been adopted to model and cope with noncooperative behaviors. For example, [9] adopted a Stackelberg decision policy for motion coordination, while [10] proposed a receding horizon, dynamic cooperative game with heuristic rules and [11] a single-vehicle, extensive-form game approach based on the prediction of the surrounding traffic.

A preliminary version of this paper with a simplified problem setup and one simplified solution algorithm is in [1]. At the time of the work, F. Fabiani was with the Department of Information Engineering, University of Pisa, Italy. Currently, the authors are with the Delft Center for Systems and Control (DCSC), TU Delft, The Netherlands. E-mail addresses: {filippo.fabiani, s.grammatico}@tudelft.nl. This work was partially supported by 3mE/TU Delft under research project Intelligent Autonomous Vehicles and by the ERC under research project COSMOS (ERC-StG 802348).

Differently from the aforementioned literature, in our multi-lane, multi-vehicle scenario, we first provide a distributed, hybrid decision-making framework that couples the decisions of all vehicles involved in the MVAD coordination problem (§II-IV). Specifically, we assume that each vehicle has a cost function that reflects its individual interests, e.g. minimize travel time or fuel consumption, given the driving decisions of the other vehicles (communicated or estimated), individual and safety constraints. The peculiarity of our approach is to equip each vehicle with both continuous and discrete decisions over a prediction horizon, i.e., longitudinal velocity, acceleration and lane selector, direction indicators, respectively. We also propose some AD rules to prevent potential sources of collision that would emerge with a standard, model-predictive, formulation, and massage each individual decision problem into a Mixed-Logical-Dynamical (MLD) system [12].

Since the dynamics of the vehicles are mutually coupled, the solution of the overall inter-dependent decision-making problem is non-trivial, as conflicts may arise [13]. For this reason, we formalize the MVAD coordination problem as a Generalized Mixed-Integer Potential Game (GMIPG) [14], [15], [16], where an equilibrium solution is a set of almost individually-optimal decisions, given the safety constraints (§V). Unfortunately, the mixed-integer nature of the proposed framework places enormous challenges to the computation of a standard equilibrium. Thus, we rely on potential games [17], where all players unconsciously minimize the same (potential) function and hence are suitable to design iterative procedures seeking a minimum of the potential function, which corresponds to an equilibrium of the game. In this context, we propose distributed algorithms to compute an approximate equilibrium of the GMIPG, also via a novel Gauss-Seidel iteration scheme. The computed equilibrium solution is then embedded within open- and closed-loop control policies (§VI). Finally, in §VII, we simulate two MVAD scenarios.

II. Hybrid motion planning

In this paper, we consider a set of vehicles $\mathcal{I} := \{1, \ldots, N\}$ driving on a multi-lane environment (highway) with lane set $\mathcal{L} := \{1, \ldots, L\}$, as illustrated in Fig. 1. We assume that each vehicle controls a set of hybrid decision variables, namely, both continuous and discrete decision

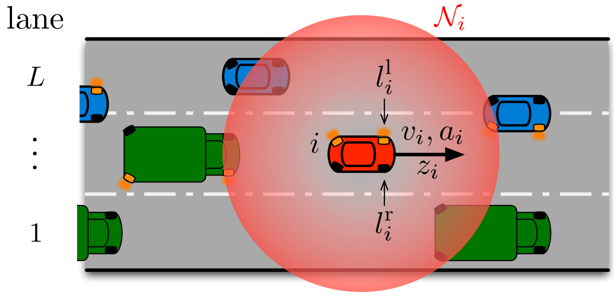


Fig. 1: A set of vehicles driving along a highway.

variables, over a prediction horizon $\mathcal{T} \coloneqq \{0, \ldots, T\}$, $T \geq 1$, as described next.

1) Continuous decision variables: each vehicle i controls its longitudinal acceleration $a_i \in \mathcal{A}_i \coloneqq [\underline{a}_i, \bar{a}_i] \subset \mathbb{R}$, with $\underline{a}_i < 0 < \bar{a}_i$, assuming that the cruise speed $v_i \in \mathcal{V}_i \subset \mathbb{R}$ follows a standard forward-Euler scheme, i.e.,

$$v_i(t+1) = v_i(t) + \tau a_i(t),$$

where $\tau > 0$ denotes the length of a predefined time interval. Hence, the continuous decision variables over the horizon $\mathcal{T}$ are $\boldsymbol{a}_i \coloneqq [a_i(0); \ldots ; a_i(T-1)] \in \mathcal{A}_i^T$ and $\boldsymbol{v}_i \coloneqq [v_i(1); \ldots ; v_i(T)] \in \mathcal{V}_i^T$.

2) Discrete decision variables: we assume that each vehicle i selects the traveling lane $z_i \in \mathcal{L}$. Inspired by a common practice in a multi-lane environment, we adopt direction indicators to allow lane change maneuvers. Specifically, we introduce two binary decision variables, namely l_i^{r}, $l_i^{\mathrm{l}} \in \mathbb{B} \coloneqq \{0, 1\}$, such that $l_i^{\mathrm{r}} = 1$ (respectively, $l_i^{\mathrm{l}} = 1$) denotes that vehicle i has its right (left) direction indicator on, hence wants to change its current lane, moving to the right (left). It seems reasonable to assume that, at each time $t \in \mathcal{T}$, the vehicles may turn on only one indicator, i.e., they shall satisfy the following constraint:

$$l_i^{\mathrm{r}}(t) + l_i^{\mathrm{l}}(t) \leq 1. \tag{1}$$

Thus, the discrete decision variables over $\mathcal{T}$ are $\boldsymbol{z}_i \coloneqq [z_i(1); \ldots ; z_i(T)] \in \mathcal{L}^T$, $\boldsymbol{l}_i^{\mathrm{r}} \coloneqq [l_i^{\mathrm{r}}(0); \ldots ; l_i^{\mathrm{r}}(T-1)] \in \mathbb{B}^T$ and $\boldsymbol{l}_i^{\mathrm{l}} \coloneqq \left[l_i^{\mathrm{l}}(0); \ldots ; l_i^{\mathrm{l}}(T-1)\right] \in \mathbb{B}^T$.

We denote by $d_{i,j}(t) \in \mathbb{R}$ the inter-vehicle distance at time $t \in \mathcal{T}$ between the pair of vehicles i and j, which we assume evolves according to a forward-Euler scheme:

$$d_{i,j}(t+1) = d_{i,j}(t) + \tau \left(v_j(t) - v_i(t)\right). \tag{2}$$

By referring to $\bar{d} > 0$ as a predefined interaction distance, which may depend, for example, on the on-board sensors, $d_{i,j}$ allows us to introduce the set of vehicles in the neighborhood of i that can affect its driving over $\mathcal{T}$ as $\mathcal{N}_i \coloneqq \{j \in \mathcal{I} \mid |d_{i,j}(t)| \leq \bar{d},\ t \in \mathcal{T}\}$. From now on, we refer to j as a generic vehicle in $\mathcal{N}_i$ so that, according to (2), if each vehicle knows the current velocity of its neighbor $v_j(t)$ (via communication or direct measurement), it can estimate the relative longitudinal distance in the next time interval.

Within the proposed framework, we assume that each road user aims to pursue its selfish interest, e.g., tracking a desired speed profile $\boldsymbol{v}_i^{\mathrm{d}} \in \mathcal{V}_i^T$ or driving along a target lane $\boldsymbol{z}_i^{\mathrm{d}} \in \mathcal{L}^T$. Therefore, each vehicle i seeks for a sequence of hybrid decisions towards its individual goals. We preliminary formulate a MPC motion planning as an optimization problem with mixed-integer variables:

$$\begin{cases} \min\limits_{\boldsymbol{v}_i, \boldsymbol{a}_i, \boldsymbol{z}_i, \boldsymbol{l}_i^{\mathrm{r}}, \boldsymbol{l}_i^{\mathrm{l}}} & J_i(\boldsymbol{v}_i, \boldsymbol{a}_i, \boldsymbol{z}_i) \\ \quad \text{s.t.} & v_i(t+1) = v_i(t) + \tau a_i(t),\ \forall t \in \mathcal{T} \\ & a_i(t) \in \mathcal{A}_i,\ v_i(t+1) \in \mathcal{V}_i(t),\ \forall t \in \mathcal{T} \\ & z_i(t+1) \in \mathcal{L}_i(t),\ \forall t \in \mathcal{T} \\ & l_i^{\mathrm{r}}(t),\ l_i^{\mathrm{l}}(t) \in \mathbb{B},\ l_i^{\mathrm{r}}(t) + l_i^{\mathrm{l}}(t) \leq 1,\ \forall t \in \mathcal{T} \end{cases} \tag{3}$$

where $J_i : \mathcal{V}_i^T \times \mathcal{A}_i^T \times \mathcal{L}^T \to \mathbb{R}$ is a continuous objective function for vehicle i. The sets $\mathcal{V}_i$ and $\mathcal{L}_i \subset \mathcal{L}$ shall be defined to limit the speed variation and the selected lane between consecutive time intervals. Given the maximum velocity of vehicle i, $\bar{v}_i > 0$, we can define for instance:

$$\mathcal{V}_i(t) \coloneqq [0,\ \bar{v}_i] \cap [v_i(t) + \tau \underline{a}_i,\ v_i(t) + \tau \bar{a}_i]\,, \tag{4}$$

$$\mathcal{L}_i(t) \coloneqq \mathcal{L} \cap \left[z_i(t) - l_i^{\mathrm{r}}(t),\ z_i(t) + l_i^{\mathrm{l}}(t)\right]. \tag{5}$$

From (5), the direction indicators allow the lane-change maneuver in the next time interval, without forcing it.

The MPC motion planning in (3) is far from being "safe", since it considers local decision variables only (i.e., those of vehicle i) and completely ignores the actions of the neighboring vehicles. In the next section, we set up some coupling constraints between vehicles i and all $j \in \mathcal{N}_i$ by exploiting their longitudinal distance $d_{i,j}$ and their discrete lateral "distance", defined in terms of relative lane difference by the integer variable $z_{i,j} \coloneqq z_j - z_i$.

By defining d_i^{s}, $\hat{d} > 0$ as the safety distance on the longitudinal direction for vehicle i and the inter-distance between vehicles that could lead to a lateral collision, respectively, let us consider the following definitions.

Definition 1 (Longitudinal safety): A pair of vehicles $(i, j) \in \mathcal{I}^2$ is longitudinally safe over the prediction horizon $\mathcal{T}$ if, for all $t \in \mathcal{T}$ such that $z_{i,j}(t) = 0$, $|d_{i,j}(t)| \geq d_i^{\mathrm{s}}$ and, furthermore, if $z_{i,j}(t+1) = 0$, $d_{i,j}(t+1) \cdot d_{i,j}(t) \geq 0$. The system is longitudinally safe over the prediction horizon $\mathcal{T}$ if any pair of vehicles $(i, j) \in \mathcal{I}^2$ is longitudinally safe. □

Definition 2 (Lateral safety): A pair of vehicles $(i, j) \in \mathcal{I}^2$ is laterally safe over the prediction horizon $\mathcal{T}$ if, for all $t \in \mathcal{T}$ such that $|d_{i,j}(t)| \leq \hat{d}$ and $|z_{i,j}(t)| = 1$, $z_i(t+1) \neq z_j(t)$ and $z_j(t+1) \neq z_i(t)$. The system is laterally safe over the prediction horizon $\mathcal{T}$ if any pair of vehicles $(i, j) \in \mathcal{I}^2$ is laterally safe. □

Finally, we aim at designing a mixed-integer decision-making framework capable for safely coordinating a set of

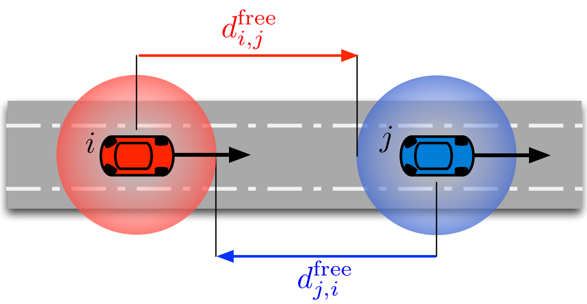
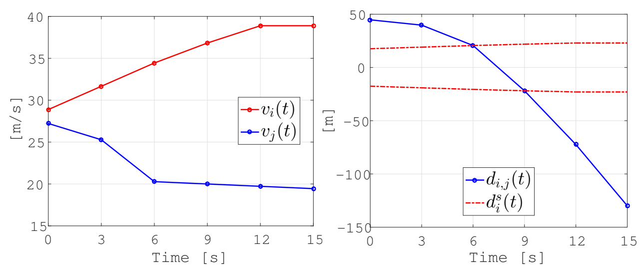


Fig. 2: Two vehicles traveling on the same lane.



(a) (b)

Fig. 3: Example of longitudinal collision: (a) Velocity profiles. (b) Relative and safety distances.

vehicles traveling on a multi-lane environment. Therefore, we do not address the issue of communication among vehicles and for the remainder we assume that: i) each vehicle is (autonomously) driven by the solution of the hybrid decision-making framework; ii) vehicles can exchange information, i.e., their decisions, without communication delays or packet losses.

III. Automated Driving rules

The hybrid motion planner in (3) requires some additional arrangements to ensure collision-free trajectories to each vehicle. A naïve formulation, indeed, may allow unsafe driving scenarios as those in Figures 2–3, 4–5.

1) Safety on a single lane: We focus on those vehicles $j \in \mathcal{N}_i$ that, in the prediction of vehicle i, travel on the same lane, i.e., $z_{i,j} = 0$, either behind or ahead of it ($|d_{i,j}| \geq 0$). We shall ensure that the relative distance $d_{i,j}(t)$ must be greater or equal than the safety distance $d_i^{\text{s}}(t)$, here conceived as a linear function of the current cruise speed, i.e., $d_i^{\text{s}}(t) = d_i^{\text{s}}(v_i(t))$. This comes from the common drive experience: compared to driving at high speed, we are induced to get closer to the vehicle ahead at low speed. Hence, by denoting with $\wedge$ and $\vee$ as the logical AND and OR, respectively, for all $t \in \mathcal{T}$ we impose:

$$[z_{i,j}(t) = 0] \wedge [|d_{i,j}(t)| \geq 0] \implies [|d_{i,j}(t)| \geq d_i^{\text{s}}(t)] \,. \quad (6)$$

However, the safety distance constraint is not sufficient to prevent collisions on a lane, see Fig. 3. A feasible scenario foresees that vehicle i accelerates, to e.g. minimize its traveling time, while vehicle j reduces its speed, to e.g. minimize fuel consumption. In terms of control decisions, an optimal strategy exists for both vehicles (Fig. 3(a)), i.e., the optimization problem in (3) with additional constraints (6) is feasible. Since the vehicles travel on the same lane, such strategies are clearly not implementable (Fig. 3(b)).

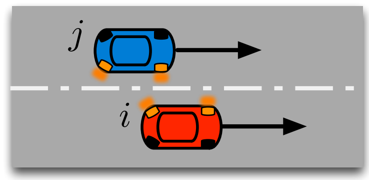


Fig. 4: Two vehicles traveling side by side along two consecutive lanes. Vehicle j (left lane) has the right indicator on, while i (right lane) has the left one on.

In view of Definition 1, we do not want to directly impose $d_{i,j}(t+1) \cdot d_{i,j}(t) \geq 0$, since it would lead to nonlinear constraints. Given a pair of vehicles $(i,j) \in \mathcal{I}^2$ traveling on the same lane across two consecutive time intervals, such a constraint demands that $d_{i,j}(t)$ and $d_{i,j}(t+1)$ shall have the same sign. Therefore, this limits their relative velocity, namely $v_{i,j} := v_j - v_i$, allowing each vehicle to (selfishly) exploit a portion of the free longitudinal space. Then, for all $j \in \mathcal{N}_i$ and $t \in \mathcal{T}$, we have:

$$\{[z_{i,j}(t) = 0] \wedge [z_{i,j}(t+1) = 0]\} \wedge [d_{i,j}(t) \geq 0] \implies \left[v_{i,j}(t) \geq -\frac{d_{i,j}(t)}{\tau} \right], \quad (7a)$$

$$\{[z_{i,j}(t) = 0] \wedge [z_{i,j}(t+1) = 0]\} \wedge [d_{i,j}(t) \leq 0] \implies \left[v_{i,j}(t) \leq -\frac{d_{i,j}(t)}{\tau} \right]. \quad (7b)$$

Proposition 1: Given a pair of vehicles $(i,j) \in \mathcal{I}^2$, assume that $v_i(0), v_j(0)$ are feasible. The hybrid MPC motion planner in (3) with safety distance constraints (6) and rule (7) guarantees the longitudinal safety. □

Proof: By referring to the case $d_{i,j}(t) \geq 0$ (the same holds for $d_{i,j}(t) \leq 0$), the constraint in (6) forces the safety distance between all pair of vehicles traveling on the same lane over $\mathcal{T}$, i.e., $d_{i,j}(t) \geq d_i^{\text{s}}(t) > 0$, while from (7a), $0 \leq d_{i,j}(t) + \tau v_i(t) = d_{i,j}(t+1)$, which premultiplied by $d_{i,j}(t)$ gives the conditions in Def. 1. ■

2) Prevent lateral collision with direction indicators: By referring to Fig. 4, due to the small relative distance between vehicles i and j, the safety distance constraint in (6) does not allow to change lane individually over $\mathcal{T}$. However, in the case that both vehicles aim to swap the lanes, they predict that the destination lane will be free during the next time intervals. Therefore, it is possible that by keeping their own speed unchanged, as well as relative distance (Fig. 5(b)), the two vehicles perform the lane change at the same time (Fig. 5(a)), causing a collision.

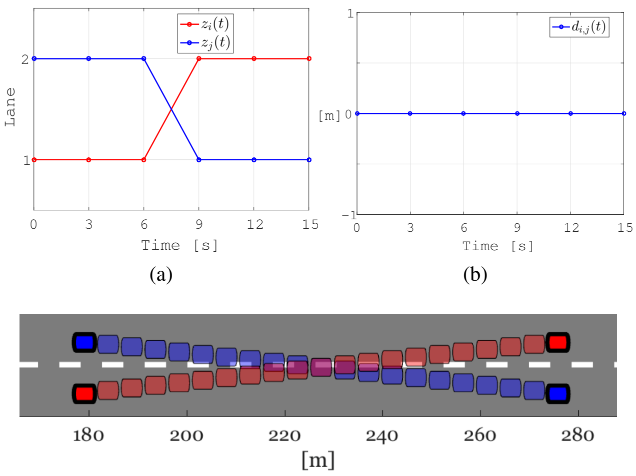


Fig. 5: Example of lateral collision between two vehicles: (a) Lane profiles. (b) Relative distance. (c) Collision during simultaneous lane change for $t \in [2T, 3T]$.

To prevent this unsafe scenario, we propose to exploit the direction indicators, by designing an additional mixed-logical rule. Two vehicles $(i,j) \in \mathcal{I}^2$ travel side by side on consecutive lanes if $|d_{i,j}(t)| \leq \hat{d}$ and $z_{i,j}(t) = 1$. If both vehicles express the will of change lane performing a swap, i.e., vehicle i turns on the left indicator l_i^{l} and vehicle j the right one l_j^{r} at the same time, we impose that the vehicle traveling on a lower lane keeps it, $z_i(t+1) = z_i(t)$. In fact, higher lanes are usually deputed for overtaking maneuvers, hence vehicles should facilitate the re-entry towards lower lanes. Thus, we design the following rule:

$$[z_{i,j}(t) = 1] \wedge [|d_{i,j}(t)| \leq \hat{d}] \wedge \{[l_i^{\mathrm{l}}(t) = 1] \wedge [l_j^{\mathrm{r}}(t) = 1]\} \implies [z_i(t+1) - z_i(t) = 0] \tag{8}$$

Proposition 2: Given a pair of vehicles $(i,j) \in \mathcal{I}^2$ and $\hat{d} > 0$ sufficiently large, assume that $[v_i(0); z_i(0)]$ and $[v_j(0); z_j(0)]$ are feasible. The hybrid MPC motion planner (3) with rule (8) guarantees the consecutive lane safety. □

Proof: By Def. 2, two vehicles may be unsafe on consecutive lanes if $z_i(t+1) = z_j(t)$ and $z_j(t+1) = z_i(t)$. In view of (5), this is possible only if $|z_{i,j}(t)| = 1$ and each vehicle turns on the proper direction indicator, i.e., $l_i^{\mathrm{l}}(t) = 1$ and $l_j^{\mathrm{r}}(t) = 1$ (or $l_i^{\mathrm{r}}(t) = 1$ and $l_j^{\mathrm{l}}(t) = 1$). If $|d_{i,j}(t)| > \hat{d}$ the vehicles may swap the lanes without collision; otherwise the condition in (8) forces the vehicle driving on a lower lane to keep it. ■

IV. Mixed-Logical-Dynamical model

We now translate the AD rules in (6), (7) and (8) into mixed-integer linear constraints to be imposed for each neighboring vehicle $j \in \mathcal{N}_i$ and for each time $t \in \mathcal{T}$.

By referring to (6), (7), we introduce three additional logical implications, to be handled with variables $\alpha, \beta, \gamma \in \mathbb{B}$. Specifically, α discriminates the vehicles that effectively travel along the same lane of the i-th at time t, either ahead ($\beta = 1$) or behind it ($\beta = 0$), while γ discriminates those vehicles that will be on the same lane at $t+1$:

$$[\alpha_{i,j}(t) = 1] \iff [z_{i,j}(t) \leq 0] \wedge [z_{i,j}(t) \geq 0], \tag{9a}$$

$$[\beta_{i,j}(t) = 1] \iff [d_{i,j}(t) \geq 0], \tag{9b}$$

$$[\gamma_{i,j}(t) = 1] \iff [z_{i,j}(t+1) \leq 0] \wedge [z_{i,j}(t+1) \geq 0]. \tag{9c}$$

Hence, equations (6), (7) can be preliminary rewritten as nonlinear inequalities:

$$\alpha_{i,j}(t)\,[\beta_{i,j}(t)\,(d_i^{\mathrm{s}}(t) - d_{i,j}(t)) + (1 - \beta_{i,j}(t))\,(d_i^{\mathrm{s}}(t) + d_{i,j}(t))] \leq 0, \tag{10a}$$

$$\alpha_{i,j}(t)\gamma_{i,j}(t)\,[-\beta_{i,j}(t)\,(\tau v_{i,j}(t) + d_{i,j}(t)) + (1 - \beta_{i,j}(t))\,(\tau v_{i,j}(t) + d_{i,j}(t))] \leq 0. \tag{10b}$$

In a similar way, it is possible to handle (8) with δ, ζ, $\eta \in \mathbb{B}$ that lead to the following logical implications:

$$[\delta_{i,j}(t) = 1] \iff [z_{i,j}(t) \leq 1] \wedge [z_{i,j}(t) \geq 1], \tag{11a}$$

$$[\zeta_{i,j}(t) = 1] \iff [l_i^{\mathrm{l}}(t) = 1] \wedge [l_j^{\mathrm{r}}(t) = 1], \tag{11b}$$

$$[\eta_{i,j}(t) = 1] \iff [d_{i,j}(t) \leq \hat{d}] \vee [d_{i,j}(t) \geq -\hat{d}], \tag{11c}$$

then (8) can be equivalently reformulated as:

$$\delta_{i,j}(t)\zeta_{i,j}(t)\eta_{i,j}(t)\,(z_i(t+1) - z_i(t)) = 0. \tag{12}$$

We henceforth rely on the pattern of inequalities summarized in Tab. I to handle both the logical implications and nonlinear constraints. For instance, let us consider the right-hand side in (9a): by introducing θ, $\kappa \in \mathbb{B}$, $[\theta_{i,j}(t) = 1] \iff [z_{i,j}(t) \leq 0]$ translates into $\mathcal{S}_{\leq}(\theta_{i,j}(t), z_{i,j}(t), 0)$, while $[\kappa_{i,j}(t) = 1] \iff [z_{i,j}(t) \geq 0]$ into $\mathcal{S}_{\geq}(\kappa_{i,j}(t), z_{i,j}(t), 0)$. Moreover, $[\alpha_{i,j}(t) = 1] \iff [\theta_{i,j}(t) = 1] \wedge [\kappa_{i,j}(t) = 1]$ corresponds to $\mathcal{S}_{\wedge}(\alpha_{i,j}(t), \theta_{i,j}(t), \kappa_{i,j}(t))$. Finally, (9a) reads as the following system of inequalities:

$$\text{(9a)} \implies \begin{cases} \mathcal{S}_{\leq}(\theta_{i,j}(t), z_{i,j}(t), 0), \\ \mathcal{S}_{\geq}(\kappa_{i,j}(t), z_{i,j}(t), 0), \\ \mathcal{S}_{\wedge}(\alpha_{i,j}(t), \theta_{i,j}(t), \kappa_{i,j}(t)). \end{cases} \tag{13}$$

Thus, it follows that:

$$\text{(9b)} \implies \mathcal{S}_{\geq}(\beta_{i,j}(t), d_{i,j}(t), 0). \tag{14}$$

$$\text{(9c)} \implies \begin{cases} \mathcal{S}_{\leq}(\lambda_{i,j}(t), z_{i,j}(t), 0), \\ \mathcal{S}_{\geq}(\mu_{i,j}(t), z_{i,j}(t), 0), \\ \mathcal{S}_{\wedge}(\gamma_{i,j}(t), \lambda_{i,j}(t), \mu_{i,j}(t)). \end{cases} \tag{15}$$

$$\text{(11a)} \implies \begin{cases} \mathcal{S}_{\leq}(\nu_{i,j}(t), z_{i,j}(t), 1), \\ \mathcal{S}_{\geq}(\xi_{i,j}(t), z_{i,j}(t), 1), \\ \mathcal{S}_{\wedge}(\delta_{i,j}(t), \nu_{i,j}(t), \xi_{i,j}(t)). \end{cases} \tag{16}$$

$$\text{(11b)} \implies \mathcal{S}_{\wedge}(\zeta_{i,j}(t), l_i^{\mathrm{l}}(t), l_j^{\mathrm{r}}(t)). \tag{17}$$

TABLE I: Basic logical implications and associated system of inequalities.
($f : \mathbb{R} \to \mathbb{R}$ linear function, $M := \max_{x\in X} f(x)$, $m := \min_{x\in X} f(x)$, X compact set; $c \in \mathbb{R}$, $\epsilon > 0$, $\delta, \sigma, \gamma \in \mathbb{B}$)

Name	**Logical Implication**	**System of Inequalities**
$\mathcal{S}_{\geq}(\delta, f(x), c)$	$[\delta = 1] \iff [f(x) \geq c]$	$\begin{cases}(c-m)\delta \leq f(x) - m\\(M-c+\epsilon)\delta \geq f(x) - c + \epsilon\end{cases}$
$\mathcal{S}_{\leq}(\delta, f(x), c)$	$[\delta = 1] \iff [f(x) \leq c]$	$\begin{cases}(M-c)\delta \leq M - f(x)\\(c+\epsilon-m)\delta \geq \epsilon + c - f(x)\end{cases}$
$\mathcal{S}_{\wedge}(\delta, \sigma, \gamma)$	$[\delta = 1] \iff [\sigma = 1] \wedge [\gamma = 1]$	$\begin{cases}-\sigma + \delta \leq 0\\-\gamma + \delta \leq 0\\\ \ \sigma + \gamma - \delta \leq 1\end{cases}$
$\mathcal{S}_{\vee}(\delta, \sigma, \gamma)$	$[\delta = 1] \iff [\sigma = 1] \vee [\gamma = 1]$	$\begin{cases}\ \ \sigma - \delta \leq 0\\\ \ \gamma - \delta \leq 0\\-\sigma - \gamma + \delta \leq 0.\end{cases}$
$\mathcal{S}_{\Rightarrow}(g, f(x), \delta)$	$[\delta = 0] \implies [g = 0],\ [\delta = 1] \implies [g = f(x)]$	$\begin{cases}m\delta \leq g \leq M\delta\\-M(1-\delta) \leq g - f(x) \leq -m(1-\delta)\end{cases}$

$$\text{(11c)} \implies \begin{cases}\mathcal{S}_{\leq}(\rho_{i,j}(t), d_{i,j}(t), \hat{d}),\\\mathcal{S}_{\geq}(\sigma_{i,j}(t), d_{i,j}(t), -\hat{d}),\\\mathcal{S}_{\vee}(\eta_{i,j}(t), \rho_{i,j}(t), \sigma_{i,j}(t)).\end{cases} \tag{18}$$

Next, we follow the procedure in [12] to recast the inequalities in (10) and (12) into a linear formulation by means of both real and binary auxiliary variables [18]. Specifically, by starting from (10), we define $\phi_{i,j} := \alpha_{i,j}\beta_{i,j}$, $\chi_{i,j} := \phi_{i,j}\gamma_{i,j}$ and $\psi_{i,j} := \alpha_{i,j}\gamma_{i,j}$ as binary variables which satisfy the following systems of inequalities

$$\mathcal{S}_{\wedge}(\phi_{i,j}(t), \alpha_{i,j}(t), \beta_{i,j}(t)), \tag{19}$$
$$\mathcal{S}_{\wedge}(\chi_{i,j}(t), \phi_{i,j}(t), \gamma_{i,j}(t)), \tag{20}$$
$$\mathcal{S}_{\wedge}(\psi_{i,j}(t), \alpha_{i,j}(t), \gamma_{i,j}(t)). \tag{21}$$

By referring to (10a), we also define the real auxiliary variables $f_{i,j} := \phi_{i,j}d_{i,j}$, $g_{i,j} := \alpha_{i,j}d_i^{\text{s}}$ and $h_{i,j} := \alpha_{i,j}d_{i,j}$ that shall satisfy $\mathcal{S}_{\Rightarrow}$ in Tab. I as follows:

$$\mathcal{S}_{\Rightarrow}(f_{i,j}(t), d_{i,j}(t), \phi_{i,j}(t)), \tag{22}$$
$$\mathcal{S}_{\Rightarrow}(g_{i,j}(t), d_i^{\text{s}}(t), \alpha_{i,j}(t)), \tag{23}$$
$$\mathcal{S}_{\Rightarrow}(h_{i,j}(t), d_{i,j}(t), \alpha_{i,j}(t)). \tag{24}$$

Thus, the nonlinear inequalities in (10a) become:

$$-2f_{i,j}(t) + g_{i,j}(t) + h_{i,j}(t) \leq 0. \tag{25}$$

Now, let us consider (10b). We define four real auxiliary variables, $k_{i,j} = \chi_{i,j}v_{i,j}$, $m_{i,j} = \chi_{i,j}d_{i,j}$, $p_{i,j} = \psi_{i,j}v_{i,j}$ and $q_{i,j} = \psi_{i,j}d_{i,j}$, that satisfy:

$$\mathcal{S}_{\Rightarrow}(k_{i,j}(t), v_{i,j}(t), \chi_{i,j}(t)), \tag{26}$$
$$\mathcal{S}_{\Rightarrow}(m_{i,j}(t), d_{i,j}(t), \chi_{i,j}(t)), \tag{27}$$
$$\mathcal{S}_{\Rightarrow}(p_{i,j}(t), v_{i,j}(t), \psi_{i,j}(t)), \tag{28}$$
$$\mathcal{S}_{\Rightarrow}(q_{i,j}(t), d_{i,j}(t), \psi_{i,j}(t)). \tag{29}$$

Hence, (10b) is rewritten in linear form:

$$-2(\tau k_{i,j}(t) + m_{i,j}(t)) + q_{i,j}(t) + \tau p_{i,j}(t) \leq 0. \tag{30}$$

Finally, we proceed with the same procedure as for (12) by introducing two auxiliary binary variables, $\upsilon_{i,j} := \delta_{i,j}\zeta_{i,j}$ and $\omega_{i,j} := \upsilon_{i,j}\eta_{i,j}$, that satisfy the systems

$$\mathcal{S}_{\wedge}(\rho_{i,j}(t), \delta_{i,j}(t), \zeta_{i,j}(t)), \tag{31}$$
$$\mathcal{S}_{\wedge}(\omega_{i,j}(t), \upsilon_{i,j}(t), \eta_{i,j}(t)), \tag{32}$$

and two discrete variables, $r_{i,j} := \omega_{i,j}z_i(t)$ and $s_{i,j} := \omega_{i,j}z_i(t+1)$, so that we obtain:

$$\begin{cases}-s_{i,j}(t) + r_{i,j}(t) \leq 0\\\ \ s_{i,j}(t) - r_{i,j}(t) \leq 0.\end{cases} \tag{33}$$

Then, the variables $r_{i,j}$ and $s_{i,j}$ satisfy the inequalities

$$\mathcal{S}_{\Rightarrow}(r_{i,j}(t), z_i(t), \omega_{i,j}(t)), \tag{34}$$
$$\mathcal{S}_{\Rightarrow}(s_{i,j}(t), z_i(t+1), \omega_{i,j}(t)). \tag{35}$$

All those mixed-integer linear inequalities are then rearranged within the final hybrid framework for each vehicle:

$$\begin{cases}\min\limits_{\boldsymbol{v}_i, \boldsymbol{a}_i, \ldots, \boldsymbol{s}_i} & J_i(\boldsymbol{v}_i, \boldsymbol{a}_i, \boldsymbol{z}_i)\\\text{s.t.} & v_i(t+1) = v_i(t) + \tau a_i(t),\ \forall t \in \mathcal{T}\\ & a_i(t) \in \mathcal{A}_i,\ v_i(t+1) \in \mathcal{V}_i(t),\ \forall t \in \mathcal{T}\\ & z_i(t+1) \in \mathcal{L}_i(t),\ \forall t \in \mathcal{T}\\ & l_i^{\text{r}}(t),\ l_i^{\text{l}}(t) \in \mathbb{B},\ l_i^{\text{r}}(t) + l_i^{\text{l}}(t) \leq 1,\ \forall t \in \mathcal{T}\\ & (13) - (35),\ \forall j \in \mathcal{N}_i,\ \forall t \in \mathcal{T}\end{cases} \tag{36}$$

The number of constraints for vehicle i is $c_i := (88|\mathcal{N}_i| + 7)T$, while for the whole neighborhood is $c := (\sum_{j\in\mathcal{N}_i} c_j) + c_i$. Note that the coupling constraints in (13) – (35) contain the strategies of the neighbors as affine, given terms. We define $\boldsymbol{x}_i := [\boldsymbol{v}_i; \boldsymbol{a}_i; \ldots; \boldsymbol{s}_i] \in \mathbb{R}^{n_i}$, $n_i := (28|\mathcal{N}_i|+5)T$, as the i-th vector of decision variables and $\boldsymbol{x} := (\boldsymbol{x}_i, \boldsymbol{x}_{-i}) \in \mathbb{R}^n$, $n := (\sum_{j\in\mathcal{N}_i} n_j) + n_i$, as the vector of all the decision variables in the neighborhood $\mathcal{N}_i$, where $\boldsymbol{x}_{-i} \in \mathbb{R}^{n-n_i}$ stacks the variables of the neighbors. Finally, the hybrid motion planner in compact form is:

$$\forall i \in \mathcal{I}: \begin{cases} \min\limits_{\boldsymbol{x}_i} & J_i(\boldsymbol{x}_i) \\ \text{s.t.} & A\boldsymbol{x} \leq b \end{cases} \tag{37}$$

for some suitable $A \in \mathbb{R}^{c\times n}$, $b \in \mathbb{R}^c$.

V. Game-theoretic perspective

In principle, by computing a solution to (37), each selfish road user $i \in \mathcal{I}$ can be driven towards its goal over the horizon $\mathcal{T}$ by a (optimal) sequence of mixed-integer decision strategies. However, the linear constraints introduced above couple the dynamics of pair of vehicles, making the strategies inter-dependent. Moreover, each control sequence is computed by assuming the strategies of the neighbors be given: if at least one of these latter changes, then the computed strategy may not be optimal anymore, or even unsafe. Thus, we aim at designing suitable sequences of decision variables that safely control each vehicle towards its own goal. To achieve such a trade-off, we propose to formalize the MVAD coordination problem as a GMIPG [14], i.e., an instance of the class of Generalized Nash Equilibrium Problems (GNEPs) [19].

Therefore, we preliminary define the feasible set of each player (i.e., vehicle), namely $\mathcal{X}_i(\boldsymbol{x}_{-i}) := \{\boldsymbol{x}_i \in \mathbb{R}^{n_i} \mid A(\boldsymbol{x}_i, \boldsymbol{x}_{-i}) \leq b\}$, and $\boldsymbol{\mathcal{X}} := \{\boldsymbol{x} \in \mathbb{R}^n \mid A\boldsymbol{x} \leq b\}$. Moreover, by noticing that each $J_i(\boldsymbol{x}_i)$ depends only on the local variable $\boldsymbol{x}_i$, we introduce the function $P : \mathbb{R}^n \to \mathbb{R}$, defined as $P(\boldsymbol{x}) := \sum_{i\in\mathcal{I}} J_i(\boldsymbol{x}_i)$, that satisfies for all $i \in \mathcal{I}$, for all $\boldsymbol{x}_{-i}$, and for all $\boldsymbol{x}_i$, $\boldsymbol{y}_i \in \mathcal{X}_i(\boldsymbol{x}_{-i})$,

$$P(\boldsymbol{x}_i, \boldsymbol{x}_{-i}) - P(\boldsymbol{y}_i, \boldsymbol{x}_{-i}) = J_i(\boldsymbol{x}_i) - J_i(\boldsymbol{y}_i).$$

By [14], P is an exact potential function for the proposed MVAD coordination game. Let us now introduce the mixed-integer best response mapping for player i, given the strategies of its neighbors $\boldsymbol{x}_{-i}$:

$$\boldsymbol{x}_i^\star(\boldsymbol{x}_{-i}) \in \begin{cases} \operatorname*{argmin}\limits_{\boldsymbol{x}_i} & J_i(\boldsymbol{x}_i) \\ \text{s.t.} & (\boldsymbol{x}_i, \boldsymbol{x}_{-i}) \in \boldsymbol{\mathcal{X}}. \end{cases} \tag{38}$$

Given the selfish nature of road users, let us define the notion of equilibrium solution.

Definition 3 (ε-Mixed-Integer Nash equilibrium): Let $\varepsilon > 0$. $\boldsymbol{x}^* \in \boldsymbol{\mathcal{X}}$ is an ε-Mixed-Integer Nash Equilibrium (ε-MINE) of the game in (38) if, for all $i \in \mathcal{I}$,

$$J_i(\boldsymbol{x}_i^*) \leq \inf_{\boldsymbol{x}_i \in \mathcal{X}_i(\boldsymbol{x}_{-i}^*)} J_i(\boldsymbol{x}_i) + \varepsilon.$$

□

In view of [16, Th. 2], the set of ε-approximated minimum over $\boldsymbol{\mathcal{X}}$ of P is a subset of the ε-MINE of the game. Then, any $\boldsymbol{x}^* \in \boldsymbol{\mathcal{X}}$ such that $P(\boldsymbol{x}^*) \leq P(\boldsymbol{x}) + \varepsilon$ for all $\boldsymbol{x} \in \boldsymbol{\mathcal{X}}$ (ε-approximated minimizer of P) is an ε-MINE of the GMIPG (the converse does not hold in general). Informally, an ε-MINE is a set of driving strategies that are almost individually optimal, given the safety constraints.

Algorithm 1: Gauss-Southwell method (open loop)

Choose a feasible starting point $\boldsymbol{x}(0) \in \boldsymbol{\mathcal{X}}$, set $k := 0$
while $\boldsymbol{x}(k)$ is not an ε-MINE **do**
  Broadcast $\boldsymbol{x}_i(k)$ to $\mathcal{N}_i$, $\forall i \in \mathcal{I}$
  Choose a player $i = i_k \in \mathcal{I}$
  $\boldsymbol{x}_i(k+1) := \begin{cases} \boldsymbol{x}_i^*(k) & \text{if } \Delta J_i(k) \geq \varepsilon \\ \boldsymbol{x}_i(k) & \text{otherwise} \end{cases}$
  Set $\boldsymbol{x}_j(k+1) := \boldsymbol{x}_j(k)$, $j \in \mathcal{I} \setminus i$, and $k := k+1$
end

VI. Solution methods

Solving a GNEP is known to be challenging, even in the presence of continuous variables only [19]. Here, we propose distributed, best-response based, algorithms for computing an ε-MINE of the MVAD game via iterative procedure, despite the mixed-integer nature of (37).

From now on, we refer to *temporal* step as a sample step of the decision variables over $\mathcal{T}$, and to *algorithmic* step as an iterative step to compute a solution of the game. $\boldsymbol{x}_i(k)$ and $\boldsymbol{x}_{-i}(k)$, $k \in \mathbb{N}$, denote the vectors of decision variables at the k-th algorithmic step of player i and of its neighbors, respectively. We refer to $\boldsymbol{x}_i(t|k)$ as the i-th vector of decision variables at time t computed at the algorithmic step k. Finally, we introduce the cost variation $\Delta J_i(k) := J_i(\boldsymbol{x}_i(k)) - J_i(\boldsymbol{x}_i^*(k))$, with $\boldsymbol{x}_i^*(k) \in \boldsymbol{x}_i^\star(\boldsymbol{x}_{-i}(k))$.

A. Open-loop Gauss-Southwell algorithm

First, we propose a Gauss-Southwell method in open loop, i.e, each vehicle directly implements the whole equilibrium sequence of decision variables over $\mathcal{T}$.

From Algorithm 1, at each iteration k, only a selected player $i = i_k \in \mathcal{I}$ computes a best response to the strategies adopted by the other players, updating its decision vector only if it leads to an ε-improvement in terms of minimization of J_i. Under suitable conditions on the sequence $\{i_k\}_{k\in\mathbb{N}}$, Algorithm 1 converges to an ε-MINE [16, Th. 4]. Thus, the overall control method reads as:

S1) At time t, set $\mathcal{T}_t := \{t, \ldots, t+T\}$;
S2) Find an ε-MINE, $\boldsymbol{x}^*$, via Algorithm 1;
S3) Implement $\boldsymbol{x}_i^*$, $\forall i \in \mathcal{I}$, set $t = t+T$, and go to S1.

Algorithm 1 ensures the feasibility of the full horizon strategy by allowing only the i_k-th player to modify the shared constraints, $A\boldsymbol{x} \leq b$. By focusing on practical aspects, as long as the convergence is not achieved, the vehicles have to broadcast in real-time each other the whole vector of strategies.

Algorithm 2: Gauss-Seidel method (open loop)

Choose a feasible starting point $\boldsymbol{x}(0) \in \boldsymbol{\mathcal{X}}$, set $k \coloneqq 0$
while $\boldsymbol{x}(k)$ is not an ε-MINE **do**
  Broadcast $\boldsymbol{x}_i(k)$ to $\mathcal{N}_i$, $\forall i \in \mathcal{O}_t$
  for all $i \in \mathcal{O}_t$ **do**
    $$\boldsymbol{x}_i(k+1) \coloneqq \begin{cases} \boldsymbol{x}_i^*(k) & \text{if } \Delta J_i(k) \geq \varepsilon \\ \boldsymbol{x}_i(k) & \text{otherwise} \end{cases}$$
    Broadcast $\boldsymbol{x}_i(k+1)$ to all $j \succ_t i$
  end
  Set $k \coloneqq k+1$
end

Algorithm 3: Gauss-Seidel method (closed loop)

Choose a feasible point $\boldsymbol{x}(0) \in \hat{\boldsymbol{\mathcal{X}}}_t$, set $k \coloneqq 0$
while $\boldsymbol{x}(k)$ is not an ε-MINE **do**
  Broadcast $\boldsymbol{x}_i(t+1|k)$ to $\mathcal{N}_i$, $\forall i \in \mathcal{O}_t$
  for all $i \in \mathcal{O}_t$ **do**
    $$\boldsymbol{x}_i(k+1) \coloneqq \begin{cases} \boldsymbol{x}_i^*(k) & \text{if } \Delta J_i(k) \geq \varepsilon \\ \boldsymbol{x}_i(k) & \text{otherwise} \end{cases}$$
    Broadcast $\boldsymbol{x}_i(t+1|k+1)$ to all $j \succ_t i$
  end
  Set $k \coloneqq k+1$
end

B. Gauss-Seidel algorithm

A typical Gauss-Seidel method is reported in Algorithm 2. To compute an equilibrium of the GMIPG, the algorithm follows a certain ordering to consecutively solve an optimization problem for each player. As an example, we adopt the same approach in [2] by defining an inter-vehicle ordering relation at time t, i.e., $\prec_t$. Given any pair of vehicles $(i, j) \in \mathcal{I}^2$, we say that j has lower order than i at time t, namely $j \prec_t i$, when

1) $d_{i,j}(t) > 0$, or
2) $d_{i,j}(t) = 0$ and $v_{i,j}(t) > 0$, or
3) $d_{i,j}(t) = 0$, $v_{i,j}(t) = 0$ and $z_{i,j}(t) > 0$.

Thus, for each temporal step $t \in \mathcal{T}$, we define the set of the ordered vehicles as $\mathcal{O}_t \subseteq \mathcal{I}$. From now on, the subscripts refer to vehicles which follow the ordering in $\mathcal{O}_t$. For any vehicle i, $\boldsymbol{x}_{-i}(k)$ is obtained by stacking $\boldsymbol{x}_j(k+1)$ for all $j \prec_t i$ and $\boldsymbol{x}_j(k)$ for $j \succ_t i$. Therefore, vehicle i computes the best-response mapping using the "new" information from the players with lower order in $\mathcal{O}_t$, and the "old" one from those with higher order.

Lemma 1: Let $\mathcal{O}_t \subseteq \mathcal{I}$ be an ordered set of vehicles. For all $i \in \mathcal{O}_t$ and $k \in \mathbb{N}$, the collective strategy $(\boldsymbol{x}_i(k), \boldsymbol{x}_{-i}(k))$ generated by Algorithm 2 is feasible. □

Proof: The proof goes by induction over k. Take an arbitrary $i \in \mathcal{O}_t$, by assuming $(\boldsymbol{x}_i(k), \boldsymbol{x}_{-i}(k))$ feasible, i.e., $(\boldsymbol{x}_i(k), \boldsymbol{x}_{-i}(k)) \in \boldsymbol{\mathcal{X}}$, we show that also $(\boldsymbol{x}_i(k+1), \boldsymbol{x}_{-i}(k))$ is feasible. The claim is true if player i keeps its strategy, i.e., $\boldsymbol{x}_i(k+1) = \boldsymbol{x}_i(k)$. On the other hand, if player i updates the strategy, by definition of best response mapping $\boldsymbol{x}_i(k+1) \in \boldsymbol{x}_i^\star(\boldsymbol{x}_{-i}(k))$, hence $\boldsymbol{x}_i(k+1) \in \mathcal{X}_i(\boldsymbol{x}_{-i}(k))$ and $(\boldsymbol{x}_i(k+1), \boldsymbol{x}_{-i}(k)) \in \boldsymbol{\mathcal{X}}$. The proof follows by noticing that $\boldsymbol{x}(0) \in \boldsymbol{\mathcal{X}}$, $(\boldsymbol{x}_1(k), \boldsymbol{x}_{-1}(k)) = \boldsymbol{x}(k)$ and that $(\boldsymbol{x}_N(k+1), \boldsymbol{x}_{-N}(k)) = \boldsymbol{x}(k+1) \in \boldsymbol{\mathcal{X}}$. ■

Proposition 3: Let $\mathcal{O}_t \subseteq \mathcal{I}$ be an ordered set of vehicles and $\varepsilon > 0$. Algorithm 2 computes an ε-MINE, $\boldsymbol{x}^* \in \boldsymbol{\mathcal{X}}$, of the GMIPG. □

Proof: By definition, if there exists some $\bar{k} \in \mathbb{N}$ such that $\boldsymbol{x}_i(\bar{k}) = \boldsymbol{x}_i(\bar{k}+1) \in \boldsymbol{\mathcal{X}}$ for all $i \in \mathcal{O}_t$, then $\boldsymbol{x}(\bar{k}) = \boldsymbol{x}^*$ is an ε-MINE. Let us introduce $\mathcal{U}(k) \subseteq \mathcal{O}_t$ as the set of players that update their strategy at the k-th iteration. In view of Lemma 1, $\boldsymbol{x}_i(k) \in \mathcal{X}_i(\boldsymbol{x}_{-i}(k))$ for every $k \in \mathbb{N}$ and $i \in \mathcal{O}_t$. Moreover, we have

$$J_i(\boldsymbol{x}_i(k)) - J_i(\boldsymbol{x}_i(k+1)) \geq \varepsilon, \ \forall i \in \mathcal{U}(k), k \in \mathbb{N}.$$

Since P is an exact potential function, we obtain, for every $k \in \mathbb{N}$, $\Delta P(k) \coloneqq P(\boldsymbol{x}(k)) - P(\boldsymbol{x}(k+1)) \geq \bar{\varepsilon}(k)$, where $\bar{\varepsilon}(k) \coloneqq |\mathcal{U}(k)|\,\varepsilon \geq 0$. Therefore, $\{P(\boldsymbol{x}(k))\}_{k\in\mathbb{N}}$ is a non-increasing, bounded from below sequence, thus it converges to some finite value $\bar{P} \geq 0$. Hence, we have

$$0 = \lim_{k\to\infty} \Delta P(k) = \lim_{k\to\infty} \sum_{i\in\mathcal{U}(k)} \Delta J_i(k) \geq \lim_{k\to\infty} \bar{\varepsilon}(k).$$

Finally, since $\varepsilon > 0$, $\lim_{k\to\infty} |\mathcal{U}(k)| = 0$, i.e., there exists some $\bar{k} \in \mathbb{N}$ such that, for all $k \geq \bar{k}$, none of the vehicles deviate from $\boldsymbol{x}(\bar{k})$. ■

Algorithm 2 is tailored for an open-loop control scheme where we shall assume that $\mathcal{O}_t$ is fixed for the horizon $\mathcal{T}_t$. Then, the overall control policy reads as:

S1) At time t, define $\mathcal{O}_t$ and set the horizon $\mathcal{T}_t$;
S2) Find an ε-MINE, $\boldsymbol{x}^*$, via Algorithm 2;
S3) Implement $\boldsymbol{x}_i^*$, $\forall i \in \mathcal{I}$, set $t = t + T$, and go to S1.

Note that also Algorithm 2 requires intensive communication efforts among vehicles. To mitigate them, we investigate a closed-loop implementation of the Gauss-Seidel procedure, where the vehicles implement only the first temporal step of the equilibrium solution and, successively, play again. Thus, each vehicle $i \in \mathcal{O}_t$ is interested in the "next" action of its neighbors, limiting the amount of communication to $\boldsymbol{x}_i(t+1|k)$, $k \in \mathbb{N}$.

With this aim, we introduce Algorithm 3, where at the generic time t, each player is assumed to have some estimates for the remaining part of the strategy, i.e., $\hat{\boldsymbol{x}}(h|k)$, $\forall h \in \mathcal{H}_t \coloneqq \{t+2, \dots, t+T\}$, $k \in \mathbb{N}$. This turns into additional linear constraints that fix some components of the collective strategy $\boldsymbol{x}$, namely $\hat{A}_t \boldsymbol{x} \leq \hat{b}_t$ for some matrix $\hat{A}_t$ and vector $\hat{b}_t$ of appropriate structure and dimensions. The MVAD coordination problem has then a restricted feasible set,

$$\hat{\boldsymbol{\mathcal{X}}}_t := \{\boldsymbol{x} \in \mathbb{R}^n \mid A_t \boldsymbol{x} \leq b_t\} \subseteq \boldsymbol{\mathcal{X}}, \quad (39)$$

where $A_t := [A; \hat{A}_t]$ and $b_t := [b; \hat{b}_t]$. We shall assume, however, that the estimates are "reasonable".

Assumption 1: For all t, the set $\hat{\boldsymbol{\mathcal{X}}}_t$ in (39) is non-empty. □

Since each best-response mapping in (38) is computed by using an estimated strategy for the neighbors, we can not guarantee to achieve an ε-MINE over $\boldsymbol{\mathcal{X}}$ and the full horizon strategies computed on the basis of possibly incorrect estimates may be unfeasible. In view of a closed-loop policy, we show that Algorithm 3 provides feasible (hence implementable) decisions at $t+1$ and returns an ε-MINE over the restricted domain $\hat{\boldsymbol{\mathcal{X}}}_t$.

Lemma 2: Let Assumption 1 holds true and let $\mathcal{O}_t \subseteq \mathcal{I}$ be an ordered set of vehicles. For all $i \in \mathcal{O}_t$ and $k \in \mathbb{N}$, Algorithm 3 provides a feasible collective strategy $(\boldsymbol{x}_i(t+1|k), \boldsymbol{x}_{-i}(t+1|k))$. □

Proof: Since the estimate $\hat{\boldsymbol{x}}(h|k)$, $\forall h \in \mathcal{H}_t$, does not affect the constraints at $t+1$, we can restrict the analysis to the case with $T = 1$, i.e., no estimates needed and $\hat{\boldsymbol{\mathcal{X}}}_t = \boldsymbol{\mathcal{X}}$. Thus, for all $i \in \mathcal{O}_t$ and $k \in \mathbb{N}$, $(\boldsymbol{x}_i(k), \boldsymbol{x}_{-i}(k)) = (\boldsymbol{x}_i(t+1|k), \boldsymbol{x}_{-i}(t+1|k))$, which is feasible in view of Lemma 1. ■

Proposition 4: Let Assumption 1 holds true and let $\mathcal{O}_t \subseteq \mathcal{I}$ be an ordered set of vehicles and $\varepsilon > 0$. Under Assumption 1, Algorithm 3 computes an ε-MINE, $\bar{\boldsymbol{x}} \in \hat{\boldsymbol{\mathcal{X}}}_t$, of the GMIPG. □

Proof: For all $i \in \mathcal{O}_t$ and $k \in \mathbb{N}$, the mechanism in Algorithm 3 leaves $\boldsymbol{x}_i(k)$ as a decision variable over the entire horizon, to be updated by "freezing" the non-communicated components in $\boldsymbol{x}_{-i}(k)$ and by negotiating $\boldsymbol{x}_i(t+1|k)$. Hence, we have $\boldsymbol{x}_i(k) \in \mathcal{X}_i(\boldsymbol{x}_{-i}(k)) \cap \{\boldsymbol{x}_i \in \mathbb{R}^{n_i} \mid S_i \hat{A}_t(\boldsymbol{x}_i, \boldsymbol{x}_{-i}(k)) \leq S_i \hat{b}_t\}$, for some suitable matrix S_i that allows to fix the appropriate elements in $\boldsymbol{x}_{-i}(k)$. If $\hat{\boldsymbol{\mathcal{X}}}_t$ is non-empty, by discarding the fixed part of $\boldsymbol{x}_i(k)$, i.e., $\boldsymbol{x}_i(h|k)$, $\forall h \in \mathcal{H}_t$, and by appending $\boldsymbol{x}_i(t+1|k)$, feasible in view of Lemma 2, with the estimates $\hat{\boldsymbol{x}}_i(h|k)$, we obtain $(\boldsymbol{x}_i(k), \boldsymbol{x}_{-i}(k)) \in \hat{\boldsymbol{\mathcal{X}}}_t$ for all $i \in \mathcal{O}_t$ and $k \in \mathbb{N}$. Thus, the proof follows the one for Prop. 3. ■

Finally, we propose the closed-loop control policy that embeds the Gauss-Seidel algorithm:

S1) At time t, define $\mathcal{O}_t$, set $\mathcal{T}_t$ and $\hat{\boldsymbol{\mathcal{X}}}_t$;
S2) Find an ε-MINE, $\bar{\boldsymbol{x}} \in \hat{\boldsymbol{\mathcal{X}}}_t$, via Algorithm 3;
S3) Implement $[\bar{v}_i(t+1); \ldots; \bar{l}_i^{\mathrm{l}}(t+1)]$, $\forall i \in \mathcal{O}_t$, set $t = t+1$, and go to S1.

VII. Numerical simulations

This section shows the solution of the MVAD coordination problem applied to two scenarios via open- and closed-loop control policies. The numerical simulations are performed in MATLAB with solver GUROBI by choosing a quadratic objective function for the problem in (37), i.e., $J_i(\boldsymbol{x}_i) = \left\|\boldsymbol{v}_i - \boldsymbol{v}_i^{\mathrm{d}}\right\|_Q^2 + \left\|\boldsymbol{z}_i - \boldsymbol{z}_i^{\mathrm{d}}\right\|_R^2$ with the numerical values summarized in Tab. II. Thus, each vehicle solves a Mixed-Integer Quadratic Programming (MIQP) to iteratively compute an ε-MINE. We consider heterogeneous vehicles, with the main parameters sampled from normal distributions, $\bar{v}_i \sim \mathcal{N}(41.7 \text{ [m/s]}, 2.9 \text{ [m}^2\text{/s}^2])$ and $\bar{a}_i \sim \mathcal{N}(1.39 \text{ [m/s}^2], 0.4 \text{ [m}^2\text{/s}^4])$ ($\underline{a}_i = -\bar{a}_i$), $\forall i \in \mathcal{I}$.

TABLE II: Parameters for simulations in §VII.

Parameter	τ	$\hat{d}$	Q	R	ε
Value	3 [s]	5 [m]	diag(1)	diag(10)	10^{-12}

A. Multi-lane traffic

A typical MVAD scenario is shown in Fig. 6, where nine vehicles are disposed side-by-side on three lanes (Fig. 6(a)). Here, a solution is computed by means of Algorithm 1 over a prediction horizon of length $T = 4$. The resolution of each MIQP takes about 70 [ms] on average, and the open-loop control policy requires 27 iterations to converge to an ε-MINE. As shown in Figs. 6(b)–(e), the collective strategy vector $\boldsymbol{x}^*$ safely drives each vehicle $i \in \mathcal{I}$ over the horizon $\mathcal{T}$ to a randomly chosen target lane, z_i^{d}, while tracking a randomly chosen reference speed, v_i^{d}.

B. Merging a platoon of vehicles

The second scenario involves six vehicles disposed as in Fig. 7(a), where vehicle 1 aims at merging the platoon of vehicles, while vehicle 3 aims at leaving the latter to accelerate. Here, we propose a comparison between the open- and closed-loop implementation of the Gauss-Seidel algorithm. In both cases, vehicle 3 moves to the second lane at the first control step. Since the closed-loop policy is based on the estimates obtained by freezing the observed strategies at time t, vehicle 3, with a lower ordering than vehicle 1, decides to accelerate at $t+1$ (Fig. 7(b)). As shown in Fig. 7(c), vehicle 1 decides to accelerate as well, and at the third step takes the free spot left by vehicle 3 within the platoon (Fig. 7(b), (e)). On the other hand, the open-loop policy exploits the communicated full horizon strategy, which induces vehicle 1 to decelerate as a feasible initial sequence of decisions. Vehicle 3, despite a lower ordering, adopts a conservative strategy to fulfill the constraint in (7b) over the full horizon. Thus, at the fourth control steps, vehicle 1 merges the queue of the platoon (Fig. 7(b), (d)) and vehicle 3 freely accelerates.

VIII. Conclusion and outlook

A hybrid decision-making framework, shouldered by some AD rules, can model and solve the multi-lane MVAD problem in highways, hence ensure a safe use of the road space-time, despite the presence of selfish vehicles. Computational game theory is the key tool for solving the MVAD coordination problem, as it intrinsically catches the selfish behaviour of each road user. Generalized potential games allow us to bypass the mixed-integer nature of the problem by computing an approximated minimum of the underlying potential function, and are suitable to coordinate vehicles via both open- and closed-loop control policies. In the former case, a minimum of the potential function corresponds to an almost Nash equilibrium that satisfies the constraints over the full prediction horizon. In the latter, by introducing estimates to limit the communication efforts, feasibility guarantees are limited to the first control step.

This work can be extended in several directions, since systems of coupled hybrid systems model many relevant applications. Control design via game theory is a promising approach to handle noncooperative agents, with several open questions, e.g. equilibrium stability and robustness.

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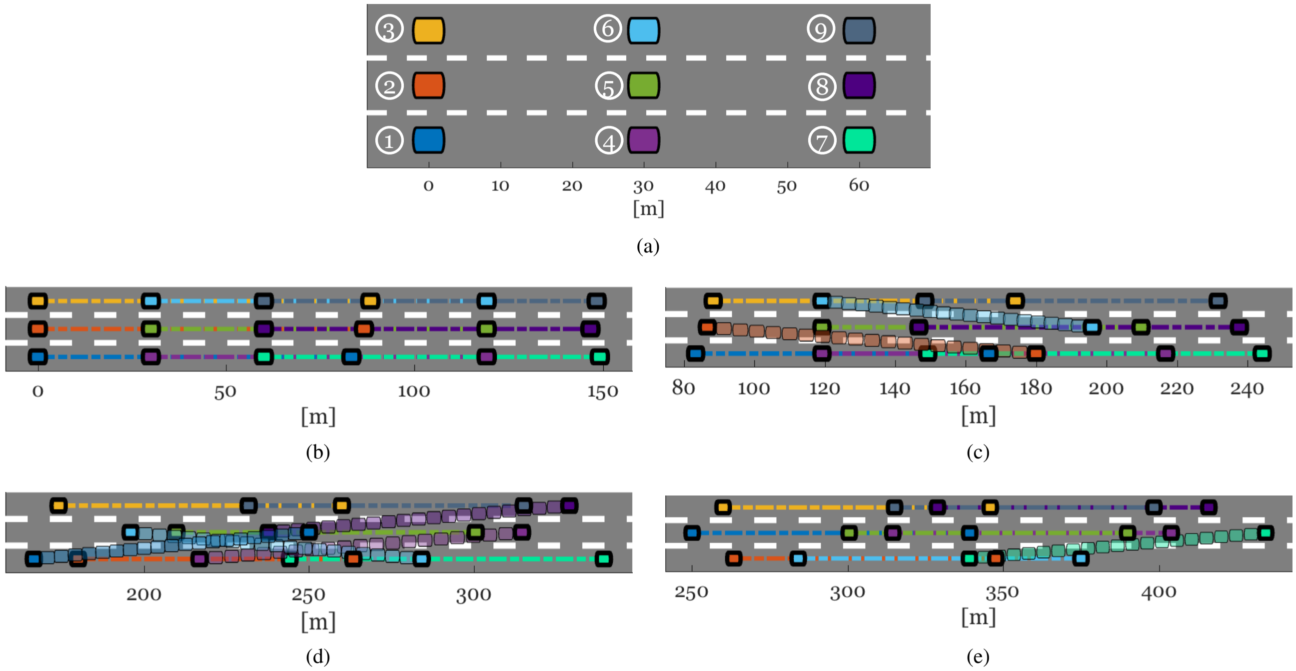

Fig. 6: (a) Initial configuration of the nine numbered vehicles. (b) 0–3 [s]. (c) 3.1–6 [s]. (d) 6.1–9 [s]. (e) 9.1–12 [s].

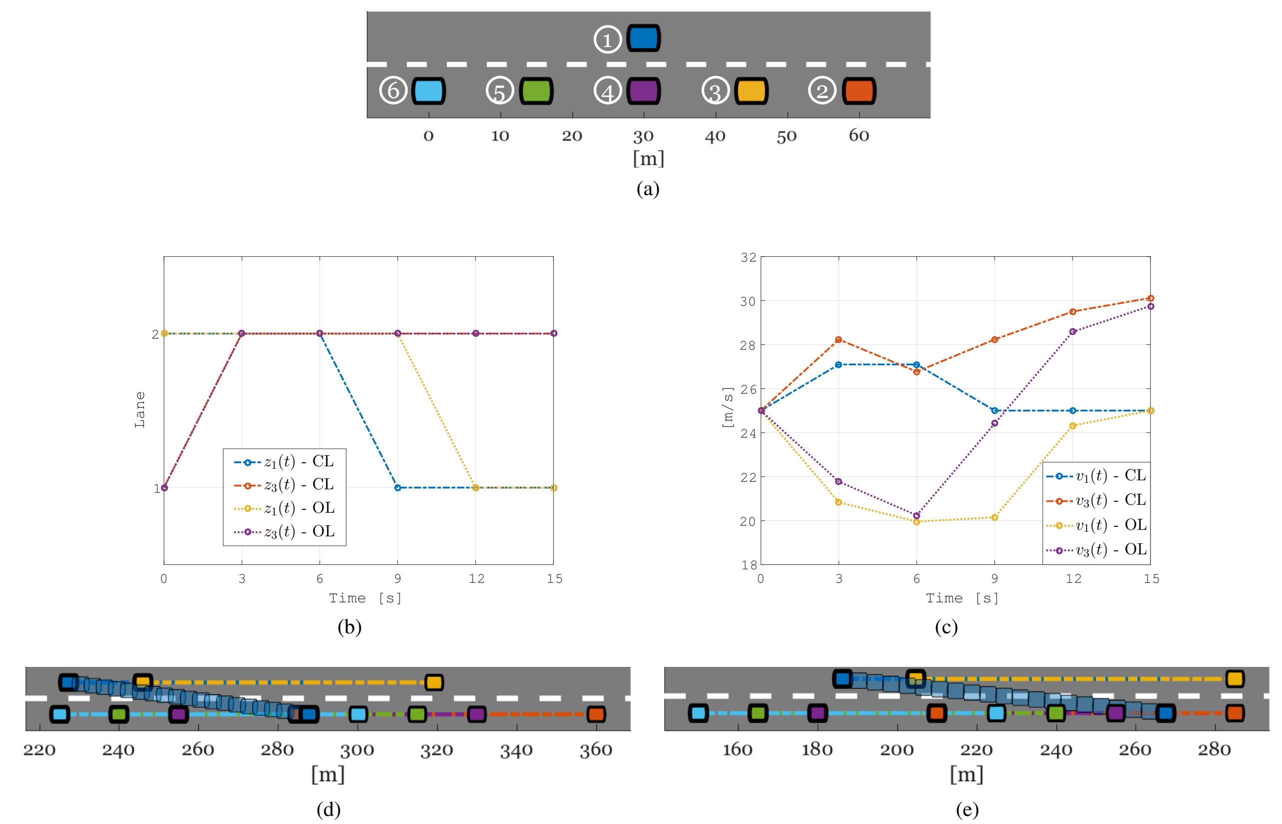

Fig. 7: (a) Initial configuration of the six numbered vehicles. (b) Comparison between lane profiles for vehicles 1 and 3. (c) Comparison between velocity profiles for vehicles 1 and 3. (d) Merging maneuver with the open-loop policy - 9.1–12 [s]. (e) Merging maneuver with the closed-loop policy - 6.1–9 [s].