A Proof of Nielsen's Conjecture on the GPS Dilution of Precision

The dilution of precision terms for relative positioning as defined in [1], are bounded from above by the corresponding dilution of precision terms for point positioning. In [1], this result is proven for the case of four satellites and conjectured to be valid for the case of more than four satellites. A proof of this conjecture is given. We also extend the result by giving two different lower bounds for the dilution of precision terms. The first lower bound depends on the receiver-satellite geometry, whereas the second does not. The proof of the bounds is based on the solution of a generalized eigenvalue problem.

I. INTRODUCTION

Double-difference processing of the NAVSTAR/Global Positioning System (GPS) satellite signals has been employed by the surveying and geodetic community for some time [2]. In analogy with HDOP and VDOP (the horizontal and vertical dilution of precision terms of point positioning), Nielsen [1] introduces corresponding dilution of precision (DOP) terms for relative positioning using double differences and demonstrates for the four-satellite case that his DOP values for relative positioning are bounded from above by the corresponding DOP values of point positioning. In this contribution we extend Nielsen's result to an arbitrary number of satellites. We also show how the relevant DOP values are bounded from below. This enables us to identify the condition for which the two types of DOP values coincide. Section II summarizes Nielsen's result and conjecture, while Section III gives the solution of a generalized eigenvalue problem. It forms the basis of our main result, which is stated and proven in Section IV.

II. NIELSEN'S CONJECTURE

Let $(x_i, y_i, z_i)^T$ be the unit direction vector between the ith satellite and the approximate receiver location, and define the two matrices

$$A_m = \begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ \vdots & \vdots & \vdots \\ x_m & y_m & z_m \end{bmatrix}, \qquad D_m = \begin{bmatrix} -I_{m-1} \\ e_{m-1}^T \end{bmatrix} \tag{1}$$

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where I_{m-1} denotes the unit matrix of order $m-1$ and e_{m-1} denotes the vector which has all its $m-1$ entries equal to one. The $m \times 3$ matrix A_m captures the receiver-satellite geometry of the m satellites and the $(m-1) \times m$ matrix D_m^T equals the differencing operator that transforms single-difference measurements to double-difference measurements having satellite m as reference. Note that the double-difference measurements are referenced to a common satellite. For Nielsen's result to be valid, the need for such a common reference was already shown in [1].

The design matrix for point positioning is denoted as H_m and its counterpart for relative positioning based on double-difference measurements is denoted as G_m. These two matrices are given as

$$H_m = [A_m, e_m], \qquad G_m = D_m^T A_m. \tag{2}$$

Since both matrices are assumed to be of full rank, we must have $m \geq 4$. For $m = 4$, H_m corresponds with [1, eq. (4)] and G_m with [1, eq. (10)].

The DOP terms for point positioning are defined as

$$\begin{aligned} \text{HDOP}_m &= \sqrt{[H_m^T H_m]_{1,1}^{-1} + [H_m^T H_m]_{2,2}^{-1}} \\ \text{VDOP}_m &= \sqrt{[H_m^T H_m]_{3,3}^{-1}} \end{aligned} \tag{3}$$

where $[H_m^T H_m]_{i,i}^{-1}$ is the ith element on the main diagonal of $[H_m^T H_m]^{-1}$. The corresponding DOP terms for relative positioning are defined in [1] as

$$\begin{aligned} \text{HDOP}_{m,DD} &= \sqrt{[G_m^T G_m]_{1,1}^{-1} + [G_m^T G_m]_{2,2}^{-1}} \\ \text{VDOP}_{m,DD} &= \sqrt{[G_m^T G_m]_{3,3}^{-1}} \end{aligned} \tag{4}$$

where $[G_m^T G_m]_{i,i}^{-1}$ is the ith element on the main diagonal of $[G_m^T G_m]^{-1}$.

The main result of [1] is the proof of the two inequalities

$$\begin{aligned} \text{HDOP}_{m,DD} &\leq \text{HDOP}_m \quad \text{for} \quad m = 4 \\ \text{VDOP}_{m,DD} &\leq \text{VDOP}_m \quad \text{for} \quad m = 4. \end{aligned} \tag{5}$$

For $m > 4$ however, the two inequalities are conjectured to be true. In order to prove this conjecture, we need to compare the two matrices $[H_m^T H_m]^{-1}$ and $[G_m^T G_m]^{-1}$. This is done in Section III by means of a generalized eigenvalue problem.

III. GENERALIZED EIGENVALUE PROBLEM

Since the two matrices $[H_m^T H_m]^{-1}$ and $[G_m^T G_m]^{-1}$ are of a different order, respectively 4 and 3, we first need to find an expression for the first three rows and columns of matrix $[H_m^T H_m]^{-1}$. It is easily verified that the inverse of

$$[H_m^T H_m] = \begin{bmatrix} A_m^T A_m & A_m^T e_m \\ e_m^T A_m & m \end{bmatrix} \tag{6}$$

is given as

$$[H_m^T H_m]^{-1} = \begin{bmatrix} [F_m^T F_m]^{-1} & -\frac{1}{m}[F_m^T F_m]^{-1} A_m^T e_m \\ -\frac{1}{m} e_m^T A_m [F_m^T F_m]^{-1} & \frac{1}{m} + \frac{1}{m^2} e_m^T A_m [F_m^T F_m]^{-1} A_m^T e_m \end{bmatrix} \tag{7}$$

where $F_m = P_m A_m$, with the orthogonal projector $P_m = I_m - (1/m) e_m e_m^T$. Since the first three rows and columns of $[H_m^T H_m]^{-1}$ are captured by the matrix $[F_m^T F_m]^{-1}$, the two DOP terms of (3) can be expressed in matrix F_m as

$$\begin{aligned} \mathrm{HDOP}_m &= \sqrt{[F_m^T F_m]^{-1}_{1,1} + [F_m^T F_m]^{-1}_{2,2}} \\ \mathrm{VDOP}_m &= \sqrt{[F_m^T F_m]^{-1}_{3,3}}. \end{aligned} \tag{8}$$

Thus in order to compare the DOP terms, we need to compare the two matrices $[F_m^T F_m]^{-1}$ and $[G_m^T G_m]^{-1}$. This comparison can be based on the following generalized eigenvalue problem.

THEOREM *Let λ_i and f_i, $i = 1,2,3$, be the eigenvalues resp. eigenvectors of the generalized eigenvalue problem*

$$[G_m^T G_m]^{-1} f = \lambda [F_m^T F_m]^{-1} f. \tag{9}$$

Then

$$\begin{cases} \lambda_1 = 1 - \frac{1}{m} e_{m-1}^T P_{G_m} e_{m-1} & \text{with} \quad f_1 = G_m^T e_{m-1} \\ \lambda_2 = \lambda_3 = 1 & \text{with} \quad f_2, f_3 \perp [G_m^T G_m]^{-1} G_m^T e_{m-1} \end{cases} \tag{10}$$

where $P_{G_m} = G_m [G_m^T G_m]^{-1} G_m^T$ is the orthogonal projector that projects onto the range space of G_m and along the null space of G_m^T.

PROOF Since P_m projects along e_m and onto the orthogonal complement of e_m, which is the range space of D_m, the projector can be represented in the following two ways

$$P_m = I_m - \frac{1}{m} e_m e_m^T = D_m [D_m^T D_m]^{-1} D_m^T. \tag{11}$$

This shows, since $F_m = P_m A_m$ and $G_m = D_m^T A_m$, that

$$F_m^T F_m = G_m^T [D_m^T D_m]^{-1} G_m. \tag{12}$$

From (1) it follows that $D_m^T D_m = I_{m-1} + e_{m-1} e_{m-1}^T$ and thus

$$[D_m^T D_m]^{-1} = I_{m-1} - \frac{1}{m} e_{m-1} e_{m-1}^T. \tag{13}$$

Substitution of (13) into (12) gives $F_m^T F_m = G_m^T G_m - (1/m) G_m^T e_{m-1} e_{m-1}^T G_m$ and after inversion

$$[F_m^T F_m]^{-1} = [G_m^T G_m]^{-1} + \frac{[G_m^T G_m]^{-1} G_m^T e_{m-1} e_{m-1}^T G_m [G_m^T G_m]^{-1}}{m - e_{m-1}^T P_{G_m} e_{m-1}}. \tag{14}$$

From substituting (14) into (9), the result (10) is now easily verified.

IV. MAIN RESULT

We are now in a position to prove the conjecture of Nielsen and to give an extension by including lower bounds on the DOP terms as well. As a direct consequence of the above theorem we have the following bounds for the Raleigh quotient

$$\left(1 - \frac{1}{m} e_{m-1}^T P_{G_m} e_{m-1}\right) \le \frac{f^T [G_m^T G_m]^{-1} f}{f^T [F_m^T F_m]^{-1} f} \le 1 \tag{15}$$

for all non-null f and $m \ge 4$. By choosing f respectively as $f = (1,0,0)^T$, $f = (0,1,0)^T$ and $f = (0,0,1)^T$, it follows that

$$\begin{aligned} &\left(1 - \frac{1}{m} e_{m-1}^T P_{G_m} e_{m-1}\right) \sum_{i=1}^{2} [F_m^T F_m]^{-1}_{i,i} \\ &\quad \le \sum_{i=1}^{2} [G_m^T G_m]^{-1}_{i,i} \le \sum_{i=1}^{2} [F_m^T F_m]^{-1}_{i,i} \\ &\left(1 - \frac{1}{m} e_{m-1}^T P_{G_m} e_{m-1}\right) [F_m^T F_m]^{-1}_{3,3} \\ &\quad \le [G_m^T G_m]^{-1}_{3,3} \le [F_m^T F_m]^{-1}_{3,3}. \end{aligned} \tag{16}$$

By taking the square roots, the corresponding bounds for the HDOP and VDOP terms follow as

$$\begin{aligned} &\sqrt{1 - \frac{1}{m} e_{m-1}^T P_{G_m} e_{m-1}}\, \mathrm{HDOP}_m \le \mathrm{HDOP}_{m,DD} \le \mathrm{HDOP}_m \\ &\sqrt{1 - \frac{1}{m} e_{m-1}^T P_{G_m} e_{m-1}}\, \mathrm{VDOP}_m \le \mathrm{VDOP}_{m,DD} \le \mathrm{VDOP}_m. \end{aligned} \tag{17}$$

This result extends (5) in two ways. Apart from the upper bounds, lower bounds are now included as well. Moreover, these bounds are not only valid for $m = 4$, but also for $m > 4$.

Note that $\mathrm{HDOP}_{m,DD} = \mathrm{HDOP}_m$ and $\mathrm{VDOP}_{m,DD} = \mathrm{VDOP}_m$, when $G_m^T e_{m-1} = 0$. From (1) and (2) it follows that this happens when

$$\begin{aligned} x_m &= \frac{1}{m-1} \sum_{i=1}^{m-1} x_i, \\ y_m &= \frac{1}{m-1} \sum_{i=1}^{m-1} y_i, \\ z_m &= \frac{1}{m-1} \sum_{i=1}^{m-1} z_i \end{aligned} \tag{18}$$

i.e., when one of the m satellites is located at the "center of gravity" of the receiver-satellite configuration.

The two lower bounds of (17) depend on the receiver-satellite geometry through the matrix G_m. Lower bounds that are independent of this geometry can be given as well. Since the eigenvalues of a projector are either 0 or 1, it follows that

$$0 \leq \frac{e_{m-1}^T P_{G_m} e_{m-1}}{e_{m-1}^T e_{m-1}} \leq 1. \tag{19}$$

With $e_{m-1}^T P_{G_m} e_{m-1} \leq m-1$, the geometry independent bounds follow from (17) as

$$\begin{aligned} \frac{1}{\sqrt{m}}\mathrm{HDOP}_m \leq \mathrm{HDOP}_{m,DD} \leq \mathrm{HDOP}_m \\ \frac{1}{\sqrt{m}}\mathrm{VDOP}_m \leq \mathrm{VDOP}_{m,DD} \leq \mathrm{VDOP}_m. \end{aligned} \tag{20}$$

V. CONCLUSION

We have proven Nielsen's conjecture by showing that his DOP terms for relative positioning are also bounded from above by the corresponding DOP terms for point positioning when more than four satellites are tracked. This result was extended by giving lower bounds as well. It was also shown that the two types of DOP coincide when one of the satellites is located at the center of gravity of the receiver-satellite configuration.

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An Approach for Accurate GPS Navigation With SA

An approach is presented for more accurate GPS navigation with selective availability. In this approach, a linearized perturbation model has been obtained. Using the measurement perturbation difference approach, the model is made suitable to formulate the Kalman filter to obtain the estimate of state perturbation and thereby the estimate of the user position and velocity. Simulation results are provided to confirm the efficacy of the approach.

I. INTRODUCTION

The Global Positioning System (GPS) is capable of easily providing 100 m level accuracy for a great number of users worldwide with almost any type of GPS receiver using built-in, real time software to convert the pseudo-range measurements into positions.

The GPS navigation task is principally to determine an unknown user position and velocity, receiver clock bias, and clock drift from at least four known satellite positions, velocities, clock corrections and measured pseudo-ranges and delta ranges to each satellite. The satellite positions as well as the user positions are all referenced to an Earth-centered and Earth-fixed (ECEF) coordinate system .

The observed GPS pseudo-range varies from the true range because of range measurement errors. All of these error sources are described and it is shown [1] that selective availability (SA) is the dominant error source in terms of sheer magnitude.

The extended Kalman filter (EKF) is a nonlinear version of the Kalman filter and is widely used for position estimation [2, 3] in GPS receivers. An approach for faster implementation of the EKF for GPS navigation is given in [4]. However, the EKF is formulated with the assumption that the measurement noise is white. Since the SA is not a white noise process the EKF cannot be used for accurate GPS navigation. Hence, it is good to derive an EKF which specifically accounts for non-white behavior of SA. The SA on GPS can be modeled as a second-order Gauss-Markov process [5]. This random process would be included in the measurement model by augmenting the state vector, and these states would then be estimated along with the other states. The second-order model introduces two additional states

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