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De L^p begrensdheid van de Riesztransformatie

(Engelse titel: The L^p boundedness of the Riesz transform)

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"De L^p begrensdheid van de Riesztransformatie"

(Engelse titel: "The L^p boundedness of the Riesz transform")

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Abstract

In this BSc project we study the convolution-type singular integral operators and mainly the Riesz transform. In the process of studying the Riesz transform, we consider the maximal function and the conditions for the L^p-boundedness of the maximal function. A special case of the Marcinkiewicz interpolation theorem is treated, which will be used quite often in the further treatment of the singular integral operators.

After this, a theorem is discussed which says something about the convolution-type singular integral operators, that is, operators of the form $(Tf)(x) = (K * f)(x)$ where K is the kernel with a singularity in the origin. In the beginning there are quite restrictive conditions on the kernel, but these will be weakened in the process.

After the study of the general singular integral operators, the Riesz transform is introduced. The L^p-boundedness of the Riesz transform is proven using previous results.

An estimation for the Laplacian is proven. With this estimation we prove a characterisation of the Sobolev space $W^{2,p}$, which is the end result of this project.

This project is based upon the first three chapters of [1].

Samenvatting

In dit BSc project bestuderen we de convolutie-type singuliere integraaloperatoren en met als hoofddoel de Riesztransformatie. In dit proces bekijken we eerst de maximale functie en de voorwaarden voor de L^p begrensdheid van deze. Een speciaal geval van de Marcinkiewicz interpolatiestelling wordt behandeld, deze wordt vaak gebruikt in de verdere behandeling van de stof.

Daarna wordt een stelling behandeld die wat zegt over de convolutie-type singuliere integraaloperator, dat wil zeggen een operator van de vorm $(Tf)(x) = (K * f)(x)$ waarbij K de kernel is met een singulariteit in de oorsprong. In het begin worden er nog een aantal strikte eisen gelegd op de kern, maar deze worden gaandeweg verzwakt.

Na de studie van de algemene singuliere integraaloperatoren wordt de Riesztransformatie geïntroduceerd. De L^p begrensdheid van de Riesztransformatie wordt bewezen gebruik makende van voorgaande resultaten.

Er wordt ook een afschatting voor de Laplaciaan bewezen. Met behulp van deze afschatting wordt een karakterisering voor de Sobolevruimte $W^{2,p}$ bewezen, wat het eindresultaat van dit project is.

Dit project is gebaseerd op de eerste drie hoofdstukken van [1].

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Preliminaries

1.1 L^p spaces and their dual

1.1.1 L^p spaces

1.1 Definition Let X be a measure space and let μ be a measure on X. For $0 < p < \infty$ we denote the set of all complex-valued μ-measurable functions such that

$$\int |f(x)|^p \, d\mu < \infty$$

by $L^p(X, \mu)$ (Or L^p if it is clear which measure space is meant). $L^\infty(X, \mu)$ will denote the set of all complex-valued μ-measurable functions f on X for which there exists a $B > 0$ such that $\mu\{x : |f(x)| > B\} = 0$.
We will consider two functions on L^p to be equal if they are equal μ-a.e.

For $0 < p < \infty$ we can define a quasi-norm on L^p of a function $f \in L^p$ by

$$\|f\|_p = \left(\int_X |f(x)|^p \, d\mu \right)^{1/p} \tag{1.1}$$

and for $p = \infty$ by

$$\|f\|_\infty = \inf\{B > 0 : \mu\{x : |f(x)| > B\} = 0\} \tag{1.2}$$

Whenever $1 \leqslant p \leqslant \infty$ these quasi-norms are in fact norms and for any $1 \leqslant p \leqslant \infty$, it can be shown that L^p are complete normed vector spaces, hence Banach spaces. These results are well known, and for a proof we refer to [2].

The distribution function

1.2 Definition The distribution function λ_f of a measurable function f is the function defined on $[0, \infty)$ as follows

$$\lambda_f(\alpha) = \mu\{x : |f(x)| > \alpha\}. \tag{1.3}$$

Note that this is a well-defined expression since f is measurable and hence $|f|$ is too.

1.3 Proposition For f in $L^p(X,\mu)$, $0<p<\infty$ we have

$$\|f\|_p^p = p\int_0^\infty \alpha^{p-1}\lambda_f(\alpha)\,d\alpha. \tag{1.4}$$

Proof.

$$\begin{aligned} p\int_0^\infty \alpha^{p-1}\lambda_f(\alpha)\,d\alpha &= p\int_0^\infty \alpha^{p-1}\int_X 1_{\{|f|>\alpha\}}\,d\mu\,d\alpha \\ &= \int_X\int_0^\infty p\alpha^{p-1}1_{\{|f|>\alpha\}}\,d\alpha\,d\mu \\ &= \int_X\int_0^{|f(x)|} p\alpha^{p-1}\,d\alpha\,d\mu(x) \\ &= \int_X |f(x)|^p\,d\mu(x) \\ &= \|f\|_p^p \end{aligned}$$

□

1.1.2 The normed dual of $L^p(\mathbb{R}^n)$

The following theorem is from [2, Theorem 6.16], where a proof can be found.

1.4 Theorem Let $1 \leqslant p < \infty$ and let q be the conjugate exponent of p (i.e. $1/p + 1/q = 1$). And let λ be a bounded linear functional on $L^p(\mathbb{R}^n)$. Then there is a unique $g \in L^q(\mathbb{R}^n)$, such that

$$\lambda(f) = \int_{\mathbb{R}^n} f(x)g(x)\,dx \text{ with } f \in L^p(\mathbb{R}^n)$$

Further we have that,

$$\|\lambda\| = \|g\|_q$$

In other words, L^q is isometrically isomorphic to the dual space of L^p.

1.2 Inequalities and equalities

1.2.1 Inner regular measures

1.5 Definition A measure μ on a σ-algebra in X containing all the Borel sets is said to be inner-regular if the following relation holds for every Borel set B,

$$\mu B = \sup\{\mu(C) : C \subset B, C \text{ compact}\}$$

1.6 Theorem The Lebesgue measure is inner-regular.

Proof. See [2][Theorem 2.14]. □

1.2.2 A measure inequality

1.7 Theorem Let f, g and h be complex functions defined on a common domain A. Suppose that

$$|f(x)| \leqslant |g(x)| + |h(x)| \tag{1.5}$$

then

$$\mu\{|f| > \alpha\} \leqslant \mu\left\{|g| > \frac{\alpha}{2}\right\} + \mu\left\{|h| > \frac{\alpha}{2}\right\} \tag{1.6}$$

Proof. Note that $\{|g| + |h| > \alpha\} \subset \{|g| > \alpha/2\} \cup \{|h| > \alpha/2\}$ because if x in $\{|g| < \alpha/2\}$ and $\{|h| < \alpha/2\}$ then by DeMorgan, x in $\{|g| + |h| < \alpha\}$. Taking complement gives the desired result. □

1.2.3 Minkowski's inequality for integrals

1.8 Theorem If $F(x, y)$ is a measurable function on the product space $\mathbb{R}^n \times \mathbb{R}^n$ and if $1 \leqslant p < \infty$, then

$$\left(\int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n}|F(x,y)|\,dx\right)^p dy\right)^{1/p} \leqslant \int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n}|F(x,y)|^p\,dy\right)^{1/p} dx$$

1.2.4 An important integral inequality

1.9 Theorem One has for $x \in \mathbb{R}^n$ and $\varepsilon > 0$ the following

$$\int_{\varepsilon \geqslant |x|} \frac{1}{|x|^a} = \varepsilon^{-a+n} \int_{1 \geqslant |x|} \frac{1}{|x|^a} \text{ when } a > n,$$

and

$$\int_{\varepsilon \leqslant |x|} \frac{1}{|x|^a} = \varepsilon^{-a+n} \int_{1 \leqslant |x|} \frac{1}{|x|^a} \text{ when } a < n,$$

and the integrals in the RHS are finite.

Proof. Since the proofs are similar, only the first one will be shown.

$$\int_{|x| \leqslant \varepsilon} \frac{1}{|x|^a}\,dx = \int_{S^{n-1}} \int_0^\infty \frac{1}{r^a} r^{n-1}\,dr\,d\sigma(\lambda)$$

Consider the inner integral with the substition $u = \frac{r}{\varepsilon}$,

$$\begin{aligned}\int_\varepsilon^\infty \frac{r^{n-1}}{r^a}\,dr &= \varepsilon \int_1^\infty \frac{1}{(u\varepsilon)^{a+1-n}}\,du \\ &= \varepsilon^{-a+n} \int_1^\infty \frac{1}{u^{a+1-n}}\,du\end{aligned}$$

So, the equality is valid. Now we will show that this integral is bounded.

$$\begin{aligned}\varepsilon^{-a+n} \int_1^\infty \frac{1}{u^{a+1-n}}\,du &= \left[\frac{r^{n-a}}{n-a}\right]_1^\infty \\ &\leqslant C \text{ if } a > n\end{aligned}$$

This means that integral is bounded by a constant times ε^{-a+n}. □

1.3 The Sobolev spaces $W^{2,p}$

For a full discussion of the Sobolev spaces $W^{m,p}$ see [3].

1.3.1 Weak derivatives

1.10 Definition (multi-index) If $\alpha = (\alpha_1, \ldots, \alpha_n)$ is an n-tuple of non-negative integers, we will call α a multi-index. We denote by x^α the monomial $x_1^{\alpha_1} \cdot \ldots \cdot x_n^{\alpha_n}$, this monomial has degree

$$|\alpha| = \sum_{j=1}^{n} \alpha_j$$

Analogously, if we write $D_j = \partial/\partial x_j$, then

$$D^\alpha = D_1^{\alpha_1} \cdots D_n^{\alpha_n}$$

1.11 Definition Let Ω be an open subset of $\mathbb{R}^n$, and let $f : \Omega \to \mathbb{R}$ and $g : \Omega \to \mathbb{R}$ be locally integrable functions on Ω and if α is a multi-index we say that g is the α-th weak derivative of f if the equality

$$\int_\Omega f D^\alpha \phi \, dx = (-1)^{|\alpha|} \int_\Omega g \phi \, dx$$

holds for all smooth functions ϕ with compact support (that is $\phi \in C_0^\infty(\Omega)$) and for all $i = 1, \ldots, n$.

1.3.2 The Sobolev spaces $W^{2,p}(\mathbb{R}^n)$

1.12 Definition We define the Sobolev space $W^{2,p}$ as follows

$$W^{2,p}(\mathbb{R}^n) := \{f \in L^p(\mathbb{R}^n) : D^\alpha f \in L^p(\mathbb{R}^n) \text{ for } 0 \leqslant |\alpha| \leqslant 2\}.$$

Where the derivates are understood to be the weak derivatives as defined in Definition 1.11.

1.3.3 The Sobolev norms

1.13 Definition We will define a functional $\|\cdot\|_{2,p}$ where $1 \leqslant p < \infty$ as follows:

$$\|f\|_{2,p} = \left(\sum_{0 \leqslant |\alpha| \leqslant 2} ||D^\alpha f||_p^p \right)^{1/p} \tag{1.7}$$

for f in $W^{2,p}(\mathbb{R}^n)$.

1.14 Theorem $W^{2,p}(\mathbb{R}^n)$ with the norm given by (1.7) is a Banach space.

Proof. For a proof see [3, Theorem 3.3] □

1.4 A special case of the Marcienkiewicz interpolation theorem

First we need a couple of definitions. Let $T : L^p(\mathbb{R}^n) \to L^q(\mathbb{R}^n)$ be a mapping with $1 \leqslant p \leqslant \infty$, $1 \leqslant q \leqslant \infty$. Then T is said to be *type* (p, q) if

$$\|T(f)\|_q \leqslant A\|f\|_q, \quad f \in L^p(\mathbb{R}^n)$$

where A is independent of f. $T : L^p(\mathbb{R}^n) \to L^q(\mathbb{R}^n)$ is said to be of *weak-type* (p, q) if

$$\mu\{x : |Tf(x)| > \alpha\} \leqslant \left(\frac{A\|f\|_p}{\alpha}\right)^q, \quad q < \infty$$

where A is independent of f and $\alpha > 0$, $f \in L^p$. If $q = \infty$ we say f is of weak-type (p, q) if it is of type (p, q).

A mapping T from a complex vector space X into the set of all complex-valued measurable functions Y is said to be sublinear if for all $x, y \in X$ and all $\alpha \in \mathbb{C}$ we have that

$$|T(x+y)| \leqslant |T(x)| + |T(y)| \text{ and } |T(\alpha x)| = |\alpha||T(x)|$$

Further, if V and W are subspaces of the vector space Z over scalar field $\mathbb{C}$, then we can form a new vector space $V + W$ over $\mathbb{C}$ contained in Z by considering all the vectors $v + w$ where $v \in V$ and $w \in W$.

1.15 Proposition If $T : L^p(\mathbb{R}^n) \to L^q(\mathbb{R}^n)$ is of type (p, q) where $1 \leqslant p \leqslant \infty$, $1 \leqslant q \leqslant \infty$, then T is of weak-type (p, q)

Proof. We can assume that $q < \infty$ because when $q = \infty$ both definitions coincide. Now, if T satisfies

$$\|T(f)\|_q \leqslant A\|f\|_q, \quad f \in L^p(\mathbb{R}^n)$$

then the goal is to show that it also satisfies

$$\mu\{x : |Tf(x)| > \alpha\} \leqslant \left(\frac{A\|f\|_p}{\alpha}\right)^q$$

Note that

$$\alpha^q \mu\{x : |Tf(x)| > \alpha\} \leqslant \int |Tf|^q \, d\mu = \|Tf\|_q^q \leqslant (A\|f\|_p)^q$$

where the first inequality follows from Chebyshev's inequality. □

Now we can state the following theorem (which is a special case of the Marcienkiewicz interpolation theorem)

1.16 Theorem Suppose that $1 < r \leqslant \infty$. If T is a sublinear mapping from $L^1(\mathbb{R}^n) + L^r(\mathbb{R}^n)$ to the space of all measurable functions on $\mathbb{R}^n$ which is simultaneously of weak-type $(1, 1)$ and weak-type (r, r), then for all p such that $1 < p < r$, T is also of (weak-)type (p, p).

Proof. First we will prove the case where $r < \infty$. Let $f \in L^p(\mathbb{R}^n)$ and let α be a positive constant. Set

$$f_1^\alpha(x) = \begin{cases} f(x) & \text{if } |f(x)| > \alpha, \\ 0 & \text{if } |f(x)| \leqslant \alpha, \end{cases}$$

and,

$$f_2^\alpha(x) = \begin{cases} f(x) & \text{if } |f(x)| \leqslant \alpha, \\ 0 & \text{if } |f(x)| > \alpha. \end{cases}$$

Now $f(x) = f_1^\alpha(x) + f_2^\alpha(x)$. It can be checked as follows that f_1^α is an L^1 function and f_2^α is an L^r function. Since $1 < p$ we have,

$$\int |f_1^\alpha| \, d\mu = \int |f_1^\alpha|^p |f_1^\alpha|^{1-p} \, d\mu \leqslant \alpha^{1-p} \int |f(x)|^p \, d\mu < \infty$$

Similarly since $p < r$,

$$\int |f_2^\alpha|^r \, d\mu \leqslant \alpha^{r-p} \int |f(x)|^p \, d\mu < \infty$$

Now, by sublinearity we have that,

$$|T(f)| \leqslant |T(f_1^\alpha)| + |T(f_2^\alpha)|.$$

This in turn implies that

$$\{x : |T(f)(x)| > \alpha\} \subset \{x : |T(f_1^\alpha)(x)| > \frac{\alpha}{2}\} \cup \{x : |T(f_2^\alpha)(x)| > \frac{\alpha}{2}\}$$

so

$$\begin{aligned}\lambda_{Tf}(\alpha) &= \mu\{x : |T(f)(x)| > \alpha\} \\ &\leqslant \mu\{x : |T(f_1^\alpha)(x)| > \frac{\alpha}{2}\} + \mu\{x : |T(f_2^\alpha)(x)| > \frac{\alpha}{2}\}\end{aligned}$$

by elementary properties of the measure μ. Therefore, we have by assumption that

$$\lambda_{Tf}(\alpha) \leqslant \frac{A_1}{\alpha/2}\int |f_1^\alpha(x)|\,d\mu + \frac{A_r^r}{(\alpha/2)^r}\int |f_2^\alpha(x)|^r\,d\mu$$

We can rewrite this using the definitions of f_1^α and f_2^α as

$$\lambda_{Tf}(\alpha) \leqslant \frac{A_1}{\alpha/2}\int_{\{|f|>\alpha\}} |f(x)|\,d\mu + \frac{A_r^r}{(\alpha/2)^r}\int_{\{|f|\leqslant\alpha\}} |f|^r\,d\mu \tag{1.8}$$

From Proposition 1.3 we know that for $f \in L^p(X,\mu)$ and $0 < p < \infty$ that

$$\|f\|_p^p = p\int_0^\infty \alpha^{p-1}\lambda_f(\alpha)\,d\alpha.$$

So, all we need to do is multiply both sides of (1.8) by $p\alpha^{p-1}$ and integrate with respect to α.

$$\|T(f)\|_p^p \leqslant \int_0^\infty p\alpha^{p-1}\left(\frac{A_1}{\alpha/2}\int_{\{|f|>\alpha\}} |f|\,d\mu + \frac{A_r^r}{(\alpha/2)^r}\int_{\{|f|\leqslant\alpha\}} |f|^r\,d\mu\right)d\alpha$$

We will treat the integrals separately.

$$\begin{aligned}\int_0^\infty p\alpha^{p-1}\frac{2A_1}{\alpha}\int_{\{|f|>\alpha\}} |f|\,d\mu\,d\alpha &= 2A_1\int_0^\infty p\alpha^{p-2}\int_{\{|f|>\alpha\}} |f|\,d\mu\,d\alpha \\ &= 2A_1\int_0^\infty\int_{\{|f|>\alpha\}} p\alpha^{p-2}|f|\,d\mu\,d\alpha \\ &= 2A_1\int\int_0^{|f(x)|} p\alpha^{p-2}|f|\,d\alpha\,d\mu(x) \\ &= \frac{2pA_1}{p-1}\int |f|^p\,d\mu \\ &= \frac{2pA_1}{p-1}\|f\|_p^p\end{aligned}$$

since $p > 1$. And similarly (assuming $p < r$)

$$\begin{aligned}\int_0^\infty p\alpha^{p-1}\frac{2A_r^r}{\alpha^r}\int_{\{|f|\leqslant\alpha\}} |f|^r\,d\mu\,d\alpha &= \int 2A_r^r\int_{|f(x)|}^\infty p\alpha^{p-1-r}|f|^r\,d\alpha\,d\mu(x) \\ &= \int 2A_r^r\frac{p|f|^p}{r-p}\,d\mu \\ &= \frac{2pA_r^r}{r-p}\|f\|_p^p.\end{aligned}$$

Putting things together yields

$$\|T(f)\|_p^p \leqslant \left(\frac{2pA_1}{p-1} + \frac{2pA_r^r}{r-p}\right)\|f\|_p^p$$

which is the requested inequality.

Now, suppose that $r = \infty$. Write $f = f_0^\alpha + f_1^\alpha$ for $f \in L^p(\mathbb{R}^n)$, where

$$f_0^\alpha(x) = \begin{cases} f(x) & \text{if } |f(x)| > \delta\alpha, \\ 0 & \text{if } |f(x)| \leqslant \delta\alpha, \end{cases}$$

and,

$$f_1^\alpha(x) = \begin{cases} f(x) & \text{if } |f(x)| \leqslant \delta\alpha, \\ 0 & \text{if } |f(x)| > \delta\alpha. \end{cases}$$

Then $\|T(f_1^\alpha)\|_\infty \leqslant A_{\infty,1}\|f_1^\alpha\| \leqslant A_{\infty,1}\delta\alpha = \alpha/2$ is we choose $\delta = \frac{1}{2A_{\infty,1}}$. So by definition $\{|T(f_1^\alpha)| > \alpha/2\}$ has measure zero.

So

$$\begin{aligned} \lambda_{Tf}(\alpha) &\leqslant \lambda_{Tf_0^\alpha}(\alpha/2) + \leqslant \lambda_{Tf_1^\alpha}(\alpha/2) \\ &= \lambda_{Tf_0^\alpha}(\alpha/2) \end{aligned}$$

from the definition of λ. Now T is of weak-type $(1,1)$ and (∞,∞) so

$$\lambda_{Tf_0^\alpha}(\alpha/2) \leqslant \frac{2A_{\infty,0}\|f_0^\alpha\|_1}{\alpha} = \frac{2A_{\infty,0}}{\alpha}\int_{|f|>\delta\alpha} |f|\,d\mu$$

Now,

$$\begin{aligned} \|Tf\|_p^p &= p\int_0^\infty \alpha^{p-1}\lambda_{Tf}(\alpha)\,d\alpha \\ &\leqslant p\int_0^\infty \alpha^{p-1}\lambda_{Tf_0^\alpha}\left(\frac{\alpha}{2}\right)\,d\alpha \\ &\leqslant p\int_0^\infty \alpha^{p-1}\frac{2A_{\infty,0}}{\alpha}\int_{|f|>\delta\alpha}|f|\,d\mu\,d\alpha \\ &= 2pA_{\infty,0}\int_{\mathbb{R}^n}|f|\int_0^\infty \alpha^{p-2}1_{\{f>\delta\alpha}\,d\alpha\,d\mu \\ &= 2pA_{\infty,0}\int_{\mathbb{R}^n}|f|\int_0^{|f|/\delta}\alpha^{p-2}\,d\alpha\,d\mu \\ &= 2pA_{\infty,0}\int_{\mathbb{R}^n}|f|\int_0^{2A_{\infty,1}|f|}\alpha^{p-2}\,d\alpha\,d\mu \\ &= C\|f\|_p^p \end{aligned}$$

which proves the case $r = \infty$. □

1.5 The maximal function

We define the maximal function Mf of a locally integrable function f by

$$M(f)(x) = \sup_{r>0}\frac{1}{\mu(B(x,r))}\int_{B(x,r)}|f(y)|\,dy. \tag{1.9}$$

Where $B(x,r)$ is the open ball of radius r centered at x.

With these definitions given, we can state the following theorem:

1.17 Theorem Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function.

1. If $f \in L^p(\mathbb{R}^n)$ for $1 \leqslant p < \infty$ then the maximal function Mf is finite a.e.

2. If $f \in L^1(\mathbb{R}^n)$ then for every $\alpha > 0$ we have

$$\mu\{x : (Mf)(x) > \alpha\} \leqslant \frac{A_n}{\alpha}\|f\|_1$$

where A_n is a constant that only depends on the dimension n. ($A_n = 3^n$ will do)

3. If $f \in L^p(\mathbb{R}^n)$ for $1 < p \leqslant \infty$, then we have that $Mf \in L^p(\mathbb{R}^n)$ and the inequality

$$\|Mf\|_p \leqslant A_{p,n}\|f\|_p$$

where $A_{p,n}$ depends only on p and the dimension n.

Before we state the proof we need a technical lemma.

1.18 Lemma (of Vitali) Let $\mathcal{B} = \{B_1, B_2, \ldots, B_N\}$ be a finite collection of open balls in $\mathbb{R}^n$. In this case, there exists a disjoint subcollection of $\mathcal{B}$, $B_{i_1}, B_{i_2}, \ldots, B_{i_k}$ that satisfies

$$\mu\left(\bigcup_{j=1}^{N} B_j\right) \leqslant 3^n \sum_{j=1}^{k} \mu(B_{i_j})$$

Proof. We will give a constructive proof.

Step 1: We pick the ball B_{i_1} in $\mathcal{B}$ with maximal radius, and then we delete from $\mathcal{B}$ this ball B_{i_1} and as well all the balls that intersect with B_{i_1}. This way we form a new set $\mathcal{B}'$ from $\mathcal{B}$

Step 2: Pick the ball B_{i_2} from $\mathcal{B}'$ with the same procedure as in Step 1. This yields a new collection of balls $\mathcal{B}''$.

Step N: Continuing the same way as before we get a collection of disjoint balls $B_{i_1}, B_{i_2}, \ldots, B_{i_k}$.

Now, let $\widehat{B}_{i_j}$ denote the ball concentric with B_{i_j} but with three times its radius. Note that if B and B' are balls that intersect, with the radius of B smaller of equal to the radius of B', then B' is contained in the ball $\widehat{B}$ that is concentric with B but has three times its radius. Since any ball B in $\mathcal{B}$ intersects a ball B_{i_j} and has smaller of equal radius than B_{i_j}, we have that $B \subset \widehat{B}_{i_j}$. So,

$$\mu\left(\bigcup_{j=1}^{N} B_j\right) \leqslant \mu\left(\bigcup_{j=1}^{k} \widehat{B}_{i_j}\right) \leqslant \sum_{j=1}^{k} m(\widehat{B}_{i_j}) = 3^d \sum_{j=1}^{k} m(B_{i_j})$$

where the last inequality follows from the fact that the Lebesgue measure of the dilation of a set in $\mathbb{R}^n$ by a factor $\delta > 0$ is equal to the multiplication of δ^n by the Lebesgue measure of that set. □

Proof of Theorem 1.17. We will first prove 2. Let

$$E_\alpha = \{Mf > \alpha\}$$

then we have for each $x \in E_\alpha$ that there exists a open ball B_x centered at x such that

$$\frac{1}{\mu(B_x)} \int_{B_x} |f(y)|\, dy > \alpha$$

by the definition of the supremum. Thus, for each $x \in E_\alpha$ we have an open ball B_x such that

$$\mu(B_x) < \frac{1}{\alpha} \int_{B_x} |f(y)|\, dy \tag{1.10}$$

Now, take a compact subset C of E_α, C is now covered by $\bigcup_{x \in E_\alpha} B_x$, so, by compactness, we can select a finite subcover of C, say $\bigcup_{j=1}^{N} B_j$. Now, the previous lemma guarantees the existence of a disjoint subcollection $B_{i_1}, B_{i_2}, \ldots, B_{i_k}$ of open balls with,

$$\mu\left(\bigcup_{j=1}^{N} B_j\right) \leqslant 3^n \sum_{j=1}^{k} \mu(B_{i_j}) \tag{1.11}$$

Now, Equation (1.10) and (1.11) ensure that

$$\begin{aligned}
\mu(C) \leqslant \mu\left(\bigcup_{j=1}^{N} B_j\right) \leqslant 3^n \sum_{j=1}^{k} \mu(B_{i_j}) &\leqslant \frac{3^n}{\alpha} \sum_{j=1}^{k} \int_{B_{i_j}} |f(y)|\, dy \\
&= \frac{3^n}{\alpha} \int_{\bigcup_{j=1}^{k} B_{i_j}} |f(y)|\, dy \\
&\leqslant \frac{3^n}{\alpha} \int_{\mathbb{R}^n} |f(y)|\, dy
\end{aligned}$$

Taking the supremum of all compact $C \subset E_\alpha$, we obtain the equality as given in 2 by the inner regularity of the Lebesgue measure.

To prove 3, note that 2 says that Mf is of weak-type (1,1), the next step in our proof is showing that Mf is also of weak-type (∞, ∞) (and hence of type (∞, ∞)). So we must show that

$$\|Mf\|_\infty \leqslant A_\infty \|f\|_\infty$$

This is true, where $A_\infty = 1$. To show this let $\varepsilon > 0$, then there exists an $M \geqslant 0$ such that $M < \|f\|_\infty + \varepsilon$ where $|f(t)| \leqslant M$ a.e. thus we have that $|f(t)| < \|f\|_\infty + \varepsilon$ a.e. Consider,

$$\frac{1}{\mu(B(x,r))} \int_{B(x,r)} |f(y)|\, dy < \frac{1}{\mu(B(x,r))} \int_{B(x,r)} (\|f\|_\infty + \varepsilon)\, dy = \|f\|_\infty + \varepsilon$$

If we now let $\varepsilon \to 0$ we obtain

$$\frac{1}{\mu(B(x,r))} \int_{B(x,r)} |f(y)|\, dy \leqslant \|f\|_\infty.$$

Since the bound is independent of r we take can the supremum over all $r > 0$, now

$$\left\{B : \left|\sup_{r>0} \frac{1}{\mu(B(x,r))} \int_{B(x,r)} |f(y)|\, dy\right| \leqslant B\right\} \supset \{B : \|f\|_\infty \leqslant B\}.$$

And, taking infimums gives us the result since $\inf\{B : \|f\|_\infty \leqslant B\} = \|f\|_\infty$.

If we now show that Mf is sublinear, then we can apply Marcinkiewicz's interpolation theorem (because Mf is defined on L^1_{loc} and $L^1 + L^\infty \subset L^1_{\text{loc}}$[1]). But this is obvious from the definition of Mf and the triangle inequality.

So, 3 follows immediately.

Now we will prove 1 as an easy corollary to 2 and 3. To this end, for $p = 1$, consider

$$\{Mf = \infty\} \subset \{Mf > \alpha\}$$

for all α. Now, take the limit of $\alpha \to \infty$ in $\mu\{x : (Mf)(x) > \alpha\} \leqslant \frac{A_n}{\alpha}\|f\|_1$. Now if $1 < p < \infty$, $\|Mf\|_p$ is finite, which implies that Mf is finite a.e. $\square$

This theorem has an important corollary which we now state

1.19 Corollary (Lebesgue's differentiation theorem) If $f \in L_1(\mathbb{R}^n)$, then

$$\lim_{r \to 0} \frac{1}{\mu(B(x,r))} \int_{B(x,r)} f(y)\,dy = f(x) \tag{1.12}$$

for almost every x.

Proof. We will show that the set for every $\alpha > 0$

$$E_\alpha = \left\{ x \in \mathbb{R}^n : \limsup_{r \to 0} \left| \frac{1}{\mu(B(x,r))} \int_{B(x,r)} f(y)\,dy - f(x) \right| > 2\alpha \right\}$$

has measure zero. Then there immediately follows that

$$E = \bigcup_{n=1}^{\infty} E_{1/n}$$

has measure zero. From this there follows that the limit in (1.12) holds for every $x \in E^c$. Next we will fix α and use the fact that the continuous functions of compact support are dense in $L_1(\mathbb{R}^n)$. So, we can find for every $\varepsilon > 0$ a continuous function g of compact support such that

$$\|f - g\|_1 < \varepsilon$$

If g is continuous then,

$$\lim_{r \to 0} \frac{1}{\mu(B(x,r))} \int_{B(x,r)} g(y)\,dy = g(x)$$

for all x. To see this, we can find for every x and every $\varepsilon > 0$, a $\delta > 0$ such that $|x - y| < \delta$ implies $|g(x) - g(y)| < \varepsilon$. Note that,

$$g(x) - \frac{1}{\mu(B(x,r))} \int_{B(x,r)} g(y)\,dy = \frac{1}{\mu(B(x,r))} \int_{B(x,r)} (g(x) - g(y))\,dy$$

So,

$$\left| g(x) - \frac{1}{\mu(B(x,r))} \int_{B(x,r)} g(y)\,dy \right| = \frac{1}{\mu(B(x,r))} \int_{B(x,r)} |(g(x) - g(y))|\,dy$$

[1] Let K be a compact set, and let $f \in L^1 + L^\infty$ be written as $f_1 + f_\infty$, then

$$\begin{aligned} \int_K |f_1 + f_\infty| &\leqslant \int |f_1 1_K| + \int |f_\infty 1_K| \\ &\leqslant \|f_1\|_1 \|1_K\|_\infty + \|f_\infty\|_\infty \|1_K\|_1 < \infty \end{aligned}$$

Then we obviously have that the RHS is smaller than ε when $r < \delta/2$. Which proves this fact. So, we can write

$$\begin{aligned}\frac{1}{\mu(B(x,r))}\int_{B(x,r)} f(y)\,dy - f(x) &= \frac{1}{\mu(B(x,r))}\int_{B(x,r)} (f(y)-g(y))\,dy \\ &+ \frac{1}{\mu(B(x,r))}\int_{B(x,r)} g(y)\,dy - g(x) \\ &+ g(x) - f(x)\end{aligned}$$

Thus,

$$\begin{aligned}\left|\frac{1}{\mu(B(x,r))}\int_{B(x,r)} f(y)\,dy - f(x)\right| &\leqslant \left|\frac{1}{\mu(B(x,r))}\int_{B(x,r)} (f(y)-g(y))\,dy\right| \\ &+ \left|\frac{1}{\mu(B(x,r))}\int_{B(x,r)} g(y)\,dy - g(x)\right| \\ &+ |g(x) - f(x)|\end{aligned}$$

So,

$$\begin{aligned}&\limsup_{r\to 0}\left|\frac{1}{\mu(B(x,r))}\int_{B(x,r)} f(y)\,dy - f(x)\right| \\ &\leqslant \limsup_{r\to 0}\frac{1}{\mu(B(x,r))}\int_{B(x,r)} |f(y)-g(y)|\,dy \\ &+ |g(x) - f(x)|\end{aligned}$$

Define

$$F_\alpha = \left\{x \in \mathbb{R}^n : \limsup_{r\to 0}\frac{1}{\mu(B(x,r))}\int_{B(x,r)} |f(y)-g(y)|\,dy > \alpha\right\}$$

and

$$G_\alpha = \{x \in \mathbb{R}^n : |f(x) - g(x)| > \alpha\}$$

Then we clearly have that $E_\alpha \subset (F_\alpha \cup G_\alpha)$, because if f_1 and f_2 are positive, then $f_1 + f_2 > \alpha$ if $f_i > \alpha$ for at least one f_i. Now we have by Chebychev's inequality that

$$\mu(G_\alpha) \leqslant \frac{1}{\alpha}\|f-g\|_1 < \frac{\varepsilon}{\alpha}$$

and due to Theorem 1.17 item 2 we have the weak-type estimate for the Hardy-Littlewood maximal function

$$\mu(F_\alpha) \leqslant \frac{A}{\alpha}\|f-g\|_1 < \frac{A\varepsilon}{\alpha}$$

So, combining these estimates yields

$$\mu(E_\alpha) \leqslant \frac{(A+1)\varepsilon}{\alpha}.$$

And since $\varepsilon > 0$ is arbitrary, there follows that $\mu(E_\alpha) = 0$. □

1.6 Integral of Marcinkiewicz

1.20 Definition Consider a closed set F. Let $\delta(x)$ denote the distance from x to F. The integral

$$I(x) = \int_{|y|\leqslant 1}\frac{\delta(x+y)}{|y|^{n+1}}\,dy$$

is called the integral of Marcinkiewicz.

1.21 Theorem

1. When $x \in F^c$, then $I(x) = \infty$,
2. For a.e. $x \in F$, $I(x)$ is finite.

We will prove the theorem using the following lemma.

1.22 Lemma Let F be a closed set. Suppose that F^c has finite measure. Let

$$I_*(x) = \int \frac{\delta(x+y)}{|y|^{n+1}}\,dy$$

then $I_*(x)$ is finite for a.e. $x \in F$. Further

$$\int_F I_*(x)\,dx \leqslant c\mu(F^c) \tag{1.13}$$

Proof of the lemma. It suffices to prove (1.13), because if the integral of a positive function is finite the integrand is finite almost everywhere. Because the integrand is positive we can apply Tonelli's theorem. Formally,

$$\begin{aligned}
\int_F I_*(x)\,dx &= \int_F \int \frac{\delta(x+y)}{|y|^{n+1}}\,dy\,dx \\
&= \int_F \int \frac{\delta(y)}{|x-y|^{n+1}}\,dy\,dx \\
&= \int_F \int_{F^c} \frac{\delta(y)}{|x-y|^{n+1}}\,dy\,dx \\
&= \int_{F^c} \delta(y) \left(\int_F \frac{1}{|x-y|^{n+1}}\,dx \right) dy
\end{aligned}$$

where the second equality follows from the fact that we integrate of the whole space. The third equality follows from the fact that $\delta(y) = 0$ for $y \in F$. We will now consider the inner integral for $y \in F^c$,

$$\int_F \frac{1}{|x-y|^{n+1}}\,dx.$$

The smallest value of $|x-y|$ (as x varies over F) is $\delta(y)$. Thus, we have that (let $z = x - y$),

$$\begin{aligned}
\int_F \frac{1}{|x-y|^{n+1}}\,dx &= \int_{F-y} \frac{1}{|z|^{n+1}}\,dz \\
&\leqslant \int_{\delta(y)\leqslant|z|} \frac{1}{|z|^{n+1}}\,dz \\
&\leqslant \frac{c}{\delta(y)}
\end{aligned}$$

where the first ineqality follows from $F - y \subset \{\delta(y) \leqslant |z|\}$ which is clear and the second inequality follows from Theorem 1.9 (which can be used without being afraid of circularities). So,

$$\int_F I_*(x)\,dx \leqslant \int_{F^c} \frac{c}{\delta(y)}\delta(y)\,dy = c\mu(F^c).$$

Which completes the present proof. $\square$

Proof of Theorem 1.21. First we will prove part 1. The complement of F is an open set. So if $x \in F^c$, then $\delta(x+y) \geqslant c > 0$ for an neighborhood of the origin in y and in this case the integral diverges.

Let B_m be the ball $B(0,m)$, and let $F_m = F \cup B_m^c$. Then F_m is closed, as it is a finite union of closed sets. By DeMorgan its complement has finite measure. So, we can apply the lemma to F_m.

To prove the second part, let $\delta_m(x)$ denote the distance of x from F_m, and let δ have its usual meaning. First observe that $\delta(x+y) = \delta_m(x+y)$ if $|y| \leqslant 1$ and $x \in B_{m-2}$ as a figure shows. Thus the lemma implies that $I(x) < \infty$ for a.e. $x \in F_m \cap B_{m-2}$. Taking the limit $m \to \infty$ gives us the desired result. □

1.7 Decomposition in cubes of open sets in $\mathbb{R}^n$

1.23 Theorem Let $F \neq \emptyset$ be a closed set in $\mathbb{R}^n$. There exist cubes Q_k whose sides are parallel to the axes, with

1. $\Omega = F^c = \displaystyle\bigcup_{k=1}^{\infty} Q_k$.
2. The interiors of Q_k are mutually disjoint.
3. $\operatorname{diameter}(Q_k) \leqslant d(Q_k, F) \leqslant 4\operatorname{diameter}(Q_k)$ for all k.

Proof. Consider the lattice of points with integer coordinates in $\mathbb{R}^n$. This lattice gives us a mesh $\mathcal{M}_0$, which is a collection of cubes, where the vertices of those cubes are the points in the lattice. So from the mesh $\mathcal{M}_0$ we can get an infinite chain of meshes $\mathcal{M}_k = 2^{-k}\mathcal{M}_0$ by bisecting the sides in 2^k parts. So, for each cube in the mesh $\mathcal{M}_k$ we get 2^n cubes in $\mathcal{M}_{k+1}$ and the cubes in the mesh $\mathcal{M}_k$ each have sides of length 2^{-k} so, they have diameter of $\sqrt{n}2^{-k}$.

Further we have the layers Ω_k defined by

$$\Omega_k = \{x : c2^{-k} < d(x,F) \leqslant c2^{-k+1}\}$$

where $c > 0$ is a fixed number which we will determine later on. So, from this definition we see that

$$\Omega = \bigcup_{k=-\infty}^{\infty} \Omega_k$$

We will now make a choice of cubes, and we will denote the resulting collection by $\mathcal{F}_0$. We define $\mathcal{F}_0$ to be equal to

$$\mathcal{F}_0 = \bigcup_k \{Q \in \mathcal{M}_k : Q \cap \Omega_k \neq \emptyset\}$$

So we have

$$\bigcup_{Q \in \mathcal{F}_0} Q = \Omega$$

since we stay away from F in the definition of $\mathcal{F}_0$. We will now prove that for the right choice of c we have that

$$\operatorname{diameter}(Q) \leqslant d(Q,F) \leqslant 4\operatorname{diameter}(Q) \tag{1.14}$$

for $Q \in \mathcal{F}_0$. We will now prove Eq. (1.14). Suppose that $Q \in \mathcal{M}_k$, then $\operatorname{diameter}(Q) = \sqrt{n}2^{-k}$. And since $Q \in \mathcal{F}_0$, there exists $x \in Q \cap \Omega_k$, thus

$$d(Q,F) \leqslant d(x,F) \leqslant c2^{-k+1}$$

and
$$d(Q,F) \geqslant d(x,F) - \text{diameter}(Q) > c2^{-k} - \sqrt{n}2^{-k}$$
So, if we choose $c = \sqrt{n}$, we obtain Eq. (1.14). So, $\mathcal{F}_0$ as our collection of cubes we obtain part 1 and part 3 of the theorem. Now, the only problem is that the collection of cubes from $\mathcal{F}_0$ are not necessarly disjoint (by disjoint here we mean disjoint interiors), so we will refine the collection $\mathcal{F}_0$ without disturbing the parts we have already proven. Suppose $Q_1 \in \mathcal{M}_{k_1}$ and $Q_2 \in \mathcal{M}_{k_2}$ are two cubes. Then, if Q_1 and $_2$ are not disjoint, one must be contained in the other, by construction. So, begin with any cube $Q \in \mathcal{F}_0$. Consider now the maximal cube in $\mathcal{F}_0$ which contains it. So, in view of Eq. (1.14), for any cube $Q' \in \mathcal{F}_0$ which contains $Q \in \mathcal{F}_0$ we have $\text{diameter}(Q') \leqslant 4\,\text{diameter}(Q)$. Further any two cubes Q' and Q'' which contain Q are obviously not disjoint. So, for any cube $Q \in \mathcal{F}_0$ there is a maximal unique cube in $\mathcal{F}$, that contains it. So, as we have seen, these cubes are disjoint. Now, let $\mathcal{F}$ be the collection of maximal cubes of $\mathcal{F}_0$, first this collection the conclusions in the theorem are valid. □

1.24 Theorem Let $f : \mathbb{R}^n \to \mathbb{R}$ be an integrable function and let $\alpha > 0$ be a constant. Then there exists a partition of $\mathbb{R}^n$ in two sets, F and Ω such that $f(x) \leqslant \alpha$ a.e. on F and Ω is the union of cubes Q_k whose interiors are disjoint and so that for each Q_k we have
$$\alpha < \frac{1}{\mu(Q_k)} \int_{Q_k} f(x)\,dx \leqslant 2^n \alpha. \tag{1.15}$$
Further we have that
$$\mu(\Omega) \leqslant \frac{1}{\alpha}\|f\|_1$$
and,
$$\frac{1}{\mu(Q_k)} \int_{Q_k} f(x)\,dx \leqslant 2^n \alpha$$

Proof. We will give a constructive proof. First we decompose $\mathbb{R}^n$ into a mesh of equal cubes Q', with mutually disjoint interiors whose common interior is so large that
$$\frac{1}{\mu(Q')} \int_{Q'} f(x)\,dx \leqslant \alpha$$
This can easily be done. Let Q' be one of the cubes in this mesh. Now, divide Q' into 2^n congruent new cubes and let Q'' denote one of these cubes. Then have two cases,

Case one: $\displaystyle\frac{1}{\mu(Q'')} \int_{Q''} f(x)\,dx \leqslant \alpha.$

Case two: $\displaystyle\frac{1}{\mu(Q'')} \int_{Q''} f(x)\,dx > \alpha.$

In case two, we do not subdivide Q'' any further and select this cube as one of the cubes in the statement. We then have inequality (1.15) for Q'' because
$$\alpha < \frac{1}{\mu(Q'')} \int_{Q''} f(x)\,dx \leqslant \frac{1}{2^{-n}\mu(Q')} \int_{Q'} f(x)\,dx \leqslant 2^n \alpha$$
In the case one, we proceed as before. Next, we claim that $f(x) \leqslant \alpha$ a.e. in $F = \Omega^c$. Due to Lebesgue's differentiation theorem we have for almost every $x \in F$ that,
$$f(x) = \lim_{Q} \frac{1}{\mu(Q)} \int_Q f(y)\,dy$$

where the limit is taken over all cubes Q centered at x and the diameter of the cube goes to zero. But for each of these cubes case one holds. This ends the proof. Note that in the Lebesgue differentiation theorem we use balls, but we could as well use cubes, it makes no difference in Vitali's covering lemma.

By Eq. (1.15) we can take as constant 2^n in the second estimate, and also

$$\mu(\Omega) = \sum_k \mu(Q_k) < \frac{1}{\alpha}\int_\Omega f(x)\,dx \leqslant \frac{1}{\alpha}\|f\|_1.$$

□

1.8 Extension of continuous functionals on dense subsets

1.25 Theorem Let X be a normed linear space and let W be a dense subspace of X. Let Y be a Banach space and let $T : W \to Y$ be a bounded linear operator then there exist a unique extension of T, $T_1 : X \to Y$ that preserves operator norm, that is $\|T\| = \|T_1\|$.

Proof. For a proof, see for example [4]. □

1.9 Fourier analysis

1.9.1 Convolutions

1.26 Theorem (Young's inequality) If $f \in L^p(\mathbb{R}^n)$ with $1 \leqslant p \leqslant \infty$, and $g \in L^1(\mathbb{R}^n)$. Then $f * g$ is well defined and lives in $L^p(\mathbb{R}^n)$. And,

$$\|f * g\|_p \leqslant \|f\|_p\|g\|_1.$$

Proof. It is easy to see that,

$$|(f*g)(x)| \leqslant \int |f(x-y)||g(y)|\,dy$$

Now,

$$\begin{aligned}
\left(\int |(f*g)(x)|^p\,dx\right)^{1/p} &\leqslant \left(\int\left(\int |f(x-y)||g(y)|\,dy\right)^p dx\right)^{1/p} \\
&= \left(\int\left(\int |f(x-y)||g(y)|\,dy\right)^p dx\right)^{1/p} \\
&\leqslant \int |g(y)|\left(\int |f(x-y)|^p\,dx\right)^{1/p} dy \\
&= \|f\|_p\|g\|_1
\end{aligned}$$

Where the second inequality follows from Minkowski's inequality for integrals (Theorem 1.8). □

1.9.2 Plancherel theorem

1.27 Theorem If $f \in L^1 \cap L^2$, then $\widehat{f} \in L^2$ (where $\widehat{f}$ is the Fourier transform of f) and $\|\widehat{f}\|_2 = \|f\|_2$.

Proof. For a proof see [5]. □

2

Singular integrals

2.1 Singular integrals: The main theorem

2.1 Theorem Let $K \in L^2(\mathbb{R}^n)$. Suppose the following:

1. The Fourier transform of K is essentially bounded by a constant B.
2. K is of class C^1 (C^1 is the class of continuous differentiable functions) outside the origin and
$$|\nabla K(x)| \leqslant \frac{B}{|x|^{n+1}}$$
3. for f in $L^1 \cap L^p$, set
$$(Tf)(x) = \int K(x-y)f(y)\,dy \tag{2.1}$$
(This integral exists because of Young's inequality (Theorem 1.26).)

Then T is bounded in the L^p norm, that is,

$$\|T(f)\|_p \leqslant A_p\|f\|_p \text{ for } 1 < p < \infty$$

Where A_p only depends on p, B and n, but not on f. So, by Theorem 1.25 we can extend T to all of L^p by continuity because $L^1 \cap L^p$ is dense in L^p.

Proof. We will begin the proof[1] by showing that $L^1 \cap L^p$ is dense in L^p. So we must show that for every $f \in L^p$ there exists a sequence $f_n \in L^1 \cap L^p$ such that $\|f_n - f\|_p \to 0$ as $n \to \infty$. Let $f_n := f \cdot 1_{[-n,n]}$, then it is obvious that f_n is in L^p, to show that f_n is in L^1 we use Hölders inequality (where q is the conjugated exponent of p),

$$\int |f_n| \leqslant \left(\int |f|^p\right)^{1/p} \mu([-n,n])^{1/q} < \infty$$

[1] In this proof we will use C as a general constant, not necessarly the same at all instances.

So, since $|f_n - f|^p \leqslant |f|^p$ and $|f|^p$ is integrable, we can use Lebesgue Dominated Convergence Theorem (LDCT) and the result follows immediately.

We proceed in three steps:

1. T is of weak-type $(2,2)$.
2. T is of weak-type $(1,1)$.
3. The L^p inequalities.

First step: T is of weak-type $(2,2)$.
If we take the Fourier transform of Equation (2.1) we obtain

$$\widehat{Tf}(y) = \widehat{K}(y)\widehat{f}(y)$$

for $f \in L^1 \cap L^2$ because $(Tf)(y)$ is actually the convolution $(K * f)(y)$. K lives in L^2, thus according to Theorem 1.26 (Tf) lives in L^2 because $f \in L^1 \cap L^2$. So we have, if we use the Plancherel theorem (Theorem 1.27) that

$$\|Tf\|_2 = \|\widehat{Tf}\|_2 = \|\widehat{K}\widehat{f}\|_2 \leqslant B\|\widehat{f}\|_2 = B\|f\|_2. \tag{2.2}$$

Because of Equation (2.2), we can extend T to all of L^2 where Equation (2.2) is still valid by Theorem 1.25. By Proposition 1.15, T is now of weak-type $(2,2)$.

Second step: T is of weak-type $(1,1)$.
We will treat Tf for $f \in L^1(\mathbb{R}^n)$ by decomposing f as $f = g + b$. Now, we need to find a constant C so that

$$\mu\{x : (Tf(x)) > \alpha\} \leqslant \frac{C}{\alpha}\|f\|_1 \text{ where } f \in L^1(\mathbb{R}^n) \tag{2.3}$$

To this end, fix $\alpha > 0$, and for this fixed α and $|f(x)|$ we can apply Theorem 1.24. If we do this, we get disjoint F and Ω so that $\mathbb{R}^n = F \cup \Omega$, and $|f(x)| \leqslant \alpha$ for $x \in F$. Further we get that

$$\Omega = \bigcup_{j=1}^{\infty} Q_j$$

where the interiors of the cubes Q_j are mutually disjoint. And finally we get that

$$\mu(\Omega) \leqslant \frac{C}{\alpha}\int |f|\,dx \text{ and } \frac{1}{\mu(Q_j)}\int_{Q_j} |f|\,dx \leqslant C\alpha. \tag{2.4}$$

So, we set,

$$g(x) = \begin{cases} f(x) & \text{for } x \in F, \\ \frac{1}{\mu(Q_j)}\int_{Q_j} f(x)\,dx & \text{for } x \in \operatorname{int} Q_j, \end{cases} \tag{2.5}$$

which defines $g(x)$ almost everywhere since the boundary of Q_j has measure zero. This, together with definition $f(x) = g(x) + b(x)$ gives

$$b(x) = 0 \text{ for } x \in F$$

(this is clear) and for $x \in Q_j$ we must have that

$$b(x) = f(x) - \frac{1}{\mu(Q_j)}\int_{Q_j} f(x)\,dx$$

So, after integrating with respect to x over Q_j there follows immediately that,

$$\int_{Q_j} b(x)\,dx = 0 \text{ for each cube } Q_j.$$

From $Tf = Tg + Tb$ it follows that

$$\mu\{|Tf| > \alpha\} \leqslant \mu\left\{|Tg| > \frac{\alpha}{2}\right\} + \mu\left\{|Tb| > \frac{\alpha}{2}\right\}.$$

So it suffices to establish inequalities similar to our desired inequality (2.3) for both terms of the RHS in the equation above. First, note that $g \in L^2(\mathbb{R}^n)$ since we have by (2.5)

$$\begin{aligned}
\|g\|_2^2 &= \int |g(x)|^2\,dx \\
&= \int_F |g(x)|^2\,dx + \int_\Omega |g(x)|^2\,dx \\
&= \int_F |f(x)|^2\,dx + \int_\Omega |g(x)|^2\,dx \\
&\leqslant \int_F \alpha|f(x)|\,dx + \int_\Omega |g(x)|^2\,dx \\
&\leqslant \int_F \alpha|f(x)|\,dx + \int_\Omega |C\alpha|^2\,dx \\
&\leqslant \int_F \alpha|f(x)|\,dx + C^2\alpha^2\mu(\Omega) \\
&= \alpha\left(\int_F |f(x)|\,dx + C^2\alpha\mu(\Omega)\right) \\
&\leqslant (C^3+1)\alpha\|f\|_1 < \infty
\end{aligned}$$

So, we can now apply the result from Step 1.

$$\mu\{x : |Tg(x)| > \alpha/2\} \leqslant \frac{D}{\alpha^2}\|g\|_2^2 \leqslant \frac{D'\alpha}{\alpha^2}\|f\|_1 = \frac{D'}{\alpha}\|f\|_1 \tag{2.6}$$

To estimate Tb, we first define $b_j(x)$ as,

$$b_j(x) = \begin{cases} b(x) & x \in Q_j, \\ 0 & x \notin Q_j. \end{cases}$$

Then we have that,

$$b(x) = \sum_{j=1}^{\infty} b_j(x)$$

and we have that for $x \in F$

$$(Tb)(x) = \sum_{j=1}^{\infty} (Tb_j)(x) \tag{2.7}$$

(convergence will be shown below) where

$$Tb_j(x) = \int_{Q_j} K(x-y)b_j(y)\,dy \tag{2.8}$$

We can rewrite $Tb_j(x)$ as,

$$Tb_j(x) = \int_{Q_j} [K(x-y) - K(x-y^j)]b_j(y)\,dy$$

where y^j is the center of the cube Q_j. This is because,

$$\int_{Q_j} K(x-y^j)b_j(y)\,dy = K(x-y^j)\int_{Q_j} b_j(y)\,dy = 0$$

Since

$$|\nabla K| \leqslant \frac{B}{|x|^{n+1}}$$

there follows that for $y \in Q_j$,

$$|K(x-y) - K(x-y^j)| \leqslant C\frac{\mathrm{diameter}(Q_j)}{|x-\bar{y}^j|^{n+1}} \tag{2.9}$$

where $\bar{y}^j$ is a point on the line segment connecting y^j and y. This can be easily seen since,

$$\begin{aligned}
|K(x-y) - K(x-y^j)| &= \left|\int_0^1 \frac{d}{dt}K(x-ty-(1-t)y^j)\,dt\right| \\
&= \left|\int_0^1 \nabla K(x-ty-(1-t)y^j)\cdot(y-y^j)\,dt\right| \\
&\leqslant \int_0^1 \left|\nabla K(x-ty-(1-t)y^j)\cdot(y-y^j)\right|\,dt \\
&\leqslant \int_0^1 \frac{C\,\mathrm{diameter}(Q_j)}{|x-(ty+(1-t)y^j)|^{n+1}}\,dt
\end{aligned}$$

for $x \in F = (\bigcup_j Q_j)^c$, and if $\bar{y}^j$ is the point on the line connecting y and y^j where the denumerator is minimal we obtain (2.9). From Theorem 1.23 we know that the diameter of Q_j is comparable to its distance from F. This means that if x is a point in F, then the set of distances $\{|x-y| : y \in Q_j\}$ are all comparable with each other. Hence,

$$|Tb_j(x)| \leqslant C\,\mathrm{diameter}(Q_j)\int_{Q_j}\frac{|b(y)|}{|x-y|^{n+1}}\,dy$$

But we have that $|b(y)| \leqslant |f(x)| + |g(x)|$ so,

$$\begin{aligned}
\int_{Q_j}|b(y)|\,d\mu &\leqslant \int_{Q_j}|f(y)|\,d\mu + \int_{Q_j}|g(y)|\,d\mu \\
&\leqslant \int_{Q_j}|f(y)|\,d\mu + \int_{Q_j}\frac{1}{\mu(Q_j)}\int_{Q_j}|f(y)|\,d\mu \\
&\leqslant \int_{Q_j}|f(y)|\,d\mu + C\alpha\int_{Q_j}d\mu
\end{aligned}$$

Where the second inequality follows from (2.5) and the last one from (2.4). So now there follows that

$$\int_{Q_j}|b(y)|\,dy \leqslant (1+C)\alpha\mu(Q_j),$$

further we have that if we define $\delta(x) := d(x,F)$, we can note that $\delta(x) \geqslant \frac{1}{2}\,\mathrm{diameter}(Q_j)$ by Theorem 1.23. So if we integrate this, we obtain

$$\mathrm{diameter}(Q_j)\mu(Q_j) \leqslant C\int_{Q_j}\delta(y)\,dy.$$

From this and the fact that $|x-y|$ are comparable, that is there exists $c_1, c_2 > 0$ such that for all $y_0 \in Q_j$

$$c_1|x-y| \leqslant |x-y_0| \leqslant c_2|x-y|$$

there follows that for $x \in F$

$$|Tb_j(x)| \leqslant C\alpha \int_{Q_j} \frac{\delta(y)}{|x-y|^{n+1}}\, dy$$

because

$$\begin{aligned} C \operatorname{diameter}(Q_j) \int_{Q_j} \frac{|b(y)|}{|x-y|^{n+1}}\, dy &\leqslant C' \operatorname{diameter}(Q_j) \int_{Q_j} \frac{|b(y)|}{|x-y_0|^{n+1}}\, dy \\ &\leqslant \alpha C'' \frac{\operatorname{diameter}(Q_j)}{|x-y_0|^{n+1}} \\ &\leqslant C\alpha \int_{Q_j} \frac{\delta(y)}{|x-y|^{n+1}}\, dy. \end{aligned}$$

So, now we can sum of all j to obtain (for $x \in F$) the convergence of the sum (2.8) and

$$|Tb(x)| \leqslant C\alpha \int_{\mathbb{R}^n} \frac{\delta(y)}{|x-y|^{n+1}}\, dy.$$

From Lemma 1.22 and Eq. (2.4) we see that

$$\int_F |Tb(x)|\, dx \leqslant C\alpha\mu(\Omega) \leqslant C\|f\|_1$$

From Chebyshev's inequality there now follows that

$$\mu\{x \in F : |Tb(x)| > \alpha/2\} \leqslant \frac{2C}{\alpha}\|f\|_1$$

But Eq. (2.4) tells us that $\mu(\Omega) = \mu(F^c) \leqslant \frac{C}{\alpha}\|f\|_1$, but, when we take the measure of $\{x \in F^c : |Tb(x)| > \alpha/2\}$ we obtain a value smaller than $\mu(\Omega)$. So, we obtain

$$\mu\{x : |Tb(x)| > \alpha/2\} \leqslant \frac{C}{\alpha}\|f\|_1 \tag{2.10}$$

When we combine this with Eq. (2.6) we obtain

$$\mu\{x : |Tf(x)| > \alpha/2\} \leqslant \frac{C}{\alpha}\|f\|_1.$$

That is, T is of weak-type $(1,1)$.

Third and final step: The L^p inequalities.

1. For $p = 2$ see the first step.
2. For $1 < p < 2$ we will use the Marcienkiewicz interpolation theorem (Theorem 1.16). To this end we must show that T is sublinear (it is linear, so certainly sublinear) and note that T is well-defined on $L^1(\mathbb{R}^n) + L^2(\mathbb{R}^n)$ because of the linearity of T and the fact that Tf is defined for both $f \in L^1$ (by Young's inequality) and $f \in L^2$ (by Step 1). Thus the interpolation theorem shows that

$$\|T(f)\|_p \leqslant A\|f\|_p \text{ for } 1 < p < 2 \text{ and } f \in L^p$$

3. For $2 < p < \infty$ we will use the duality between L^p and L^q where q is the conjugate exponent of p: Now we claim that if ϕ is locally integrable and if

$$\sup \left| \int \psi(x)\phi(x)\, dx \right| = A < \infty$$

where the supremum is taken over all continuous ψ with compact support and $\|\psi\|_q \leqslant 1$, $1 < q < 2$. In this case $\phi \in L^p$ and $\|\psi\|_p = A$. For a proof of this fact see [2].

Now take $f \in L^1 \cap L^p$, $(2 < p < \infty)$ and let ψ be of the type described above, then

$$I = \iint K(x-y)f(y)\phi(x)\,dx\,dy$$

converges absolutely since,

$$\begin{aligned}
\iint |K(x-y)f(y)\phi(x)|\,dx\,dy &= \iint |K(x-y)f(y)\phi(x)|\,dy\,dx \\
&= \int |\phi(x)| \int |K(x-y)||f(y)|\,dy\,dx \\
&= \int (|K| * |f|)(x)|\phi(x)|\,dx \\
&= \int_{\operatorname{supp}\phi} (|K| * |f|)(x)|\phi(x)|\,dx
\end{aligned}$$

$(|K| * |f|)(x)$ is in L^2 by Theorem 1.26, and by Cauchy-Schwartz $(|K| * |f|)(x)$ is then in $L^1(\operatorname{supp}\phi)$. So by Tonelli-Fubini, the order of integration can be switched. Now, the theorem is valid for $1 < q < 2$, so by Step 2 we have that

$$\int K(x-y)\psi(x)\,dx$$

belongs to L^q and is bounded in L^q norm by $A_q\|\psi\|_q = A_q$.

Let $\tilde{K}(x) = K(-x)$, then $\tilde{K}$ obviously satisfies the same conditions as K. Then write $\tilde{T}f(x)$ for $\tilde{K} * f(x)$, now Hölders inequality gives us that

$$\begin{aligned}
|I| &= \left|\int (Tf)\psi\,dx\right| \\
&= \left|\int f\tilde{T}\psi\,dx\right| \\
&\leqslant \int |f(\tilde{T}\psi)|\,dx \\
&\leqslant \|f\|_p\|\tilde{T}\psi\|_q \\
&\leqslant A_q\|f\|_p\|\psi\|_q = A_q\|f\|_p
\end{aligned}$$

Now taking the supremum of all the ψ as indicated above gives us that

$$\|Tf\|_p \leqslant A_q\|f\|_p$$

for $2 < p < \infty$ completing the proof of the theorem.

□

2.2 Singular integrals: extensions and variants of the preceding

2.2 Corollary The results of Theorem 2.1 hold with the second condition replaced by

$$\int_{|x|\geqslant 2|y|} |K(x-y) - K(x)|\,dx \leqslant B'$$

for $0 < |y|$, and with the bound B' replacing the bound B, indepedent of y.

Proof. First we will show that this condition is implied by the "older" condition.

$$
\begin{aligned}
|K(x-y)-K(x)| &= \left|\int_0^1 \frac{d}{dt}K(x-ty)\,dt\right| \\
&= \left|\int_0^1 \nabla K(x-ty)\cdot y\,dt\right| \\
&\leqslant \int_0^1 |\nabla K(x-ty)\cdot y|\;dt \\
&\leqslant \int_0^1 \frac{B|y|}{|x-ty|^{n+1}}\,dt \text{ if } |x|\neq|ty| \text{ for all } 0\leqslant t\leqslant 1
\end{aligned}
$$

Note that $|x|\neq|ty|$ is certainly satisfied (with $0\leqslant t\leqslant 1$) if $|x|\geqslant 2|y|$. First note that if $|x|\geqslant 2|y|$ and $0\leqslant t\leqslant 1$, then $|x-ty|\geqslant|x|-|ty|\geqslant|x|-|y|\geqslant\frac{1}{2}|x|$. So,

$$
\begin{aligned}
\int_{|x|\geqslant 2|y|}|K(x-y)-K(x)|\,dx &\leqslant \int_{|x|\geqslant 2|y|}\int_0^1 \frac{B|y|}{|x-ty|^{n+1}}\,dt\,dx \\
&= \int_0^1\int_{|x|\geqslant 2|y|}\frac{B|y|}{|x-ty|^{n+1}}\,dx\,dt \\
&\leqslant \int_0^1\int_{|x|\geqslant 2|y|}\frac{B|y|}{|x|^{n+1}}\,dx\,dt \\
&\leqslant \int_0^1\int_{|x|\geqslant 1}\frac{B}{|x|^{n+1}}\,dx\,dt \\
&\leqslant \int_0^1 C'B\,dt \\
&= C'B
\end{aligned}
$$

where we have use Theorem 1.9 to simplify the integral in the third step. The rest of the argument is as in the proof of Theorem 2.1 except that the second step (the proof of the weak-type $(1,1)$ inequality) is different. Consider for each cube Q_j the cube Q_j^* with the same center y^j but which is expanded $2\sqrt{n}$ times. Then we have

1. $Q_j\subset Q_j^*$. If $\Omega^*=\bigcup_j Q_j^*$ then we have $\Omega\subset\Omega^*$, and $\mu(\Omega^*)\leqslant(2\sqrt{n})^n\mu(\Omega)$. And if $F^*=(\Omega^*)^c$ then $F^*\subset F$.

2. If $x\notin Q_j^*$ then we have that $|x-y^j|\geqslant\operatorname{diameter}(Q_j)$ and $2|y-y^j|\leqslant\operatorname{diameter}(Q_j)$ for all $y\in Q_j$. So, for $y\in Q_j$ we have $|x-y^j|\geqslant 2|y-y^j|$.

As before

$$Tb_j(x)=\int_{Q_j}[K(x-y)-K(x-y^j)]b_j(y)\,dy.$$

So

$$|Tb_j(x)|\leqslant\int_{Q_j}|K(x-y)-K(x-y^j)||b_j(y)|\,dy.$$

As before we can sum over all j

$$\sum_j|Tb_j(x)|\leqslant\sum_j\int_{Q_j}|K(x-y)-K(x-y^j)||b_j(y)|\,dy.$$

From this there follows that (writing $F^* = (\bigcup_j Q_j^*)^c = \bigcap_j (Q_j^*)^c$)

$$\begin{aligned}\int_{F^*} |Tb(x)|\,dx &\leqslant \sum_j \int_{x\in F^*}\int_{Q_j} |K(x-y)-K(x-y^j)||b_j(y)|\,dy\,dx \\ &\leqslant \sum_j \int_{x\in (Q_j^*)^c}\int_{Q_j} |K(x-y)-K(x-y^j)||b_j(y)|\,dy\,dx \\ &\leqslant \sum_j \int_{x\notin Q_j^*}\int_{Q_j} |K(x-y)-K(x-y^j)||b_j(y)|\,dy\,dx\end{aligned}$$

Because the integrand is positive we can switch the order of integration by Tonelli's theorem. So if we substitute $x' = x - y^j$ and $y' = y - y^j$, then we have that if $y \in Q_j$ that

$$\int_{x\notin Q_j^*} |K(x-y)-K(x-y^j)|\,dx \leqslant \int_{|x'|\geqslant 2|y'|} |K(x'-y')-K(x')|\,dx' \leqslant B'$$

where the bound is given by the hypothesis. So

$$\int_{F^*} |Tb(x)|\,dx \leqslant B' \sum_j \int_{Q_j} |b(y)|\,dy \tag{2.11}$$

The RHS is bounded by $C\|f\|_1$ as derived in the proof of the preceding theorem. This takes us back to Eq. (2.10) and then rest of the proof is the same as in the proof of the preceding theorem. □

Now, we want to remove the restrictive conditions on K, especially the condition that $K \in L^2$. In the process we will improve the bound on $|\nabla K|$ too.

2.3 Theorem Suppose the kernel $K(x)$ satisfies the following conditions,

1. $|K(x)| \leqslant B/|x|^n$ for $0 < |x|$.
2. $$\int_{|x|\geqslant 2|y|} |K(x-y)-K(x)|\,dx \leqslant B$$
   for $0 < |y|$ and,
3. $$\int_{R_1<|x|<R_2} K(x)\,dx = 0 \tag{2.12}$$
   for $0 < R_1 < R_2 < \infty$

And for $f \in L^p(\mathbb{R}^n)$ where $1 < p < \infty$ let (for $\varepsilon > 0$)

$$T_\varepsilon(f)(x) = \int_{|y|\geqslant\varepsilon} f(x-y)K(y)\,dy. \tag{2.13}$$

Then we have the bound

$$\|T_\varepsilon(f)\|_p \leqslant A_p\|f\|_p \tag{2.14}$$

where A_p is independent of f and ε. Further, we have for each $f \in L^p(\mathbb{R}^n)$ that,

$$\lim_{\varepsilon\to 0} T_\varepsilon(f) = T(f)$$

exists in the L^p-norm. The operator T so defined also satisfies

$$\|T(f)\|_p \leqslant A_p\|f\|_p$$

We will use the following lemma to prove Theorem 2.3

2.4 Lemma Suppose K satisfies the conditions of the above theorem with bound B. Let

$$K_\varepsilon(x) = \begin{cases} K(x) & \text{if } |x| \geqslant \varepsilon \\ 0 & \text{if } |x| < \varepsilon. \end{cases}$$

Then we have that $K_\varepsilon \in L^2(\mathbb{R}^n)$. For the Fourier transform of K_ε we have the following estimate

$$\sup_y |\widehat{K}_\varepsilon(y)| \leqslant CB, \quad \varepsilon > 0 \tag{2.15}$$

where C only depends on n.

Proof. K_ε obviously lives in L^2 since

$$\begin{aligned} \int |K_\varepsilon(x)|^2\, dx &= \int_{|x|\geqslant\varepsilon} |K(x)|^2\, dx \\ &\leqslant \int_{|x|\geqslant\varepsilon} \frac{B^2}{|x|^{2n}}\, dx \end{aligned}$$

And this integral is finite (since $n \geqslant 1$) by Theorem 1.9.

We will prove the estimate (2.15) first for the special case $\varepsilon = 1$. First we will prove that $K_\varepsilon(x)$ satisfies the same conditions as $K(x)$ except that the bound B must be replaced by CB. First note that

$$|K_\varepsilon(x)| \leqslant |K(x)| \leqslant \frac{B}{|x|^{n+1}}.$$

And

$$\int_{R_1<|x|<R_2} K_\varepsilon(x)\, dx = 0 \text{ for all } 0 < R_1 < R_2$$

this is clear from the same integral equality for $K(x)$, since it is valid for **all** $0 < R_1 < R_2$. And finally, consider the four cases

1. $|x-y| < \varepsilon, |x| < \varepsilon$;
2. $|x-y| < \varepsilon, |x| \geqslant \varepsilon$;
3. $|x-y| \geqslant \varepsilon, |x| < \varepsilon$;
4. $|x-y| \geqslant \varepsilon, |x| \geqslant \varepsilon$.

In the first case the integral from condition 2 is bounded by 0, in the second and third case by CB and in the last case by B. We will show the second case, the third case is similar. Since $|x| \geqslant 2|y|$, $|x| \geqslant \varepsilon$ and $|x-y| < \varepsilon$ we have that $|x|/2 \leqslant |x| - |y| \leqslant |x-y| < \varepsilon$ so $\varepsilon < |x| < 2\varepsilon$, So,

$$\int_{|x|\geqslant 2|y|} |K_\varepsilon(x-y) - K_\varepsilon(x)|\, dx \leqslant \int_{\varepsilon<|x|<2\varepsilon} |K_\varepsilon(x)|\, dx \leqslant CB$$

Now,

$$\begin{aligned} \widehat{K}_1(y) &= \lim_{R\to\infty} \int_{|x|\leqslant R} e^{2\pi i x\cdot y} K_1(x)\, dx \\ &= \int_{|x|\leqslant 1/|y|} e^{2\pi i x\cdot y} K_1(x)\, dx + \lim_{R\to\infty} \int_{1/|y|\leqslant x\leqslant R} e^{2\pi i x\cdot y} K_1(x)\, dx \\ &:= I_1 + \lim_{R\to\infty} I_2 \end{aligned}$$

But,

$$\int_{|x|\leqslant 1/|y|} e^{2\pi i x\cdot y} K_1(x)\,dx = \int_{|x|\leqslant 1/|y|} \left[e^{2\pi i x\cdot y} - 1\right] K_1(x)\,dx$$

because of the cancellation equation (2.12). Thus,

$$\begin{aligned}
|I_1| &\leqslant \int_{|x|\leqslant 1/|y|} \left|\left[e^{2\pi i x\cdot y} - 1\right] K_1(x)\right|\,dx \\
&= \int_{|x|\leqslant 1/|y|} \left|\left[e^{2\pi i x\cdot y} - 1\right]\right| |K_1(x)|\,dx \\
&= |y| \int_{|x|\leqslant 1/|y|} |x| \underbrace{\frac{\left|\left[e^{2\pi i x\cdot y} - 1\right]\right|}{|x||y|}}_{\leqslant C} |K_1(x)|\,dx \\
&\leqslant C|y| \int_{|x|\leqslant 1/|y|} |x||K_1(x)|\,dx \\
&\leqslant BC|y| \int_{|x|\leqslant 1/|y|} \frac{1}{|x|^{n-1}}\,dx \\
&= BC\frac{|y|}{|y|} \int_{|x|\leqslant 1} \frac{1}{|x|^{n-1}}\,dx \\
&\leqslant BC'
\end{aligned}$$

where the last equality and inequality follows from Theorem 1.9.
To estimate I_2 take $z = y/(2|y|^2)$, then $e^{2\pi i y\cdot z} = -1$, thus

$$\begin{aligned}
I_2 &= \int_{1/|y|\leqslant|x-z|\leqslant R} e^{-2\pi i(x-z)\cdot y} K_1(x-z)\,dx \\
&= -\int_{1/|y|\leqslant|x-z|\leqslant R} e^{-2\pi i x\cdot y} K_1(x-z)\,dx \\
&= -\int_{1/|y|\leqslant|x|\leqslant R} e^{-2\pi i x\cdot y} K_1(x-z)\,dx \\
&+ \left(\int_{1/|y|\leqslant|x|\leqslant R} e^{-2\pi i x\cdot y} K_1(x-z)\,dx\right. \\
&- \left.\int_{1/|y|\leqslant|x-z|\leqslant R} e^{-2\pi i x\cdot y} K_1(x-z)\,dx\right).
\end{aligned}$$

Thus, we have

$$\begin{aligned}
I_2 &= \frac{1}{2}\int_{1/|y|\leqslant|x|\leqslant R} e^{-2\pi i x\cdot y}[K_1(x) - K_1(x-z)]\,dx \\
&+ \frac{1}{2}\left(\int_{1/|y|\leqslant|x|\leqslant R} e^{-2\pi i x\cdot y} K_1(x-z)\,dx\right. \\
&- \left.\int_{1/|y|\leqslant|x-z|\leqslant R} e^{-2\pi i x\cdot y} K_1(x-z)\,dx\right).
\end{aligned}$$

$|z| = \frac{1}{2|y|}$, thus we have that

$$\begin{aligned}
&\left|\frac{1}{2}\int_{1/|y|\leqslant|x|\leqslant R} e^{-2\pi i x\cdot y}[K_1(x) - K_1(x-z)]\,dx\right| \\
&\leqslant \int_{|x|\geqslant 2|z|} |K_1(x) - K_1(x-z)|\,dx \leqslant CB
\end{aligned}$$

On the other hand, let E be the symmetric difference of the sets $\{x : \frac{1}{|y|} < |x| \leqslant R\}$ and $\{x : \frac{1}{|y|} < |x-z| \leqslant R\}$, then

$$\begin{aligned}&\left|\left(\int_{1/|y|\leqslant|x|\leqslant R} e^{-2\pi i x\cdot y} K_1(x-z)\,dx\right.\right.\\ &\left.\left.-\int_{1/|y|\leqslant|x-z|\leqslant R} e^{-2\pi i x\cdot y} K_1(x-z)\,dx\right)\right|\\ &\leqslant \int_E |K_1(x-z)|\,dx.\end{aligned}$$

By using that $|z| \leqslant \frac{1}{2|y|}$, it follows that

$$E \subset \left\{x : \frac{1}{2|y|} \leqslant |x| \leqslant \frac{2}{|y|}\right\} \cup \left\{x : \frac{R}{2} \leqslant |x| \leqslant 2R\right\}.$$

for R sufficiently large ($R \geqslant \frac{1}{2|y|}$). Thus by the condition $|K(x)| \leqslant B/|x|^n$ for $0 < |x|$ and Theorem 1.9. this implies that

$$\begin{aligned}&\left|\left(\int_{1/|y|\leqslant|x|\leqslant R} e^{-2\pi i x\cdot y} K_1(x-z)\,dx\right.\right.\\ &\left.\left.-\int_{1/|y|\leqslant|x-z|\leqslant R} e^{-2\pi i x\cdot y} K_1(x-z)\,dx\right)\right|\\ &\leqslant \int_{\frac{1}{|y|}\leqslant|x|\leqslant\frac{2}{|y|}} |K_1(x-z)|\,dx + \int_{\frac{R}{2}\leqslant|x|\leqslant 2R} |K_1(x-z)|\,dx\\ &\leqslant CB\end{aligned}$$

Summing all the above yields $\sup_y |\widehat{K}_1(y)| \leqslant CB$.

So, the present theorem is proven for K_1. To pass to the general case K_ε we introduce the dilation operator τ_ε, that is $(\tau_\varepsilon f)(x) = f(\varepsilon x)$. Note that, now let K be given and define $K'(x) = \varepsilon^n K(\varepsilon x)$, then we know that K' satisfies the hypothesis of the lemma with the same bound B. Now, let

$$K_1'(x) = \begin{cases} K'(x) & \text{if } |x| \geqslant 1,\\ 0 & \text{if } |x| < 1,\end{cases}$$

then we have found that,

$$|\widehat{K_1'(y)}| \leqslant CB.$$

Further we can easily show that the Fourier transform of $\varepsilon^{-n}K_1'(\frac{x}{\varepsilon})$ is $\widehat{K_1'(\varepsilon x)}$ by elementary properties of the Fourier transform, see for example [6, Proposition 1.2, Ch. 5]. $\widehat{K_1'(\varepsilon x)}$ this is obviously bounded by CB. But $\varepsilon^{-n}K_1'(\frac{x}{\varepsilon}) = K_\varepsilon(x)$ proving the lemma. □

Proof of Theorem 2.3. K satisfies hypotheses 1, 2 and 3 from Theorem 2.3. Then K_ε satisfies the same conditions with bounds not greater than CB where C depends only on the dimension n. This was pointed out in the lemma. As we have pointed out in the previous lemma $K_\varepsilon \in L^2(\mathbb{R}^n)$, $\varepsilon > 0$. Now we can apply Corollary 2.2 to K_ε since

$$(T_\varepsilon f)(x) = \int K_\varepsilon(x-y)f(y)\,dy = \int_{|y|\geqslant\varepsilon} K(y)f(x-y)\,dy$$

then the corollary gives us

$$\|T_\varepsilon(f)\|_p \leqslant A_p\|f\|_p.$$

We will now prove that $\lim_{\varepsilon\to 0} T_\varepsilon(f) = T(f)$ exists in the L^p norm. Suppose that f_1 is a function from C_c^1, then

$$\begin{aligned} T_\varepsilon(f_1)(x) &= \int_{|y|\geqslant\varepsilon} K(y)f_1(x-y)\,dy \\ &= \int_{|y|\geqslant 1} K(y)f_1(x-y)\,dy + \int_{1\geqslant|y|\geqslant\varepsilon} K(y)[f_1(x-y)-f(x)]\,dy \end{aligned}$$

where the second integral follows from the condition

$$\int_{1\geqslant|y|\geqslant\varepsilon} K(y)\,dy = 0.$$

Now, f_1 is an L^1 function, since the continuous compacted supported functions are bounded, say by some constant M[2], thus

$$\int |f_1|\,d\mu \leqslant \int_{\operatorname{supp} f} |M|\,d\mu = M\cdot\mu(\operatorname{supp} f)$$

So, by Theorem 1.26 we know that the first integral lives in L^p (since $K_1 \in L^2(\mathbb{R}^n)$). Because f_1 which has compact support has a continuous derivative we can say that the derivative of f_1 is bounded because the derivative is continuous by assumption. So let A denote the bound of the derivative of f_1. Then

$$|f_1(x-y) - f_1(x)| \leqslant A|y|.$$

But we clearly have that the second integral is supported in as fixed compact set since the integrand is zero for large $|x|$. Together with the bound, we have that the second integral converges uniformly in x as $\varepsilon\to 0$. Now we note that the second integral converges in the L^p norm as $\varepsilon\to 0$. Suppose that $f_\varepsilon(x)$ which is compactly supported in a fixed set K converges uniformly in x as $\varepsilon\to 0$ to $f(x)$, then

$$\int_K |f_\varepsilon - f| \leqslant \int_K \|f-f_\varepsilon\|_\infty = \|f-f_\varepsilon\|_\infty \mu(K) < \infty$$

so after taking the $1/p$-th power of both sides, we obtain the desired assertion.

An arbitrary $f\in L^p$ can be written as $f = f_1 + f_2$ where f_1 is continuous and has compact support with one continuous derivative, and where f_2 is small, that is, $\|f_2\|_p \leqslant \delta$ for some chosen $\delta > 0$ because the C^1 functions with compact support are dense in L^p. So we have that $\|T_\varepsilon(f_2)\|_p \leqslant A_p\|f_2\|_p$.

First we will show that the limiting operator T exists. First we will show that for $f\in C_c^\infty(\mathbb{R}^n)$ and for any $y\neq 0$ we have

$$\int |f(x-y)-f(x)|^p\,dx \leqslant C|y|^p$$

So,

$$\begin{aligned} f(x-y)-f(x) &= \int_0^1 \frac{d}{dt} f(x-ty)\,dt \\ &= \int_0^1 \nabla f(x-ty)\cdot -y\,dt \\ &= \int_0^{|y|} \nabla f(x-t'y')\cdot -y'\,dt' \end{aligned}$$

[2]Because the compactly supported functions are zero outside a compact set (hence bounded outside that set) and bounded in the compact set because a continuous function on a compact set is compact, and thus in the $\mathbb{R}^n$ case bounded.

where $y' = y/|y|$. Thus

$$\begin{aligned}\left(\int |f(x-y)-f(x)|^p\,dx\right)^{1/p} &= \left(\int \left|\int_0^{|y|} \nabla f(x-t'y')\cdot -y'\,dt'\right|^p\,dx\right)^{1/p}\\ &\leqslant \int_0^{|y|}\left(\int |\nabla f(x-t'y')\cdot -y'|^p\,dx\right)^{1/p}dt'\\ &\leqslant |y|\sum_{j=1}^n \left\|\frac{\partial f}{\partial x_j}\right\|_p .\end{aligned}$$

So, for $0<\eta<\varepsilon$ then we have

$$\begin{aligned}\|T_\eta f - T_\varepsilon f\|_p &\leqslant \int_{\eta<|y|\leqslant\varepsilon} |K(y)|\left(\int |f(x-y)-f(x)|^p\,dx\right)^{1/p} dy\\ &\leqslant C\int_{\eta<|y|\leqslant\varepsilon} |y||K(y)|\\ &\leqslant CB\varepsilon \to 0 \text{ as } \eta,\varepsilon\to 0\end{aligned}$$

So, for every function $f\in C_c^\infty$, $\{T_\varepsilon f\}$ is a Cauchy sequence in $L^p(\mathbb{R}^n)$ so the limits $Tf\in L^p$ exists and $T_\varepsilon f\to Tf$ in the L^p-norm

Now, the limiting operator T satisfies $\|Tf\|_p\leqslant A_p\|f\|_p$ because we have for every $\delta>0$ ($\|T_\varepsilon f - Tf\|_p<\delta$ for ε small enough),

$$\|Tf\|_p = \|(T+T_\varepsilon - T_\varepsilon)f\|_p \leqslant \|T_\varepsilon f\| + \|T_\varepsilon f - Tf\|_p \leqslant \|T_\varepsilon f\| + \delta$$

□

2.3 Singular integral operators which commute with dilations

2.3.1 L^p limit

2.5 Definition A function $K:\mathbb{R}^n\to\mathbb{C}$ is said to be homogeneous of degree k if $K(\varepsilon x)=\varepsilon^k K(x)$ for $\varepsilon>0$ and $x\in\mathbb{R}^n$.

2.6 Theorem Let Ω be homogeneous of degree 0. Suppose that Ω is bounded and suppose that Ω satisfies the condition

$$\int_{S^{n-1}} \Omega \, d\sigma = 0$$

where S^{n-1} is the unit sphere and $d\sigma$ the induced Euclidean measure on S^{n-1}. Further suppose that if

$$\sup_{|x-x'|\leqslant\delta, |x|=|x'|=1} |\Omega(x) - \Omega(x')| = \omega(\delta)$$

then

$$\int_0^1 \frac{\omega(\delta)}{\delta} \, d\delta < \infty.$$

For $1 < p < \infty$, $f \in L^p(\mathbb{R}^n)$ and $\varepsilon > 0$ let

$$T_\varepsilon(f)(x) = \int_{\varepsilon\leqslant|y|} \frac{\Omega(y)}{|y|^n} f(x-y) \, dy$$

Then,

1. There exists a bound A_p which is independent of f and ε such that

$$\|T_\varepsilon(f)\|_p \leqslant A_p \|f\|_p;$$

2. $\lim_{\varepsilon\to 0} T_\varepsilon(f) = T(f)$ exists in the L^p norm and

$$\|T(f)\|_p \leqslant A_p \|f\|_p;$$

Proof. For item 1 and 2 we merely need to show that

$$K(x) = \frac{\Omega(x)}{|x|^n}$$

satisfies the condition

$$\int_{|x|\geqslant 2|y|} |K(x-y) - K(x)| \, dx \leqslant B$$

by Theorem 2.3. So,

$$\begin{aligned} K(x-y) - K(x) &= \frac{\Omega(x-y) - \Omega(x)}{|x-y|^n} + \Omega(x)\left(\frac{1}{|x-y|^n} - \frac{1}{|x|^n}\right) \\ &:= I_1 + I_2. \end{aligned}$$

When $|x| \geqslant 2|y|$ with $0 \leqslant \theta \leqslant 1$, we have that

$$|x - \theta y| \leqslant |x| + |y| \leqslant \frac{3}{2}|x|$$

and

$$|x - y| \geqslant ||x| - |y|| \geqslant |x| - |y| \geqslant \frac{1}{2}|x|$$

So, first define $f(x) = |x|^n$ for $x \in \mathbb{R}^n - \{0\}$, then the mean value theorem tells us that there exists $\theta \in [0,1]$ such that

$$|f(x) - f(x-y)| \leqslant |\nabla f(x - \theta y)||y|$$

Now we will determine $\nabla f(x)$, note that

$$\frac{\partial f}{\partial x_i} = \frac{x_i n |x|^n}{|x|^2} = n x_i |x|^{n-2}$$

So, $|\nabla f(z)|$ is equal to

$$|\nabla f(z)| = \sqrt{\sum_{i=1}^{n} z_i^2 n^2 |z|^{2n-4}} = n|z|^{n-1}$$

Finally, we get the requested inequality if we substitute $z = x - \theta y$.

$$\begin{aligned}
\left|\frac{1}{|x-y|^n} - \frac{1}{|x|^n}\right| = \left|\frac{|x|^n - |x-y|^n}{|x-y|^n|x|^n}\right| \\
&\leqslant C'' \frac{|y||x-\theta y|^{n-1}}{|x-y|^n|x|^n} \\
&\leqslant C' \frac{|y||x|^{n-1}}{|x-y|^n|x|^n} \\
&\leqslant C \frac{|y||x|^{n-1}}{|x|^n|x|^n} \\
&\leqslant C \frac{|y|}{|x|^{n+1}}.
\end{aligned}$$

and on the other hand, when $|x| \geqslant 2|y|$

$$\left|\frac{x-y}{|x-y|} - \frac{x}{|x|}\right| \leqslant C \frac{|y|}{|x|}.$$

because we can take without loss of generality that $|x| = 1$ since we can divide both x and y by $|x|$. Then rotate both vectors until $x = (1, 0, 0, \dots, 0)$, now rotate around the x-axis until y is of the form $y = (a, b, 0, \dots, 0)$. Then a geometric argument shows that requested inequality.

Therefore it follows that

$$\begin{aligned}
\int_{|x|\geqslant 2|y|} |K(x-y) - K(x)|\, dx &\leqslant \int_{|x|\geqslant 2|y|} |I_1|\, dx + \int_{|x|\geqslant 2|y|} |I_2|\, dx \\
&\leqslant \int_{|x|\geqslant 2|y|} \frac{1}{|x-y|^n} \left|\Omega\left(\frac{x-y}{|x-y|}\right)\right. \\
&\left. -\Omega\left(\frac{x}{|x|}\right)\right| dx \\
&+ C\|\Omega\|_\infty |y| \int_{|x|\geqslant 2|y|} \frac{1}{|x|^{n+1}}\, dx \\
&\leqslant C \int_{|x|\geqslant 2|y|} \frac{1}{|x|^{n+1}} \omega\left(2\frac{|y|}{|x|}\right) dx + C'\|\Omega\|_\infty \\
&= C \int_{2|y|}^{\infty} \int_{S^{n-1}} \frac{1}{r} \omega\left(2\frac{|y|}{r}\right) d\sigma(x')\, dr + C'\|\Omega\|_\infty \\
&\leqslant C \int_0^1 \frac{\omega(\delta)}{\delta}\, d\delta + C'\|\Omega\|_\infty \\
&\leqslant B
\end{aligned}$$

□

3

Riesz transforms and a bound for the Laplacian

3.1 Riesz transforms

3.1 Definition Define for $f \in L^p(\mathbb{R}^n)$, $1 \leqslant p < \infty$ (where $y = (y_1, y_2, \ldots, y_n)$)

$$R_j(f)(x) = \lim_{\varepsilon \to 0} c_n \int_{\varepsilon \leqslant |y|} \frac{y_j}{|y|^{n+1}} f(x-y)\, dy \text{ for } j = 1, \ldots, n.$$

Where

$$c_n = \frac{\Gamma\left(\frac{n+1}{2}\right)}{\pi^{\frac{n+1}{2}}}.$$

Those transforms are called the n Riesz transforms.

Thus R_j is completely defined by the kernel $K_j(x) = \Omega_j(x)|x|^{-n}$ where $\Omega_j(x) = c_n x_j |x|^{-1}$. But, it is not totally clear if this integral exists in the first place, but this will be shown, together with the L^p boundedness of the Riesz transform in the following theorem.

3.2 Theorem The limit in the definition of the Riesz transforms exists and the Riesz transforms are L^p bounded for $1 < p < \infty$, that is $\|R_j(f)(x)|_p \leqslant A_p \|f\|_p$

Proof. When we show that $\Omega_j(x) = c_n \frac{x_j}{|x|}$ satisfies the hypotheses of Theorem **??**. It is clear that Ω_j is homogeneous of degree 0 i.e. $\Omega_j(\varepsilon x) = \Omega_j(x)$. Now,

$$\int_{S^{n-1}} \frac{x_j}{|x|}\, d\sigma(x) = 0$$

is obviously true since $|x| = 1$ on the unit sphere. So clearly $\omega(\delta) = \delta$, so

$$\int_0^1 1\, d\delta = 1 < \infty$$

Which completes the present proof. $\square$

3.3 Theorem If $f \in C_c(\mathbb{R}^n)$ then for $j = 1, \ldots, n$ we have

$$(\widehat{R_j f})(x) = i\frac{x_j}{|x|}\widehat{f}(x).$$

where $\widehat{f}$ denotes the Fourier transform of f

3.2 A bound for the Laplacian

3.4 Theorem Suppose $f \in C^2$ and suppose that f has compact support. Let Δf denote the Laplacian of f. Then we have the bound

$$\left\| \frac{\partial^2 f}{\partial x_j \partial x_k} \right\|_p \leqslant A_p \|\Delta f\|_p \quad 1 < p < \infty \tag{3.1}$$

Proof. This follows from the L^p boundedness of the Riesz transforms (Theorem 3.2) and the identity

$$\frac{\partial^2 f}{\partial x_j \partial x_k} = -R_j R_k \Delta f \tag{3.2}$$

Before we prove (3.2) we show how this implies (3.1)

$$\begin{aligned} \left\| \frac{\partial^2 f}{\partial x_j \partial x_k} \right\|_p &= \| - R_j R_k \Delta f\|_p \\ &= \|R_j R_k \Delta f\|_p \\ &\leqslant B_p \|R_k \Delta f\|_p \\ &\leqslant B_p C_p \|\Delta f\|_p \\ &= A_p \|\Delta f\|_p. \end{aligned}$$

Now we'll prove (3.2), it is a well know fact that,

$$\widehat{\frac{\partial f}{\partial x_j}} = -2\pi i x_j \widehat{f}(x)$$

Thus,

$$\begin{aligned} \widehat{\frac{\partial^2 f}{\partial x_j \partial x_k}} &= -2\pi i x_k \widehat{\frac{\partial f}{\partial x_j}} \\ &= -4\pi^2 x_j x_k \widehat{f}(x) \\ &= -\left(\frac{i x_j}{|x|}\right)\left(\frac{i x_k}{|x|}\right)(-4\pi |x|^2)\widehat{f}(x) \\ &= -\left(\frac{i x_j}{|x|}\right)\left(\frac{i x_k}{|x|}\right)\widehat{\Delta f}(x) \\ &= -(R_j R_k \Delta f)^{\widehat{}} \end{aligned}$$

Taking the inverse Fourier transform of this equation yields (3.2). This completes the proof. $\square$

4

The domain of the Laplacian in $L^p(\mathbb{R}^n)$

4.1 The domain of the Laplacian in $L^p(\mathbb{R}^n)$

We can now formulate our main result.

4.1 Theorem

$$W^{2,p}(\mathbb{R}^n) = \{f \in L^p(\mathbb{R}^n) : \Delta f \in L^p\}$$

Proof. Define $|||f||| = \|f\|_p + \|\Delta f\|_p$, this is clearly a norm. Now, assume that $|||.|||$ is equivalent to $\|.\|_{W^{2,p}}$ for $f \in C_c^2$, that is there exist C_1, C_2 such that

$$C_1 |||f||| \leqslant \|f\|_{W^{2,p}} \leqslant C_2 |||f||| \text{ for } f \in C_c^2 \tag{4.1}$$

Define $D(\Delta) = \{f \in L^p(\mathbb{R}^n) : \Delta f \in L^p\}$ where the derivative is weak derivative. It is clear that $W^{2,p} \subset D(\Delta)$, since $W^{2,p}$ is just more restrictive subset of L^p than $D(\Delta)$.

Now we will prove that we have an L^p bound on our first order partial derivatives in terms L^p-bound on the second order partial derivatives. To this end let f be a function from $C_c^2(\mathbb{R})$, then let K be the support of a chosen $f \in C_c^2$, then define $t = \inf K$ and $t + d = \sup K$, then

$$f(t+h) = \int_0^h f'(t+r)\,dr$$

then

$$\begin{aligned}
|f(t+h)|^p &= \left| \int_0^h f'(t+r)\,dr \right|^p \\
&\leqslant \left(\int_0^h |f'(t+r)| \right)^p \\
&\leqslant \|f'\|_p^p \|1_{[t,t+h]}\|_q \\
&= h^{p/q} \|f'\|_p^p.
\end{aligned}$$

So,

$$\begin{aligned}\|f\|_p^p &= \int_0^d |f(t+h)|^p\,dh \\ &\leqslant \int_0^d \left(\int_0^h h^{p/q}|f'(t+r)|^p\,dr\right)\,dh \\ &= \int_0^d \int_r^d h^{p/q}|f'(t+r)|^p\,dh\,dr \\ &\leqslant C\int_0^d |f'(t+r)|^p\,dr \\ &= C\|f'\|_p^p.\end{aligned}$$

For the case that $f:\mathbb{R}^n \to \mathbb{R}$ we fix the other variables and apply the same result.

Now, consider the space $(D(\Delta), |||.|||)$ and consider a function f from this space, then since C_c^2 is dense in this space, we can pick $(f_n) \subset C_c^2$ that converges to f in the $|||.|||$ norm. So in particular $f_n \to f$ in the L^p norm. Since (f_n) converges, the (f_n) form a Cauchy sequence in $|||.|||$, by the equivalence of the norms (Eq. 4.1) this is a Cauchy sequence in $W^{2,p}$ too. So, since $W^{2,p}$ is a Banach space, the f_n converge to a function $\tilde{f}$ in the $W^{2,p}$-norm, so in particular in L^p. That $f = \tilde{f}$ now follows from the fact that L^p is a Hausdorff space, so the limit is unique. So we can take the limit $n \to \infty$ in

$$C_1|||f_n||| \leqslant \|f_n\|_{W^2,p} \leqslant C_2|||f_n||| \text{ for } f_n \in C_c^2$$

to obtain the same inequality for functions $f \in W^{2,p}$. So, now we have by Theorem 3.4 a bound on the second-order mixed derivatives and by a previous remark on the first partial derivatives too (where the bound is still valid for the same reason). So if $f \in D(\Delta)$, then $f \in W^{2,p}(\mathbb{R}^n)$.

Finally, we need to prove the equivalence of the norms. Now, $C_1 = 1$ because we just remove some terms, since

$$\|f\|_W^{2,p} = \sum_{0\leqslant|\alpha|\leqslant 2} \|D^\alpha f\|_p.$$

So,

$$\|f\|_p + \|\Delta\|_p \leqslant \|f\|_{W^{2,p}}.$$

C_2 exists by Poincaré's inequality. A question that might arise is 'what do the derivatives become in the limit?', as we will see these become the weak derivatives, so then the proof is complete. Define $T_{ij} = \frac{\partial^2}{\partial x_i \partial x_j}$, and let g be a test function ($g \in C_c^\infty(\mathbb{R}^n)$), and let f_n be the sequence of continuous functions from C_c^2 that converge to $f \in L^p$ in the L^p-norm (by density), then

$$\begin{aligned}\int T_{ij}(f)g\,d\mu &= \int \lim_n \frac{\partial^2 f_n}{\partial x_i \partial x_j} g\,d\mu \\ &= \lim_n \int \frac{\partial^2 f_n}{\partial x_i \partial x_j} g\,d\mu \\ &= \lim_n \int f_n \frac{\partial^2 g}{\partial x_i \partial x_j}\,d\mu \\ &= \int f \frac{\partial^2 g}{\partial x_i \partial x_j}\,d\mu\end{aligned}$$

So $T_{ij}f$ is the weak mixed order derivative of f.

□

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