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# Non-criticality criteria for Abelian sandpile models with sources and sinks

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We prove that the Abelian sandpile model on a random binary and binomial tree, as introduced in Redig, Ruszel, and Saada [J. Stat. Phys. **147**, 653–677 (2012)], is not critical for all branching probabilities  $p < 1$ ; by estimating the tail of the annealed survival time of a random walk on the binary tree with randomly placed traps, we obtain some more information about the exponential tail of the avalanche radius. Next we study the sandpile model on  $\mathbb{Z}^d$  with some additional dissipative sites: we provide examples and sufficient conditions for non-criticality; we also make a connection with the parabolic Anderson model. Finally we initiate the study of the sandpile model with both sources and sinks and give a sufficient condition for non-criticality in the presence of a finite number of sources, using a connection with the homogeneous pinning model. *Published by AIP Publishing.* <https://doi.org/10.1063/1.5022128>

## I. INTRODUCTION

The Abelian sandpile model was introduced in Refs. 2 and 4 as a toy model displaying self-organized criticality. This means that in the thermodynamic limit, the sandpile model has features of models of statistical physics at the critical point, such as the power law decay of avalanches, but without any fine-tuning of parameters. In short, an Abelian sandpile model is a discrete dynamical system defined as follows. Assign to each vertex of some finite graph a discrete variable (number of particles) representing a height, and call some special vertices sinks. At each unit of time, we add an extra particle to the system uniformly at random. If the resulting height exceeds a certain threshold, then the vertex topples by distributing the particles among its neighbours. This toppling can cause other vertices to topple which can lead to an avalanche. The self-organized criticality comes into play in the fact that the avalanche distribution obeys in certain cases a power law. In order to stabilize, the sandpile model needs sinks, which in the standard setting are associated with the boundary. On the contrary, if grains are lost upon toppling in the bulk, we call the model dissipative. It is well-known that this leads to non-criticality (i.e., the exponential decay of avalanche sizes). In this paper, we are interested in how much dissipation is needed in order to lose criticality. In the language of random walks, this amounts to create traps (associated with dissipative sites) and the expected avalanche size corresponds to an expected survival time, which depends on the configuration of traps. Second, we also introduce sites where upon toppling mass is gained (source sites). In the random walk picture, this corresponds to branching and the combination of sources and sinks to random walk with branching and killing. We associate non-criticality with finite expected avalanche size which then, in the setting of models with sources and sinks, corresponds to the finite expected total mass, when the random walk with killing and branching is started with unit mass.

In Ref. 24, we introduced the Abelian sandpile model on a random binary (or binomial) tree and proved that in a small region of the supercritical regime of the associated branching process, the model is not critical: that is, avalanche sizes decay exponentially. For the full Bethe lattice, avalanche sizes decay as a power law (Ref. 6), and so it remained an open question whether the absence of criticality persists in the whole supercritical regime, except for the degenerate case of

the full Bethe lattice. In the present paper, we study this question, thereby solving an open issue of Ref. 24.

In relation to this problem, we also tackle the following more general question: how much dissipation can be added to the sandpile model in order to make it non-critical? Therefore we consider various other settings and examples where we obtain criteria for non-criticality, such as  $\mathbb{Z}^d$  with a (possible random) set of dissipative sites. Furthermore we also start the study of the sandpile model with both sources and sinks. Assuming that the height variables do not grow indefinitely, i.e., if the system is stabilizable, we show that a finite number of source sites surrounded by enough dissipative sites produces a non-critical system. Even this at first sight intuitively obvious fact turns out to be non-trivial to prove; indeed, in the presence of source sites, large deviations come into play, and therefore, the simple intuition that the system is non-critical when there are more dissipative sites than source sites is wrong. For example, the system with one source site and all other sites “standard” is not stable.

The first result in this direction obtained in our paper also creates a link between the sandpile and the parabolic Anderson model, which is a model of a random walk in a random potential landscape where sites with negative potential are interpreted as killing and sites with positive potential as branching (see Ref. 19 for an up-to-date introduction to this model): we show that the non-criticality of the sandpile model with randomly placed sinks and sources is equivalent to the finiteness of the first moment of the total mass in the corresponding parabolic Anderson model. The source sites act as sites where the random walk (of the corresponding parabolic Anderson model) is branching, whereas the dissipative sites act as (possibly soft) traps. Therefore, whether or not the expected total mass has a first moment is related to random walk local time large deviation properties.

The rest of the paper is organized as follows. In Sec. II, we prove the exponential tail of the probability of survival of a random walk on a  $q$ -ary tree with traps. In Sec. III, we apply this result to prove non-criticality of the sandpile model on a random binomial tree and more general supercritical branching processes with bounded offspring distribution. In Sec. IV, we study the sandpile model on  $\mathbb{Z}^d$  and prove, guided by examples, sufficient conditions for non-criticality in the presence of dissipative sites. In Sec. V, we consider the model with sources and sinks and prove the first result of non-criticality in that context, making use of a connection with the homogeneous pinning model.

## II. PROBABILITY OF SURVIVAL OF A RANDOM WALK ON THE $q$ -ARY TREE WITH TRAPS

In this section, we consider a random walk on a  $q$ -ary tree with randomly placed traps and prove that the probability of its survival decays exponentially. There is a large literature on the probability of survival asymptotics for the walk on  $\mathbb{Z}^d$ , which is related to the large deviations of the Wiener sausage (see, e.g., Refs. 3 and 7). Surprisingly, we could not find the corresponding result for random walks on trees, although some results on trapped random walks on trees exist (see, e.g., Ref. 19). Intuitively, the probability of survival should decay exponentially because the random walk is strongly transient; hence, it has a range which typically grows linearly in the number of steps. This intuition is exactly what we make rigorous in this section.

Consider a rooted infinite  $q$ -ary tree  $\mathcal{T}_q$  with root  $o$ ; i.e., each vertex  $x \in \mathcal{T}_q$  different from the root has degree  $q + 1$  and the root has degree  $q$ .

For two vertices  $x, y \in \mathcal{T}_q$  denoted by  $d(x, y)$ , the graph distance between  $x$  and  $y$  is on  $\mathcal{T}_q$ . We place at every site  $x \in \mathcal{T}_q$  a trap with probability  $p_x > 0$ , independently for different sites. We call  $\omega : \mathcal{T}_q \rightarrow \{0, 1\}$  a trap configuration. More precisely, we set  $\omega(x) = 1$  if  $x$  is a trap with probability  $p_x$ , respectively  $\omega(x) = 0$  otherwise. Note that we allow the trapping probability to depend on the location  $x$ .

Denote by  $(S_n)_{n \geq 0}$  a simple random walk on  $\mathcal{T}_q$ . This random walk is killed upon hitting a trap with probability 1, and we call  $T(\omega)$  its survival time, i.e.,

$$T(\omega) = \inf \{n \in \mathbb{N} : \omega(S_n) = 1\}. \quad (1)$$

We denote by  $\mathbf{P}_o(\cdot)$  the annealed law, the joint probability of the random walk  $(S_n)_{n \geq 0}$  on  $\mathcal{T}_q$  starting at  $o$  together with the trap configuration  $\omega$  conditioned on the root not being a trap, and we denote by  $\mathbf{E}_o(\cdot)$  the corresponding expectation. We condition on the fact that the root is not a trap in order to make the link to branching processes in Sec. III. In Proposition 2.1 below, we prove that the probability of survival decays exponentially in the annealed setting. This is in contrast to the analogous problem on  $\mathbb{Z}^d$ , where the optimal strategy to survive for a long time is to stay for a long time in a trapless ball around the origin.<sup>19</sup>

*Proposition 2.1.* Assume that there exists  $p > 0$  such that the trapping probabilities  $(p_x)_{x \in \mathcal{T}_q}$  satisfy  $p_x > p$  for all  $x$ . Then there exists  $c = c(p, q) > 0$  such that for all  $n$

$$\mathbf{P}_o(T(\omega) > n) \leq e^{-cn}, \quad (2)$$

and as a consequence all moments of  $T$  exist.

*Proof.* We have  $S_0 = o$ ; call  $d(o, S_n) =: X_n$ . Then,  $(X_n)_{n \geq 0}$  is a one-dimensional random walk making a +1 step with probability  $q/(q+1)$  and a -1 step with probability  $1/(q+1)$ , reflected at the origin.

Let us denote by  $R_n(\omega)$  the range of the random walk  $(S_n)_{n \geq 0}$ , i.e., the number of points visited before time  $n$ . Then we have that on the event  $\{T(\omega) > n\}$ , all the points counted in  $R_n(\omega)$  should be trapless so that

$$\begin{aligned} \mathbf{P}_o(T(\omega) > n) &\leq \mathbf{E}_o((1-p)^{R_n(\omega)}) \\ &\leq \mathbf{E}_o\left((1-p)^{\epsilon n} \mathbb{1}_{\{R_n(\omega) \geq \epsilon n\}}\right) + \mathbf{E}_o(\mathbb{1}_{\{R_n(\omega) < \epsilon n\}}) \\ &\leq (1-p)^{\epsilon n} + \mathbf{P}_o(R_n(\omega) < \epsilon n). \end{aligned} \quad (3)$$

To estimate  $\mathbf{P}_o(R_n(\omega) < \epsilon n)$ , realize that if  $X_n \geq \epsilon n$  then necessarily  $R_n(\omega) \geq \epsilon n$ . Hence if  $R_n(\omega) < \epsilon n$ , then also  $X_n < \epsilon n$ . Therefore, because the random walk  $(X_n)_{n \geq 0}$  has a drift

$$\frac{q}{q+1} - \frac{1}{q+1} = \frac{q-1}{q+1},$$

by applying the Chernoff bound for  $t > 0$ , we have

$$\mathbf{P}_o(X_n < \epsilon n) \leq e^{\epsilon n t} \left( \frac{q}{1+q} e^{-t} + \frac{1}{q+1} e^t \right)^n.$$

Choose

$$t := \frac{1}{2} \log \left( \frac{2q}{1+\epsilon} - q \right) > 0$$

and then there exists  $\epsilon := \epsilon(q)$  such that for all  $c < \epsilon$  and all  $n$  we have that

$$\mathbf{P}_o(R_n(\omega) < \epsilon n) \leq \mathbf{P}_o(X_n < \epsilon n) \leq e^{-cn}, \quad (4)$$

which, combined with (3), yields the claim.  $\square$

Let us consider  $\{S_n, 0 \leq n \leq T(\omega)\}$ , i.e., the random walk killed upon hitting a trap. First note that for every given environment  $\omega$  of traps, we can write the survival time  $T(\omega)$  of the random walk as

$$T(\omega) = \sum_{x \in \mathcal{T}_q} \sum_{n=0}^{T(\omega)} \mathbb{1}_{\{S_n=x\}}. \quad (5)$$

We denote by  $\mathbb{E}_o^\omega(\cdot)$  the quenched expectation over the random walk started at  $o$  with a fixed trap configuration  $\omega$  and by  $E(\cdot)$  the average over  $\omega$ . We have by taking the expectation first over the random walk and second over the traps,

$$\mathbf{E}_o(T(\omega)) = \mathbf{E}_o \left( \sum_{x \in \mathcal{T}_q} \sum_{n=0}^{T(\omega)} \mathbb{1}_{\{S_n=x\}} \right) = E \left( \sum_{x \in \mathcal{T}_q} \mathcal{G}_{T(\omega)}(o, x) \right), \quad (6)$$

where

$$\mathcal{G}_{T(\omega)}(o, x) = \mathbb{E}_o^\omega \left( \sum_{n=0}^{T(\omega)} \mathbb{1}_{\{S_n=x\}} \right) \quad (7)$$

is Green's function of the random walk  $(S_n)_{n \geq 0}$  started at  $o$ , for a given trap configuration  $\omega$ .

The following annealed bound will be useful in Sec. III to estimate the avalanche diameter of the sandpile model.

*Proposition 2.2.* *There exists a constant  $C := C(p, q) > 0$  such that*

$$E \left( \sum_{x:d(0,x)>n} \mathcal{G}_{T(\omega)}(o, x) \right) \leq e^{-Cn}. \quad (8)$$

*Proof.*

$$\begin{aligned} \sum_{x:d(0,x)>n} \mathcal{G}_{T(\omega)}(o, x) &= \sum_{x:d(0,x)>n} \mathbb{E}_o^\omega \left( \sum_{n=0}^{T(\omega)} \mathbb{1}_{\{S_n=x\}} \right), \\ &= \sum_{x:d(0,x)>n} \mathbb{E}_o^\omega \left( \sum_{k=1}^{T(\omega)} \mathbb{1}_{\{S_k=x\}} \mathbb{1}_{\{T(\omega)>n\}} \right) \\ &= \mathbb{E}_o^\omega \left( \mathbb{1}_{\{T(\omega)>n\}} \sum_{x:d(0,x)>n} \sum_{k=1}^{T(\omega)} \mathbb{1}_{\{S_k=x\}} \right) \\ &\leq \mathbb{E}_o^\omega \left( \mathbb{1}_{\{T(\omega)>n\}} T(\omega) \right), \end{aligned} \quad (9)$$

where the first equality comes from (7) and the inequality from (5). As a consequence, taking the expectation over all the trap configurations provides

$$E \left( \sum_{x:d(0,x)>n} \mathcal{G}_{T(\omega)}(o, x) \right) \leq E \left( \mathbb{E}_o^\omega (T(\omega) \mathbb{1}_{\{T(\omega)>n\}}) \right).$$

By using first the Cauchy-Schwarz inequality and then Proposition 2.1, we have for some  $C > 0$ ,

$$\mathbf{E}_o \left( T(\omega) \mathbb{1}_{\{T(\omega)>n\}} \right) \leq \sqrt{\mathbf{E}_o(T^2(\omega))} \sqrt{\mathbf{P}_o(T(\omega) > n)} \leq e^{-Cn},$$

and the claim follows.  $\square$

### III. THE SANDPILE MODEL ON THE RANDOM BINOMIAL TREE

In this section, we will first describe how we can construct a realization of a Galton-Watson branching process  $\mathbb{T}^q$  starting from a single individual and having at most  $q$  descendants from a realization of a  $q$ -ary tree  $\mathcal{T}_q$  with some appropriate deletion, and second we will define the sandpile model on this realization.

We identify any vertex  $x$  (apart from the root) of a  $q$ -ary tree  $\mathcal{T}_q$  with a trap with probability  $p > 0$  and no trap with probability  $1 - p$  (thus there is no dependency on the location  $x \neq o$ , but the root  $o$  is trapless). This defines a trap configuration  $\omega$ . If a vertex is a trap, then we delete it, as well as all its descendants and the edge to its parent. Note that the remaining tree is in distribution equal to a realization of a Galton-Watson branching process with offspring distribution given by  $\text{Bin}(q, 1 - p)$ . We will refer to this tree as a random binomial tree with at most  $q$  descendants. Remark that for this construction to hold we had to assume that the root is not a trap.

We now consider a sandpile model on a fixed realization of the binomial tree. We first define the model in finite volume and then give its infinite volume construction. Standard references are Refs. 14, 15, 21, and 23.

For some  $n \in \mathbb{N}$ , denote by  $\mathbb{T}_n^q = \{x \in \mathbb{T}^q : d(o, x) \leq n\}$  the random binomial tree with root  $o$  up to generation  $n$ . We assign to each vertex  $x$  a height  $\eta_x \in \mathbb{N}$ . A height configuration  $\eta = (\eta_x)_{x \in \mathbb{T}_n^q}$  is *stable* if for all sites  $x \in \mathbb{T}_n^q$ ,  $\eta_x \in \{0, 1, 2, \dots, q\}$ ; otherwise, we call it *unstable*. If a site  $x$  is unstable (i.e., if  $\eta_x > q$ ), it topples: this means that it ejects  $q + 1$  particles out of which it redistributes one particle to each of its neighbours. If there exist particles ejected that are not redistributed, then the site is said *dissipative*. Note that the root is always losing 1 particle upon toppling. The effect of toppling of site  $x$  on the height configuration  $\eta$  at some site  $y \in \mathbb{T}_n^q$  is given by

$$(T_x \eta)_y = \eta_y - (\Delta^{\mathbb{T}_n^q})_{x,y}.$$

Here  $\Delta^{\mathbb{T}_n^q}$ , the toppling matrix, is the discrete Laplacian with Dirichlet boundary conditions, i.e.,

$$(\Delta^{\mathbb{T}_n^q})_{x,y} = \begin{cases} q + 1 & \text{if } x = y \in \mathbb{T}_n^q \\ -1 & \text{if } x \sim y \in \mathbb{T}_n^q \\ 0 & \text{otherwise,} \end{cases} \quad (11)$$

where  $x \sim y$  means  $d(x, y) = 1$ .

We denote by  $\Omega_n$  the set of stable configurations on  $\mathbb{T}_n^q$ . We call  $y$  a boundary site of  $\mathbb{T}_n^q$  if  $d(o, y) = n$ . Note that in this model every site which has either a non-maximal degree or is a boundary site is dissipative. As a consequence, because not all vertices have a maximal degree with positive probability, there is a positive density of dissipative sites outside the boundary, which intuitively should lead to a non-critical model. Note that this setup, similar to what was done in Ref. 24, is in contrast to the setup of Ref. 18, where the random toppling matrix depends on the realization of the tree in such a way that the only dissipative sites are the boundary sites so that the model there is critical.

A toppling at  $x \in \mathbb{T}_n^q$  in configuration  $\eta$  is called *legal* if  $\eta_x > q$ . A sequence of legal topplings is a composition  $T_{x_\ell} \circ \dots \circ T_{x_1}(\eta)$  such that for all  $k = 1, \dots, \ell$ , we have  $x_k \in \mathbb{T}_n^q$  and the toppling at  $x_k$  is legal in  $T_{x_{k-1}} \circ \dots \circ T_{x_1}(\eta)$ .

We denote by  $\mathcal{S}_n$  the stabilization with toppling matrix (11) and by  $a_{x,n}$  the addition operator defined on  $\Omega_n$  via

$$a_{x,n}(\eta) = \mathcal{S}_n(\eta + \delta_x), \quad (12)$$

where  $\delta_x$  denotes a unit mass at  $x$  and zero mass everywhere else. In other words,  $\mathcal{S}_n(\eta + \delta_x)$  is the unique stable configuration that arises from  $\eta + \delta_x$  by a sequence of legal topplings. Notice that both  $\mathcal{S}_n$  and  $a_{x,n}$  are quenched, i.e., for a given realization  $\mathbb{T}^q$  of the random tree. Thus the model is Abelian so that the stabilization does not depend on the toppling order and indeed leads to a unique stable configuration. The set of recurrent configurations corresponding to the addition operators  $a_{x,n}$  is denoted by  $\mathcal{R}_n$ . The unique measure  $\mu_n$  on  $\Omega_n$  invariant under all addition operators  $a_{x,n}$  is the uniform measure on the set of recurrent configurations

$$\mu_n = \frac{1}{|\mathcal{R}_n|} \sum_{\eta \in \mathcal{R}_n} \delta_\eta.$$

By Theorem 3 of Ref. 16, an infinite volume sandpile measure  $\mu = \lim_{n \rightarrow \infty} \mu_n$  on  $\mathbb{T}^q$  exists because each tree in the wired uniform spanning forest on  $\mathbb{T}^q$  has one end almost surely. See Ref. 20 for the background on wired spanning forests. The one end property for Galton-Watson trees with bounded degree distribution was proven in Ref. 1 (Theorem 7.2).

For  $\mu$ -a.e.  $\eta$ , the addition operator

$$a_x(\eta) = \lim_{n \rightarrow \infty} a_{x,n}(\eta)$$

is well defined (see Ref. 14), where with a small abuse of the notation we denoted by  $a_{x,n}(\eta)$  the concatenation of the addition operator applied in finite volume  $\mathbb{T}_n^q$  and the identity, i.e., fixing  $\eta$  outside  $\mathbb{T}_n^q$ , that is,  $a_{x,n}(\eta) = a_{x,n}(\eta|_{\mathbb{T}_n^q})\eta|_{(\mathbb{T}_n^q)^c}$ .

Moreover, we have

$$a_x(\eta) = \eta + \delta_x - \sum_{y \in \mathbb{T}^q} (\Delta^{\mathbb{T}^q})_{x,y} N(x, y, \eta), \quad (13)$$

where  $N(x, y, \eta)$  denotes the number of topplings needed at  $y$  to stabilize  $\eta + \delta_x$ , also known as the odometer function, and  $\Delta^{\mathbb{T}^q}$  is defined analogously in (11), with  $\mathbb{T}_n^q$  replaced by  $\mathbb{T}^q$ . We then define the avalanche at  $x$  in configuration  $\eta$  by

$$Av(x, \eta) = \{y : N(x, y, \eta) > 0\}. \quad (14)$$

Integrating (13) with respect to  $\mu$  (we denote by  $\mathbb{E}_\mu$  the corresponding expectation), we obtain Dhar's formula,<sup>5</sup>

$$\mathbb{E}_\mu(N(x, y, \eta)) = G_{\mathbb{T}^q}(x, y), \quad (15)$$

where  $G_{\mathbb{T}^q}$  denotes Green's function of the *Dirichlet random walk* on  $\mathbb{T}^q$ .

The latter is defined as follows: at every vertex  $x$  in  $\mathbb{T}^q$ , except the root, the random walk is killed with probability  $(q + 1 - \deg(x))/(q + 1)$  and is moving to every neighbouring vertex with probability  $1/(q + 1)$ . At the root, the Dirichlet random walk is killed with probability  $(q - \deg(o))/q$  and is moving to a descendant with probability  $1/q$ .

By the identification between branching processes and  $q$ -ary trees with traps described at the beginning of this section, Green's function  $G_{\mathbb{T}^q}$  of the Dirichlet random walk on  $\mathbb{T}^q$  is in distribution equal to Green's function of the random walk on  $\mathcal{T}_q$  killed upon hitting a trap (for a given trap configuration  $\omega$ , with trapping probabilities  $p_x = p$  for all  $x \in \mathcal{T}_q$ ,  $x \neq o$ , and the root  $o$  is trapless); i.e., Green's function  $\mathcal{G}_{T(\omega)}$  is defined in (7) and estimated in Proposition 2.2. We denote by  $\mathbb{P}_{\mathbb{T}^q}$  the distribution over the realizations of the binomial tree and by  $\mathbb{E}_{\mathbb{T}^q}$  the corresponding expectation.

This leads to the following Theorem.

**Theorem 3.1.** *We have for all  $x \in \mathbb{T}^q$ ,*

- (a) *annealed exponential bound on the diameter of the avalanche: there is a constant  $c := c(q) > 0$  such that*

$$\mathbb{P}_{\mathbb{T}^q}(\mu(\text{diam}(Av(x, \eta)) > n)) \leq e^{-cn}. \quad (16)$$

- (b) *and annealed finite expected avalanche size*

$$\mathbb{E}_{\mathbb{T}^q}(\mathbb{E}_\mu(|Av(x, \eta)|)) < \infty. \quad (17)$$

*Proof.* In Part (a), notice that Markov's inequality and Dhar's formula give

$$\begin{aligned} \mu(y \in Av(x, \eta)) &= \mu(N(x, y, \eta) \geq 1) \\ &\leq \mathbb{E}_\mu(N(x, y, \eta)) = G_{\mathbb{T}^q}(x, y). \end{aligned}$$

Hence

$$\begin{aligned} \mathbb{P}_{\mathbb{T}^q}(\mu(\text{diam}(Av(x, \eta)) > n)) &= \mathbb{P}_{\mathbb{T}^q}(\mu(\exists y : d(x, y) > n, Av(x, \eta) \ni y)) \\ &\leq \mathbb{P}_{\mathbb{T}^q} \left( \sum_{y: d(x, y) > n, y \in \mathbb{T}^q} \mu(Av(x, \eta) \ni y) \right) \\ &\leq \mathbb{E}_{\mathbb{T}^q} \left( \sum_{y: d(x, y) > n, y \in \mathbb{T}^q} G_{\mathbb{T}^q}(x, y) \right) \end{aligned}$$

and the result follows by (8).

For part (b), averaging over all tree realizations  $\mathbb{T}^q$ , using the equality in distribution of Green's functions and (6), we get

$$\begin{aligned}\mathbb{E}_{\mathbb{T}^q}(\mathbb{E}_\mu(|Av(x, \eta)|)) &= \mathbb{E}_{\mathbb{T}^q}\left(\sum_{y \in \mathbb{T}^q} \mu(y \in Av(x, \eta))\right) \\ &\leq \mathbb{E}_{\mathbb{T}^q}\left(\sum_{y \in \mathbb{T}^q} G_{\mathbb{T}^q}(x, y)\right) \\ &= E\left(\sum_{x \in \mathcal{T}_q} \mathcal{G}_{T(\omega)}(x, y)\right) \\ &= \mathbf{E}_x(T(\omega)),\end{aligned}$$

which is finite as given in Proposition 2.1.  $\square$

### A. Other random trees

In this section, we want to stress that Theorem 3.1 also applies to other related models, namely, the sandpile model on the random binary tree as in Ref. 24 and a general Galton-Watson tree with bounded offspring distribution.

- (i) In the *random binary tree*, each vertex has either 2 descendants with probability  $p$  or no descendant with probability  $1 - p$ . Green's function of the Dirichlet random walk on this random binary tree can be then dominated by Green's function of a random walk on a full binary tree with traps defined as follows. Each vertex of the full binary tree  $\mathcal{T}_2$  is a trap with probability  $p$ , independently of the other vertices. The trap is effective with probability  $2/3$ , i.e., the random walk is killed with probability  $2/3$  upon hitting a trap. Then Green's function of the latter random walk is dominating Green's function of the Dirichlet random walk since the traps are not perfect and the random walk can survive upon hitting a trap.
- (ii) A *non-homogeneous branching process*  $\mathbb{T}^q$  starts from a single individual. It has offspring probabilities  $(p_x(k))_{k \in \{0, \dots, q\}, x \in \mathbb{T}^q}$  to have  $0, \dots, q$  descendants (so they possibly depend on the vertex  $x$ ) such that  $p_x(k) > p > 0$  uniformly in  $k$  and  $x$ . Dirichlet's random walk Green's function can be again dominated by a trapped random walk on the full  $q$ -ary tree  $\mathcal{T}_q$  as follows. Every vertex  $x \in \mathcal{T}_q$  is a trap with probability  $1 - p_x(q)$ . Upon hitting a trap, the random walk is killed with probability  $1/(q + 1)$ .

## IV. THE ABELIAN SANDPILE MODEL WITH ADDITIONAL DISSIPATIVE SITES ON THE LATTICE $\mathbb{Z}^d$

In this section, we consider the sandpile model on the lattice  $\mathbb{Z}^d$  and add to it the dissipative sites. More precisely, for the finite box

$$\Lambda_n := [-n, n]^d \cap \mathbb{Z}^d, \quad (18)$$

we consider the toppling matrix indexed by sites  $x, y \in \Lambda_n$  and additionally parametrized by a set  $D \subset \mathbb{Z}^d$  (finite or infinite) of dissipative sites. We further denote  $D_n := D \cap \Lambda_n$  and  $D_n^c := D^c \cap \Lambda_n$ ,

$$\Delta_{x,y}^{D_n} = \begin{cases} -1 & \text{for } x, y \in \Lambda_n, x \sim y, \\ 2d + 1 & \text{for } x = y, x \in D_n, \\ 2d & \text{for } x = y, x \in D_n^c. \end{cases} \quad (19)$$

As before, we denote by  $\mathcal{S}_n$  stabilization with toppling matrix (19) within  $\Lambda_n$  and  $Av(x, \eta)$  the set of sites of  $\Lambda_n$  which have to be toppled at least once upon the addition of one grain at  $x$ . If



$D = \emptyset$ , then we will simply write  $\Delta^{D_n} = \Delta^{\Lambda_n}$ . Furthermore, we call  $\mu_n$  the uniform measure on recurrent configurations corresponding to the toppling matrix (19) and by  $\mu$  as before the weak limit

$$\mu = \lim_{n \rightarrow \infty} \mu_n,$$

which exists; see, for example, Ref. 13.

**Definition 4.1.** We call the sandpile model with dissipative sites  $D$  on  $\mathbb{Z}^d$  non-critical if for all  $x \in \mathbb{Z}^d$

$$\limsup_{n \rightarrow \infty} \mathbb{E}_{\mu_n}(|Av(x, \eta)|) < \infty. \quad (20)$$

Otherwise, we call it critical.

Whether the model is critical or not will of course depend on the choice of the set of dissipative sites. Let us comment on Definition 4.1. The sandpile model was introduced as a toy model displaying self-organized criticality, which is characterized by the power-law behavior of certain quantities like the avalanche distribution or two-point correlation functions. It is known in the mean-field setting<sup>12</sup> and on homogeneous trees<sup>6</sup> that we have the asymptotics

$$\mu_n(|Av(0, \eta)| > k) \approx k^{-1/2}, \quad \text{as } k, n \rightarrow \infty. \quad (21)$$

This is also conjectured to hold above the critical dimension  $d \geq 5$ .<sup>22</sup> A consequence of (21) is

$$\limsup_{n \rightarrow \infty} \mathbb{E}_{\mu_n}(|Av(x, \eta)|) = \infty. \quad (22)$$

Our definition is inspired by the analogous situation in percolation theory, where precisely at criticality the cluster is finite with probability 1 and has infinite expectation, while in the sub-critical regime it has a finite expectation; see chapter 1 in Ref. 10. We define the finiteness of the avalanche cluster cardinality as the signature of non-criticality. In what follows, we will be studying sufficient criteria on the set of dissipative sites ensuring that the model is no longer critical.

We will first characterize non-criticality by Green's function associated with the toppling matrix and give sufficient conditions. Let us denote  $G_n = (\Delta^{D_n})^{-1}$ . Note that for any given  $x \in \mathbb{Z}^d$ , there exists  $n_0$  such that for all  $n \geq n_0$ ,  $x \in \Lambda_n$  so that  $G_n(x, y)$  makes sense for  $y \in \mathbb{Z}^d$ .

**Theorem 4.1.** Consider a sandpile model on  $\Lambda_n$  with a set  $D \subset \mathbb{Z}^d$  of dissipative sites. Then the model is non-critical if either condition (a) or (b) is satisfied and critical if condition (c) is satisfied:

(a) For all  $x \in \mathbb{Z}^d$ ,

$$\limsup_{n \rightarrow \infty} \sum_{y \in \Lambda_n} G_n(x, y) < \infty. \quad (23)$$

(b) For all  $x \in \mathbb{Z}^d$ , the infinite volume Green's function  $G(x, y) = \lim_{n \rightarrow \infty} G_n(x, y)$  exists. Furthermore we have for all  $x \in \mathbb{Z}^d$ ,

$$\sum_{y \in \mathbb{Z}^d} G(x, y) < \infty.$$

(c) For all  $x \in \mathbb{Z}^d$ ,  $\lim_{n \rightarrow \infty} G_n(x, y) = G(x, y)$  is well-defined and there exists a dissipative site  $z$  such that

$$\sum_{y \in \mathbb{Z}^d} G(z, y) = \infty.$$

*Proof.* This follows directly from the upper and lower bounds of Theorem 6.1 (b) in Ref. 17.  $\square$

Recall that as before we will use that there is a particular random walk associated with Green's function, hence we can characterize non-criticality of the sandpile model via expected values of survival times of a corresponding random walk. Theorem 4.2 below describes sufficient conditions for non-criticality of the dissipative sandpile model in terms of corresponding random walk estimates.

Given a set of dissipative sites  $D \subset \mathbb{Z}^d$ , define a random walk  $\widehat{X} = (\widehat{X}_k)_{k \in \mathbb{N}}$  on  $\mathbb{Z}^d$  starting at some point  $x \in \mathbb{Z}^d$  as follows. If the random walk is at a non-dissipative site  $x$ , then it moves in the next step to one of its neighbours with probability  $1/(2d)$ . In the other case, it moves to one of its neighbours with probability  $1/(2d+1)$  or is killed with probability  $1/(2d+1)$ . The simple random walk on  $\mathbb{Z}^d$  will be denoted by  $X = (X_k)_{k \in \mathbb{N}}$ ,  $X_0 = x$  with corresponding expectation  $\mathbb{E}_x(\cdot)$ . Call

$$\tau_x(D) = \inf\{k \geq 1 : X_k \in D\}$$

the hitting or return time of the simple random walk to  $D$ . We denote by  $l_k(x)$  the corresponding local time, i.e., the number of visits to  $x$  of the simple random walk before time  $k$ . The following Theorem provides a number of sufficient conditions for non-criticality.

**Theorem 4.2.** *Let  $\widehat{T}$  denote the survival time of the random walk  $(\widehat{X}_k)_{k \in \mathbb{N}}$  defined above with respect to some set of dissipative sites  $D \subset \mathbb{Z}^d$ . Then the model is non-critical if either of the following conditions is satisfied:*

- (a) *For all  $x \in \mathbb{Z}^d$ , we have that  $\mathbb{E}_x(\widehat{T}) < \infty$ .*
- (b) *For all  $x \in \mathbb{Z}^d$ ,*

$$\mathbb{E}_x \left( \sum_{k=0}^{\infty} \prod_{z \in D} \left( \frac{2d}{2d+1} \right)^{l_k(z)} \right) < \infty.$$

- (c)  *$|D^c| < \infty$ .*
- (d)  *$D$  is such that there exists  $\varphi : \mathbb{N} \rightarrow \mathbb{R}$  such that  $\sum_{k=0}^{\infty} \left( \frac{2d}{2d+1} \right)^{\varphi(k)} < \infty$  and for all  $x \in \mathbb{Z}^d$*

$$\sum_{k=0}^{\infty} \mathbb{P}_x(l_k(D^c) \geq k - \varphi(k)) < \infty.$$

- (e) *For all  $x \in \mathbb{Z}^d$  :  $\mathbb{E}_x(\sup_{y \in \mathbb{Z}^d} \tau_y(D)) < \infty$ .*
- (f) *There exists a constant  $R > 0$  such that*

$$\sup_{x \in \mathbb{Z}^d} \inf_{y \in D} |x - y| = R + 1 < \infty,$$

*i.e., every point in  $\mathbb{Z}^d$  is at most at distance  $R + 1$  from a point in  $D$ .*

*Proof.* (a) For this proof, we introduce  $(\overline{X}_t^{\Lambda_n})_{t \geq 0}$ , the continuous-time random walk in  $\Lambda_n$  jumping at rate  $2d$ , starting at  $x \in \Lambda_n$  and killed upon leaving  $\Lambda_n$ ; we denote by  $\overline{\mathbb{E}}_x^{\Lambda_n}(\cdot)$  the corresponding expectation; moreover we denote by  $\overline{X} = (\overline{X}_t)_{t \geq 0}$  the continuous-time random walk on  $\mathbb{Z}^d$  jumping at rate  $2d$ , by  $\overline{\mathbb{E}}(\cdot)$  the corresponding expectation and by  $\bar{l}_t(z) = \int_0^t \mathbb{1}_{\{\overline{X}_s = z\}} ds$  the corresponding local time in  $z$ .

Let us fix the set of dissipative sites  $D$  and consider the associated toppling matrix  $\Delta^{D_n}$  defined as in (19). Note that

$$\begin{aligned} G_n &= (\Delta^{D_n})^{-1} \\ &= (\Delta^{\Lambda_n} + \mathbb{1}_{D_n} Id)^{-1} =: (\Delta^{\Lambda_n} + V_{D_n} Id)^{-1}, \end{aligned} \tag{24}$$

where  $Id$  denotes the identity matrix in  $\Lambda_n$  and  $V_{D_n} = \mathbb{1}_{D_n}$  can be interpreted as a *potential*. Finally fix  $x \in \mathbb{Z}^d$  and  $\Lambda_n$  for  $n$  large enough such that  $x \in \Lambda_n$ . In order to prove non-criticality: by Theorem 4.1 (a), we have to show that

$$\limsup_{n \rightarrow \infty} \sum_{y \in \Lambda_n} G_n(x, y) < \infty.$$

By the Feynman-Kac formula and using the exponential distribution of the jump times of the continuous time random walk, we can write

$$\begin{aligned}
 G_n(x, y) &= \int_0^\infty \left( e^{-t\Delta^{D_n}} \right)_{x,y} dt \\
 &= \int_0^\infty \overline{\mathbb{E}}_x^{\Lambda_n} \left( e^{-\int_0^t V_{D_n}(\bar{X}_s) ds} \mathbb{1}_{\{\bar{X}_t = y\}} \right) dt \\
 &\leq \int_0^\infty \overline{\mathbb{E}}_x \left( e^{-\int_0^t V_D(\bar{X}_s) ds} \mathbb{1}_{\{\bar{X}_t = y\}} \right) dt \\
 &= \int_0^\infty \overline{\mathbb{E}}_x \left( e^{-\sum_{z \in D} \bar{l}_t(z)} \mathbb{1}_{\{\bar{X}_t = y\}} \right) dt \\
 &= \frac{1}{2d} \mathbb{E}_x \left( \sum_{k=0}^\infty \prod_{z \in D} \left( \frac{2d}{2d+1} \right)^{l_k(z)} \mathbb{1}_{\{X_k = y\}} \right). \tag{25}
 \end{aligned}$$

Then we have

$$\mathbb{E}_x \left[ \prod_{z \in D} \left( \frac{2d}{2d+1} \right)^{l_k(z)} \mathbb{1}_{\{X_k = y\}} \right] = \widehat{\mathbb{E}}_x \left( \mathbb{1}_{\{\widehat{X}_k = y\}} \mathbb{1}_{\{\widehat{T} > k\}} \right). \tag{26}$$

Summing over all  $y \in \mathbb{Z}^d$  gives

$$\sum_{y \in \mathbb{Z}^d} \sum_{k \in \mathbb{N}} \widehat{\mathbb{E}}_x \left( \mathbb{1}_{\{\widehat{X}_k = y\}} \mathbb{1}_{\{\widehat{T} > k\}} \right) = \sum_{k=0}^\infty \widehat{\mathbb{E}}_x \left( \mathbb{1}_{\{\widehat{T} > k\}} \right) \leq \widehat{\mathbb{E}}_x(\widehat{T}).$$

Therefore,

$$\limsup_n \sum_{y \in \mathbb{Z}^d} G_n(x, y) \leq \frac{1}{2d} \mathbb{E}_x \left[ \sum_{k=0}^\infty \prod_{z \in D} \left( \frac{2d}{2d+1} \right)^{l_k(z)} \right] \leq \frac{1}{2d} \widehat{\mathbb{E}}_x(\widehat{T}). \tag{27}$$

The claim follows using Theorem 4.1 (a).

- (b) Condition (b) implies that the rhs of (27) is finite and hence again the condition in Theorem 4.1 (a) is satisfied.
- (c) We will show that condition (c) implies condition (b). It is enough to consider  $x = o$ . Call  $l_k(G) = \sum_{x \in G} l_k(x)$  for  $G \subset \mathbb{Z}^d$ . For some  $\alpha \in (0, 1)$ , we write

$$\begin{aligned}
 &\mathbb{E}_o \left( \left( \frac{2d}{2d+1} \right)^{l_k(D)} \right) \\
 &= \mathbb{E}_o \left( \left( \frac{2d}{2d+1} \right)^{l_k(D)} \mathbb{1}_{\{l_k(D) > \alpha k\}} \right) + \mathbb{E}_o \left( \left( \frac{2d}{2d+1} \right)^{l_k(D)} \mathbb{1}_{\{l_k(D) \leq \alpha k\}} \right) \\
 &\leq \left( \frac{2d}{2d+1} \right)^{\alpha k} + \mathbb{E}_o \left( \mathbb{1}_{\{l_k(D) \leq \alpha k\}} \right). \tag{28}
 \end{aligned}$$

It now suffices to estimate the probability  $\mathbb{P}_o(l_k(D) \leq \alpha k)$  and show that it is summable in  $k$ . Notice that because  $l_k(D) + l_k(D^c) = k$  this amounts to estimate the probability

$$\mathbb{E}_o \left( \mathbb{1}_{\{l_k(D^c) \geq (1-\alpha)k\}} \right).$$

When  $D^c$  is finite, this is simple. We have the following bound on local time tails from lemma 1 Sec. 3 of Ref. 11: there exists  $a, b > 0$  such that for all  $\delta > 0$ ,

$$\mathbb{P}_o \left( \sup_{x \in \mathbb{Z}^d} l_k(x) > k^{1/2+\delta} \right) \leq a e^{-b k^{\delta/2}}.$$

As a consequence, using  $l_k(D^c) < |D^c| \sup_{x \in \mathbb{Z}^d} l_k(x)$ , we obtain for  $\delta = \frac{1}{2}$ ,

$$\mathbb{P}_o(l_k(D^c) \geq (1 - \alpha)k) \leq \mathbb{P}\left(\sup_{x \in \mathbb{Z}^d} l_k(x) \geq \frac{(1 - \alpha)k}{|D^c|}\right) \leq a' e^{-b' k^{1/4}}, \quad (29)$$

which is summable in  $k$ . Non-criticality now follows from arguments in part (a).

- (d) This follows from the proof in (c) by replacing  $\alpha k$  by  $\varphi(k)$ .  
 (e) Let us call  $T_1, T_2, \dots$  the successive hitting times of the set  $D$  of the simple random walk  $X$ . Every time that  $D$  is visited, the corresponding trapped walk  $\hat{X}$  is killed with probability  $\frac{1}{2d+1}$ . The survival time  $\hat{T}$  of the killed walk  $\hat{X}$  starting from  $x$  is a sum,

$$\hat{T} \leq \sum_{i=1}^N \tau_{X_{T_i}}(D),$$

where  $N$  is a geometric random variable with parameter  $\frac{1}{2d+1}$ , independent of the simple random walk  $X$ . Therefore,

$$\widehat{\mathbb{E}}_x(\hat{T}) \leq (2d + 1) \mathbb{E}_x\left(\sup_{y \in \mathbb{Z}^d} \tau_y(D)\right) < \infty,$$

then the claim follows from (a).

- (f) Denote for  $x \in \mathbb{Z}^d$  the ball with radius  $R$ ,

$$B(x, R) := \{y \in \mathbb{Z}^d : |x - y| \leq R\}.$$

Upon exiting  $B(x, R)$ , there is a strictly positive probability that a point of  $D$  is hit. This probability is bounded from below by a number  $\kappa_R$  only depending on  $R$ . Indeed denote  $\sigma_{B(x, R)}$  the exit time of  $B(x, R)$ . Note that by translation invariance the distribution of  $\sigma$  does not depend on  $x$ . Then we can choose  $\kappa_R = \inf_{z \in B(x, R)} \inf_y \mathbb{P}_z(X_{\sigma_{B(x, R)}} = y)$ . If no point of  $D$  is hit upon exiting  $B(x, R)$ , then start from the exit point and look at the exit time of the ball with radius  $R$  from that point. We now see that from  $x$  the hitting time of  $D$  is bounded from above by a geometric sum of exit times of balls of radius  $R$  of which the distribution does not depend on  $x$ . Therefore the condition of item (e) is satisfied.  $\square$

*Remark 4.1.* One can also perform the sum over  $y$  in the continuous-time expression appearing in the third line of (25). This leads to the sufficient criterion of non-criticality

$$\int_0^\infty \overline{\mathbb{E}}_x \left( e^{-\int_0^t V_D(\bar{X}_s) ds} \right) dt < \infty \quad (30)$$

for all  $x \in \mathbb{Z}^d$ . Notice that this corresponds to the total mass in the parabolic Anderson model,<sup>19</sup> integrated over time. The parabolic Anderson model is a random walk model in a random potential  $V$ . It also corresponds to the expected survival time of a continuous-time random walk which is trapped at rate 1 on dissipative sites and moves to every neighbouring site at rate one.

The sufficient conditions for non-criticality provided in items (c)–(f) of Theorem 4.2 are not necessary. We provide two further examples illustrating critical versus non-critical behavior as a function of the “size” of the set of dissipative sites which are not covered by items (c)–(f) of Theorem 4.2.

- (i) First, if we make all sites of the  $x$ -axis of the two-dimensional square lattice dissipative, and all other sites ordinary, then the model is critical because the expected hitting time of the  $x$ -axis of a two-dimensional simple random walk started from a point outside the  $x$ -axis is infinite. By the same argument, if the set of dissipative sites is a lower dimensional subset of  $\mathbb{Z}^d$  (such as a hyperplane intersected with  $\mathbb{Z}^d$ ), the model is still critical.  
 (ii) As a second example, consider the sandpile model on  $\mathbb{Z}^2$  where we put dissipative sites on a sequence of horizontal lines  $y = 0, y = -r_1, y = r_1, y = -r_2, y = r_2, \dots$ , where the distances  $r_n$

form an increasing sequence such that the gaps  $r_{n+1} - r_n$  diverge in the limit  $n \rightarrow \infty$ . Then the expected survival time starting at the origin is bounded by

$$\frac{1}{4} \sum_{i=1}^N (r_{i+1} - r_i)^2,$$

where  $N$  is an independent geometric random variable with success probability  $1/(d+1)$  [see also the proof of Theorem 4.2 (e)]. This can be seen from the fact that the expected hitting time of the set  $\{a, b\}$  of the simple random walk starting at  $x \in (a, b)$  is bounded by  $(b-a)^2/4$ . Therefore, as long as the sum

$$\sum_{k=0}^{\infty} \left( \frac{d}{d+1} \right)^k (r_{k+1} - r_k)^2$$

is convergent, the model is non-critical. Indeed one can choose the distances between two successive lines  $r_{k+1} - r_k$  to grow faster than any polynomial in  $k$ . So this includes cases where the set of dissipative sites does not have a positive density in  $\mathbb{Z}^2$  and hence cases not covered by the sufficient conditions provided in items (c)–(f) of Theorem 4.2.

## V. THE ABELIAN AVALANCHE MODEL WITH SINKS AND SOURCES

In this section, we study a model with sources and sinks. Sources will be sites where upon toppling mass is gained, whereas sinks are sites where upon toppling mass is lost. To study such a system, it is more convenient to work in the continuous-height setting and therefore put ourselves in the context of the Abelian avalanche model (which was introduced in Ref. 8 and then studied in Ref. 18), which is the natural continuous-height counterpart of the Abelian sandpile model.

More precisely, we will consider a model which has dissipative sites in a set  $D \subset \mathbb{Z}^d$  and source sites in a set  $S \subset \mathbb{Z}^d$  and in which the amount of mass transferred to neighbors upon a toppling is governed by a parameter  $\gamma > 0$ , the amount of mass lost upon toppling a (bulk) dissipative site is governed by a parameter  $\alpha$ , and the amount of mass gained upon toppling a source site is governed by a parameter  $\beta$ . This is what we call the *Abelian avalanche model with sources and sinks*, with parameters  $(D, S, \alpha, \beta, \gamma)$ . Note that one can retrieve the Abelian sandpile model from the Abelian avalanche model by  $\gamma, \beta \rightarrow 0$ , as in Ref. 1.

To define such a model, we first put ourselves in finite volume, and we consider the following toppling matrix indexed by sites  $x, y \in \Lambda_n = [-n, n]^d \cap \mathbb{Z}^d$ , with  $D_n = D \cap \Lambda_n$ ,  $S_n = S \cap \Lambda_n$ ,

$$\Delta_{x,y}^{D_n, S_n, \alpha, \beta} = \begin{cases} -\gamma & \text{for } x, y \in \Lambda_n, x \sim y, \\ 2d\gamma + \alpha & \text{for } x = y, x \in D_n, \\ 2d\gamma & \text{for } x = y, x \in \Lambda_n \setminus (D_n \cup S_n), \\ 2d\gamma - \beta & \text{for } x = y, x \in S_n. \end{cases} \quad (31)$$

This defines a continuous-height sandpile model, called the *Abelian avalanche model* with dissipative sites in  $D_n$  and source sites in  $S_n$ . The interpretation of the toppling matrix is, as already announced before, that a site is stable if its height is respectively below  $2d\gamma$  for an “ordinary” site  $x \in \Lambda_n \setminus (D_n \cup S_n)$ , below  $2d\gamma + \alpha$  for a dissipative site  $x \in D_n$ , and below  $2d - \beta$  for a source site  $x \in S_n$ .

Upon toppling of a site  $x \in \Lambda_n$ , a mass  $\gamma$  is transferred to each neighbor of  $x$  in  $\Lambda_n$ . This means that upon toppling, for a dissipative site which is not on the boundary, mass  $\alpha$  is *lost*, whereas for a source site which is not on the boundary, mass  $\beta$  is *gained*.

For a height configuration  $\eta: \Lambda_n \rightarrow [0, \infty)$ , we define as before  $\mathcal{S}_n(\eta)$  to be its stabilization according to the toppling matrix (31), provided  $\eta$  is stabilizable, i.e., provided there exists a sequence of legal topplings with a stable final result. If this is the case, then by the same argument as in Ref. 23, stabilization is unique and well-defined. We therefore assume first that  $D, S, \alpha, \beta, \gamma$  are chosen in such a way that all  $\eta$  are stabilizable for the toppling matrix (31) for all  $n \in \mathbb{N}$ . We then say that the model with parameters  $(D, S, \alpha, \beta, \gamma)$  is *well-defined*.

In that case, we are in the same setting as the standard Abelian avalanche model, i.e., there exists a unique stationary measure  $\mu_n$  which is the uniform measure on recurrent configurations (i.e., configurations which are “burnable”—through the burning algorithm; see Ref. 17) and we have Dhar’s formula

$$\mathbb{E}_n(N(x, y, \eta)) = G_n(x, y),$$

where  $G_n(x, y)$  is again  $(\Delta^{D_n, S_n, \alpha, \beta})_{x, y}^{-1}$  and where  $N(x, y, \eta)$  denotes the number of topplings at  $y$  needed to stabilize  $\eta + \delta_x$ . As before, we then define the infinite volume model to be *non-critical* as in Definition 4.1. A sufficient condition for non-criticality is then cf. (23)

$$\limsup_{n \rightarrow \infty} \sum_{y \in \Lambda_n} G_n(x, y) < \infty.$$

In analogy with the “potential”  $V_D$  defined in (24), in our setting, we define the potential

$$V_{D,S}(x) = \begin{cases} +\alpha & \text{if } x \in D, \\ 0 & \text{if } x \text{ ordinary,} \\ -\beta & \text{if } x \in S. \end{cases} \quad (32)$$

In analogy with (30), we then have the following sufficient criterion of non-criticality of the infinite-volume model with parameters  $(D, S, \alpha, \beta, \gamma)$ .

**Proposition 5.1.** *Provided that the model with parameters  $(D, S, \alpha, \beta, \gamma)$  is well-defined, it is non-critical if for all  $x \in \mathbb{Z}^d$  we have*

$$\int_0^\infty \overline{\mathbb{E}}_{\gamma, x} \left( e^{-\int_0^t V_{D,S}(\bar{X}_s) ds} \right) dt < \infty, \quad (33)$$

where  $\overline{\mathbb{E}}_{\gamma, x}(\cdot)$  is the expectation with respect to the continuous-time random walk  $(\bar{X}_s)_{s \geq 0}$  with rate  $2d\gamma$  starting at  $x$ .

The expectation  $\overline{\mathbb{E}}_{\gamma, x}(e^{-\int_0^t V_{D,S}(\bar{X}_s) ds})$  can be interpreted as the total mass at time  $t > 0$ , when starting from a unit mass at time zero which is splitting at rate  $\beta$  (in two unit masses) on source sites and is killed at rate  $\alpha$  at dissipative sites and besides is moving according to the continuous-time random walk at rate  $2d\gamma$ . Notice that

$$\overline{\mathbb{E}}_{\gamma, x} \left( e^{-\int_0^t V_{D,S}(\bar{X}_s) ds} \right) = \overline{\mathbb{E}}_x \left( e^{-\frac{1}{\gamma} \int_0^{t\gamma} V_{D,S}(\bar{X}_s) ds} \right), \quad (34)$$

where we remind the reader that  $\overline{\mathbb{E}}$  denoted the expectation with respect to the continuous time random walk. We will look at a finite number of source sites and everywhere else dissipative sites and show that for  $\gamma$  large enough, the model is *not critical*.

**Theorem 5.1.** *Let  $S$  be finite and  $D := \mathbb{Z}^d \setminus S$ . Then for  $\gamma$  large enough, the sandpile model with toppling matrix (31) is not critical.*

*Proof.* We start with a single source site at the origin,  $S_n = \{o\}$ . In that case, we have using (34)

$$\overline{\mathbb{E}}_{\gamma, o} \left( e^{-\int_0^t V_{D,S}(\bar{X}_s) ds} \right) = \overline{\mathbb{E}}_o \left( e^{-\alpha t} e^{\frac{\alpha+\beta}{\gamma} l_{t\gamma}(\{o\})} \right), \quad (35)$$

where as before  $l_t(G) = \int_0^t \mathbb{1}_{\{\bar{X}_s \in G\}} ds$  denotes the local time associated with the random walk  $(\bar{X}_s)_{s \geq 0}$ . Denote

$$F(m) = \lim_{t \rightarrow \infty} \frac{1}{t} \log \overline{\mathbb{E}}_o \left( e^{m l_t(\{o\})} \right). \quad (36)$$

Notice that this is exactly the free energy of the homogeneous pinning model; see Ref. 9. The homogeneous pinning model is a random polymer model which is penalized or rewarded upon touching the point 0. From Theorem 2.10 in Ref. 9, we conclude that around  $m \approx 0$  the behavior is

$$F(m) = O(m^2)$$

in  $d = 1, 3$ . In particular,

$$\lim_{m \rightarrow 0} \frac{1}{m} F(m) = 0. \quad (37)$$

Therefore,

$$\overline{\mathbb{E}}_o \left( e^{\frac{\alpha+\beta}{\gamma} l_{\gamma}(\{o\})} \right) \approx e^{(\gamma F(\frac{\alpha+\beta}{\gamma}) + o(1))}$$

by (37), and as  $\gamma \rightarrow \infty$ , we have

$$\gamma F\left(\frac{\alpha+\beta}{\gamma}\right) \rightarrow 0 \text{ as } \gamma \rightarrow \infty. \quad (38)$$

As a consequence, the right hand side of (35) is integrable as a function of  $t$  for  $\gamma$  large enough. In  $d = 2$ , (37) still holds. In  $d \geq 4$ ,  $F(m) = 0$  for  $m \in [0, m_c]$  with  $m_c > 0$ ; so also in that case the rhs of (35) is integrable for  $\gamma$  large enough.

Finally, if we have a finite number of source sites  $S$ , then we need to estimate

$$\overline{\mathbb{E}}_o \left( e^{\frac{\alpha+\beta}{\gamma} l_{\gamma}(S)} \right),$$

which by iteratively using the Cauchy-Schwarz inequality can be estimated by

$$e^{\gamma t F(2^{n-1} \frac{\alpha+\beta}{\gamma}) + o(t)},$$

where  $n = |S|$  is the number of source sites. The result then follows again from (38) as before.  $\square$

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