Vector-valued harmonic analysis with applications to SPDE

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VECTOR-VALUED HARMONIC ANALYSIS WITH APPLICATIONS TO SPDE

VECTOR-VALUED HARMONIC ANALYSIS WITH APPLICATIONS TO SPDE

Dissertation

for the purpose of obtaining the degree of doctor
at Delft University of Technology
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chair of the Board for Doctorates
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by

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PREFACE

This thesis is based on (parts of) the following eight papers:

[1] E. Lorist. Space-time weighted maximal regularity for parabolic (S)PDE. In preparation.

[3] E. Lorist and B. Nieraeth. Sparse domination implies vector-valued sparse domination. arXiv:2003.02233.

[4] N.J. Kalton, E. Lorist, and L. Weis. Euclidean structures and operator theory in Banach spaces. *To appear in Mem. Amer. Math. Soc.*, 2021.

[5] E. Lorist. On pointwise ℓ^r-sparse domination in a space of homogeneous type. *J. Geom. Anal.*, 2020. Online first.

[6] E. Lorist and M.C. Veraar. Singular stochastic integral operators. *To appear in Anal. PDE*, 2020.

[9] T.S. Hänninen and E. Lorist. Sparse domination for the lattice Hardy–Littlewood maximal operator. *Proc. Amer. Math. Soc.*, 147(1):271–284, 2019.

[10] A. Amenta, E. Lorist, and M.C. Veraar. Fourier multipliers in Banach function spaces with UMD concavifications. *Trans. Amer. Math. Soc.*, 371(7):4837–4868, 2019.

[11] A. Amenta, E. Lorist, and M.C. Veraar. Rescaled extrapolation for vector-valued functions. *Publ. Mat.*, 63(1):155–182, 2019.

It is complemented by a few unpublished results. These works form a selection of my research output during my appointment as a PhD candidate in the Analysis Group of the Delft Institute of Applied Mathematics at Delft University of Technology from October 2016 to January 2021. A full list of my research output can be found in the list of publications at the end of this dissertation. These will be referenced by [1-12] throughout this dissertation.

During my PhD I was advised by prof.dr.ir. M.C. Veraar (daily supervisor and promotor) and prof. dr. J.M.A.M. van Neerven (promotor). This PhD position was part of prof. dr. ir. M.C. Veraar's Vidi Project "Harmonic Analysis for Stochastic Partial Differential Equations" subsidized by the Dutch Organisation for Scientific Research (NWO) under project number 639.032.427.

CONTENTS

1

INTRODUCTION

In the study of partial differential equations from a functional analytic viewpoint, harmonic analysis methods, like the theory of singular integral and Fourier multiplier operators, have been developed hand in hand with well-posedness and regularity theory for such equations over the past decades. In contrast, harmonic analysis has not yet fully made its entrance in the study of the stochastic counterparts of these partial differential equations. In this dissertation we will develop new methods in vector-valued harmonic analysis to treat stochastic partial differential equations from a functional analytic viewpoint.

In this first chapter we will provide a stand-alone introduction to the results that can be found in this dissertation, comment on works omitted from this dissertation and give a brief outline of the rest of this dissertation. The subsequent chapters each have their own, more elaborate introduction and can be read independently.

1.1. STOCHASTIC EVOLUTION EQUATIONS

Many naturally occurring phenomena can be mathematically modelled by partial differential equations. Think for example of the flow of water, the transfer of heat in a room, the spread of a virus through a population, a chemical reaction or the formation of a weather system. In many of these phenomena a source of noise is present, like thermal fluctuations, turbulence or random interactions. When one wants to incorporate this noise into the mathematical model, one obtains a stochastic partial differential equation.

As a motivating example let us zoom in on a polycrystalline material, for example a metal. As depicted in Figure 1.1, the material is not uniform, but consists of microscopic grains or crystals. In each of these grains the atoms form a lattice, but the lattices of different grains are not compatible (see Figure 1.2). A prototypical equation that models the growth of such grains is the Allen–Cahn equation. It is a phase field model with two phases, which in the case of grain growth corresponds to two lattice orientations. Starting with a mixture of these two phases, the Allen–Cahn equation models the division into two phase regions within a short timescale. Moreover, on a longer timescale, it models the minimization of the energy in the system, which corresponds to the length

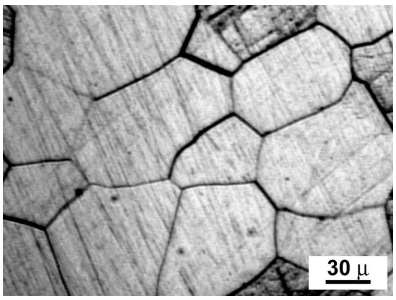


Figure 1.1: Photograph of a polycrystalline metal taken through a microscope. The grain boundaries are made visible by acid etching.

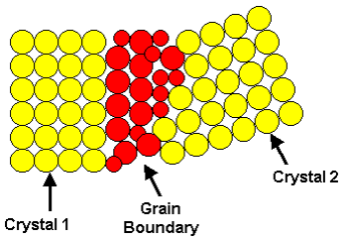


Figure 1.2: Differently oriented crystals and their grain boundary in a polycrystalline material.

of the boundary between the two phases. As with any mathematical model, there is not a one-to-one correspondence between the grain growth in a metal and the Allen-Cahn equation, but it does capture the characteristic dynamics of grain growth very well.

Turning to the mathematics, let us consider the Allen–Cahn equation on $\mathbb{R}^d$. Given an initial state $u_0 \colon \mathbb{R}^d \to \mathbb{R}$, we look for a function $u \colon \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}$ satisfying

$$\begin{cases} \dfrac{\mathrm{d}u}{\mathrm{d}t} - \Delta u = -\Psi'(u) & \text{in } \mathbb{R}_+ \times \mathbb{R}^d, \\ u(0,\cdot) = u_0, \end{cases}$$

where $\Psi \colon \mathbb{R} \to \mathbb{R}$ is a double well potential as depicted in Figure 1.3. The evolution of u over time is driven by a force towards the two stable states at the bottom of the wells at ± 1 respectively, which correspond to the two crystal orientations in the material. This force is in competition with a diffusion process, which smoothens the transition between the two phases. The width of the transition layer is determined by the depth of the wells. We refer to [Emm03, Appendix 4.C] for a physical deduction of the Allen-Cahn equation.

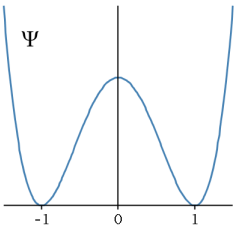


Figure 1.3: An example of a double well potential

To account for thermal fluctuations in the material, one can add a noise term in the mathematical model. This gives rise to the stochastic Allen-Cahn equation. Given an initial state $u_0 \colon \mathbb{R}^d \to \mathbb{R}$, we look for a function $u \colon \Omega \times \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}$ satisfying

$$\begin{cases} \mathrm{d}u - \Delta u \, \mathrm{d}t = -\Psi'(u) \, \mathrm{d}t + B(u) \, \mathrm{d}W & \text{in } \mathbb{R}_+ \times \mathbb{R}^d, \\ u(0,\cdot) = u_0, \end{cases}$$

where W is a Brownian motion on a filtered probability space $(\Omega, \mathscr{F}, \mathbb{P})$ and B describes the nature of the noise term. In the simplest case of additive noise one takes e.g. $B(u) = \varepsilon$

for some $\varepsilon > 0$. This stochastic partial differential equation should be interpreted as the integral equation

$$u(t,x) = u_0(x) + \int_0^t \Delta u(s,x) - \Psi'(u(s,x))\,\mathrm{d}s + \int_0^t B(u(s,x))\,\mathrm{d}W(s), \qquad (t,x) \in \mathbb{R}_+ \times \mathbb{R}^d.$$

To study SPDEs like the stochastic Allen–Cahn equation from a functional analysis viewpoint, one 'hides' the space variable in a Banach space X in order to obtain a stochastic differential equation. One then looks for a function $u\colon \Omega \times \mathbb{R}_+ \to X$ satisfying

$$\begin{cases} \mathrm{d}u + Au\,\mathrm{d}t = F(u)\,\mathrm{d}t + G(u)\,\mathrm{d}W & \text{in } \mathbb{R}_+, \\ u(0) = u_0, \end{cases} \tag{1.1.1}$$

where in general we have that A is a closed operator on X, $F(u), G(u)\colon \Omega \times \mathbb{R}_+ \to X$ are adapted nonlinear forcing terms and the initial state u_0 is for example an element of the domain $D(A)$. Many SPDEs fit into this abstract framework and specifically for the stochastic Allen-Cahn equation on $\mathbb{R}^d$ one takes

$$A = -\Delta, \qquad F(u) = -\Psi'(u), \qquad G(u) = B(u).$$

For the Banach space X there are various choices, each with advantages and disadvantages:

- A space of Hölder continuous functions (see [Lun95]).
- A Hilbert space, for example a Gelfand triple $V \hookrightarrow H \hookrightarrow V^*$ (see [Lio69, LR15, Roz90]) or $X = L^2(\mathbb{R}^d)$ (see [DZ14]).
- A Lebesgue space, i.e. $X = L^q(\mathbb{R}^d)$ for $q \in [1,\infty]$ (see [Kry08, NVW15c, PS16]).

In this dissertation we will focus on the Lebesgue space case and look for solutions u in the space $L^p(\mathbb{R}_+; L^q(\mathbb{R}^d))$ with $p, q \in (1,\infty)$, or in the stochastic case actually $p, q \in [2,\infty)$. From a harmonic and stochastic analysis viewpoint these spaces have much better geometric properties than spaces of smooth functions, but the price we pay is that we can no longer work with classical solutions to (1.1.1) having classical smoothness, but rather work with mild or strong solutions living in a Sobolev space. Of course, the Hilbert space $L^2(\mathbb{R}_+ \times \mathbb{R}^d)$ has even better geometric properties, but in applications one often requires large p and q to obtain better classical smoothness of u from Sobolev embeddings. Moreover $p \neq q$ is often necessary due to criticality or scaling invariance, see e.g. [AV20a, AV20b, KPW10, PSW18].

One approach to obtain existence and uniqueness of a solution to (1.1.1) is to prove sharp estimates for the linear problem

$$\begin{cases} \mathrm{d}u + Au\,\mathrm{d}t = f\,\mathrm{d}t + g\,\mathrm{d}W & \text{in } \mathbb{R}_+, \\ u(0) = 0, \end{cases} \tag{1.1.2}$$

where $f, g\colon \Omega \times \mathbb{R}_+ \to X$ are adapted linear forcing terms. Such sharp estimates imply that there exists an isomorphism between the data f, g and the solution u in suitable function spaces, which is called *maximal L^p-regularity* of A. Having established maximal L^p-regularity, the nonlinear problem can often be treated with quite simple tools, like the Banach contraction mapping theorem and the implicit function theorem (see [AV20a, AV20b, PS16]). Moreover initial conditions $u(0) = u_0 \neq 0$ can be established by trace theory.

By the linear nature of (1.1.2), we can split the maximal L^p-regularity problem into a deterministic and a stochastic part. Indeed, if $u_1, u_2\colon \Omega \times \mathbb{R}_+ \to X$ satisfy

$$\begin{cases} du_1 + Au_1 \,\mathrm{d}t = f \,\mathrm{d}t, & \text{in } \mathbb{R}_+, \\ u_1(0) = 0, \end{cases} \tag{1.1.3}$$

and

$$\begin{cases} \mathrm{d}u_2 + Au_2 \,\mathrm{d}t = g \,\mathrm{d}W, & \text{in } \mathbb{R}_+, \\ u_2(0) = 0, \end{cases} \tag{1.1.4}$$

respectively, then $u = u_1 + u_2$ satisfies (1.1.2). The mild solutions u_1 and u_2 are given by the variation of constants formulas

$$\begin{aligned} u_1(t) &= \int_0^t \mathrm{e}^{-(t-s)A} f(s) \,\mathrm{d}s, \\ u_2(t) &= \int_0^t \mathrm{e}^{-(t-s)A} g(s) \,\mathrm{d}W(s), \end{aligned}$$

where $(\mathrm{e}^{-tA})_{t\geq 0}$ is a semigroup of bounded operators on X. In our motivating example of the Allen–Cahn equation we have that $A = -\Delta$ on $X = L^q(\mathbb{R}^d)$ for $q \in (1,\infty)$ and thus $(\mathrm{e}^{t\Delta}))_{t\geq 0}$ is the heat semigroup on $L^q(\mathbb{R}^d)$, which for $h \in L^q(\mathbb{R}^d)$ is given by

$$\mathrm{e}^{t\Delta} h(x) = \int_{\mathbb{R}^d} \frac{1}{(4\pi t)^{d/2}} \mathrm{e}^{-|x-y|/4t} h(y) \,\mathrm{d}y, \qquad (t,x) \in \mathbb{R}_+ \times \mathbb{R}^d. \tag{1.1.5}$$

For the deterministic part we can work pointwise in Ω and we can therefore omit Ω. We say that A has *deterministic maximal L^p-regularity* if for each $f \in L^p(\mathbb{R}_+; X)$, the mild solution u_1 to (1.1.3) satisfies $Au \in L^p(\mathbb{R}_+; X)$. For our motivating example this means that u_1 needs to be in the second order Sobolev space $W^{2,q}(\mathbb{R}^d)$. We can reformulate deterministic maximal L^p-regularity in terms of the boundedness of an integral operator. Indeed, A has maximal L^p-regularity if and only if

$$T_K f(t) := \int_0^\infty K(t,s) f(s) \,\mathrm{d}s \tag{1.1.6}$$

with

$$K(t,s) = A\mathrm{e}^{-(t-s)A} \mathbf{1}_{t>s}, \qquad t,s \in \mathbb{R}_+ \tag{1.1.7}$$

defines a bounded operator on $L^p(\mathbb{R}_+;X)$. We refer to [DHP03, KW04, PS16] for a detailed discussion on the history of deterministic maximal L^p-regularity and to [KPW10, PS16, PSW18] for applications to nonlinear PDE.

The kernel in (1.1.7) is singular in $s=t$. Indeed, we have

$$\|K(t,s)\| \lesssim \tfrac{1}{|t-s|}, \qquad |t-s|\leq 1$$

since tAe^{-tA} is uniformly bounded on $[0,1]$. This means that $\|K(t,s)\|$ is not integrable around $t=s$ and therefore one needs sophisticated arguments that rely on cancellative properties of K to prove the boundedness of T_K. Operators T_K with such kernels have been studied thoroughly in harmonic analysis and are called *Calderón–Zygmund operators*. Using operator-valued Calderón–Zygmund theory (see [RRT86]), it was shown by Dore [Dor00] that one can deduce deterministic maximal L^p-regularity of T_K for all $p\in(1,\infty)$ from deterministic maximal L^{p_0}-regularity for some $p_0\in[1,\infty]$. Moreover, in the breakthrough paper by Weis [Wei01b], operator-valued Fourier multiplier theory was developed to give a sufficient condition for the deterministic maximal L^p-regularity of A on UMD Banach spaces, i.e. on Banach spaces such that the Hilbert transform is a bounded operator on $L^p(\mathbb{R};X)$ (see [HNVW16]).

Turning to the stochastic version of maximal regularity, let $L^p_{\mathscr{F}}(\Omega\times\mathbb{R}_+;X)$ denote the space of all $g\in L^p(\Omega\times\mathbb{R}_+;X)$ adapted to the filtration $\mathscr{F}$. We say that A has *stochastic maximal L^p-regularity* if for each $g\in L^p_{\mathscr{F}}(\Omega\times\mathbb{R}_+;X)$, the mild solution u_2 to (1.1.3) satisfies $A^{\frac{1}{2}}u_2\in L^p(\mathbb{R}_+;X)$. For our motivating example this means that u_2 needs to be in the first order Sobolev space $W^{1,q}(\mathbb{R}^d)$. Note that we only have half the regularity of u_2 compared to the regularity of u_1, which is caused by the roughness of the involved Brownian motion. We can reformulate stochastic maximal L^p-regularity in terms of the boundedness of a singular stochastic integral operator. Indeed, A has stochastic maximal L^p-regularity if and only if

$$S_K g(t) := \int_0^\infty K(t,s)g(s)\,\mathrm{d}W(s), \qquad t\in\mathbb{R}_+ \tag{1.1.8}$$

with

$$K(t,s) = A^{\frac{1}{2}}\mathrm{e}^{-(t-s)A}\mathbf{1}_{t>s}, \qquad t,s\in\mathbb{R}_+ \tag{1.1.9}$$

defines a bounded operator from $L^p_{\mathscr{F}}(\Omega\times\mathbb{R}_+;X)$ to $L^p(\Omega\times\mathbb{R}_+;X)$. Note that the X-valued stochastic integral in (1.1.8) only makes sense under certain geometric assumptions on the Banach space X, e.g. if X has the UMD property (see [NVW07, NVW15c]). Moreover A needs to be e.g. *sectorial* for $A^{\frac{1}{2}}$ to be well-defined (see [KW04]). We refer to [AV20c, NVW12b, NVW15c] for a detailed discussion of stochastic maximal L^p-regularity and to [Agr18, AV20a, AV20b, Brz95, Hor19, KK18, Kry99, NVW12a, PV19] for applications to nonlinear SPDE.

The kernel in (1.1.9) is again singular in $s=t$ with

$$\|K(t,s)\| \lesssim \tfrac{1}{|t-s|^{1/2}}, \qquad |t-s|\leq 1$$

and $\|K(t,s)\|$ is therefore not integrable with respect to W in $t = s$. However, unlike the deterministic setting, there is no general theory for the L^p-boundedness of *singular stochastic integral operators* of the form (1.1.8). For the specific kernel K in (1.1.9) the L^p-boundedness of S_K for $p \in (2,\infty)$ was obtained by van Neerven, Veraar and Weis in [NVW12b]. They assumed that A has a so-called *bounded H^∞-functional calculus* (see [Haa06]) and X satisfies a certain geometric assumption, which is fulfilled for L^q, $W^{s,q}$, etc. as long as $q \in [2,\infty)$. In the case that $A = -\Delta$ on $L^q(\mathbb{R}^d)$ and $p \geq q \geq 2$ this result was already obtained by Krylov in [Kry94b, Kry99, Kry00, Kry08] using sharp estimates for stochastic integrals and sophisticated real analysis arguments. Moreover, by using PDE arguments, the operator $-\Delta$ can be replaced by a second order elliptic operator with coefficients depending on $(\omega, t, x) \in \Omega \times \mathbb{R}_+ \times \mathbb{R}^d$.

The dichotomy between the available methods to analyse the deterministic and the stochastic part of (1.1.2) leads us to the main goals of this dissertation:

- The first goal of this dissertation is to develop harmonic analysis methods to treat singular stochastic integral operators. In particular we will develop an extrapolation theory for stochastic singular integral operators resembling Calderón–Zygmund theory for deterministic singular integral operators.

- Motivated by the use of the tensor extension of various classical operators prevalent in harmonic analysis in the study of (S)PDEs, the second goal of this dissertation is to develop general sufficient conditions for a bounded operator on $L^p(\mathbb{R}^d)$ to have a bounded tensor extension on $L^p(\mathbb{R}^d; X)$ when X is a Banach function space.

1.2. VECTOR-VALUED HARMONIC ANALYSIS FOR SPDE

The behavior of the stochastic singular integral operators (1.1.8) is quite different from their deterministic counterpart in (1.1.6). Due to the Itô isomorphism the stochastic integrals converge absolutely, whereas in the deterministic case prototypical examples of singular integral operators, like the Hilbert transform and the Riesz projections, have kernels that rely on principle value integrals in their definition. As a consequence, in contrast with the deterministic setting, the scalar-valued setting for stochastic singular integral operators can easily be characterized using the Schur lemma for positive kernel operators (see [Gra14b, Appendix A]).

In the operator-valued setting we may have $\|K(\cdot,s)\| \notin L^2(\mathbb{R}_+)$ for $s \in \mathbb{R}_+$, but still have cancellation of the form

$$\Big(\int_0^\infty \|K(t,s)x\|_X^2 \,\mathrm{d}t\Big)^{1/2} \lesssim \|x\|_X, \qquad s \in \mathbb{R}_+, x \in X. \tag{1.2.1}$$

If the kernel indeed has cancellation of this form, one can check that S_K is L^2-bounded using a simple Fubini argument. In particular, this method was used for the kernel in

(1.1.9) in [DZ14, Section 6.3] for Hilbert spaces X. For $X = L^q(\mathbb{R}^d)$ the cancellation in (1.2.1) does often not hold. For example it fails for the important case $A = -\Delta$. However, cancellation in this setting takes the form

$$\Big\| \Big(\int_0^\infty |K(t,s)x|^2 \, \mathrm{d}t \Big)^{1/2} \Big\|_{L^q(\mathbb{R}^d)} \lesssim \|x\|_{L^q(\mathbb{R}^d)}, \qquad s \in \mathbb{R}_+, \, x \in X, \tag{1.2.2}$$

which in a general Banach space can be reformulated using *γ-radonifying operators* (see [HNVW17, Chapter 9].

Despite the rather different behaviour of stochastic singular integral operators compared to their deterministic counterparts, as our first theorem on the boundedness of singular stochastic integral operators we obtain a stochastic version of the classical extrapolation result for Calderón–Zygmund operators.

Theorem 1.2.1. *Let X be a* UMD *Banach space with type* 2. *Let $K : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathcal{L}(X)$ be strongly measurable and assume that for every interval $I \subseteq \mathbb{R}_+$ we have the following L^2-Hörmander condition*

$$\Big(\int_{\mathbb{R}_+ \setminus I} \|K(t,s) - K(t',s)\|^2 \, \mathrm{d}s \Big)^{1/2} \leq C \qquad t, t' \in \tfrac{1}{2} I$$
$$\Big(\int_{\mathbb{R}_+ \setminus I} \|K(t,s) - K(t,s')\|^2 \, \mathrm{d}t \Big)^{1/2} \leq C \qquad s, s' \in \tfrac{1}{2} I$$

for some constant $C > 0$ independent of I. Suppose that the mapping S_K as defined in (1.1.8) is bounded from $L^p_{\mathscr{F}}(\Omega \times \mathbb{R}_+; X)$ into $L^p(\Omega \times \mathbb{R}_+; X)$ for some $p \in [2, \infty)$. Then the mapping

$$S_K \colon L^q_{\mathscr{F}}(\Omega \times \mathbb{R}_+; X) \to L^q(\Omega \times \mathbb{R}_+; X)$$

is bounded for all $q \in (2, \infty)$.

The *type 2* assumption on X (see [HNVW17, Chapter 7]) in Theorem 1.2.1 is natural in the stochastic setting and is actually necessary for the boundedness of S_K for many nontrivial kernels K (see [NVW15b]). By proving a general extrapolation result for so-called *singular γ-integral operators* and using the Itô isomorphism for X-valued stochastic integrals from [NVW07], a slightly more general version of Theorem 1.2.1 will be proven in Chapter 4. In the γ-integral operator setting we also obtain the endpoint estimates $L^2 \to L^{2,\infty}$ and $L^\infty \to \mathrm{BMO}$.

1.2.1. SPARSE DOMINATION

For Calderón–Zygmund operators weighted bounds with weights in the so-called *Muckenhoupt A_p-class* are classical (see e.g. [Gra14a, Chapter 7]). Sharp dependence of the estimates on the weight characteristic in this setting is known as the A_2-theorem, which was obtained in [Hyt12] by Hytönen. It settles the so-called A_2-conjecture for standard Calderón-Zygmund operators and states that under standard assumptions on the kernel K one has for all $p \in (1, \infty)$ that

$$\|T_K\|_{L^p(\mathbb{R}^d, w) \to L^p(\mathbb{R}^d, w)} \lesssim [w]_{A_p}^{\max\{\frac{1}{p-1}, 1\}}. \tag{1.2.3}$$

Originally the A_2-conjecture was formulated for the Beurling–Ahlfors transform [AIS01], where it was shown to imply quasiregularity of certain complex functions. Shortly afterwards it was settled for this operator in [PV02] and subsequently many other operators were treated, which eventually led to [Hyt12].

A new proof of the A_2-theorem was obtained by Lerner in [Ler13], where it was shown that any standard Calderón-Zygmund operator can be dominated by a so-called *sparse operator* of the form

$$\sum_{Q\in\mathcal{S}} \langle |f| \rangle_{1,Q} \mathbf{1}_Q, \qquad f \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$$

for a sparse collection of cubes $\mathcal{S}$ in $\mathbb{R}^d$. Here we denote $\langle |f| \rangle_{1,Q} := \frac{1}{|Q|}\int_Q |f(t)|\,\mathrm{d}t$ and we call a family of cubes $\mathcal{S}$ in $\mathbb{R}^d$ sparse if for every $Q \in \mathcal{S}$ there exists a measurable set $E_Q \subseteq Q$ such that $|E_Q| \geq \eta |Q|$ for some $\eta \in (0,1)$ and such that the E_Q's are pairwise disjoint. Such sparse operators are easily shown to be bounded on $L^p(\mathbb{R}^d, w)$ for all $p \in (1,\infty)$ and $w \in A_p$ and for this reason the technique of controlling various operators by such sparse operators has proven to be a very useful tool to obtain (sharp) weighted norm inequalities for various operators in the past decade.

To be able to apply this approach to stochastic singular integral operators and obtain a stochastic analogue of the A_2-theorem, we generalize the sparse domination framework to also include these operators. Let (S,d,μ) be a space of homogeneous type, i.e. a quasi-metric measure space satisfying a doubling condition (see [AM15]), and let X and Y be Banach spaces. For a bounded linear operator T from $L^{p_0}(S;X)$ to $L^{p_0,\infty}(S;Y)$ and $\alpha \geq 1$ we define the *sharp grand maximal truncation operator*

$$\mathcal{M}^{\#}_{T,\alpha} f(s) := \sup_{B \ni s} \operatorname*{ess\,sup}_{s',s'' \in B} \left\| T(f\mathbf{1}_{S\setminus \alpha B})(s') - T(f\mathbf{1}_{S\setminus \alpha B})(s'') \right\|_Y, \qquad s \in S,$$

where the supremum is taken over all balls $B \subseteq S$ containing s. In Chapter 3 we will prove the following generalization of a sparse domination result of Lerner and Ombrosi [LO20], which builds upon the efforts of various authors over the past decade.

Theorem 1.2.2. *Let (S,d,μ) be a space of homogeneous type and let X and Y be Banach spaces. Take $p_0, r \in [1,\infty)$ and take $\alpha \geq 1$ large enough. Assume the following conditions:*

- *T is a bounded linear operator from $L^{p_0}(S;X)$ to $L^{p_0,\infty}(S;Y)$.*
- *$\mathcal{M}^{\#}_{T,\alpha}$ is a bounded operator from $L^{p_0}(S;X)$ to $L^{p_0,\infty}(S)$.*
- *For any disjointly and boundedly supported $f_1,\dots,f_n \in L^{p_0}(S;X)$ we have*

$$\Big\| T\Big(\sum_{k=1}^n f_k\Big)(s) \Big\|_Y \lesssim \Big(\sum_{k=1}^n \big\| T f_k(s) \big\|_Y^r \Big)^{1/r}, \qquad s \in S.$$

Then for any boundedly supported $f \in L^{p_0}(S;X)$ there is a sparse collection of cubes $\mathcal{S}$ such that

$$\|Tf(s)\|_Y \lesssim \Big(\sum_{Q\in\mathcal{S}} \langle \|f\|_X \rangle_{p_0,Q}^r \mathbf{1}_Q(s) \Big)^{1/r}, \qquad s \in S.$$

Moreover, for all $p \in (p_0, \infty)$ and $w \in A_{p/p_0}$ we have

$$\|T\|_{L^p(S,w;X)\to L^p(S,w;Y)} \lesssim [w]_{A_{p/p_0}}^{\max\left\{\frac{1}{p-p_0}, \frac{1}{r}\right\}},$$

The key novelty is the introduction of the parameter $r \in [1,\infty)$ in the third bullet of Theorem 1.2.2, which expresses a form of sublinearity of the operator T when $r = 1$. As r increases, this assumption becomes more restrictive and the sparse domination and weighted bounds in the conclusion become stronger. Applying Theorem 1.2.2 with $p_0 = r = 2$ to a stochastic singular integral operator, we obtain the following stochastic variant of the A_2-theorem.

Theorem 1.2.3. *Let X be a* UMD *Banach space with type* 2. *Let $K : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathcal{L}(X,Y)$ be strongly measurable and assume that for some $\epsilon \in (0,1]$*

$$\|K(s,t) - K(s',t)\| \le \Big(\frac{|s-s'|}{|s-t|}\Big)^{\epsilon} \frac{1}{|s-t|^{1/2}} \qquad |s-s'| \le \frac{1}{2}|s-t|,$$
$$\|K(s,t) - K(s,t')\| \le \Big(\frac{|t-t'|}{|s-t|}\Big)^{\epsilon} \frac{1}{|s-t|^{1/2}} \qquad |t-t'| \le \frac{1}{2}|s-t|.$$

Suppose S_K as defined in (1.1.8) is bounded from $L^{p_0}_{\mathscr{F}}(\Omega \times \mathbb{R}_+; X)$ into $L^{p_0}(\Omega \times \mathbb{R}_+; X)$ for some $p_0 \in [2,\infty)$. Then S_K is bounded from $L^p_{\mathscr{F}}(\Omega \times \mathbb{R}_+, w; X)$ to $L^p(\Omega \times \mathbb{R}_+, w; X)$ for all $p \in (2,\infty)$ and $w \in A_{p/2}$ with

$$\|S_K\|_{L^p(\Omega\times\mathbb{R}_+,w;X)\to L^p(\Omega\times\mathbb{R}_+,w;X)} \lesssim [w]_{A_{p/2}}^{\max\{\frac{1}{2}, \frac{1}{p-2}\}}.$$

In Chapter 4 we will prove a more general version of this theorem using an L^2-variant of the Dini conditions from Calderón–Zygmund theory. We also prove that the above estimate is sharp in terms of the dependence on the weight characteristic.

Although the main motivation for Theorem 1.2.2 comes from its applications to singular stochastic integral operators, it also has various interesting, new applications in harmonic analysis. In particular, in Chapter 3 and Section 6.4 we will use a version of Theorem 1.2.2 to prove:

- The A_2-theorem for vector-valued Calderón–Zygmund operators with operator-valued kernel in a space of homogeneous type. We use this A_2-theorem to prove an operator-valued, weighted, anisotropic, mixed-norm Mihlin multiplier theorem.

- Sparse domination and quantitative weighted norm inequalities for both the lattice Hardy–Littlewood and the Rademacher maximal operator.

- Sharp weighted norm inequalities for Littlewood–Paley operators.

1.2.2. SUFFICIENT CONDITIONS FOR THE L^p-BOUNDEDNESS OF S_K

In Theorem 1.2.1 and Theorem 1.2.3 one needs to starts with an L^p-bounded singular stochastic integral operator. Only in the the Hilbert space setting in the convolution case we obtain a full characterization of the boundedness of S_K in terms of kernel conditions. Outside the Hilbert space setting or for non-convolution kernels we do not have abstract theory to ensure L^p-boundedness, so this has to be established on a case-by-case basis. It would be interesting to find general sufficient conditions from which L^p-boundedness can be derived, like a stochastic version of the $T(1)$ and $T(b)$-theorems (see e.g. [HW06, Hyt06, Hyt20, HH16]) or Fourier multiplier theory (see [HNVW16]).

In the important special case that K is the kernel from (1.1.9) , i.e.

$$K(t,s) = A^{\frac{1}{2}} \mathrm{e}^{-(t-s)A} \mathbf{1}_{t>s}, \qquad t,s \in \mathbb{R}_+,$$

on $X = L^q(\mathcal{O})$ for some domain $\mathcal{O} \subseteq \mathbb{R}^d$, we can push our approach further. Indeed, in this case the operators $K(s,t) \in \mathcal{L}(L^q(\mathcal{O}))$ for $s,t \in \mathbb{R}_+$ often have a kernel representation of their own. For example if $A = -\Delta$ on $L^q(\mathbb{R}^d)$, the heat semigroup $(\mathrm{e}^{t\Delta})_{t\geq 0}$ is given by (1.1.5). Therefore we can write S_K as a stochastic-deterministic singular integral operator

$$S_K g(t,x) = \int_0^\infty \int_{\mathcal{O}} k(t,x,s,y) g(s,y) \, \mathrm{d}y \, \mathrm{d}W(s), \qquad (t,x) \in \mathbb{R}_+ \times \mathcal{O}$$

for a kernel $k\colon \mathbb{R}_+ \times \mathcal{O} \times \mathbb{R}_+ \times \mathcal{O} \to \mathbb{C}$. To establish $L^p(\Omega \times \mathbb{R}_+, w; L^q(\mathcal{O}))$-boundedness with $p,q \in (2,\infty)$ and $w \in A_{p/2}$ for S_K, we have seen in Theorem 1.2.1 and Theorem 1.2.3 that it suffices to have $L^q(\Omega \times \mathbb{R}_+ \times \mathcal{O})$-boundedness for S_K and certain assumptions on the kernel K. In applications it is easier to establish boundedness for S_K on the Hilbert space $L^2(\Omega \times \mathbb{R}_+ \times \mathcal{O})$. It is therefore desirable to deduce $L^q(\Omega \times \mathbb{R}_+ \times \mathcal{O})$-boundedness for S_K from $L^2(\Omega \times \mathbb{R}_+ \times \mathcal{O})$-boundedness for S_K. In the deterministic case, this can be done using Calderón–Zygmund theory in the space $\mathbb{R}_+ \times \mathcal{O}$ with a parabolic metric. For a class of elliptic operators of fractional order this theory was developed in [KKL15, KKL16] under a parabolic Hörmander assumption on k. Using a parabolic stochastic Hörmander condition on k, a stochastic version of these results was obtained in [Kim15, KK20] and for the moments of S_K a Calderón–Zygmund theory approach was recently employed in [Kim20].

In Chapter 4 we will extend the results from [KK20] using the abstract sparse domination result in Theorem 1.2.2. We will use the space of homogeneous type $\mathbb{R}_+ \times \mathcal{O}$ with a parabolic metric, which is the main motivation to formulate Theorem 1.2.2 in a space of homogeneous type. Under a $(2,1)$-Dini condition on the kernel k, which we will define in Section 4.3, we obtain the following result:

Theorem 1.2.4. *Let $\mathcal{O} \subseteq \mathbb{R}^d$ be a smooth domain, equip $\mathbb{R}_+ \times \mathcal{O}$ with the parabolic metric*

$$d\big((t,x)(s,y)\big) = \max\big\{|t-s|^{1/m}, |x-y|\big\}, \qquad (t,x),(s,y) \in \mathbb{R}_+ \times \mathcal{O}$$

for some $m > 0$ and let $k\colon \mathbb{R}_+ \times \mathcal{O} \times \mathbb{R}_+ \times \mathcal{O} \to \mathbb{C}$ be a $(2,1)$-Dini kernel. Suppose that

$$S_k g(t,x) := \int_0^\infty \int_{\mathcal{O}} k(t,x,s,y) g(s,y) \, \mathrm{d}y \, \mathrm{d}W(s), \qquad (t,x) \in \mathbb{R}_+ \times \mathcal{O}$$

is a well-defined, bounded operator from $L^2_{\mathscr{F}}(\Omega \times \mathbb{R}_+ \times \mathcal{O})$ to $L^2(\Omega \times \mathbb{R}_+ \times \mathcal{O})$. Then S_k is bounded from $L^p_{\mathscr{F}}(\Omega \times \mathbb{R}_+, v; L^q(\mathcal{O}, w)))$ to $L^p(\Omega \times \mathbb{R}_+, v; L^q(\mathcal{O}, w)))$ for all $p, q \in (2,\infty)$, $v \in A_{p/2}(\mathbb{R}_+)$ and $w \in A_q(\mathcal{O})$.

We are also able to reverse the integration order of space and time in the conclusion of Theorem 1.2.4, i.e. we also show $L^r(\Omega; L^q(\mathcal{O}, w; L^p(\mathbb{R}_+, w)))$-boundedness for S_k with $p \in (2,\infty)$ and $q, r \in (1,\infty)$. This reversed integration order allows one to deduce additional regularity results in applications to SPDE, see [Ant17, NVW15a]. Moreover, we are able to put the expectation on the inside, i.e. we obtain estimates for the moments of S_k as in [Kim20].

1.2.3. APPLICATIONS TO SPDE

Since stochastic maximal regularity can be reformulated in terms of the boundedness of a stochastic singular integral operator, it follows from Theorem 1.2.1 that in many instances stochastic maximal L^p-regularity for some $p \in [2,\infty)$ implies stochastic maximal L^q-regularity for all $q \in (2,\infty)$. Moreover Theorem 1.2.3 gives us weighted estimates for the mild solution u. We will discuss various applications of this principle in Chapter 5. A typical example of the results that we will obtain reads as follows:

Theorem 1.2.5. *Assume $-A$ is the generator of a bounded C_0-semigroup on a* UMD *Banach space X with type 2 and suppose A has stochastic maximal L^p-regularity for some $p \in [2,\infty)$. Then A has stochastic maximal L^q-regularity for all $q \in (2,\infty)$ and the mild solution u to (1.1.4) satisfies the following weighted estimates for all $w \in A_{q/2}$*

$$\|A^{1/2} u\|_{L^q(\Omega \times \mathbb{R}_+, w; X)} \lesssim [w]_{A_{q/2}}^{\max\{\frac{1}{2}, \frac{1}{q-2}\}} \|g\|_{L^q(\Omega \times \mathbb{R}_+, w; X)},$$

The use of temporal $A_{q/2}$-weights in stochastic maximal L^p-regularity is new. In most of the results in [NVW12b, NVW15c] such weights can also be added without causing major difficulties, but it is very natural to deduce this from extrapolation theory. Moreover with our method we actually obtain sharp dependence on the $A_{q/2}$-characteristic. Power weights of the form t^α have already been considered before in both the deterministic (see [KPW10, PSW18]) and stochastic (see [AV20a, AV20b, AV20c, PV19]) evolution equations and can be used to allow for rough initial data. General A_p-weights in deterministic parabolic PDEs have used in [DK18, DK19b, GV17a, GV17b] to derive mixed $L^p(L^q)$-regularity estimates using Rubio de Francia extrapolation (see e.g. [GR85, CMP11]).

For more concrete SPDEs, for example for the stochastic heat equation on a domain $\mathcal{O} \subseteq \mathbb{R}^d$, S_K can be written as a stochastic-deterministic singular integral operator with

kernel k as in Theorem 1.2.4. The assumed $(2,1)$-Dini kernel assumption then translates to Green's function estimates or heat kernel estimates, which are available in quite general settings (see e.g. [EI70, KN14]). As a consequence we obtain time-weighted stochastic maximal L^p-regularity on $L^q(\mathcal{O},w)$ for $p,q\in(2,\infty)$ and $w\in A_q(\mathcal{O})$ from unweighed stochastic maximal L^2-regularity on $L^2(\mathcal{O})$. Power weights in space can be used to allow for rough boundary conditions (see e.g. [HL19, Lin18, Lin20, LV20]), treat singularities due to corners in the domain (see e.g. [Cio20, CKL19, CKLL18, KN14, Naz01, Sol01, PS04]), and handle the incompatibility of the boundary conditions and the noise term (see e.g. [Kim04, KK04, KL99a, KL99b, Kry94a]).

1.3. BANACH FUNCTION SPACE-VALUED EXTENSIONS OF OPERATORS

For a bounded linear operator T on $L^p(\mathbb{R}^d)$ and a Banach space X we define a linear operator $\widetilde{T}$ on $L^p(\mathbb{R}^d)\otimes X$ by setting

$$\widetilde{T}(f\otimes x):=Tf\otimes x,\qquad f\in L^p(\mathbb{R}^d),\,x\in X,$$

and extending by linearity. For $p\in[1,\infty)$ the space $L^p(\mathbb{R}^d)\otimes X$ is dense in the Bochner space $L^p(\mathbb{R}^d;X)$ and it thus makes sense to ask whether the tensor extension $\widetilde{T}$ extends to a bounded operator on $L^p(\mathbb{R}^d;X)$. Motivated by the use of the boundedness of the tensor extension of various classical operators prevalent in harmonic analysis in the study of (S)PDE from a functional analytic viewpoint, we will develop general sufficient conditions for the boundedness of $\widetilde{T}$ on $L^p(\mathbb{R}^d;X)$ in the final part of this dissertation.

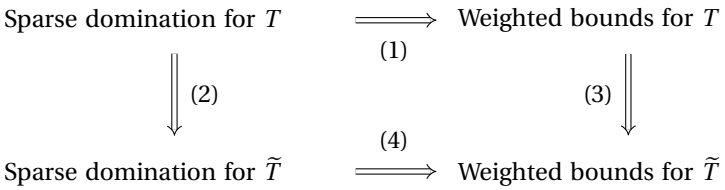
Tensor extensions of operators have been actively studied in the past decades. A centerpoint of the theory is the result of Burkholder [Bur83] and Bourgain [Bou83] that the tensor extension of the Hilbert transform is bounded on $L^p(\mathbb{R};X)$ if and only if the Banach space X has the UMD property. From this connection one can derive the boundedness of the vector-valued extension of many operators in harmonic analysis, like Fourier multipliers and Littlewood–Paley operators.

When X is a Banach function space, very general extension theorems are known. These follow from the connection between the boundedness of the *lattice Hardy–Littlewood maximal operator* on $L^p(\mathbb{R}^d;X)$, which is given by

$$M_{\mathrm{Lat}}f:=\sup_{B\subseteq\mathbb{R}^d\text{ a ball}}\langle|f|\rangle_{1,B}\mathbf{1}_B,\qquad f\in L^1_{\mathrm{loc}}(\mathbb{R}^d;X),$$

and the UMD property of X, shown by Bourgain [Bou84] and Rubio de Francia [Rub86]. The boundedness of the lattice Hardy–Littlewood maximal operator allows one to use scalar-valued arguments to show the boundedness of the vector-valued extension of an operator. Moreover it connects the extension problem to the theory of Muckenhoupt weights. Combined this enabled Rubio de Francia to show a very general extension principle in [Rub86], yielding vector-valued extensions of operators on $L^p(\mathbb{T})$ satisfying weighted bounds. This result was subsequently extended by Amenta, Veraar and the author in [11], replacing $\mathbb{T}$ by $\mathbb{R}^d$ and adding weights in the conclusion.

As we have previously discussed, weighted bounds for operators in harmonic analysis are nowadays often obtained through sparse domination. So, to deduce the weighted boundedness of the vector-valued extension $\widetilde{T}$ of an operator T using [Rub86] and its generalization in [11], one typically goes through implications (1) and (3) in the following diagram



In this diagram implications (1) and (4) are well-known and unrelated to the operator T. Another approach to obtain the weighted boundedness of the vector-valued extension $\widetilde{T}$ of an operator T, through implications (2) and (4) in this diagram, was obtained by Culiuc, Di Plinio, and Ou in [CDO17] for $X = \ell^q$.

The advantage of the route through implications (2) and (4) over the route through implications (1) and (3) is that the Fubini-type techniques needed for implication (2) are a lot less technical than the ones needed for implication (3). Moreover implication (4) yields quantitative and in many cases sharp weighted estimates for $\widetilde{T}$, while the weight dependence in the arguments used for implication (3) is certainly not sharp. A downside of the approach through implications (2) and (4) is the fact that one needs sparse domination for T as a starting point, while one only needs weighted bounds in order to apply (3).

1.3.1. EXTENSION OF OPERATORS USING FACTORIZATION

Implication (3) for $X = \ell^q$ with $q \in (1,\infty)$ follows easily from Rubio de Francia extrapolation and Fubini's theorem (see e.g. [CMP11]). A generalization of this result to general UMD Banach function spaces was first proven by Rubio de Francia in [Rub86, Theorem 5]. Extended in [11] by Amenta, Veraar and the author, this theorem reads as follows:

Theorem 1.3.1. *Let T be a bounded linear operator on $L^{p_0}(\mathbb{R}^d, v)$ for some $p_0 \in (1,\infty)$ and all $v \in A_{p_0}$ and let X be a* UMD *Banach function space. Then $\widetilde{T}$ extends uniquely to a bounded linear operator on $L^p(\mathbb{R}^d, w; X)$ for all $p \in (1,\infty)$ and $w \in A_p$.*

The proof of Theorem 1.3.1 in [Rub86] is based on the factorization of ℓ^r-bounded families of operators on a r-convex Banach function space X through a weighted L^r-space. The classical approach for this factorization comes from the work of Nikišin [Nik70], Maurey [Mau73] and Rubio de Francia [Rub82, Rub86, Rub87] (see also [GR85]).

In Chapter 6 we will give a alternative approach to the proof of Theorem 1.3.1, which will be based on the factorization of an ℓ^2-bounded family of operators on a (not necessarily 2-convex!) Banach function space X through a weighted L^2-space. This fac-

torization result is a special case of a representation and factorization theory based on Euclidean structures, which was developed in [4] by Kalton, Weis and the author.

Our approach yields quantitative bounds, allowing us to estimate the operator norm of $\widetilde{T}$ by a power of the UMD constant $\beta_{p,X}$ of X. Moreover, the original approach relies upon the boundedness of the lattice Hardy-Littlewood maximal operator on $L^p(\mathbb{R}^d;X)$ whereas this will not be used in our approach. Since we will also prove a version of Theorem 1.3.1 for sublinear operators, we will be able to use Theorem 1.3.1 to give a quantitative proof of the boundedness of the lattice Hardy-Littlewood maximal operator on UMD Banach function spaces.

Theorem 1.3.2. *Let X be Banach function space. If X has the* UMD *property, then M_{Lat} is bounded on $L^p(\mathbb{R}^d;X)$ for all $p\in(1,\infty)$ with*

$$\|M_{\mathrm{Lat}}\|_{L^p(\mathbb{R}^d;X)\to L^p(\mathbb{R}^d;X)}\lesssim \beta_{p,X}^2.$$

Combined with the sparse domination principle from Chapter 3, we also obtain sparse domination and sharp weighted estimates for M_{Lat} in Chapter 6.

1.3.2. EXTENSION OF OPERATORS USING SPARSE DOMINATION

The proof of the sparse domination-based extension theorem depicted by implication (2) relies on the following two key ingredients:

- The equivalence between sparse forms and the L^1-norm of the bisublinear maximal function.
- A sparse domination result for the bisublinear lattice maximal operator on UMD Banach function spaces.

Combining these two ingredients in Chapter 6, we will obtain the following theorem.

Theorem 1.3.3. *Let T be a linear operator such that for any $f,g\in L_c^\infty(\mathbb{R}^d)$ there exists a sparse collection of cubes $\mathcal{S}$ such that*

$$\int_{\mathbb{R}^d}|Tf|\cdot|g|\,\mathrm{d}t\lesssim\sum_{Q\in\mathcal{S}}\langle|f|\rangle_{1,Q}\langle|g|\rangle_{1,Q}|Q|.$$

Let X be a UMD *Banach function space. Then for all simple functions $f\in L_c^\infty(\mathbb{R}^d;X)$ and $g\in L_c^\infty(\mathbb{R}^d)$ there exists a sparse collection of cubes $\mathcal{S}$ such that*

$$\int_{\mathbb{R}^d}\|\widetilde{T}f\|_X\cdot|g|\,\mathrm{d}t\lesssim\sum_{Q\in\mathcal{S}}\big\langle\|f\|_X\big\rangle_{1,Q}\langle|g|\rangle_{1,Q}|Q|.$$

In particular, $\widetilde{T}$ extends uniquely to a bounded linear operator on $L^p(\mathbb{R}^d,w;X)$ for all $p\in(1,\infty)$ and $w\in A_p$ with

$$\|\widetilde{T}\|_{L^p(\mathbb{R}^d,w;X)\to L^p(\mathbb{R}^d,w;X)}\lesssim [w]_{A_p}^{\max\left\{\frac{1}{p-1},1\right\}}.$$

Note that the sparse form domination for T in the assumption of Theorem 1.3.3 is in particular satisfied if we have pointwise sparse domination for T as in Theorem 1.2.2 with $p_0 = r = 1$, which follows by integrating against a $g \in L^\infty_c(\mathbb{R}^d)$. We remark that in Theorem 1.3.3 it actually suffices to assume that M_{Lat} is bounded on both $L^p(\mathbb{R}^d; X)$ and $L^{p'}(\mathbb{R}^d; X^*)$, which is implied by the UMD property of X and Theorem 1.3.2. This observation allows us to also prove the converse of Theorem 1.3.2 in Chapter 6.

1.3.3. LITTLEWOOD–PALEY–RUBIO DE FRANCIA ESTIMATES IN BANACH FUNCTION SPACES

Theorems 6.1.1 and 6.1.4 and their multilinear, limited range counterparts in [3, 8] have various interesting applications. They can for example be applied to obtain Banach function space-valued boundedness of:

- The bilinear Hilbert transform.
- The variational Carleson operator.
- Multilinear Calderón–Zygmund operators.
- Bochner–Riesz multipliers.
- Spherical maximal operators.

For the details of these applications, we refer to [3, Section 6], [8, Section 5] and [11, Section 5]. In this dissertation we will focus on one specific, quite elaborate application. We will use Theorem 6.1.1 to deduce a vector-valued version of so-called Littlewood–Paley–Rubio de Francia estimates and use these estimates to deduce operator-valued Fourier multipliers on Banach function spaces.

To introduce these Littlewood–Paley–Rubio de Francia estimates, let S_I denote the Fourier projection onto the interval $I \subseteq \mathbb{R}$, defined by $S_I f := (\mathbf{1}_I \widehat{f}\,)^\vee$ for Schwartz functions $f \in \mathcal{S}(\mathbb{R})$. For a collection $\mathcal{I}$ of pairwise disjoint intervals in $\mathbb{R}$ and $q \in (0,\infty)$ we consider the operator

$$\mathcal{S}_{\mathcal{I},q}(f) := \Big(\sum_{I\in\mathcal{I}} |S_I f|^q\Big)^{1/q}.$$

When $\Delta := \{\pm[2^k, 2^{k+1}), k \in \mathbb{Z}\}$ is the dyadic decomposition of $\mathbb{R}$, the classical Littlewood–Paley inequality states that for $p \in (1,\infty)$

$$\big\|\mathcal{S}_{\Delta,2} f\big\|_{L^p(\mathbb{R})} \simeq \|f\|_{L^p(\mathbb{R})}, \qquad f \in \mathcal{S}(\mathbb{R}).$$

A surprising extension of this classical Littlewood–Paley square function estimate was shown by Rubio de Francia in [Rub85]: for all $q \in [2,\infty)$ and $p \in (q',\infty)$ and *any* collection $\mathcal{I}$ of mutually disjoint intervals in $\mathbb{R}$ we have

$$\big\|\mathcal{S}_{\mathcal{I},q} f\big\|_{L^p(\mathbb{R})} \lesssim \|f\|_{L^p(\mathbb{R})}, \qquad f \in \mathcal{S}(\mathbb{R}). \tag{1.3.1}$$

This result (in particular the $q = 2$ case) is now known as the Littlewood–Paley–Rubio de Francia inequality.

The definition of S_I extends directly to the X-valued Schwartz functions $f \in \mathcal{S}(\mathbb{R}; X)$ for a Banach space X. Vector-valued extensions of Littlewood–Paley–Rubio de Francia estimates for the case $q = 2$ case are studied in [BGT03, GT04, HP06, HTY09, PSX12] via a reformulation in terms of random sums, i.e.

$$\mathbb{E}\Big\|\sum_{I\in\mathcal{I}} \varepsilon_I S_I f\Big\|_{L^p(\mathbb{R};X)} \lesssim \|f\|_{L^p(\mathbb{R};X)}, \qquad f \in \mathcal{S}(\mathbb{R}; X),$$

where $(\varepsilon_I)_{I\in\mathcal{I}}$ is a Rademacher sequence. If this estimate holds then we say that X has the LPR_p property. By the Khintchine inequalities and the result of Rubio de Francia it follows that $\mathbb{C}$ has the LPR_p property for all $p \in [2,\infty)$.

When $q \neq 2$, no analogue of the boundedness of $\mathcal{S}_{\mathcal{I},q}$ for general Banach spaces is known. However, when X is a Banach function space over a measure space (Ω,μ), the operator $\mathcal{S}_{\mathcal{I},q}$ is well-defined for $f \in \mathcal{S}(\mathbb{R}; X)$ by interpreting the ℓ^q-sum pointwise in Ω. Therefore one may wonder whether (1.3.1) holds for $f \in \mathcal{S}(\mathbb{R}; X)$. In Chapter 7 we will show that this is indeed the case if the q'-concavification

$$X^{q'} = \{|x|^{q'} \operatorname{sgn} x : x \in X\} = \{x : |x|^{1/q'} \in X\}$$

has the UMD property.

Theorem 1.3.4. *Let $q \in [2,\infty)$, and suppose X is a q'-convex Banach function space such that $X^{q'}$ has the* UMD *property. Then there exists a increasing function $\phi\colon \mathbb{R}_+ \to \mathbb{R}_+$ such that for all $p \in (q',\infty)$, and $w \in A_{p/q'}$*

$$\|\mathcal{S}_{\mathcal{I},q} f\|_{L^p(\mathbb{R},w;X)} \leq \phi([w]_{A_{p/q'}})\|f\|_{L^p(\mathbb{R},w;X)}, \qquad f \in \mathcal{S}(\mathbb{R}; X).$$

We deduce this result directly from the scalar case $X = \mathbb{C}$ and the factorization-based extension theorem in Theorem 1.3.1. The case $q = 2$ has previously been obtained by Potapov, Sukochev and Xu in [PSX12, Theorem 3] using an ad hoc argument.

1.3.4. FOURIER MULTIPLIER OPERATORS ON BANACH FUNCTION SPACES

As a consequence of the Littlewood–Paley–Rubio de Francia estimates, Coifman, Rubio de Francia and Semmes [CRdFS88] showed that if $p \in (1,\infty)$ and

$$\tfrac{1}{s} > \big|\tfrac{1}{p} - \tfrac{1}{2}\big|,$$

then every $m\colon \mathbb{R} \to \mathbb{C}$ of bounded s-variation uniformly on dyadic intervals induces a bounded Fourier multiplier operator

$$T_m f := \big(m \cdot \widehat{f}\,\big)^\vee, \qquad f \in \mathcal{S}(\mathbb{R}).$$

This is analogous to the situation for the Marcinkiewicz multiplier theorem, which is the $s = 1$ case of the Coifman–Rubio de Francia–Semmes theorem and which follows from the classical Littlewood–Paley theorem.

An operator-valued analogue of the Coifman–Rubio de Francia–Semmes theorem was obtained by Hytönen and Potapov in [HP06], where the Banach space X was assumed to satisfy the LPR_p property. The main goal of Chapter 7 is to prove a wider range of operator-valued Coifman–Rubio de Francia–Semmes type results when X is a Banach function space. We will use Theorem 1.3.4 to prove such results under a UMD assumption on a q-concavification X^q of X. This naturally leads to an '$\ell^2(\ell^{q'})$-boundedness' condition on the range of m, which is a strengthening of uniform boundedness.

The following multiplier theorem is the fundamental result of Chapter 7. Let Δ again denote the standard dyadic partition of $\mathbb{R}$. Let X and Y be Banach function spaces, and for a set of bounded linear operators $\Gamma \subseteq \mathcal{L}(X,Y)$ let $V^s(\Delta;\Gamma)$ denote the space of functions $m\colon \mathbb{R} \to \mathrm{span}(\Gamma)$ with bounded s-variation uniformly on dyadic intervals $J \in \Delta$, measured with respect to the Minkowski norm on $\mathrm{span}(\Gamma)$.

Theorem 1.3.5. *Let $q \in (1,2]$, $p \in (q,\infty)$, $s \in [1,q)$, and let $w \in A_{p/q}$. Let X and Y be Banach function spaces such that X^q and Y have the* UMD *property. Suppose that $m \in V^s(\Delta;\Gamma)$ for some absolutely convex, $\ell^2(\ell^{q'})$-bounded $\Gamma \subseteq \mathcal{L}(X,Y)$. Then the Fourier multiplier operator T_m is bounded from $L^p(\mathbb{R},w;X)$ to $L^p(\mathbb{R},w;Y)$.*

The case $q = 2$ and $w = 1$ of Theorem 1.3.5 was considered in [HP06, Theorem 2.3] for Banach spaces $X = Y$ with the LPR_p property. Our approach only works for Banach function spaces, but these are currently the only known examples of Banach spaces with LPR_p. Note that as the parameter q decreases, we assume less of X, but more of Γ and m.

We will also various extensions and modifications of Theorem 1.3.5 in Chapter 7. For example, we will give sufficient conditions for the bounded s-variation assumption in terms of Hölder regularity of m, give sufficient conditions for the $\ell^2(\ell^{q'})$-boundedness in terms of weighted estimates and prove a variant of Theorem 1.3.5 for general Banach spaces which are complex interpolation space between a Hilbert space and a UMD Banach space.

1.4. WORK NOT INCLUDED IN THIS DISSERTATION

To keep this dissertation coherent and at a reasonable length, not all results obtained during the PhD period are presented. We will sketch the content of the omitted results below.

1.4.1. MULTILINEAR BANACH FUNCTION SPACE-VALUED EXTENSIONS OF OPERATORS

The results in Chapter 6 have been shown in a more general setting by Nieraeth and the author in [3, 8]. In [8] the factorization-based extension theorem is shown in a multilinear, limited range setting. In this setting we show that a bounded operator

$$T\colon L^{p_1}(\mathbb{R}^d,w_1) \times \cdots \times L^{p_1}(\mathbb{R}^d,w_1) \to L^p(\mathbb{R}^d,w)$$

for $p_1,\dots,p_n \in (0,\infty)$, weights $w_1,\cdots,w_n$ in certain Muckenhoupt classes, $\frac{1}{p} = \sum_{k=1}^n \frac{1}{p_k}$ and $w = \prod_{k=1}^n w_k^{p/p_k}$ extends to a bounded operator

$$\widetilde{T}\colon L^{p_1}(\mathbb{R}^d, w_1; X_1) \times \cdots \times L^{p_n}(\mathbb{R}^d, w_n; X_n) \to L^p(\mathbb{R}^d, w; X)$$

for quasi-Banach function spaces $X_1,\dots,X_n$ satisfying a rescaled UMD condition and $X = \prod_{k=1}^n X_k$. The proof is an extension of the original proof in the linear, full range setting by Rubio de Francia [Rub86].

In [3] the sparse domination-based extension theorem is also shown in this multilinear, limited range setting. In this result we use the multilinear structure to its fullest, i.e. we use a weight condition on the tuple $(w_1,\cdots,w_n)$ and a UMD condition on the tuple $(X_1,\dots,X_n)$ rather than a condition on each individual weight and quasi-Banach function space respectively. It is an interesting open problem whether the factorization-based extension theorem can also be generalized to this fully multilinear setting.

1.4.2. EUCLIDEAN STRUCTURES AND OPERATOR THEORY IN BANACH SPACES

In [4], which could be a dissertation in itself, Kalton, Weis and the author developed a general method to extend results on Hilbert space operators to the Banach space setting by representing certain sets of Banach space operators Γ on a Hilbert space. The assumption on Γ is expressed in terms of α-boundedness for a Euclidean structure α on the underlying Banach space X. α-Boundedness is originally motivated by $\mathcal{R}$- or ℓ^2-boundedness of sets of operators and this representation result explains why $\mathcal{R}$- or ℓ^2-boundedness assumptions make their appearance in many results in vector-valued harmonic analysis.

By choosing the Euclidean structure α accordingly, a unified and more general approach to the factorization theory of Kwapień and Maurey and the factorization theory of Maurey, Nikišin and Rubio de Francia is obtained. The factorization theorem we use for our factorization-based extension theorem in Chapter 6 is a special case of this theory. Furthermore Euclidean structures are used to build vector-valued function spaces, which enjoy the nice property that any bounded operator on L^2 extends to a bounded operator on these vector-valued function spaces, which is in stark contrast to the extension problem for Bochner spaces that we discuss in Chapter 6. Moreover, the representation theorem is used to prove a quite general transference principle for sectorial operators on a Banach space, which extends Hilbert space results for sectorial operators to the Banach space setting. Moreover some sophisticated counterexamples for sectorial operators are constructed.

1.4.3. THE ℓ^s-BOUNDEDNESS OF A FAMILY OF INTEGRAL OPERATORS

In [12] Gallarati, Veraar and the author proved the ℓ^s-boundedness of a family of integral operators with an operator-valued kernel on L^q. The proof is based on Rubio de Francia extrapolation and the factorization theory of Maurey, Nikishin and Rubio de

Francia as discussed in Chapter 6. The results have been applied by Gallarati and Veraar in [GV17b], where a new approach to maximal L^p-regularity for parabolic problems with time-dependent generator is developed. An extension of the ℓ^s-boundedness result in [12], in which L^q is replaced by a UMD Banach function space X, has been obtained by the author in [7] using the boundedness of the lattice Hardy–Littlewood maximal operator.

1.5. OVERVIEW

After discussing the necessary preliminaries in Chapter 2, this dissertation consists of two parts. Part I consists of Chapters 3-5 and is concerned with harmonic analysis methods to treat singular stochastic integral operators. Part II, consisting of Chapters 6 and 7, is devoted to the boundedness of Banach function space-valued extensions of operators.

In Part I we first develop the necessary harmonic analysis. In particular, we prove the abstract sparse domination result in Theorem 1.2.2 and give some applications of this result in harmonic analysis in Chapter 3. Afterwards, we develop the extrapolation theory for singular stochastic integral operators with operator-valued kernel in Chapter 4. In Chapter 5 we apply the results of Chapter 4 to obtain p-independence and weighted bounds for stochastic maximal L^p-regularity.

We develop sufficient conditions for a bounded operator on $L^p(\mathbb{R}^d)$ to have a bounded Banach function space-valued extension in Chapter 6. In particular, we will prove Theorems 1.3.1 and 1.3.3 and their consequences. Using Theorem 1.3.1, we prove Banach function space-valued Littlewood–Paley–Rubio de Francia-type estimates and the operator-valued analogues of the Coifman–Rubio de Francia–Semmes Fourier multiplier theorem in Chapter 7.

2

PRELIMINARIES

In this chapter we will present the background material that will be used throughout this dissertation. We start by introducing some basic notation.

- We denote the Lebesgue measure on $\mathbb{R}^n$ by $\mathrm{d}t$ and we denote the Lebesgue measure of a Borel set $E \subseteq \mathbb{R}^n$ as $|E|$.

- For $p \in [1,\infty]$ we let p' be the Hölder conjugate of p, i.e. $p' \in [1,\infty]$ such that $\frac{1}{p} + \frac{1}{p'} = 1$.

- For $s, t \in \mathbb{R}$ we define $s \vee t = \max\{s,t\}$ and $s \wedge t = \min\{s,t\}$ and for vectors $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{C}^n$ we write
$$\boldsymbol{a} \cdot \boldsymbol{b} := \sum_{k=1}^{n} a_k b_k, \qquad \boldsymbol{a}/\boldsymbol{b} := \sum_{k=1}^{n} \frac{a_k}{b_k}.$$
For a multi-index $\alpha \in \mathbb{N}^n$ we write $|\alpha| = \sum_{k=1}^n \alpha_k$ and for $t \in \mathbb{R}^n$ we write $t^\alpha := \prod_{k=1}^n t_k^{\alpha_k}$. Moreover we define the partial derivatives $\partial^\alpha := \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}$.

- Let X, Y be a Banach spaces and (S,μ) a measure space. We write $\mathcal{L}(X,Y)$ for the bounded linear operators from X to Y and we write $\mathcal{L}(X) := \mathcal{L}(X,X)$. If we say that a function $f : S \to \mathcal{L}(X,Y)$ is strongly measurable, we mean that f is strongly measurable in the strong operator topology on $\mathcal{L}(X,Y)$, i.e. $s \mapsto f(s)x$ is strongly measurable for all $x \in X$. We denote the space of all strongly measurable functions $f\colon S \to X$ by $L^0(S;X)$.

  For $p \in [1,\infty]$ we define the Bochner space $L^p(S;X)$ as the space of all $f \in L^0(S;X)$ such that
$$\|f\|_{L^p(S;X)} := \Big(\int_S \|f\|_X^p \,\mathrm{d}\mu\Big)^{1/p} < \infty, \qquad p < \infty,$$
$$\|f\|_{L^\infty(S;X)} := \operatorname*{ess\,sup}_{s\in S} \|f(s)\|_X < \infty, \qquad p = \infty.$$
  For $p \in [1,\infty)$ and $q \in [1,\infty]$ we define the X-valued Lorentz space $L^{p,q}(S;X)$ as the space of all $L^0(S;X)$ such that
$$\|f\|_{L^{p,q}(S;X)} := \Big\| t \mapsto t \cdot \mu\big(\{s \in S : \|f(s)\|_X > t\}\big)^{1/p} \Big\|_{L^q(\mathbb{R}_+, \frac{\mathrm{d}t}{t})} < \infty.$$

Note that $\|\cdot\|_{L^{p,q}(S;X)}$ is only a quasi-norm. For $p \in (1,\infty)$ these spaces are normable, i.e. there exists an equivalent *norm* on $L^{p,q}(S;X)$. For $p \in [1,\infty)$ we have $L^{p,p}(S;X) = L^p(S;X)$ with equivalent norms and if $\mu(S) < \infty$ we have the continuous embedding $L^{p,\infty}(S) \hookrightarrow L^1(S)$ with

$$\|f\|_{L^1(S)} \lesssim_p \mu(S)^{1/p'} \|f\|_{L^{p,\infty}(S)}, \qquad f \in L^{p,\infty}(S). \tag{2.0.1}$$

- Let S be a quasi-metric space with a Borel measure μ. We denote a ball around $s \in S$ with radius r by $B(s,r)$. For a Banach space X and $p \in [1,\infty)$ we denote by $L^p_{\mathrm{loc}}(S;X)$ the space of all $f \in L^0(S;X)$ such that $f\mathbf{1}_B \in L^p(S;X)$ for all balls $B \subseteq S$. Moreover let $L^\infty_c(S;X)$ be the space of all $f \in L^\infty(S;X)$ such that the support of f, denoted by $\operatorname{supp} f$, is contained in some ball $B \subseteq S$. For $f \in L^1_{\mathrm{loc}}(S;X)$ and a Borel set $E \subseteq S$ with finite positive measure we write

$$\langle f \rangle_{1,E} := \fint_E f \,\mathrm{d}\mu := \frac{1}{\mu(E)} \int_E f \,\mathrm{d}\mu.$$

  and if $f \in L^p_{\mathrm{loc}}(S)$ is positive, we write $\langle f \rangle_{p,E} := \big(\langle f^p \rangle_{1,E}\big)^{1/p}$.

- For an interpolation couple of Banach spaces (X_0, X_1), $\theta \in (0,1)$ and $q \in [1,\infty]$, we denote the real and complex interpolation spaces by $(X_0,X_1)_{\theta,q}$ and $[X_0,X_1]_\theta$ respectively. For $p_0, p_1 \in [1,\infty]$ and $\theta \in [0,1]$, we define the interpolation exponent $[p_0,p_1]_\theta$ by

$$\frac{1}{[p_0,p_1]_\theta} := \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$$

  with the interpretation $1/0 := \infty$. This lets us write interpolation results such as

$$[L^{p_0}(S), L^{p_1}(S)]_\theta = L^{[p_0,p_1]_\theta}(S)$$

  in a pleasing compact form. For details on the real and complex interpolation method we refer to [BL76, Tri78] and [HNVW16, Appendix C].

- We write $C_{a,b,\dots}$ to denote a constant which only depends on the parameters $a, b, \dots$ and which may change from line to line. By $\lesssim_{a,b,\dots}$ we mean that there is a constant $C_{a,b,\dots}$ such that inequality holds and by $\simeq_{a,b,\dots}$ we mean that $\lesssim_{a,b,\dots}$ and $\gtrsim_{a,b,\dots}$ hold.

2.1. SPACES OF HOMOGENEOUS TYPE

A space of homogeneous type (S,d,μ), originally introduced by Coifman and Weiss in [CW71], is a set S equipped with a quasi-metric d and a doubling Borel measure μ. That is, a metric d which instead of the triangle inequality satisfies

$$d(s,t) \le c_d \big(d(s,u) + d(u,t)\big), \qquad s,t,u \in S$$

for some $c_d \geq 1$, and a Borel measure μ that satisfies the doubling property

$$\mu\big(B(s,2r)\big) \leq c_\mu\, \mu\big(B(s,r)\big), \qquad s \in S, \quad r > 0$$

for some $c_\mu \geq 1$. Taking the least admissible c_μ, we define the *doubling dimension* by $\nu := \log_2 c_\mu$. Then there is a $C > 0$ such that

$$\mu(B(s,R)) \leq C\left(\frac{R}{r}\right)^{\nu} \mu(B(s,r)), \qquad s \in S, \qquad R > r > 0, \tag{2.1.1}$$

$$\mu(B(s,r)) \leq C\left(1 + \frac{d(s,t)}{r}\right)^{\nu} \mu(B(t,r)), \qquad s,t \in S, \qquad r > 0. \tag{2.1.2}$$

Throughout this dissertation we will assume additionally that all balls $B \subseteq S$ are Borel sets and that we have $0 < \mu(B) < \infty$. We will write that an estimate depends on S if it depends on c_d and c_μ.

It was shown in [Ste15, Example 1.1] that it can indeed happen that balls are not Borel sets in a quasi-metric space. This can be circumvented by taking topological closures and adjusting the constants c_d and c_μ accordingly. However, to simplify matters we just assume all balls to be Borel sets and leave the necessary modifications if this is not the case to the reader. The size condition on the measure of a ball ensures that taking the average $\langle f \rangle_{p,B}$ of a positive function $f \in L^p_{\mathrm{loc}}(S)$ over a ball $B \subseteq S$ is always well-defined.

As μ is a Borel measure, i.e. a measure defined on the Borel σ-algebra of the quasi-metric space (S,d), the Lebesgue differentiation theorem holds and as a consequence the continuous functions with bounded support are dense in $L^p(S)$ for all $p \in [1,\infty)$. The Lebesgue differentiation theorem (and consequently our results) remain valid if μ is a measure defined on a σ-algebra Σ that contains the Borel σ-algebra as long as the measure space (S,Σ,μ) is Borel semi-regular, see [AM15, Theorem 3.14] for the details. For a thorough introduction to and a list of examples of spaces of homogeneous type we refer to the monographs of Christ [Chr90] and Alvarado and Mitrea [AM15].

2.1.1. DYADIC CUBES

Let $0 < c_0 \leq C_0 < \infty$ and $0 < \delta < 1$. Suppose that for $k \in \mathbb{Z}$ we have an index set J_k, a pairwise disjoint collection $\mathscr{D}_k = \{Q_k^j\}_{j \in J_k}$ of measurable sets and a collection of points $\{z_j^k\}_{j \in J_k}$. We call $\mathscr{D} := \bigcup_{k \in \mathbb{Z}} \mathscr{D}_k$ a *dyadic system* with parameters c_0, C_0 and δ if it satisfies the following properties:

(i) For all $k \in \mathbb{Z}$ we have $S = \bigcup_{j \in J_k} Q_j^k$.

(ii) For $k \geq l$, $Q \in \mathscr{D}_k$ and $Q' \in \mathscr{D}_l$ we either have $Q \cap Q' = \varnothing$ or $Q \subseteq Q'$.

(iii) For each $k \in \mathbb{Z}$ and $j \in J_k$ we have

$$B(z_j^k, c_0\delta^k) \subseteq Q_j^k \subseteq B(z_j^k, C_0\delta^k).$$

We will call the elements of a dyadic system $\mathscr{D}$ cubes. For a cube $Q \in \mathscr{D}$ we define the *restricted dyadic system* $\mathscr{D}(Q) := \{P \in \mathscr{D} : P \subseteq Q\}$. We will say that an estimate depends on $\mathscr{D}$ if it depends on the parameters c_0, C_0 and δ.

One can view z_j^k and δ^k as the center and side length of a cube $Q_j^k \in \mathscr{D}_k$. These have to be with respect to a specific $k \in \mathbb{Z}$, as this k may not be unique. We therefore think of a cube $Q \in \mathscr{D}$ to also encode the information of its center z and generation k. The structure of individual dyadic cubes $Q \in \mathscr{D}$ in a space of homogeneous type can be very messy and consequently the dilations of such cubes do not have a canonical definition. Therefore for a cube $Q \in \mathscr{D}$ with center z and of generation k we define the *dilations* αQ for $\alpha \geq 1$ as

$$\alpha Q := B\big(z, \alpha \cdot C_0 \delta^k\big),$$

which are actually dilations of the ball that contains Q by property (iii) of a dyadic system.

When $S = \mathbb{R}^n$ and d is the Euclidean distance, the standard dyadic cubes form a dyadic system and, combined with its translates over $\alpha \in \{0, \frac{1}{3}, \frac{2}{3}\}^n$, it holds that any ball in $\mathbb{R}^n$ is contained in a cube of comparable size from one of these dyadic systems (see e.g. [HNVW16, Lemma 3.2.26]). We will rely on the following proposition for the existence of dyadic systems with this property in a general space of homogeneous type. For the proof and a more detailed discussion we refer to [HK12].

Proposition 2.1.1. *Let (S, d, μ) be a space of homogeneous type. There exist constants $0 < c_0 \leq C_0 < \infty$, $\gamma \geq 1$, $0 < \delta < 1$ and $m \in \mathbb{N}$ such that there are dyadic systems $\mathscr{D}^1, \dots, \mathscr{D}^m$ with parameters c_0, C_0 and δ, and with the property that for each $s \in S$ and $r > 0$ there is a $j \in \{1, \dots, m\}$ and a $Q \in \mathscr{D}^j$ such that*

$$B(s, r) \subseteq Q, \qquad \text{and} \qquad \operatorname{diam}(Q) \leq \gamma r.$$

As an example of a space of homogeneous type with a dyadic system, we now introduce the anisotropic Euclidean spaces, which are for example used when one considers parabolic equations in space-time $\mathbb{R}^n \times \mathbb{R}_+$.

Example 2.1.2 (Anisotropic Euclidean spaces). For $\boldsymbol{a} \in (0, \infty)^n$ let $|\cdot|_{\boldsymbol{a}}$ be the anisotropic quasi-norm

$$|s|_{\boldsymbol{a}} := \Big(\sum_{k=1}^n |s_k|^{2/a_k} \Big)^{1/2}, \qquad s \in \mathbb{R}^n. \tag{2.1.3}$$

and define

$$\mathbb{R}^n_{\boldsymbol{a}} := (\mathbb{R}^n, |\cdot - \cdot|_{\boldsymbol{a}}, \mathrm{d}t).$$

Then $\mathbb{R}^n_{\boldsymbol{a}}$ is a space of homogeneous type and e.g.

$$\mathscr{D}_{\boldsymbol{a}} := \Big\{ \prod_{k=1}^n \big(2^{-j a_k}([0,1) + m_k)\big) : \boldsymbol{m} \in \mathbb{Z}^n, j \in \mathbb{Z} \Big\}$$

is a dyadic system in $\mathbb{R}^n_{\boldsymbol{a}}$.

We end this section with a covering lemma, which shows that we can partition a space of homogeneous type S with a dyadic system $\mathscr{D}$ such that a certain set is contained in a dilation of every element of the partition. This will be useful to turn our abstract local sparse domination result in Chapter 3 into a global sparse domination result.

Lemma 2.1.3. *Let (S,d,μ) be a space of homogeneous type and $\mathscr{D}$ a dyadic system with parameters c_0, C_0 and δ. Suppose that* $\operatorname{diam}(S)=\infty$, *take $\alpha \geq 3c_d^2/\delta$ and let $E \subseteq S$ satisfy* $0<\operatorname{diam}(E)<\infty$. *Then there exists a partition $\mathcal{D} \subseteq \mathscr{D}$ of S such that $E \subseteq \alpha Q$ for all $Q \in \mathcal{D}$.*

Proof. For $s \in S$ and $k \in \mathbb{Z}$ let $Q_s^k \in \mathscr{D}_k$ be the unique cube such that $s \in Q_s^k$ and denote its center by z_s^k. Define

$$K_s := \{k \in \mathbb{Z} : E \not\subseteq 2c_d Q_s^k\},$$

where c_d is the quasi-metric constant. If $k \in \mathbb{Z}$ is such that

$$\operatorname{diam}(2c_d Q_s^k) \leq 4c_d^2 C_0 \delta^k < \operatorname{diam}(E),$$

then $E \not\subseteq 2c_d Q_s^k$, i.e. $k \in K_s$, so K_s is non-empty. On the other hand if $k \in \mathbb{Z}$ is such that $C_0\delta^k > \sup_{s' \in E} d(s,s')$, then

$$\sup_{s' \in E} d(s', z_s^k) \leq c_d \left(\sup_{s' \in E} d(s,s') + d(s, z_s^k)\right) \leq 2c_d C_0 \delta^k,$$

so $E \subseteq 2c_d Q_s^k$ and thus $k \notin K_s$. Therefore K_s is bounded from below.

Define $k_s := \min K_s$ and set $\mathcal{D} := \{Q_s^{k_s} : s \in S\}$. Then $\mathcal{D}$ is a partition of S. Indeed, suppose that for $s, s' \in S$ we have $Q_s^{k_s} \cap Q_{s'}^{k_{s'}} \neq \varnothing$. Then using property (ii) of a dyadic system we may assume without loss of generality that $Q_s^{k_s} \subseteq Q_{s'}^{k_{s'}}$. Property (ii) of a dyadic system then implies that $k_s \geq k_{s'}$. In particular $s \in Q_{s'}^{k_{s'}}$, so by the minimality of k_s we must have $k_s = k_{s'}$. Therefore since the elements of $\mathscr{D}_{k_s}$ are pairwise disjoint we can conclude $Q_s^{k_s} = Q_{s'}^{k_{s'}}$.

To conclude note that $z_s^{k_s} \in Q_s^{k_s} \subseteq Q_s^{k_s-1}$ by property (ii) of a dyadic system, which implies $d(z_s^{k_s-1}, z_s^{k_s}) \leq C_0\delta^{k_s-1}$. Therefore, using the minimality of k_s, we obtain

$$E \subseteq 2c_d Q_s^{k_s-1} = B(z_s^{k_s-1}, 2c_d C_0 \delta^{k_s-1}) \subseteq B\left(z_s^{k_s}, \frac{3c_d^2}{\delta} \cdot C_0\delta^{k_s}\right) \subseteq \alpha Q_s^{k_s},$$

which finishes the proof. □

2.2. MAXIMAL OPERATORS

Let (S,d,μ) be a space of homogeneous type with a dyadic system $\mathscr{D}$ and take $q \in (0,\infty)$. We define the *Hardy–Littlewood maximal operator* M_q for an $f \in L^q_{\mathrm{loc}}(S)$ by

$$M_q f(s) := \sup_{B \ni s} \langle |f| \rangle_{q,B}, \qquad s \in S,$$

where the supremum is taken over all balls $B \subseteq S$ containing s. We define the *dyadic Hardy–Littlewood maximal operator* $M_q^{\mathscr{D}}$ by

$$M_q^{\mathscr{D}} f(s) := \sup_{Q \in \mathscr{D} : s \in Q} \langle |f| \rangle_{q,Q}, \qquad s \in S$$

When $q = 1$ we write $M := M_1$ and $M^{\mathscr{D}} := M_1^{\mathscr{D}}$ respectively. Obviously we have

$$M_q^{\mathscr{D}} f(s) \lesssim_{S,\mathscr{D},q} M_q f(s), \qquad s \in S.$$

Conversely, by Proposition 2.1.1 there are m dyadic systems $\mathscr{D}^1, \dots, \mathscr{D}^m$ such that

$$M_q f(s) \lesssim_{S,q} \sum_{j=1}^m M_q^{\mathscr{D}} f(s), \qquad s \in S. \tag{2.2.1}$$

The Hardy–Littlewood maximal operator satisfies the following bounds:

Proposition 2.2.1. *Let (S, d, μ) be a space of homogeneous type and $0 < q < p < \infty$. Then*

$$\begin{aligned} \|M_q f\|_{L^p(S)} &\lesssim_{S,p,q} \|f\|_{L^p(S)}, && f \in L^p(S), \\ \|M_q f\|_{L^{p,\infty}(S)} &\lesssim_{S,p,q} \|f\|_{L^{p,\infty}(S)}, && f \in L^{p,\infty}(S), \\ \|M_q f\|_{L^{q,\infty}(S)} &\lesssim_{S,q} \|f\|_{L^q(S)}, && f \in L^q(S). \end{aligned}$$

The case $q = 1$ for the dyadic Hardy–Littlewood maximal operator follow from Doob's maximal inequalities (see [HNVW16, Theorem 3.2.3]). The same estimates for the non-dyadic Hardy–Littlewood maximal operator then follow from (2.2.1). The case $q \neq 1$ follows by rescaling.

Remark 2.2.2. In $\mathbb{R}^n$ one can also consider the Hardy–Littlewood maximal operator over cubes, defined by

$$M^{\text{cubes}} f(t) := \sup_{Q \ni t} \langle |f| \rangle_{1,Q}, \qquad t \in \mathbb{R}^n,$$

where the supremum is taken over all cubes Q in $\mathbb{R}^n$ with sides parallel to the axes containing t. Then $M^{\text{cubes}} f(t)$ is pointwise comparable with $Mf(t)$

Let X be a Banach space. We define the *sharp maximal operator* for an $f \in L^1_{\text{loc}}(S; X)$ by

$$M^{\#} f(s) := \sup_{B \ni s} \fint_B \big\| f(t) - \langle f \rangle_{1,B} \big\|_X \, \mathrm{d}\mu(t), \qquad s \in S,$$

where the supremum is again taken over all balls $B \subseteq S$ containing s. Note that it is immediate from this definition that $M^{\#} f \leq 2M(\|f\|_X)$, so by Proposition 2.2.1 we have in particular that

$$\|M^{\#} f\|_{L^p(S)} \lesssim_{S,p} \|f\|_{L^p(S;X)}, \qquad f \in L^p(S; X).$$

There is a partial converse to this statement, which is known as the Fefferman-Stein inequality:

Proposition 2.2.3 (Feffermann-Stein). *Let (S,d,μ) be a space of homogeneous type, X be a Banach space, $1<p<\infty$ and $f\in L^p(S;X)$. Then*

$$\|M^\# f\|_{L^p(S)} \lesssim_{S,p} \|f\|_{L^p(S;X)} \lesssim_{S,p} \begin{cases} \|M^\# f\|_{L^p(S)}, & \mu(S)=\infty, \\ \|M^\# f\|_{L^p(S)} + \mu(S)^{-1/p'}\|f\|_{L^1(S)}, & \mu(S)<\infty. \end{cases}$$

For $X=\mathbb{C}$ the proof can be found in [Mar04, Proposition 3.1 and Theorem 4.2] or [DK18, Theorem 2.3]. The general case follows analogously replacing absolute values by norms.

Proposition 2.2.3 is not valid for $p=\infty$. In this case the space of all $f\in L^1_{\mathrm{loc}}(S;X)$ such that $M^\# f\in L^\infty(S;X)$ is strictly larger than $L^\infty(S;X)$. It includes all functions $f\in L^0(S;X)$ which have *bounded mean oscillation*. We define $\mathrm{BMO}(S;X)$ to be the space of all $f\in L^1_{\mathrm{loc}}(S;X)$ such that

$$\|f\|_{\mathrm{BMO}(S;X)} := \sup_B \inf_{c\in X} \fint_B \|f(s)-c\|_X \,\mathrm{d}\mu(s) < \infty$$

where the supremum is taken over all balls $B\subseteq S$. Note that $\|\cdot\|_{\mathrm{BMO}(S;X)}$ is only a seminorm, since $\|c\,\mathbf{1}_S\|_{\mathrm{BMO}(S;X)}=0$ for any $c\in X$. In analogy with Proposition 2.2.3 we have

$$\frac{1}{2}\|M^\# f\|_{L^\infty(S)} \le \|f\|_{\mathrm{BMO}(S;X)} \le \|M^\# f\|_{L^\infty(S)}.$$

We refer to [Gra14b, Chapter 3] for an introduction to BMO.

2.3. MUCKENHOUPT WEIGHTS

Let (S,μ) be a measure space. A *weight* is a function $w\colon S\to[0,\infty)$. For $p\in[1,\infty)$, a weight w and a Banach space X we let $L^p(S,w;X)$ be the subspace of all $f\in L^0(S;X)$ such that

$$\|f\|_{L^p(S,w;X)} := \Big(\int_S \|f\|_X^p\, w\,\mathrm{d}\mu\Big)^{1/p} < \infty.$$

If (S,d,μ) is a space of homogeneous type we will say that a locally integrable weight w lies in the *Muckenhoupt class* $A_p(S)$ and write $w\in A_p(S)$ if it satisfies

$$[w]_{A_p(S)} := \sup_B \langle w\rangle_{1,B}\langle w^{-1}\rangle_{\frac{1}{p-1},B} < \infty,$$

where the supremum is taken over all balls $B\subseteq S$ and the second factor is replaced by $(\operatorname{ess\,inf}_B w)^{-1}$ if $p=1$. We will omit the space of homogeneous type S in our notation if no confusion may arise.

Example 2.3.1 (Power weights). Let $\alpha\in(-n,\infty)$ and define $w:\mathbb{R}^n\to(0,\infty)$ by $w(x):=|x|^\alpha$. Then $w\in A_p(\mathbb{R}^n)$ if and only $\alpha\in(-n,n(p-1))$.

Let us note some basic properties of Muckenhoupt weights, the proofs of which can be found in [Gra14a, Chapter 7] in the Euclidean setting and carry over to spaces of homogeneous type (see also [HPR12]).

Proposition 2.3.2. *Let (S,d,μ) be a space of homogeneous type and let w be a weight.*

(i) Let $p\in(1,\infty)$. We have $w\in A_p$ if and only if $w^{-\frac{1}{p-1}}\in A_{p'}$ with $\big[w^{-\frac{1}{p-1}}\big]_{A_{p'}}=[w]_{A_p}^{\frac{1}{p-1}}$.

(ii) Let $p\in(1,\infty)$ and $q\in[1,p)$. For $w\in A_q$ we have $w\in A_p$ with $[w]_{A_p}\le[w]_{A_q}$.

(iii) Let $p\in(1,\infty)$. For all $w\in A_p$ there exists an $\varepsilon>0$ such that $w\in A_{p-\varepsilon}$.

(iv) Let $p\in[1,\infty)$. For all $w\in A_p$ there is a $\delta>0$ such that $w^{1+\delta}\in A_p$.

(v) For all $p\in(1,\infty)$ and all weights w we have

$$\|M\|_{L^p(S,w)\to L^p(S,w))}\lesssim_{S,p}[w]_{A_p}^{\frac{1}{p-1}}\lesssim_{S,p}\|M\|_{L^{p'}(S,w)\to L^p(S,w))}^{p'}$$

Related to property (i) in Proposition 2.3.2 we define the dual weight $w':=w^{-\frac{1}{p-1}}$ for $p\in(1,\infty)$. We then have $L^p(S,w)^*=L^{p'}(S,w'))$ under the duality pairing

$$\langle f,g\rangle=\int_S fg\,\mathrm{d}\mu,\qquad f\in L^p(S,w),\,g\in L^{p'}(S,w').$$

One of the most important features of the Muckenhoupt weight classes is the celebrated Rubio de Francia extrapolation theorem (see e.g. [GR85, Chapter IV]). This allows one to deduce estimates for *all* $p\in(1,\infty)$ and all $w\in A_p$ from the corresponding estimates for a *single* $p_0\in(1,\infty)$ and all $w\in A_{p_0}$. For a nice exposition of the proof in the Euclidean setting we refer to [CMP12]. The proof carries over directly to spaces of homogeneous type, see e.g. [DK18, Theorem 2.5].

Theorem 2.3.3 (Rubio de Francia extrapolation)**.** *Let (S,d,μ) a space of homogeneous type. Let $f,g\in L^0(S)$ and suppose that there is a $p_0\in(1,\infty)$ and a nondecreasing function $\phi\colon\mathbb{R}_+\to\mathbb{R}_+$ such that for all $w\in A_{p_0}$*

$$\|f\|_{L^{p_0}(S,w)}\le\phi\big([w]_{A_{p_0}}\big)\|g\|_{L^{p_0}(S,w)}\tag{2.3.1}$$

Then for all $p\in(1,\infty)$ there exists a nondecreasing $\psi:\mathbb{R}_+\to\mathbb{R}_+$, depending on S,ϕ,p,p_0, such that for all $w\in A_p$

$$\|f\|_{L^p(S,w)}\le\psi\big([w]_{A_p}\big)\|g\|_{L^p(S,w)}\tag{2.3.2}$$

Note that if (2.3.1) in Theorem 2.3.3 holds for a fixed function ϕ, all $w\in A_{p_0}$ and all $(f,g)\in\mathcal{F}$ for some $\mathcal{F}\subseteq L^0(S)\times L^0(S)$, then (2.3.2) also holds for a fixed ψ, all $w\in A_{p_0}$ and all $(f,g)\in\mathcal{F}$. For a further introduction to the theory of Muckenhoupt weights and Rubio de Francia extrapolation we refer to [Gra14a, Chapter 7] and [CMP12].

2.4. BANACH SPACE GEOMETRY

A random variable ε on a probability space $(\Omega,\mathbb{P})$ is called a *Rademacher* if it is uniformly distributed in $\{z\in\mathbb{C}:|z|=1\}$. A random variable γ on $(\Omega,\mathbb{P})$ is called a *Gaussian* if its distribution has density

$$f(z)=\frac{1}{\pi}\mathrm{e}^{-|z|^2},\qquad z\in\mathbb{C},$$

with respect to the Lebesgue measure on $\mathbb{C}$. A *Rademacher sequence* (respectively *Gaussian sequence*) is a sequence of independent Rademachers (respectively Gaussians). For all our purposes we could equivalently use real-valued Rademacher and Gaussians, see e.g. [HNVW17, Section 6.1.c].

Let X be a Banach space and let $(\varepsilon_k)_{k=1}^\infty$ be a Rademacher sequence and $(\gamma_k)_{k=1}^\infty$ a Gaussian sequence. For $p\in(0,\infty)$ and $x_1,\dots,x_n\in X$ the random sums

$$\Big\|\sum_{k=1}^n \varepsilon_k x_k\Big\|_{L^p(\Omega;X)}\qquad\text{and}\qquad\Big\|\sum_{k=1}^n \gamma_k x_k\Big\|_{L^p(\Omega;X)},$$

play a major role in the study of the geometry of X. It is immediate from Hölder's inequality that for $0<q<p$ we have

$$\Big\|\sum_{k=1}^n \varepsilon_k x_k\Big\|_{L^q(\Omega;X)}\le\Big\|\sum_{k=1}^n \varepsilon_k x_k\Big\|_{L^p(\Omega;X)}$$

and a similar estimate for Gaussian random sums. The converse of these inequalities are known as the Kahane-Khintchine inequalities. For the proof we refer to [HNVW17, Theorem 6.2.4 and 6.2.6].

Proposition 2.4.1 (Kahane-Khintchine inequalities)**.** *Let X be a Banach space and let $(\varepsilon_k)_{k=1}^\infty$ be a Rademacher sequence and $(\gamma_k)_{k=1}^\infty$ a Gaussian sequence. For $p,q\in(0,\infty)$ and $x_1,\dots,x_n\in X$ we have*

$$\Big\|\sum_{k=1}^n \varepsilon_k x_k\Big\|_{L^p(\Omega;X)}\lesssim_{p,q}\Big\|\sum_{k=1}^n \varepsilon_k x_k\Big\|_{L^q(\Omega;X)}$$
$$\Big\|\sum_{k=1}^n \gamma_k x_k\Big\|_{L^p(\Omega;X)}\lesssim_{p,q}\Big\|\sum_{k=1}^n \gamma_k x_k\Big\|_{L^q(\Omega;X)}$$

2.4.1. TYPE AND COTYPE

Let X be a Banach space and let $(\varepsilon_k)_{k=1}^\infty$ be a Rademacher sequence on a probability space $(\Omega,\mathbb{P})$. We say that X has *type* $p\in[1,2]$ if there exists a constant $C\ge 0$ such that for $x_1,\dots,x_n\in X$ we have

$$\Big\|\sum_{k=1}^n \varepsilon_k x_k\Big\|_{L^p(\Omega;X)}\le C\Big(\sum_{k=1}^n \|x_k\|_X^p\Big)^{1/p}. \tag{2.4.1}$$

We say that X has *cotype* $q \in [2,\infty]$ if there exists a constant $C \geq 0$ such that for $x_1,\dots,x_n \in X$ we have

$$\Big(\sum_{k=1}^n \|x_k\|_X^q\Big)^{1/q} \leq C \Big\|\sum_{k=1}^n \varepsilon_k x_k\Big\|_{L^q(\Omega;X)} \tag{2.4.2}$$

with the usual modification if $q=\infty$. The least admissible constants C will be denoted by $\tau_{p,X}$ and $c_{q,X}$ respectively. By Proposition 2.4.1 one may replace the $L^p(\Omega;X)$-norm in (2.4.1) and the $L^q(\Omega;X)$-norm in (2.4.2) by the $L^2(\Omega;X)$-norm if $q<\infty$.

Any Banach space has type 1 and cotype ∞. Moreover if X has type $p_0 \in [1,2]$ and cotype $q_0 \in [2,\infty]$, it also has type $p \in [1,p_0)$ and cotype $q \in (q_0,\infty]$. We say that X has *nontrivial type* if X has type $p \in (1,2]$ and finite cotype if X has cotype $q \in [2,\infty)$. Any space with nontrivial type has finite cotype (see [HNVW17, Theorem 7.1.14]).

Type and cotype are dual notions. For the proof of the following proposition we refer to [HNVW17, Proposition 7.1.3, 7.4.10 and 7.4.12].

Proposition 2.4.2. *Let X be a Banach space.*

(i) If X has type $p \in [1,2]$, then X^* has cotype p'.

(ii) If X has cotype $q \in [2,\infty]$ and nontrivial type, then X^* has type q'.

As examples we note that for $p \in [1,\infty)$ the Lebesgue spaces $L^p(\mathbb{R}^n)$ and Sobolev spaces $W^{k,p}(\mathbb{R}^n)$ have type $p\wedge 2$ and cotype $p\vee 2$. Any Hilbert space has type and cotype 2. Conversely, any Banach space with type and cotype 2 is isomorphic to a Hilbert space (see [HNVW17, Theorem 7.3.1]).

If X has finite cotype, Rademacher and Gaussian random sums are comparable. For the proof we refer to [HNVW17, Corollary 7.2.10]

Proposition 2.4.3. *Let X be a Banach space and let $(\varepsilon_k)_{k=1}^\infty$ be a Rademacher sequence and $(\gamma_k)_{k=1}^\infty$ a Gaussian sequence. For $p \in (0,\infty)$ and $x_1,\dots,x_n \in X$ we have*

$$\Big\|\sum_{k=1}^n \varepsilon_k x_k\Big\|_{L^p(\Omega;X)} \lesssim \Big\|\sum_{k=1}^n \gamma_k x_k\Big\|_{L^p(\Omega;X)},$$

If X has finite cotype, then

$$\Big\|\sum_{k=1}^n \gamma_k x_k\Big\|_{L^p(\Omega;X)} \lesssim_{X,p} \Big\|\sum_{k=1}^n \varepsilon_k x_k\Big\|_{L^p(\Omega;X)}.$$

As a direct consequence of Proposition 2.4.3 we note that (2.4.1) and (2.4.2) imply the same estimates with the Rademacher sequence replaced by a Gaussian sequence. For a further introduction to type and cotype we refer to [HNVW17, Chapter 7].

2.4.2. THE UMD PROPERTY

We say that a Banach space X has the *UMD property*, and write $X \in \mathrm{UMD}$, if the martingale difference sequence of any finite martingale in $L^p(S;X)$ on a σ-finite measure space (S,μ) is unconditional for some (equivalently all) $p \in (1,\infty)$. That is, if there exists a constant $C > 0$ such that for all finite martingales $(f_k)_{k=0}^n$ in $L^p(S;X)$ and scalars $|\epsilon_k| = 1$, we have

$$\Big\| \sum_{k=1}^n \epsilon_k d f_k \Big\|_{L^p(S;X)} \leq \Big\| \sum_{k=1}^n d f_k \Big\|_{L^p(S;X)}, \tag{2.4.3}$$

where $(df_k)_{k=1}^n$ denotes the difference sequence of $(f_k)_{k=0}^n$. The least admissible implicit constant in (2.4.3) will be denoted by $\beta_{p,X}$. It is equivalent to assume (2.4.3) only for Paley-Walsh martingales (see [HNVW16, Theorem 4.2.5]).

Any Banach space with the UMD property is reflexive, has nontrivial type and finite cotype (see [HNVW16, Theorem 4.3.3] and [HNVW17, Proposition 7.3.15]). Standard examples of Banach spaces with the UMD property include reflexive Lebesgue, Lorentz, (Musielak)-Orlicz, Sobolev, Bessel potential and Besov spaces.

We will also use randomized versions of the UMD property. We say that a Banach space X has the UMD^+ (respectively UMD^-) property if for some (equivalently all) $p \in (1,\infty)$ there exists a constant $\beta^+ > 0$ (respectively $\beta^- > 0$) such that for all finite martingales $(f_k)_{k=1}^n$ in $L^p(S;X)$ we have

$$\frac{1}{\beta^-} \Big\| \sum_{k=1}^n d f_k \Big\|_{L^p(S;X)} \leq \Big\| \sum_{k=1}^n \epsilon_k d f_k \Big\|_{L^p(S\times\Omega;X)} \leq \beta^+ \Big\| \sum_{k=1}^n d f_k \Big\|_{L^p(S;X)}, \tag{2.4.4}$$

where $(\epsilon_k)_{k=1}^n$ is a Rademacher sequence on $(\Omega,\mathbb{P})$. The least admissible constants in (2.4.4) will be denoted by $\beta_{p,X}^+$ and $\beta_{p,X}^-$. If (2.4.4) holds for Paley-Walsh martingales on a probability space S we say that X has the dyadic UMD^+ or UMD^- property respectively and denote the least admissible constants by $\beta_{p,X}^{\Delta,+}$ and $\beta_{p,X}^{\Delta,-}$. As for the UMD property, the (dyadic) UMD^+ and UMD^- properties are independent of $p \in (1,\infty)$ (see [Gar90]). However, in contrast to the situation for the UMD property, it is not clear whether the dyadic versions of the UMD^+ and UMD^- properties are equivalent to their non-dyadic counterparts. We do have that $\beta_{p,X}^{\Delta,+} \leq \beta_{p,X}^+$ and $\beta_{p,X}^{\Delta,-} \leq \beta_{p,X}^-$. Furthermore X has the UMD property if and only if it has the UMD^+ and UMD^- properties with

$$\max\{\beta_{p,X}^-, \beta_{p,X}^+\} \leq \beta_{p,X} \leq \beta_{p,X}^- \beta_{p,X}^+,$$

see [HNVW16, Proposition 4.1.16]. For a thorough introduction to the theory of UMD Banach spaces we refer the reader to [HNVW16, Pis16].

2.5. BANACH LATTICES AND FUNCTION SPACES

A partially ordered vector space X is called a *vector lattice* if any two elements $x, y \in X$ have a least upper bound $x \vee y$ and a greatest lower bound $x \wedge y$. A *Banach lattice* X is a

complete normed vector lattice such that order and norm are compatible, i.e.

$$|x| \leq |y| \Rightarrow \|x\|_X \leq \|y\|_X, \qquad x, y \in X,$$

where $|x| = x \vee -x$ for $x \in X$. We refer to [Mey91] or [Zaa67] for an introduction to Banach lattices.

On a Banach lattice we can compare Rademacher sums and Gaussian sums with square sums of the form

$$\Big(\sum_{k=1}^n |x_k|^2\Big)^{1/2}, \qquad x_1, \dots, x_n \in X,$$

which are defined through the Krivine calculus, see e.g. [LT79, Theorem 1.d.1]. For the proof of the following proposition we refer to [HNVW17, Proposition 7.2.13].

Proposition 2.5.1 (Khintchine–Maurey inequalities)**.** *Let X be a Banach lattice and let $(\varepsilon_k)_{k=1}^\infty$ be a Rademacher sequence on a probability space $(\Omega, \mathbb{P})$. For $p \in (0, \infty)$ and $x_1, \dots, x_n \in X$ we have*

$$\Big\| \Big(\sum_{k=1}^n |x_k|^2\Big)^{1/2} \Big\|_X \eqsim_p \Big\| \sum_{k=1}^n \varepsilon_k x_k \Big\|_{L^p(\Omega;X)},$$

If X has finite cotype $q \in [2, \infty)$, then

$$\Big\| \sum_{k=1}^n \varepsilon_k x_k \Big\|_{L^p(\Omega;X)} \eqsim_p \sqrt{q}\, c_{q,X} \Big\| \Big(\sum_{k=1}^n |x_k|^2\Big)^{1/2} \Big\|_X.$$

A class of Banach lattices that we will frequently use is the class of Banach function spaces. Let (Ω, μ) be a σ-finite measure space. A Banach lattice $X \subseteq L^0(\Omega)$ with the partial order given by $x \geq 0$ if and only if $x(\omega) \geq 0$ for a.e. $\omega \in \Omega$ is called a *Banach function space* if it satisfies the following two additional properties

- *Weak order unit:* There is an $x \in X$ with $x(\omega) > 0$ for a.e. $\omega \in \Omega$.
- *Fatou property:* If $0 \leq x_n \uparrow x$ for $(x_n)_{n=1}^\infty$ in X and $\sup_{n \in \mathbb{N}} \|x_n\|_X < \infty$, then $x \in X$ and $\|x\|_X = \sup_{n \in \mathbb{N}} \|x_n\|_X$.

A Banach function space X is called *order-continuous* if for any sequence satisfying $0 \leq x_n \uparrow x \in X$ we have $\|x_n - x\|_X \to 0$. As an example we note that all reflexive Banach function spaces are order-continuous (see e.g. [Mey91, Section 2.4]). Order-continuity of X ensures that its the dual X^* is also a Banach function space. In this case the duality pairing is given by

$$\langle x, x^* \rangle = \int_\Omega x(\omega) \cdot x^*(\omega)\, \mathrm{d}\mu(\omega), \qquad x \in X,\, x^* \in X^*.$$

For a Banach function space X and a Banach space Y we write $X(Y)$ for the *Köthe-Bochner space* of all $f \in L^0(\Omega; Y)$ such that $\omega \mapsto \|f(\omega)\|_Y \in X$ and define its norm by

$$\|f\|_{X(Y)} := \big\| \omega \mapsto \|f(\omega)\|_Y \big\|_X.$$

For an introduction to Banach function spaces we refer the reader to [LT79, Section 1.b] or [BS88].

2.5.1. p-CONVEXITY AND q-CONCAVITY

On a Banach lattice X, the notions p-convexity and q-concavity are closely related to type and cotype. We say X is *p-convex* with $p \in [1,\infty]$ if for $x_1,\dots,x_n \in X$

$$\Big\|\Big(\sum_{k=1}^n |x_k|^p\Big)^{1/p}\Big\|_X \lesssim_{X,p} \Big(\sum_{k=1}^n \|x_k\|_X^p\Big)^{1/p}$$

and X is called *q-concave* with $q \in [1,\infty]$ if for $x_1,\dots,x_n \in X$

$$\Big(\sum_{k=1}^n \|x_k\|_X^q\Big)^{1/q} \lesssim_{X,q} \Big\|\Big(\sum_{k=1}^n |x_k|^q\Big)^{1/q}\Big\|_X.$$

By renorming we may assume without loss of generality that the implicit constants are equal to 1 (see [LT79, Theorem 1.d.8].

Any Banach lattice is 1-convex and ∞-concave and if X is p-concave and q-concave for $1 \le p \le q \le \infty$, then it is p_0-convex and q_0-concave for $p_0 \in [1,p)$ and $q_0 \in (q,\infty]$. The duality of p-convexity and q-concavity is simpler than the duality of type and cotype, for the proof we refer to [LT79, Theorem 1.d.4].

Proposition 2.5.2. *Let X be a Banach lattice and $p,q \in [1,\infty]$. If X is p-convex and q-concave, then X^* is q'-convex and p'-concave.*

The connection between type, cotype, p-convexity and q-concavity is captured in the following proposition. For the proof we refer to [LT79, Proposition 1.f.3 and Corollary 1.f.9].

Proposition 2.5.3. *Let X be a Banach lattice and $1 < p < r < q < \infty$.*

(i) If X has type r, then it is p-convex.

(ii) If X is p-convex and q-concave, then it has type $p \wedge 2$.

(iii) If X has cotype r, then it is q-concave.

(iv) If X is q-concave, then it has cotype $q \vee 2$.

If X is p-convex Banach function space for some $p \in [1,\infty]$, we can define its p-concavification X^p by

$$X^p = \{|x|^p \operatorname{sgn} x : x \in X\} = \{x : |x|^{1/p} \in X\}$$

with norm

$$\|x\|_{X^p} = \big\||x|^{1/p}\big\|_X^p.$$

This definition can be extended to Banach lattices, see [LT79, Section 1.d]. In Chapter 7 we will often use p-convex Banach function spaces such that X^p has the UMD property, which we will abbreviate as $X^p \in \mathrm{UMD}$. We refer to [LT79, Chapter 1] for a further introduction to p-convexity and q-concavity.

2.6. $\mathcal{R}$- AND ℓ^r-BOUNDEDNESS

For a family of bounded operators Γ from a Banach space X to a Banach space Y, we will use the notions $\mathcal{R}$- and ℓ^r-boundedness, which are both a strengthening of uniform boundedness for Γ.

Definition 2.6.1. Let X and Y be Banach spaces and $\Gamma \subseteq \mathcal{L}(X,Y)$.

- Let $(\varepsilon_k)_{k=1}^\infty$ be a Rademacher sequence on a probability space $(\Omega,\mathbb{P})$. We say that Γ is $\mathcal{R}$-*bounded* if for all $T_1,\cdots,T_n \in \Gamma$ and $x_1,\cdots,x_n \in X$,

$$\Big\|\sum_{k=1}^n \varepsilon_k T_k x_k\Big\|_{L^2(\Omega;Y)} \lesssim \Big\|\sum_{k=1}^n \varepsilon_k x_k\Big\|_{L^2(\Omega;X)}.$$

  The least admissible implicit constant is called the $\mathcal{R}$-bound of Γ and is denoted by $\|\Gamma\|_{\mathcal{R}}$.

- If X and Y are Banach lattices, we say that Γ is ℓ^r-*bounded* for $r \in [1,\infty]$ if for all $T_1,\cdots,T_n \in \Gamma$ and $x_1,\cdots,x_n \in X$,

$$\Big\|\Big(\sum_{k=1}^n |T_k x_k|^r\Big)^{1/r}\Big\|_Y \lesssim \Big\|\Big(\sum_{k=1}^n |x_k|^r\Big)^{1/r}\Big\|_X.$$

  The least admissible implicit constant is called the ℓ^r-bound of Γ, and is denoted by $\|\Gamma\|_{\ell^r}$.

For $\mathcal{R}$- and ℓ^2-boundedness it suffices to consider *subsets* of Γ in the defining inequality (see [CPSW00, KVW16]). For ℓ^r-boundedness with $r \neq 2$ this is not the case: one must allow repeated elements. A singleton $\{T\}$ can fail to be ℓ^r-bounded, as the defining estimate may fail for arbitrarily long constant sequences $(T,\dots,T)$ (see [KU14, Example 2.16]).

If a set $\Gamma \subseteq \mathcal{L}(X,Y)$ is $\mathcal{R}$- or ℓ^r-, then so is its closure in the strong operator topology, and likewise its absolutely convex hull $\operatorname{absco}(\Gamma)$. This was proven in [KW04] for $\mathcal{R}$-boundedness and [KU14] for ℓ^r-boundedness. Moreover, if $\Gamma_1,\Gamma_2 \subseteq \mathcal{L}(X)$ are $\mathcal{R}$ or ℓ^2-bounded respectively, then $\Gamma_1 \cup \Gamma_2$ is $\mathcal{R}-$ or ℓ^2-bounded respectively. For duality we have the following result, for the proof of (i) we refer to [HNVW17, Proposition 8.4.1] and for (ii) follows from the duality $X(\ell^r_n)^* = X^*(\ell^{r'}_n)$ (see [LT79, Section 1.d.]).

Proposition 2.6.2. *Let X and Y be Banach spaces and let $\Gamma \subseteq \mathcal{L}(X,Y)$. Define the adjoint family $\Gamma^* := \{T^* : T \in \Gamma\} \subseteq \mathcal{L}(X^*,Y^*)$.*

(i) If X has nontrivial type and Γ is $\mathcal{R}$-bounded, then Γ^* is $\mathcal{R}$-bounded.

(ii) If X and Y are Banach lattices and Γ is ℓ^r-bounded for some $r \in [1,\infty]$, then Γ^* is $\ell^{r'}$-bounded.

If X has cotype 2 and Y has type 2, then it is direct from the definitions that any uniformly bounded $\Gamma \subseteq \mathcal{L}(X,Y)$ is $\mathcal{R}$-bounded with

$$\|\Gamma\|_{\mathcal{R}} \leq c_{X,2}\tau_{Y,2}\sup_{T\in\Gamma}\|T\|.$$

The converse is also true, i.e. if every uniformly bounded $\Gamma \subseteq \mathcal{L}(X,Y)$ is $\mathcal{R}$-bounded, then X has cotype 2 and Y has type 2 (see [HNVW17, Proposition 8.6.1]). In particular, in the case $X = Y$, we have that $\mathcal{R}$-boundedness coincides with uniform boundedness if and only if X is isomorphic with a Hilbert space. Similar statements can be made for ℓ^r-boundedness and r-convex and r-concave Banach lattices.

If X and Y are Banach lattices and X and Y have finite cotype, then $\mathcal{R}$- and ℓ^2-boundedness are equivalent by Proposition 2.5.1.

Proposition 2.6.3. *Let X and Y be Banach lattices and let $\Gamma \subseteq \mathcal{L}(X,Y)$.*

- *If X has finite cotype $q \in [2,\infty)$ and Γ is $\mathcal{R}$-bounded, then Γ is ℓ^2-bounded with*

$$\|\Gamma\|_{\ell^2} \lesssim \sqrt{q}c_{q,X}\|\Gamma\|_{\mathcal{R}}.$$

- *If Y has finite cotype $q \in [2,\infty)$ and Γ is ℓ^2-bounded, then Γ is $\mathcal{R}$-bounded with*

$$\|\Gamma\|_{\mathcal{R}} \lesssim \sqrt{q}c_{q,Y}\|\Gamma\|_{\ell^2}.$$

For a thorough discussion on the connection between $\mathcal{R}$ and ℓ^2-boundedness we refer to [KVW16]. For a further introduction to $\mathcal{R}$-boundedness we refer the reader to [HNVW17, KW04], and for a further introduction to ℓ^r-boundedness see [KU14, Wei01a].

2.7. FOURIER MULTIPLIERS

In this section we will introduce operator-valued Fourier multiplier theory. For a detailed historical description of vector-valued and operator-valued Fourier multiplier theory we refer to [HNVW16, HNVW17] and for an introduction to scalar-valued Fourier multiplier theory we refer to [Gra14a, Chapter 6].

Let X be a Banach space. The Fourier transform on Bochner spaces is defined similarly to the scalar-valued case, i.e. for $f \in L^1(\mathbb{R}^n;X)$ we define

$$\widehat{f}(\xi) = \mathcal{F}f(\xi) := \int_{\mathbb{R}^n} f(s)\mathrm{e}^{-2\pi i s\cdot\xi}\,\mathrm{d}s, \qquad \xi \in \mathbb{R}^n.$$
$$\check{f}(\xi) = \mathcal{F}^{-1}f(\xi) := \int_{\mathbb{R}^n} f(s)\mathrm{e}^{2\pi i s\cdot\xi}\,\mathrm{d}s, \qquad \xi \in \mathbb{R}^n.$$

We denote the space of X-valued Schwartz functions by $\mathcal{S}(\mathbb{R}^n;X)$ and the space of X-valued tempered distributions by $\mathcal{S}'(\mathbb{R}^n;X) := \mathcal{L}(\mathcal{S}(\mathbb{R}^n);X)$. The space $\mathcal{S}(\mathbb{R}^n;X)$ is dense in $L^p(\mathbb{R}^n,w;X)$ (see [Gra14a, Exercise 7.4.1]) and $L^p(\mathbb{R}^n,w;X)$ is continuously embedded in $\mathcal{S}'(\mathbb{R}^n;X)$ for $w \in A_p$ and $p \in [1,\infty)$.

Let X and Y be Banach spaces. To a bounded, strongly measurable $m\colon \mathbb{R}^n \to \mathcal{L}(X,Y)$ we associate the Fourier multiplier operator

$$T_m\colon \mathcal{S}(\mathbb{R}^n;X) \to \mathcal{S}'(\mathbb{R}^n;Y), \qquad T_m f = (m\widehat{f})^\vee.$$

One may ask under which conditions on m the operator T_m extends to a bounded operator from $L^p(\mathbb{R}^n,w;X)$ to $L^p(\mathbb{R}^n,w;Y)$ for $L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and $p\in[1,\infty)$. If this is the case we call m a bounded Fourier multiplier and T_m a Fourier multiplier operator.

The UMD property is intimately connected to the boundedness of Fourier multiplier operators. Indeed, for prototypical examples of Fourier multiplier operators like the Hilbert transform

$$Hf := \big(\xi \mapsto -i\operatorname{sgn}(\xi)\cdot \widehat{f}(\xi)\big)^\vee = \frac{1}{\pi}\,\mathrm{p.v.}\int_{\mathbb{R}} \frac{f(t)}{s-t}\,\mathrm{d}t, \qquad s\in\mathbb{R},\ f\in\mathcal{S}(\mathbb{R};X)$$

and the Riesz projections for $1,\cdots,k$

$$R_k f(s) := \big(\xi\mapsto -i\frac{\xi_k}{|\xi|}\cdot\widehat{f}(\xi)\big)^\vee(s) = c_n\cdot \mathrm{p.v.}\int_{\mathbb{R}^n}\frac{s_j-t_j}{|s-t|^{n+1}}f(t)\,\mathrm{d}t, \qquad s\in\mathbb{R}^n, f\in\mathcal{S}(\mathbb{R}^n;X),$$

we have the following result, which for the Hilbert transform was first proven by Burkholder [Bur83] and Bourgain [Bou83].

Theorem 2.7.1. *Let X be a Banach space. The following are equivalent:*

(i) $X\in$ UMD.

(ii) *The Hilbert transform H is bounded on $L^p(\mathbb{R};X)$ for some (all) $p\in(1,\infty)$.*

(iii) *The Riesz projections R_k for $k=1,\dots,n$ are bounded on $L^p(\mathbb{R}^n;X)$ for some (all) $p\in(1,\infty)$.*

Another major breakthrough was given in [McC84], [Bou86] and [Zim89], where the Marcinkiewicz–Mihlin multiplier theorem and Littlewood–Paley decomposition have been obtained on $L^p(\mathbb{R}^n;X)$ for UMD Banach spaces X and $p\in(1,\infty)$.

A necessary condition for boundedness of an operator-valued Fourier multiplier $m\colon\mathbb{R}^n\to\mathcal{L}(X,Y)$ is that the range of m is $\mathcal{R}$-bounded. Following the breakthrough papers [Wei01b, CPSW00] there has been an extensive study of operator-valued multiplier theory, in which $\mathcal{R}$-boundedness techniques are central, see e.g. [AB02, HHN02, ŠW07, Wei01b]. In Section 3.5 we will discuss a generalization of the operator-valued Mihlin multiplier theorem and in Chapter 7 we will develop a generalization of the operator-valued Marcinkiewicz multiplier theorem.

2.8. γ-RADONIFYING OPERATORS

We introduce the definition and some basic properties of γ-radonifying operators, for details we refer to [HNVW17, Chapter 9]. Let X be a Banach space and H be a Hilbert

space. We say that an operator $T \in \mathcal{L}(H,X)$ is γ-summing and write $T \in \gamma_\infty(H,X)$ if

$$\|T\|_{\gamma_\infty(H,X)} := \sup \Big\| \sum_{k=1}^n \gamma_k T\varphi_k \Big\|_{L^2(\Omega;X)} < \infty,$$

where $(\gamma_k)_{k=1}^n$ is a Gaussian sequence on a probability space $(\Omega,\mathbb{P})$ and the supremum is taken over all finite orthonormal systems $(\varphi_k)_{k=1}^n$ in H. Any finite rank operator T, i.e. $T = \sum_{k=1}^n e_k \otimes x_k$, with $e_1,\dots,e_n \in H$ and $x_1,\dots,x_n \in X$, belongs to $\gamma_\infty(H,X)$. We denote the closure of the finite rank operators in $\gamma_\infty(H,X)$ by $\gamma(H,X)$. We thus have $\gamma(H,X) \hookrightarrow \gamma_\infty(H,X) \hookrightarrow \mathcal{L}(H,X)$.

When X does not contain a subspace isomorphic to c_0, so in particular if X has finite cotype, we have $\gamma_\infty(H,X) = \gamma(H,X)$. When H is separable with orthonormal basis $(\varphi_k)_{k=1}^\infty$ we have

$$\|T\|_{\gamma_\infty(H,X)} = \sup_{n\in\mathbb{N}} \Big\| \sum_{k=1}^n \gamma_k T\varphi_k \Big\|_{L^2(\Omega;X)}$$

and if $T \in \gamma(H,X)$, then $\sum_{k=1}^\infty \gamma_k T\varphi_k$ converges in $L^2(\Omega;X)$ and we have

$$\|T\|_{\gamma(H,X)} = \Big\| \sum_{k=1}^\infty \gamma_k T\varphi_k \Big\|_{L^2(\Omega;X)}.$$

The spaces $\gamma(H,X)$ satisfy the following domination property:

Proposition 2.8.1 (Domination)**.** *Let X be a Banach space and let H_1 and H_2 be Hilbert spaces. If $T_1 \in \gamma(H_1,X)$ and $T_2 \in \mathcal{L}(H_2,X)$ with*

$$\|T_2^* x^*\|_{H_2} \le \|T_1^* x^*\|_{H_1}, \qquad x^* \in X^*,$$

then $T_2 \in \gamma(H_2,X)$ with $\|T_2\|_{\gamma(H_2,X)} \le \|T_1\|_{\gamma(H_1,X)}$.

We also have γ-versions of Fatou's lemma and the dominated convergence theorem, which we state next. For the γ-Fatou lemma we assume finite cotype in order to avoid ending up in $\gamma_\infty(H,X)$.

Proposition 2.8.2 (γ-Fatou)**.** *Let X be a Banach space with finite cotype and H a Hilbert space. If $(T_n)_{n=1}^\infty$ is a bounded sequence in $\gamma(H,X)$ and $T \in \mathcal{L}(H,X)$ with*

$$\lim_{n\to\infty} \langle T_n\varphi, x^*\rangle = \langle T\varphi, x^*\rangle, \qquad \varphi \in H,\ x^* \in X^*,$$

then $T \in \gamma(H,X)$ with $\|T\|_{\gamma(H,X)} \le \liminf_{n\to\infty} \|T_n\|_{\gamma(H,X)}$.

Proposition 2.8.3 (γ-Dominated convergence)**.** *Let X be a Banach space and H a Hilbert space. Let $(T_n)_{n=1}^\infty$ be a sequence in $\mathcal{L}(H,X)$ and $T \in \mathcal{L}(H,X)$ such that $\lim_{n\to\infty} T_n^* x^* = T^* x^*$ for all $x^* \in X^*$. If there exists a $U \in \gamma(H,X)$ such that for $n \in \mathbb{N}$*

$$\|T_n^* x^*\|_H \le \|U^* x^*\|_H, \qquad x^* \in X^*,$$

then $T_n, T \in \gamma(H,X)$ and $T_n \to T$ in $\gamma(H,X)$.

For a measure space (S,μ), we write $\gamma(S;H,X) := \gamma(L^2(S;H),X)$ and in particular $\gamma(S;X) := \gamma(L^2(S),X)$. Any strongly measurable $f\colon S \to X$ for which $\langle f, x^*\rangle \in L^2(S)$ for all $x^* \in X^*$ defines a bounded linear operator $T_f\colon L^2(S) \to X$ by

$$T_f\varphi := \int_S f\varphi \,\mathrm{d}\mu, \qquad \varphi \in L^2(S),$$

where the integral is well-defined in the Pettis sense (see [HNVW16, Theorem 1.2.37]). If $T_f \in \gamma(S;X)$ we say that f represents T_f and write $f \in \gamma(S;X)$.

For a Hilbert space K we have $\gamma(S;K) = L^2(S;K)$ isometrically (see [HNVW17, Theorem 9.2.10]). More generally, if the Banach space X has type 2 we have the following embedding properties for the γ-spaces, which follow directly from [HNVW17, Theorem 9.2.10 and Proposition 7.1.20].

Lemma 2.8.4. *Let X be a Banach space with type 2, H a Hilbert space and (S,μ) a σ-finite measure space. Then we have the following embeddings*

$$L^2(S;\gamma(H;X)) \hookrightarrow \gamma(S;\gamma(H;X)) \hookrightarrow \gamma(S;H,X)$$

with both embedding constants bounded by $\tau_{2,X}$.

We also note that for disjointly supported functions we have the following square function estimate, which follows from [HNVW17, Proposition 9.4.13].

Lemma 2.8.5. *Let X be a Banach space with type 2 and let $f_1,\dots,f_n \in \gamma(S;X)$ be disjointly supported. Then we have*

$$\Big\|\sum_{k=1}^n f_k\Big\|_{\gamma(S;X)} \le \tau_{2,X}\Big(\sum_{k=1}^n \|f_k\|_{\gamma(S;X)}^2\Big)^{1/2}.$$

To conclude the introduction of γ-radonifying operators we extend the γ-Fubini theorem for Lebesgue spaces in [HNVW17, Theorem 9.4.8] to Banach function spaces.

Proposition 2.8.6 (γ-Fubini)**.** *Let X be a Banach space, H a Hilbert space and let E be a Banach function space over a measure space (S,μ). Then we have the embedding*

$$\gamma(H,E(X)) \hookrightarrow E(\gamma(H,X)).$$

If in addition E is q-concave for some $q \in [1,\infty)$, then we have the embedding

$$E(\gamma(H,X)) \hookrightarrow \gamma(H,E(X)).$$

Proof. We make two preliminary observations. Since E is a Banach space, the triangle inequality in E implies that for all simple functions $\xi : \Omega \to E$

$$\|\xi\|_{E(L^1(\Omega))} \le \|\xi\|_{L^1(\Omega;E)}. \tag{2.8.1}$$

By density this extends to a contractive embedding $L^1(\Omega;E) \hookrightarrow E(L^1(\Omega))$. The second observation is that if E is q-concave for some $q \in [1,\infty)$, then we have for all simple functions $\xi : S \to L^q(\Omega)$,

$$\|\xi\|_{L^q(\Omega;E)} \lesssim_{X,q} \|\xi\|_{E(L^q(\Omega))}. \tag{2.8.2}$$

By density this can be extended to a contractive embedding $E(L^q(\Omega)) \hookrightarrow L^q(\Omega;E)$.

Let $(h_j)_{j=1}^n$ be an orthonormal system in H and let $f = \sum_{j=1}^n h_j \otimes \xi_j$ with $\xi_j \in E(X)$. Now setting $\xi = \big\|\sum_{j=1}^n \gamma_j \xi_j\big\|_X$, where $(\gamma_j)_{j=1}^n$ is a Gaussian sequence, we can write

$$\|f\|_{\gamma(H,E(X))} = \Big\|\sum_{j=1}^n \gamma_j \xi_j\Big\|_{L^2(\Omega;E(X))} = \|\xi\|_{L^2(\Omega;E)},$$
$$\|f\|_{E(\gamma(H,X))} = \Big\|\sum_{j=1}^n \gamma_j \xi_j\Big\|_{E(L^2(\Omega;X))} = \|\xi\|_{E(L^2(\Omega))}$$

By the Kahane–Khintchine inequalities (see Proposition 2.4.1) replacing the $L^2(\Omega)$-norm on the right-hand sides of the above identities with $L^r(\Omega)$ with $r \in [1,\infty)$ leads to an equivalent norm. Taking $r = 1$ we have by (2.8.1) that

$$\|f\|_{E(\gamma(H,X))} \lesssim \|f\|_{\gamma(H,E(X))},$$

which by density proves $\gamma(H,E(X)) \hookrightarrow E(\gamma(H,X))$.

For the second embedding note that by the above with $r = q$ we find by (2.8.2) that

$$\|f\|_{\gamma(H,E(X))} \lesssim_{X,q} \|f\|_{E(\gamma(H,X))}.$$

Again by density this gives $E(\gamma(H,X)) \hookrightarrow \gamma(H,E(X))$. □

Remark 2.8.7. The result of Proposition 2.8.6 can also be extended to quasi-Banach function spaces which are p-convex and q-concave. For the definition of $\gamma(H,X)$ for quasi-Banach spaces we refer to [CCV18].

2.9. STOCHASTIC INTEGRATION IN BANACH SPACES

The γ-radonifying operators play a pivotal role in the development of stochastic integration in Banach spaces, which we will introduce now. For details of the introduced notions we refer to [NVW07, NVW15c].

Let X be a Banach space and H a Hilbert space. Let $(\Omega,\mathcal{A},\mathbb{P})$ be a probability space with filtration $(\mathscr{F}_t)_{t\geq 0}$. Functions $G\colon \Omega \times (0,T) \to \mathcal{L}(H,X)$ of the form

$$G = \mathbf{1}_{(a,b]} \otimes h \otimes \xi, \tag{2.9.1}$$

where $0 \leq a < b < \infty$, $h \in H$ and $\xi \in L^\infty(\Omega;X)$ is strongly $\mathscr{F}_a$-measurable, are called a *rank-one adapted step processes* and functions in the linear span of the rank-one adapted

step processes are called *finite rank adapted step processes*. For $p \in [1,\infty)$, $T \in (0,\infty]$ and a weight w on $(0,T)$ we let

$$L^p_{\mathscr{F}}(\Omega;\gamma((0,T);H,X)),$$
$$L^p_{\mathscr{F}}(\Omega\times(0,T),w;\gamma(H,X))$$

denote the closure of the finite rank adapted step processes $G\colon \Omega\times(0,T)\to\mathcal{L}(H,X)$ in $L^p(\Omega;\gamma((0,T);H,X))$ and $L^p(\Omega\times(0,T),w;\gamma(H,X))$ respectively. We omit the weight w if $w\equiv 1$. One has that $f\in L^p_{\mathscr{F}}(\Omega;\gamma((0,T);H,X))$ if and only if $f\in L^p(\Omega;\gamma((0,T);H,X))$ and $f(\mathbf{1}_{[0,t]}\otimes h)$ is strongly $\mathscr{F}_t$-measurable and for all $t\in(0,T)$ and $h\in H$.

Let (S,μ) be a measure space, take $p,q,r\in[1,\infty)$, $T\in(0,\infty]$, let a v be weight on $(0,T)$ and w a weight on S. For the specific case that $X=L^q(S;w)$ we let

$$L^r_{\mathscr{F}}(\Omega;L^q(S,w;L^2((0,T);H))),$$
$$L^r_{\mathscr{F}}(\Omega;L^p((0,T),v;L^q(S,w;H))),$$
$$L^r_{\mathscr{F}}(\Omega;L^q(S,w;L^p((0,T),v;H)))$$

denote the closure of the finite rank adapted step processes in the respective spaces, where we omit the weights if $v,w\equiv 1$. In the case $p=r$ there is some overlap with the definitions for abstract Banach spaces X by the γ-Fubini theorem in Proposition 2.8.6 and the identification

$$\gamma((0,T);H)=L^2((0,T);H).$$

Let $\mathcal{W}\in\mathcal{L}(L^2(\mathbb{R}_+;H),L^2(\Omega))$ denote an *isonormal mapping* (see [Kal02]) such that $\mathcal{W}f$ is $\mathscr{F}_t$-measurable if $f\in L^2(\mathbb{R}_+;H)$ with $f=0$ on (t,∞). Define a *cylindrical Brownian motion* $(W_H(t))_{t\geq 0}$ by $W_H(t)h:=\mathcal{W}(\mathbf{1}_{[0,t]}h)$. For a rank-one adapted step process $G\colon\Omega\times(0,T)\to\mathcal{L}(H,X)$ as in (2.9.1) we define

$$\int_0^s G(t)\,\mathrm{d}W_H(t):=(W_H(b\wedge s)-W_H(a\wedge s))h\otimes\xi\qquad s\in\mathbb{R}_+,$$

which is an element of $L^p(\Omega;X)$. We extend the definition of the stochastic integral by linearity.

The following result provides two-sided estimates for the stochastic integral with respect to a H-cylindrical Brownian motion $(W_H(t))_{t\geq 0}$. This in particular allows us to define the stochastic integral $\int_0^T f(t)\,\mathrm{d}W_H(t)$ for $f\in L^p_{\mathscr{F}}(\Omega;\gamma(0,T;H,X))$.

Theorem 2.9.1 (Itô isomorphism)**.** *Let X be a* UMD *Banach space, H a Hilbert space, let $p\in(1,\infty)$ and $T\in(0,\infty]$. For every adapted finite rank step process $G:(0,T)\times\Omega\to\mathcal{L}(H,X)$, one has*

$$\Big\|\int_0^T G(t)\,\mathrm{d}W_H(t)\Big\|_{L^p(\Omega;X)}\eqsim_{p,X}\|G\|_{L^p(\Omega;\gamma(0,T;H,X))}.$$

In particular, $G\mapsto\int_0^T G(t)\,\mathrm{d}W_H(t)$ extends to an isomorphism from $L^p_{\mathscr{F}}(\Omega;\gamma((0,T);H,X))$ to $L^p(\Omega;X)$.

I

VECTOR-VALUED HARMONIC ANALYSIS FOR SPDE

3

ℓ^r-SPARSE DOMINATION IN A SPACE OF HOMOGENEOUS TYPE

This chapter is based on the paper

[5] E. Lorist. On pointwise ℓ^r-sparse domination in a space of homogeneous type. *J. Geom. Anal.*, 2020. Online first.

It is complemented by a few additional applications. In particular, the application to Littlewood–Paley operators in Section 3.7 and the application to the unconditionality of the Haar decomposition in Section 3.8 are unpublished.

Abstract. *We prove a general sparse domination theorem in a space of homogeneous type, in which a vector-valued operator is controlled pointwise by a positive, local expression called a sparse operator. We use the structure of the operator to get sparse domination in which the usual ℓ^1-sum in the sparse operator is replaced by an ℓ^r-sum.*

This sparse domination theorem is applicable to various operators from both harmonic analysis and (S)PDE. Using our main theorem, we prove the A_2-theorem for vector-valued Calderón–Zygmund operators in a space of homogeneous type, from which we deduce an anisotropic, mixed norm Mihlin multiplier theorem. Furthermore we show quantitative weighted norm inequalities for Littlewood–Paley operators and the Rademacher maximal operator. In the latter application the geometry of the underlying Banach space plays a major role. Applications to (S)PDE will be given in Chapters 4 and 5.

3.1. INTRODUCTION

The technique of controlling various operators by so-called sparse operators has proven to be a very useful tool to obtain (sharp) weighted norm inequalities in the past decade. The key feature in this approach is that a typically signed and non-local operator is dominated, either in norm, pointwise or in dual form, by a positive and local expression.

The sparse domination technique comes from Lerner's work towards an alternative proof of the A_2-theorem, which was first proven by Hytönen in [Hyt12]. In [Ler13] Lerner applied his local mean oscillation decomposition approach to the A_2-theorem, estimating the norm of a Calderón-Zygmund operator by the norm of a sparse operator. This was later improved to a pointwise estimate independently by Conde-Alonso and Rey [CR16] and by Lerner and Nazarov [LN18]. Afterwards, Lacey [Lac17] obtained the same result for a slightly larger class of Calderón-Zygmund operators by a stopping cube argument instead of the local mean oscillation decomposition approach. This argument was further refined by Hytönen, Roncal and Tapiola [HRT17] and afterwards made strikingly clear by Lerner [Ler16], where the following abstract sparse domination principle was shown:

If T is a bounded sublinear operator from $L^{p_1}(\mathbb{R}^n)$ to $L^{p_1,\infty}(\mathbb{R}^n)$ and the *grand maximal truncation operator*

$$\mathcal{M}_T f(s) := \sup_{Q \ni s} \operatorname*{ess\,sup}_{s' \in Q} |T(f \mathbf{1}_{\mathbb{R}^n \setminus 3Q})(s')|, \qquad s \in \mathbb{R}^n$$

is bounded from $L^{p_2}(\mathbb{R}^n)$ to $L^{p_2,\infty}(\mathbb{R}^n)$ for some $1 \le p_1, p_2 < \infty$, then there is an $\eta \in (0,1)$ such that for every compactly supported $f \in L^{p_0}(\mathbb{R}^n)$ with $p_0 := \max\{p_1, p_2\}$ there exists an η-sparse family of cubes $\mathcal{S}$ such that

$$|Tf(s)| \lesssim \sum_{Q \in \mathcal{S}} \langle |f| \rangle_{p_0,Q} \mathbf{1}_Q(s), \qquad s \in \mathbb{R}^n. \tag{3.1.1}$$

We call a family of cubes $\mathcal{S}$ η-sparse if for every $Q \in \mathcal{S}$ there exists a measurable set $E_Q \subseteq Q$ such that $|E_Q| \ge \eta |Q|$ and such that the E_Q's are pairwise disjoint.

This sparse domination principle was further generalized in the recent paper [LO20] by Lerner and Ombrosi, in which the authors showed that the weak L^{p_2}-boundedness of the more flexible operator

$$\mathcal{M}^{\#}_{T,\alpha} f(s) := \sup_{Q \ni s} \operatorname*{ess\,sup}_{s', s'' \in Q} |T(f \mathbf{1}_{\mathbb{R}^n \setminus \alpha Q})(s') - T(f \mathbf{1}_{\mathbb{R}^n \setminus \alpha Q})(s'')|, \qquad s \in \mathbb{R}^n$$

for some $\alpha \ge 3$ is already enough to deduce the pointwise sparse domination as in (3.1.1). Furthermore they relaxed the weak L^{p_1}-boundedness condition on T to a condition in the spirit of the $T(1)$-theorem.

3.1.1. MAIN RESULT

Our main result is a generalization of the main result in [LO20] in the following four directions:

(i) We replace $\mathbb{R}^n$ by a space of homogeneous type (S,d,μ).

(ii) We let T be an operator from $L^{p_1}(S;X)$ to $L^{p_1,\infty}(S;Y)$, where X and Y are Banach spaces.

(iii) We use structure of the operator T and geometry of the Banach space Y to replace the ℓ^1-sum in the sparse operator by an ℓ^r-sum for $r \geq 1$.

(iv) We replace the truncation $T(f\mathbf{1}_{\mathbb{R}^n\setminus\alpha Q})$ in the grand maximal truncation operator by an abstract localization principle.

The extensions (i) and (ii) are relatively straightforward. The main novelty is (iii), which controls the weight characteristic dependence that can be deduced from the sparse domination. Generalization (iv) will make its appearance in Theorem 3.2.2 and can be used to make the associated grand maximal truncation operator easier to estimate in specific situations.

Let (S,d,μ) be a space of homogeneous type and let X and Y be Banach spaces. For a bounded linear operator T from $L^{p_1}(S;X)$ to $L^{p_1,\infty}(S;Y)$ and $\alpha \geq 1$ we define the following *sharp grand maximal truncation operator*

$$\mathcal{M}^{\#}_{T,\alpha} f(s) := \sup_{B\ni s} \operatorname*{ess\,sup}_{s',s''\in B} \big\| T(f\mathbf{1}_{S\setminus\alpha B})(s') - T(f\mathbf{1}_{S\setminus\alpha B})(s'') \big\|_Y, \qquad s\in S,$$

where the supremum is taken over all balls $B \subseteq S$ containing $s \in S$. Our main theorem reads as follows.

Theorem 3.1.1. *Let (S,d,μ) be a space of homogeneous type and let X and Y be Banach spaces. Take $p_1,p_2,r \in [1,\infty)$ and set $p_0 := \max\{p_1,p_2\}$. Take $\alpha \geq 3c_d^2/\delta$, where c_d is the quasi-metric constant and δ is as in Proposition 2.1.1. Assume the following conditions:*

- *T is a bounded linear operator from $L^{p_1}(S;X)$ to $L^{p_1,\infty}(S;Y)$.*
- *$\mathcal{M}^{\#}_{T,\alpha}$ is a bounded operator from $L^{p_2}(S;X)$ to $L^{p_2,\infty}(S)$.*
- *There is a $C_r > 0$ such that for disjointly and boundedly supported $f_1,\dots,f_n \in L^{p_0}(S;X)$*

$$\Big\| T\Big(\sum_{k=1}^n f_k\Big)(s)\Big\|_Y \leq C_r \Big(\sum_{k=1}^n \big\| Tf_k(s)\big\|_Y^r\Big)^{1/r}, \qquad s\in S.$$

Then there is an $\eta \in (0,1)$ such that for any boundedly supported $f \in L^{p_0}(S;X)$ there is an η-sparse collection of cubes $\mathcal{S}$ such that

$$\|Tf(s)\|_Y \lesssim_{S,\alpha} C_T\, C_r \Big(\sum_{Q\in\mathcal{S}} \langle \|f\|_X\rangle_{p_0,Q}^r \mathbf{1}_Q(s)\Big)^{1/r}, \qquad s\in S,$$

where $C_T = \|T\|_{L^{p_1}\to L^{p_1,\infty}} + \|\mathcal{M}^{\#}_{T,\alpha}\|_{L^{p_2}\to L^{p_2,\infty}}$.

As the assumption in the third bullet of Theorem 3.1.1 expresses a form of sublinearity of the operator T when $r = 1$, we will call this assumption *r-sublinearity*. Note that it is crucial that the constant C_r is independent of $n \in \mathbb{N}$. If $C_r = 1$ it suffices to consider $n = 2$.

3.1.2. SHARP WEIGHTED NORM INEQUALITIES

One of the main reasons to study sparse domination of an operator is the fact that sparse bounds yield weighted norm inequalities and these weighted norm inequalities are sharp for many operators. Here sharpness is meant in the sense that for $p \in (p_0, \infty)$ we have a $\beta \geq 0$ such that

$$\|T\|_{L^p(S,w;X)\to L^p(S,w;Y)} \lesssim [w]^{\beta}_{A_{p/p_0}}, \qquad w \in A_{p/p_0} \tag{3.1.2}$$

and (3.1.2) is false for any $\beta' < \beta$.

The first result of this type was obtained by Buckley [Buc93], who showed that $\beta = \frac{1}{p-1}$ for the Hardy–Littlewood maximal operator. A decade later, the quest to find sharp weighted bounds attracted renewed attention because of the work of Astala, Iwaniec and Saksman [AIS01]. They proved sharp regularity results for the solution to the Beltrami equation under the assumption that $\beta = 1$ for the Beurling–Ahlfors transform for $p \geq 2$. This linear dependence on the A_p characteristic for the Beurling–Ahlfors transform was shown by Petermichl and Volberg in [PV02]. Another decade later, after many partial results, sharp weighted norm inequalities were obtained for general Calderón–Zygmund operators by Hytönen in [Hyt12] as discussed before.

In Proposition 3.2.4 we prove weighted L^p-boundedness for the sparse operators appearing in Theorem 3.1.1. As a direct corollary from Theorem 3.1.1 we then have:

Corollary 3.1.2. *Under the assumptions of Theorem 3.1.1 we have for all $p \in (p_0, \infty)$ and $w \in A_{p/p_0}$*

$$\|T\|_{L^p(S,w;X)\to L^p(S,w;Y)} \lesssim C_T\, C_r\, [w]^{\max\left\{\frac{1}{p-p_0}, \frac{1}{r}\right\}}_{A_{p/p_0}},$$

where the implicit constant depends on S, p_0, p, r and α.

As noted before the main novelty in Theorem 3.1.1 is the introduction of the parameter $r \in [1, \infty)$. The r-sublinearity assumption in Theorem 3.1.1 becomes more restrictive as r increases and the conclusions of Theorem 3.1.1 and Corollary 3.1.2 consequently become stronger. In order to check whether the dependence on the weight characteristic is sharp one can employ e.g. [LPR15, Theorem 1.2], which provides a lower bound for the best possible weight characteristic dependence in terms of the operator norm of T from $L^p(S;X)$ to $L^p(S;Y)$. For some operators, like Littlewood–Paley or maximal operators, sharpness in the estimate in Corollary 3.1.2 is attained for $r > 1$ and thus Theorem 3.1.1 can be used to show sharp weighted bounds for more operators than precursors like [LO20, Theorem 1.1].

3.1.3. HOW TO APPLY OUR MAIN RESULT

Let us outline the typical way how one applies Theorem 3.1.1 (or the local and more general version in Theorem 3.2.2) to obtain (sharp) weighted L^p-boundedness for an operator T:

(i) If T is not linear it is often *linearizable*, which means that we can linearize it by putting part of the operator in the norm of the Banach space Y. For example if T is a Littlewood-Paley square function we take $Y = L^2$ and if T is a maximal operator we take $Y = \ell^\infty$. Alternatively one can apply Theorem 3.2.2, which is a local and more abstract version of Theorem 3.1.1 that does not assume T to be linear.

(ii) The weak L^{p_1}-boundedness of T needs to be studied separately and is often already available in the literature.

(iii) The operator $\mathcal{M}^\#_{T,\alpha}$ reflects the non-localities of the operator T. The weak L^{p_2}-boundedness of $\mathcal{M}^\#_{T,\alpha}$ requires an intricate study of the structure of the operator. In many examples $\mathcal{M}^\#_{T,\alpha}$ can be pointwise dominated by the Hardy–Littlewood maximal operator M_{p_2}, which is weak L^{p_2}-bounded by Proposition 2.2.1. This is exemplified for Calderón–Zygmund operators in the proof of Theorem 3.4.1. Sometimes one can choose a suitable localization in Theorem 3.2.2 such that the sharp maximal truncation operator is either zero (see e.g. Section 3.6 on the Rademacher maximal operator) or pointwise dominated by T (see e.g. Section 3.7 on Littlewood–Paley operators).

(iv) The r-sublinearity assumption on T is trivial for $r = 1$, which suffices if one is not interested in quantitative weighted bounds. To check the r-sublinearity for some $r > 1$ one needs to use the structure of the operator and often also the geometric properties of the Banach space Y like type r. See, for example, the proofs of Theorems 3.6.1 and Theorem 4.4.11 how to check r-sublinearity in concrete cases.

3.1.4. APPLICATIONS

The main motivation to generalize the results in [LO20] comes from the applications to stochastic singular integral operators in Chapter 4. Indeed, we will use Theorem 3.1.1 with $p_1 = p_2 = r = 2$ to prove a stochastic version of the vector-valued A_2-theorem for Calderón–Zygmund operators. Moreover using $S = \mathbb{R}_+ \times \mathbb{R}^n$ equipped with the parabolic metric and the Lebesgue measure we will develop Calderón–Zygmund theory for singular mixed stochastic-deterministic integral operators. The fact that $r = 2$ and spaces of homogeneous type like $\mathbb{R}_+ \times \mathbb{R}^n$ are needed in these applications are the key motivations to incorporate these generalizations in this chapter.

In this chapter we will focus on applications in harmonic analysis. We will provide a few examples that illustrate the sparse domination principle nicely, and comment on further potential applications in Section 3.8.

- As a first application of Theorem 3.1.1 we prove an A_2-theorem for vector-valued Calderón–Zygmund operators with operator-valued kernel in a space of homogeneous type. The A_2-theorem for vector-valued Calderón–Zygmund operators with operator-valued kernel in Euclidean space has previously been proven in [HH14]

and the A_2-theorem for scalar-valued Calderón–Zygmund operators in spaces of homogeneous type in [NRV13, AV14]. Our theorem unifies these two results.

- Using the A_2-theorem, we prove a weighted, anisotropic, mixed norm Mihlin multiplier theorem, which is a natural supplement to the recent results in [FHL20] and is particularly useful in the study of spaces of smooth, vector-valued functions.
- In our second application of Theorem 3.1.1 we prove sparse domination and quantitative weighted norm inequalities for the Rademacher maximal operator, extending the qualitative bounds in Euclidean space in [Kem13]. The proof demonstrates how one can use the geometry of the Banach space to deduce r-sublinearity for an operator.
- Thirdly we give a short proof of the sharp weighted norm inequalities of Littlewood–Paley operators, recovering the result in [Ler11]. The proof illustrates nicely how the structure of these operators yields 2-sublinearity.

Moreover, in Section 6.4 we will apply Theorem 3.1.1 to the lattice Hardy–Littlewood maximal operator, which again demonstrates how one can use the geometry of the Banach space to deduce r-sublinearity for an operator.

3.2. POINTWISE ℓ^r-SPARSE DOMINATION

In this section we will prove a local version of the sparse domination result in Theorem 3.1.1, from which we will deduce Theorem 3.1.1 by a covering argument using Lemma 2.1.3. This local version will use an abstract localization of the operator T, since it depends upon the operator at hand as to the most effective localization. For example in the study of a Calderón–Zygmund operator it is convenient to localize the function inserted into T, for a maximal operator it is convenient to localize the supremum in the definition of the maximal operator and for a Littlewood–Paley operator it is most suitable to localize the defining integral.

Definition 3.2.1. Let (S, d, μ) be a space of homogeneous type with a dyadic system $\mathscr{D}$, let X and Y be Banach spaces, $p \in [1, \infty)$ and $\alpha \geq 1$. For a bounded operator

$$T\colon L^p(S;X) \to L^{p,\infty}(S;Y)$$

we say that a family of operators $\{T_Q\}_{Q\in\mathscr{D}}$ from $L^p(S;X)$ to $L^{p,\infty}(Q;Y)$ is an *α-localization family of T* if for all $Q \in \mathscr{D}$ and $f \in L^p(S;X)$ we have

$$T_Q(f\,\mathbf{1}_{\alpha Q})(s) = T_Q f(s), \qquad s \in Q, \qquad \text{(Localization)}$$
$$\big\|T_Q(f\,\mathbf{1}_{\alpha Q})(s)\big\|_Y \leq \big\|T(f\,\mathbf{1}_{\alpha Q})(s)\big\|_Y, \qquad s \in Q, \qquad \text{(Domination)}$$

For $Q, Q' \in \mathscr{D}$ with $Q' \subseteq Q$ we define the difference operator

$$T_{Q\setminus Q'} f(s) := T_Q f(s) - T_{Q'} f(s), \qquad s \in Q'.$$

and for $Q \in \mathscr{D}$ the *localized sharp grand maximal truncation operator*

$$\mathcal{M}^{\#}_{T,Q} f(s) := \sup_{\substack{Q' \in \mathscr{D}(Q):\\ s \in Q'}} \operatorname*{ess\,sup}_{s',s'' \in Q'} \big\| (T_{Q \setminus Q'}) f(s') - (T_{Q \setminus Q'}) f(s'') \big\|_Y, \qquad s \in S.$$

In order to obtain interesting results, one needs to be able to recover the boundedness of T from the boundedness of T_Q uniformly in $Q \in \mathscr{D}$. The canonical example of an α-localization family is

$$T_Q f(s) := T(f \mathbf{1}_{\alpha Q})(s), \qquad s \in Q.$$

for all $Q \in \mathscr{D}$, which allows one to recover weighted boundedness of T by the density of boundedly supported functions. Furthermore it is exactly this choice of an α-localization family that will lead to our global sparse domination result in the introduction.

Theorem 3.2.2. *Let (S,d,μ) be a space of homogeneous type with dyadic system $\mathscr{D}$ and let X and Y be Banach spaces. Take $p_1, p_2, r \in [1,\infty)$, set $p_0 := \max\{p_1,p_2\}$ and take $\alpha \geq 1$. Suppose that*

- *T is a bounded operator from $L^{p_1}(S;X)$ to $L^{p_1,\infty}(S;Y)$ with α-localization family $\{T_Q\}_{Q \in \mathscr{D}}$.*
- *$\mathcal{M}^{\#}_{T,Q}$ is bounded from $L^{p_2}(S;X)$ to $L^{p_2,\infty}(S)$ uniformly in $Q \in \mathscr{D}$.*
- *For all $Q_1, \dots, Q_n \in \mathscr{D}$ with $Q_n \subseteq \cdots \subseteq Q_1$ and any $f \in L^p(S;X)$*

$$\big\| T_{Q_1} f(s) \big\|_Y \leq C_r \Big(\big\| T_{Q_n} f(s) \big\|_Y^r + \sum_{k=1}^{n-1} \big\| T_{Q_k \setminus Q_{k+1}} f(s) \big\|_Y^r \Big)^{1/r}, \quad s \in Q_n.$$

Then for any $f \in L^{p_0}(S;X)$ and $Q \in \mathscr{D}$ there exists a $\frac{1}{2}$-sparse collection of dyadic cubes $\mathcal{S} \subseteq \mathscr{D}(Q)$ such that

$$\big\| T_Q f(s) \big\|_Y \lesssim_{S,\mathscr{D},\alpha} C_T\, C_r \Big(\sum_{P \in \mathcal{S}} \langle \|f\|_X \rangle^r_{p_0,\alpha P}\, \mathbf{1}_P(s) \Big)^{1/r}, \qquad s \in Q,$$

with $C_T := \|T\|_{L^{p_1} \to L^{p_1,\infty}} + \sup_{P \in \mathscr{D}} \|\mathcal{M}^{\#}_{T,P}\|_{L^{p_2} \to L^{p_2,\infty}}$.

The assumption in the third bullet in Theorem 3.2.2 replaces the r-sublinearity assumption in Theorem 3.1.1. We will call this assumption a *localized ℓ^r-estimate*.

Proof. Fix $f \in L^p(S,X)$ and $Q \in \mathscr{D}$. We will prove the theorem in two steps: we will first construct the $\frac{1}{2}$-sparse family of cubes $\mathcal{S}$ and then show that the sparse expression associated to $\mathcal{S}$ dominates $T_Q f$ pointwise.

Step 1: We will construct the $\frac{1}{2}$-sparse family of cubes $\mathcal{S}$ iteratively. Given a collection of pairwise disjoint cubes $\mathcal{S}^k$ for some $k \in \mathbb{N}$ we will first describe how to construct

$\mathcal{S}^{k+1}$. Afterwards we can inductively define $\mathcal{S}^k$ for all $k \in \mathbb{N}$ starting from $\mathcal{S}^1 = \{Q\}$ and set $\mathcal{S} := \bigcup_{k\in\mathbb{N}} \mathcal{S}^k$.

Fix a $P \in \mathcal{S}^k$ and for $\lambda \geq 1$ to be chosen later define

$$\Omega_P^1 := \Big\{ s \in P : \|T_P f(s)\|_Y > \lambda\, C_T \big\langle \|f\|_X \big\rangle_{p_0,\alpha P} \Big\}$$
$$\Omega_P^2 := \Big\{ s \in P : \mathcal{M}^{\#}_{T,P}(f)(s) > \lambda\, C_T \big\langle \|f\|_X \big\rangle_{p_0,\alpha P} \Big\}$$

and $\Omega_P := \Omega_P^1 \cup \Omega_P^2$. Let $c_1 \geq 1$, depending on S, $\mathscr{D}$ and α, be such that $\mu(\alpha P) \leq c_1\, \mu(P)$. By the domination property of the α-localization family we have

$$\|T_P f(s)\|_Y \leq \|T(f\,\mathbf{1}_{\alpha P})(s)\|_Y, \qquad s \in P,$$

and by the localization property

$$\mathcal{M}^{\#}_{T,P}(f)(s) = \mathcal{M}^{\#}_{T,P}(f\,\mathbf{1}_{\alpha P})(s), \qquad s \in P.$$

Thus by the weak boundedness assumptions on T and $\mathcal{M}^{\#}_{T,P}$ and Hölder's inequality we have for $i = 1, 2$

$$\mu(\Omega_P^i) \leq \Big(\frac{\|f\,\mathbf{1}_{\alpha P}\|_{L^{p_i}(S;X)}}{\lambda \big\langle \|f\|_X \big\rangle_{p_0,\alpha P}} \Big)^{p_i} = \frac{\big\langle \|f\|_X \big\rangle^{p_i}_{p_i,\alpha P}}{\lambda^{p_i} \big\langle \|f\|_X \big\rangle^{p_i}_{p_0,\alpha P}} \mu(\alpha P) \leq \frac{c_1}{\lambda}\, \mu(P). \tag{3.2.1}$$

Therefore it follows that

$$\mu(\Omega_P) \leq \frac{2c_1}{\lambda} \mu(P). \tag{3.2.2}$$

To construct the cubes in $\mathcal{S}^{k+1}$ we will use a local Calderón–Zygmund decomposition (see e.g. [FN19, Lemma 4.5]) on

$$\Omega_{P,\rho} := \{ s \in P : M^{\mathscr{D}(P)}(\mathbf{1}_{\Omega_P}) > \tfrac{1}{\rho} \}, \qquad \rho > 0$$

which will be a proper subset of P for our choice of λ and ρ. Here $M^{\mathscr{D}(P)}$ is the dyadic Hardy–Littlewood maximal operator with respect to the restricted dyadic system $\mathscr{D}(P)$. The local Calderón–Zygmund decomposition yields a pairwise disjoint collection of cubes $\mathcal{S}_P \subseteq \mathscr{D}(P)$ and a constant $c_2 \geq 2$, depending on S and $\mathscr{D}$, such that $\Omega_{P,c_2} = \bigcup_{P' \in \mathcal{S}_P} P'$ and

$$\tfrac{1}{c_2}\, \mu(P') \leq \mu(P' \cap \Omega_P) \leq \tfrac{1}{2}\, \mu(P'), \qquad P' \in \mathcal{S}_P. \tag{3.2.3}$$

Then by (3.2.2), (3.2.3) and the disjointness of the cubes in $\mathcal{S}_P$ we have

$$\sum_{P' \in \mathcal{S}_P} \mu(P') \leq c_2 \sum_{P' \in \mathcal{S}_P} \mu(P' \cap \Omega_P) \leq c_2\, \mu(\Omega_P) \leq \frac{2c_1 c_2}{\lambda} \mu(P).$$

Therefore, by choosing $\lambda = 4c_1c_2$, we have $\sum_{P' \in \mathcal{S}_P} \mu(P') \leq \frac{1}{2}\mu(P)$. This choice of λ also ensures that Ω_{P,c_2} is a proper subset of P as claimed before. We define $S^{k+1} := \bigcup_{P \in \mathcal{S}^k} \mathcal{S}_P$.

Now take $\mathcal{S}^1 = \{Q\}$, iteratively define $\mathcal{S}^k$ for all $k \in \mathbb{N}$ as described above and set $\mathcal{S} := \bigcup_{k\in\mathbb{N}} \mathcal{S}^k$. Then $\mathcal{S}$ is $\frac{1}{2}$-sparse family of cubes, since for any $P \in \mathcal{S}$ we can set

$$E_P := P \setminus \bigcup_{P'\in\mathcal{S}_P} P',$$

which are pairwise disjoint by the fact that $\bigcup_{P'\in\mathcal{S}^{k+1}} P' \subseteq \bigcup_{P\in\mathcal{S}^k} P$ for all $k \in \mathbb{N}$ and we have

$$\mu(E_P) = \mu(P) - \sum_{P'\in\mathcal{S}_P} \mu(P') \geq \frac{1}{2}\mu(P).$$

Step 2: We will now check that the sparse expression corresponding to $\mathcal{S}$ constructed in Step 1 dominates $T_Q f$ pointwise. Since

$$\lim_{k\to\infty} \mu\Big(\bigcup_{P\in\mathcal{S}^k} P\Big) \leq \lim_{k\to\infty} \frac{1}{2^k}\mu(Q) = 0,$$

we know that there is a set N_0 of measure zero such that for all $s \in Q \setminus N_0$ there are only finitely many $k \in \mathbb{N}$ with $s \in \bigcup_{P\in\mathcal{S}^k} P$. Moreover by the Lebesgue differentiation theorem we have for any $P \in \mathcal{S}$ that $\mathbf{1}_{\Omega_P}(s) \leq M^{\mathscr{D}(P)}(\mathbf{1}_{\Omega_P})(s)$ for a.e. $s \in P$. Thus

$$\Omega_P \setminus N_P \subseteq \Omega_{P,1} \subseteq \Omega_{P,c_2} = \bigcup_{P'\in\mathcal{S}_P} P' \tag{3.2.4}$$

for some set N_P of measure zero. We define $N := N_0 \cup \bigcup_{P\in\mathcal{S}} N_P$, which is a set of measure zero.

Fix $s \in Q \setminus N$ and take the largest $n \in \mathbb{N}$ such that $s \in \bigcup_{P\in\mathcal{S}^n} P$, which exists since $s \notin N_0$. For $k = 1,\dots,n$ let $P_k \in \mathcal{S}^k$ be the unique cube such that $s \in P_k$ and note that by construction we have $P_n \subseteq \dots \subseteq P_1 = Q$. Using the localized ℓ^r-estimate of T we split $\|T_Q f(s)\|_Y^r$ into two parts

$$\begin{aligned}\big\|T_Q f(s)\big\|_Y^r &\leq C_r^r\Big(\big\|T_{P_n} f(s)\big\|_Y^r + \sum_{k=1}^{n-1} \big\|T_{P_k\setminus P_{k+1}} f(s)\big\|_Y^r\Big)\\ &=: C_r^r\Big(\boxed{\text{A}} + \boxed{\text{B}}\Big).\end{aligned}$$

For $\boxed{\text{A}}$ note that $s \notin N_{P_n}$ and $s \notin \bigcup_{P'\in\mathcal{S}^{n+1}} P'$ and therefore by (3.2.4) we know that $s \in P_n \setminus \Omega_{P_n}$. So by the definition of $\Omega^1_{P_n}$

$$\boxed{\text{A}} \leq \lambda^r\, C_T^r \big\langle \|f\|_X\big\rangle^r_{p_0,\alpha P_n}.$$

For $1 \leq k \leq n-1$ we have by (3.2.2) and (3.2.3) that

$$\begin{aligned}\mu\big(P_{k+1} \setminus (\Omega_{P_{k+1}} \cup \Omega_{P_k})\big) &\geq \mu(P_{k+1}) - \mu(\Omega_{P_{k+1}}) - \mu(P_{k+1} \cap \Omega_{P_k})\\ &\geq \mu(P_{k+1}) - \frac{1}{2c_2}\mu(P_{k+1}) - \frac{1}{2}\mu(P_{k+1}) > 0,\end{aligned} \tag{3.2.5}$$

so $P_{k+1} \setminus (\Omega_{P_{k+1}} \cup \Omega_{P_k})$ is non-empty. Take $s' \in P_{k+1} \setminus (\Omega_{P_{k+1}} \cup \Omega_{P_k})$, then we have

$$\begin{aligned}\big\|T_{P_k\setminus P_{k+1}}f(s)\big\|_Y &\le \big\|T_{P_k\setminus P_{k+1}}f(s)-T_{P_k\setminus P_{k+1}}f(s')\big\|_Y+\big\|T_{P_k\setminus P_{k+1}}f(s')\big\|_Y\\ &\le \mathcal{M}^{\#}_{T,P_k}f(s')+\big\|T_{P_k}(s')\big\|_Y+\big\|T_{P_{k+1}}(s')\big\|_Y\\ &\le 2\lambda C_T\big(\langle\|f\|_X\rangle_{p_0,\alpha P_k}+\langle\|f\|_X\rangle_{p_0,\alpha P_{k+1}}\big),\end{aligned}$$

where we used the definition of $\mathcal{M}^{\#}_{T,P_k}$ and $T_{P_{k+1}\setminus P_k}$ in the second inequality and $s'\notin \Omega_{P_{k+1}}\cup\Omega_{P_k}$ in the third inequality. Using $(a+b)^r\le 2^{r-1}(a^r+b^r)$ for any $a,b>0$ this implies that

$$\begin{aligned}\boxed{\text{B}} &\le \sum_{k=1}^{n-1}2^r2^{r-1}\lambda^r C_T^r\Big(\langle\|f\|_X\rangle^r_{p_0,\alpha P_k}+\langle\|f\|_X\rangle^r_{p_0,\alpha P_{k+1}}\Big)\\ &\le \sum_{k=1}^{n}4^r\lambda^r C_T^r\langle\|f\|_X\rangle^r_{p_0,\alpha P_k}.\end{aligned}$$

Combining the estimates for $\boxed{\text{A}}$ and $\boxed{\text{B}}$ we obtain

$$\begin{aligned}\big\|T_Qf(s)\big\|_Y &\le 5\lambda C_T C_r\Big(\sum_{k=1}^{n}\langle\|f\|_X\rangle^r_{p_0,\alpha P_k}\Big)^{1/r}\\ &= 5\lambda C_T C_r\Big(\sum_{P\in\mathcal{S}}\langle\|f\|_X\rangle^r_{p_0,\alpha P}\mathbf{1}_P(s)\Big)^{1/r}.\end{aligned}$$

Since $s\in Q\setminus N$ was arbitrary and N has measure zero, this inequality holds for a.e. $s\in Q$. Noting that $\lambda=4c_1c_2$ and c_1 and c_2 only depend on S, α and $\mathscr{D}$ finishes the proof of the theorem. □

As announced Theorem 3.1.1 now follows directly from Theorem 3.2.2 and a covering argument with Lemma 2.1.3.

Proof of Theorem 3.1.1. We will prove Theorem 3.1.1 in three steps: we will first show that the assumptions of Theorem 3.1.1 imply the assumptions of Theorem 3.2.2, then we will improve the local conclusion of Theorem 3.2.2 to a global one and finally we will replace the averages over the dilation αP in the conclusion of Theorem 3.2.2 by the average over larger cubes P'.

To start let $\mathscr{D}^1,\dots,\mathscr{D}^m$ be as in Proposition 2.1.1 with parameters c_0, C_0, δ and γ, which only depend on S.

Step 1: For any $Q\in\mathscr{D}^1$ define T_Q by $T_Qf(s):=T(f\mathbf{1}_{\alpha Q})(s)$ for $s\in Q$. Then:

- $\{T_Q\}_{Q\in\mathscr{D}^1}$ is an α-localization family of T.
- For any $Q\in\mathscr{D}^1$ and $f\in L^{p_1}(S;X)$ we have

$$\mathcal{M}^{\#}_{T,Q}f(s)\le\mathcal{M}^{\#}_{T,\alpha}(f\mathbf{1}_{\alpha Q})(s),\qquad s\in Q.$$

  So by the weak L^{p_2}-boundedness of $\mathcal{M}^{\#}_{T,\alpha}$ it follows that $\mathcal{M}^{\#}_{T,Q}f$ is weak L^{p_2}-bounded uniformly in $Q\in\mathscr{D}^1$.

- For any $f \in L^p(S;X)$ and $Q_1,\dots,Q_n \in \mathscr{D}^1$ with $Q_n \subseteq \dots \subseteq Q_1$ the functions $f_k := f\,\mathbf{1}_{\alpha Q_k \setminus \alpha Q_{k+1}}$ for $k = 1,\dots,n-1$ and $f_n := f\,\mathbf{1}_{\alpha Q_n}$ are disjointly supported. Thus by the r-sublinearity of T

$$\big\|T_{Q_1} f(s)\big\|_Y \le C_r\Big(\big\|T_{Q_n} f(s)\big\|_Y^r + \sum_{k=1}^{n-1} \big\|T_{Q_k\setminus Q_{k+1}} f(s)\big\|_Y^r\Big)^{1/r}, \qquad s \in Q_n.$$

So the assumptions of Theorem 3.2.2 follow from the assumptions of Theorem 3.1.1.

Step 2: Let $f \in L^p(S;X)$ be boundedly supported. First suppose that $\operatorname{diam}(S) = \infty$ and let E be a ball containing the support of f. By Lemma 2.1.3 there is a partition $\mathcal{D} \subseteq \mathscr{D}^1$ such that $E \subseteq \alpha Q$ for all $Q \in \mathcal{D}$. Thus by Theorem 3.2.2 we can find a $\frac{1}{2}$-sparse collection of cubes $\mathcal{S}_Q \subseteq \mathscr{D}^1(Q)$ for every $Q \in \mathcal{D}$ with

$$\big\|Tf(s)\big\|_Y \lesssim_{S,\alpha} C_T\, C_r \Big(\sum_{P\in\mathcal{S}_Q} \langle\|f\|_X\rangle_{p_0,\alpha P}^r\, \mathbf{1}_P(s)\Big)^{1/r}, \qquad s \in Q,$$

where we used that $T_Q f = T(f\,\mathbf{1}_{\alpha Q}) = Tf$ as $\operatorname{supp} f \subseteq \alpha Q$. Since $\mathcal{D}$ is a partition, $\mathcal{S} := \bigcup_{Q\in\mathcal{D}} \mathcal{S}_Q$ is also a $\frac{1}{2}$-sparse collection of cubes with

$$\big\|Tf(s)\big\|_Y \lesssim_{S,\alpha} C_T\, C_r \Big(\sum_{P\in\mathcal{S}} \langle\|f\|_X\rangle_{p_0,\alpha P}^r\, \mathbf{1}_P(s)\Big)^{1/r}, \qquad s \in S, \tag{3.2.6}$$

If $\operatorname{diam}(S) < \infty$, then (3.2.6) follows directly from Theorem 3.2.2 since $S \in \mathscr{D}$ in that case.

Step 3: For any $P \in \mathcal{S}$ with center z and side length δ^k we can find a $P' \in \mathscr{D}^j$ for some $1 \le j \le m$ such that

$$\alpha P = B(z, \alpha C_0 \cdot \delta^k) \subseteq P', \qquad \operatorname{diam}(P') \le \gamma\alpha C_0 \cdot \delta^k.$$

Therefore there is a $c_1 > 0$ depending on S and α such that

$$\mu(P') \le \mu\big(B(z, \gamma\alpha C_0 \cdot \delta^k)\big) \le c_1\, \mu\big(B(z, c_0 \cdot \delta^k)\big) \le c_1\, \mu(P).$$

So by defining $E_{P'} := E_P$ we can conclude that the collection of cubes $\mathcal{S}' := \{P' : P \in \mathcal{S}\}$ is $\frac{1}{2c_1}$-sparse. Moreover since $\alpha P \subseteq P'$ and $\mu(P') \le c_1\,\mu(P) \le c_1\,\mu(\alpha P)$ for any $P \in \mathcal{S}$, we have

$$\langle\|f\|_X\rangle_{p_0,\alpha P} \le c_1 \langle\|f\|_X\rangle_{p_0,P'}.$$

Combined with (3.2.6) this proves the sparse domination in the conclusion of Theorem 3.1.1. □

Remark 3.2.3. The assumption $\alpha \ge 3c_d^2/\delta$ in Theorem 3.1.1 arises from the use of Lemma 2.1.3, which transfers the local sparse domination estimate of Theorem 3.2.2 to the global statement of Theorem 3.1.1. To deduce weighted estimates the local sparse domination estimate of Theorem 3.2.2 suffices by testing against boundedly supported functions. However the operator norm of $\mathcal{M}^{\#}_{T,\alpha}$ usually becomes easier to estimate for larger α, so the lower bound on α is not restrictive.

To conclude this section we will prove weighted bounds for the sparse operators in Theorems 3.1.1 and 3.2.2, from which Corollary 3.1.2 follows directly. In the Euclidean case such bounds are thoroughly studied and most of the arguments extend directly to spaces of homogeneous type. For the convenience of the reader we will give a self-contained proof of the strong weighted L^p-boundedness of these sparse operators in spaces of homogeneous type, following the proof of [Ler16, Lemma 4.5]. For further results for various special cases of the sparse operators in Theorems 3.1.1 and 3.2.2 we refer to:

- Weak weighted L^p-boundedness (including the endpoint $p = p_0$) can be found [HL18, FN19].
- More precise bounds in terms of two-weight A_p-A_∞-characteristics can be found in e.g. [FH18, HL18, HP13, LL16].

Proposition 3.2.4. *Let (S,d,μ) be a space of homogeneous type, let $\mathcal{S}$ be an η-sparse collection of cubes and take $p_0, r \in [1,\infty)$. For $p \in (p_0,\infty)$, $w \in A_{p/p_0}$ and $f \in L^p(S,w)$ we have*

$$\Big\| \Big(\sum_{Q\in\mathcal{S}} \langle |f| \rangle_{p_0,Q}^r \mathbf{1}_Q \Big)^{1/r} \Big\|_{L^p(S,w)} \lesssim [w]_{A_{p/p_0}}^{\max\{\frac{1}{p-p_0},\frac{1}{r}\}} \|f\|_{L^p(S,w)}$$

where the implicit constant depends on S, p_0, p, r and η.

Proof. We first note that by Proposition 2.1.1 we may assume without loss of generality that $\mathcal{S} \subseteq \mathscr{D}$, where $\mathscr{D}$ is an arbitrary dyadic system in (S,d,μ). If $p - p_0 \le r$ we have $\max\{\frac{1}{p-p_0}, \frac{1}{r}\} = \frac{1}{p-p_0}$. Since $\ell^{p-p_0} \hookrightarrow \ell^r$, the case $p-p_0 \le r$ follows from the case $p-p_0 = r$, so without loss of generality we may also assume $p \ge p_0 + r$.

For a weight u and a measurable set E we define $u(E) := \int_E u \,\mathrm{d}\mu$ and we denote dyadic Hardy–Littlewood maximal operator with respect to the measure $u\,\mathrm{d}\mu$ by

$$M^{\mathscr{D},u} f(s) := \sup_{Q\in\mathscr{D}} \frac{1}{u(Q)} \int_Q |f| u \,\mathrm{d}\mu \cdot \mathbf{1}_Q(s), \qquad s \in S,$$

which is bounded on $L^p(S,u)$ for all $p \in (1,\infty)$ by Doob's maximal inequality (see e.g. [HNVW16, Theorem 3.2.2]). Take $f \in L^p(S,w)$, set $q := (p/r)' = \frac{p}{p-r}$ and take

$$g \in L^q(S, w^{1-q}) = \big(L^{p/r}(S,w)\big)^*.$$

Then we have by the disjointness of the E_Q's associated to each $Q \in \mathcal{S}$

$$\begin{aligned} \sum_{Q\in\mathcal{S}} w(E_Q) \Big(\frac{\mu(Q)}{w(Q)} \Big)^q \langle |g| \rangle_{1,Q}^q &\le \sum_{Q\in\mathcal{S}} \int_{E_Q} M^{\mathscr{D},w}(g w^{-1})^q w \,\mathrm{d}\mu \\ &\le \big\| M^{\mathscr{D},w}(g w^{-1}) \big\|_{L^q(S,w)}^q \\ &\lesssim_{p,r} \|g\|_{L^q(S,w^{1-q})}^q \end{aligned} \tag{3.2.7}$$

and similarly, setting $\sigma := w^{1-(p/p_0)'}$, we have

$$\begin{aligned}\sum_{Q\in\mathcal{S}} \sigma(E_Q)\Big(\frac{\mu(Q)}{\sigma(Q)}\Big)^{\frac{p}{p_0}} \langle |f|^{p_0}\rangle_{1,Q}^{p/p_0} &\le \big\| M^{\mathscr{D},\sigma}(|f|^{p_0}\sigma^{-1})\big\|_{L^{p/p_0}(S,\sigma)}^{p/p_0} \\ &\lesssim_{p,p_0} \|f\|_{L^p(S,w)}^p\end{aligned} \tag{3.2.8}$$

using $\sigma\cdot\sigma^{-p_0/p} = w$. Define the constant

$$c_w := \sup_{Q\in\mathscr{D}} \frac{w(Q)^{1/r}}{w(E_Q)^{\frac{1}{r}-\frac{1}{p}}} \frac{\sigma(Q)^{1/p_0}}{\sigma(E_Q)^{1/p}} \frac{1}{\mu(Q)^{1/p_0}},$$

Then by Hölders inequality, (3.2.7) and (3.2.8) we have

$$\begin{aligned}\int_S \Big(\sum_{Q\in\mathcal{S}} \langle |f|\rangle_{p_0,Q}^r \mathbf{1}_Q\Big)\cdot g\,\mathrm{d}\mu &= \sum_{Q\in\mathcal{S}} \mu(Q)\langle |f|^{p_0}\rangle_{1,Q}^{r/p_0}\langle |g|\rangle_{1,Q} \\ &\le c_w^r \sum_{Q\in\mathcal{S}} \Big(\sigma(E_Q)^{r/p}\Big(\frac{\mu(Q)}{\sigma(Q)}\Big)^{r/p_0}\langle |f|^{p_0}\rangle_{1,Q}^{r/p_0}\Big) \\ &\qquad\qquad \times \Big(w(E_Q)^{1/q}\frac{\mu(Q)}{w(Q)}\langle |g|\rangle_{1,Q}\Big) \\ &\lesssim_{p,p_0,r} c_w^r \|f\|_{L^p(S,w)}^r \|g\|_{L^q(S,w^{1-q})}.\end{aligned} \tag{3.2.9}$$

So by duality it remains to show $c_w \lesssim [w]_{A_{p/p_0}}^{\max\{\frac{1}{p-p_0},\frac{1}{r}\}}$. Fix a $Q\in\mathscr{D}$ and note that by Hölders's inequality we have

$$\mu(Q)^{p/p_0} \le \eta^{p/p_0}\Big(\int_{E_Q} w^{p_0/p} w^{-p_0/p}\,\mathrm{d}\mu\Big)^{p/p_0} \le \eta^{p/p_0}\, w(E_Q)\,\sigma(E_Q)^{p/p_0-1}.$$

and thus

$$\frac{w(Q)}{w(E_Q)}\Big(\frac{\sigma(Q)}{\sigma(E_Q)}\Big)^{p/p_0-1} \le \eta^{p/p_0}\frac{w(Q)}{\mu(Q)}\Big(\frac{\sigma(Q)}{\mu(Q)}\Big)^{p/p_0-1} \lesssim_S \eta^{p/p_0}[w]_{A_{p/p_0}}.$$

Therefore we can estimate

$$\begin{aligned}c_w &= \sup_{Q\in\mathscr{D}}\Big[\frac{w(Q)}{\mu(Q)}\Big(\frac{\sigma(Q)}{\mu(Q)}\Big)^{\frac{p}{p_0}-1}\Big]^{\frac{1}{p}}\cdot\Big[\Big(\frac{w(Q)}{w(E_Q)}\Big)^{\frac{1}{r}-\frac{1}{p}}\Big(\frac{\sigma(Q)}{\sigma(E_Q)}\Big)^{\frac{1}{p}}\Big] \\ &\lesssim_S [w]_{A_{p/p_0}}^{\frac{1}{p}} \sup_{Q\in\mathscr{D}}\Big[\frac{w(Q)}{w(E_Q)}\Big(\frac{\sigma(Q)}{\sigma(E_Q)}\Big)^{\frac{p}{p_0}-1}\Big]^{\max\{\frac{1}{r}-\frac{1}{p},\frac{1}{p}\frac{p_0}{p-p_0}\}} \\ &\lesssim_{S,\eta} [w]_{A_{p/p_0}}^{\frac{1}{p}+\max\{\frac{1}{r}-\frac{1}{p},\frac{1}{p}\frac{p_0}{p-p_0}\}} = [w]_{A_{p/p_0}}^{\max\{\frac{1}{p-p_0},\frac{1}{r}\}},\end{aligned}$$

which finishes the proof. □

3.3. GENERALIZATIONS OF ℓ^r-SPARSE DOMINATION

Our main sparse domination theorems, Theorem 3.1.1 and Theorem 3.2.2, allow for various further generalizations. One can for instance change the boundedness assumptions on T and $\mathcal{M}^{\#}_{T,\alpha}$, treat multilinear operators, or deduce domination by sparse forms for operators that do not admit a pointwise sparse estimate. We end this section by sketching some of these possible generalizations.

In [LO20, Section 3] various variations and extensions of the main result in [LO20] are outlined. In particular, they show:

- The sparse domination for an individual function follows from assumptions on the same function. This can be exploited to prove a sparse $T(1)$-type theorem, see [LO20, Section 4].
- One can use certain Orlicz estimates to deduce sparse domination with Orlicz averages.
- The method of proof extends to the multilinear setting (see also [Li18]).

Our sparse domination results can also be extended in these directions, which we leave to the interested reader. In the remainder of this section, we will explore some further directions in which our results can be extended.

Sparse domination techniques have been successfully applied to *fractional integral operators*, see e.g. [CM13a, CM13b, Cru17, IRV18]. In these works sparse domination and sharp weighted estimates are deduced for e.g. the Riesz potentials, which for $0 < \alpha < n$ and a Schwartz function $f\colon \mathbb{R}^n \to \mathbb{C}$ are given by

$$I_\alpha f(s) := \int_{\mathbb{R}^n} \frac{f(t)}{|s-t|^{d-\alpha}}\,\mathrm{d}t, \qquad s \in \mathbb{R}^n,$$

A key feature of such operators is that they are not (weakly) L^p-bounded, but bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$, where $p, q \in (1,\infty)$ are such that $\frac{1}{p} = \frac{1}{q} + \frac{\alpha}{d}$. The sparse domination that one obtains in this case involves fractional sparse operators, in which the usual averages $\langle |f| \rangle_{p,Q}$ are replaced by fractional averages.

These operators fit in our framework with minimal effort. Indeed, upon inspection of the proof of Theorem 3.2.2 it becomes clear that the only place where we use the boundedness of T and $\mathcal{M}^{\#}_{T,\alpha}$ is in (3.2.1). Replacing the bounds with the off-diagonal bounds arising from fractional integral operators, we obtain the following variant of Theorem 3.1.1. Weighted bounds for the fractional sparse operators in Theorem 3.3.1 can be found in [FH18]

Theorem 3.3.1. *Let (S,d,μ) be a space of homogeneous type and let X and Y be Banach spaces. Take $p_0, q_0, r \in [1,\infty)$. Take $\alpha \geq 3c_d^2/\delta$, where c_d is the quasi-metric constant and δ is as in Proposition 2.1.1. Assume the following conditions:*

- *T is a bounded linear operator from $L^{p_0}(S;X)$ to $L^{q_0,\infty}(S;Y)$.*
- *$\mathcal{M}^{\#}_{T,\alpha}$ is a bounded operator from $L^{p_0}(S;X)$ to $L^{q_0,\infty}(S)$.*
- *T is r-sublinear.*

Then there is an $\eta\in(0,1)$ such that for any boundedly supported $f\in L^{p_0}(S;X)$ there is an η-sparse collection of cubes $\mathcal{S}$ such that

$$\|Tf(s)\|_Y \lesssim_{S,\alpha} C_T C_r \Big(\sum_{Q\in\mathcal{S}} \mu(\alpha P)^{\frac{r}{p_0}-\frac{r}{q_0}} \big\langle \|f\|_X\big\rangle^r_{p_0,Q} \mathbf{1}_Q(s)\Big)^{1/r}, \qquad s\in S,$$

where $C_T = \|T\|_{L^{p_0}\to L^{p_0,\infty}} + \|\mathcal{M}^{\#}_{T,\alpha}\|_{L^{p_0}\to L^{p_0,\infty}}$ and C_r is the r-sublinearity constant.

Proof. The proof is the same as the proof of Theorem 3.1.1, using an adapted version of Theorem 3.2.2 with the canonical α-localization family

$$T_Q f(s) = T(\mathbf{1}_{\alpha Q} f)(s), \qquad s\in Q.$$

The only thing that changes in the proof of Theorem 3.2.2 is the definition of Ω^1_P and Ω^2_P and the computation in (3.2.2). Indeed, we define

$$\Omega^1_P := \Big\{s\in P : \|T_P f(s)\|_Y > \lambda C_T\, \mu(\alpha P)^{\frac{1}{p_0}-\frac{1}{q_0}} \big\langle \|f\|_X\big\rangle_{p_0,\alpha P}\Big\}$$
$$\Omega^2_P := \Big\{s\in P : \mathcal{M}^{\#}_{T,P}(f)(s) > \lambda C_T\, \mu(\alpha P)^{\frac{1}{p_0}-\frac{1}{q_0}} \big\langle \|f\|_X\big\rangle_{p,\alpha P}\Big\}$$

and then by the assumptions on T and $\mathcal{M}^{\#}_{T,P}$ we have for $i=1,2$

$$\mu(\Omega^i_P) \le \Big(\frac{\|f\mathbf{1}_{\alpha P}\|_{L^{p_0}(S;X)}}{\lambda\,\mu(\alpha P)^{\frac{1}{p_0}-\frac{1}{q_0}} \big\langle \|f\|_X\big\rangle_{p_0,\alpha P}}\Big)^{q_0} = \frac{\big\langle \|f\|_X\big\rangle^{q_0}_{p_0,\alpha P}}{\lambda^{q_0}\big\langle \|f\|_X\big\rangle^{q_0}_{p_0,\alpha P}}\mu(\alpha P) \le \frac{c_1}{\lambda}\,\mu(P).$$

which proves (3.2.2). In Step 2 of the proof of Theorem 3.2.2 one needs to keep track of the factor $\mu(\alpha P)^{\frac{1}{p_0}-\frac{1}{q_0}}$ in the estimates. □

In the celebrated paper [BFP16] by Bernicót, Frey and Petermichl, domination by *sparse forms* was introduced to treat operators falling outside the scope of Calderón–Zygmund theory. This method was later adopted by Lerner in [Ler19] into his framework to prove sparse domination for rough homogeneous singular integral operators. As our methods are based on Lerner's sparse domination framework, our main result can also be generalized to the sparse form domination setting.

Let (S,d,μ) be a space of homogeneous type with a dyadic system $\mathscr{D}$, let X and Y be Banach spaces, $q\in(1,\infty)$, $p\in[1,q)$ and $\alpha\ge 1$. For a bounded operator

$$T\colon L^p(S;X)\to L^{p,\infty}(S;Y)$$

with an α-localization family $\{T_Q\}_{Q\in\mathscr{D}}$ we define the *localized sharp grand q-maximal truncation operator* for $Q\in\mathscr{D}$ by

$$\mathcal{M}^{\#}_{T,Q,q}f(s) := \sup_{\substack{Q'\in\mathscr{D}(Q):\\ s\in Q'}} \Big(\fint_{Q'}\fint_{Q'} \big\|(T_{Q\setminus Q'})f(s') - (T_{Q\setminus Q'})f(s'')\big\|_Y^q \,\mathrm{d}\mu(s')\,\mathrm{d}\mu(s'')\Big)^{1/q}.$$

Note that for $q=\infty$ one formally recovers the operator $\mathcal{M}^{\#}_{T,Q}$.

We will prove a version of Theorem 3.2.2 for operators for which the truncation operators $\mathcal{M}^{\#}_{T,Q,q}$ are bounded uniformly in $Q\in\mathscr{D}$ using sparse forms. Of course taking

$$T_Q f(s) := T(f\mathbf{1}_{\alpha Q})(s), \qquad s\in Q.$$

for $Q\in\mathscr{D}$ as the α-localization family one can easily deduce a statement like Theorem 3.1.1 in this setting, which we leave to the interested reader. Weighted bounds for the sparse forms in the following theorem can be found in [BFP16, FN19, Nie19]

Theorem 3.3.2. *Let (S,d,μ) be a space of homogeneous type with dyadic system $\mathscr{D}$ and let X and Y be Banach spaces. Take $q_0\in(1,\infty)$, $r\in(0,q_0)$, $p_1,p_2\in[1,q_0)$, set $p_0 := \max\{p_1,p_2\}$ and take $\alpha\geq 1$. Suppose that*

- *T is a bounded operator from $L^{p_1}(S;X)$ to $L^{p_1,\infty}(S;Y)$ with an α-localization family $\{T_Q\}_{Q\in\mathscr{D}}$.*
- *$\mathcal{M}^{\#}_{T,Q,q_0}$ is bounded from $L^{p_2}(S;X)$ to $L^{p_2,\infty}(S)$ uniformly in $Q\in\mathscr{D}$.*
- *T satisfies a localized ℓ^r-estimate.*

Then for any $f\in L^{p_0}(S;X)$, $g\in L^{(\frac{1}{r}-\frac{1}{q_0})^{-1}}(S)$ and $Q\in\mathscr{D}$ there exists a $\frac{1}{2}$-sparse collection of dyadic cubes $\mathcal{S}\subseteq\mathscr{D}(Q)$ such that

$$\Big(\int_Q \big\|T_Q f\big\|_Y^r\cdot|g|^r\,\mathrm{d}\mu\Big)^{1/r} \lesssim_{S,\mathscr{D},\alpha,r} C_T\,C_r\Big(\sum_{P\in\mathcal{S}}\mu(P)\big\langle\|f\|_X\big\rangle_{p_0,\alpha P}^r\big\langle|g|\big\rangle_{\frac{1}{\frac{1}{r}-\frac{1}{q_0}},P}^r\Big)^{1/r}$$

with $C_T := \|T\|_{L^{p_1}\to L^{p_1,\infty}} + \sup_{P\in\mathscr{D}}\|\mathcal{M}^{\#}_{T,P,q_0}\|_{L^{p_2}\to L^{p_2,\infty}}$ and C_r the constant from the localized ℓ^r-estimate.

Proof. We construct the sparse collection of cubes $\mathcal{S}$ exactly as in Step 1 of the proof of Theorem 3.2.2, using $\mathcal{M}^{\#}_{T,P,q_0}$ instead of $\mathcal{M}^{\#}_{T,P}$ in the definition of Ω^2_P. We will check that sparse form corresponding to $\mathcal{S}$ satisfies the claimed domination property, which will roughly follow the same lines as Step 2 of the proof of Theorem 3.2.2.

Fix $f\in L^{p_0}(S;X)$ and $g\in L^{(\frac{1}{r}-\frac{1}{q_0})^{-1}}(S)$. Note that for a.e. $s\in Q$ there are only finitely many $k\in\mathbb{N}$ with $s\in\bigcup_{P\in\mathcal{S}^k}P$. So we can use the localized ℓ^r-estimate of T to split

$$\begin{aligned}\int_Q \big\|T_Q f\big\|_Y^r\cdot|g|^r &\leq C_r^r\sum_{k\in\mathbb{N}}\sum_{P\in\mathcal{S}^k}\Big(\int_{P\setminus\bigcup_{P'\in\mathcal{S}^{k+1}}P'}\big\|T_P f\big\|_Y^r\cdot|g|^r \\ &\qquad + \sum_{P'\in\mathcal{S}^{k+1}:P'\subseteq P}\int_{P'}\big\|T_{P\setminus P'}f\big\|_Y^r\cdot|g|^r\Big) \qquad (3.3.1)\\ &=: C_r^r\sum_{k\in\mathbb{N}}\sum_{P\in\mathcal{S}^k}\Big(\boxed{\mathrm{A}_P}+\boxed{\mathrm{B}_P}\Big).\end{aligned}$$

Fix $k \in \mathbb{N}$ and $P \in \mathcal{S}^k$. As in the estimate for $\boxed{\text{A}}$ in Step 2 of the proof of Theorem 3.2.2, we have

$$\boxed{\text{A}_P} \leq \lambda^r C_T^r \langle \|f\|_X \rangle_{p_0,\alpha P}^r \int_P |g|^r \leq \lambda^r C_T^r \mu(P) \langle \|f\|_X \rangle_{p_0,\alpha P}^r \langle |g| \rangle_{\frac{1}{\frac{1}{r}-\frac{1}{q_0}}}^r,$$

using Hölder's inequality in the second inequality. For $P' \in \mathcal{S}^{k+1}$ such that $P' \subseteq P$ we have as in (3.2.5) that

$$\mu\big(P' \setminus (\Omega_{P'} \cup \Omega_P)\big) \geq \frac{1}{4}\mu(P').$$

Therefore we can estimate each of the terms in the sum in $\boxed{\text{B}_P}$ as follows

$$\begin{aligned}
\int_{P'} \big\| T_{P\setminus P'} f \big\|_Y^r \cdot |g|^r &\leq 2^r \int_{P'} \fint_{P'\setminus(\Omega_P \cup \Omega_{P'})} \big\| T_{P\setminus P'} f(s) - T_{P\setminus P'} f(s') \big\|_Y^r \cdot |g(s)|^r \, \mathrm{d}\mu(s') \, \mathrm{d}\mu(s) \\
&\quad + 2^r \int_{P'} \fint_{P'\setminus(\Omega_P \cup \Omega_{P'})} \big\| T_{P\setminus P'} f(s') \big\|_Y^r \cdot |g(s)|^r \, \mathrm{d}\mu(s') \, \mathrm{d}\mu(s) \\
&\leq 2^{r+2} \mu(P') \inf_{s'' \in P'} \mathcal{M}^{\#}_{T,P,q_0} f(s'')^r \cdot \langle |g| \rangle_{\frac{1}{\frac{1}{r}-\frac{1}{q_0}},P'}^r \\
&\quad + 2^{2r} \mu(P') \fint_{P'\setminus(\Omega_P \cup \Omega_{P'})} \big\| T_P f \big\|_Y^r + \big\| T_{P'} f \big\|_Y^r \, \mathrm{d}\mu \cdot \langle |g| \rangle_{r,P'}^r \\
&\leq 4^{r+2} \lambda^r C_T^r \mu(P') \big(\langle \|f\|_X \rangle_{p_0,\alpha P}^r + \langle \|f\|_X \rangle_{p_0,\alpha P'}^r \big) \langle |g| \rangle_{\frac{1}{\frac{1}{r}-\frac{1}{q_0}},P'}^r
\end{aligned}$$

where we used Hölder's inequality and the definitions of $\mathcal{M}^{\#}_{T,P,q_0}$ and $T_{P\setminus P'}$ in the second inequality and the definitions of Ω_P and $\Omega_{P'}$ in the third inequality. Furthermore we note that by Hölders inequality we have

$$\begin{aligned}
\sum_{\substack{P' \in \mathcal{S}^{k+1}: \\ P' \subseteq P}} \mu(P') \langle |g| \rangle_{\frac{1}{\frac{1}{r}-\frac{1}{q_0}},P'}^r &\leq \Big(\sum_{\substack{P' \in \mathcal{S}^{k+1}: \\ P' \subseteq P}} \int_{P'} |g|^{\frac{1}{\frac{1}{r}-\frac{1}{q_0}}} \, \mathrm{d}\mu \Big)^{1-\frac{r}{q_0}} \cdot \Big(\sum_{\substack{P' \in \mathcal{S}^{k+1}: \\ P' \subseteq P}} \mu(P') \Big)^{r/q_0} \\
&\leq \Big(\int_P |g|^{\frac{1}{\frac{1}{r}-\frac{1}{q_0}}} \, \mathrm{d}\mu \Big)^{1-\frac{r}{q_0}} \cdot \mu(P)^{r/q_0} = \mu(P) \langle |g| \rangle_{\frac{1}{\frac{1}{r}-\frac{1}{q_0}},P}^r
\end{aligned}$$

Thus for $\boxed{\text{B}_P}$ we obtain

$$\begin{aligned}
\boxed{\text{B}_P} \leq 4^{r+2} \lambda^r C_T^r \Big(& \mu(P) \langle \|f\|_X \rangle_{p_0,\alpha P}^r \langle |g| \rangle_{\frac{1}{\frac{1}{r}-\frac{1}{q_0}},P}^r \\
& + \sum_{P' \in \mathcal{S}^{k+1}: P' \subseteq P} \mu(P') \langle \|f\|_X \rangle_{p_0,\alpha P'}^r \langle |g| \rangle_{\frac{1}{\frac{1}{r}-\frac{1}{q_0}},P'}^r \Big)
\end{aligned}$$

Plugging this estimate and the estimate for $\boxed{\text{A}_P}$ into (3.3.1) yields

$$\int_Q \big\| T_Q f \big\|_Y^r \cdot |g|^r \, \mathrm{d}\mu \leq 4^{r+3} \lambda^r C_T^r C_r^r \sum_{P \in \mathcal{S}} \mu(P) \langle \|f\|_X \rangle_{p_0,\alpha P}^r \langle |g| \rangle_{\frac{1}{\frac{1}{r}-\frac{1}{q_0}},P}^r.$$

Since $\lambda = 4c_1c_2$ and c_1 and c_2 only depend on S, α and $\mathscr{D}$, this finishes the proof of the theorem. □

3.4. THE A_2-THEOREM FOR OPERATOR-VALUED CALDERÓN–ZYGMUND OPERATORS

The A_2-theorem, first proved by Hytönen in [Hyt12] as discussed in the introduction, states that a Calderón–Zygmund operator is bounded on $L^2(\mathbb{R}^n, w)$ with a bound that depends linearly on the A_2-characteristic of w. This bound is sharp, and by sharp Rubio de Francia extrapolation (see [DGPP05]) one can obtain sharp weighted bounds for all $p \in (1,\infty)$. Originally the A_2-conjecture was formulated for the Beurling–Ahlfors transform in [AIS01] where it is shown to imply quasiregularity of certain complex functions. Shortly afterwards it was settled for this operator in [PV02] and subsequently many other operators were treated, which eventually led to [Hyt12].

Since its first proof by Hytönen, the A_2-theorem has been extended in various directions. We mention two of these extensions relevant for the current discussion:

- The A_2-theorem for Calderón–Zygmund operators on a geometric doubling metric space was first proven by Nazarov, Reznikov and Volberg [NRV13], afterwards it was proven on a space of homogeneous type by Anderson and Vagharshakyan [AV14] (see also [And15]) using Lerner's mean oscillation decomposition method. It was further extended to the setting of ball bases by Karagulyan [Kar19].

- The A_2-theorem for vector-valued Calderón–Zygmund operators with operator-valued kernel was proven by Hänninen and Hytönen [HH14], using a suitable adapted version of Lerner's median oscillation decomposition.

In this section we will prove sparse domination for vector-valued Calderón–Zygmund operators with operator-valued kernel on a space of homogeneous type. This yields the A_2-theorem for these Caldeŕon–Zygmund operators, unifying the results from [AV14] and [HH14].

As an application of this theorem, we will prove a weighted, anisotropic, mixed norm Mihlin multiplier theorem in the next section. Moreover, we will use it to study maximal regularity for parabolic partial differential equations in Chapters 4 and 5. In these applications S is (a subset of) the anisotropic Euclidean space $\mathbb{R}^n_{\boldsymbol{a}}$ as introduced in Example 2.1.2.

In a different direction our A_2-theorem can be applied in the study of fundamental harmonic analysis operators associated with various discrete and continuous orthogonal expansions, started by Muckenhoupt and Stein [MS65]. In the past decade there has been a surge of results in which such operators are proven to be vector-valued Calderón–Zygmund operators on concrete spaces of homogeneous type. Weighted bounds are then often concluded using [RRT86, Theorem III.1.3] or [RT88]. With our A_2-theorem these results can be made quantitative in terms of the A_p-characteristic. We refer to [BCN12, BMT07, CGR$^+$17, NS12, NS07] and the references therein for an overview of the recent developments in this field.

Let (S,d,μ) be a space of homogeneous type, X and Y be Banach spaces and let

$$K\colon (S\times S)\setminus\{(s,s)\colon s\in S\}\to\mathcal{L}(X,Y)$$

be strongly measurable in the strong operator topology. We say that K is a *Dini kernel* if there is a $c_K\geq 2$ such that

$$\|K(s,t)-K(s,t')\|\leq\omega\Big(\frac{d(t,t')}{d(s,t)}\Big)\frac{1}{\mu\big(B(s,d(s,t))\big)},\qquad 0<d(t,t')\leq\frac{1}{c_K}d(s,t),$$
$$\|K(s,t)-K(s',t)\|\leq\omega\Big(\frac{d(s,s')}{d(s,t)}\Big)\frac{1}{\mu\big(B(s,d(s,t))\big)},\qquad 0<d(s,s')\leq\frac{1}{c_K}d(s,t),$$

where $\omega\colon[0,1]\to[0,\infty)$ is increasing, subadditive, $\omega(0)=0$ and

$$\|K\|_{\mathrm{Dini}}:=\int_0^1\omega(t)\frac{\mathrm{d}t}{t}<\infty.$$

We will introduce variants of this condition and discuss their properties in Section 4.3.

Take $p_0\in[1,\infty)$ and let

$$T\colon L^{p_0}(S;X)\to L^{p_0,\infty}(S;Y)$$

be a bounded linear operator. We say that T has Dini kernel K if for every boundedly supported $f\in L^{p_0}(S;X)$ and a.e. $s\in S\setminus\overline{\operatorname{supp} f}$ we have

$$Tf(s)=\int_S K(s,t)f(t)\,\mathrm{d}t.$$

Theorem 3.4.1. *Let (S,d,μ) be a space of homogeneous type and let X and Y be Banach spaces. Let $p_0\in[1,\infty)$ and suppose T is a bounded linear operator from $L^{p_0}(S;X)$ to $L^{p_0,\infty}(S;Y)$ with Dini kernel K. Then for every boundedly supported $f\in L^1(S;X)$ there exists an η-sparse collection of cubes $\mathcal{S}$ such that*

$$\|Tf(s)\|_Y\lesssim_{S,p_0}C_T\sum_{Q\in\mathcal{S}}\langle\|f\|_X\rangle_{1,Q}\mathbf{1}_Q(s),\qquad s\in S.$$

Moreover, for all $p\in(1,\infty)$ and $w\in A_p$ we have

$$\|T\|_{L^p(S,w;X)\to L^p(S,w;Y)}\lesssim_{S,p,p_0}C_T\,[w]_{A_p}^{\max\{\frac{1}{p-1},1\}}$$

with $C_T:=\|T\|_{L^{p_0}(S;X)\to L^{p_0,\infty}(S;Y)}+\|K\|_{\mathrm{Dini}}$.

Proof. We will check the assumptions of Theorem 3.1.1 with $p_1=p_2=r=1$. The weak L^1-boundedness of T with

$$\|T\|_{L^1(S;X)\to L^{1,\infty}(S;Y)}\lesssim_{S,p}C_T.$$

follows from the classical Calderón-Zygmund argument, see e.g. [RRT86, Theorem III.1.2]. The 1-sublinearity assumption on T follows from the triangle inequality, so the only thing left to check is the weak L^1-boundedness of $\mathcal{M}^{\#}_{T,\alpha}$. Let

$$\alpha := 3\, c_d^2 \max\{\delta^{-1}, c_K\}$$

with c_d the quasi-metric constant, δ as in Proposition 2.1.1 and c_K the constant from the definition of a Dini kernel. Fix $s \in S$ and a ball $B = B(z,\rho)$ such that $s \in B$. Then for any $s', s'' \in B$ and $t \in S \setminus \alpha B$ we have

$$\begin{aligned} d(s',t) &\geq \frac{1}{c_d} d(z,t) - d(z,s') \geq \frac{\alpha\rho}{c_d} - \rho \geq 2\, c_K\, c_d\, \rho =: \varepsilon \\ d(s',s'') &\leq 2\, c_d\, \rho = c_K^{-1}\varepsilon. \end{aligned}$$

Therefore we have for any boundedly supported $f \in L^1(S;X)$

$$\begin{aligned} &\|T(\mathbf{1}_{S\setminus\alpha B} f)(s') - T_K(\mathbf{1}_{S\setminus\alpha B} f)(s'')\|_Y \\ &\quad\leq \int_{S\setminus\alpha B} \big\|\big(K(s',t) - K(s'',t)\big) f(t)\big\|_Y \,\mathrm{d}\mu(t) \\ &\quad\leq \int_{d(s',t)>\varepsilon} \omega\Big(\frac{d(s',s'')}{d(s',t)}\Big) \frac{1}{\mu\big(B(s',d(s',t))\big)} \|f(t)\|_X \,\mathrm{d}\mu(t) \\ &\quad\leq \sum_{j=0}^{\infty} \omega\big(c_K^{-1} 2^{-j}\big) \int_{2^j\varepsilon < d(s',t) \leq 2^{j+1}\varepsilon} \frac{1}{\mu\big(B(s',d(s',t))\big)} \|f(t)\|_X \,\mathrm{d}\mu(t) \\ &\quad\lesssim_S \sum_{j=0}^{\infty} \omega\big(2^{-j-1}\big) \fint_{B(s',2^{j+1}\varepsilon)} \|f(t)\|_X \,\mathrm{d}\mu(t) \\ &\quad\leq \|K\|_{\mathrm{Dini}}\, M\big(\|f\|_X\big)(s), \end{aligned}$$

where the last step follows from $s \in B(s', 2^{j+1}\varepsilon)$ for all $j \in \mathbb{N}$ and

$$\sum_{j=0}^{\infty} \omega\big(2^{-j-1}\big) \leq \sum_{j=0}^{\infty} \omega\big(2^{-j-1}\big) \int_{2^{-j-1}}^{2^{-j}} \frac{\mathrm{d}t}{t} \leq \sum_{j=0}^{\infty} \int_{2^{-j-1}}^{2^{-j}} \omega(t) \frac{\mathrm{d}t}{t} = \|K\|_{\mathrm{Dini}}. \tag{3.4.1}$$

So, taking the supremum over all $s', s'' \in B$ and all balls B containing s, we find that $\mathcal{M}^{\#}_{T,\alpha} f(s) \lesssim_S \|K\|_{\mathrm{Dini}}\, M\big(\|f\|_X\big)(s)$. Thus by the weak L^1-boundedness of the Hardy–Littlewood maximal operator and the density of boundedly supported functions in $L^1(S;X)$ we get

$$\big\|\mathcal{M}^{\#}_{T,\alpha}\big\|_{L^1(S;X)\to L^{1,\infty}(S;Y)} \lesssim_S \|K\|_{\mathrm{Dini}}.$$

The pointwise sparse domination now follows from Theorem 3.1.1 and the weighted bounds from Proposition 3.2.4. □

Remark 3.4.2. In the proof of Theorem 3.4.1 it actually suffices to use the so-called L^r-Hörmander condition for some $r > 1$, which is implied by the Dini condition. See [Li18, Section 3] for the definition of the L^r-Hörmander condition and a comparison between the L^r-Hörmander and the Dini condition.

Note that Theorem 3.4.1 does not assume anything about the Banach spaces X and Y and is therefore applicable in situations where for example $Y = \ell^\infty$. However, in various applications X and Y will need to have the UMD property in order to check the assumed weak L^{p_0}-boundedness of T for some $p_0 \in [1,\infty)$. For instance, for a large class of operators the weak L^{p_0}-boundedness of T can be checked using theorems like the $T(1)$-theorem or $T(b)$-theorem. See [Fig90] and [Hyt14] for these theorems in the vector-valued setting, which assume the UMD property for the underlying Banach space.

3.5. THE WEIGHTED ANISOTROPIC MIXED-NORM MIHLIN MULTIPLIER THEOREM

One of the main Fourier multiplier theorems is the Mihlin multiplier theorem, which was first proven in the operator-valued setting by Weis in [Wei01b]. The operator-valued Mihlin multiplier theorem of Weis has since been extended in many directions. Recently Fackler, Hytönen and Lindemulder extended the operator-valued Mihlin multiplier theorem to a weighted, anisotropic, mixed norm setting in [FHL20]. This is for example useful in the study of spaces of smooth, vector-valued functions and has applications to parabolic PDEs with inhomogeneous boundary conditions, see [Lin20]. In [FHL20] the Mihlin multiplier theorem is shown using the following two approaches:

- Using a weighted Littlewood–Paley decomposition, they show a weighted, anisotropic, mixed-norm Mihlin multiplier theorem for rectangular A_p-weights, i.e. A_p-weights for which the defining supremum is taken over rectangles with sides parallel to the coordinate axes instead of balls.

- Using Calderón–Zygmund theory, they show a weighted, isotropic, non-mixed-norm Mihlin multiplier theorem for cubicular A_p-weights, i.e. A_p-weights for which the defining supremum is taken over cubes with sides parallel to the coordinate axes instead of balls, which is equivalent to the definition using balls in Section 2.3

Both approaches have their pros and cons. The result using a Littlewood–Paley decomposition only requires estimates of $\partial^\theta m$ for $\theta \in \{0,1\}$, whereas the approach using Calderón–Zygmund theory also requires estimates of higher-order derivatives. On the other hand, the class of rectangular A_p-weights is a proper subclass of the class of cubicular A_p-weights.

In applications it is be desirable to have the Mihlin multiplier theorem for cubicular A_p-weights in the anisotropic, mixed-norm setting as well. This would remove the need to distinguish between the isotropic and anisotropic setting in e.g. [Lin20, (6) on p.64]. In order to obtain the Mihlin multiplier theorem for cubicular A_p-weights in the anisotropic, mixed-norm setting, one needs Calderón–Zygmund theory in the anisotropic Euclidean spaces of Example 2.1.2. Since these are a special cases of spaces

of homogeneous type, we can use Theorem 3.4.1 to supplement the results of [FHL20], which will be the main result of this section.

Let us introduce the anisotropic, mixed-norm setting. For $\boldsymbol{a} \in (0,\infty)^d$ we let $\mathbb{R}^d_{\boldsymbol{a}}$ be the anisotropic Euclidean space as introduced in Example 2.1.2. We write $|\boldsymbol{a}|_1 := \sum_{j=1}^d a_j$ and $|\boldsymbol{a}|_\infty := \max_{j=1,\dots,d} a_j$. Take $n \in \mathbb{N}$ and $\vec{d} \in \mathbb{N}^n$ with $\sum_{j=1}^n d_j = d$ and consider the $\vec{d}$-decomposition of $\mathbb{R}^d$:

$$\mathbb{R}^d_{\vec{d}} := \mathbb{R}^{d_1} \times \ldots \times \mathbb{R}^{d_n}.$$

For a $t \in \mathbb{R}^d_{\vec{d}}$ we write $t = (t_1,\dots,t_n)$ with $t_j \in \mathbb{R}^{d_j}$ for $j = 1,\dots,n$ and similarly we write $\boldsymbol{a} = (\boldsymbol{a}_1,\dots,\boldsymbol{a}_n)$. For $\boldsymbol{p} \in [1,\infty)^n$, a vector of weights $\boldsymbol{w} \in \prod_{j=1}^n A_p(\mathbb{R}^{d_j}_{\boldsymbol{a}_j})$ and a Banach space X we define the weighted mixed-norm Bochner space $L^{\boldsymbol{p}}(\mathbb{R}^d_{\vec{d}}, \boldsymbol{w}; X)$ as the space of all strongly measurable $f : \mathbb{R}^d_{\vec{d}} \to X$ such that

$$\|f\|_{L^{\boldsymbol{p}}(\mathbb{R}^d_{\vec{d}},\boldsymbol{w};X)} := \Big(\int_{\mathbb{R}^{d_1}} \ldots \Big(\int_{\mathbb{R}^{d_n}} \|f\|_X^{p_n} w_n \,\mathrm{d}t_n\Big)^{\frac{p_{n-1}}{p_n}} \ldots w_1 \,\mathrm{d}t_1\Big)^{\frac{1}{p_1}}$$

is finite.

Theorem 3.5.1. *Let X and Y be* UMD *Banach spaces, set $N = |\boldsymbol{a}|_1 + |\boldsymbol{a}|_\infty + 1$ and let $m \in L^\infty(\mathbb{R}^d; \mathcal{L}(X,Y))$. Suppose that for all $\theta \in \mathbb{N}^d$ with $\boldsymbol{a}\cdot\theta \le N$ the distributional derivative $\partial^\theta m$ coincides with a continuous function on $\mathbb{R}^d \setminus \{0\}$ and we have the $\mathcal{R}$-bound*

$$\big\|\{|\xi|_{\boldsymbol{a}}^{\boldsymbol{a}\cdot\theta} \cdot \partial^\theta m(\xi) : \xi \in \mathbb{R}^d \setminus \{0\}\}\big\|_{\mathcal{R}} \le C_m.$$

for some $C_m > 0$. Then for every compactly supported $f \in L^1(\mathbb{R}^d; X)$ there exists an η-sparse collection of anisotropic cubes $\mathcal{S}$ such that

$$\|T_m f(s)\|_Y \lesssim_{X,Y,\boldsymbol{a}} C_m \sum_{Q\in\mathcal{S}} \langle \|f\|_X \rangle_{1,Q} \mathbf{1}_Q(s), \qquad s \in \mathbb{R}^d.$$

Moreover, for all $\boldsymbol{p} \in (1,\infty)^n$ there is a function $\phi : \mathbb{R}^n_+ \to \mathbb{R}_+$, depending on $X, Y, \vec{d}, \boldsymbol{a}, \boldsymbol{p}$ and nondecreasing in every variable, such that for all $\boldsymbol{w} \in \prod_{j=1}^n A_{p_j}(\mathbb{R}^{d_j}_{\boldsymbol{a}_j})$ we have

$$\|T_m\|_{L^{\boldsymbol{p}}(\mathbb{R}^d_{\vec{d}},\boldsymbol{w};X)\to L^{\boldsymbol{p}}(\mathbb{R}^d_{\vec{d}},\boldsymbol{w};Y)} \le C_m \cdot \phi\Big([w_1]_{A_{p_1}(\mathbb{R}^{d_1}_{\boldsymbol{a}_1})}, \ldots, [w_n]_{A_{p_n}(\mathbb{R}^{d_n}_{\boldsymbol{a}_n})}\Big).$$

Proof. We will check the conditions of Theorem 3.4.1. By [Hyt07, Theorem 3], which trivially extends to the case $X \neq Y$, we know that T_m is bounded from $L^2(\mathbb{R}^d; X)$ to $L^2(\mathbb{R}^d; Y)$ with

$$\|T_m\|_{L^2(\mathbb{R}^d;X)\to L^2(\mathbb{R}^d;Y)} \lesssim_{X,Y,\vec{d},\boldsymbol{a}} C_m.$$

By [Lin14a, Lemma 4.4.6 and 4.4.7] we know that $\check{m}$ coincides with a continuous function on $\mathbb{R}^d \setminus \{0\}$, which is bounded away from 0 and moreover,

$$K(t,s) := \check{m}(t-s), \qquad t \neq s$$

is a Dini kernel on the space of homogeneous type $\mathbb{R}^d_{\boldsymbol{a}}$ with

$$\omega(r) = C_{\boldsymbol{a}} \cdot C_m \cdot r^{\epsilon}, \qquad r \in [0,1]$$

with $\epsilon = \min_{1\le k\le n} a_k$. Now let $f \in L^p(\mathbb{R}^d; X)$ with compact support. Fix a $c \in \mathbb{R}^d \setminus \overline{\operatorname{supp} f}$ and take $r > 0$ such that $B(c,2r) \cap \overline{\operatorname{supp} f} = \varnothing$. Take a sequence $(f_k)_{k=1}^{\infty}$ in $\mathcal{S}(\mathbb{R}^d; X)$ such that $\overline{\operatorname{supp} f_k} \cap B(c,r) = \varnothing$ and $f_k \to f$ in $L^2(\mathbb{R}^d; X)$. Then $Tf_k \to Tf$ in $L^2(\mathbb{R}^d; X)$ and, by passing to a subsequence if necessary, we have $f_k(t) \to f(t)$ and $Tf_k(t) \to Tf(t)$ for a.e. $t \in \mathbb{R}^d$. Fix $k \in \mathbb{N}$, then we have for all $\varphi \in C_c^{\infty}(\mathbb{R}^d \setminus \overline{\operatorname{supp} f_k})$

$$\begin{aligned}
\langle T_m f_k, \varphi\rangle &= \int_{\mathbb{R}^d} m(s)\widehat{f_k}(s)\check{\varphi}(s)\,\mathrm{d}s \\
&= \int_{\mathbb{R}^d} \check{m}(s) \int_{\mathbb{R}^d} f_k(t-s)\varphi(t)\,\mathrm{d}t\,\mathrm{d}s \\
&= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} K(t,s) f_k(s)\,\mathrm{d}s\,\varphi(t)\,\mathrm{d}t
\end{aligned}$$

from which we obtain for a.e. $t \in B(c,r)$

$$T_m f(t) = \lim_{k\to\infty} T_m f_k(t) = \lim_{n\to\infty} \int_{\mathbb{R}^d} K(t,s) f_k(s)\,\mathrm{d}s = \int_{\mathbb{R}^d} K(t,s) f(s)\,\mathrm{d}s.$$

Covering $\mathbb{R}^d \setminus \overline{\operatorname{supp} f}$ by countably many such balls, we conclude that T_m has kernel K. Therefore the sparse domination, as well as the weighted estimate in case $n = 1$, follows from Theorem 3.4.1.

To conclude the proof we will show the case $n = 2$, the general case follows by iterating the argument. Take $\boldsymbol{p} \in (1,\infty)^2$ and $\boldsymbol{w} \in A_{p_1}(\mathbb{R}^{d_1}_{\boldsymbol{a}_1}) \times A_{p_2}(\mathbb{R}^{d_2}_{\boldsymbol{a}_2})$. For $v_1 \in A_{p_2}(\mathbb{R}^{d_1}_{\boldsymbol{a}_1})$ note that

$$v(t) := v_1(t_1) \cdot w_2(t_2), \qquad t \in \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$$

belongs to $A_{p_2}(\mathbb{R}^d_{\boldsymbol{a}})$, so by the case $n = 1$ we have

$$\|T_m f\|_{L^{p_2}(\mathbb{R}^d, v; Y)} \lesssim_{X,Y,d,\boldsymbol{a},p_2,v} C_m \cdot \|f\|_{L^{p_2}(\mathbb{R}^d, v; X)}$$

for all $f \in L^{p_2}(\mathbb{R}^d, v; X)$. Since $\mathbb{R}^{d_2}_{\boldsymbol{a}_2}$ is a space of homogeneous type, we can use Rubio de Francia extrapolation (Theorem 2.3.3) on the pairs of functions

$$(f,g) \in \left\{ \left(\|T_m f\|_{L^{p_2}(\mathbb{R}^{d_2}, w_2; Y)}, \|f\|_{L^{p_2}(\mathbb{R}^{d_2}, w_2; X)} \right) : f \in \mathcal{S}(\mathbb{R}^d; X) \right\}$$

to deduce the existence of a function $\phi : \mathbb{R}^2_+ \to \mathbb{R}_+$, depending on $X, Y, d, \boldsymbol{a}, \boldsymbol{p}$ and non-decreasing in both variables, such that

$$\|T_m f\|_{L^{\boldsymbol{p}}(\mathbb{R}^d_d, \boldsymbol{w}; Y)} \le C_m \cdot \phi\Big([w_1]_{A_{p_1}(\mathbb{R}^{d_1}_{\boldsymbol{a}_1})}, [w_2]_{A_{p_2}(\mathbb{R}^{d_2}_{\boldsymbol{a}_2})} \Big) \|f\|_{L^{\boldsymbol{p}}(\mathbb{R}^d_d, \boldsymbol{w}; X)}$$

for all $f \in \mathcal{S}(\mathbb{R}^d; X)$, which implies the result by density. □

Remark 3.5.2.

(i) The nondecreasing function in Theorem 3.5.1 in the case $n = 1$ is

$$\phi(t) = C_{X,Y,d,\boldsymbol{a},\boldsymbol{p}} \cdot t^{\max\{\frac{1}{p-1},1\}}, \qquad t \in [1,\infty),$$

which is sharp. For $n \geq 2$ the weight dependence our proof yields is more complicated and not sharp for all choices of $\boldsymbol{p} \in (1,\infty)^n$.

(ii) In the proof of Theorem 3.5.1 we only use the $\mathcal{R}$-boundedness of the set

$$\{|\xi|_{\boldsymbol{a}}^{\boldsymbol{a}\cdot\theta} \cdot \partial^\theta m(\xi) : \xi \in \mathbb{R}^d \setminus \{0\}\}$$

for $\theta \in \{0,1\}^d$. For all other $\theta \in \mathbb{N}^d$ with $\boldsymbol{a}\cdot\theta \leq N$ it suffices to know uniform boundedness of this set.

(iii) One could reduce the number of derivatives necessary in Theorem 3.5.1 by arguing as in [Hyt04] instead of using [Lin14a, Lemma 4.4.6 and 4.4.7]. See also [FHL20, Section 6].

(iv) Using the sparse domination in Theorem 3.5.1, one can also deduce two-weight estimates for T_m as in [FHL20, Section 6].

3.6. THE RADEMACHER MAXIMAL FUNCTION

In this section we will apply Theorem 3.2.2 to the Rademacher maximal function. The proofs will illustrate nicely how the geometry of the Banach space plays a role in deducing the localized ℓ^r-estimate for this operator. In particular, we will use the type of a Banach space X to deduce the localized ℓ^r-estimate for the Rademacher maximal function.

The Rademacher maximal function was introduced by Hytönen, McIntosh and Portal in [HMP08] as a vector-valued generalization of Doob's maximal function that takes into account the different "directions" in a Banach space. They used the Rademacher maximal function to prove a Carleson's embedding theorem for vector-valued functions in connection to Kato's square root problem in Banach spaces. The Carleson's embedding theorem for vector-valued functions has since found many other applications, like the local vector-valued $T(b)$ theorem (see [HV15]) and vector-valued multilinear multiplier theorems (see [DO18]).

Let (S,d,μ) be a space of homogeneous type with a dyadic system $\mathscr{D}$ and let X be a Banach space. For $f \in L^1_{\mathrm{loc}}(S;X)$ we define the *Rademacher maximal function* by

$$M^{\mathscr{D}}_{\mathrm{Rad}} f(s) := \sup\Big\{\Big\| \sum_{Q\in\mathscr{D}:s\in Q} \varepsilon_Q \lambda_Q \langle f\rangle_{1,Q} \Big\|_{L^2(\Omega;X)} : (\lambda_Q)_{Q\in\mathscr{D}} \text{ finitely non-zero with } \sum_{Q\in\mathscr{D}} |\lambda_Q|^2 \leq 1\Big\},$$

where $(\varepsilon_Q)_{Q\in\mathscr{D}}$ is a Rademacher sequence on Ω. One can interpret this maximal function as Doob's maximal function

$$f^*(s) := \sup_{Q\in\mathscr{D}:s\in Q} \big\|\langle f\rangle_{1,Q}\big\|_X, \qquad s\in S,$$

with the uniform bound over the $\langle f\rangle_{1,Q}$'s replaced by the $\mathcal{R}$-bound. Here the $\mathcal{R}$-bound of a set $U\subseteq X$ is the $\mathcal{R}$-bound of the family of operators $T_x:\mathbb{C}\to X$ given by $\lambda\mapsto\lambda x$ for $x\in U$.

We say that the Banach space X has the RMF property if $M^{\mathscr{D}}_{\mathrm{Rad}}$ is a bounded operator on $L^p([0,1);X)$ for some $p\in(1,\infty)$, where $\mathscr{D}$ is the standard dyadic system in $[0,1)$. It was shown by Hytönen, McIntosh and Portal [HMP08, Proposition 7.1] that this implies boundedness for all $p\in(1,\infty)$ and by Kemppainen [Kem11, Theorem 5.1] that this implies boundedness of $M^{\mathscr{D}}_{\mathrm{Rad}}$ on $L^p(S;X)$ for any space of homogeneous type (S,d,μ) with a dyadic system $\mathscr{D}$.

The relation of RMF property to other Banach space properties is not yet fully understood. However, we do have some necessary and sufficient conditions:

- If X has type 2, the $\mathcal{R}$-bound of a set $U\subseteq X$ is equivalent to the uniform bound of U. Therefore if X has type 2 we have for any $f\in L^1_{\mathrm{loc}}([0,1);X)$ that $M^{\mathscr{D}}_{\mathrm{Rad}}f\lesssim M^{\mathscr{D}}(\|f\|_X)$, so X has the RMF property.
- Any UMD Banach function space has the RMF property, which follows from the boundedness of the *lattice Hardy–Littlewood maximal operator* on UMD Banach function spaces. We will introduce the lattice Hardy–Littlewood maximal operator and discuss this connection in Section 6.4
- Non-commutative L^p-spaces for $p\in(1,\infty)$ have the RMF property, see [HMP08, Corollary 7.6].
- The RMF property implies nontrivial type, see [Kem11, Proposition 4.2]. Therefore e.g. ℓ^1 does not have the RMF property.

It is an open problem whether nontrivial type or even the UMD property implies the RMF property in a general Banach space. We refer to [HNVW16, Section 3.6.b] for a further introduction to the RMF property.

Weighted bounds for the Rademacher maximal function in the Euclidean setting were studied by Kemppainen [Kem13, Theorem 1]. The proof was based on a good-λ inequality, which does not give sharp quantitative estimates in terms of the weight characteristic. Using Theorem 3.2.2 we can prove sharp quantitative weighted estimates for the Rademacher maximal function through sparse domination. We will not consider the situation in which X has type 2, as this case follows directly from $M^{\mathscr{D}}_{\mathrm{Rad}}f\lesssim M^{\mathscr{D}}(\|f\|_X)$ and the well-known sparse domination for the Hardy–Littlewood maximal operator.

We will need a version of the Rademacher maximal function for finite collections of cubes. For a subcollection of cubes $\mathcal{D}\subseteq\mathscr{D}$ we define $M^{\mathcal{D}}_{\mathrm{Rad}}$ analogous to $M^{\mathscr{D}}_{\mathrm{Rad}}$.

Theorem 3.6.1. *Let (S,d,μ) be a space of homogeneous type with a dyadic system $\mathscr{D}$ and let X be a Banach space with the* RMF *property. Assume that X has type r for $r\in[1,2)$. For any finite collection of cubes $\mathcal{D}\subseteq\mathscr{D}$ and $f\in L^1(S;X)$ there exists an $\frac{1}{2}$-sparse collection of cubes $\mathcal{S}\subseteq\mathscr{D}$ such that*

$$M^{\mathcal{D}}_{\mathrm{Rad}}f(s)\lesssim_{X,S,\mathscr{D},r}\Big(\sum_{Q\in\mathcal{S}}\langle\|f\|_X\rangle_{1,Q}^{(\frac{1}{r}-\frac{1}{2})^{-1}}\mathbf{1}_Q(s)\Big)^{\frac{1}{r}-\frac{1}{2}},\qquad s\in S$$

Moreover, for all $p\in(1,\infty)$ and $w\in A_p$ we have

$$\big\|M^{\mathscr{D}}_{\mathrm{Rad}}\big\|_{L^p(S,w;X)\to L^p(S,w;X)}\lesssim_{X,S,\mathscr{D},p,r}[w]_{A_p}^{\max\{\frac{1}{p-1},\frac{1}{r}-\frac{1}{2}\}}.$$

Proof. Fix a finite collection of cubes $\mathcal{D}\subseteq\mathscr{D}$. By [Kem11, Proposition 6.1] $M^{\mathcal{D}}_{\mathrm{Rad}}$ is weak L^1-bounded. We will view $M^{\mathcal{D}}_{\mathrm{Rad}}$ as a bounded operator

$$M^{\mathcal{D}}_{\mathrm{Rad}}:L^1(S;X)\to L^{1,\infty}(S;\mathcal{L}(\ell^2(\mathcal{D}),L^2(\Omega;X)))$$

given by

$$M^{\mathcal{D}}_{\mathrm{Rad}}f(s)=\Big((\lambda_Q)_{Q\in\mathcal{D}}\mapsto\sum_{Q\in\mathcal{D}:s\in Q}\varepsilon_Q\lambda_Q\langle f\rangle_{1,Q}\Big),\qquad s\in S,$$

where $(\varepsilon_Q)_{Q\in\mathcal{D}}$ is a Rademacher sequence on Ω.

For $Q\in\mathscr{D}$ set

$$\mathcal{D}(Q):=\{P\in\mathcal{D}:P\subseteq Q\}$$

and define $T_Q:=M^{\mathcal{D}(Q)}_{\mathrm{Rad}}$. Then $\{T_Q\}_{Q\in\mathscr{D}}$ is a 1-localization family for $M^{\mathcal{D}}_{\mathrm{Rad}}$. Furthermore we have for $f\in L^1(S;X)$ and $s\in Q\in\mathscr{D}$ that

$$\begin{aligned}\mathcal{M}^{\#}_{M^{\mathcal{D}}_{\mathrm{Rad}},Q}f(s)&=\sup_{\substack{Q'\in\mathscr{D}(Q):\\ s\in Q'}}\operatorname*{ess\,sup}_{s',s''\in Q'}\big\|T_{Q\setminus Q'}f(s')-T_{Q\setminus Q'}f(s'')\big\|_{\mathcal{L}(\ell^2(\mathscr{D}),L^2(\Omega;X))}\\&=0\end{aligned}$$

where the last step follows from the fact that $T_{Q\setminus Q'}f=M^{\mathcal{D}(Q)\setminus\mathcal{D}(Q')}_{\mathrm{Rad}}f$ is constant on Q'. So $\mathcal{M}^{\#}_{M^{\mathcal{D}}_{\mathrm{Rad}},Q}$ is trivially bounded from $L^1(S;X)$ to $L^{1,\infty}(S)$.

Set $q:=(\frac{1}{r}-\frac{1}{2})^{-1}$. To check the localized ℓ^q-estimate for $M^{\mathcal{D}}_{\mathrm{Rad}}$ take $Q_1,\dots,Q_n\in\mathscr{D}$ with $Q_n\subseteq\ldots\subseteq Q_1$. Let $(\lambda_Q)_{Q\in\mathcal{D}}\in\ell^2(\mathcal{D})$ be of norm one and let $(\varepsilon_Q)_{Q\in\mathcal{D}}$ and $(\varepsilon'_k)_{k=1}^n$ be Rademacher sequences on Ω and Ω' respectively. Define for $k=1,\dots,n-1$

$$\lambda_k:=\Big(\sum_{Q\in\mathcal{D}(Q_{k+1})\setminus\mathcal{D}(Q_k)}|\lambda_Q|^2\Big)^{1/2},\qquad\lambda_n:=\Big(\sum_{Q\in\mathcal{D}(Q_n)}|\lambda_Q|^2\Big)^{1/2}$$

Then for $f\in L^1(S;X)$, setting $f_Q:=\varepsilon_Q\lambda_Q\langle f\rangle_{1,Q}$, we have

$$\Big\|\sum_{Q\in\mathcal{D}(Q_1)}\varepsilon_Q\lambda_Q\langle f\rangle_{1,Q}\Big\|_{L^2(\Omega;X)}$$

$$
\begin{aligned}
&= \Big\| \varepsilon'_n \sum_{Q\in\mathcal{D}(Q_n)} f_Q + \sum_{k=1}^{n-1} \varepsilon'_k \sum_{Q\in\mathcal{D}(Q_{k+1})\setminus\mathcal{D}(Q_k)} f_Q \Big\|_{L^2(\Omega\times\Omega';X)} \\
&\lesssim_{X,r} \Big(\lambda_n^r \Big\| \sum_{Q\in\mathcal{D}(Q_n)} \lambda_n^{-1} f_Q \Big\|_{L^2(\Omega;X)}^r + \sum_{k=1}^{n-1} \lambda_k^r \Big\| \sum_{Q\in\mathcal{D}(Q_{k+1})\setminus\mathcal{D}(Q_k)} \lambda_k^{-1} f_Q \Big\|_{L^2(\Omega;X)}^r \Big)^{1/r} \\
&\le \Big(\Big\| \sum_{Q\in\mathcal{D}(Q_n)} \varepsilon_Q \lambda_n^{-1} \lambda_Q \langle f\rangle_{1,Q} \Big\|_{L^2(\Omega;X)}^q + \sum_{k=1}^{n-1} \Big\| \sum_{Q\in\mathcal{D}(Q_{k+1})\setminus\mathcal{D}(Q_k)} \varepsilon_Q \lambda_k^{-1} \lambda_Q \langle f\rangle_{1,Q} \Big\|_{L^2(\Omega;X)}^q \Big)^{1/q},
\end{aligned}
$$

using randomization (see [HNVW17, Proposition 6.1.11]) in the first step, type r of X in the second step, and Hölder's inequality and $\sum_{k=1}^n \lambda_k^2 = 1$ in the last step. Noting that for $k = 1,\dots,n-1$

$$
\sum_{Q\in\mathcal{D}(Q_{k+1})\setminus\mathcal{D}(Q_k)} |\lambda_k^{-1}\lambda_Q|^2 = 1, \qquad \sum_{Q\in\mathcal{D}(Q_n)} |\lambda_n^{-1}\lambda_Q|^2 = 1,
$$

this implies the localized ℓ^q-estimate for $M_{\mathrm{Rad}}^{\mathcal{D}}$.

Having checked all assumptions of Theorem 3.2.2 for $M_{\mathrm{Lat}}^{\mathcal{D}}$ it follows that for any $Q\in\mathcal{D}$ there is a $\frac{1}{2}$-sparse collection of cubes $\mathcal{S}_Q \subseteq \mathscr{D}(Q)$ such that

$$
\big\| T_Q(s) \big\|_Y \lesssim_{X,S,\mathscr{D},r} \Big(\sum_{P\in\mathcal{S}} \langle \|f\|_X \rangle_{p,\alpha P}^r \mathbf{1}_P(s) \Big)^{1/r}, \qquad s\in Q.
$$

Let $\mathcal{D}'$ be the maximal cubes (with respect to set inclusion) of $\mathcal{D}$, which are pairwise disjoint. Then $\mathcal{S} := \bigcup_{Q\in\mathcal{D}'} \mathcal{S}_Q$ is a $\frac{1}{2}$-sparse collection of cubes that satisfies the claimed sparse domination as $T_Q(s) = M_{\mathrm{Rad}}^{\mathcal{D}} f(s)$ for any $s\in Q\in\mathcal{D}'$ and $M_{\mathrm{Rad}}^{\mathcal{D}} f$ is zero outside $\bigcup_{Q\in\mathcal{D}'} Q$. The weighted bounds follow from Proposition 3.2.4 and the monotone convergence theorem. □

We finish this section by showing that the weighted estimate in Theorem 3.6.1, and consequently also the sparse domination in Theorem 3.6.1, is sharp. We take $X = \ell^r$ for $r\in(1,2]$, a prototypical Banach space with type r.

Proposition 3.6.2. *Let $p\in(1,\infty)$ and $r\in(1,2]$. Suppose that for some $\beta\ge 0$ we have*

$$
\big\| M_{\mathrm{Rad}}^{\mathscr{D}} \big\|_{L^p([0,1);\ell^r)\to L^p([0,1);\ell^r)} \lesssim_{p,r} [w]_{A_p}^{\beta}.
$$

for all $w\in A_p$. Then

$$
\beta \ge \max\Big\{ \frac{1}{p-1}, \frac{1}{r} - \frac{1}{2} \Big\},
$$

Proof. Since $\mathcal{R}$-bounds are stronger than uniform bounds, we note that for any strongly measurable $f\colon [0,1)\to\ell^r$ we have

$$
f^*(s) \le M_{\mathrm{Rad}}^{\mathscr{D}} f(s), \qquad s\in[0,1).
$$

Thus by the corresponding result for Doob's maximal operator (see [HNVW16, Proposition 3.2.4]), we have for $p\in(1,\infty)$

$$
\big\| M_{\mathrm{Rad}}^{\mathscr{D}} \big\|_{L^p([0,1);\ell^r)\to L^p([0,1);\ell^r)} \ge \frac{p}{p-1}, \tag{3.6.1}
$$

which implies

$$\alpha_M := \sup\Big\{\alpha \geq 0 : \forall \varepsilon > 0, \limsup_{p\to 1} \frac{\|M^{\mathscr{D}}_{\mathrm{Rad}}\|_{L^p([0,1);\ell^r)\to L^p([0,1);\ell^r)}}{(p-1)^{-\alpha+\varepsilon}} = \infty\Big\} = 1.$$

Now let $(e_n)_{n=1}^\infty$ be the canonical basis of ℓ^r and define

$$f(s) := \sum_{n=1}^\infty \mathbf{1}_{[2^{-n},2^{-n+1})}(s) e_n, \qquad s \in [0,1).$$

For $p \in (1,\infty)$ we have

$$\|f\|_{L^p([0,1);\ell^r)} = 1.$$

To compute $\|M^{\mathscr{D}}_{\mathrm{Rad}} f\|_{L^p([0,1);\ell^r)}$ set $I_j := [0, 2^{-j+1}]$, take $s \in$ and let $m \in \mathbb{N}$ be such that $2^{-m} \leq s \leq 2^{-m+1}$. Then we have, using $\lambda_{I_j} = m^{-1/2}$ for $j = 1, \dots, m$ and Proposition 2.5.1, that

$$\begin{aligned} M^{\mathscr{D}}_{\mathrm{Rad}} f(s) &\geq \frac{1}{m^{1/2}} \Big\| \sum_{j=1}^m \varepsilon_j \langle f \rangle_{1,I_j} \Big\|_{L^2(\Omega;\ell^r)} \gtrsim \frac{1}{m^{1/2}} \Big\| \Big(\sum_{j=1}^m \langle f \rangle_{1,I_j}^2 \Big)^{1/2} \Big\|_{\ell^r} \\ &\gtrsim \frac{1}{m^{1/2}} \Big\| \sum_{j=1}^m e_j \Big\|_{\ell^r} \gtrsim m^{1/r-1/2} \gtrsim \log(1/s)^{1/r-1/2}. \end{aligned}$$

Therefore we obtain

$$\begin{aligned} \|M^{\mathscr{D}}_{\mathrm{Rad}} f\|_{L^p([0,1);\ell^r)} &\gtrsim \Big(\int_0^1 \log(1/s)^{p/r-p/2} \, \mathrm{d}s \Big)^{1/p} \\ &= \Big(\int_1^\infty x^{p/r-p/2} \mathrm{e}^{-x} \, \mathrm{d}x \Big)^{1/p} \\ &\geq \Big(\sum_{n=2}^\infty n^{p/r-p/2} \mathrm{e}^{-n} \Big)^{1/p} \\ &\gtrsim p^{1/r-1/2}, \end{aligned}$$

where we drop all terms except $n = \lceil p \rceil$ in the last step. Thus, we find

$$\gamma_M = \sup\Big\{\gamma \geq 0 : \forall \varepsilon > 0, \limsup_{p\to\infty} \frac{\|M^{\mathscr{D}}_{\mathrm{Rad}}\|_{L^p([0,1);\ell^r)\to L^p([0,1);\ell^r)}}{p^{-\gamma+\varepsilon}} = \infty\Big\} = \frac{1}{r} - \frac{1}{2}.$$

The proposition now follows from [LPR15, Theorem 1.2]. □

3.7. LITTLEWOOD–PALEY OPERATORS

As a third application of our main sparse domination result we prove sparse domination and consequently sharp weighted norm estimates for Littlewood–Paley operators. Sharp weighted norm inequalities for Littlewood–Paley operators were obtained

by Lerner [Ler11], who used his local mean oscillation decomposition to deduce sparse domination for various Littlewood–Paley operators S. This implies

$$\|S\|_{L^p(\mathbb{R}^n,w)\to L^p(\mathbb{R}^n,w)} \lesssim [w]_{A_p}^{\max\left\{\frac{1}{p-1},\frac{1}{2}\right\}}$$

for all $p\in(1,\infty)$ and $w\in A_p$ and the dependence on the weight characteristic is sharp (see [Ler08]). The goal of this section is to show that these sharp weighted norm inequalities are an almost direct corollary from Theorem 3.2.2 with $r=2$ and the well-known weak L^1-boundedness of Littlewood–Paley operators.

In [Wil07] (see also [Wil08, Chapter 6]) Wilson introduced the so-called *intrinsic square function*, which pointwise dominates the Lusin area integral, the Littlewood–Paley g-function and their more modern, real-variable variants. Therefore it suffices to show sparse domination for this intrinsic square function, which we will now introduce. For $\alpha\in(0,1]$ let $\mathcal{C}_\alpha$ be the family of functions $\varphi:\mathbb{R}^n\to\mathbb{R}$ supported in $\{x\in\mathbb{R}^n:|x|\le 1\}$, satisfying $\int_{\mathbb{R}^n}\varphi(x)\,\mathrm{d}x=0$ and

$$|\varphi(x)-\varphi(x')|\le|x-x'|^\alpha,\qquad x,x'\in\mathbb{R}^n.$$

Let $\mathbb{R}^{n+1}_+:=\mathbb{R}^n\times\mathbb{R}_+$ and define the cone of aperture $\beta>0$ by

$$\Gamma_\beta(x):=\{(y,t)\in\mathbb{R}^{n+1}_+:|x-y|<\beta t\},\qquad x\in\mathbb{R}^n.$$

For $f\in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ set

$$A_\alpha(f)(y,t)=\sup_{\varphi\in\mathcal{C}_\alpha}|f*\varphi_t(y)|,\qquad (y,t)\in\mathbb{R}^{n+1}_+,$$

where $\varphi_t(x):=t^{-n}\varphi(x/t)$. We define the intrinsic square function of order $\alpha\in(0,1]$ and aperture $\beta>0$ by

$$G_{\alpha,\beta}(f)(x):=\Big(\int_{\Gamma_\beta(x)}A_\alpha(f)(y,t)^2\frac{\mathrm{d}y\,\mathrm{d}t}{t^{n+1}}\Big)^{1/2},\qquad x\in\mathbb{R}^n.$$

We will prove sparse domination for a local variant of the intrinsic square function, from which weighted bounds for $G_{\alpha,\beta}$ will follow by an approximation argument.

Theorem 3.7.1. *Let $\alpha\in(0,1]$ and $\beta>0$. For all $p\in(1,\infty)$ and $w\in A_p$ we have*

$$\|G_{\alpha,\beta}\|_{L^p(\mathbb{R}^n,w)\to L^p(\mathbb{R}^n,w)}\lesssim_{\alpha,\beta,p,n}[w]_{A_p}^{\max\left\{\frac{1}{p-1},\frac{1}{2}\right\}}.$$

Proof. We will first check the assumptions of Theorem 3.2.2 for $G_{\alpha,\beta}$. The weak L^1-bound follows from [Wil07, Section 1]. We will interpret $G_{\alpha,\beta}$ as a linear bounded operator

$$G_{\alpha,\beta}\colon L^1(\mathbb{R}^n)\to L^{1,\infty}\Big(\mathbb{R}^n;L^2\big(\mathbb{R}^{n+1}_+,\frac{\mathrm{d}y\,\mathrm{d}t}{t^{n+1}}\big)\Big)$$

given by

$$G_{\alpha,\beta}(f)(x) := (t,y) \mapsto \mathbf{1}_{\Gamma_\beta(x)}(y,t) A_\alpha(f)(y,t), \qquad x \in \mathbb{R}^n.$$

Fix a cube $Q \subseteq \mathbb{R}^n$ and let $\mathscr{D}$ be a dyadic system in $\mathbb{R}^n$ containing Q. For $P \in \mathscr{D}$ we define the restricted cone of aperture $\beta > 0$ by

$$\Gamma_\beta^P(x) := \big\{(y,t) \in \mathbb{R}^{n+1}_+ : |x-y| < \beta t,\ t \le \operatorname{diam}(P)\big\}, \qquad x \in \mathbb{R}^n$$

and let $G^P_{\alpha,\beta}$ be defined analogously to $G_{\alpha,\beta}$ with Γ_β replaced by Γ^P_β. Then $\{T_P\}_{P\in\mathscr{D}}$ with $T_P := G^P_{\alpha,\beta}$ is a $(3+2\beta)$-localization family for $G_{\alpha,\beta}$. Indeed the localization property follows from

$$A_\alpha(f)(y,t) = A_\alpha(f\,\mathbf{1}_E)(y,t), \qquad (y,t) \in \mathbb{R}^{n+1}_+$$

for any $E \subseteq \mathbb{R}^n$ containing $B(y,t)$ and

$$\bigcup \{B(y,t) : (y,t) \in \Gamma^P_\beta(x) \text{ for some } x \in P\} \subseteq (1+2(1+\beta))P.$$

The domination property follows from $\Gamma^\rho_\beta(x) \subseteq \Gamma_\beta(x)$ for any $x \in \mathbb{R}^n$. The localized ℓ^2-estimate in Theorem 3.2.2 for this localization family follows from the pairwise disjointness of

$$\Gamma^{P_n}_\beta(x) \quad \text{and} \quad \Gamma^{P_k}_\beta(x) \setminus \Gamma^{P_{k+1}}_\beta(x) \qquad k = 1,\dots,n-1$$

for $P_1,\dots,P_n \in \mathscr{D}$ with $P_n \subseteq \cdots \subseteq P_1$.

Now let $x \in \mathbb{R}^n$ and $P, P' \in \mathscr{D}$ such that $x \in P' \subseteq P$. Then we have for any $x' \in P'$ that

$$\Gamma^P_\beta(x') \setminus \Gamma^{P'}_\beta(x') \subseteq \Gamma_{1+\beta}(x)$$

and thus

$$T_{P\setminus P'} f(x') \le G_{\alpha,\beta+1}(f)(x)$$

So $\mathcal{M}^\#_{G_{\alpha,\beta},P} f \le 2 G_{\alpha,1+\beta} f$ and since $G_{\alpha,1+\beta}$ is weak L^1-bounded (see [Wil07, Section 1]), it follows that $\mathcal{M}^\#_{G_{\alpha,\beta},P}$ is weak L^1-bounded uniformly in $P \in \mathscr{D}$.

We can conclude by Theorem 3.2.2 that for any cube $Q \subseteq \mathbb{R}^n$ and $f \in L^1(S;X)$ there is a $\frac{1}{2}$-sparse family of cubes $\mathcal{S}$ such that

$$G^Q_{\alpha,\beta}(f)(x) \lesssim_{\alpha,\beta,n} \Big(\sum_{P\in\mathcal{S}} \langle |f| \rangle^2_{1,P} \mathbf{1}_P(x)\Big)^{1/2}, \qquad x \in Q.$$

Combined with Proposition 3.2.4 this yields the claimed weighted bounds for $G^Q_{\alpha,\beta}$ for any cube $Q \subseteq \mathbb{R}^n$. Taking an increasing sequence of cubes $(Q_k)_{k\in\mathbb{N}}$ with $\bigcup_{k\in\mathbb{N}} Q_k = \mathbb{R}^n$ and using the monotone convergence theorem yields the same weighted bounds for $G_{\alpha,\beta}$, finishing the proof. □

Remark 3.7.2.

- Using similar arguments as in the proof of Theorem 3.7.1 one can also treat the dyadic square function with Theorem 3.2.2. This yields sharp weighted norm inequalities for the dyadic square function as obtained by Cruz-Uribe, Martell and Perez [CMP12].
- Recently Bui and Duong [BD20] extended the result of Lerner [Ler11] to square functions of a general operator L which has a Gaussian heat kernel bound and a bounded holomorphic functional calculus on $L^2(S)$, where (S,d,μ) is a space of homogeneous type. The arguments they present can also be used to estimate our sharp grand maximal truncation operator.

3.8. FURTHER APPLICATIONS

In this final section provide some further applications of our main theorems. We start with a sparse domination result for the Haar decomposition of $L^p(\mathbb{R};X)$ for a UMD Banach space X. We will use this result in Chapter 6 to connect the UMD constant to a sparsely dominated operator.

Let X be a Banach space, let $\mathscr{D}$ be the standard dyadic system in $\mathbb{R}$ and for $I \in \mathscr{D}$ define the Haar function h_I by

$$h_I := |I|^{\frac{1}{2}}(\mathbf{1}_{I_-} - \mathbf{1}_{I_+}),$$

where I_+ and I_- are the left and right half of I. For $f \in L^1_{\mathrm{loc}}(\mathbb{R};X)$ define the Haar projection D_I by

$$D_I f(t) := h_I(t) \int_{\mathbb{R}} f(s) h_I(s)\, \mathrm{d}s, \qquad t \in \mathbb{R}. \tag{3.8.1}$$

Theorem 3.8.1. *Let X be a* UMD *Banach space, $p \in (1,\infty)$ and let $\mathscr{D}$ be the standard dyadic system in $\mathbb{R}$. Take $\epsilon_I \in \{-1,1\}$ for all $I \in \mathscr{D}$. Then for any compactly supported $f \in L^1(\mathbb{R};X)$ there exists a $\frac{1}{12}$-sparse collection of intervals $\mathcal{S}$ such that*

$$\Big\| \sum_{I\in\mathscr{D}} \epsilon_I D_I f(t) \Big\|_X \lesssim \beta_{p,X} \Big(\sum_{Q\in\mathcal{S}} \langle \|f\|_X \rangle_{1,Q} \mathbf{1}_Q(t) \Big), \qquad t \in \mathbb{R}.$$

Moreover, for any $w \in A_p$ and $f \in L^p(\mathbb{R},w;X)$ we have

$$\Big\| \sum_{I\in\mathscr{D}} \epsilon_I D_I f \Big\|_{L^p(\mathbb{R},w;X)} \lesssim_p \beta_{p,X}\, [w]_{A_p}^{\max\{\frac{1}{p-1},1\}} \|f\|_{L^p(\mathbb{R},w;X)}$$

Proof. Define for $f \in L^1(\mathbb{R};X)$ the operator

$$Tf(t) := \sum_{I\in\mathscr{D}} \epsilon_I D_I f(t), \qquad t \in \mathbb{R}.$$

Then T can be interpreted as a martingale transform and is therefore a bounded operator from $L^1(\mathbb{R};X)$ to $L^{1,\infty}(\mathbb{R};X)$ with

$$\|T\|_{L^1(\mathbb{R};X)\to L^{1,\infty}(\mathbb{R};X)} \lesssim \beta_{p,X}$$

by [HNVW16, Proposition 3.5.16 and Theorem 4.2.25]. For $Q \in \mathscr{D}$ define $T_Q f = T(f\,\mathbf{1}_{6Q})$. Then $\{T_Q\}_{Q\in\mathscr{D}}$ is a 6-localization family of T and moreover for $Q' \in \mathscr{D}$ we have that

$$T_{Q\setminus Q'} f = \sum_{I\in\mathscr{D}} \epsilon_I D_I(f\,\mathbf{1}_{6Q\setminus 6Q'}) = \sum_{I\in\mathscr{D}:Q'\subsetneq I} \epsilon_I D_I(f\,\mathbf{1}_{6Q\setminus 6Q'})$$

is constant on Q', so $\mathcal{M}^{\#}_{T,Q} = 0$.

Now fix a compactly supported $f \in L^1(\mathbb{R};X)$, set $E = \operatorname{supp} f$ and let $\mathcal{D} \subseteq \mathscr{D}$ be a partition of $\mathbb{R}$ as in Lemma 2.1.3. Then by Theorem 3.2.2 with $p_1 = p_2 = r = 1$ we obtain a $\frac{1}{2}$-sparse collection of dyadic intervals $\mathcal{S}_Q \subseteq \mathscr{D}(Q)$ for every $Q \in \mathcal{D}$ such that

$$\Big\| \sum_{I\in\mathscr{D}} \epsilon_I D_I f(t) \Big\|_X \lesssim \beta_{p,X} \Big(\sum_{P\in\mathcal{S}} \langle \|f\|_X \rangle_{1,6P}\, \mathbf{1}_P(t) \Big), \qquad t \in Q.$$

The claimed sparse domination now follows by taking

$$\mathcal{S} := \bigcup_{Q\in\mathcal{D}} \{6P : P \in \mathcal{S}_Q\}.$$

The second claim follows from Proposition 3.2.4 □

To conclude this chapter we comment on some further potential applications of our main theorems. We leave the details to the interested reader.

- Sparse domination and weighted bounds for variational truncations of Calderón–Zygmund operators were studied in [HLP13, MTX15, MTX17, Zor20]. The arguments presented in these references also imply the boundedness of our sharp grand maximal truncation operator and thus by Theorem 3.1.1 yield sparse domination of the variational truncations of Calderón–Zygmund operators.

- In [LOR17] Lerner, Ombrosi and Rivera-Ríos show sparse domination for commutators of a BMO function b with a Calderón–Zygmund operator, using sparse operators adapted to the function b. By a slight adaptation of the arguments presented in the proof of Theorem 3.2.2, one can prove the main result of [LOR17] in our framework and extend it to the vector-valued setting and to spaces of homogeneous type.

- Hörmander–Mihlin type conditions as in [GR85, Theorem IV.3.9] imply the weak L^{p_1}-boundedness of our maximal truncation operator for $p_1 > n/a$ and thus sparse domination for the associated Fourier multiplier operator by Theorem 3.1.1. Vector-valued extensions under Fourier type assumptions can be found in [GW03, Hyt04] and Theorem 3.1.1 may also be used to prove weighted results in that setting.

- Fackler, Hytönen and Lindemulder [FHL20] proved weighted vector-valued Littlewood-Paley theory on a UMD Banach space in order to prove their weighted, anisotropic, mixed-norm Mihlin multiplier theorems. Using Theorem 3.1.1 and Proposition 3.2.4 on the Littlewood–Paley square function with smooth cut-offs,

one can prove sparse domination and weighted estimates in the smooth cut-off case. This can then be transferred to sharp cut-offs by standard arguments, recovering [FHL20, Theorem 3.4].

- Theorem 3.3.1 can be used to show sparse domination and sharp weighted estimates for fractional integral operators as in [CM13a, CM13b, Cru17, IRV18]. The boundedness of the sharp grand maximal truncation operator associated to these operators can be shown using a similar argument as we used in the proof of Theorem 3.4.1.

- In [BFP16] Bernicot, Frey and Petermichl showed that the sparse domination principle is also applicable to non-integral singular operators falling outside the scope of Calderón–Zygmund operators. Sparse domination for square functions related to these operators was studied in [BBR20]. The methods developed in these papers actually show the boundedness of the localized sharp grand q-maximal truncation operator used in Theorem 3.3.2, so these results also fit in our framework.

4

SINGULAR STOCHASTIC INTEGRAL OPERATORS

This chapter is based on the first half of the paper

[6] E. Lorist and M.C. Veraar. Singular stochastic integral operators. *To appear in Anal. PDE*, 2020.

It is complemented by a discussion on γ-Fourier multiplier operators from an unpublished manuscript in Section 4.5 and extrapolation theory for stochastic-deterministic singular integral operators from

[1] E. Lorist. Space-time weighted maximal regularity for parabolic (S)PDE. In preparation.

in Section 4.6.

Abstract. *In this chapter we develop extrapolation theory for singular stochastic integrals with operator-valued kernel. In particular, we prove L^p-extrapolation results under a Hörmander condition on the kernel. Sparse domination and sharp weighted bounds are obtained under a Dini condition on the kernel, leading to a stochastic version of the solution to the A_2-conjecture. We also discuss the closely related γ-Fourier multiplier operators and develop an extrapolation theory for singular stochastic-deterministic integral operators.*

4.1. INTRODUCTION

In the study of stochastic partial differential equations (SPDEs), one often needs sharp regularity results for the linear equations. Together with fixed point arguments, this can be used to obtain existence, uniqueness and regularity for the solution to nonlinear SPDEs. In the last decades so-called maximal regularity results for SPDEs have been obtained in many papers. We refer to [DZ14, Section 6.3] for an overview on the subject in the Hilbert space setting. In the L^q-setting sharp regularity results have been obtained in [Kry99] by real analysis and PDE methods, and in [NVW12b] by functional calculus techniques.

In the above approaches one needs to prove sharp regularity estimates for singular stochastic integral operators of the form

$$S_K G(s) := \int_0^T K(s,t)G(t)\,\mathrm{d}W_H(t), \qquad s \in (0,T), \tag{4.1.1}$$

where X and Y are Banach spaces, G is an adapted process taking values in X, W_H is a cylindrical Brownian motion and K is a given operator-valued kernel $K\colon (0,T)\times(0,T) \to \mathcal{L}(X,Y)$ for some $T \in (0,\infty]$. An important example of a kernel K is

$$K(s,t) = e^{-(s-t)A}\mathbf{1}_{t<s}, \qquad s,t \in (0,T) \tag{4.1.2}$$

where $-A$ is the generator of an analytic semigroup on X and Y is either the real interpolation space $(X,D(A))_{1/2,2}$, the complex interpolation space $[X,D(A)]_{1/2}$ or the fractional domain space $D((\lambda + A^{1/2}))$, where $\lambda \in \rho(-A)$. This kernel has a singularity of the form $\|K(s,t)\| \le C(s-t)^{-1/2}$ for $|s-t| \le 1$. The L^p-boundedness of singular stochastic integrals with this kernel leads to *stochastic maximal L^p-regularity*, which we will discuss in the next chapter..

For deterministic PDEs one analogously obtains *deterministic maximal L^p-regularity* from the L^p-boundedness of

$$T_K f(s) := \int_0^T K(s,t)f(t)\,\mathrm{d}t, \qquad s \in (0,T),$$

where K is as in (4.1.2) with $Y = D(A)$. This kernel satisfies

$$\|K(s,t)\| \le C(s-t)^{-1}, \qquad |t-s| \le 1.$$

Operators T_K with such kernels have been studied thoroughly in the field of harmonic analysis. For example, using operator-valued Calderón–Zygmund theory as in Section 3.4, it was shown in [Dor00] that one can deduce that the L^p-boundedness of T_K for all $p \in (1,\infty)$ from maximal L^{p_0}-regularity for some $p_0 \in [1,\infty]$. Moreover, with the Mihlin multiplier theorem as in Section 3.5, one can obtain the L^p-boundedness of T_K for $p \in (1,\infty)$ from the $\mathcal{R}$-boundedness of the family of bounded operators

$$\{A(it+A)^{-1} : t \in \mathbb{R}\} \subseteq \mathcal{L}(X).$$

The necessity of this $\mathcal{R}$-boundedness condition was shown in [CP01].

Unlike in the deterministic setting, there is no general theory for the L^p-boundedness of singular stochastic integral operators. The aim of this chapter is to provide a version of this theory, which we will use to obtain new regularity results for abstract classes of SPDEs and more concrete examples, such as the heat equation, in the next chapter.

4.1.1. SINGULAR STOCHASTIC INTEGRALS

The behavior of the stochastic singular integral operators in (4.1.1) is quite different from the deterministic setting. Due to the Itô L^2-isometry the integrals convergence absolutely and thus no principal values are needed. As a consequence, in contrast with the deterministic setting, the scalar-valued setting can easily be characterized, see Subsection 4.2.3. In the operator-valued setting cancellation can for example occur in the following form:

$$\Big(\int_0^T \|K(s,t)x\|_Y^2 \,\mathrm{d}t\Big)^{1/2} \leq C\|x\|_X, \qquad s \in (0,T), \quad x \in X. \tag{4.1.3}$$

If the kernel is of this form, then using a simple Fubini argument one can check that S_K is L^2-bounded (see Propositions 4.2.3 and 4.2.10(i)). In particular, this method was used for the kernel in (4.1.2) in the classical monograph [DZ14, Section 6.3]. A sophisticated extension of this type of argument was used in [Brz95], [BH09] and [DL98] to cover L^p-boundedness in the scale of real interpolation spaces $(X, D(A))_{\theta,p}$.

The complex interpolation scale is more complicated. In particular, for $X = L^p(\mathbb{R}^d)$ (4.1.3) is often not true. For example it fails for $A = -\Delta$. To obtain L^p-estimates in this case, [Kry94b, Kry99, Kry08] use sharp estimates for stochastic integrals and sophisticated real analysis arguments. Moreover, using PDE arguments, the operator A can be replaced by a second order elliptic operator with coefficients depending on $(t,\omega,x) \in (0,T) \times \Omega \times \mathbb{R}^d$, where some regularity in x is assumed, but only progressive measurability is assumed in (t,ω). By an elaborate trick in [Kry00] the estimates were extended to an $L^p(L^q)$-setting with $p \geq q \geq 2$. There are many variations of the above methods in the literature, in which different operators than Δ are considered and equations on different domains $D \subseteq \mathbb{R}^d$ are treated (see e.g. [CKLL18, CKL19, Du20, Kim05, KK18, Kry09, Lin14b] and references therein).

On the scale of tent spaces stochastic maximal regularity for elliptic operators in divergence form is shown in [ANP14]. This is done through extrapolation using off-diagonal estimates, which are substitutes for the classical pointwise kernel estimates of Calderón--Zygmund theory. See also [AKMP12] for the more general harmonic analysis framework developed to analyse this scale.

In [NVW12b, NVW15c] the L^p-boundedness of stochastic singular integrals with kernel (4.1.2) was obtained using the boundedness of the H^∞-functional calculus together with the sharp two-sided estimates for stochastic integrals in UMD Banach spaces developed in [NVW07]. One of the advantages of this approach is that it can be used for an

abstract operator A as long as it has an H^∞-calculus. Secondly, the stochastic integral operator is automatically L^p-bounded for any $p \in (2,\infty)$. Some geometric restrictions on X are required, but these are fulfilled for L^q, $W^{s,q}$, etc. as long as $q \in [2,\infty)$ (see Section 5.5). In particular, mixed $L^p(L^q)$-regularity can be obtained for all $q \in [2,\infty)$ and $p \in (2,\infty)$, where $p = q = 2$ is allowed as well. The results of [NVW12b, NVW15c] have been applied to semilinear equations in [NVW12a], to quasilinear equations in [Hor19, AV20a, AV20b] and to fully nonlinear equations in [Agr18].

Recently, in [PV19] the framework of [NVW12b, NVW15c] has been extended to cover the case where A depends on time and Ω, as long as $D(A(t,\omega))$ is constant. The method is based on a reduction to the time and Ω-independent setting and gives a new approach to [Kry99], which additionally includes new optimal space-time regularity estimates and is applicable to a large class of SPDEs.

4.1.2. EXTRAPOLATION FOR SINGULAR STOCHASTIC INTEGRAL OPERATORS

A large part of the theory of maximal L^p-regularity for deterministic PDEs was developed after the Calderón-Zygmund theory for operator-valued kernels was founded. In the stochastic case such a Calderón–Zygmund theory is not available yet, and our goal motivation is to build such a theory and discover its potential for stochastic maximal L^p-regularity (see Chapter 5). Despite the rather different behaviour of stochastic singular integral operators compared to their deterministic counterparts, our first main theorem on the boundedness of singular stochastic integral operators is a stochastic version of the classical extrapolation result for Calderón–Zygmund operators, see [Hör60] for the scalar case and e.g. [BCP62, GR85, HNVW2x, RRT86] for the operator-valued case.

Theorem 4.1.1 (L^p-boundedness of stochastic Calderón-Zygmund operators)**.** *Let X and Y be Banach spaces with type 2 and assume Y has the* UMD *property. Take $T \in (0,\infty]$, let $K : (0,T) \times (0,T) \to \mathcal{L}(X,Y)$ be strongly measurable and assume that for every interval $I \subseteq (0,T)$ we have the following Hörmander condition*

$$\Big(\int_{(0,T)\setminus I} \|K(s,t) - K(s',t)\|^2 \,\mathrm{d}t\Big)^{1/2} \leq C \qquad s,s' \in \tfrac{1}{2}I \tag{4.1.4}$$

$$\Big(\int_{(0,T)\setminus I} \|K(s,t) - K(s,t')\|^2 \,\mathrm{d}s\Big)^{1/2} \leq C \qquad t,t' \in \tfrac{1}{2}I \tag{4.1.5}$$

for some constant $C > 0$ independent of I. Fix $p \in [2,\infty)$ and suppose that the mapping S_K as defined in (4.1.1) is bounded from $L^p_{\mathscr{F}}(\Omega \times (0,T); \gamma(H,X))$ into $L^p(\Omega \times (0,T); Y)$. Then the mapping

$$S_K \colon L^q_{\mathscr{F}}(\Omega \times (0,T); \gamma(H,X)) \to L^q(\Omega \times (0,T); Y)$$

is bounded for all $q \in (2,\infty)$.

In Theorems 4.4.2 and 4.4.4 we prove a general extrapolation result for so-called singular γ-integral operators. In this setting we also obtain the end point estimates $L^2 \to L^{2,\infty}$ and $L^\infty \to \mathrm{BMO}$. Singular γ-integral operators are connected to singular

stochastic integral operators by the Itô isomorphism, see Proposition 4.2.3 and Proposition 4.2.5. Theorem 4.1.1 follows by combining the aforementioned results.

The conditions (4.1.4) and (4.1.5) are L^2-variants of what is usually called the Hörmander condition. The L^r-variant for $r \in [1,\infty]$ also appears in [Hör60, Definition 2.1] in the scalar case and in [RV17, Section 5.1] in the vector-valued case, where it was used to extrapolate (deterministic) boundedness of operators from L^p into L^q with $\frac{1}{p} - \frac{1}{q} = \frac{1}{r}$ to other pairs (u, v) satisfying $1 < u \le v < \infty$ and $\frac{1}{u} - \frac{1}{v} = \frac{1}{r}$.

For Calderón–Zygmund operators weighted bounds are classical (see e.g. [Gra14a, Chapter 7]) and the sharp dependence of the estimates on the weight characteristic in this setting is known as the A_2-theorem of Hytönen [Hyt12], which we discussed in Section 3.4. By design, the abstract sparse domination principle in Theorem 3.1.1 is also applicable to singular stochastic integral operators, which yields a stochastic version of the A_2-theorem.

Theorem 4.1.2 (Sharp weighted bounds)**.** *Let X and Y be Banach spaces with type 2 and assume Y has the* UMD *property. Take $T \in (0,\infty]$, let $K : (0,T) \times (0,T) \to \mathcal{L}(X,Y)$ be strongly measurable and assume that*

$$\|K(s,t) - K(s',t)\| \le \omega\Big(\frac{|s-s'|}{|s-t|}\Big)\frac{1}{|s-t|^{1/2}} \qquad |s-s'| \le \frac{1}{2}|s-t|,$$
$$\|K(s,t) - K(s,t')\| \le \omega\Big(\frac{|t-t'|}{|s-t|}\Big)\frac{1}{|s-t|^{1/2}} \qquad |t-t'| \le \frac{1}{2}|s-t|,$$

where $\omega : [0,1] \to [0,\infty)$ is increasing and subadditive, $\omega(0) = 0$ and

$$\Big(\int_0^1 \omega(r)^2 \frac{\mathrm{d}r}{r}\Big)^{1/2} < \infty.$$

Suppose S_K as defined in (4.1.1) is bounded from $L^p_{\mathscr{F}}(\Omega \times (0,T); \gamma(H,X))$ into $L^p(\Omega \times (0,T); Y)$ for some $p \in [2,\infty)$. Then the mapping

$$S_K : L^q_{\mathscr{F}}(\Omega \times (0,T), w; \gamma(H,X)) \to L^q(\Omega \times (0,T), w; Y)$$

is bounded for all $q \in (2,\infty)$ and $w \in A_{q/2}$ with

$$\|S_K\|_{L^q(\Omega\times(0,T),w;\gamma(H,X)) \to L^q(\Omega\times(0,T),w;Y)} \lesssim [w]_{A_{q/2}}^{\max\{\frac{1}{2},\frac{1}{q-2}\}}.$$

The above result follows from Proposition 4.2.3, Proposition 4.2.5 and Theorem 4.4.11. We also prove that the above estimate is sharp in terms of the dependence on the weight characteristic. Note that the difference with the A_2-theorem for Calderón–Zygmund operators (Theorem 3.4.1) occurs because the L^p-norm of (4.1.1) is equivalent to a certain generalized square function. The conditions on the kernel are L^2-versions of the Dini condition. The integrability condition on ω holds in particular if $\omega(t) = Ct^\epsilon$ for some $C > 0$ and $\epsilon \in (0,1]$. We will discuss these kernel conditions thoroughly in Section 4.3.

4.1.3. SUFFICIENT CONDITIONS FOR L^p-BOUNDEDNESS

In Theorem 4.1.1 and Theorem 4.1.2 one always needs to starts with an L^p-bounded stochastic integral operator. Only in the the Hilbert space setting in the convolution case we obtain a full characterization of the boundedness of S_K in terms of kernel conditions, see Corollary 4.4.9 and Corollary 4.4.13. Outside the Hilbert space setting or for non-convolution kernels we do not have abstract theory to ensure L^p-boundedness, so this has to be established on a case-by-case basis. It would be interesting to find general sufficient conditions from which L^p-boundedness can be derived.

In the deterministic case L^p-boundedness can e.g. be derived using $T(1)$ and $T(b)$-theorems (see e.g. [HW06, Hyt06, Hyt20, HH16] for the operator-valued case). A stochastic version of these theorems could have significant implications for the regularity theory of SPDEs.

In the deterministic convolution case, i.e. if T_K is a Calderón–Zygmund operator with kernel $K(s,t) = k(s-t)$ for some $k\colon \mathbb{R}^d \setminus \{0\} \to \mathcal{L}(X,Y)$, one can equivalently study the Fourier multiplier operator T_m with with $m = \widehat{k}$. To deduce L^p-boundedness for T one can then use the operator-valued Mihlin multiplier theorem, see Section 3.5. This theorem relies on the boundedness of the Hilbert transform on $L^p(\mathbb{R};X)$, which is equivalent to X having the UMD by Theorem 2.7.1. The stochastic analog of the Hilbert transform, i.e. the stochastic integral operator with kernel $K(s,t) = \frac{1}{|s-t|^{1/2}}$, does not define a bounded stochastic singular integral operator (see Example 4.2.13(ii)), so a stochastic version of the Mihlin multiplier theorem would require a proper replacement of the Hilbert transform.

Although we do not have a stochastic Mihlin multiplier theorem, we can use the smoothness and decay of Fourier transform of k to check the L^2-versions of the Hörmander and Dini conditions in Theorem 4.1.1 and Theorem 4.1.2. For the L^1-variant of Hörmander's condition this is classical and for the L^r-variant of Hörmander's condition for $r \in [1,\infty]$ this has been done in [RV17, Section 5.2]. In Section 4.5 we will check the L^2-variant of Dini's condition in terms of smoothness and decay of m, following the arguments for the L^1-variant of Hörmander's condition in [HNVW2x]. Moreover, we will prove a γ-Fourier multiplier extrapolation theorem, which is a consequence of the connection with singular γ-integral operators and Theorem 4.4.11.

4.1.4. STOCHASTIC-DETERMINISTIC INTEGRAL OPERATORS

As noted before, an important kernel K for applications is given by

$$K(t,s) = A^{\frac{1}{2}} e^{-(t-s)A} \mathbf{1}_{t>s}, \qquad t,s \in (0,T)$$

where $-A$ is the generator of an analytic semigroup on $X = Y$. If $X = L^q(\mathcal{O})$ for some domain $\mathcal{O} \subseteq \mathbb{R}^d$, the operators $K(s,t) \in \mathcal{L}(L^q(\mathcal{O}))$ for $s,t \in (0,T)$ often have a kernel representation of their own. For example if $A = -\Delta$ on $L^q(\mathbb{R}^d)$, the heat semigroup $(e^{t\Delta})_{t\geq 0}$

is for $h \in L^q(\mathbb{R}^d)$ and $t \in \mathbb{R}_+$ given by

$$e^{t\Delta}h(x) = \int_{\mathbb{R}^d} \frac{1}{(4\pi t)^{d/2}} e^{-|x-y|/4t} h(y)\, dy, \qquad x \in \mathbb{R}^d.$$

If the operators $K(s,t) \in \mathcal{L}(L^q(\mathcal{O}))$ for $s,t \in (0,T)$ indeed have a kernel representation, we can write S_K as

$$S_K G(t,x) = \int_0^T \int_{\mathcal{O}} k(t,x,s,y) G(s,y)\, dy\, dW_H(s), \qquad (t,x) \in (0,T) \times \mathcal{O} \tag{4.1.6}$$

for a kernel

$$k\colon (0,T) \times \mathcal{O} \times (0,T) \times \mathcal{O} \to \mathbb{C}$$

and adapted processes $G\colon \Omega \times (0,T) \times \mathcal{O} \to H$. To establish $L^p(\Omega \times \mathbb{R}_+, w; L^q(\mathcal{O}))$-boundedness with $p,q \in (2,\infty)$ and $w \in A_{p/2}((0,T))$ for S_K, we have seen in Theorem 4.1.1 and Theorem 4.1.2 that it suffices to prove $L^q(\Omega \times \mathbb{R}_+ \times \mathcal{O})$-boundedness for S_K and certain assumptions on the kernel K. In applications it is easier to establish boundedness for S_K on the Hilbert space $L^2(\Omega \times \mathbb{R}_+ \times \mathcal{O})$. It is therefore desirable to deduce $L^q(\Omega \times \mathbb{R}_+ \times \mathcal{O})$-boundedness for S_K from $L^2(\Omega \times \mathbb{R}_+ \times \mathcal{O})$-boundedness for S_K. In the deterministic case, this can be done using Calderón–Zygmund theory (see Section 3.4) in the space $(0,T) \times \mathcal{O}$ with a parabolic metric, which is a space of homogeneous type. For a class of elliptic operators of fractional order this theory was developed in [KKL15, KKL16] under a parabolic Hörmander assumption on k and this can be extended to the weighted and $p \neq q$ setting under a Dini condition on k using Theorem 3.4.1 and Rubio de Francia extrapolation. Using a parabolic stochastic Hörmander condition on k, a stochastic version of these results was obtained in [Kim15, KK20] and for the moments of S_K a Calderón–Zygmund theory approach was recently employed in [Kim20].

In Section 4.6 we will extend the abstract result in [KK20] to the weighted and $p \neq q$ setting, using a parabolic stochastic Dini condition. For this we will once again use the sparse domination framework developed in Chapter 3 and Rubio de Francia extrapolation. This will allow us to deduce time- and space-weighted $L^p(\mathbb{R}_+, v; L^q(\mathcal{O}, w))$-boundedness for S_K from $L^2(\mathbb{R}_+ \times \mathcal{O})$-boundedness for S_K. Moreover, we are obtain time- and space-weighted $L^q(\mathcal{O}, w; L^p(\mathbb{R}_+), v)$-boundedness for S_K, in which case the space integrability parameters $q \in (1,\infty)$ are allowed. In applications to SPDEs this reversed integration order allows one to deduce additional regularity results, see also [Ant17, NVW15a].

To formulate our result fix $T \in (0,\infty]$, $m > 0$ and a space of homogeneous type $(\mathcal{O}, d, \mu)$. We define

$$\begin{aligned} \mathcal{O}_T &:= (0,T) \times \mathcal{O} \\ d_T\big((t,x),(s,y)\big) &:= \max\big\{|t-s|^{1/m}, d(x,y)\big\} \\ \mu_T &:= dt \otimes d\mu, \end{aligned}$$

which is also a space of homogeneous type. In applications m will often be an even integer and $\mathcal{O}$ a domain in $\mathbb{R}^d$, equipped with the Euclidean distance and the Lebesgue measure. We refer to Section 4.3 for the definition of a $(2,1)$-Dini kernel.

Theorem 4.1.3. *Let $K\colon \mathcal{O}_T \times \mathcal{O}_T \to \mathbb{C}$ be a $(2,1)$-Dini kernel. Suppose that*

$$S_K G(t,x) := \int_0^T \int_{\mathcal{O}} K\big((t,x),(s,y)\big) G(s,y)\, \mathrm{d}\mu(y)\, \mathrm{d}W_H(s), \qquad (t,x) \in \mathcal{O}_T$$

is a well-defined, bounded operator from $L^2_{\mathscr{F}}(\Omega \times \mathcal{O}_T; H)$ to $L^2(\Omega \times \mathcal{O}_T)$. For $p \in (2,\infty)$ and $q, r \in (1,\infty)$ the following hold:

(i) The operator

$$S_K\colon L^r_{\mathscr{F}}(\Omega; L^p(\mathcal{O}_T, w; H)) \to L^r(\Omega; L^p(\mathcal{O}_T, w))$$

is bounded for all $w \in A_{p/2}(\mathcal{O}_T)$ with

$$\|S_K\|_{L^r(\Omega;L^p(\mathcal{O}_T,w;H)) \to L^r(\Omega;L^p(\mathcal{O}_T,w))} \lesssim [w]_{A_p(\mathcal{O}_T)}^{\max\{\frac{1}{p-2},1\}}.$$

(ii) If $q > 2$, the operator

$$S_K\colon L^r_{\mathscr{F}}(\Omega; L^p((0,T), v; L^q(\mathcal{O}, w; H))) \to L^r(\Omega; L^p((0,T), v; L^q(\mathcal{O}, w)))$$

is bounded for all $v \in A_{p/2}((0,T))$ and $w \in A_q(\mathcal{O})$.

(iii) The operator

$$S_K\colon L^r_{\mathscr{F}}(\Omega; L^q(\mathcal{O}, w; L^p((0,T), v; H))) \to L^r(\Omega; L^q(\mathcal{O}, w; L^p((0,T), v)))$$

is bounded for all $v \in A_{p/2}((0,T))$ and $w \in A_q(\mathcal{O})$.

Theorem 4.1.3 will be proven in Section 4.6 using deterministic Calderón–Zygmund theory, the abstract sparse domination result from Chapter 3 and Rubio de Francia extrapolation.

Remark 4.1.4. By Fubini's theorem and Rubio de Francia extrapolation, Theorem 4.1.3(i) implies that the operator

$$S_K\colon L^p((0,T), v; L^q(\mathcal{O}, w; L^r(\Omega; H))) \to L^p((0,T), v; L^q(\mathcal{O}, w; L^r(\Omega)))$$

is bounded for $p, q, r \in (2,\infty)$, $v \in A_{p/2}((0,T))$ and $w \in A_{q/2}(\mathcal{O})$ under suitable measurability conditions. The case $p = q$ and $v \equiv w \equiv 1$ of this result for second-order elliptic operators has been treated in [Kim20]. In [Kim20] the solvability of the resulting parabolic SPDEs, including a deterministic term, has also been obtained for $p = q \geq r$. In [1] we will extend Theorem 4.1.3 to also include such a deterministic term and obtain e.g. the solvability of the parabolic SPDEs from [Kim20] for $p, q, r \in (2,\infty)$, $v \in A_{p/2}((0,T))$ and $w \in A_{q/2}(\mathcal{O})$.

4.2. STOCHASTIC INTEGRAL OPERATORS

We start by introducing stochastic integral operators S_K associated to a kernel K. The reason we consider $p \in [2,\infty)$ will become clear in Subsection 4.2.3. Although we will not assume Y to have type 2 for the moment, it follows from [NVW15b, Proposition 6.2] that, already for very easy kernels K, in order to have boundedness of S_K, a type 2 condition on Y is necessary.

Definition 4.2.1 (Stochastic integral operator)**.** Let X be a Banach space and Y a UMD Banach space. Let $p \in [2,\infty)$, $T \in (0,\infty]$, let w be a weight on $(0,T)$ and let

$$K\colon (0,T)\times(0,T) \to \mathcal{L}(X,Y)$$

be strongly measurable. We write $K \in \mathcal{K}^H_W(L^p((0,T),w))$ if for $f \in L^p_{\mathscr{F}}(\Omega\times(0,T),w;\gamma(H,X))$ and a.e. $s\in(0,T)$ the mapping $t \mapsto K(s,t)f(t)$ is in $L^p_{\mathscr{F}}(\Omega;\gamma(0,T;H,Y))$ and the operator S_K given by

$$S_K G(s) := \int_0^T K(s,t)G(t)\,\mathrm{d}W_H(t), \qquad s\in(0,T)$$

is bounded from $L^p_{\mathscr{F}}(\Omega\times(0,T),w;\gamma(H,X))$ into $L^p(\Omega\times(0,T),w;Y)$. We norm the space $\mathcal{K}^H_W(L^p((0,T),w))$ by

$$\|K\|_{\mathcal{K}^H_W(L^p((0,T),w))} := \|S_K\|_{L^p(\Omega\times(0,T),w;\gamma(H,X))\to L^p(\Omega\times(0,T),w;Y)}.$$

We omit the weight if $w\equiv 1$ and we omit the Hilbert space if $H=\mathbb{R}$.

We want to study the boundedness properties of S_K. In the next results we will reformulate this problem in a the deterministic setting using square functions (γ-norms in time) and reduce considerations to the case $H=\mathbb{R}$.

Definition 4.2.2 (γ-integral operator)**.** Let X and Y be a Banach spaces. Let (S,μ) be a measure space, $p\in[2,\infty)$, w be a weight on S and let

$$K : S\times S \to \mathcal{L}(X,Y)$$

be strongly measurable. We say that $K\in\mathcal{K}^H_\gamma(L^p(S,w))$ (resp. $K\in\mathcal{K}^H_\gamma(L^{p,\infty}(S,w))$) if for $f\in L^p(S,w;\gamma(H,X))$ and a.e $s\in S$ the mapping $t\mapsto K(s,t)f(t)$ is in $\gamma(S;H,Y)$ and the operator T_K given by

$$T_K f(s) := K(s,\cdot)f(\cdot), \qquad s\in S$$

is bounded from $L^p(S,w;\gamma(H,X))$ into $L^p(S,w;\gamma(S;H,Y))$ (resp. from L^p into $L^{p,\infty}$). We norm these spaces by

$$\|K\|_{\mathcal{K}^H_\gamma(L^p(S,w))} := \|T_K\|_{L^p(S,w;\gamma(H,X))\to L^p(S,w;\gamma(S;H,Y))},$$
$$\|K\|_{\mathcal{K}^H_\gamma(L^{p,\infty}(S,w))} := \|T_K\|_{L^p(S,w;\gamma(H,X))\to L^{p,\infty}(S,w;\gamma(S;H,Y))}.$$

We omit the weight if $w\equiv 1$ and we omit the Hilbert space if $H=\mathbb{R}$.

We start by connecting the definitions of stochastic and γ-integral operators.

Proposition 4.2.3 (Deterministic characterization)**.** *Let X be a Banach space and Y a UMD Banach space. Let $p \in [2,\infty)$, $T \in (0,\infty]$ and let w be a weight on $(0,T)$. Then*

$$\mathcal{K}_W^H(L^p((0,T),w)) = \mathcal{K}_\gamma^H(L^p((0,T),w))$$

isomorphically.

Proof. The proof follows directly from Theorem 2.9.1. Indeed, if $K \in \mathcal{K}_\gamma^H(L^p((0,T),w))$, then for $G \in L^p_{\mathscr{F}}(\Omega \times (0,T), w; \gamma(H,X))$ one has

$$\|S_K G\|_{L^p(\Omega;L^p((0,T),w;Y))} \eqsim_{p,Y} \|T_K G\|_{L^p(\Omega;\gamma((0,T);H,L^p((0,T),w;Y)))}.$$

Therefore, by Fubini's theorem and the γ-Fubini theorem (Proposition 2.8.6), we have

$$\begin{aligned}
\|S_K G\|_{L^p(\Omega\times(0,T),w;Y))} &\eqsim_{p,Y} \|T_K G\|_{L^p(\Omega;\gamma((0,T);H,L^p((0,T),w;Y)))} \\
&\eqsim_p \|T_K G\|_{L^p(\Omega;L^p((0,T),w;\gamma(0,T;H,Y)))} \\
&\leq \|K\|_{\mathcal{K}_\gamma^H(L^p((0,T),w))} \|G\|_{L^p(\Omega;L^p((0,T)w;\gamma(H,X)))} \\
&= \|K\|_{\mathcal{K}_\gamma^H(L^p((0,T),w))} \|G\|_{L^p(\Omega\times(0,T),w;\gamma(H,X))}.
\end{aligned}$$

Conversely, taking $f \in L^p(\Omega \times (0,T), w; \gamma(H,X))$ independent of Ω, a similar argument yields that $K \in \mathcal{K}_W^H(L^p((0,T),w))$ implies $K \in \mathcal{K}_\gamma^H(L^p((0,T),w))$. □

Remark 4.2.4. For simplicity we took the Ω-integrability parameter equal to the time-integrability parameter in the definition of $\mathcal{K}_W^H(L^p((0,T),w))$. As can be seen from the proof of Proposition 4.2.3, for $K \in \mathcal{K}_W^H(L^p((0,T),w))$ one actually has that S_K is bounded from $L^r_{\mathscr{F}}(\Omega; L^p((0,T),w;\gamma(H,X)))$ into $L^r(\Omega; L^p((0,T),w;Y))$ for any $r \in (1,\infty)$.

In the next result we show that one can take $H = \mathbb{R}$. The result extends [AV20c, Theorem 5.4], where a particular kernel was considered.

Proposition 4.2.5 (Independence of H)**.** *Let X and Y be a Banach spaces and (S,μ) a measure space. Assume Y has type 2, let $p \in [2,\infty)$ and let w be a weight on S. Then*

$$\begin{aligned}
\mathcal{K}_\gamma^H(L^p(S,w)) &= \mathcal{K}_\gamma(L^p(S,w)) \\
\mathcal{K}_\gamma^H(L^{p,\infty}(S,w)) &= \mathcal{K}_\gamma(L^{p,\infty}(S,w))
\end{aligned}$$

isomorphically.

Proof. By considering a 1-dimensional subspace of H, we immediately see that $\subseteq$ holds. For the converse let T_K^H and $T_K^{\mathbb{R}}$ be the γ-integral operators on $L^p(S,w;\gamma(H,X))$ and $L^p(S,w;X)$ respectively. By Lemma 2.8.4 one has

$$\|T_K^H f(s)\|_{\gamma(S;H,Y)} \lesssim_Y \|T_K^{\mathbb{R}} f(s)\|_{\gamma(S;\gamma(H,Y))} = \|T_K^{\mathbb{R}} f(s)\|_{\gamma(H,\gamma(S,Y))}$$

Taking $L^p(S,w)$-norms and using Proposition 2.8.6 with $E = L^p(S,w)$ we obtain

$$\begin{aligned}\|T_K^H f\|_{L^p(S,w;\gamma(S;H,Y))} &\lesssim_Y \|T_K^{\mathbb{R}} f\|_{L^p(S,w;\gamma(H,\gamma(S;Y)))}\\ &\simeq_p \|T_K^{\mathbb{R}} f\|_{\gamma(H,L^p(S,w;\gamma(S;Y)))}\\ &\leq \|K\|_{\mathcal{K}_\gamma(L^p(S,w))}\|f\|_{\gamma(H,L^p(S,w;X))}\\ &\simeq_p \|K\|_{\mathcal{K}_\gamma(L^p(S,w))}\|f\|_{L^p(S,w;\gamma(H,X))}.\end{aligned}$$

The $L^{p,\infty}$-case follows analogously. □

4.2.1. TRUNCATIONS

We will now illustrate a major difference between stochastic and deterministic integral operators. Indeed, we will show that even when the kernel K has a singularity, the "γ-integrals" converge absolutely. In particular, we will show that if we truncate the singularity of K, then the operators associated to these truncations converge back to the operator associated to K without any regularity assumptions on K. This in contrast to the deterministic setting (cf. [Gra14a, Section 5.3]). Let (S,d,μ) be a space of homogeneous type, let X and Y be Banach spaces and suppose that $K\colon S\times S\to\mathcal{L}(X,Y)$ is strongly measurable. We define for $\varepsilon>0$

$$K_\varepsilon(s,t) := K(s,t)\mathbf{1}_{A_\varepsilon}(s,t), \qquad s,t\in S,$$

where $A_\varepsilon := \{(s,t)\in S\times S : \varepsilon < d(s,t) < \varepsilon^{-1}\}$. Let $p\in[2,\infty)$ and let w be a weight on S. If $K_\varepsilon\in\mathcal{K}_\gamma(L^p(S,w))$ for all $\varepsilon>0$ we define for $f\in L^p(S,w;X)$ the maximal truncation operator

$$T_K^\star f(s) := \sup_{\varepsilon>0}\|T_{K_\varepsilon}f(s)\|_{\gamma(S;Y)} \qquad s\in S.$$

Proposition 4.2.6 (Truncations)**.** *Let X and Y be Banach spaces, assume that Y has finite cotype and let (S,d,μ) be a space of homogeneous type. Let $p\in[2,\infty)$ and let w be a weight on S. Let*

$$K\colon S\times S\to\mathcal{L}(X,Y)$$

be strongly measurable such that $K_\varepsilon\in\mathcal{K}_\gamma(L^{p,\infty}(S,w))$ for all $\varepsilon>0$. Then for $f\in L^p(S,w;X)$ we have

$$T_K^\star f(s) = \|T_K f(s)\|_{\gamma(S;Y)}, \qquad s\in S,$$

and in particular

$$\begin{aligned}\|K\|_{\mathcal{K}_\gamma(L^p(S,w))} &= \sup_{\varepsilon>0}\|K_\varepsilon\|_{\mathcal{K}_\gamma(L^p(S,w))},\\ \|K\|_{\mathcal{K}_\gamma(L^{p,\infty}(S,w))} &= \sup_{\varepsilon>0}\|K_\varepsilon\|_{\mathcal{K}_\gamma(L^{p,\infty}(S,w))}.\end{aligned}$$

Furthermore if $K\in\mathcal{K}_\gamma(L^p(S,w))$, then $T_{K_\varepsilon}\to T_K$ in the strong operator topology.

Proof. Fix $f \in L^p(S;X)$ and $s \in S$. Assume that $\|T_K f(s)\|_{\gamma(S;Y)} < \infty$ and take $\varepsilon > 0$. Then by Proposition 2.8.1

$$\|T_{K_\varepsilon} f(s)\|_{\gamma(S;Y)} \le \|T_K f(s)\|_{\gamma(S;Y)} \tag{4.2.1}$$

which yields $T_K^\star f(s) \le \|T_K f(s)\|_{\gamma(S;Y)}$.

Conversely assume that $T_K^\star f(s) < \infty$. Note that since $\gamma(S,Y) \hookrightarrow \mathcal{L}(L^2(S),Y)$, we have

$$\begin{aligned}\int_S |\langle K(s,t)f(t), y^*\rangle|^2 \,\mathrm{d}t &\le \sup_{\varepsilon>0} \int_S |\langle K_\varepsilon(s,t)f(t), y^*\rangle|^2 \,\mathrm{d}t \\ &\le \sup_{\varepsilon>0} \|t \mapsto K_\varepsilon(s,t)f(t)\|^2_{\gamma(S;Y)} \|y^*\|^2 < \infty.\end{aligned}$$

Therefore, $t \mapsto K(s,t)f(t)$ is weakly in L^2 and thus $T_K f(s)$ is a bounded operator from $L^2(S)$ into Y. Moreover, for all $\varphi \in L^2(S)$ and $y^* \in Y^*$, the dominated convergence theorem yields that

$$\langle T_K f(s)\varphi, y^*\rangle = \lim_{\varepsilon\to 0} \langle T_{K_\varepsilon} f(s)\varphi, y^*\rangle.$$

Now the γ-Fatou lemma (Proposition 2.8.2) yields

$$\|T_K f(s)\|_{\gamma(S;Y)} \le \lim_{\varepsilon\to 0} \|T_{K_\varepsilon} f(s)\|_{\gamma(S;Y)} = \sup_{\varepsilon>0} \|T_{K_\varepsilon} f(s)\|_{\gamma(S;Y)},$$

where the equality follows again by domination. This concludes the proof of the equality

$$T_K^\star f(s) = \|T_K f(s)\|_{\gamma(S;Y)}.$$

By taking L^p-norms we directly obtain

$$\|K\|_{\mathcal{K}(L^p(S,w))} = \|T_K^\star\|_{L^p(S,w)} \le \sup_{\varepsilon>0} \|K_\varepsilon\|_{\mathcal{K}(L^p(S,w))},$$

and the converse inequality follows from (4.2.1). The estimate for $L^{p,\infty}$ follows analogously. Finally, the strong convergence follows from (4.2.1), the dominated convergence theorem and the γ-dominated convergence theorem (Proposition 2.8.3). □

Next we prove a version of the above result for stochastic integral operators. For this let X and Y be Banach spaces, $p \in [2,\infty)$ and w a weight on $\mathbb{R}_+$. If $K_\varepsilon \in \mathcal{K}_W^H(L^p(\mathbb{R}_+,w))$ for all $\varepsilon > 0$, we define for $f \in L^p_{\mathscr{F}}(\Omega\times\mathbb{R}_+;\gamma(H,Y))$ the operator

$$S_K^\star f(s) = \sup_{\varepsilon>0} \|S_{K_\varepsilon} f(s)\|_Y, \qquad s \in \mathbb{R}_+.$$

Theorem 4.2.7. *Let X and Y Banach spaces and assume that Y has the* UMD *property. Let $p \in [2,\infty)$ and let w be a weight on $\mathbb{R}_+$. Let*

$$K\colon \mathbb{R}_+ \times \mathbb{R}_+ \to \mathcal{L}(X,Y)$$

be strongly measurable such that $K_\varepsilon \in \mathcal{K}_W^H(L^p(\mathbb{R}_+,w))$ for all $\varepsilon > 0$. Then

$$\|S_K^\star\|_{L^p_{\mathscr{F}}(\Omega\times\mathbb{R}_+,w;\gamma(H,Y))\to L^p(\Omega\times\mathbb{R}_+,w)} \eqsim_p \sup_{\varepsilon>0} \|K_\varepsilon\|_{\mathcal{K}_W^H(L^p(\mathbb{R}_+,w))} \eqsim_{Y,p} \|K\|_{\mathcal{K}_W^H(L^p(\mathbb{R}_+w))}$$

Furthermore if $K \in \mathcal{K}_W^H(L^p(\mathbb{R}_+w))$, then $S_{K_\varepsilon} \to S_K$ in the strong operator topology.

Proof. It is clear from Propositions 4.2.3, 4.2.5 and 4.2.6 that the second and third expression are norm equivalent. Moreover, it is clear that

$$\|S_K^\star\|_{L^p_{\mathscr{F}}(\Omega\times\mathbb{R}_+,w;\gamma(H,Y))\to L^p(\Omega\times\mathbb{R}_+,w)} \geq \sup_{\varepsilon>0}\|K_\varepsilon\|_{\mathcal{K}^H_W(L^p(\mathbb{R}_+,w))}.$$

Thus it remains to prove the converse estimate. In order to show this let $f \in L^p_{\mathscr{F}}(\Omega\times\mathbb{R}_+,w;\gamma(H,Y))$ and $\varepsilon\in(0,1)$. Since $K\in\mathcal{K}^H_W(L^p(\mathbb{R}_+,w))$, by Doob's maximal inequality we can write

$$\begin{aligned}\|S_K^\star f(s)\|_{L^p(\Omega)} &\leq \Big(\mathbb{E}\sup_{\varepsilon>0}\Big\|\int_{\max\{s-\varepsilon,0\}}^{\max\{s-1/\varepsilon,0\}} K(s,t)f(t)\,\mathrm{d}W_H(t)\Big\|_Y^p\Big)^{1/p}\\ &\qquad + \Big(\mathbb{E}\sup_{\varepsilon>0}\Big\|\int_{s+\varepsilon}^{s+1/\varepsilon} K(s,t)f(t)\,\mathrm{d}W_H(t)\Big\|_Y^p\Big)^{1/p}\\ &\leq \frac{4p}{p-1}\|S_K f(s)\|_{L^p(\Omega;Y)}.\end{aligned}$$

Taking $L^p(\mathbb{R}_+,w)$-norms the desired estimate follows.

For the strong convergence note that by the proof of Proposition 4.2.3 we have

$$\|S_K f - S_{K_\varepsilon} f\|_{L^p(\Omega\times\mathbb{R}_+,w;Y)} \eqsim_{p,Y} \|T_K f - T_{K_\varepsilon} f\|_{L^p(\Omega\times\mathbb{R}_+,w;\gamma(S;H,Y))}.$$

Here the right-hand side for fixed $\omega\in\Omega$ is independent of H by Proposition 4.2.5, so the strong convergence follows by Proposition 4.2.6 and the dominated convergence theorem. □

4.2.2. NECESSARY AND SUFFICIENT CONDITIONS

Before we turn to more involved results in the subsequent sections, we first analyse the boundedness of γ-integral operators in a few special cases. We start with a necessary condition for T_K to be bounded if S is $\mathbb{R}^d$ or $\mathbb{R}_+$ and K is of convolution type.

Proposition 4.2.8 (Necessary condition for convolution type)**.** *Let X and Y be Banach spaces, assume that Y has type 2 and let $p\in[2,\infty)$. Let $k:\mathbb{R}^d\to\mathcal{L}(X,Y)$ be strongly measurable and set $K(s,t):=k(s-t)$. If $K\in\mathcal{K}_\gamma(L^{p,\infty}(\mathbb{R}^d))$, then for all $x\in X$*

$$\|t\mapsto k(t)x\|_{\gamma(\mathbb{R}^d;Y)} \leq C_d\|K\|_{\mathcal{K}_\gamma(L^{p,\infty}(\mathbb{R}^d))}\|x\|_X.$$

The same holds for $\mathbb{R}_+$ instead of $\mathbb{R}^d$, where we set $K(s,t)=0$ if $s\leq t$.

Proof. We start with the $\mathbb{R}^d$-case. Let $r>0$, $x\in X$ and set $f=\mathbf{1}_{B(0,2r)}\otimes x$. Then for all $s\in B(0,r)$,

$$\begin{aligned}L_r := \|t\mapsto k(t)x\|_{\gamma(B(0,r);Y)} &= \|t\mapsto k(s-t)x\|_{\gamma(B(s,r);Y)}\\ &= \|T_K f(s)\|_{\gamma(B(s,r);Y)} \leq \|T_K f(s)\|_{\gamma(\mathbb{R}^d;Y)}.\end{aligned}$$

Therefore, for any $0 < \lambda < L_r$ we find that

$$\begin{aligned}\lambda &\leq \lambda |B(0,r)|^{-1/p} \big|\{s \in B(0,r) : \|Tf(s)\|_{\gamma(\mathbb{R}^d;Y)} > \lambda\}\big|^{1/p} \\ &\leq |B(0,r)|^{-1/p} \|K\|_{\mathcal{K}_\gamma(L^{p,\infty}(\mathbb{R}^d))} \|f\|_{L^p(\mathbb{R}^d;X)} \\ &= C_d \|K\|_{\mathcal{K}_\gamma(L^{p,\infty}(\mathbb{R}^d))} \|x\|_X.\end{aligned}$$

Taking $\lambda = \frac{1}{2} L_r$, we find that $L_r \leq C_d \|K\|_{\mathcal{K}_\gamma(L^{p,\infty}(\mathbb{R}^d))} \|x\|$. Now the proposition follows by letting $r \to \infty$ and applying the γ-Fatou lemma (see Proposition 2.8.2). The proof for $\mathbb{R}_+$ is analogous, taking $s \in (r, 2r)$ instead. □

Remark 4.2.9. If we replace $\mathbb{R}^d$ by $(0,T)$ with $T \in (0,\infty)$ in Proposition 4.2.8, we can deduce that

$$\|t \mapsto k(t)x\|_{\gamma((0,\frac{1}{2}T);Y)} \leq C_d \|K\|_{\mathcal{K}_\gamma(L^{p,\infty}(0,T))} \|x\|_X.$$

For specific kernels one can stretch this estimate to the whole interval $(0,T)$ with a constant dependent on T, see [AV20c, Lemma 4.2].

Next we provide some simple sufficient conditions on K for T_K to be bounded using Fubini's theorem and Young's inequality:

Proposition 4.2.10 (Simple sufficient conditions)**.** *Let X and Y be Banach spaces, assume that Y has type 2 and suppose that $K \colon \mathbb{R}^d \times \mathbb{R}^d \to \mathcal{L}(X,Y)$ is strongly measurable. Then the following hold:*

(i) If there is an $A_0 > 0$ such that

$$\|s \mapsto K(s,t)x\|_{L^2(\mathbb{R}^d;Y)} \leq A_0 \|x\|_X, \qquad t \in \mathbb{R}^d, \quad x \in X,$$

then $K \in \mathcal{K}_\gamma(L^2(\mathbb{R}^d))$ with $\|K\|_{\mathcal{K}_\gamma(L^2(\mathbb{R}^d))} \leq \tau_{2,Y} A_0$.

(ii) If $\|K(s,t)\| \leq k(s-t)$ for some $k \in L^2(\mathbb{R}^d)$, then $K \in \mathcal{K}_\gamma(L^p(\mathbb{R}^d))$ for all $p \in [2,\infty)$ with $\|K\|_{\mathcal{K}_\gamma(L^p(\mathbb{R}^d))} \leq \tau_{2,Y} \|k\|_{L^2(\mathbb{R}^d)}$.

The same holds for $(0,T)$ with $T \in (0,\infty]$ instead of $\mathbb{R}^d$, where $K(s,t) = 0$ if $s \leq t$.

Proof. For (i) we have by Lemma 2.8.4 that

$$\|T_K f(s)\|_{\gamma(\mathbb{R}^d;Y)} \leq \tau_{2,Y} \Big(\int_{\mathbb{R}^d} \|K(s,t)f(t)\|_Y^2 \, \mathrm{d}t \Big)^{1/2}, \qquad s \in \mathbb{R}^d.$$

Taking L^2-norms on both sides and applying Fubini's theorem we obtain

$$\begin{aligned}\|T_K f\|_{L^2(\mathbb{R}^d;\gamma(\mathbb{R}^d;Y))} &\leq \tau_{2,Y} \Big(\int_{\mathbb{R}^d} \|s \mapsto K(s,t)f(t)\|_{L^2(\mathbb{R}^d;Y)}^2 \, \mathrm{d}t \Big)^{1/2} \\ &\leq \tau_{2,Y} \|f\|_{L^2(\mathbb{R}^d;X)}.\end{aligned}$$

For (ii) we have by Lemma 2.8.4

$$\|T_K f(s)\|_{\gamma(\mathbb{R}^d;Y)} \le \tau_{2,Y}\Big(\int_{\mathbb{R}^d} |k(s-t)|^2 \|f(t)\|_Y^2 \,\mathrm{d}t\Big)^{1/2}, \qquad s \in \mathbb{R}^d.$$

Taking L^p-norms on both sides and applying Young's inequality we obtain

$$\|T_K f\|_{L^p(\mathbb{R}^d;\gamma(\mathbb{R}^d;Y))} \le \tau_{2,Y} \|k\|_{L^2(\mathbb{R}^d)} \|f\|_{L^p(\mathbb{R}^d;X)}.$$

The $(0,T)$ case follows similarly, where we extend K and f by 0 outside $(0,T)$ to apply Young's inequality for (ii). □

If Y is a Hilbert space and K is of convolution type, we can actually characterize the boundedness of T_K, since in this case $\gamma(\mathbb{R}^d;Y) = L^2(\mathbb{R}^d;Y)$. In Corollaries 4.4.9 and 4.4.13 the following result will be improved under regularity conditions on K.

Corollary 4.2.11. *Let X be a Banach space and Y be a Hilbert space. Let $k : \mathbb{R}^d \to \mathcal{L}(X,Y)$ be strongly measurable and set $K(s,t) := k(s-t)$. Then the following hold:*

(i) $K \in \mathcal{K}_\gamma(L^2(\mathbb{R}^d))$ if and only if $\|t \mapsto k(t)x\|_{L^2(\mathbb{R}^d;Y)} \lesssim \|x\|_X$.

(ii) If $K \in \mathcal{K}_\gamma(L^{p,\infty}(\mathbb{R}^d))$ for some $p \in [2,\infty)$, then $K \in \mathcal{K}_\gamma(L^q(\mathbb{R}^d))$ for all $q \in [2,p)$.

The same hold for $(0,T)$ with $T \in (0,\infty]$ instead of $\mathbb{R}^d$, where we set $K(s,t) = 0$ if $s \le t$.

Proof. One has for all $t \in \mathbb{R}^d$ that

$$\|s \mapsto K(s,t)x\|_{L^2(\mathbb{R}^d;Y)} = \|s \mapsto k(s)x\|_{L^2(\mathbb{R}^d;Y)} = \|s \mapsto k(s)x\|_{\gamma(\mathbb{R}^d;Y)},$$

from which (i) follows using Proposition 4.2.8 and 4.2.10(i). Part (ii) follows by combining Proposition 4.2.8, part (i) and Marcinkiewicz interpolation theorem (see [HNVW16, Theorem 2.23]). □

4.2.3. SCALAR KERNELS

If we allow X to be any Banach space with type 2, but restrict K to be scalar-valued, we can easily characterize the boundedness of T_K if K is of convolution type. This explains why we study the more interesting operator-valued case.

Proposition 4.2.12. *Let X be a Banach space with type 2, let $p \in [2,\infty)$, let $k : \mathbb{R}^d \to \mathbb{K}$ be measurable and set $K(s,t) := k(s-t)$. Then T_K is bounded from $L^p(\mathbb{R}^d;X)$ to $L^p(\mathbb{R}^d;\gamma(\mathbb{R}^d;X))$ if and only if $k \in L^2(\mathbb{R}^d)$. Moreover, in this case $\|K\|_{\mathcal{K}_\gamma(L^p(\mathbb{R}^d))} \le \tau_{2,X}\|k\|_{L^2(\mathbb{R}^d)}$.*

Proof. Since k is scalar-valued, we have for $x \in X$

$$\|s \mapsto k(s)x\|_{\gamma(\mathbb{R}^d;X)} = \|x\|_X \|k\|_{L^2(\mathbb{R}^d)}.$$

Therefore the result follows from Proposition 4.2.8 and Proposition 4.2.10(ii). □

In the scalar case, i.e. $X = Y = \mathbb{K}$, the L^p-boundedness of T_K can also be well-understood from existing theory for non-convolution kernels. Indeed, in this case $K \in \mathcal{K}_\gamma(L^p(\mathbb{R}^d))$ is equivalent to

$$\int_{\mathbb{R}^d} \Big(\int_{\mathbb{R}^d} |K(s,t)|^2 g(t)\,\mathrm{d}t \Big)^{p/2} \mathrm{d}s \le C^p \|g\|^p_{L^{p/2}(\mathbb{R}^d)}, \tag{4.2.2}$$

where we have set $g(t) = |f(t)|^2$. The validity of the above estimate is completely characterized by the optimality of Schur's lemma (see [Gra14b, Appendix A.2]) applied to the positive kernel $|K(s,t)|^2$. Moreover, in this case T_K is also bounded in the vector-valued setting when $X = Y$ has type 2, since by Lemma 2.8.4

$$\|T_K f\|_{L^p(\mathbb{R}^d;\gamma(\mathbb{R}^d;X))} \le \tau_{2,X} \Big(\int_{\mathbb{R}^d} \Big(\int_{\mathbb{R}^d} |K(s,t)|^2 g(t)\,\mathrm{d}t \Big)^{p/2} \mathrm{d}s \Big)^{1/p},$$

where $g(t) = \|f(t)\|_X^2$. Conversely, by considering a one-dimensional subspace of X, one obtains that (4.2.2) is also necessary.

Example 4.2.13.

(i) Let $d = 1$ and $K(s,t) = \frac{1}{(s+t)^{1/2}} \mathbf{1}_{s,t>0}$. Then by [Gar07, Theorem 5.10.1] we know that $K \in \mathcal{K}_\gamma(L^p(\mathbb{R}))$ if and only if $p \in (2,\infty)$. More generally for $1 \le j \le d$ set

$$K_j(s,t) := \frac{(s_j + t_j)^{1/2}}{|s+t|^{(d+1)/2}} \mathbf{1}_{s_j,t_j>0}, \qquad s,t \in \mathbb{R}^d.$$

Then we know by [Oseę17, Theorem 1] that $K_j \in \mathcal{K}_\gamma(L^p(\mathbb{R}^d))$ if and only if $p \in (2,\infty)$.

(ii) If $K(s,t) = \frac{1}{|s-t|^{1/2}}$, then for all $p \in [2,\infty)$, $K \notin \mathcal{K}_\gamma(L^p(\mathbb{R}))$, which is immediate from Proposition 4.2.8.

Example 4.2.13(ii) can be seen as the analog of the Hilbert transform. It is not bounded for any $p \in [2,\infty)$ due to the lack of cancellation in the stochastic, scalar-valued setting. This further exemplifies the difference between the deterministic and the stochastic theory.

Remark 4.2.14. The scalar case also shows why we only consider $p \in [2,\infty)$. Boundedness for $p < 2$ holds if and only if $K \equiv 0$ (see [Kal78]). This also holds for the operator-valued case since L^p-boundedness with $p < 2$ would imply that $\langle K(t,s)x, y^* \rangle = 0$ a.e. for all $x \in X$ and $y^* \in Y^*$. By strong measurability of $(t,s) \mapsto K(t,s)x$ this implies that for all $x \in X$, $K(t,s)x \equiv 0$. Thus by density of $L^p(\mathbb{R}^d) \otimes X$ in $L^p(\mathbb{R}^d;X)$, we find that $K(s,t)f(t) = 0$.

4.3. SINGULAR KERNELS

Motivated by the connection between stochastic integral operators and γ-integral operators proven in Proposition 4.2.3 and Proposition 4.2.5, we want to systematically study

of the $\mathcal{K}_\gamma$-classes for more involved kernels than treated in Subsection 4.2.2. In particular, we want to study kernels that have a singularity in $s = t$. Let us first introduce the abstract kernel classes that we will use.

We say that $(\boldsymbol{S}, \boldsymbol{d}, \boldsymbol{\mu})$ is an n-product space of homogeneous type if it is the product of n spaces of homogeneous type, i.e.

$$\begin{aligned} \boldsymbol{S} &= S_1 \times \cdots \times S_n \\ \boldsymbol{d}(s,t) &= \max_{1\le k\le n} d_k(s_k,t_k), \qquad s,t \in \boldsymbol{S} \\ \boldsymbol{\mu} &= \mu_1 \otimes \cdots \otimes \mu_n. \end{aligned} \tag{4.3.1}$$

for spaces of homogeneous type (S_k, d_k, μ_k) for $k = 1,\dots,n$. For $s \in \boldsymbol{S}$ we write $s = (s_1,\dots,s_n)$ with $s_k \in S_k$ for $k = 1,\dots,n$. Note that by the choice of the metric $\boldsymbol{d}$ we have for $s \in \boldsymbol{S}$ and $r > 0$

$$B(s,r) = B(s_1,r) \times \cdots \times B(s_n,r).$$

Important examples of n-product spaces of homogeneous type are (subsets of) the anisotropic Euclidean spaces introduced in Example 2.1.2.

For $\boldsymbol{p} \in [1,\infty)^n$ we define the $\boldsymbol{p}$-volume

$$V_{\boldsymbol{p}}(s,t) := \prod_{k=1}^n \mu_k\big(B(s_k, \boldsymbol{d}(s,t))\big)^{1/p_k}, \qquad s,t \in \boldsymbol{S}$$

and set $V(s,t) := V_{(1,\dots,1)}(s,t)$. By (2.1.2) we have

$$V_{\boldsymbol{p}}(s,t) \simeq V_{\boldsymbol{p}}(t,s), \qquad s,t \in \boldsymbol{S} \tag{4.3.2}$$

with implicit constant depending on $\boldsymbol{S}$ and $\boldsymbol{p}$.

Definition 4.3.1. Let X, Y be a Banach spaces, $(\boldsymbol{S}, \boldsymbol{d}, \boldsymbol{\mu})$ a n-product space of homogeneous type, $\boldsymbol{p} \in [1,\infty)^n$ and let

$$K\colon (S \times S) \setminus \{(s,s) : s \in S\} \to \mathcal{L}(X,Y)$$

be strongly measurable.

(i) We say that K is an $\boldsymbol{p}$-Hörmander kernel if there is a $c_K \ge 2$ such that for every ball $B \subseteq S$

$$\big\| \mathbf{1}_{\boldsymbol{S}\setminus B}\big(K(\cdot,t) - K(\cdot,t')\big)\big\|_{L^{\boldsymbol{p}}(\boldsymbol{S};\mathcal{L}(X,Y))} \le C, \qquad t,t' \in \tfrac{1}{c_K}B, \tag{4.3.3}$$

$$\big\| \mathbf{1}_{\boldsymbol{S}\setminus B}\big(K(s,\cdot) - K(s',\cdot)\big)\big\|_{L^{\boldsymbol{p}}(\boldsymbol{S};\mathcal{L}(X,Y))} \le C, \qquad s,s' \in \tfrac{1}{c_K}B \tag{4.3.4}$$

for some constant $C > 0$ independent of s, t and B. The least admissible C will be denoted by $\|K\|_{\boldsymbol{p}\text{-Hörm}}$.

(ii) We say that K is a $\boldsymbol{p}$-Dini kernel if there is a $c_K \geq 2$ such that

$$\|K(s,t) - K(s,t')\| \leq \omega\left(\frac{d(t,t')}{d(s,t)}\right)\frac{1}{V_{\boldsymbol{p}}(s,t)}, \qquad 0 < d(t,t') \leq \frac{1}{c_K} d(t,s), \tag{4.3.5}$$

$$\|K(s,t) - K(s',t)\| \leq \omega\left(\frac{d(s,s')}{d(s,t)}\right)\frac{1}{V_{\boldsymbol{p}}(s,t)}, \qquad 0 < d(s,s') \leq \frac{1}{c_K} d(s,t), \tag{4.3.6}$$

where $\omega : [0,1] \to [0,\infty)$ is increasing, subadditive, submultiplicative, $\omega(0) = 0$ and

$$\|K\|_{\boldsymbol{p}\text{-Dini}} := \max_{1\leq k\leq n}\left(\int_0^1 \omega(t)^{p_k}\frac{\mathrm{d}t}{t}\right)^{1/p_k} < \infty.$$

If $n = 1$ the submultiplicativity of ω can be omitted.

(iii) We say that K is an $\boldsymbol{p}$-standard kernel if K is a $\boldsymbol{p}$-Dini kernel with $\omega(t) = C\,t^{\epsilon}$ for some $C, \epsilon > 0$ and set

$$\|K\|_{\boldsymbol{p}\text{-std}} := \|K\|_{\boldsymbol{p}\text{-Dini}}.$$

We do not track dependence on c_K in our estimates.

Various special cases of Definition 4.3.1 are already present in the literature:

- For $n = 1$ and $\boldsymbol{p} = 1$, Definition 4.3.1 contains the standard kernel assumption from Calderón–Zygmund theory. In particular, Definition 4.3.1(ii) was already used in Section 3.4 to prove the (deterministic) A_2-theorem for operator-valued Calderón–Zygmund operators in a space of homogeneous type. When $(\boldsymbol{S}, \boldsymbol{d}, \boldsymbol{\mu})$ is $\mathbb{R}^d$ with Euclidean distance and the Lebesgue measure, (4.3.5) takes the more familiar form
$$\|K(s,t) - K(s,t')\| \leq \frac{|t-t'|^{\epsilon}}{|s-t|^{d+\epsilon}}.$$
- If $n = 1$, $(\boldsymbol{S}, \boldsymbol{d}, \boldsymbol{\mu})$ is $\mathbb{R}^d$ with Euclidean distance and the Lebesgue measure and K is of convolution type, i.e. $K(s,t) = k(s-t)$ for some $k\colon \mathbb{R}^d \to \mathcal{L}(X,Y)$, Definition 4.3.1 can be reformulated using a change of variables. Indeed, (4.3.3) and (4.3.4) both simplify to
$$\left(\int_{\frac{1}{2}|s|\geq|t|} \|k(s-t) - k(s)\|^p \,\mathrm{d}s\right)^{1/p} \leq C \qquad t \in \mathbb{R}^d, \tag{4.3.7}$$
which goes back to the work of Hörmander himself (see [Hör60]), where it was used to extrapolate off-diagonal boundedness for integral operators. The operator-valued version has been used in [RV17, Section 5.1].
- For $n = 2$ and $\boldsymbol{p} = (2,1)$, Definition 4.3.1(i) was introduced in [KK20, Kim20] to study parabolic SPDE.

For our purposes the two main examples will be:

- In our analysis of the $\mathcal{K}_\gamma$-classes, we will use $n = 1$, $\boldsymbol{p} = 2$. Moreover in applications of these results to SPDE, we will take $\boldsymbol{S} = (0,T)$ for $T \in (0,\infty]$. In this setting we of course have $V_{\boldsymbol{p}}(s,t) = \frac{1}{|s-t|^{1/2}}$.
- For our mixed-norm extrapolation results in Section 4.6, we will use $n = 2$, $\boldsymbol{p} = (2,1)$ and let $(\boldsymbol{S},\boldsymbol{d},\boldsymbol{\mu})$ be $(0,T)\times\mathcal{O}$ for $T \in (0,\infty]$ and a domain $\mathcal{O} \subseteq \mathbb{R}^d$ equipped with an anisotropic metric and the Lebesgue measure.

By definition a $\boldsymbol{p}$-standard kernel is also an $\boldsymbol{p}$-Dini kernel. As in the case $n = 1$, $\boldsymbol{p} = 1$, a $\boldsymbol{p}$-Dini kernel is also a $\boldsymbol{p}$-Hörmander kernel. The proof is an adaptation of the proof in the case $n = 1$, $\boldsymbol{p} = 1$.

Lemma 4.3.2. *Let X, Y be a Banach spaces, $(\boldsymbol{S},\boldsymbol{d},\boldsymbol{\mu})$ a n-product space of homogeneous type, $\boldsymbol{p} \in [1,\infty)^n$ and suppose that*

$$K\colon (\boldsymbol{S}\times\boldsymbol{S})\setminus\{(s,s) : s \in \boldsymbol{S}\} \to \mathcal{L}(X,Y)$$

is a $\boldsymbol{p}$-Dini kernel. Then K is a $\boldsymbol{p}$-Hörmander kernel with

$$\|K\|_{\boldsymbol{p}\text{-Hörm}} \lesssim_{\boldsymbol{S},\boldsymbol{p}} \|K\|_{\boldsymbol{p}\text{-Dini}}.$$

Proof. We will show that K satisfies (4.3.3), the proof of (4.3.4) is analogous. Let $B = B(t,r) \subseteq \boldsymbol{S}$ be a ball and take $u^1, u^2 \in \frac{1}{c_K}B$. Set

$$B_j := B(t,2^{j+1}r)\setminus B(t,2^j r), \qquad j \in \mathbb{N},$$

$p_{\min} = \min_{1\le k\le n} p_k$ and $\boldsymbol{q} = (p_1/p_{\min},\dots,p_n/p_{\min})$. Since $d(t,u^k) \le \frac{1}{c_K}r \le \frac{1}{c_K}d(s,t)$ for any $s \in S\setminus B$ and $c_K \ge 2$, we have by the $\boldsymbol{p}$-Dini condition

$$\begin{aligned}
&\big\|\mathbf{1}_{\boldsymbol{S}\setminus B}K(\cdot,u^1) - K(\cdot,u^2)\big\|_{L^{\boldsymbol{p}}(\boldsymbol{S};\mathcal{L}(X,Y))}\\
&\quad\le \sum_{k=1}^{2}\big\|\mathbf{1}_{\boldsymbol{S}\setminus B}\big(K(\cdot,t) - K(\cdot,u^k)\big)\big\|_{L^{\boldsymbol{p}}(\boldsymbol{S};\mathcal{L}(X,Y))}\\
&\quad\le 2\Big\|s\mapsto \mathbf{1}_{\boldsymbol{S}\setminus B}(s)\cdot\omega\Big(\frac{r/2}{d(s,t)}\Big)\frac{1}{V_{\boldsymbol{p}}(s,t)}\Big\|_{L^{\boldsymbol{p}}(S)}\\
&\quad\le 2\Big(\sum_{j=0}^{\infty}\Big\|s\mapsto \mathbf{1}_{B_j}(s)\cdot\omega\Big(\frac{r/2}{d(s,t)}\Big)^{p_{\min}}\frac{1}{V_{\boldsymbol{q}}(s,t)}\Big\|_{L^{\boldsymbol{q}}(S)}\Big)^{1/p_{\min}}\\
&\quad\le 2\Big(\sum_{j=0}^{\infty}\omega(2^{-j-1})^{p_{\min}}\Big\|s\mapsto\prod_{k=1}^{n}\frac{1}{\mu_k(B(s_k,2^j r))^{q_k}}\Big\|_{L^{\boldsymbol{p}}(B_j)}\Big)^{1/p_{\min}}\\
&\quad\lesssim_{\boldsymbol{S},\boldsymbol{p}} \|K\|_{\boldsymbol{p}\text{-Dini}}
\end{aligned}$$

using (2.1.2) and a similar computation as in (3.4.1) in the final step. □

If (S,d,μ) is an anisotropic Euclidean space (see Example 2.1.2), we can check the $\boldsymbol{p}$-standard kernel conditions in terms of the derivatives of the kernel.

Lemma 4.3.3. *Let X, Y be a Banach spaces, let $\boldsymbol{a}, \boldsymbol{p} \in (0,\infty)^d$ and suppose that*

$$K \in C^1\big((\mathbb{R}^d \times \mathbb{R}^d) \setminus \{(s,s) : s \in \mathbb{R}^d\}; \mathcal{L}(X,Y)\big)$$

is a kernel satisfying for some $A_0 > 0$

$$\begin{aligned} \big\|\partial_{s_k} K(s,t)\big\| &\le A_0 |s-t|_{\boldsymbol{a}}^{-(\boldsymbol{a}/\boldsymbol{p}+a_k)} \qquad s \neq t,\ k = 1,\dots,d \\ \big\|\partial_{t_k} K(s,t)\big\| &\le A_0 |s-t|_{\boldsymbol{a}}^{-(\boldsymbol{a}/\boldsymbol{p}+a_k)} \qquad s \neq t,\ k = 1,\dots,d. \end{aligned}$$

Then K is a $\boldsymbol{p}$-standard kernel on $\mathbb{R}^d_{\boldsymbol{a}}$ with $\|K\|_{\boldsymbol{p}\text{-std}} \lesssim_{\boldsymbol{a}} A_0$.

Proof. We will only prove (4.3.5), as the proof of (4.3.6) is analogous. For (4.3.5) we need to show

$$\|K(s,t) - K(s,u)\| \lesssim \left(\frac{|t-u|_{\boldsymbol{a}}}{|t-s|_{\boldsymbol{a}}}\right)^{\epsilon} \frac{1}{|t-s|_{\boldsymbol{a}}^{\boldsymbol{a}/\boldsymbol{p}}}$$

for all $0 < |t-u|_{\boldsymbol{a}} \le \frac{1}{c_K}|t-s|_{\boldsymbol{a}}$. Set $c_K := 2c_{\boldsymbol{a}}$, where $c_{\boldsymbol{a}}$ is the constant in the triangle inequality for $|\cdot|_{\boldsymbol{a}}$. Fix $s,t,u \in \mathbb{R}^d$ with $0 < |t-u|_{\boldsymbol{a}} \le \frac{1}{c_K}|t-s|_{\boldsymbol{a}}$. Then we have for all $\lambda \in [0,1]$

$$\begin{aligned} |t-s-\lambda(t-u)|_{\boldsymbol{a}} &\ge \frac{1}{c_{\boldsymbol{a}}}|t-s|_{\boldsymbol{a}} - |\lambda(t-u)|_{\boldsymbol{a}} \\ &\ge \frac{1}{c_{\boldsymbol{a}}}|t-s|_{\boldsymbol{a}} - |t-u|_{\boldsymbol{a}} \\ &\ge \frac{1}{2c_{\boldsymbol{a}}}|t-s|_{\boldsymbol{a}}. \end{aligned}$$

Therefore, using the fundamental theorem of calculus, we obtain

$$\begin{aligned} \big\|K(s,t) - K(s,u)\big\| &= \left\| \int_0^1 \frac{\partial}{\partial\lambda} K\big(s, t-\lambda(t-u)\big)\,\mathrm{d}\lambda \right\| \\ &\le \sum_{k=1}^d \int_0^1 \big\| (\partial_{t_k} K)\big(s, t-\lambda(t-u)\big)\cdot(t_k - u_k)\big\|\,\mathrm{d}\lambda \\ &\le A_0 \sum_{k=1}^d \int_0^1 \frac{|t-u|_{\boldsymbol{a}}^{a_k}}{|s-t+\lambda(t-u)|_{\boldsymbol{a}}^{\boldsymbol{a}/\boldsymbol{p}+a_k}}\,\mathrm{d}\lambda \\ &\lesssim_{\boldsymbol{a}} A_0 \left(\frac{|t-u|_{\boldsymbol{a}}}{|t-s|_{\boldsymbol{a}}}\right)^{\epsilon} \frac{1}{|t-s|_{\boldsymbol{a}}^{\boldsymbol{a}/\boldsymbol{p}}} \end{aligned}$$

with $\epsilon = \min_{1\le k\le d} a_k$, proving the lemma. □

Remark 4.3.4. Lemma 4.3.3 remains valid if we replace $\mathbb{R}^d$ by a convex subset of $\mathbb{R}^d$ with the Euclidean distance and the Lebesgue measure. Moreover it is also valid on a smooth domain in $\mathbb{R}^d$, as one can then locally reduce to the $\mathbb{R}^d_+$ case. Combining these observations, we note that Lemma 4.3.3 remains valid on $[0,T] \times D$, with $T \in (0,\infty)$ and $D \subseteq \mathbb{R}^d$ a smooth domain.

4.4. EXTRAPOLATION FOR γ-INTEGRAL OPERATORS

Having introduced the assumptions on our kernels, we will now extrapolate the L^p-boundedness of an γ-integral operator T_K to the (weighted) L^q-boundedness of T_K for all $q \in (2,\infty)$. We will first consider the unweighted setting under a 2-Hörmander assumption on K, from which we will also obtain a weak L^2- and a BMO-endpoint result. This will follow by an adaptation of the arguments for singular integral operators as in [HNVW2x], which we need to combine with ideas from [DM99] when $2 < q < p$. Afterwards we will study sparse domination and weighted boundedness of T_K under a 2-Dini assumption on K, which will follow from the sparse domination framework developed in Chapter 3.

4.4.1. EXTRAPOLATION FOR $2 < q < p$

Let us start our analysis with an extrapolation result downwards. We will show that if $K \in \mathcal{K}_\gamma(L^{p,\infty}(S))$ satisfies the 2-Hörmander condition, then also $K \in \mathcal{K}_\gamma(L^q(S))$ for all $q \in (2,p)$ and $K \in \mathcal{K}_\gamma(L^{2,\infty}(S))$. For this we will adapt the Calderón-Zygmund decomposition technique for singular integral operators to the γ-case. Our main tool will be the following L^q-Calderón–Zygmund decomposition. A similar statement in the case $X = \mathbb{C}$ can for example be found in [BK03, Theorem 3.1], which caries over verbatim to the vector-valued setting, replacing absolute values by norms.

Proposition 4.4.1 (L^2-Calderón–Zygmund decomposition)**.** *Let X be a Banach space and (S,d,μ) a space of homogeneous type and $q \in [1,\infty)$. For every $f \in L^q(S;X)$ and*

$$\lambda > \begin{cases} 0 & \mu(S) = \infty, \\ \langle \|f\|_X \rangle_{2,S} & \mu(S) < \infty. \end{cases}$$

there exists a decomposition $f = g + b$ with

$$\|g\|_{L^\infty(S;X)} \le \lambda, \qquad \|g\|_{L^q(S;X)} \le \|f\|_{L^q(S;X)}$$

and $b = \sum_j b_j$ with

$$\operatorname{supp} b_j \subseteq Q_j \qquad \Big(\sum_j \mu(Q_j)\Big)^{1/q} \lesssim \lambda^{-1} \|f\|_{L^q(S;X)}$$

$$\|b_j\|_{L^q(S;X)} \lesssim \lambda\, \mu(Q_j)^{1/q}, \qquad \Big(\sum_j \|b_j\|^q_{L^q(S;X)}\Big)^{1/q} \lesssim \|f\|_{L^q(S;X)}$$

for disjoint dyadic cubes $\{Q_j\}$. All implicit constants depend on S and q.

Proof. Let $\mathscr{D}$ be a dyadic system in S, which exists by Proposition 2.1.1. Let $\{Q_j\} \subseteq \mathscr{D}$ be the maximal dyadic cubes such that

$$\lambda > \langle \|f\|_X \rangle_{2,Q},$$

which exist by our choice of λ. By their maximality these cubes are pairwise disjoint and their dyadic parents $\{\widehat{Q}_j\}$ satisfy

$$\langle \|f\|_X \rangle_{2,\widehat{Q}} \leq \lambda. \tag{4.4.1}$$

By Proposition 2.2.1 we have

$$\sum_j \mu(Q_j) = \Big\| \sum_j \mathbf{1}_{Q_j} \Big\|_{L^q(S)}^q = \big\| \mathbf{1}_{\{M_q^{\mathscr{D}}(\|f\|_X) > \lambda\}} \big\|_{L^q(S)}^q \lesssim_{S,q} \lambda^{-q} \|f\|_{L^q(S;X)} \tag{4.4.2}$$

Define $b_j := \mathbf{1}_{Q_j} f$, for which we have by (4.4.1)

$$\|b_j\|_{L^q(S;X)} \leq \|\mathbf{1}_{\widehat{Q}_j} f\|_{L^q(S;X)} \lesssim \mu(Q_j)^{1/q} \lambda,$$

which combined with (4.4.2) yields

$$\Big(\sum_j \|b_j\|_{L^q(S;X)}^q \Big)^{1/q} \lesssim \|f\|_{L^q(S;X)}.$$

Set $g = \mathbf{1}_{S \setminus \bigcup_j Q_j} f$, which trivially implies $\|g\|_{L^q(S;X)} \leq \|f\|_{L^q(S;X)}$. For $s \in S \setminus \bigcup_j Q_j$ we have that all dyadic cubes $Q \in \mathscr{D}$ containing s satisfy $\langle \|f\|_X \rangle_{2,Q} \leq \lambda$. Thus by the Lebesgue differentiation theorem and Jensen's inequality we have

$$\|g(s)\|_X = \|f(s)\|_X = \lim_{\substack{Q \in \mathscr{D} : s \in Q \\ \operatorname{diam}(Q) \to 0}} \big\langle \|f\|_X \big\rangle_{1,Q} \leq \lim_{\substack{Q \in \mathscr{D} : s \in Q \\ \operatorname{diam}(Q) \to 0}} \big\langle \|f\|_X \big\rangle_{q,Q} \leq \lambda$$

for a.e. $s \in S \setminus \bigcup_j Q_j$. Thus $\|g\|_{L^\infty(S;X)} \leq \lambda$, which completes the proof. □

In the deterministic setting the functions b_j in a Calderón–Zygmund decomposition are usually also taken such that $\int_{Q_k} b_j = 0$, but we will not be able to use this property for γ-integral operators. Instead we use the L^2-Calderón–Zygmund decomposition in a way that is inspired by [DM99], which builds upon ideas developed in [DR96, Fef70, Heb90].

Theorem 4.4.2 (Extrapolation downwards)**.** *Let X and Y be Banach spaces with type 2 and (S, d, μ) a space of homogeneous type. Let $p \in [2, \infty)$ and suppose that $K \in \mathcal{K}_\gamma(L^{p,\infty}(S))$ satisfies the 2-Hörmander condition. Then*

(i) $K \in \mathcal{K}_\gamma(L^q(S))$ for all $q \in (2, p)$ with

$$\|K\|_{\mathcal{K}_\gamma(L^q(S))} \lesssim_{S,p,q} \Big(\tau_{2,X} \tau_{2,Y} \|K\|_{\mathcal{K}_\gamma(L^{p,\infty}(S))} + \tau_{2,Y} \|K\|_{2\text{-Hörm}} \Big).$$

(ii) $K \in \mathcal{K}_\gamma(L^{2,\infty}(S))$ with

$$\|K\|_{\mathcal{K}_\gamma(L^{2,\infty}(S))} \lesssim_{S,p} \Big(\tau_{2,X} \tau_{2,Y} \|K\|_{\mathcal{K}_\gamma(L^{p,\infty}(S))} + \tau_{2,Y} \|K\|_{2\text{-Hörm}} \Big).$$

Proof. It suffices to show (ii), as (i) then follows directly from the Marcinkiewicz interpolation theorem, see e.g. [HNVW16, Theorem 2.2.3].

Let $f \in L^2(S;X) \cap L^p(S;X)$ be boundedly supported, $\lambda > 0$ and set $A_0 := \|K\|_{\mathcal{K}_\gamma(L^{p,\infty}(S))}$. Let $f = g + b$ be the L^2-Calderón–Zygmund decomposition of f at level $\kappa\lambda$ for some $\kappa > 0$ to be chosen later. Then we have

$$\|g\|_{L^p(S;X)}^p \le \|g\|_{L^\infty(S;X)}^{p-2} \|g\|_{L^2(S;X)}^2 \le (\kappa\lambda)^{p-2} \|f\|_{L^2(S;X)}^2, \tag{4.4.3}$$

so in particular $g \in L^p(S;X)$. It follows that $b = f - g \in L^p(S;X)$, and thus

$$T_K f = T_K g + T_K b$$

is well-defined.

To estimate the $L^{2,\infty}(S;\gamma(S;Y))$-norm of $T_K f$ we need to analyse the size of the upper level set $\{\|T_K f\|_{\gamma(S;Y)} > \lambda\}$. We split as follows:

$$\mu\big(\{\|T_K f\|_{\gamma(S;Y)} > \lambda\}\big) \le \mu\big(\{\|T_K g\|_{\gamma(S;Y)} > \tfrac{\lambda}{2}\}\big) + \mu\big(\{\|T_K b\|_{\gamma(S;Y)} > \tfrac{\lambda}{2}\}\big). \tag{4.4.4}$$

For the term with the "good" part g we have by our assumption on T_K and (4.4.3) that

$$\begin{aligned}\mu\big(\{\|T_K g\|_{\gamma(S;Y)} > \lambda/2\}\big) &\le \frac{A_0^p}{(\lambda/2)^p} \|g\|_{L^p(S;X)}^p \\ &\le 2^p A_0^p \kappa^{p-2} \frac{\|f\|_{L^2(S;X)}^2}{\lambda^2}\end{aligned}$$

For the term with the "bad" part b, let Q_j be the dyadic cube corresponding to b_j. Let B_j be the ball with the same center as Q_j and radius $c_K \cdot \operatorname{diam}(Q_j)$. Then $Q_j \subseteq B_j$ and $\mu(B_j) \lesssim_S \mu(Q_j)$. Set $O := \bigcup_j B_j$.

As a preparation for our estimates we will define some auxiliary operators. Let

$$S_j \colon L^2(S;X) \to L^2(S;\gamma(S;X))$$

be the γ-integral operator given by

$$S_j h(s) := \frac{\mathbf{1}_{Q_j}(s)}{\mu(Q_j)^{1/2}} \cdot h, \qquad s \in S, \quad h \in L^2(S;X)$$

which is bounded by Lemma 2.8.4. We claim that $\sum_j S_j b_j$ converges in $L^p(S;\gamma(S;X))$. To prove this we first estimate for fixed j and a.e. $s \in S$

$$\|S_j b_j(s)\|_{\gamma(S;X)}^2 \le \tau_{2,X}^2 \frac{\mathbf{1}_{Q_j}(s)}{\mu(Q_j)} \int_{Q_j} \|b_j(t)\|_X^2 \, \mathrm{d}\mu(t) \lesssim_S \tau_{2,X}^2 (\kappa\lambda)^2 \mathbf{1}_{Q_j}(s)$$

using Lemma 2.8.4 and the norm estimate of b_j in terms of $\mu(Q_j)$. So summing over j we get, using the disjointness of the Q_j's, that

$$\Big\| \sum_j \|S_j b_j\|_{\gamma(S;X)} \Big\|_{L^p(S)} \lesssim_S \tau_{2,X} \kappa\lambda \Big(\sum_j \mu(Q_j) \Big)^{1/p}.$$

Since $\sum_j \mu(Q_j) \le (\kappa\lambda)^{-2}\|f\|^2_{L^2(S;X)}$ it follows that $\sum_j S_j b_j$ converges in $L^p(S;\gamma(S;X))$ as claimed and in particular we have

$$\Big\|\sum_j S_j b_j\Big\|_{L^p(S;\gamma(S;X))} \lesssim_S \tau_{2,X}\,(\kappa\lambda)^{1-2/p}\|f\|^{2/p}_{L^2(S;X)}. \tag{4.4.5}$$

Next set

$$\psi(t',t) := \sum_j \frac{1}{\mu(Q_j)^{1/2}}\,\mathbf{1}_{Q_j}(t')\,\mathbf{1}_{Q_j}(t), \qquad t,t' \in S$$

and define for a.e. $s \in S$

$$T_\psi b(s) := \Big((t,t') \mapsto K(s,t)\psi(t',t)b(t)\Big).$$

Since $\|\psi(\cdot,t)\|_{L^2(S)} = 1$ for $t \in \operatorname{supp} b$ we have

$$\big\|\langle T_\psi b(s), y^*\rangle\big\|_{L^2(S\times S)} = \big\|\langle Tb(s), y^*\rangle\big\|_{L^2(S)}$$

for every $y^* \in Y^*$. Thus by Proposition 2.8.1 it follows that $T_\psi b(s) \in \gamma(S\times S;Y)$ with

$$\big\|T_K b(s)\big\|_{\gamma(S;Y)} = \big\|T_\psi b(s)\big\|_{\gamma(S\times S;Y)}.$$

Finally let

$$\widetilde{T}_K\colon \gamma(S;L^p(S;X)) \to \gamma(S;L^{p,\infty}(S;\gamma(S;Y)))$$

be the canonical extension of T_K, which is trivially bounded with norm A_0. By Lemma 2.8.4 and the γ-Fubini embedding in Proposition 2.8.6, $\widetilde{T}_K$ is also bounded as an operator

$$\widetilde{T}_K\colon L^p(S;\gamma(S;X)) \to L^{p,\infty}(S;\gamma(S\times S;Y))$$

with norm $C_p\,\tau_{2,Y}\,A_0$. Combined with (4.4.5) this implies that $\sum_j \widetilde{T}_K S_j b_j$ is well-defined.

Using these auxiliary operators we now decompose as follows:

$$\begin{aligned}
\mu\big(\{\|T_K b\|_{\gamma(S;Y)} > \lambda/2\}\big) &= \mu\big(\{\|T_\psi b\|_{\gamma(S\times S;Y)} > \lambda/2\}\big)\\
&\le \mu\Big(\Big\{\Big\|T_\psi b - \sum_j \widetilde{T}_K S_j b_j\Big\|_{\gamma(S\times S;Y)} > \lambda/4\Big\}\setminus O\Big)\\
&\quad + \mu\Big(\Big\{\Big\|\sum_j \widetilde{T}_K S_j b_j\Big\|_{\gamma(S\times S;Y)} > \lambda/4\Big\}\Big) + \mu(O)\\
&=: \boxed{\text{A}} + \boxed{\text{B}} + \boxed{\text{C}}
\end{aligned}$$

To estimate $\boxed{\text{A}}$ we first note that by Chebyshev's inequality and Lemma 2.8.4 we have

$$\boxed{\text{A}} \le \tau^2_{2,Y}\,\frac{16}{\lambda^2}\int_{S\setminus O}\Big\|T_\psi b - \sum_j \widetilde{T}_K S_j b_j\Big\|^2_{L^2(S\times S;Y)}\,\mathrm{d}\mu$$

Using the fact that the b_j's are disjointly supported on the cubes $Q_j \subseteq B_j$, Fubini's theorem and the 2-Hörmander condition we deduce

$$\begin{aligned}
&\int_{S\setminus O} \big\| T_\psi b - \sum_j \widetilde{T}_K S_j b_j \big\|^2_{L^2(S\times S;Y)} \,\mathrm{d}\mu \\
&\quad\le \sum_j \frac{1}{\mu(Q_j)} \int_{S\setminus B_j} \int_S \int_S \big\| \big(K(s,t) - K(s,t')\big) \mathbf{1}_{Q_j}(t') b_j(t) \big\|^2_Y \,\mathrm{d}\mu(t') \,\mathrm{d}\mu(t) \,\mathrm{d}\mu(s) \\
&\quad\le \sum_j \frac{1}{\mu(Q_j)} \int_{Q_j} \int_{Q_j} \int_{S\setminus B_j} \|K(s,t) - K(s,t')\|^2 \|b_j(t)\|^2_X \,\mathrm{d}\mu(s) \,\mathrm{d}\mu(t) \,\mathrm{d}\mu(t') \\
&\quad\le \|K\|^2_{2\text{-Hörm}} \sum_j \|b_j\|^2_{L^2(S;X)}.
\end{aligned}$$

Therefore by the norm estimate of the b_j's in terms of f we have

$$\boxed{\text{A}} \lesssim_S \tau_{2,Y}^2 \|K\|^2_{2\text{-Hörm}} \frac{\|f\|^2_{L^2(S;X)}}{\lambda^2}.$$

For $\boxed{\text{B}}$ we use the boundedness of $\widetilde{T}_K$ and (4.4.5) to obtain

$$\begin{aligned}
\boxed{\text{B}} &\lesssim_{S,p} \frac{(\tau_{2,Y} A_0)^p}{\lambda^p} \Big\| \sum_j S_j b_j \Big\|^p_{L^p(S;\gamma(S;X)))} \\
&\lesssim_{S,p} \big(\tau_{2,X}\tau_{2,Y} A_0\big)^p \kappa^{p-2} \frac{\|f\|^2_{L^2(S;X)}}{\lambda^2}
\end{aligned}$$

and for $\boxed{\text{C}}$ we have by the estimate of $\mu(Q_j)$ in terms of f that

$$\boxed{\text{C}} \le \sum_j \mu(B_j) \lesssim_S \sum_j \mu(Q_j) \lesssim_S \kappa^{-2} \frac{\|f\|^2_{L^2(S;X)}}{\lambda^2}.$$

Plugging the estimate for g and the estimates for b into (4.4.4) and choosing $\kappa := (\tau_{2,X}\tau_{2,Y} A_0)^{-1}$, we now have

$$\begin{aligned}
\lambda \cdot \mu\big(\{\|T_K f\|_{\gamma(S;Y)} > \lambda\}\big)^{1/2} &\lesssim_{S,p} \Big(\frac{\big(\tau_{2,X}\tau_{2,Y} A_0 \kappa\big)^p + 1}{\kappa} + \tau_{2,Y} \|K\|_{2\text{-Hörm}} \Big) \|f\|_{L^2(S;X)} \\
&= \big(2\,\tau_{2,X}\tau_{2,Y} \|K\|_{\mathcal{K}_\gamma(L^{p,\infty}(S))} + \tau_{2,Y} \|K\|_{2\text{-Hörm}} \big) \|f\|_{L^2(S;X)}.
\end{aligned}$$

for all $\lambda > 0$ and boundedly supported $f \in L^2(S;X) \cap L^p(S;X)$, except when $\mu(S) < \infty$ and $\kappa\lambda \le \langle \|f\|_X \rangle_{2,S}$. However, this case is trivial, since

$$\mu\big(\{\|T_K f\|_{\gamma(\mathbb{R}^d;Y)} > \lambda\}\big)^{1/2} \le \mu(S)^{1/2} \le \frac{1}{\kappa\lambda} \|f\|_{L^2(S;X)}.$$

By density this estimate extends to all $f \in L^2(S;X)$, which finishes the proof of the weak L^2-endpoint. □

Remark 4.4.3. In general one can not expect $T_K \in \mathcal{K}_\gamma(L^2(S))$ in Theorem 4.4.2 , which is already clear from the scalar case and $S = \mathbb{R}$. For instance the kernel $K(s,t) = \frac{1}{(s+t)^{1/2}} \mathbf{1}_{s,t>0}$ of Example 4.2.13 is a 2-Hörmander kernel. However, L^p-boundedness holds only for $p \in (2,\infty)$.

4.4.2. EXTRAPOLATION FOR $p < q < \infty$

We now turn our attention to extrapolation upwards for γ-integral operator. We will show that if $K \in \mathcal{K}_\gamma(L^{p,\infty}(S))$ satisfies the 2-Hörmander condition, then also $K \in \mathcal{K}_\gamma(L^q(S))$ for all $q \in (p,\infty)$ and we will prove a BMO-endpoint result. For this we will adapt the arguments in [HNVW2x] for singular integral operators to the case of singular γ-integral operators.

Theorem 4.4.4 (Extrapolation upwards)**.** *Let X and Y be Banach spaces, assume that Y has type 2 and let (S,d,μ) be a space of homogeneous type. Let $p \in [2,\infty)$ and suppose $K \in \mathcal{K}_\gamma(L^{p,\infty}(S))$ satisfies the 2-Hörmander condition. Then*

(i) $K \in \mathcal{K}_\gamma(L^q(S))$ for all $q \in (p,\infty)$ with

$$\|K\|_{\mathcal{K}_\gamma(L^q(S))} \lesssim_{S,p,q} \Big(\|K\|_{\mathcal{K}_\gamma(L^{p,\infty}(S))} + \tau_{2,Y}\|K\|_{2\text{-Hörm}}\Big).$$

(ii) There exists a $\widetilde{T}_K \in \mathcal{L}(L^\infty(S;X), \mathrm{BMO}(S;\gamma(S;Y)))$ such that

$$\|\widetilde{T}_K\|_{L^\infty \to \mathrm{BMO}} \lesssim_{S,p} \Big(\|K\|_{\mathcal{K}_\gamma(L^{p,\infty}(S))} + \tau_{2,Y}\|K\|_{2\text{-Hörm}}\Big).$$

and $\widetilde{T}_K f - T_K f$ is constant for all $f \in L^p(S;X) \cap L^\infty(S;X)$.

Remark 4.4.5. The extension of T_K to all $f \in L^\infty(S;X)$ in Theorem 4.4.4(ii) is not in the traditional sense, as even for $f \in L^p(S;X) \cap L^\infty(S;X)$ the extension $\widetilde{T}_K f$ may not coincide with $T_K f$. However, as $\widetilde{T}_K f$ and $T_K f$ only differ by a constant in this case, they represent the same function in the Banach space

$$\mathrm{BMO}(S;\gamma(S;Y))/\gamma(S;Y).$$

Furthermore, we can not claim uniqueness, as $L^p(S;X) \cap L^\infty(S;X)$ is not dense in $L^\infty(S;X)$

In order to prove Theorem 4.4.4, we need to introduce local versions of the operator T_K. For any cube Q in S we define the local operator

$$T_K^Q : L^\infty(S;X) \to L^p(Q;\gamma(S;Y))$$

for $s \in Q$ and $\varphi \in L^2(S)$ by

$$T_K^Q f(s)\varphi := T_K(\mathbf{1}_B f)(s)\varphi + \fint_Q \int_{S\setminus B} \big(K(s,t) - K(s',t)\big) f(t)\varphi(t)\, \mathrm{d}\mu(t)\, \mathrm{d}\mu(s'),$$

where B is the ball with the same center as Q and radius $c_K \cdot \mathrm{diam}(Q)$. Note that T_K^Q is well-defined since $\mathbf{1}_B f \in L^p(S;X)$ and for a.e. $s, s' \in Q$ we have

$$\big\|\big(K(s,\cdot) - K(s',\cdot)\big)\mathbf{1}_{S\setminus B} f\big\|_{\gamma(S;Y)} \le \tau_{2,Y}\|K\|_{2\text{-Hörm}}\|f\|_{L^\infty(S;X)}. \tag{4.4.6}$$

by Lemma 2.8.4. Heuristically one may think about T_K^Q as

$$T_K^Q f(s) = T_K f(s) + \fint_Q \big(K(s',\cdot)\big) f(\cdot)\mathbf{1}_{S\setminus B}(\cdot)\, \mathrm{d}\mu(s'),$$

which is, of course, not well-defined in general. These operators satisfy the following properties:

Lemma 4.4.6. *Let X and Y and be Banach spaces, assume that Y has type 2 and let (S,d,μ) be a space of homogeneous type with dyadic systems $\mathscr{D}$ and $\mathscr{D}'$. Let $p \in [2,\infty)$ and suppose $K \in \mathcal{K}_\gamma(L^{p,\infty}(S))$ satisfies the 2-Hörmander condition. For dyadic cubes $Q \in \mathscr{D}$ and $Q' \in \mathscr{D}'$ the following hold*

(i) For all $f \in L^\infty(S;X)$ we have

$$\big\| T_K^Q f \big\|_{L^{p,\infty}(Q;\gamma(S;Y))} \lesssim_{S,\mathscr{D},p} \big(\|K\|_{\mathcal{K}_\gamma(L^{p,\infty}(S))} + \tau_{2,Y}\|K\|_{2\text{-Hörm}}\big)\mu(Q)^{1/p}\|f\|_{L^\infty(S;X)}.$$

(ii) For all $f \in L^p(S;X) \cap L^\infty(S;X)$ there exists a $c \in \gamma(S;Y)$ such that

$$T_K f(s) - T_K^Q f(s) = c, \qquad s \in Q.$$

(iii) For all $f \in L^\infty(S;X)$ there exists a $c \in \gamma(S;Y)$ such that

$$T_K^Q f(s) - T_K^{Q'} f(s) = c, \qquad s \in Q \cap Q'.$$

Proof. Let $B \subseteq S$ be the ball with the same center as Q and radius $c_K \cdot \operatorname{diam}(Q)$. Define $B' \subseteq S$ similarly. Take $f \in L^\infty(S;X)$, then by the assumption on T_K we have, using $\|\mathbf{1}_Q\|_{L^{p,\infty}(S)} = \mu(Q)^{1/p}$, that

$$\begin{aligned}\|T_K(\mathbf{1}_B f)\|_{L^{p,\infty}(S;\gamma(S;Y))} &\le \|K\|_{\mathcal{K}_\gamma(L^{p,\infty}(S))}\|\mathbf{1}_B f\|_{L^p(S;X)} \\ &\lesssim_{S,\mathscr{D},p} \|K\|_{\mathcal{K}_\gamma(L^{p,\infty}(S))}\mu(Q)^{1/p}\|f\|_{L^\infty(S;X)}.\end{aligned}$$

The estimate in (i) now readily follows using the definition of $T_K^Q f$ and (4.4.6).

Next take $f \in L^p(S;X) \cap L^\infty(S;X)$ and let $s,s' \in Q$. Define $c := \fint_Q T_K(\mathbf{1}_{S\setminus B} f)(s')\,\mathrm{d}s'$. Then we have for a.e. $s \in Q$ that

$$\begin{aligned}T_K f(s) &= T_K(\mathbf{1}_B f)(s) + T_K(\mathbf{1}_{S\setminus B} f)(s) - \fint_Q T_K(\mathbf{1}_{S\setminus B} f)(s')\,\mathrm{d}\mu(s') + c \\ &= T_K^Q f(s) + c\end{aligned}$$

proving (ii).

For (iii) by considering a larger cube $Q'' \in \mathscr{D}$ containing both Q and Q' we may assume without loss of generality that $Q' \subseteq Q$ and $B' \subseteq B$. Fix $\varphi \in L^2(S)$ and define

$$g(s,s',t) := \big(K(s,t) - K(s',t)\big)f(t)\varphi(t).$$

Then we have for a.e. $s \in Q' = Q \cap Q'$ by Fubini's theorem

$$T_K^Q f(s)\varphi - T_K^{Q'} f(s)\varphi$$

$$
\begin{aligned}
&= T_K(\mathbf{1}_{B\setminus B'} f)(s)\varphi + \Big(\fint_Q \int_{S\setminus B} - \fint_{Q'} \int_{B\setminus B'} - \fint_{Q'} \int_{S\setminus B}\Big) g(s,s',t)\,\mathrm{d}\mu(t)\,\mathrm{d}\mu(s') \\
&= \fint_{Q'} T_K(\mathbf{1}_{B\setminus B'} f)(s')\varphi\,\mathrm{d}\mu(s') + \int_{S\setminus B}\Big(\fint_Q - \fint_{Q'}\Big) g(s,s',t)\,\mathrm{d}\mu(s')\,\mathrm{d}\mu(t) \\
&= \fint_{Q'} T_K(\mathbf{1}_{B\setminus B'} f)(s')\varphi\,\mathrm{d}\mu(s') - \int_{S\setminus B}\Big(\fint_Q - \fint_{Q'}\Big) K(s',t) f(t)\varphi(t)\,\mathrm{d}\mu(s')\,\mathrm{d}\mu(t).
\end{aligned}
$$

As the final right-hand side does not depend on s, this proves (iii). □

Using the properties of these local operators T_K^Q we can prove an L^∞-estimate of T_K involving the sharp maximal operator, introduced in Section 2.2.

Proposition 4.4.7. *Let X and Y be Banach spaces, assume that Y has type 2 and let (S,d,μ) be a space of homogeneous type. Let $p \in [2,\infty)$ and suppose $K \in \mathcal{K}_\gamma(L^{p,\infty}(S))$ satisfies the 2-Hörmander condition. Then we have for all $f \in L^p(S;X) \cap L^\infty(S;X)$*

$$
\big\|M^\#\big(T_K f\big)\big\|_{L^\infty(S)} \lesssim_{S,p} \big(\|K\|_{\mathcal{K}_\gamma(L^{p,\infty}(S))} + \tau_{2,Y}\|K\|_{2\text{-Hörm}}\big)\|f\|_{L^\infty(S;X)}.
$$

Proof. Let $B \subseteq S$ be a ball and let $\mathscr{D}$ be a dyadic system in (S,d,μ) such that there is a $Q \in \mathscr{D}$ with $B \subseteq Q$ and $\operatorname{diam}(Q) \lesssim_S \operatorname{diam}(B)$, which exists by Proposition 2.1.1. Let $f \in L^p(S;X) \cap L^\infty(S;X)$ and, using Lemma 4.4.6(ii), choose $c \in \gamma(S;Y)$ such that

$$
T_K f(s) - T_K^Q f(s) = c, \qquad s \in Q.
$$

Then, using (2.0.1) and Lemma 4.4.6(i), we have

$$
\begin{aligned}
\fint_B \big\|T_K f(s) - c\big\|\,\mathrm{d}\mu(s) &\lesssim_S \fint_Q \big\|T_K^Q f\big\|_{\gamma(S;Y)}\,\mathrm{d}\mu \\
&\lesssim_p \mu(Q)^{-1+1/p'}\big\|T_K^Q f\big\|_{L^{p,\infty}(Q;\gamma(S;Y))} \\
&\lesssim_{S,p} \big(\|K\|_{\mathcal{K}_\gamma(L^{p,\infty}(S))} + \tau_{2,Y}\|K\|_{2\text{-Hörm}}\big)\|f\|_{L^\infty(S;X)}.
\end{aligned}
$$

It follows that

$$
\big\|M^\#\big(T_K f\big)\big\|_{L^\infty(S)} \lesssim_{S,p} (\|K\|_{\mathcal{K}_\gamma(L^{p,\infty}(S))} + \tau_{2,Y}\|K\|_{2\text{-Hörm}})\|f\|_{L^\infty(S;X)},
$$

which proves the proposition. □

Using Proposition 4.4.7, the proof of Theorem 4.4.4(i) is now a straightforward application of Stampacchia interpolation (see e.g. [GR85, Theorem II.3.7]).

Proof of Theorem 4.4.4(i). Let $f \in L^p(S;X) \cap L^\infty(S;X)$. Since $M^\# f \le 2M(\|f\|_{\gamma(S;Y)})$, we know by Proposition 2.2.1 that $M^\#$ is bounded from $L^{p,\infty}(S;\gamma(S;Y))$ to $L^{p,\infty}(S)$ and thus

$$
\begin{aligned}
\big\|M^\#(T_K f)\big\|_{L^{p,\infty}(S)} &\lesssim_{S,p} \big\|T_K f\big\|_{L^{p,\infty}(S;\gamma(S;Y))} \\
&\lesssim_{S,p} \|K\|_{\mathcal{K}_\gamma(L^{p,\infty}(S))}\|f\|_{L^p(S;X)}.
\end{aligned}
$$

Moreover, by Proposition 4.4.7, we know that

$$\left\|M^{\#}\left(T_K f\right)\right\|_{L^\infty(S)} \lesssim_{S,p} \left(\|K\|_{\mathcal{K}_\gamma(L^{p,\infty}(S))} + \tau_{2,Y}\|K\|_{2\text{-Hörm}}\right)\|f\|_{L^\infty(S;X)},$$

We can therefore apply the Marcinkiewicz interpolation theorem (see e.g. [HNVW16, Theorem 2.2.3]), to conclude that for all $f \in L^p(S;X) \cap L^\infty(S;X)$ we have

$$\left\|M^{\#}(T_K f)\right\|_{L^q(S)} \lesssim_{S,p,q} \left(\|K\|_{\mathcal{K}_\gamma(L^{p,\infty}(S))} + \tau_{2,Y}\|K\|_{2\text{-Hörm}}\right)\|f\|_{L^q(S;X)}.$$

We consider two cases:

(i) If $\mu(S) = \infty$, we deduce by Proposition 2.2.3

$$\begin{aligned}\|T_K f\|_{L^q(S;\gamma(S;Y))} &\lesssim_{S,q} \left\|M^{\#}(T_K f)\right\|_{L^q(S)} \\ &\lesssim_{S,p,q} \left(\|K\|_{\mathcal{K}_\gamma(L^{p,\infty}(S))} + \tau_{2,Y}\|K\|_{2\text{-Hörm}}\right)\|f\|_{L^q(S;X)}.\end{aligned}$$

for all $f \in L^p(S;X) \cap L^\infty(S;X)$.

(ii) If $\mu(S) < \infty$, we deduce by Proposition 2.2.3, (2.0.1), $K \in \mathcal{K}_\gamma(L^{p,\infty}(S))$ and Hölder's inequality

$$\begin{aligned}\|T_K f\|_{L^q(S;\gamma(S;Y))} &\lesssim_{S,q} \left\|M^{\#}(T_K f)\right\|_{L^q(S)} + \mu(S)^{-1/q'}\|T_K f\|_{L^1(S)} \\ &\lesssim_{S,p,q} \left(\|K\|_{\mathcal{K}_\gamma(L^{p,\infty}(S))} + \tau_{2,Y}\|K\|_{2\text{-Hörm}}\right)\|f\|_{L^q(S;X)} \\ &\qquad + \mu(S)^{1/p'-1/q'}\|K\|_{\mathcal{K}_\gamma(L^{p,\infty}(S;X))}\|f\|_{L^p(S;X)} \\ &\lesssim \left(\|K\|_{\mathcal{K}_\gamma(L^{p,\infty}(S))} + \tau_{2,Y}\|K\|_{2\text{-Hörm}}\right)\|f\|_{L^q(S;X)}\end{aligned}$$

for all $f \in L^p(S;X) \cap L^\infty(S;X)$.

As $L^p(S;X) \cap L^\infty(S;X)$ is a dense subspace of $L^q(S;X)$, assertion (i) of Theorem 4.4.4 follows. □

Assertion (ii) of Theorem 4.4.4 does not follow directly from Proposition 4.4.7, since $L^p(S;X) \cap L^\infty(S;X)$ is not dense in $L^\infty(S;X)$ and therefore the extension of T_K to all functions in $L^\infty(S;X)$ is a nontrivial matter.

Proof of Theorem 4.4.4(ii). Let $\mathscr{D}^1, \dots, \mathscr{D}^m$ be a dyadic systems in (S,d,μ) as in Proposition 2.1.1 and let $(Q_k)_{k=1}^\infty \subseteq \bigcup_{j=1}^m \mathscr{D}^j$ be an increasing sequence of dyadic cubes such that $\bigcup_{k=1}^\infty Q_k = S$. For $f \in L^\infty(S;X)$ define

$$\widetilde{T}_K f(s) := T_K^{Q_k} f(s) - \fint_{Q_1} T_K^{Q_k} f \, \mathrm{d}\mu \qquad \text{if } s \in Q_k.$$

Then $\widetilde{T}_K f \in L^1_{\mathrm{loc}}(S;\gamma(S;Y))$ is well-defined. Indeed, by Lemma 4.4.6(ii) we have $T_K^{Q_k} f \in L^1(Q_k;\gamma(S;Y))$, so in particular the average over Q_1 is well-defined. Moreover if $j > k$,

then by Lemma 4.4.6(iii) there is a $c \in \gamma(S;Y)$ such that $T_K^{Q_j} f(s) - T_K^{Q_k} f(s) = c$ for a.e. $s \in Q_k$. Therefore

$$\begin{aligned} T_K^{Q_k} f(s) - \fint_{Q_1} T_K^{Q_k} f \, \mathrm{d}\mu &= \big(T_K^{Q_j} f(s) - c\big) - \fint_{Q_1} \big(T_K^{Q_j} f - c\big) \, \mathrm{d}\mu \\ &= T_K^{Q_j} f(s) - \fint_{Q_1} T_K^{Q_j} f \, \mathrm{d}\mu, \end{aligned}$$

thus the definition of $\widetilde{T}_K f(s)$ is independent of the choice of $Q_k \ni s$.

If $f \in L^p(S;X) \cap L^\infty(S;X)$, then for any $k \in \mathbb{N}$ there exist $c_1, c_2 \in \gamma(S;Y)$ such that for a.e. $s \in Q_k$

$$\begin{aligned} T_K^{Q_k} f(s) - T_K f(s) &= c_1, \\ T_K^{Q_k} f(s) - \widetilde{T}_K f(s) &= c_2 \end{aligned}$$

by Lemma 4.4.6(ii) and the definition of $\widetilde{T}_K f$. As $(Q_k)_{k=1}^\infty$ is increasing and $\bigcup_{k=1}^\infty Q_k = S$, we see that c_1 and c_2 are independent of k, so $\widetilde{T}_K f - T_K f$ is indeed constant.

It remains to show that $\widetilde{T}_K f \in \mathrm{BMO}(S;X)$ with the claimed norm estimate. Let $B \subseteq S$ be any ball and fix $k \in \mathbb{N}$ such that $B \subseteq Q_k$. Take $Q \in \bigcup_{j=1}^m \mathscr{D}^j$ such that $B \subseteq Q$ and $\operatorname{diam}(Q) \lesssim_S \operatorname{diam}(B)$. By Lemma 4.4.6(iii) there exists a $c_3 \in \gamma(S;Y)$ such that for a.e. $s \in Q$

$$T_K^{Q_k} f(s) - T_K^Q f(s) = c_3.$$

Therefore

$$\big\| \widetilde{T}_K f \big\|_{\mathrm{BMO}(S;\gamma(S;Y))} \lesssim_S \fint_Q \big\| \widetilde{T}_K f - (c_3 - c_2) \big\|_{\gamma(S;Y)} \, \mathrm{d}\mu = \fint_Q \big\| T_K^Q f \big\|_{\gamma(S;Y)} \, \mathrm{d}\mu.$$

Now $\fint_Q \| T_K^Q f \|_{\gamma(S;Y)}$ can be estimated exactly as in the proof of Proposition 4.4.7, which yields the claimed norm estimate in Theorem 4.4.4(ii). □

Remark 4.4.8. By inspection of the proof it can easily be seen that for the extrapolation down in Theorem 4.4.2 one only needs

$$\Big(\int_{S \setminus B} \| (K(s,t) - K(s,t'))x \|_Y^2 \, \mathrm{d}\mu(t) \Big)^{1/2} \leq C \|x\|_X, \qquad s, s' \in \tfrac{1}{2}B, \quad x \in X$$

which is implied by (4.3.4) of the 2-Hörmander condition. For the extrapolation up in Theorem 4.4.4 one only needs the left hand side of (4.4.6) to be bounded, which is implied by (4.3.3) of the 2-Hörmander condition.

Corollary 4.4.9 (γ-convolution operator with values in a Hilbert space)**.** *Let X be a Banach space and let Y be a Hilbert space. Suppose $k : \mathbb{R}^d \to \mathcal{L}(X,Y)$ is strongly measurable and satisfies the 2-Hörmander condition in (4.3.7). Let $K(s,t) = k(s-t)$. Then the following are equivalent:*

(i) $\|t \mapsto k(t)x\|_{L^2(\mathbb{R}^d;Y)} \leq A_0 \|x\|$ *for some* $A_0 > 0$.

(ii) $K \in \mathcal{K}_\gamma(L^p(\mathbb{R}^d))$ *for all* $p \in [2,\infty)$.

(iii) $K \in \mathcal{K}_\gamma(L^{p,\infty}(\mathbb{R}^d))$ *for some* $p \in [2,\infty)$.

In particular we have for all $p \in [2,\infty)$ *and* A_0 *as in (i):*

$$\|K\|_{\mathcal{K}_\gamma(L^p(\mathbb{R}^d))} \leq C_{p,d}\,(A_0 + \|K\|_{2\text{-Hörm}}).$$

Proof. The implication (i) $\Rightarrow$ (ii) for $p = 2$ follows from Proposition 4.2.10(i) and for $p \in (2,\infty)$ we can apply Theorem 4.4.4. The implication (ii) $\Rightarrow$ (iii) is trivial and (iii) $\Rightarrow$ (i) follows from Proposition 4.2.8. □

4.4.3. SPARSE DOMINATION FOR γ-INTEGRAL OPERATORS

In this section we will obtain weighted bounds for a γ-integral operator T_K under an 2-Dini condition on K. We will deduce these weighted bounds from the abstract sparse domination principle obtained in Chapter 3, which will lead to a stochastic analogue of the A_2-theorem.

In order to apply this abstract sparse domination principle on a $K \in \mathcal{K}_\gamma(L^p(S))$ we need to check weak L^2-boundedness of T_K and $\mathcal{M}^\#_{T_K,\alpha}$ and we need to check the 2-sublinearity of T_K. The weak L^2-boundedness of T_K was already obtained in Theorem 4.4.2. For a 2-Dini kernel the boundedness of $\mathcal{M}^\#_{T_K,\alpha}$ is quite easy to check:

Lemma 4.4.10 (Boundedness of grand maximal truncation operator)**.** *Let X and Y be a Banach spaces, assume that Y has type 2 and let (S,d,μ) be a space of homogeneous type. Let $p \in [2,\infty)$ and suppose $K \in \mathcal{K}_\gamma(L^{p,\infty}(S))$ satisfies the 2-Dini condition. Then for any $f \in L^p(S;X)$ and $\alpha \geq 3\,c_d^2\,c_K$ we have*

$$\mathcal{M}^\#_{T_K,\alpha} f \lesssim_S \tau_{2,Y}\,\|K\|_{2\text{-Dini}}\,M_2(\|f\|_X).$$

In particular, $\mathcal{M}^\#_{T_K,\alpha}$ is bounded from $L^2(S;X)$ to $L^{2,\infty}(S)$ with

$$\|\mathcal{M}^\#_{T_K,\alpha}\|_{L^2(S;X)\to L^{2,\infty}(S)} \lesssim_S \tau_{2,Y}\,\|K\|_{2\text{-Dini}}.$$

Proof. Let $f \in L^p(S;X) \cap L^2(S;X)$, $s \in S$ and fix a ball $B \ni s$ with radius r. Take $s', s'' \in B$ and let $\varepsilon = 2\,c_K\,c_d\,r$. Then

$$d(s',t) \geq \frac{1}{c_d}\,d(z,t) - d(z,s') \geq \frac{\alpha r}{c_d} - \rho \geq 2\,c_K\,c_d\,r = \varepsilon$$
$$d(s',s'') \leq 2\,c_d\,r = c_K^{-1}\varepsilon$$

Therefore, applying Lemma 2.8.4 and using the 2-Dini condition, we obtain

$$\|T_K(\mathbf{1}_{S\setminus \alpha B} f)(s') - T_K(\mathbf{1}_{S\setminus \alpha B} f)(s'')\|_{\gamma(S;Y)}$$

$$
\begin{aligned}
&\leq \tau_{2,Y}\Big(\int_{S\setminus \alpha B}\big\|\big(K(s',t)-K(s'',t)\big)f(t)\big\|_Y^2\,\mathrm{d}t\Big)^{1/2}\\
&\leq \tau_{2,Y}\Big(\int_{d(s',t)>\varepsilon}\omega\Big(\frac{\varepsilon/2}{d(s',t)}\Big)^2\frac{1}{\mu(B(s',d(s',t)))}\|f(t)\|_X^2\,\mathrm{d}t\Big)^{1/2}\\
&\leq \tau_{2,Y}\Big(\sum_{j=0}^{\infty}\omega(2^{-j-1})^2\frac{1}{\mu(B(s',2^j\varepsilon))}\int_{2^j\varepsilon<d(s',t)\leq\varepsilon 2^{j+1}}\|f(t)\|_X^2\,\mathrm{d}t\Big)^{1/2}\\
&\lesssim_S \tau_{2,Y}\Big(\sum_{j=0}^{\infty}\omega(2^{-j-1})^2\fint_{B(s',2^{j+1}\varepsilon)}\|f(t)\|_X^2\,\mathrm{d}t\Big)^{1/2}\\
&\leq \tau_{2,Y}\,\|K\|_{2\text{-Dini}}M_2\big(\|f\|_X\big)(s).
\end{aligned}
$$

where the last step follows from $s\in B(s',2^{j+1}\varepsilon)$ for all $j\in\mathbb{N}$ and a similar computation as in (3.4.1). Now, taking the essential supremum over $s',s''\in B$ and the supremum over all balls $B\ni s$, we see that

$$
\mathcal{M}^{\#}_{T_K,\alpha}f(s)\lesssim_S \tau_{2,Y}\,\|K\|_{2\text{-Dini}}M_2\big(\|f\|_X\big)(s),\qquad s\in S.
$$

The weak L^2-boundedness follows from the corresponding bound for M_2 in Proposition 2.2.1 and the density of $L^p(S;X)\cap L^2(S;X)$ in $L^2(S;X)$. □

With only the 2-sublinearity of T_K left to check, we will now prove sparse domination, and thus also weighted boundedness, for the γ-integral operators

Theorem 4.4.11 (Sparse domination for γ-integral operators)**.** *Let X and Y be Banach spaces with type 2 and let (S,d,μ) be a space of homogeneous type. Let $p\in[2,\infty)$ and suppose $K\in\mathcal{K}_\gamma(L^{p,\infty}(S))$ satisfies the 2-Dini condition. Then there is an $\eta\in(0,1)$ such that for every compactly supported $f\in L^2(S;X)$ there exists an η-sparse collection of cubes $\mathcal{S}$ such that*

$$
\|Tf(s)\|_{\gamma(S;Y)}\lesssim_{X,Y,S,p} C_K\Big(\sum_{Q\in\mathcal{S}}\langle\|f\|_X\rangle_{2,Q}^2\mathbf{1}_Q(s)\Big)^{1/2},\qquad s\in S
$$

with $C_K:=\|K\|_{\mathcal{K}_\gamma(L^{p,\infty}(S))}+\|K\|_{2\text{-Dini}}$. In particular, $K\in\mathcal{K}_\gamma(L^q(S,w))$ for all $q\in(2,\infty)$ and $w\in A_{q/2}$ with

$$
\|K\|_{\mathcal{K}_\gamma(L^q(S,w))}\lesssim_{X,Y,S,p,q} C_K\,[w]_{A_{q/2}}^{\max\left\{\frac{1}{2},\frac{1}{q-2}\right\}}.
$$

Proof. Since K is an 2-Dini kernel, it is also a 2-Hörmander kernel by Lemma 4.3.2 with

$$
\|K\|_{2\text{-Hörm}}\lesssim_S\|K\|_{2\text{-Dini}}.
$$

Therefore by Theorem 4.4.2 we know that T is bounded from $L^2(S;X)$ to $L^{2,\infty}(S;\gamma(S;Y))$ with norm

$$
\|T\|_{L^2\to L^{2,\infty}}\lesssim_{S,p}\Big(\tau_{2,X}\tau_{2,Y}\|K\|_{\mathcal{K}_\gamma(L^{p,\infty}(S))}+\tau_{2,Y}\|K\|_{2\text{-Dini}}\Big).
$$

By Lemma 4.4.10 we also know that $\mathcal{M}^{\#}_{T,\alpha}$ is bounded from $L^2(S;X)$ to $L^{2,\infty}(S)$ with norm

$$
\|\mathcal{M}^{\#}_{T,\alpha}\|_{L^2\to L^{2,\infty}}\lesssim_S,\tau_{2,Y}\|K\|_{\omega\text{-Dini}2}
$$

for $\alpha > 0$ large enough. Moreover for $f_1, \dots, f_n \in L^2(S;X)$ with disjoint support we have for a.e. $s \in S$ that $T_K f_1(s), \dots, T_K f_n(s)$ have disjoint support as well and thus the 2-sublinearity with constant $\tau_{2,Y}$ follows from Lemma 2.8.5. The sparse domination therefore follows by applying Theorem 3.1.1 to T_K. The weighted bounds follow directly from Proposition 3.2.4 and the density of boundedly supported L^2-functions in $L^q(S,w;X)$ for all $q \in [2,\infty)$. □

Remark 4.4.12.

(i) If we omit the type 2 assumption for X in Theorem 4.4.11 we can still conclude that T_K is sparsely dominated by larger sparse operator

$$f \mapsto \Big(\sum_{Q \in \mathcal{S}} \langle \|f\|_X \rangle_{p,Q}^2 \mathbf{1}_Q \Big)^{1/2}$$

In the proof one then has to skip the step where Theorem 4.4.2 is applied. This is in particular useful when $p = 2$.

(ii) $A_{p/2}$ is the largest class of weights one can expect in Theorem 4.4.11, since in the case that $X = Y = \mathbb{K}$, $S = \mathbb{R}^d$ and $K(s,t) = k(s-t)$, Theorem 4.4.11 can be reduced to a statement about deterministic convolution operators with positive kernel (see Subsection 4.2.3). It is standard to check that the weighted boundedness of for example

$$Tf(s) := \int_S \lambda^d \mathrm{e}^{-\lambda|s-t|} f(t)\,\mathrm{d}t, \qquad s \in S,$$

for all $\lambda \in \mathbb{R}_+$ implies the A_p-condition, see e.g. [Gra14a, Section 7.1.1]. Also the dependence on the weight characteristic is sharp, see Proposition 4.4.14 below

Under a Dini type condition we obtain the following further characterization if Y is a Hilbert space. The proof is immediate from Corollary 4.4.9, Theorem 4.4.11 and Remark 4.4.12(i).

Corollary 4.4.13. *Let X be a Banach space and Y be a Hilbert space. Suppose $k : \mathbb{R}^d \to \mathcal{L}(X,Y)$ is strongly measurable and satisfies the 2-Dini condition. Let $K(s,t) := k(s-t)$. Then statements (i)–(iii) in Corollary 4.4.9 are equivalent to*

(iv) $K \in \mathcal{K}_\gamma(L^p(\mathbb{R}^d, w))$ for all $p \in (2,\infty)$ and all $w \in A_{p/2}$.

In particular we have for all $p \in (2,\infty)$, $w \in A_{p/2}$ and A_0 as in (i) of Corollary 4.4.9:

$$\|K\|_{\mathcal{K}_\gamma(L^p(\mathbb{R}^d))} \le C_{p,d}\,(A_0 + \|K\|_{\omega\text{-Dini}2})\,[w]_{A_{q/2}}^{\max\{\frac{1}{2}, \frac{1}{q-2}\}}.$$

We will show next that the dependence on the weight characteristic $[w]_{A_{p/2}}$ in the bounds for T_K in Theorem 4.4.11 is actually optimal. Therefore Theorem 4.4.11 can be thought of as a γ-analog of the A_2-theorem in the deterministic setting.

Proposition 4.4.14. *Let X and Y be Banach spaces, $p \in (2,\infty)$ and $\beta \geq 0$. There exists a kernel*

$$K\colon \mathbb{R}^d \times \mathbb{R}^d \setminus \{(s,s) : s \in \mathbb{R}^d\} \to \mathcal{L}(X,Y)$$

satisfying the assumptions of Theorem 4.4.11 such that if for all $w \in A_{p/2}$ we have

$$\|K\|_{\mathcal{K}_\gamma(L^p(\mathbb{R}^d,w))} \lesssim [w]^\beta_{A_{p/2}},$$

then $\beta \geq \max\{\frac{1}{2}, \frac{1}{q-2}\}$.

Proof. By considering one dimensional subspaces, we may assume without loss of generality that $X = Y = \mathbb{K}$. Define

$$K\big((s_1,\bar{s}),(t_1,\bar{t})\big) := \frac{(|s_1|+|t_1|)^{1/2}}{\big|(|s_1|,\bar{s})+(|t_1|,\bar{t})\big|^{(d+1)/2}}, \qquad (s_1,\bar{s}),(t_1,\bar{t}) \in \mathbb{R}\times\mathbb{R}^{d-1}.$$

Then by Lemma 4.3.3 we know that K is a 2-standard kernel.

Set $\mathbb{R}^d_+ := \{(s_1,\bar{s}) \in \mathbb{R}\times\mathbb{R}^{d-1} : s_1 \geq 0\}$, $\mathbb{R}^d_- := \mathbb{R}^d \setminus \mathbb{R}^d_+$ and define for $q \in (1,\infty)$ and $f \in L^q(\mathbb{R}^d_+)$

$$T_1 f(s) := \frac{\Gamma((d+1)/2)}{\pi^{(d+1)/2}} \int_{\mathbb{R}^d_+} \frac{s_1+t_1}{|s+t|^{d+1}} f(t)\,\mathrm{d}t, \qquad s \in \mathbb{R}^d_+.$$

Then T_1 is a bounded operator on $L^q(\mathbb{R}^d_+)$ for all $q \in (1,\infty)$ with

$$\|T_1\|_{L^q(\mathbb{R}^d_+)\to L^q(\mathbb{R}^d_+)} = \frac{1}{\sin(\pi/q)}$$

by [Osę17, Theorem 1]. For $g \in L^p(\mathbb{R}^d)$ we have

$$\begin{aligned}\|T_K g\|^p_{L^p(\mathbb{R}^d)} &= \|T_K(g\mathbf{1}_{\mathbb{R}^d_+} + g\mathbf{1}_{\mathbb{R}^d_-})\|^p_{L^p(\mathbb{R}^d_+)} + \|T_K(g\mathbf{1}_{\mathbb{R}^d_+} + g\mathbf{1}_{\mathbb{R}^d_-})\|^p_{L^p(\mathbb{R}^d_-)}\\ &\simeq_d \|T_1 h\|^{p/2}_{L^{p/2}(\mathbb{R}^d_+)}\end{aligned}$$

where $h(s) := |g(s)+g(-s)|^2$ for $s \in \mathbb{R}^d_+$. Therefore

$$\|K\|_{\mathcal{K}_\gamma(L^p(\mathbb{R}^d))} \simeq_d \|T_1\|^{1/2}_{L^{p/2}(\mathbb{R}^d_+)\to L^{p/2}(\mathbb{R}^d_+)} = \frac{1}{\sin(2\pi/p)^{1/2}},$$

so K satisfies the assumptions of Theorem 4.4.11. Moreover

$$\alpha_K := \sup\Big\{\alpha \geq 0 : \forall \varepsilon > 0, \limsup_{p\to 2} \frac{\|K\|_{\mathcal{K}_\gamma(L^p(\mathbb{R}^d))}}{(p-2)^{-\alpha+\varepsilon}} = \infty\Big\} = \frac{1}{2}$$

$$\gamma_K := \sup\Big\{\gamma \geq 0 : \forall \varepsilon > 0, \limsup_{p\to\infty} \frac{\|K\|_{\mathcal{K}_\gamma(L^p(\mathbb{R}^d))}}{p^{-\gamma+\varepsilon}} = \infty\Big\} = \frac{1}{2}.$$

Thus by [FN19, Theorem 5.2] it follows that if

$$\|K\|_{\mathcal{K}_\gamma(L^p(\mathbb{R}^d,w))} \lesssim [w]^\beta_{A_{p/2}},$$

then

$$\beta \geq \max\{\alpha_K \tfrac{2}{q-2}, \gamma_K\} = \max\{\tfrac{1}{q-2}, \tfrac{1}{2}\}.$$

□

4.5. γ-FOURIER MULTIPLIER OPERATORS

If $S = \mathbb{R}^d$ and K is of convolution type, i.e. $K(s,t) = k(s-t)$ for some $k\colon \mathbb{R}^d \setminus \{0\} \to \mathcal{L}(X,Y)$, a sufficient condition for the 2-Hörmander, 2-Dini and 2-standard kernel assumptions can also be formulated in terms of smoothness and decay of the Fourier transform of k. For the 1-Hörmander, 1-Dini and 1-standard kernels assumptions this is classical, see e.g. [HNVW2x, Gra14a, Ste93] and Section 3.5. The r-Hörmander kernel assumptions for $r \in [1,\infty]$ have been treated by similar methods in e.g. [RV17, Section 5.1]. In this section we will check the 2-Dini kernel assumption for k in terms of smoothness and decay of the Fourier transform of k, for which we will adapt the approach for 1-Dini kernels and singular integral operators in [HNVW2x] to the 2-Dini kernel and γ-integral operator setting. Using Theorem 4.4.11 this leads to an extrapolation theorem for γ-*Fourier multiplier operators*, which we will now introduce:

Definition 4.5.1. Let X, Y be a Banach spaces. For $m \in L^{2,\infty}(\mathbb{R}^d; \mathcal{L}(X,Y))$ we define the γ-Fourier multiplier operator T_m for $f \in \mathcal{S}(\mathbb{R}^d; X)$ and $\varphi \in \mathcal{S}(\mathbb{R}^d)$ by

$$T_m f(s)\varphi = \mathcal{F}^{-1}\big(m \cdot (\widehat{f} * \widehat{\varphi})\big)(s), \qquad s \in \mathbb{R}^d.$$

Let $p \in [1,\infty)$ and let w be a weight. We let $\mathcal{M}_\gamma(L^p(\mathbb{R}^d, w))$ (respectively $\mathcal{M}_\gamma(L^{p,\infty}(\mathbb{R}^d, w))$) be the space of all $m \in L^{2,\infty}(\mathbb{R}^d; \mathcal{L}(X,Y))$ such that T_m extends to a bounded operator from $L^p(\mathbb{R}^d, w; X)$ to $L^p(\mathbb{R}^d, w; \gamma(\mathbb{R}^d; Y))$ (respectively $L^{p,\infty}(\mathbb{R}^d, w; \gamma(\mathbb{R}^d; Y))$). We norm these spaces by

$$\|m\|_{\mathcal{M}_\gamma(L^p(\mathbb{R}^d,w))} := \|T_m\|_{L^p(\mathbb{R}^d,w;X)\to L^p(\mathbb{R}^d,w;\gamma(\mathbb{R}^d;Y))},$$
$$\|m\|_{\mathcal{M}_\gamma(L^{p,\infty}(\mathbb{R}^d,w))} := \|T_m\|_{L^p(\mathbb{R}^d,w;X)\to L^{p,\infty}(\mathbb{R}^d,w;\gamma(\mathbb{R}^d;Y))}.$$

If $w \equiv 1$ we omit it.

Remark 4.5.2.

- Note that the inverse Fourier transform of $m \cdot (\widehat{f} * \widehat{\varphi})$ in Definition 4.5.1 is a well-defined function, as we can estimate

$$\|m \cdot \widehat{f} * \widehat{\varphi}\|_{L^1(\mathbb{R}^d;Y)} \le \|m\|_{L^{2,\infty}(\mathbb{R}^d;\mathcal{L}(X,Y))} \big\|\widehat{f} * \widehat{\varphi}\big\|_{L^{2,1}(\mathbb{R}^d;X)} < \infty \tag{4.5.1}$$

  where we used Hölder's inequality for Lorentz spaces in the first step and the inclusion $\widehat{f} * \widehat{\varphi} \in \mathcal{S}(\mathbb{R}^d; X) \hookrightarrow L^{2,1}(\mathbb{R}^d; X)$ in the second step.

- An $m \in \mathcal{M}_\gamma(L^{p,\infty}(\mathbb{R}^d, w))$ will typically not be a bounded function, but rather satisfy an estimate of the form

$$\|m(\xi)\| \le A_0 |\xi|^{-d/2}, \qquad \xi \in \mathbb{R}^d \setminus \{0\}$$

  for some $A_0 > 0$. This implies in particular that $m \in L^{2,\infty}(\mathbb{R}^d; \mathcal{L}(X,Y))$, which we included in our definition of $\mathcal{M}_\gamma(L^{p,\infty}(\mathbb{R}^d, w))$ to ensure that T_m is well-defined on Schwartz functions.

For compactly supported m we can easily connect γ-integral operators and γ-Fourier multiplier operators, as we will show in the following proposition.

Proposition 4.5.3. *Let X and Y be Banach spaces and let $m \in L^{2,\infty}(\mathbb{R}^d;\mathcal{L}(X,Y))$ be compactly supported. Then for all $f \in \mathcal{S}(\mathbb{R}^d;X)$ and $\varphi \in \mathcal{S}(\mathbb{R}^d)$ we have*

$$T_m f(s)\varphi = \int_{\mathbb{R}^d} k(s-t)f(t)\varphi(t), \qquad s \in \mathbb{R}^d,$$

where $k = \check{m}$.

Proof. Since m is compactly supported, we have $m \in L^1(\mathbb{R}^d;\mathcal{L}(X,Y))$, so $\check{m}$ is well-defined. Fix $f \in \mathcal{S}(\mathbb{R}^d;X)$. Using Fubini's theorem, we can directly compute that for any $\varphi \in \mathcal{S}(\mathbb{R}^d)$ and a.e. $s \in \mathbb{R}^d$

$$\begin{aligned} T_m f(s)\varphi &= \int_{\mathbb{R}^d} m(\xi)\big(\widehat{f} * \widehat{\varphi}\big)(\xi)\mathrm{e}^{2\pi i s\cdot\xi}\,\mathrm{d}\xi \\ &= \int_{\mathbb{R}^d} m(\xi)\Big(\int_{\mathbb{R}^d} f(t)\varphi(t)\mathrm{e}^{-2\pi i t\cdot\xi}\,\mathrm{d}t\Big)\mathrm{e}^{2\pi i s\cdot\xi}\,\mathrm{d}\xi \\ &= \int_{\mathbb{R}^d}\Big(\int_{\mathbb{R}^d} m(\xi)\mathrm{e}^{2\pi i(s-t)\cdot\xi}\,\mathrm{d}\xi\Big) f(t)\varphi(t)\,\mathrm{d}t \\ &= \int_{\mathbb{R}^d} \check{m}(s-t)f(t)\varphi(t)\,\mathrm{d}t. \end{aligned}$$

□

Using smooth Littlewood–Paley functions we will reduce considerations to the compactly supported case. For this fix a Schwartz function $\phi \in \mathcal{S}(\mathbb{R}^d)$ such that $\mathbf{1}_{B(0,1)} \le \widehat{\phi} \le \mathbf{1}_{B(0,2)}$ and set

$$\widehat{\psi}(\xi) := \widehat{\phi}(\xi) - \widehat{\phi}(2\xi), \qquad \xi \in \mathbb{R}^d.$$

Then $\widehat{\psi} \in \mathcal{S}(\mathbb{R}^d)$ is nonnegative and evidently

(i) $\operatorname{supp}\widehat{\psi} \subseteq \big\{\xi \in \mathbb{R}^d : \frac{1}{2} \le |\xi| \le 2\big\}$,

(ii) $\sum_{j\in\mathbb{Z}} \widehat{\psi}(2^{-j}\xi) = 1$ for all $\xi \in \mathbb{R}^d \setminus \{0\}$.

For any $m \in L^{2,\infty}(\mathbb{R}^d;\mathcal{L}(X,Y))$, $j \in \mathbb{Z}$ and $N \in \mathbb{N}$ we define

$$m_j(\xi) := \widehat{\psi}(2^{-j}\xi)m(\xi), \qquad \xi \in \mathbb{R}^d, \tag{4.5.2}$$

$$m^N(\xi) := \sum_{-N<j\le N} m_j(\xi) = \big(\widehat{\phi}(2^{-N}\xi) - \widehat{\phi}(2^N\xi)\big)m(\xi), \qquad \xi \in \mathbb{R}^d. \tag{4.5.3}$$

Then both the m_j's and the m_N's are compactly supported away from the origin, which in particular implies that $m_j, m^N \in L^1(\mathbb{R}^d;\mathcal{L}(X,Y))$.

We will now show that the L^p-boundedness of a Fourier γ-multiplier operator T_m is equivalent to the uniform boundedness of the truncations T_{m^N}.

Proposition 4.5.4. *Let X and Y be Banach spaces, assume that Y has finite cotype, let $p \in [2,\infty)$ and let $w \in A_p$. Then the following hold:*

(i) If $m \in \mathcal{M}_\gamma(L^p(\mathbb{R}^d))$, then $m^N \in \mathcal{M}_\gamma(L^p(\mathbb{R}^d))$ for all $N \in \mathbb{N}$ with

$$\|m^N\|_{\mathcal{M}_\gamma(L^p(\mathbb{R}^d))} \leq \|\phi\|_{L^1(\mathbb{R}^d)} \|m\|_{\mathcal{M}_\gamma(L^p(\mathbb{R}^d))}.$$

(ii) If $m^N \in \mathcal{M}_\gamma(L^p(\mathbb{R}^d, w))$ uniformly for $N \in \mathbb{N}$, then $m \in \mathcal{M}_\gamma(L^p(\mathbb{R}^d, w))$ with

$$\|m\|_{\mathcal{M}_\gamma(L^p(\mathbb{R}^d, w))} \leq \|m_N\|_{\mathcal{M}_\gamma(L^p(\mathbb{R}^d, w))}.$$

Moreover these statements are valid with L^p replaced by $L^{p,\infty}$.

Proof. We will only prove the lemma for L^p, the proof for $L^{p,\infty}$ is similar. For (i) we note that for $f \in \mathcal{S}(\mathbb{R}^d; X)$ and all $\varphi \in \mathcal{S}(\mathbb{R}^d)$ we have

$$\begin{aligned} T_{m^N} f(s)\varphi &= \mathcal{F}^{-1}\big(\big(\widehat{\phi}(2^{-N}\cdot) - \widehat{\phi}(2^N\cdot)\big) m (\widehat{f} * \widehat{\varphi})\big)(s) \\ &= \int_{\mathbb{R}^d} \big(\phi_{2^{-N}}(s-t) - \phi_{2^N}(s-t)\big) T_m f(t)\varphi \, \mathrm{d}t \\ &= \big(\phi_{2^{-N}} * T_m f - \phi_{2^N} * T_m f\big)(s)\varphi, \end{aligned}$$

where $\phi_\lambda(s) := \lambda^{-d}\phi(\lambda^{-1}s)$. By Young's inequality we have

$$\big\|\phi_\lambda * T_m f\big\|_{L^p(\mathbb{R}^d;\gamma(\mathbb{R}^d;Y))} \leq \|\phi\|_{L^1(\mathbb{R}^d)} \big\|T_m f\big\|_{L^p(\mathbb{R}^d;\gamma(\mathbb{R}^d;Y))},$$

so by density we deduce that $m^N \in \mathcal{M}_\gamma(L^p(\mathbb{R}^d))$ with the claimed estimate.

For (ii) we know by the properties of our smooth Littlewood-Paley functions that $m^N(\xi) \to m(\xi)$ for all $\xi \in \mathbb{R}^d \setminus \{0\}$. Using (4.5.1) we can apply the Dominated convergence theorem to obtain for $f \in \mathcal{S}(\mathbb{R}^d; X)$, $\varphi \in \mathcal{S}(\mathbb{R}^d)$ and a.e. $s \in \mathbb{R}^d$ that

$$\begin{aligned} \lim_{N\to\infty} T_{m^N} f(s)\varphi &= \lim_{N\to\infty} \int_{\mathbb{R}^d} m^N(\xi) \cdot (\widehat{f} * \widehat{\varphi})(\xi) \, \mathrm{e}^{2\pi i s\cdot\xi} \, \mathrm{d}\xi \\ &= \int_{\mathbb{R}^d} m(\xi) \cdot (\widehat{f} * \widehat{\varphi})(\xi) \, \mathrm{e}^{2\pi i s\cdot\xi} \, \mathrm{d}\xi = T_m f(s)\varphi. \end{aligned}$$

Let $(\varphi_k)_{k=1}^\infty \subseteq \mathcal{S}(\mathbb{R}^d)$ be an orthonormal basis of $L^2(\mathbb{R}^d)$ and $(\gamma_k)_{k=1}^\infty$ a Gaussian sequence. Then, using that $\gamma_\infty(\mathbb{R}^d; Y) = \gamma(\mathbb{R}^d; Y)$ as Y has finite cotype, we have by Fatou's lemma for a.e. $s \in \mathbb{R}^d$

$$\begin{aligned} \|T_m f(s)\|_{\gamma(\mathbb{R}^d;Y)} &= \sup_{n\in\mathbb{N}} \Big\| \sum_{k=1}^n \gamma_k T_m f(s)\varphi_k \Big\|_{L^2(\Omega;Y)} \\ &\leq \sup_{n\in\mathbb{N}} \liminf_{N\to\infty} \Big\| \sum_{k=1}^n \gamma_k T_{m^N} f(s)\varphi_k \Big\|_{L^2(\Omega;Y)} \\ &\leq \liminf_{N\to\infty} \big\| T_{m^N} f(s) \big\|_{\gamma(\mathbb{R}^d;Y)}. \end{aligned}$$

Using Fatou's lemma once more, we see that

$$\|T_m f\|_{L^p(\mathbb{R}^d,w;\gamma(\mathbb{R}^d;Y))} \leq \liminf_{N\to\infty} \|T_{m^N} f\|_{L^p(\mathbb{R}^d,w;\gamma(\mathbb{R}^d;Y))}.$$

As $\mathcal{S}(\mathbb{R}^d; X)$ is a dense subspace of $L^p(\mathbb{R}^d, w; X)$, the proposition follows. □

In view of Proposition 4.5.4 we can focus on the truncations m_j and m^N as defined in (4.5.2) and (4.5.3). We start with a lemma that transfers the decay of m to decay of the Fourier inverse of the truncated multipliers m_j.

Lemma 4.5.5. *Let X and Y be Banach spaces and let $m\colon \mathbb{R}^d \setminus \{0\} \to \mathcal{L}(X,Y)$ be strongly measurable such that*

$$\|\partial^\alpha m(\xi)\| \le A_0\,|\xi|^{-d/2-|\alpha|}, \qquad \xi \in \mathbb{R}^d \setminus \{0\}, \quad |\alpha| \le \lfloor d/2 \rfloor + 1$$

for some $A_0 > 0$. Then for any $k_j := \check{m}_j$, where m_j is defined as in (4.5.2), we have for $0 \le n \le \lfloor d/2 \rfloor + 1$

$$|s|^n \|k_j(s)\| \lesssim_d A_0\,(2^j)^{d/2-n}, \qquad s \in \mathbb{R}^d, \tag{4.5.4}$$

$$|s|^n \|k_j(s-t) - k_j(s)\| \lesssim_d A_0\,(2^j)^{d/2-n} \min\{2^j|t|, 1\}, \qquad |t| \le \frac{1}{2}|s|. \tag{4.5.5}$$

Proof. Fix $j \in \mathbb{N}$ and $|\alpha| \le \lfloor d/2 \rfloor + 1$, then we have for all $\xi \in \mathbb{R}^d \setminus \{0\}$ that

$$\partial^\alpha m_j(\xi) = \partial^\alpha\big(\widehat{\psi}(2^{-j}\xi)m(\xi)\big) = \sum_{\beta \le \alpha} \binom{\alpha}{\beta} 2^{-j|\beta|} \partial^\beta \widehat{\psi}(2^{-j}\xi)\partial^{\alpha-\beta} m(\xi).$$

Therefore it follows for all $\xi \in \mathbb{R}^d \setminus \{0\}$

$$\begin{aligned} \big\|\partial^\alpha m_j(\xi)\big\| &\le A_0 \sum_{\beta \le \alpha} \binom{\alpha}{\beta} 2^{-j|\beta|}\, \mathbf{1}_{2^{j-1} \le |\xi| \le 2^{j+1}} |\xi|^{-d/2-|\alpha-\beta|} \\ &\le A_0 \sum_{\beta \le \alpha} \binom{\alpha}{\beta} 2^{-j|\beta|} \big(2^{j-1}\big)^{-d/2-|\alpha|+|\beta|} \\ &\lesssim_d A_0\,(2^j)^{-d/2-|\alpha|}. \end{aligned} \tag{4.5.6}$$

Now take $0 \le n \le \lfloor d/2 \rfloor + 1$ and define $\alpha := ne_l$ for some $1 \le l \le d$. Then, using (4.5.6) and the fact that m_j is supported in the ball $B(0, 2^{j+1})$, we have

$$\begin{aligned} \sup_{s \in \mathbb{R}^d} \|s^\alpha k_j(s)\| &\lesssim_d \|\partial^\alpha m_j\|_{L^1(\mathbb{R}^d;\mathcal{L}(X,Y))} \\ &\lesssim_d \|\partial^\alpha m_j\|_{L^\infty(\mathbb{R}^d;\mathcal{L}(X,Y))} \|\mathbf{1}_{B(0,2^{j+1})}\|_{L^1(\mathbb{R}^d)} \\ &\lesssim_d A_0\,(2^j)^{-d/2-n}\,(2^{j+1})^d. \end{aligned} \tag{4.5.7}$$

So $|s_l|^n \|k_j(s)\| \lesssim_d A_0\,(2^j)^{d/2-n}$ for all $s \in \mathbb{R}^d$ and $1 \le l \le d$, from which (4.5.4) follows readily.

Now fix $t \in \mathbb{R}^d$ and note that $k_j(\cdot - t) - k_j(\cdot)$ is the Fourier transform of the function $\xi \mapsto (\mathrm{e}^{2\pi i t\cdot\xi} - 1)m_j(\xi)$. Suppose that $|t| \le 2^{-j}$, then since

$$\partial^\alpha\big((\mathrm{e}^{2\pi i\xi\cdot t} - 1)m_j(\xi)\big) = (\mathrm{e}^{2\pi i\xi\cdot t} - 1)\partial^\alpha m_j(\xi) + \sum_{0 \ne \beta \le \alpha} \binom{\alpha}{\beta} (2\pi i)^{|\beta|} t^\beta \mathrm{e}^{2\pi i\xi\cdot t} \partial^{\alpha-\beta} m_j(\xi),$$

we have that

$$\sup_{\xi\in\mathbb{R}^d}\big\|\partial^\alpha\big((\mathrm{e}^{2\pi i\xi\cdot t}-1)m_j(\xi)\big)\big\| \lesssim_d A_0\Big(2^j|t|(2^j)^{-d/2-|\alpha|}+\sum_{0\neq\beta\leq\alpha}\binom{\alpha}{\beta}|t|^\beta(2^j)^{-d/2-|\alpha-\beta|}\Big)$$
$$\lesssim_d A_0\,2^j|t|\cdot(2^j)^{-d/2-|\alpha|}.$$

So reasoning analogously as in (4.5.7), we obtain that

$$|s|^n\|k_j(s-t)-k_j(s)\|\lesssim_d A_0\,2^j|t|\cdot(2^j)^{d/2-n}.$$

Now if $|t|>2^{-j}$ we simply use the triangle inequality and (4.5.4) to deduce for all $|s|\geq 2|t|$ that

$$\begin{aligned}\|k_j(s-t)-k_j(s)\|&\leq\|k_j(s-t)\|+\|k_j(s)\|\\&\lesssim_d A_0\,(2^j)^{d/2-n}\big(|s-t|^{-n}+|s|^{-n}\big)\\&\lesssim_d A_0\,(2^j)^{d/2-n}|s|^{-n}.\end{aligned}$$

Combining the estimates for $|t|\leq 2^{-j}$ and $|t|>2^{-j}$ yields (4.5.5). □

Using Lemma 4.5.5 we can use estimates on the derivatives of m to check the 2-standard kernel estimates for the kernels associated to m^N independent of $N\in\mathbb{N}$, as we announced at the start of this section.

Proposition 4.5.6. *Let X and Y be Banach spaces and let $m\colon\mathbb{R}^d\setminus\{0\}\to\mathcal{L}(X,Y)$ be strongly measurable such that*

$$\|\partial^\alpha m(\xi)\|\leq A_0\,|\xi|^{-d/2-|\alpha|},\qquad\qquad \xi\in\mathbb{R}^d\setminus\{0\},\quad |\alpha|\leq\lfloor d/2\rfloor+1$$

for some $A_0>0$. Then for any $N\in\mathbb{N}$ the kernel $K^N(s,t):=\check{m}^N(s-t)$, where m^N is defined as in (4.5.3), is a 2-standard kernel with

$$\|K^N\|_{2\text{-std}}\lesssim_d A_0.$$

Proof. Fix $\epsilon\in(0,\frac{1}{2})$ and define $\ell=\lfloor d/2\rfloor+1$. For $j\in\mathbb{N}$ set $k_j:=\check{m}_j$, where m_j is defined as in (4.5.2). Since the k_j's satisfy (4.5.5) by Lemma 4.5.5 and using $\ell-d/2\in\{\frac{1}{2},1\}$, we have for all $|s|\geq 2|t|$ that

$$\begin{aligned}&\|k^N(s-t)-k^N(s)\|\\&\quad\lesssim_d A_0\Big(\sum_{2^j\leq|s|^{-1}}(2^j)^{d/2+1}|t|+\sum_{|s|^{-1}\leq 2^j\leq|t|^{-1}}\frac{(2^j)^{d/2-\ell+1}}{|s|^\ell}|t|+\sum_{2^j\geq|t|^{-1}}\frac{(2^j)^{d/2-\ell}}{|s|^\ell}\Big)\\&\quad\lesssim_d A_0\Big(\frac{|t|}{|s|}\frac{1}{|s|^{d/2}}+\Big(\frac{|t|}{|s|}\Big)^{\ell-d/2}\frac{1}{|s|^{d/2}}\log_2\Big(\frac{|s|}{|t|}\Big)+\Big(\frac{|t|}{|s|}\Big)^{\ell-d/2}\frac{1}{|s|^{d/2}}\Big)\\&\quad\lesssim_{\epsilon,d} A_0\Big(\frac{|t|}{|s|}\Big)^\epsilon\frac{1}{|s|^{d/2}},\end{aligned}$$

from which (4.3.5) and (4.3.6) for K follow by a change of variables. □

Remark 4.5.7. If in Proposition 4.5.6 either one of the following assumptions hold:

- d is even
- d is odd and we have smoothness of m up to order $\lfloor d/2 \rfloor + 2$

a slightly simpler argument than the one presented could be employed. On the other hand, if d is even one can deduce a strong operator topology version of the 2-Hörmander condition for K^N using only $\lfloor d/2 \rfloor$ derivatives of m and *nontrivial Fourier type* of Y. We refer to e.g. [FHL20, Hyt04, HNVW2x] for such results for classical Fourier multiplier operators and [RV17, Section 5.2] for the r-Hörmander condition of K^N for $r \in [1,\infty]$.

With the kernel estimates of Proposition 4.5.6, the approximation result of Proposition 4.5.4 and the connection between γ-Fourier multiplier operators and γ-integral operators of Proposition 4.5.3 we can now use the sparse domination result in the previous section to deduce a weighted extrapolation theorem for Fourier γ-multiplier operators.

Theorem 4.5.8. *Let X and Y be Banach spaces with type 2 and let $p \in [2,\infty)$. Let $m \in \mathcal{M}_\gamma(L^{p,\infty}(\mathbb{R}^d))$ be such that*

$$\|\partial^\alpha m(\xi)\| \leq A_0 |\xi|^{-d/2-|\alpha|}, \qquad \xi \in \mathbb{R}^d \setminus \{0\}, \quad |\alpha| \leq \lfloor d/2 \rfloor + 1$$

for some $A_0 > 0$. Then $m \in \mathcal{M}_\gamma(L^q(\mathbb{R}^d, w))$ for all $q \in (2,\infty)$ and $w \in A_{q/2}$ with

$$\|m\|_{\mathcal{M}_\gamma(L^q(\mathbb{R}^d,w))} \lesssim_{X,Y,p,q,d} [w]_{A_{q/2}}^{\max\left\{\frac{1}{2}, \frac{1}{q-2}\right\}} \left(\|m\|_{\mathcal{M}_\gamma(L^{p,\infty}(\mathbb{R}^d))} + A_0\right)$$

Proof. Let m^N be defined as in (4.5.3). Then by Proposition 4.5.4 we know that $m^N \in \mathcal{M}_\gamma(L^{p,\infty}(\mathbb{R}^d))$ for all $N \in \mathbb{N}$ with

$$\|m^N\|_{\mathcal{M}_\gamma(L^{p,\infty}(\mathbb{R}^d))} \lesssim \|m\|_{\mathcal{M}_\gamma(L^{p,\infty}(\mathbb{R}^d))}.$$

Moreover by Proposition 4.5.3 we know that T_{m^N} is a γ-integral operator with kernel $K^N(s,t) := \check{m}^N(s-t)$, which is a 2-standard kernel by Proposition 4.5.6. Therefore, by Theorem 4.4.11, it follows that $m^N \in \mathcal{M}_\gamma(L^q(\mathbb{R}^d, w))$ for all $q \in (2,\infty)$ and $w \in A_{q/2}$ uniformly for $N \in \mathbb{N}$. So we also have that $m \in \mathcal{M}_\gamma(L^q(\mathbb{R}^d, w))$ for all $q \in (2,\infty)$ and $w \in A_{q/2}$ by Proposition 4.5.4. The norm estimate follows from the norm estimate in Theorem 4.4.11 combined with the estimate $\|K^N\|_{2\text{-std}} \lesssim A_0$ from Proposition 4.5.6. □

Remark 4.5.9. In Theorem 4.5.8 we assume a priori that T_m is weak L^p-bounded. For (classical) Fourier multiplier operators one can deduce a priori L^p-boundedness using the operator-valued Mihlin multiplier theorem (see also Section 3.5). For γ-Fourier multiplier operators such a theorem is not (yet) available. It would be very interesting to be able to give sufficient conditions on a multiplier $m\colon \mathbb{R}^d \setminus \{0\} \to \mathcal{L}(X,Y)$ such that $m \in \mathcal{M}_\gamma(L^{p,\infty}(\mathbb{R}^d))$ for some $p \in [2,\infty)$.

4.6. EXTRAPOLATION FOR STOCHASTIC-DETERMINISTIC INTEGRAL OPERATORS

In this final section we will study singular stochastic integral operators in the important special case $X = Y = L^q(\mathcal{O})$ for $q \in (1,\infty)$ and a space of homogeneous type $(\mathcal{O}, d, \mu)$. In this setting a stochastic singular integral operator S_K with kernel $K\colon (0,T)\times(0,T) \to \mathcal{L}(L^q(\mathcal{O}))$ can often be represented by a kernel $k\colon (0,T)\times\mathcal{O}\times(0,T)\times\mathcal{O}\to\mathbb{C}$ in the form

$$S_K G(t)(x) = S_k G(t,x) = \int_0^T \int_{\mathcal{O}} k(t,x,s,y) G(s,y)\,\mathrm{d}y\,\mathrm{d}W_H(s), \qquad (t,x)\in\mathbb{R}_+\times\mathcal{O}$$

for adapted processes $G\colon \Omega\times(0,T)\times\mathcal{O}\to\mathbb{C}$. To study the boundedness of these operators, as in Proposition 4.2.5, we can reduce to the case $H=\mathbb{R}$. Furthermore, as in Proposition 4.2.3, we may equivalently study the boundedness of γ-integral operators, or in this setting rather L^2-integral operators, of the form

$$T_k f(t,x) := \Big(s \mapsto \int_{\mathcal{O}} k(t,x,s,y) f(s,y)\,\mathrm{d}\mu(y)\Big), \qquad (t,x)\in(0,T)\times\mathcal{O}.$$

from $L^p((0,T)\times\mathcal{O})$ to $L^p((0,T)\times\mathcal{O};L^2(\mathcal{O}))$. The main result of this section will be the weighted $L^p((0,T),v;L^q(\mathcal{O},w))$- and $L^q(\mathcal{O},w;L^p((0,T,v)))$-boundedness of T_k, assuming unweighted $L^2((0,T)\times\mathcal{O})$-boundedness and a $(2,1)$-Dini condition on k. From this result we will deduce Theorem 4.1.3 in the introduction.

In the remainder of this section we will fix a 2-product space of homogeneous type

$$(\boldsymbol{S},\boldsymbol{d},\boldsymbol{\mu}) = (S_1,d_1,\mu_1)\times(S_2,d_2,\mu_2)$$

in the sense of (4.3.1). For the applications we have in mind it suffices to take $S_1=(0,T)$ with metric $|\cdot-\cdot|^{1/m}$ for some $m\in\mathbb{N}$ and $(S_2,d_2,\mu_2)=(\mathcal{O},d,\mu)$ as e.g. a Lipschitz domain in $\mathbb{R}^d$. Our main result is a follows:

Theorem 4.6.1. *Let $K\colon \boldsymbol{S}\times\boldsymbol{S}\to\mathbb{C}$ be a $(2,1)$-Dini kernel. Suppose that*

$$T_K f(s) := \Big(t_1 \mapsto \int_{S_2} K(s,t) f(t)\,\mathrm{d}\mu_2(t_2)\Big), \qquad s\in\boldsymbol{S}.$$

is a well-defined, bounded operator from $L^2(\boldsymbol{S})$ to $L^{2,\infty}(\boldsymbol{S};L^2(S_1))$. Then for $p\in(2,\infty)$, $q\in(1,\infty)$ and $C_T := \|T_K\|_{L^2(\boldsymbol{S})\to L^{2,\infty}(\boldsymbol{S};L^2(S_1))} + \|K\|_{(2,1)\text{-Dini}}$ the following hold:

(i) For $w\in A_{p/2}(\boldsymbol{S})$ we have

$$\|T_K\|_{L^p(\boldsymbol{S},w)\to L^p(\boldsymbol{S},w;L^2(S_1))} \lesssim_{\boldsymbol{S},p} C_T\,[w]_{A_{p/2}(\boldsymbol{S})}^{\max\{\frac{1}{p-2},1\}}.$$

(ii) For $v\in A_{p/2}(S_1)$ and $w\in A_q(S_2)$ we have

$$\|T_K\|_{L^p(S_1,v;L^q(S_2,w))\to L^p(S_1,v;L^q(S_2,w;L^2(S_1)))} \le C_T\,\phi([v]_{A_{p/2}(S_1)},[w]_{A_q(S_2)}), \quad q>2,$$
$$\|T_K\|_{L^q(S_2,w;L^p(S_1,v))\to L^q(S_2,w;L^p(S_1,v;L^2(S_1)))} \le C_T\,\phi([v]_{A_{p/2}(S_1)},[w]_{A_q(S_2)}),$$

where $\phi\colon\mathbb{R}_+^2\to\mathbb{R}_+$ depends on $\boldsymbol{S}$, p, q and is nondecreasing in both variables.

We start by proving Theorem 4.6.1(i), which is a consequence of the abstract domination principle in Corollary 3.1.2.

Proof of Theorem 4.6.1(i). In order to apply Corollary 3.1.2 with $p_1 = p_2 = 2$ and $r = 1$ it suffices to show that the sharp grand maximal truncation operator $\mathcal{M}^{\#}_{T_K,\alpha}\colon L^2(\boldsymbol{S}) \to L^{2,\infty}(\boldsymbol{S})$ is bounded for sufficiently large $\alpha > 0$. Fix a boundedly supported $f \in L^2(\boldsymbol{S})$, take $s \in \boldsymbol{S}$ and $\alpha \geq 3c_{\boldsymbol{d}}^2 c_K$ with $c_{\boldsymbol{d}}$ the quasi-metric constant and c_K the constant from the definition of a $(2,1)$-standard kernel. Take a ball $B = B(z,r)$ containing s. For $s', s'' \in B$ and $t \in \boldsymbol{S} \setminus \alpha B$ we have

$$\begin{aligned}
\boldsymbol{d}(s',t) &\geq \frac{1}{c_{\boldsymbol{d}}}\boldsymbol{d}(z,t) - \boldsymbol{d}(z,s') \geq \frac{\alpha r}{c_{\boldsymbol{d}}} - r \geq 2c_{\boldsymbol{d}}c_K r := \rho,\\
\boldsymbol{d}(s',s'') &\leq 2c_{\boldsymbol{d}} r = \frac{\rho}{c_K}.
\end{aligned}$$

So defining

$$B_j = B(s', 2^{j+1}\rho) \setminus B(s', 2^j \rho),$$

we have by the $(2,1)$-Dini kernel assumption on K, a similar computation as in (3.4.1) and Hölder's inequality

$$\begin{aligned}
&\big\| T_K(f\,\mathbf{1}_{\boldsymbol{S}\setminus\alpha B})(s') - T_K(f\,\mathbf{1}_{S\setminus\alpha B})(s'')\big\|_{L^2(S_1)}\\
&\quad\leq \Big(\int_{S_1}\Big(\int_{S_2}\big|K(s',t) - K(s'',t)f(t)\big|\,\mathbf{1}_{\boldsymbol{S}\setminus\alpha B}(t)\,\mathrm{d}\mu_2(t_2)\Big)^2\,\mathrm{d}\mu_1(t_1)\Big)^{1/2}\\
&\quad\leq \sum_{j=0}^{\infty}\Big(\int_{S_1}\Big(\int_{S_2}\omega\Big(\frac{d(s',s'')}{d(s',t)}\Big)\frac{|f(t)|\,\mathbf{1}_{B_j}(t)}{V_{(2,1)}(s',t)}\,\mathrm{d}\mu_2(t_2)\Big)^2\,\mathrm{d}\mu_1(t_1)\Big)^{1/2}\\
&\quad\leq \sum_{j=0}^{\infty}\omega(2^{-j-1})\Big(\int_{S_1}\Big(\int_{S_2}\frac{|f(t)|\,\mathbf{1}_{B(s',2^{j+1}\rho)}(t)}{\mu_1\big(B(s_1',2^j\rho)\big)^{1/2}\mu_2\big(B(s_2',2^j\rho)\big)}\,\mathrm{d}\mu_2(t_2)\Big)^2\,\mathrm{d}\mu_1(t_1)\Big)^{1/2}\\
&\quad\lesssim_{\boldsymbol{S}} \|K\|_{(2,1)\text{-Dini}}\Big(\int_{B(s',2^{j+1}\rho)}\frac{1}{\boldsymbol{\mu}\big(B(s',2^j\rho)\big)}|f(t)|^2\,\mathrm{d}\boldsymbol{\mu}(t)\Big)^{1/2}\\
&\quad\lesssim_{\boldsymbol{S}} \|K\|_{(2,1)\text{-Dini}}\,M_2 f(s)
\end{aligned}$$

So taking the supremum over all $s', s'' \in B$ and all balls B containing s we find that

$$\mathcal{M}^{\#}_{T,\alpha} f(s) \lesssim_{\boldsymbol{S}} \|K\|_{(2,1)\text{-Dini}}\,M_2\big(\|f\|_X\big)(s).$$

Thus, by the weak $L^2(\boldsymbol{S})$-boundedness of M_2 (see Proposition 2.2.1) and the density of boundedly supported functions in $L^2(\boldsymbol{S})$, we deduce that $\mathcal{M}^{\#}_{T,\alpha}$ is weak L^2-bounded, which proves (i) of Theorem 4.6.1 □

If we would now employ Rubio de Francia extrapolation to deduce (ii) of Theorem 4.6.1, we would be constrained to the case $q > 2$ and $w \in A_{q/2}(S_2)$. In order to obtain the full statement of Theorem 4.6.1, we will employ Calderón–Zygmund theory once more, this time only in the S_1-variable. For this we will need the following lemma.

Lemma 4.6.2. *Let $K\colon \boldsymbol{S}\times\boldsymbol{S}\to\mathbb{C}$ be a $(2,1)$-Dini kernel. Suppose that*

$$T_K f(t) := \Big(s_1 \mapsto \int_{S_2} K(s,t) f(t)\,\mathrm{d}\mu_2(t_2)\Big), \qquad s\in\boldsymbol{S}.$$

is a well-defined bounded operator from $L^2(\boldsymbol{S})$ to $L^{2,\infty}(\boldsymbol{S};L^2(S_1))$. Then for $p\in(2,\infty)$ and $w\in A_{p/2}(S_1)$ the kernel

$$K_2\colon S_2\times S_2\setminus\{(s_2,s_2): s_2\in S_2\}\to\mathcal{L}\big(L^p(S_1,w),L^p(S_1,w;L^2(S_1))\big)$$

given by

$$K_2(s_2,t_2)f(s_1) := K((s_1,s_2),(\cdot,t_2))f(\cdot), \qquad s_1\in S_1.$$

is a 1-Dini kernel with

$$\|K_2\|_{1\text{-Dini}} \lesssim_{S_1,p} [w]_{A_{p/2}(S_1)}^{\frac{1}{p-2}} \|K\|_{(2,1)\text{-Dini}}.$$

Proof. We will first show the 1-Dini condition for K_2, which we will afterwards use to check that we have

$$K_2(s_2,t_2)\in\mathcal{L}\big(L^p(S_1,w),L^p(S_1,w;L^2(S_1))\big), \qquad s_2\neq t_2. \tag{4.6.1}$$

Fix $s_2,t_2,t_2'\in S_2$ such that $0<d_2(t_2,t_2')\le\frac{1}{c_K}d(s_2,t_2)$, set $r:=d_2(s_2,t_2)$ and define for $f\in L^p(S_1,w)$

$$g(s_1,t_1) := K((s_1,s_2),(t_1,t_2))f(t_1) - K((s_1,s_2),(t_1,t_2'))f(t_1), \qquad s_1,t_1\in S_1.$$

Then we have for any $s_1\in S_1$, using the $(2,1)$-Dini kernel assumption on K, the submultiplicativity of ω and a similar computation as in (3.4.1),

$$\begin{aligned}
&\|g(s_1,\cdot)\|_{L^2(S_1)}^2\\
&\le \int_{d_1(s_1,t_1)\le r}\omega\Big(\frac{d_2(t_2,t_2')}{r}\Big)^2\frac{|f(t_1)|^2}{\mu_1(B(s_1,r))\cdot\mu_2(B(s_2,r))^2}\,\mathrm{d}\mu_1(t_1)\\
&\qquad+\sum_{j=0}^{\infty}\int_{2^j r<d_1(s_1,t_1)\le 2^{j+1}r}\omega\Big(\frac{d_2(t_2,t_2')}{2^j r}\Big)^2\frac{|f(t_1)|^2}{\mu_1(B(s_1,2^j r))\cdot\mu_2(B(s_2,2^j r))^2}\,\mathrm{d}\mu_1(t_1)\\
&\lesssim_{S_1}\Big(\omega(1)^2+\sum_{j=0}^{\infty}\omega(2^{-j})^2\Big)\cdot\omega\Big(\frac{d_2(t_2,t_2')}{r}\Big)^2\cdot\frac{M_2 f(s_1)^2}{\mu_1(B(s_2,r))^2}\\
&\lesssim \|K\|_{(2,1)\text{-Dini}}\cdot\omega\Big(\frac{d_2(t_2,t_2')}{d_2(s_2,t_2)}\Big)^2\frac{1}{V(s_2,t_2)^2}\cdot M_2 f(s_1)^2
\end{aligned}$$

Thus, taking $L^p(S_1,w)$-norms for $p\in(2,\infty)$ and using Proposition 2.3.2(v) we obtain (4.3.5). The proof of (4.3.6) is similar and an inspection of the involved constants yields

$$\|K_2\|_{1\text{-Dini}} \lesssim_{S_1,p} [w]_{A_{p/2}(S_1)}^{\frac{1}{p-2}} \|K\|_{(2,1)\text{-Dini}}.$$

We conclude the proof by checking (4.6.1). Take $s_2 \neq t_2 \in S$, $f \in L^p(S_1, w)$, and $r > 0$. By Theorem 4.6.1(i) we have for

$$g(u) := f(u_1) \cdot \mathbf{1}_{B(t_2,r)}(u_2), \qquad u \in \boldsymbol{S}$$

that

$$\begin{aligned} \big\| T_K g \cdot \mathbf{1}_{B(s_2,r)} \big\|_{L^p(\boldsymbol{S},w;L^2(S_1))} &\lesssim_{\boldsymbol{S},T_K,p,w} \|g\|_{L^p(\boldsymbol{S},w)} \\ &= \mu_2\big(B(t_2,r)\big)^{1/p} \|f\|_{L^p(S_1,w)}. \end{aligned}$$

Therefore, if r is such that $r < \frac{1}{c_K} d_2(s_2', t_2)$ for all $s_2' \in B(s_2, r)$, we have

$$\begin{aligned} &\|K_2(s_2,t_2) f\|_{L^p(S_1,w;L^2(S_1))} \\ &\quad\le \Big\| \fint_{B(s_2,r)} \fint_{B(t_2,r)} K_2(s_2', t_2') f \, \mathrm{d}\mu_2(t_2') \, \mathrm{d}\mu_2(s_2') \Big\|_{L^p(S_1,w;L^2(S_1))} \\ &\qquad + \Big\| \fint_{B(s_2,r)} \big(K_2(s_2,t_2) - K_2(s_2',t_2)\big) f \, \mathrm{d}\mu_2(s_2') \Big\|_{L^p(S_1,w;L^2(S_1))} \\ &\qquad + \Big\| \fint_{B(s_2,r)} \fint_{B(t_2,r)} \big(K_2(s_2',t_2) - K_2(s_2',t_2')\big) f \, \mathrm{d}\mu_2(t_2') \, \mathrm{d}\mu_2(s_2') \Big\|_{L^p(S_1,w;L^2(S_1))} \\ &\quad\le \frac{\big\| T_K g \cdot \mathbf{1}_{B(s_2,r)} \big\|_{L^p(\boldsymbol{S},w;L^2(S_1))}}{\mu_2(B(s_2,r))^{1/p} \cdot \mu_2(B(t_2,r))} \\ &\qquad + \|K_2\|_{1\text{-Dini}} \cdot \|f\|_{L^p(S_1,w)} \cdot \omega\Big(\frac{r}{d_2(s_2,t_2)}\Big) \frac{1}{V(s_2,t_2)} \\ &\qquad + \|K_2\|_{1\text{-Dini}} \cdot \|f\|_{L^p(S_1,w)} \cdot \fint_{B(s_2,r)} \omega\Big(\frac{r}{d_2(s_2',t_2)}\Big) \frac{1}{V(s_2',t_2)} \, \mathrm{d}\mu_2(s_2') \\ &\quad\lesssim_{S,T_K,p,w,r,s_2,t_2} \|f\|_{L^p(S_1,w)}. \end{aligned}$$

It follows that $K_2(s_2, t_2)$ is indeed a bounded operator from $L^p(S_1, w)$ to $L^p(S_1, w; L^2(S_1))$. □

Now using the A_2-theorem proven in Section 3.4 and Rubio de Francia extrapolation, we can prove the second part of Theorem 4.6.1.

Proof of Theorem 4.6.1(ii). We will first prove the second inequality. Take $v \in A_{p/2}(S_1)$ and $f \in L^p(\boldsymbol{S}, v)$ with bounded support and set

$$\begin{aligned} Y_0 &:= L^p(S_1, v), \\ Y_1 &:= L^p(S_1, v; L^2(S_1)). \end{aligned}$$

We view f as a function in $L^p(S_2; Y_0)$ and note that T_K is bounded from $L^p(S_2; Y_0)$ to $L^p(S_2; Y_1)$ by part (i) and the fact that $v \cdot \mathbf{1}_{S_2} \in A_{p/2}(\boldsymbol{S})$. For $s_2 \in S_2$ we have

$$T_K f(s_2) = \Big((s_1, t_1) \mapsto \int_{S_2} K((s_1, s_2), (t_1, t_2)) f(t_2) \, \mathrm{d}t_2 \Big)$$

$$=: \int_{S_2} K_2(s_2,t_2) f(t_2)\,\mathrm{d}t_2.$$

By Lemma 4.6.2 we know that

$$K_2 \colon S_2 \times S_2 \setminus \{(s_2,s_2) : s_2 \in S_2\} \to \mathcal{L}(Y_0,Y_1)$$

is a 1-Dini kernel with $\|K_2\|_{1\text{-Dini}} \lesssim_{S_1,p} [v]_{A_{p/2}(S_1)}^{\frac{1}{p-2}} \|K\|_{(2,1)\text{-std}}$. Thus by Theorem 3.4.1 we deduce for $w \in A_q(S_2)$

$$\|T_K\|_{L^q(S_2,w;Y_0)\to L^q(S_2,w;Y_1)} \lesssim_{\boldsymbol{S},p,q} C_T\, [v]_{A_{p/2}(S_1)}^{\max\{\frac{1}{p-2},1\}} [w]_{A_q(S_2)}^{\max\{\frac{1}{q-1},1\}}, \tag{4.6.2}$$

which proves the second inequality of (ii).

For the first inequality of (ii) we note that by (4.6.2) and Fubini's theorem we have for $q>2$, $v \in A_{q/2}(S_1)$ and $w \in A_q(S_2)$ that

$$\|T_K\|_{L^q(S_1,v;Z_0)\to L^q(S_1,v;Z_1)} \le C_T\, [v]_{A_{q/2}(S_1)}^{\max\{\frac{1}{q-2},1\}} [w]_{A_q(S_2)}^{\max\{\frac{1}{q-1},1\}}.$$

with $Z_0 = L^q(S_2,w)$ and $Z_1 = L^q(S_2,w;L^2(S_1))$. Therefore the claim follows from the Rubio de Francia extrapolation in Theorem 2.3.3. □

Remark 4.6.3.

- Note that, even if we are only interested in estimates with the time variable S_1 on the outside, in the proof of Theorem 4.6.1(ii) we need to first put the time variable S_1 on the inside to deduce the optimal result with S_1 on the outside. This is due to the fact that we would otherwise only be able to obtain weighted estimates for $w \in A_{q/2}(S_2)$.
- We need to start with a weak L^2-estimate in Theorem 4.6.1, whereas results like Theorem 4.4.11 require a weak L^p-estimate for some $p \in [2,\infty)$. The reason for this dichotomy is that we do not see a way to extrapolate weak L^p- to weak L^2-boundedness for the operators in Theorem 4.6.1. Fortunately, in applications the L^2-estimate is the easiest to establish.

Remark 4.6.4. The weight dependence in 4.6.1(i) is sharp, which for $2<p<3$ follows from Proposition 4.4.14 by taking $S_2=\varnothing$ and for $p\ge 3$ by taking $S_1=\varnothing$ and the sharpness of the A_2-theorem. However, the weight dependence one obtains in the proof of the first inequality in Theorem 4.6.1(ii) is not sharp, due to the use of Rubio de Francia extrapolation. One could also do the extrapolation with Calderón-Zygmund theory, i.e. using similar arguments as in Lemma 4.6.2 and the first part of the proof of Theorem 4.6.1(ii). This would yield the first inequality in Theorem 4.6.1(ii) with

$$\phi(s,t) = C_{\boldsymbol{S},p,q} \cdot s^{\max\{\frac{1}{p-2},\frac{1}{2}\}} \cdot t^{\max\{\frac{1}{q-1},1\}}, \qquad s,t \in \mathbb{R}_+,$$

which is sharp in terms of the weight dependence. The weight dependence one obtains in the proof of the second inequality in Theorem 4.6.1(ii) (see (4.6.2)) is sharp for $2<p\le 3$, but it is not sharp for $p>3$ as can be seen from Theorem 4.4.11 by taking $S_2=\varnothing$.

Theorem 4.1.3 for mixed stochastic-deterministic integral operators now follows from Theorem 4.6.1 using a similar argument as in Propositions 4.2.3 and 4.2.5.

Proof of Theorem 4.1.3. We will only prove Theorem 4.1.3(i) using Theorem 4.6.1(i), the deduction of Theorem 4.1.3(ii) and (iii) from Theorem 4.6.1(ii) is similar. For $f \in L^2(\mathcal{O}_T)$ define

$$T_K f(t,x) := \Big(s \mapsto \int_{\mathcal{O}} K\big((t,x),(s,y)\big) f(s,y)\, \mathrm{d}\mu(y)\Big), \qquad (t,x).$$

Then by applying the Itô isomorphism (Theorem 2.9.1) twice and the L^2-boundedness of S_K in between, we have for any $h \in H$ with $\|h\|_H = 1$

$$\begin{aligned}\|T_K f\|_{L^2(\mathcal{O}_T;L^2(0,T))} &\simeq \|S_K(f \otimes h)\|_{L^2(\Omega\times\mathcal{O}_T)} \\ &\leq \|S_K\|_{L^2_{\mathscr{F}}(\Omega\times\mathcal{O}_T;H)\to L^2(\Omega\times\mathcal{O}_T)} \|f\otimes h\|_{L^2(\Omega\times\mathcal{O}_T;H)} = \|f\|_{L^2(\mathcal{O}_T)}.\end{aligned}$$

Thus, T_K is a bounded operator from $L^2(\mathcal{O}_T)$ to $L^2(\mathcal{O}_T;L^2(0,T))$, which implies by Theorem 4.6.1(i) that

$$T_K \colon L^p(\mathcal{O}_T,w) \to L^p(\mathcal{O}_T,w;L^2(0,T))$$

is bounded for $p \in (2,\infty)$ and $w \in A_{p/2}(\mathcal{O}_T)$ with norm A_0 as in Theorem 4.6.1(i). Now take $g \in L^r_{\mathscr{F}}(\Omega;L^p(\mathcal{O}_T,w;\gamma(H,X)))$, then by applying the Itô isomorphism (Theorem 2.9.1), the γ-Fubini theorems (Proposition 2.8.6 and [HNVW17, Proposition 9.4.9]) and Lemma 2.8.4, we obtain

$$\begin{aligned}\|S_K g\|_{L^r(\Omega;L^p(\mathcal{O}_T,w))} &\eqsim_{p,r} \Big\|\Big(s\mapsto \int_{\mathcal{O}} K(\cdot,(s,y))g(s,y)\,\mathrm{d}\mu(y)\Big)\Big\|_{L^r(\Omega;\gamma((0,T);H,L^p(\mathcal{O}_T,w)))} \\ &\lesssim_{p,r} \Big\|\Big(s\mapsto \int_{\mathcal{O}} K(\cdot,(s,y))g(s,y)\,\mathrm{d}\mu(y)\Big)\Big\|_{\gamma(H,L^r(\Omega;L^p(\mathcal{O}_T,w;L^2(0,T))))} \\ &\leq A_0 \|g\|_{\gamma(H,L^r(\Omega;L^p(\mathcal{O}_T,w)))} \\ &\eqsim_{p,r} A_0 \|g\|_{L^r(\Omega;L^p(\mathcal{O}_T,w;H))},\end{aligned}$$

proving the theorem. □

Remark 4.6.5. We could also allow the kernel in Theorem 4.1.3 to be operator-valued, i.e. $K \colon \mathcal{O}_T \times \mathcal{O}_T \to \mathcal{L}(X,Y)$ for Banach spaces X and Y, which would e.g. allow one to study a system of SPDEs with constants independent of the size of the system by using $X = Y = \mathbb{C}^n$.

5

STOCHASTIC MAXIMAL REGULARITY

This chapter is based on the second half of the paper

[6] E. Lorist and M.C. Veraar. Singular stochastic integral operators. *To appear in Anal. PDE*, 2020.

It has been edited to make full use of the stochastic-deterministic extrapolation theory developed in Section 4.6.

Abstract. *In this chapter we apply the results of Chapter 4 to obtain p-independence and weighted bounds for stochastic maximal L^p-regularity both in the complex and real interpolation scale. As a consequence, we obtain several new regularity results for the stochastic heat equation and its time-dependent variants on $\mathbb{R}^d$ and on smooth and angular domains. We also treat applications to Volterra equations and show the p-independence of the $\mathcal{R}$-boundedness of stochastic convolution operators*

5.1. INTRODUCTION

In this chapter we will apply the Calderón–Zygmund theory for stochastic singular integral operators we developed in Chapter 4 in the study of stochastic partial differential equations (SPDEs). In particular we will study maximal regularity estimates for stochastic evolution equations. Many SPDEs can be analysed as stochastic evolution equations by using functional analytic tools. We refer to the monograph [DZ14] and the papers [Brz97, NVW08].

Let $T \in (0,\infty]$ and consider the following linear stochastic evolution equation on a Banach space X:

$$\begin{cases} du + Au\,\mathrm{d}t = G\,\mathrm{d}W_H & \text{on } (0,T), \\ u(0) = 0. \end{cases} \tag{5.1.1}$$

Here $(A(t))_{t\in(0,T)}$ is a family of closed operators on X, H is a Hilbert space, W_H is H-cylindrical Brownian motion and $G : (0,T)\times\Omega \to \gamma(H,X)$ is adapted to the filtration $\mathscr{F}$ associated to W_H. In this chapter we will focus on these linear equations. Nonlinear stochastic evolution equations can be studied by using suitable estimates for the linear case (see [Brz97, DZ14]). In particular, stochastic maximal regularity estimates have been applied to nonlinear SPDEs in [Agr18, AV20a, AV20b, Brz95, Hor19, KK18, Kry99, NVW12a, PV19].

The mild solution to (5.1.1) is given by

$$u(t) = \int_0^t S(t,s)G(s)\,dW_H(s), \qquad t \in (0,T),$$

where we have assumed that $-A$ generates the strongly continuous evolution family $(S(t,s))_{0\le s\le t}$. In the case A does not depend on time, one has that $S(t,s) = e^{-(t-s)A}$ is a strongly continuous semigroup. For details and unexplained terminology on semigroups and evolution families we refer to [EN00, Lun95, Paz83, Tan79, Yag10].

Definition 5.1.1 (Stochastic maximal regularity)**.** Let X and Y be UMD Banach spaces with type 2, H a Hilbert space, $p \in [2,\infty)$ and let w be a weight on $(0,T)$. We say that A has *stochastic maximal* $L^p((0,T),w;Y)$*-regularity* and write $A \in \mathrm{SMR}(L^p((0,T),w;Y))$ if for all $G \in L^p_{\mathscr{F}}(\Omega\times(0,T),w;\gamma(H,X))$ the mild solution u to (5.1.1) satisfies

$$\|u\|_{L^p(\Omega\times(0,T),w;Y)} \lesssim \|G\|_{L^p(\Omega\times(0,T),w;\gamma(H,X))}. \tag{5.1.2}$$

We omit the weight if $w \equiv 1$.

Abstract properties of stochastic maximal regularity have been studied in [AV20c] and extensions to the case of time-dependent A have been obtained in [PV19]. An important choice for Y is the homogenous fractional domain space $\dot{D}(A^{1/2})$ with norm

$$\|x\|_{\dot{D}(A^{1/2})} = \|A^{1/2}x\|_X.$$

In [NVW12b] it has been shown that under certain geometric restrictions on X (see also Section 5.5), the boundedness of the H^∞-calculus of angle $< \pi/2$ of A (see [Haa06, HNVW17]) implies

$$A \in \mathrm{SMR}(L^p(\mathbb{R}_+; \dot{D}(A^{\frac{1}{2}}))).$$

Stochastic maximal regularity can be reformulated using the stochastic integral operators of Definition 4.2.1. Indeed, written out explicitly, the estimate (5.1.2) becomes

$$\Big\| t \mapsto \int_0^t S(t,s)G(s)\, dW_H(s) \Big\|_{L^p(\Omega\times(0,T),w;Y)} \lesssim \|G\|_{L^p(\Omega\times(0,T),w;\gamma(H,X))},$$

so $A \in \mathrm{SMR}(L^p(w;Y))$ if and only if $K \in \mathcal{K}_W^H(L^p((0,T),w))$ for

$$K(t,s) := S(t,s)\,\mathbf{1}_{0\le s<t} \in \mathcal{L}(X,Y),$$

where we implicitly assume that $S(t,s)$ maps X into Y. From Theorem 4.1.1 we find that in many instances stochastic maximal L^p-regularity for some $p \in [2,\infty)$ implies stochastic maximal L^q-regularity for all $q \in (2,\infty)$. In the time-independent setting we obtain the following result:

Theorem 5.1.2. *Assume $-A$ is the generator of a bounded C_0-semigroup on a UMD Banach space X with type 2. Suppose $A \in \mathrm{SMR}(L^p(\mathbb{R}_+, \dot{D}(A^{1/2})))$ for some $p \in [2,\infty)$. Then for all $q \in (2,\infty)$ and $w \in A_{q/2}(\mathbb{R}_+)$ one has $A \in \mathrm{SMR}(L^q(\mathbb{R}_+, w; \dot{D}(A^{1/2})))$. In particular, the mild solution u to (5.1.1) satisfies*

$$\|A^{1/2}u\|_{L^q(\Omega\times\mathbb{R}_+,w;X)} \lesssim [w]_{A_{q/2}(\mathbb{R}_+)}^{\max\{\frac{1}{2},\frac{1}{q-2}\}} \|G\|_{L^q(\Omega\times\mathbb{R}_+,w;\gamma(H,X))}.$$

A more general result is contained in Theorem 5.2.1 below. For this we should note that the above L^p-boundedness assumption implies sectoriality of A of angle $< \pi/2$ (see [AV20c, Theorem 4.1]). Theorem 5.1.2 with $w \equiv 1$ can be seen as the stochastic analogue of a similar statement for deterministic maximal regularity in [Dor00, Theorem 7.1]. The weighted estimates are a stochastic version of [CF14, Corollary 4] and [CK18, Theorem 5.1].

For many differential operators A one can directly apply the results in [NVW12b, NVW15c] to obtain stochastic maximal L^p-regularity. However, there are numerous situations where this is not the case, for example if:

(i) A does not have a bounded H^∞-calculus.

(ii) There is no explicit characterization of $\dot{D}(A^{1/2})$ known.

(iii) $A(t)$ and its domain $D(A(t))$ are time-dependent.

(iv) X does not satisfy the $\mathcal{R}$-boundedness condition of [NVW12b, NVW15c].

In Corollary 5.2.3 and Remark 5.2.4 we give a situation where (i) occurs, i.e. we give an example of an operator A without a bounded H^∞-calculus which has stochastic maximal L^p-regularity. In Example 5.2.15 both (i) and (ii) are open problems. In Section 5.3 we present applications to certain non-autonomous problems where (iii) occurs and in Theorem 5.2.5 we have avoided the geometric restriction mentioned in (iv).

The use of temporal $A_{q/2}$-weights in stochastic maximal L^p-regularity is new. In most of the results in [NVW12b, NVW15c] such weights can also be added without causing major difficulties, but it is very natural to deduce this from extrapolation theory. Moreover with our method we actually obtain sharp dependence on the $A_{q/2}$-characteristic. Power weights of the form t^α have already been considered before in both the deterministic (see [KPW10, PSW18]) and stochastic (see [AV20a, AV20b, AV20c, PV19]) evolution equations and can be used to allow for rough initial data. General A_p-weights in deterministic parabolic PDEs have used in [DK18, DK19b, GV17a, GV17b] to derive mixed $L^p(L^q)$-regularity estimates by Rubio de Francia extrapolation (see [GR85, CMP11]).

5.1.1. SPACE-TIME EXTRAPOLATION USING GREEN FUNCTION ESTIMATES

In the important special case that $X = L^q(\mathcal{O})$ for a domain $\mathcal{O} \subseteq \mathbb{R}^d$, we can also employ the Calderón–Zygmund theory for stochastic-deterministic integral operators in Theorem 4.1.3, see Examples 5.2.8 and 5.3.5. The assumed kernel estimates in Theorem 4.1.3 then correspond to so-called Green's function estimates or heat kernel estimates for the studied (parabolic) SPDE. Such estimates are available in quite general settings, see e.g. [EI70, KN14]. The advantage of this approach is that one reduces the study of stochastic maximal L^p-regularity problem on $L^q(\mathcal{O})$ to the study of stochastic maximal L^2-regularity problem on $L^2(\mathcal{O})$, for which one can employ Hilbert space techniques. Moreover, one obtains space-time weights in the conclusion. Power weights in space can e.g. be used to allow for rough boundary conditions (see e.g. [HL19, Lin20, Lin18, LV20]), treat singularities due to corners in the domain (see e.g. [Cio20, CKL19, CKLL18, KN14, Naz01, Sol01, PS04]), and handle the incompatibility of the boundary conditions and the noise term W_H (see e.g. [Kim04, KK04, KL99a, KL99b, Kry94a]). One can also use the obtained space-time weighted estimates to derive mixed $L^p(L^q)$- and $L^q(L^p)$-regularity estimates by Rubio de Francia's weighted extrapolation theorem [GR85, CMP11], which is already included in the conclusion of Theorem 4.1.3. The reversed integration order in stochastic maximal $L^q(L^p)$-regularity estimates allows one to deduce additional regularity results for the mild solution of (5.1.1), see also [Ant17, NVW15a]. Moreover one can obtain estimates for the moments of the mild solution of (5.1.1) as in [Kim20], see also Remark 4.1.4.

5.2. AUTONOMOUS CASE

We first turn to the time-independent case, in which we assume A to be the generator of a strongly continuous semigroup $(e^{-tA})_{t\in\mathbb{R}_+}$. In fact, the maximal regularity estimates

that we will assume imply that A is a sectorial operator of angle $< \pi/2$, so without loss of generality we may include this in our assumptions. Let us recall the definition of a sectorial operator, for their properties we refer to [KW04, Haa06, HNVW17, Yag10].

Let X be a Banach space and define for $0 < \sigma < \pi$

$$\Sigma_\sigma = \{z \in \mathbb{C} \setminus \{0\} : |\arg(z)| < \sigma\}.$$

A closed operator A with domain $D(A)$ on X will be called *sectorial* if there is a $\sigma \in (0,\pi)$ such that $\mathbb{C} \setminus \overline{\Sigma}_\sigma \subseteq \rho(A)$ and there is a constant $C > 0$ such that

$$\|\lambda(\lambda - A)^{-1}\| \leq C, \qquad \lambda \in \mathbb{C} \setminus \overline{\Sigma}_\sigma.$$

The infimum over all such σ is called the *angle of sectoriality* of A and is denoted by $\omega(A)$. A sectorial operator with $\omega(A) < \pi/2$ generates an analytic semigroup $(\mathrm{e}^{-zA})_{z \in \Sigma_{\pi/2-\sigma}}$ for $\omega(A) < \sigma < \pi/2$.

Theorem 5.2.1 (Extrapolation in the semigroups case)**.** *Suppose X is a* UMD *Banach space with type* 2*. Let A be a sectorial operator on X with $\omega(A) < \pi/2$. Take $r \in [2,\infty)$ and assume that Y is one of the following spaces*

$$D(A^{1/2}), \quad \dot{D}(A^{1/2}), \quad [X, D(A)]_{\frac{1}{2}}, \quad \text{or} \quad (X, D(A))_{\frac{1}{2},r}, \tag{5.2.1}$$

Suppose $A \in \mathrm{SMR}(L^p(\mathbb{R}_+; Y))$ for some $p \in [2,\infty)$. Then for all $q \in (2,\infty)$ and $w \in A_{q/2}(\mathbb{R}_+)$ one has $A \in \mathrm{SMR}(L^q(\mathbb{R}_+, w; Y))$. In particular, the mild solution u to (5.1.1) satisfies

$$\|u\|_{L^q(\Omega \times \mathbb{R}_+, w; Y)} \lesssim [w]_{A_{q/2}(\mathbb{R}_+}^{\max\{\frac{1}{2}, \frac{1}{q-2}\}} \|G\|_{L^q(\Omega \times \mathbb{R}_+, w; \gamma(H,X))},$$

where the implicit constant only depends on X, A, p, q.

Proof. The space Y has type 2 with $\tau_{2,Y} \leq \tau_{2,X}$, which is trivial for $D(A^{1/2})$ and $\dot{D}(A^{1/2})$, follows from [HNVW17, Proposition 7.1.3] for $[X, D(A)]_{\frac{1}{2}}$ and follows from [Cob83, Corollary 1] for $(X, D(A))_{\frac{1}{2},r}$. In all cases except for $\dot{D}(A^{1/2})$ it follows from the proof of [AV20c, Proposition 4.8] that A is invertible. We claim that in all cases

$$\|x\|_Y \leq C \|x\|_X^{\frac{1}{2}} \|Ax\|_X^{\frac{1}{2}}, \qquad x \in D(A).$$

Indeed, this standard interpolation estimate follows from [Lun95, Corollary 1.2.7 and Proposition 2.2.15], [Tri78, Theorem 1.10.3] and [Haa06, Proposition 6.6.4]. Since

$$t\|Ae^{-tA}\|_{\mathcal{L}(X)} \leq M, \qquad t \geq 0$$

for some $M > 0$ (see [EN00, Theorem II.4.6]), the above interpolation estimate implies that

$$\|Ae^{-tA}\|_{\mathcal{L}(X,Y)} \leq CM^{3/2} t^{-3/2}, \qquad t \geq 0.$$

Define $K \colon \mathbb{R}_+ \times \mathbb{R}_+ \to \mathcal{L}(X,Y)$ by $K(t,s) = \mathrm{e}^{-(t-s)A} \mathbf{1}_{t \geq s}$. Then by assumption we have $K \in \mathcal{K}_W^H(L^p(\mathbb{R}_+))$. Applying Propositions 4.2.3 and 4.2.5 we obtain that $K \in \mathcal{K}_\gamma(L^p(\mathbb{R}_+))$.

Next we will check the conditions of Theorem 4.4.11 for the space of homogeneous type $\mathbb{R}_+$. By the analyticity of the semigroup and the above estimate, we find that for $t \neq s \in \mathbb{R}_+$

$$\|\partial_t K(t,s)\|_{\mathcal{L}(X,Y)} = \|\partial_s K(t,s)\|_{\mathcal{L}(X,Y)} = \|Ae^{-(t-s)A}\|_{\mathcal{L}(X,Y)} \leq C\, M^{3/2}\, |t-s|^{-3/2}.$$

Therefore, by Lemma 4.3.3 and Remark 4.3.4, we know that K is a 2-standard kernel, so Theorem 4.4.11 gives that $K \in \mathcal{K}_\gamma(L^q(\mathbb{R}_+, w))$. Propositions 4.2.3 and 4.2.5 then imply that $K \in \mathcal{K}_W^H(L^q(\mathbb{R}_+, w))$ with the claimed estimate. □

Remark 5.2.2.

(i) Combining Theorem 5.2.1 with [AV20c, Section 5], similar results as in Theorem 5.2.1 hold on finite time intervals $(0, T)$. Alternatively, this can be deduced by applying Theorem 4.4.11 on $(0, T)$.

(ii) In general the result of Theorem 5.2.1 does not hold in the endpoint $q = 2$. A counterexample can be found in [NVW12b, Section 6].

(iii) Arguing as in the proof of Theorem 5.2.1 but with

$$K(t,s) = (t-s)^{-\theta} A^{\frac{1}{2}-\theta} e^{-(t-s)A} \mathbf{1}_{t\geq s},$$

and $Y = X$ it follows that for any $\theta \in (0, \frac{1}{2})$ the property $A \in \mathrm{SMR}_\theta(p, \infty)$ introduced in [AV20c] is p-independent.

(iv) From the proof it is clear that Theorem 5.2.1 holds for any Banach space Y with type 2 such that $e^{-tA} : X \to Y$ with

$$\|e^{-tA}\|_{\mathcal{L}(X,Y)} \leq Ct^{-\frac{1}{2}}, \qquad t > 0.$$

We have the following corollary in the case that X is a Hilbert space.

Corollary 5.2.3. *Let X be a Hilbert space and let Y be any of the spaces in (5.2.1) with $r = 2$. Suppose that A is a sectorial operator on X with $\omega(A) < \pi/2$. Then the following are equivalent:*

(i) There exists a constant $C > 0$ such that

$$\|t \mapsto e^{-tA}x\|_{L^2(\mathbb{R}_+;Y)} \leq C\|x\|_X, \qquad x \in X.$$

(ii) For all $p \in (2, \infty)$ and $w \in A_{p/2}(\mathbb{R}_+)$ (and $p = 2$, $w \equiv 1$) we have $A \in \mathrm{SMR}(L^p(\mathbb{R}_+, w; Y))$.

(iii) $A \in \mathrm{SMR}(L^p(\mathbb{R}_+, Y))$ for some $p \in [2, \infty)$.

Proof. Note that Y is a Hilbert space. For (i)⇒(ii) define $K(t,s) \in \mathcal{L}(X,Y)$ by

$$K(t,s) = e^{-(t-s)A}\mathbf{1}_{t\geq s}.$$

From Proposition 4.2.10(i) we obtain $K \in \mathcal{K}_\gamma(L^2(\mathbb{R}_+))$. Therefore $K \in \mathcal{K}_W^H(L^2(\mathbb{R}_+))$ by Propositions 4.2.3 and 4.2.5, so the result follows from Theorem 5.2.1. (ii)⇒(iii) is trivial and (iii)⇒(i) follows from Proposition 4.2.8 combined with Propositions 4.2.3 and 4.2.5. □

Remark 5.2.4.

(i) Corollary 5.2.3(i) is equivalent to the *admissability* of $A^{1/2}$ and is connected to the *Weiss conjecture*, which was solved negatively (See [JZ04], [LM03, Theorem 5.5] and references therein).

(ii) It is well-known that there exist operators A on a Hilbert space X such that $-A$ generates an analytic semigroup which is exponentially stable and

$$\|t \mapsto A^{\frac{1}{2}} e^{-tA} x\|_{L^2(\mathbb{R}_+;X)} \leq C\|x\|_X,$$

but

$$c\|x\|_X \not\leq \|t \mapsto A^{\frac{1}{2}} e^{-tA} x\|_{L^2(\mathbb{R}_+;X)}.$$

Such A can be constructed as in [LM03, Theorem 5.5] (see [AV20c, Section 5.2] for details), and does not have a bounded H^∞-calculus. On the other hand, Corollary 5.2.3 implies $A \in \mathrm{SMR}(L^p(\mathbb{R}_+, w; D(A^{1/2})))$ for all $p \in [2,\infty)$ and $w \in A_{p/2}$ (with $w = 1$ if $p = 2$), which shows that having a bounded H^∞-calculus is not necessary for stochastic maximal regularity.

For $\theta > 0$ and $p \in [1,\infty]$ we define the real interpolation spaces $D_A(\theta, p)$ by

$$D_A(\theta, p) = (X, D(A^n))_{\theta/n,p},$$

where $n \in \mathbb{N}$ is the least integer larger than θ. From Theorem 5.2.1 we obtain the following result for stochastic maximal regularity with $Y = D_A(\theta, p)$.

Theorem 5.2.5 (Real interpolation scale)**.** *Let E be a* UMD *Banach space with type* 2*. Let A be a sectorial operator on E with $\omega(A) < \pi/2$ and assume $0 \in \rho(A)$. Let $\theta \in (0,1)$ and $q \in [2,\infty)$. Define $X = D_A(\theta, q)$ and $Y = D_A(\theta + \frac{1}{2}, q)$. Then for all $p \in (2,\infty)$ and $w \in A_{p/2}(\mathbb{R}_+)$, one has $A \in \mathrm{SMR}(L^p(\mathbb{R}_+, w, Y))$ (the case $p = q = 2$ and $w = 1$ is allowed as well). In particular, the solution u to (5.1.1) satisfies*

$$\|A^{\frac{1}{2}} u\|_{L^p(\Omega\times\mathbb{R}_+,w;D_A(\theta,q))} \lesssim [w]_{A_{p/2}(\mathbb{R}_+)}^{\max\{\frac{1}{2},\frac{1}{p-2}\}} \|G\|_{L^p(\Omega\times\mathbb{R}_+,w;\gamma(H,D_A(\theta,q)))}, \tag{5.2.2}$$

where the implicit constant only depends on E, A, θ, p, q.

First proof. Note that X is a UMD Banach space with type 2 by [HNVW16, Proposition 4.2.17] and $-A$ is the generator of an exponentially stable analytic semigroup on X with domain $D_A(\theta + 1, q)$ by [Lun95, Proposition 2.2.7]. Moreover, we have

$$Y = (X, D_A(\theta + 1, q))_{\frac{1}{2},q}.$$

It follows from [DL98] (see also [BH09, Theorem 5.1]) and [AV20c, Theorem 5.2] that $A \in \mathrm{SMR}(L^q(\mathbb{R}_+;Y))$. Therefore, the required result follows from Theorem 5.2.1. The claimed norm estimate follows since $A^{1/2}$ maps $D_A(\theta+\frac{1}{2},q)$ isomorphically to $D_A(\theta,q)$ (see [Tri78, Theorem 1.15.2]). □

Next we give a proof that only uses elementary properties of the real interpolation spaces $D_A(\theta,p)$.

Second proof. First consider the case $p=q=2$. By Propositions 4.2.10(i), 4.2.3 and 4.2.5 and [Tri78, Theorem 1.15.2] it suffices to show

$$\boxed{\mathrm{A}} := \Big(\int_0^\infty \|A^{\frac{1}{2}} e^{-tA}x\|^2_{D_A(\theta,2)}\,\mathrm{d}t\Big)^{\frac{1}{2}} \le C\|x\|_{D_A(\theta,2)}. \tag{5.2.3}$$

Since $D_A(\theta,2) = D_{A^2}(\theta/2,2)$ (see [Tri78, Theorem 1.15.2]), by [Tri78, Theorem 1.14.5] we can write

$$\begin{aligned}
\boxed{\mathrm{A}}^2 &\simeq \int_0^\infty \|A^{\frac{1}{2}} e^{-tA}x\|^2_{D_{A^2}(\theta/2,2)}\,\mathrm{d}t \\
&\simeq \int_0^\infty \int_0^\infty r^{4(1-\frac{\theta}{2})}\|A^{\frac{5}{2}} e^{-(r+t)A}x\|_E^2 \frac{\mathrm{d}r}{r}\,\mathrm{d}t \\
&\lesssim \int_0^\infty \int_0^\infty (t+r)^{-3} r^{4(1-\frac{\theta}{2})}\|Ae^{-rA}x\|_E^2 \frac{\mathrm{d}r}{r}\,\mathrm{d}t \\
&= 2\int_0^\infty r^{2(1-\theta)}\|Ae^{-rA}x\|_E^2 \frac{\mathrm{d}r}{r} \simeq \|x\|^2_{D_A(\theta,2)}
\end{aligned}$$

which gives the required estimate (5.2.3).

From the previous case and Theorem 5.2.1 we obtain stochastic maximal L^p-regularity for $p \in [2,\infty)$ in the case $q=2$. Thus, using Propositions 4.2.3 and 4.2.5 to take $H=\mathbb{R}$, the mapping

$$SG(t) := \int_0^t A^{\frac{1}{2}} e^{-(t-s)A} G(s)\,\mathrm{d}W(s), \qquad t \in \mathbb{R}_+$$

is bounded from $L^p(\mathbb{R}_+;D_A(\theta,2))$ to $L^p(\mathbb{R}_+\times\Omega;D_A(\theta,2))$ for all $\theta\in(0,1)$ and $p\in[2,\infty)$. By [Tri78, 1.10 and 1.18.4] one has

$$(L^q(\mathbb{R}_+;D_A(\theta-\varepsilon,2)), L^q(\mathbb{R}_+;D_A(\theta+\varepsilon,2)))_{\frac{1}{2},q} = L^q(\mathbb{R}_+;D_A(\theta,q))$$

for $\varepsilon\in(0,\min\{\theta,1-\theta\})$ and the same holds with $\mathbb{R}_+$ replaced by $\mathbb{R}_+\times\Omega$. It follows from [Tri78, Theorem 1.3.3] that S is bounded from $L^q(\mathbb{R}_+;D_A(\theta,q))$ into $L^q(\mathbb{R}_+\times\Omega;D_A(\theta,q))$. Applying Propositions 4.2.3 and 4.2.5 once more to recover a general cylindrical Brownian motion W_H, we obtain the stochastic maximal regularity for $p=q\in[2,\infty)$. Now another application of Theorem 5.2.1 gives the result for all required p, q and weights $w\in A_{p/2}$. The claimed norm estimate again follows since $A^{1/2}$ maps $D_A(\theta+\frac{1}{2},q)$ isomorphically to $D_A(\theta,q)$ (see [Tri78, Theorem 1.15.2]). □

Remark 5.2.6.

(i) By carefully checking the proofs of Theorems 5.2.1 and 5.2.5 (and in particular Proposition 4.2.3) one sees that Theorem 5.2.5 actually holds for all martingale type 2 spaces E. As mentioned in Remark 5.2.2(i), Theorem 5.2.5 holds on finite time intervals as well and in this case we only need that $A+\lambda$ is a sectorial operator with $\omega(A+\lambda) < \pi/2$ for some $\lambda \in \mathbb{R}$.

(ii) Theorem 5.2.5 extends [BH09, Theorem 5.1] and [DL98] to the case where $p \neq q$ and to the weighted setting. Note that even for $w = 1$ one cannot obtain Theorem 5.2.5 from the case $p = q$ and a real interpolation argument. Indeed, in general for an interpolation couple (X_0, X_1) one has (see [Cwi74])

$$(L^{p_0}(\Omega \times \mathbb{R}_+; X_0), L^{p_1}(\Omega \times \mathbb{R}_+; X_1))_{\theta,q} \neq L^{[p_0,p_1]_\theta}(\Omega \times \mathbb{R}_+; (X_0, X_1)_{\theta,q}).$$

The equality does hold if $q = [p_0, p_1]_\theta$.

(iii) The assumption $0 \in \rho(A)$ in Theorem 5.2.5 is needed in general. Indeed, there exists a bounded sectorial operator A on a Hilbert space E such that (5.2.3) does not hold (see [HNVW16, Corollary 10.2.29 and Theorem 10.4.21]). Since in this case $D_A(\theta, p) = E$ for all $\theta \in (0,2)$ and $p \in [1,\infty]$, Propositions 4.2.3, 4.2.5 and 4.2.8 imply that (5.2.2) cannot hold.

We conclude with another result for real interpolation spaces. It extends [Brz95, (4.10)] to the case $p \in (2,\infty)$ and to the setting of infinite time intervals.

Theorem 5.2.7. *Let E be a* UMD *Banach space with type* 2 *and let A be a sectorial operator on E with $\omega(A) < \pi/2$. Let $X = D_A(\frac{1}{2}, 2)$ and $Y = \dot{D}(A)$. Then for all $p \in (2,\infty)$ and $w \in A_{p/2}(\mathbb{R}_+)$, one has $A \in \mathrm{SMR}(L^p(\mathbb{R}_+, w, Y))$ (the case $p = 2$ and $w \equiv 1$ is allowed as well). In particular, the solution u to (5.1.1) satisfies*

$$\|Au\|_{L^p(\Omega\times\mathbb{R}_+,w;E)} \leq C\,[w]_{A_{p/2}(\mathbb{R}_+)}^{\max\{\frac{1}{2},\frac{1}{p-2}\}} \|G\|_{L^p(\Omega\times\mathbb{R}_+,w;\gamma(H,D_A(\frac{1}{2},2)))},$$

where C only depends on E, A, p.

Proof. Note that, as in the first proof of Theorem 5.2.5, A is a sectorial operator on X with $\omega(A) < \pi/2$. For $p = 2$, as in the second proof of Theorem 5.2.5, it suffices to prove the following variant of (5.2.3)

$$\Big(\int_0^\infty \|Ae^{-tA}x\|_E^2 \,\mathrm{d}t\Big)^{\frac{1}{2}} \leq C\|x\|_{D_A(\frac{1}{2},2)}.$$

The latter estimate is immediate from the definition of $D_A(\frac{1}{2},2)$. It remains to apply Theorem 5.2.1. For this (see Remark 5.2.2(iv)) it suffices to check $\|e^{-tA}\|_{\mathcal{L}(X,Y)} \leq Ct^{-\frac{1}{2}}$, which follows from

$$\sup_{t>0} \|t^{\frac{1}{2}} Ae^{-tA}x\|_E \leq \|x\|_{D_A(\frac{1}{2},\infty)} \lesssim \|x\|_{D_A(\frac{1}{2},2)},$$

where we used [Tri78, Theorems 1.3.3(d) and 1.14.5]. □

5.2.1. THE STOCHASTIC HEAT EQUATION ON $\mathbb{R}^d$

As a first concrete application, we will now use our abstract extrapolation results to the stochastic heat equation on $\mathbb{R}^d$. We will show that, using only extrapolation results for stochastic singular integrals, one can deduce the stochastic maximal $L^p(L^q)$-regularity results for $-\Delta$ in [Kry00] and [NVW12b]. Moreover we actually obtain results with space-time weights. One can check that the proof of [NVW12b], based on the boundedness H^∞-calculus of $-\Delta$, also gives the result with weights in time, and moreover weights in space could be added. Still we find it illustrative to show in the example below that the $L^2(L^2)$-case can be combined with extrapolation arguments to deduce the weighted $L^p(L^q)$-case for all $p \in (2,\infty)$ and $q \in [2,\infty)$.

We start with a result in Bessel potential spaces, for which we use the stochastic-deterministic extrapolation developed in Section 4.6. For details on (weighted) Bessel potential spaces we refer to [MV12, Tri78].

Example 5.2.8 (Stochastic heat equation in Bessel-potential spaces). Let $m \in \mathbb{N}$, $s \in \mathbb{R}$, $p \in (2,\infty)$, $q, r \in (1,\infty)$, $v \in A_{p/2}(\mathbb{R}_+)$ and $w \in A_q(\mathbb{R}^d)$. On $\mathbb{R}^d$ consider

$$\begin{cases} du + (-\Delta)^m u \, dt = G \, dW_H, & \text{on } \mathbb{R}_+, \\ u(0) = 0, \end{cases} \tag{5.2.4}$$

where $G\colon \Omega \times \mathbb{R}_+ \times \mathbb{R}^d \to H$ is an adapted process. Then the mild solution u to (5.2.4) satisfies

$$\begin{aligned} \big\| (-\Delta)^{\frac{m}{2}} u \big\|_{L^r(\Omega; L^p(\mathbb{R}_+, v; H^{s,q}(\mathbb{R}^d, w)))} &\le C \|G\|_{L^r(\Omega; L^p(\mathbb{R}_+, v; H^{s,q}(\mathbb{R}^d, w; H)))}, \qquad q > 2, \\ \big\| (-\Delta)^{\frac{m}{2}} u \big\|_{L^r(\Omega; H^{s,q}(\mathbb{R}^d, w; L^p(\mathbb{R}_+, v)))} &\le C \|G\|_{L^r(\Omega; H^{s,q}(\mathbb{R}^d, w; L^p(\mathbb{R}_+, v; H)))}, \end{aligned}$$

where C is an increasing function of $[v]_{A_{p/2}(\mathbb{R}_+)}$ and $[w]_{A_q(\mathbb{R}^d)}$ and depends on p, q, r, m, d.

Proof. By lifting we may assume $s = 0$ (see [MV12, Proposition 3.9]). First suppose $p = q = 2$ and $v = w \equiv 1$. It suffices to check Corollary 5.2.3(i). Note for any $f \in L^2(\mathbb{R}^d)$ by Plancherel's theorem

$$\begin{aligned} \int_{\mathbb{R}_+} \int_{\mathbb{R}^d} |(-\Delta)^{\frac{m}{2}} e^{-t(-\Delta)^m} f(x)|^2 \, dx \, dt &= \int_{\mathbb{R}_+} \int_{\mathbb{R}^d} (2\pi|\xi|)^{2m} e^{-2(2\pi|\xi|)^{2m} t} |\widehat{f}(\xi)|^2 \, d\xi \, dt \\ &= \int_{\mathbb{R}^d} \int_{\mathbb{R}_+} (2\pi|\xi|)^{2m} e^{-2(2\pi|\xi|)^{2m} t} |\widehat{f}(\xi)|^2 \, dt \, d\xi \\ &= \frac{1}{2} \int_{\mathbb{R}^d} |\widehat{f}(\xi)|^2 \, d\xi = \frac{1}{2} \|f\|^2_{L^2(\mathbb{R}^d)}. \end{aligned}$$

Therefore, by Corollary 5.2.3, we find

$$\big\| (-\Delta)^{\frac{m}{2}} u \big\|_{L^2(\Omega \times \mathbb{R}_+ \times \mathbb{R}^d)} \le C \|G\|_{L^2(\Omega \times \mathbb{R}_+ \times \mathbb{R}^d; H)}. \tag{5.2.5}$$

Now fix $|\alpha| = m$ and define

$$S_K G(t,x) := \partial_x^\alpha u(t,x) = \int_{\mathbb{R}_+} \int_{\mathbb{R}^d} K\big((t,x),(s,y)\big) G(s,y) \, dy \, dW_H(s), \quad (t,x) \in \mathbb{R}_+ \times \mathbb{R}^d$$

with

$$K\big((t,x),(s,y)\big) := \mathscr{F}^{-1}\big(\xi \mapsto (2\pi)^m \xi^\alpha e^{-(2\pi|\xi|)^{2m}(t-s)}\big)(x-y), \qquad (t,x),(s,y) \in \mathbb{R}_+ \times \mathbb{R}^d,$$

From (5.2.5) it follows that S_K is bounded from $L^2_{\mathscr{F}}(\Omega \times \mathbb{R}_+ \times \mathbb{R}^d; H)$ to $L^2(\Omega \times \mathbb{R}_+ \times \mathbb{R}^d)$, so by Theorem 4.1.3 it suffices to show that K is a $(2,1)$-standard kernel on the space of homogeneous type $\mathbb{R}_+ \times \mathbb{R}^d$ with the anisotropic metric $|\cdot - \cdot|_{\boldsymbol{a}}$ as in Example 2.1.2 with $\boldsymbol{a} = (2m, 1, \dots, 1)$. Since K is of convolution type, by a change of variables and Lemma 4.3.3 (see also Remark 4.3.4) it suffices to show

$$|\partial_t k(t,x)| \le A_0 \frac{1}{|(t,x)|_{\boldsymbol{a}}^{d+3m}} \qquad (t,x) \in (\mathbb{R}_+ \times \mathbb{R}) \setminus \{0\}. \tag{5.2.6}$$

$$|\partial_x^\beta k(t,x)| \le A_0 \frac{1}{|(t,x)|_{\boldsymbol{a}}^{d+m+1}} \qquad (t,x) \in (\mathbb{R}_+ \times \mathbb{R}) \setminus \{0\},\ |\beta| = 1, \tag{5.2.7}$$

with

$$k(t,x) := \mathscr{F}^{-1}\big(\xi \mapsto (2\pi)^m \xi^\alpha e^{-(2\pi|\xi|)^{2m} t}\big)(x), \qquad (t,x) \in \mathbb{R}_+ \times \mathbb{R}^d.$$

Fix $(t,x) \in (\mathbb{R}_+ \times \mathbb{R}^d) \setminus \{0\}$, let $|\gamma| \le d+3m$ and set $I := \big\{k \in \mathbb{N} : \frac{|\gamma|-3m}{2m} \le k \le |\gamma|\big\}$. Then we have

$$\begin{aligned}
|x^\gamma \partial_t k(t,x)| &\le (2\pi)^{3m-|\gamma|} \int_{\mathbb{R}^d} \big|\partial_\xi^\gamma \big(\xi^\alpha |\xi|^{2m} \cdot e^{-(2\pi|\xi|)^{2m} t}\big)\big| \,\mathrm{d}\xi \\
&\lesssim_{m,d} \sum_{k\in I} \int_{\mathbb{R}^d} |\xi|^{3m-(|\gamma|-k)} \cdot (|\xi|^{2m-1} t)^k \cdot e^{-(2\pi|\xi|)^{2m} t} \,\mathrm{d}\xi \\
&\simeq_d \sum_{k\in I} \int_0^\infty r^{3m+2mk-|\gamma|+d-1} \cdot t^k \cdot e^{-(2\pi r)^{2m} t} \,\mathrm{d}r \\
&\simeq_m t^{-\frac{d+3m-|\gamma|}{2m}} \sum_{k\in I} \int_0^\infty s^{3m+2mk-|\gamma|+d-1} \cdot e^{-(2\pi)^{2m} s} \,\mathrm{d}s \\
&\lesssim_m t^{-\frac{d+3m-|\gamma|}{2m}}.
\end{aligned}$$

Thus combining the cases $|\gamma| = 0$ and $|\gamma| = d+3m$ with the equivalence

$$|x|^{d+3m} \simeq_{m,d} \sum_{|\gamma|=d+3m} |x^\gamma|, \qquad x \in \mathbb{R}^d,$$

we obtain (5.2.6). (5.2.7) follows from a similar computation, so K is a $(2,1)$-standard kernel. Therefore the claimed estimates for $(-\Delta)^{\frac{m}{2}} u$ follow from Theorem 4.1.3 and the norm equivalence

$$\sum_{|\alpha|=m} \|\partial^\alpha f\|_{L^p(\mathbb{R}^d,v;X)} \simeq_{X,p,m,d} \|(-\Delta)^{\frac{m}{2}} f\|_{L^p(\mathbb{R}^d,v;X)}, \qquad f \in \mathcal{S}'(\mathbb{R}^d;X)$$

for any UMD Banach space X, which is a consequence of the Mihlin multiplier theorem (see Section 3.5). □

Remark 5.2.9. As seen from the proof, the case $p = q = r = 2$ and $v = w \equiv 1$ is also allowed in Example 5.2.8. From this one can also obtain the case $p \in (2,\infty)$, $q = 2$, $r \in (1,\infty)$, $v \in A_{p/2}(\mathbb{R}_+)$ and $w \equiv 1$ using Theorem 5.2.1 and Remark 4.2.4. Moreover, using Fubini's theorem and Rubio de Francia extrapolation, we can also obtain results with the integral over Ω on the inside, which will be done in [1] (see also Remark 4.1.4).

Next we prove a similar result on Besov spaces using the extrapolation result in Theorem 5.2.5. For details on Besov spaces we refer to [Tri78].

Example 5.2.10 (Stochastic heat equation in Besov spaces). Let $m \in \mathbb{N}$, $s \in \mathbb{R}$, $p \in (2,\infty)$, $q, r \in [2,\infty)$, $v \in A_{p/2}(\mathbb{R}_+)$ and $w \in A_q(\mathbb{R}^d)$. On $\mathbb{R}^d$ consider

$$\begin{cases} du + (1-\Delta)^m u \, dt = G \, dW_H, & \text{on } \mathbb{R}_+, \\ u(0) = 0, \end{cases} \tag{5.2.8}$$

where $G \in L^p_{\mathscr{F}}(\Omega \times \mathbb{R}_+, v; B^s_{q,r}(\mathbb{R}^d, w; H))$. Then the mild solution u to (5.2.8) satisfies

$$\|u\|_{L^p(\Omega\times\mathbb{R}_+, v; B^{s+m}_{q,r}(\mathbb{R}^d,w))} \leq C \|G\|_{L^p(\Omega\times\mathbb{R}_+, v; B^s_{q,r}(\mathbb{R}^d,w;H))},$$

where C is an increasing function of $[v]_{A_{p/2}(\mathbb{R}_+)}$ and $[w]_{A_q(\mathbb{R}^d)}$ and depends on p, q, r, s, m, d.

Proof. Again by lifting (see [MV12, Proposition 3.9]) we may assume $s = 2m\theta \in (0, 2m)$. Let $E = L^q(\mathbb{R}^d, w)$ and define

$$(A, D(A)) := \big((1-\Delta)^m, W^{2m,q}(\mathbb{R}^d, w)\big).$$

Then A is sectorial with $\omega(A) = 0$ and $0 \in \rho(A)$ on E. Since $D_A(\theta, r) = B^s_{q,r}(\mathbb{R}^d, w)$ (see [MV12, Proposition 6.1]), the result follows from Theorem 5.2.5 and another application of [MV12, Proposition 3.9]. □

Remark 5.2.11.

(i) There is an inconsistency between the equations (5.2.4) and (5.2.8) ($-\Delta$ vs. $1-\Delta$). The reason to consider $1-\Delta$ is that one has the restriction $0 \in \rho(A)$ in Theorem 5.2.5. With a different proof one can also consider Example 5.2.10 with $1-\Delta$ replaced by $-\Delta$. For example one can obtain this by a real interpolation argument in Example 5.2.8. To avoid adaptedness problems in the interpolation argument, one can first consider deterministic G and afterwards apply Proposition 4.2.3.

(ii) The results of Examples 5.2.8 and 5.2.10 are incomparable except if $q = r = 2$ (see [Tri83, Theorem 2.3.9]). A similar example could be proved for Triebel–Lizorkin spaces, by using [NVW12b] and the boundedness of the H^∞-calculus of $(1-\Delta)^m$ on $F^s_{q,r}(\mathbb{R}^d, w)$, which can be proved as in [HNVW17] with the Mihlin multiplier theorem (Theorem 3.5.1). Alternatively one could use the ℓ^r-interpolation method developed in [Kun15] on Example 5.2.8, again avoiding adaptedness problems using Proposition 4.2.3.

5.2.2. STOCHASTIC HEAT EQUATION ON A WEDGE

Our next application is an $L^p(L^q)$-version of the stochastic maximal regularity result in [CKLL18] for the stochastic heat equation on an angular domain. The deterministic setting was considered in [Sol01, Theorem 1.1] and later improved in [Naz01, Theorem 1.1] and [PS07, Corollary 5.2]. At the moment it is unclear whether the Dirichlet Laplacian $-\Delta$ on an angular domain has a bounded H^∞-calculus, and how to characterize $D((-\Delta)^{1/2})$ in terms of weighted Sobolev spaces. Therefore, we can not apply [NVW12b] and instead we will use [CKLL18] and Theorem 5.2.1 to derive $L^p(L^q)$-regularity results.

We will first need some properties of the heat semigroup on a wedge. Let $\kappa \in (0, 2\pi)$ and define the wedge

$$\mathcal{O} := \{x \in \mathbb{R}^2 : x = (r\cos(\varphi), r\sin(\varphi)),\ r > 0,\ \varphi \in (0, \kappa)\}.$$

Consider heat equation:

$$\begin{cases} u_t - \Delta u = 0, & \text{on } \mathbb{R}_+ \times \mathcal{O}, \\ u(0,x) = f(x), & x \in \mathcal{O}. \end{cases} \tag{5.2.9}$$

Let Γ denote the Green kernel of the heat semigroup associated to (5.2.9). The solution to (5.2.9) is given by (see [KN14, Lemma 3.7])

$$u(t,x) = e^{t\Delta} f(x) = \int_{\mathcal{O}} \Gamma(x,y,t) f(y)\, \mathrm{d}y, \qquad (t,x) \in \mathbb{R}_+ \times \mathcal{O}.$$

In the next proposition we collect some properties of the heat semigroup $(e^{t\Delta})_{t\geq 0}$ on the wedge $\mathcal{O}$. We let $\dot{W}^{1,q}(\mathcal{O}, |\cdot|^{\theta-2})$ denote the usual homogenous Sobolev space of distributions u such that $\partial_j u \in L^q(\mathcal{O}, |\cdot|^{\theta-2})$.

Proposition 5.2.12. *Assume $\kappa \in (0, 2\pi)$, $q \in (1, \infty)$ and $\theta \in \mathbb{R}$. The following hold:*

(i) If $-\frac{\pi}{\kappa} < \frac{\theta}{q} < 2 + \frac{\pi}{\kappa}$, then $-\Delta$ is a sectorial operator on $L^q(\mathcal{O}, |\cdot|^{\theta-2})$ with $\omega(-\Delta) < \pi/2$. In particular, $(e^{t\Delta})_{t\geq 0}$ is a bounded analytic semigroup on $L^q(\mathcal{O}, |\cdot|^{\theta-2})$.

(ii) If $1 - \frac{\pi}{\kappa} < \frac{\theta}{q} < 2 + \frac{\pi}{\kappa}$, then

$$\sup_{t>0} t^{\frac{1}{2}} \|e^{t\Delta}\|_{L^q(\mathcal{O}, |\cdot|^{\theta-2}) \to \dot{W}^{1,q}(\mathcal{O}, |\cdot|^{\theta-2})} < \infty,$$
$$\sup_{t>0} t^{\frac{1}{2}} \|e^{t\Delta}\|_{L^q(\mathcal{O}, |\cdot|^{\theta-2}) \to L^q(\mathcal{O}, |\cdot|^{\theta-2-q})} < \infty.$$

Remark 5.2.13. Although $-\Delta$ is sectorial with $\omega(-\Delta) < \pi/2$ for a large range of values of θ, we do not know its domain on the full range of θ. If $2 - \frac{\pi}{\kappa} < \frac{\theta}{q} < 2 + \frac{\pi}{\kappa}$, then by [PS07, Corollary 5.2]

$$D(\Delta) = \{u : u, u/|\cdot|^2, \partial^\alpha u \in L^q(\mathcal{O}, |\cdot|^{\theta-2}) \text{ for } |\alpha| = 2\}.$$

The domain for other values of θ seems more difficult to characterize.

Proof of Proposition 5.2.12(i). First suppose that $2-\frac{\pi}{\kappa}<\frac{\theta}{q}<2+\frac{\pi}{\kappa}$. Then it follows from [PS07, Corollary 5.2] that $-\Delta$ has deterministic maximal regularity. Thus in this case (i) follows from [Dor00, Section 4]. The case $-\frac{\pi}{\kappa}<\frac{\theta}{q}<\frac{\pi}{\kappa}$ follows by a duality argument, since

$$\big(L^q(\mathcal{O},|\cdot|^{\theta-2})\big)^* = L^{q'}(\mathcal{O},|\cdot|^{\widetilde{\theta}-2})$$

with $\widetilde{\theta}=(2q-\theta)/(q-1)$. The remaining case $\frac{\pi}{\kappa}\le\frac{\theta}{q}\le 2-\frac{\pi}{\kappa}$ follows by complex interpolation (see [Tri78, Theorem 1.18.5]). □

For Proposition 5.2.12(ii) we will need the following technical lemma.

Lemma 5.2.14. *Assume $\kappa\in(0,2\pi)$, $q\in(1,\infty)$, $\sigma>0$ and $\theta\in\mathbb{R}$. For $\frac{1}{2}<\mu<\frac{\pi}{\kappa}$ and $t>0$ let $k_t:\mathbb{R}^2\to[0,\infty)$ be defined by*

$$k_t(x,y)=\zeta^{\mu-1}(t,x)\zeta^{\mu}(t,y)\,t^{-1}\exp(-\sigma|x-y|^2/t),\qquad x,y\in\mathbb{R},$$

where $\zeta(t,x)=\frac{|x|}{|x|+\sqrt{t}}$. For $1-\mu<\frac{\theta}{q}<2+\mu$ one has

$$\sup_{t>0,y\in\mathbb{R}^2}\int_{\mathbb{R}^2}k_t^j(x,y)|x|^{\frac{\theta}{q}}|y|^{2-\frac{\theta}{q}}\frac{\mathrm{d}x}{|x|^2}<\infty,$$
$$\sup_{t>0,x\in\mathbb{R}^2}\int_{\mathbb{R}^2}k_t^j(x,y)|x|^{\frac{\theta}{q}}|y|^{2-\frac{\theta}{q}}\frac{\mathrm{d}y}{|y|^2}<\infty.$$

Proof. By a substitution replacing x and y by $x\sqrt{t}$ and $y\sqrt{t}$, one can check that it suffices to consider $t=1$, and we set $k(x,y)=k_1(x,y)$. It suffices to consider $\sigma\in(0,1]$. Moreover, since $\zeta(1,x)\le\zeta(1,x/\sigma)\le\frac{1}{\sigma}\zeta(1,x)$, by a substitution one can reduce to $\sigma=1$. Let $a=\mu+2-\frac{\theta}{q}$. Then $a>0$ by the assumptions in the lemma, and a simple rewriting shows that

$$k(x,y)|x|^{\frac{\theta}{q}}|y|^{2-\frac{\theta}{q}}=\frac{|x|^{2\mu+2-1-a}}{(|x|+1)^{\mu-1}}\frac{|y|^a}{(|y|+1)^{\mu}}e^{-|x-y|^2}.$$

Step 1: First consider the integral with respect to x. One has

$$\begin{aligned}\int_{\mathbb{R}^2}k(x,y)|x|^{\frac{\theta}{q}}|y|^{2-\frac{\theta}{q}}\frac{\mathrm{d}x}{|x|^2}&=\int_{\mathbb{R}^2}\frac{|x|^{2\mu-1}}{(|x|+1)^{\mu-j}(|y|+1)^{\mu}}\Big(\frac{|y|}{|x|}\Big)^a\mathrm{e}^{-|x-y|^2}\,\mathrm{d}x\\&=S_1+S_2+S_3,\end{aligned}$$

where S_1 is the integral over $|x|\le\frac{1}{2}|y|$, S_2 is the integral over $\frac{1}{2}|y|<|x|<\frac{3}{2}|y|$ and S_3 is the integral over $|x|\ge\frac{3}{2}|y|$.

For S_1 note that $|x-y|\ge|y|-|x|\ge\frac{1}{2}|y|$. Therefore, $e^{-|x-y|^2}\le\mathrm{e}^{-\frac{1}{4}|y|^2}$ and we find

$$\begin{aligned}S_1&\le|y|^a\mathrm{e}^{-\frac{1}{4}|y|^2}\int_{|x|\le\frac{1}{2}|y|}|x|^{2\mu-1-a}(|x|+1)^{1-\mu}\,\mathrm{d}x\\&\le 2\pi(|y|+1)^{|1-\mu|+a}\mathrm{e}^{-\frac{1}{4}|y|^2}\int_0^{\frac{1}{2}|y|}r^{2\mu-a}\,\mathrm{d}r\end{aligned}$$

$$\simeq (|y|+1)^{|1-\mu|+a}|y|^{2\mu-a+1}\mathrm{e}^{-\frac{1}{4}|y|^2} \le C,$$

where we used $2\mu - a + 1 = \mu - 1 + \frac{\theta}{q} > 0$. For S_2 if $|y| \le 1$, then

$$\begin{aligned} S_2 &\le \int_{\frac{1}{2}|y|<|x|<\frac{3}{2}|y|} \Big(\frac{|x|}{|x|+1}\Big)^{\mu-1}\Big(\frac{|x|}{|y|+1}\Big)^{\mu}\,\mathrm{d}x \\ &\simeq \int_{\frac{1}{2}|y|<|x|<\frac{3}{2}|y|} |x|^{2\mu-1}\,\mathrm{d}x \simeq |y|^{2\mu} \le C, \end{aligned}$$

where we used $2\mu > 0$. If $|y| > 1$, then $S_1 \simeq \int_{\mathbb{R}^2} e^{-|x-y|^2}\,\mathrm{d}x = C$. For S_3, note that $|x-y| \ge |x| - |y| \ge \frac{1}{3}|x|$. Thus $e^{-|x-y|} \le e^{-\frac{1}{9}|x|^2}$. Now if $|y| > 1$, then

$$\begin{aligned} S_3 &\le |y|^{a-\mu}\int_{|x|>\frac{3}{2}|y|} |x|^{\mu-a}\mathrm{e}^{-\frac{1}{9}|x|^2}\,\mathrm{d}x \\ &= 2\pi|y|^{a-\mu}\int_{\frac{3}{2}|y|}^{\infty} r^{\mu-a+1}\mathrm{e}^{-\frac{1}{9}r^2}\,\mathrm{d}r \\ &= 2\pi\int_{\frac{3}{2}}^{\infty} s^{\mu-a+1}|y|^2\mathrm{e}^{-\frac{1}{9}|y|^2s^2}\,\mathrm{d}s \\ &\le 2\pi|y|^2\mathrm{e}^{-\frac{1}{36}|y|^2}\int_{\frac{3}{2}}^{\infty} s^{\mu-a+1}\mathrm{e}^{-\frac{1}{36}|s|^2}\,\mathrm{d}s \le C, \end{aligned}$$

where we used $|y|s \ge \frac{1}{2}(|y|+s)$ for $s, y > 1$. If $|y| \le 1$, then

$$\begin{aligned} S_3 &\le \int_{|x|>\frac{3}{2}|y|} |x|^{2\mu-1-a}(1+|x|)^{1-\mu}\mathrm{e}^{-\frac{1}{9}|x|^2}\,\mathrm{d}x \\ &\le 2\pi\int_0^{\infty} r^{2\mu-a}(1+r)^{|\mu-1|}\mathrm{e}^{-\frac{1}{9}r^2}\,\mathrm{d}x < \infty, \end{aligned}$$

because $2\mu - a + 1 = \mu - 1 + \frac{\theta}{q} > 0$.

Step 2: Next consider the integral with respect to y. One has

$$\begin{aligned} \int_{\mathbb{R}^2} k(x,y)|x|^{\frac{\theta}{q}}|y|^{2-\frac{\theta}{q}}\frac{\mathrm{d}y}{|y|^2} &= \int_{\mathbb{R}^2} \frac{|x|^{2\mu-1}}{(|x|+1)^{\mu-1}(|y|+1)^{\mu}}\Big(\frac{|y|}{|x|}\Big)^{a-2}\mathrm{e}^{-|x-y|^2}\,\mathrm{d}y \\ &= T_1 + T_2 + T_3, \end{aligned}$$

where T_1 is the integral over $|y| \le \frac{1}{2}|x|$, T_2 is the integral over $\frac{1}{2}|x| < |y| < \frac{3}{2}|x|$ and T_3 is the integral over $|y| \ge \frac{3}{2}|x|$.

For T_1 note that $|x-y| \ge |x| - |y| \ge \frac{1}{2}|x|$. Therefore, $e^{-|x-y|^2} \le e^{-\frac{1}{4}|x|^2}$ and we find

$$\begin{aligned} T_1 &\le |x|^{2\mu+1-a}(|x|+1)^{|\mu-1|}\mathrm{e}^{-\frac{1}{4}|x|^2}\int_{|y|\le\frac{1}{2}|x|} |y|^{a-2}\,\mathrm{d}y \\ &= 2\pi|x|^{2\mu+1-a}(|x|+1)^{|\mu-1|}\mathrm{e}^{-\frac{1}{4}|x|^2}\int_0^{\frac{1}{2}|x|} r^{a-1}\,\mathrm{d}r \\ &\simeq |x|^{2\mu+1}(|x|+1)^{|\mu-1|}\mathrm{e}^{-\frac{1}{4}|x|^2} \le C, \end{aligned}$$

where we used $a>0$ and $2\mu+1>0$. For T_2 if $|x|\le 1$ we can write

$$T_2 \lesssim \Big(\frac{|x|}{|x|+1}\Big)^{2\mu-1}\int_{\frac{1}{2}|x|<|y|<\frac{3}{2}|x|} \mathrm{e}^{-|x-y|^2}\,\mathrm{d}y \lesssim |x|^{2\mu+1}\le C.$$

where we used $2\mu-1>0$. If $|x|\ge 1$, then

$$T_2 \lesssim \int_{\mathbb{R}^2} e^{-|x-y|^2}\,\mathrm{d}y = C.$$

For T_3, note that $|x-y|\ge |y|-|x|\ge \frac{1}{3}|y|$. Thus $e^{-|x-y|}\le e^{-\frac{1}{9}|y|^2}$. If $|x|>1$ we can write

$$\begin{aligned} T_3 &\lesssim \int_{|y|>\frac{3}{2}|x|}\Big(\frac{|y|}{|x|}\Big)^{a-2-\mu} e^{-\frac{1}{9}|y|^2}\,\mathrm{d}y \\ &= 2\pi\int_{\frac{3}{2}|x|}^{\infty}\Big(\frac{r}{|x|}\Big)^{a-2-\mu} e^{-\frac{1}{9}r^2} r\,\mathrm{d}r \\ &= 2\pi\int_{\frac{3}{2}}^{\infty} s^{a-1-\mu}|x|^2\mathrm{e}^{-\frac{1}{9}|x|^2 s^2}\,\mathrm{d}s \\ &\le 2\pi|x|^2 e^{-\frac{1}{36}|x|^2}\int_{\frac{3}{2}}^{\infty} s^{a-1-\mu}\mathrm{e}^{-\frac{1}{36}s^2}\,\mathrm{d}s \le C. \end{aligned}$$

If $|x|\le 1$, then since $2\mu-a+1\ge 0$,

$$T_3 \lesssim \int_{|y|>\frac{3}{2}|x|} |y|^{a-2}\mathrm{e}^{-\frac{1}{9}|y|^2}\,\mathrm{d}y \le 2\pi\int_0^{\infty} r^{a-1}e^{-\frac{1}{9}r^2}\,\mathrm{d}r<\infty.$$

This finishes the proof. □

Proof of Proposition 5.2.12(ii). Let $\frac{1}{2}<\mu<\frac{\pi}{\kappa}$ be such that $1-\mu<\frac{\theta}{q}<2+\mu$. We use the following estimates for Γ (see [KN14, Theorem 3.10]):

$$|\partial_x^\alpha \Gamma(x,y,t)| \le C\zeta^{\mu-|\alpha|}(t,x)\zeta^{\mu}(t,y)\,t^{-\frac{2+|\alpha|}{2}}\exp\Big(-\frac{\sigma|x-y|^2}{t}\Big), \qquad |\alpha|\le 1$$

where $\zeta(t,x)=\frac{|x|}{|x|+\sqrt{t}}$ and $\sigma>0$. Therefore it suffices to prove for $f\in L^q(\mathcal{O},|\cdot|^{\theta-2})$

$$\sup_{t\in\mathbb{R}_+}\Big\|x\mapsto\int_{\mathcal{O}} k_t(x,y)f(y)\,\mathrm{d}y\Big\|_{L^q(\mathcal{O},|\cdot|^{\theta-2})} \le C\,\|f\|_{L^q(\mathcal{O},|\cdot|^{\theta-2})},$$

where $k_t(x,y)$ is either

$$k_t(x,y)=\zeta^{\mu-1}(t,x)\zeta^{\mu}(t,y)\,t^{-1}\exp\Big(-\frac{\sigma|x-y|^2}{t}\Big), \tag{5.2.10}$$

or

$$k_t(x,y)=\zeta^{\mu}(t,x)\zeta^{\mu}(t,y)|x|^{-1}\,t^{-1/2}\exp\Big(-\frac{\sigma|x-y|^2}{t}\Big). \tag{5.2.11}$$

where (5.2.10) and (5.2.11) correspond to the bound in $\dot{W}^{1,q}(\mathcal{O},|x|^{\theta-2})$ and $L^q(\mathcal{O},|x|^{\theta-2-q})$ respectively. Since (5.2.11)≤(5.2.10) it suffices to prove the boundedness for the case (5.2.10). A simple rewriting shows that it is enough to prove for $g \in L^q(\mathcal{O},|\cdot|^{-2})$

$$\Big\| x \mapsto \int_{\mathcal{O}} k_t(x,y)|x|^{\frac{\theta}{q}}|y|^{2-\frac{\theta}{q}} g(y)\frac{\mathrm{d}y}{|y|^2}\Big\|_{L^q(\mathcal{O},|\cdot|^{-2})} \leq C\|g\|_{L^q(\mathcal{O},|\cdot|^{-2})}.$$

To prove the latter by Schur's lemma it suffices to show

$$\sup_{t>0,y\in\mathbb{R}^2}\int_{\mathcal{O}} k_t(x,y)|x|^{\frac{\theta}{q}}|y|^{2-\frac{\theta}{q}}\frac{\mathrm{d}x}{|x|^2}<\infty,$$
$$\sup_{t>0,x\in\mathbb{R}^2}\int_{\mathcal{O}} k_t(x,y)|x|^{\frac{\theta}{q}}|y|^{2-\frac{\theta}{q}}\frac{\mathrm{d}y}{|y|^2}<\infty,$$

which follows from Lemma 5.2.14. □

We are now ready to study the stochastic heat equation on the wedge $\mathcal{O}$.

Example 5.2.15. On the wedge $\mathcal{O}$ consider the stochastic heat equation:

$$\begin{cases} du-\Delta u\,\mathrm{d}t = G\,\mathrm{d}W_H, \quad \text{on } \mathbb{R}_+,\\ u(0)=0. \end{cases} \tag{5.2.12}$$

Let $q\in[2,\infty)$ and assume θ is such that

$$(1-\frac{\pi}{\kappa})q<\theta<(1+\frac{\pi}{\kappa})q.$$

Then for all $p\in(2,\infty)$ and $w\in A_{p/2}$ (the case $p=q=2$ and $w\equiv 1$ is allowed as well) the mild solution u to (5.2.12) satisfies

$$\begin{aligned} \|u\|_{L^p(\Omega\times\mathbb{R}_+,w;\dot{W}^{1,q}(\mathcal{O},|\cdot|^{\theta-2}))} &\leq C\|G\|_{L^p(\Omega\times\mathbb{R}_+,w;L^q(\mathcal{O},|\cdot|^{\theta-2};H))}\\ \|u\|_{L^p(\Omega\times\mathbb{R}_+,w;L^q(\mathcal{O},|\cdot|^{\theta-2-q}))} &\leq C\|G\|_{L^p(\Omega\times\mathbb{R}_+,w;L^q(\mathcal{O},|\cdot|^{\theta-2};H))}, \end{aligned} \tag{5.2.13}$$

where C is an increasing function of $[w]_{A_{p/2}}$ and depends on p,q,θ,κ.

Proof. In [CKLL18] (5.2.13) was proved for $p=q$ and $w=1$, where it was stated for bounded intervals $(0,T)$. Since it holds with T-independent constants one can let $T\to\infty$ to find the result on $\mathbb{R}_+$. In order to prove the result for $p\neq q$ and $w\in A_p(\mathbb{R}_+)$ we will use Theorem 5.2.1 with

$$\begin{aligned} X &:= L^q(\mathcal{O},|\cdot|^{\theta-2}))\\ Y &:= \dot{W}^{1,q}(\mathcal{O},|\cdot|^{\theta-2})\cap L^q(\mathcal{O},|\cdot|^{\theta-2-q}). \end{aligned}$$

By Proposition 5.2.12 $-\Delta$ is sectorial on X with $\omega(-\Delta)<\pi/2$ and $\|e^{t\Delta}\|_{\mathcal{L}(X,Y)}\leq Ct^{-1/2}$ for $t>0$, so that Y is allowed in Theorem 5.2.1 (see Remark 5.2.2(iv)), and hence the result follows. □

5.3. NON-AUTONOMOUS CASE WITH TIME-DEPENDENT DOMAINS

We now turn to the time-dependent case. We will prove extrapolation results under the conditions introduced by Acquistapace and Terreni [AT87] (see also [Acq88, AT92, Ama95, Sch04, Tan97] and references therein). In the deterministic case extrapolation of maximal L^p-regularity was proved in [CF14, CK18] under the Acquistapace–Terreni conditions and the Kato–Tanabe conditions. Here the authors consider maximal L^p-regularity on $\mathbb{R}$ and $\mathbb{R}_+$ respectively. Below we will consider maximal regularity results on finite intervals $(0,T)$ for $T \in (0,\infty)$ in order to avoid exponential stability assumptions.

Next we introduce the (AT)-conditions due to Acquistapace and Terreni on a family of closed operators $(A(t))_{t\in[0,T]}$ on a Banach space X. Let us write $A_\rho(t) = A(t) + \rho$ for $\rho \geq 0$. We start with a uniform sectoriality condition:

(AT1) There exists a $\sigma \in (0,\pi/2)$, $\rho \geq 0$ and $M > 0$ such that for every $t \in [0,T]$, one has $\sigma(A_\rho(t)) \subseteq \Sigma_\sigma$ and

$$\|R(\lambda, A_\rho(t))\| \leq \frac{M}{|\lambda|+1}, \quad \lambda \in \mathbb{C}\setminus\Sigma_\sigma.$$

The next condition is a Hölder continuity assumption, which depends on the change of the domains $D(A(t))$.

(AT2) There exist $0 < \mu, \nu \leq 1$ with $\mu + \nu > 1$ and $M \geq 0$ such that for all $s,t \in [0,\infty)$ and $\lambda \in \mathbb{C}\setminus\Sigma_\sigma$,

$$|\lambda|^\nu \left\| A_\rho(t)R(\lambda, A_\rho(t))(A_\rho(t)^{-1} - A_\rho(s)^{-1})\right\|_{\mathcal{L}(X)} \leq M|t-s|^\mu.$$

When $(A(t))_{t\in[0,T]}$ satisfies both (AT1) and (AT2) we say that it satisfies (AT).

If the domains $D(A(t))$ all equal a fixed Banach space X_1 and

$$\|A(t) - A(s)\|_{\mathcal{L}(X_1,X)} \leq C|t-s|^\mu$$

for some $\mu > 0$, then $(A(t))_{t\in[0,T]}$ satisfies (AT2) with $\nu = 1$. Indeed, this follows directly from the equation $A_\rho(t)^{-1} - A_\rho(s)^{-1} = A_\rho(t)^{-1}(A_\rho(s) - A_\rho(t))A_\rho(s)^{-1}$.

The following generation result is due to Acquistapace and Terreni (see [Acq88, AT92, Sch04] for details).

Proposition 5.3.1 (Evolution family)**.** *Assume (AT) for $(A(t))_{t\in[0,T]}$. There exists a unique strongly continuous map*

$$S\colon \{(t,s) \in [0,T]^2 :\ t \geq s\} \to \mathcal{L}(X)$$

such that

$$S(t,t) = I, \qquad\qquad t \in [0,T],$$

$$S(t,s)S(s,r) = S(t,r), \qquad t \geq s \geq r \geq 0,$$
$$\tfrac{d}{dt}S(t,s) = A(t)S(t,s), \qquad t > s \geq 0.$$

Moreover for all $0 \leq \alpha \leq 1$ *there exists a constant* $C > 0$ *such that*

$$\|A_\rho(t)^\alpha S(t,s)\|_{\mathcal{L}(X)} \leq C(t-s)^{-\alpha}, \qquad t \geq s \geq 0.$$

Given S as in Proposition 5.3.1, we call $(S(t,s))_{t\geq s}$ the evolution family generated by $(A(t))_{t\in[0,T]}$. In order to state our extrapolation result we will need some notation. For $0 < \alpha \leq 1$ and $t \in \mathbb{R}$ define

$$X^t_\alpha := D(A_\rho(t)^\alpha).$$

endowed with the graph norm. Moreover set $X^t_0 = \overline{D(A_\rho(t))}^{\|\cdot\|_X}$. Note that since $-\rho \in \rho(A(t))$ we have

$$\|x\|_{X^t_\alpha} \leq C\|A_\rho(t)^\alpha x\|_X, \qquad x \in D(A_\rho(t)^\alpha). \tag{5.3.1}$$

Lemma 5.3.2. *Let* $0 < \alpha \leq 1$. *Let* $(\widetilde{X}_\beta)_{\beta\in[0,\alpha]}$ *be an interpolation scale and assume for* $\beta \in [0,\alpha]$ *one has* $X^t_\beta \hookrightarrow \widetilde{X}_\beta$ *uniformly in* $t \in \mathbb{R}$. *Then*

$$\|S(t,s) - I\|_{\mathcal{L}(X^s_\alpha, \widetilde{X}_\beta)} \leq C(t-s)^{\alpha-\beta}, \qquad t \geq s \geq 0.$$

Proof. The result for $\beta = \alpha$ is clear from the assumption and [Sch04, (2.19)]. For $\beta = 0$, the result follows from [Sch04, (2.16)]. The result for $0 < \alpha < \beta$ follows by interpolation. □

We can now prove our extrapolation theorem for $(A(t))_{t\in[0,T]}$ in the setting of Acquistapace and Terreni:

Theorem 5.3.3 (Extrapolation in the evolution family case)**.** *Let* $\alpha \in (\frac{1}{2}, 1]$ *and let* $(\widetilde{X}_\beta)_{\beta\in[0,\alpha]}$ *be an interpolation scale. Assume the following conditions:*

- *Both* $(A(t))_{t\in[0,T]}$ *and* $(A(t)^*)_{t\in[0,T]}$ *satisfy (AT).*
- *For* $\beta \in [0,\alpha]$ *one has* $X^t_\beta \hookrightarrow \widetilde{X}_\beta$ *uniformly in* $t \in [0,T]$.
- $\widetilde{X}_{\frac{1}{2}}$ *is a UMD Banach space with type* 2

Suppose $A \in \mathrm{SMR}(L^p(0,T;\widetilde{X}_{\frac{1}{2}}))$ *for some* $p \in [2,\infty)$. *Then for all* $q \in (2,\infty)$ *and* $w \in A_{q/2}$ *one has* $A \in \mathrm{SMR}(L^q((0,T),w;\widetilde{X}_{\frac{1}{2}}))$.

Proof. Set $Y := \widetilde{X}_{\frac{1}{2}}$ and let $K\colon [0,T]^2 \to \mathcal{L}(X,Y)$ be the kernel given by

$$K(t,s) = S(t,s)\mathbf{1}_{t\geq s}.$$

Then by our assumptions, Proposition 5.3.1 and Propositions 4.2.3 and 4.2.5 we know that $K \in \mathcal{K}_\gamma(L^p(0,T))$. Therefore by Theorem 4.4.11 it suffices to check the 2-standard kernel conditions for K.

To do so take $t > s$ and note that by Proposition 5.3.1 and (5.3.1) for $0 \le s < t \le T$,

$$\|K(t,s)\|_{\mathcal{L}(X,Y)} \le C\|A_\rho(t)^{\frac{1}{2}} S(t,s)\|_{\mathcal{L}(X)} \le C(t-s)^{-1/2}.$$

We first check (4.3.5) on $[0,T]$. By [AT92, Theorem 6.4] we have for $0 \le s < t \le T$ that

$$\|S(t,s)A(s)\|_{\mathcal{L}(X)} \le C(t-s)^{-1}. \tag{5.3.2}$$

Therefore, using Proposition 5.3.1, we have

$$\begin{aligned}
\big\|\tfrac{d}{ds}K(t,s)\big\|_{\mathcal{L}(X,Y)} &= \|S(t,s)A(s)\|_{\mathcal{L}(X,Y)} \\
&\le C\|A(t)^{1/2}S(t,s)A(s)\|_{\mathcal{L}(X)} \\
&\le C\|A(s)^{1/2}S(s,\tfrac{s+t}{2})\|_{\mathcal{L}(X)}\|S(\tfrac{s+t}{2},t)A(t)\|_{\mathcal{L}(X)} \\
&\le C(s-t)^{-\frac{3}{2}}.
\end{aligned}$$

As in the proof of Lemma 4.3.3 we obtain that (4.3.5) holds with $\omega(r) = Cr$.

To check (4.3.6) on $[0,T]$ let $\alpha \in (\frac{1}{2}, 1]$ be such that the conclusion of Lemma 5.3.2 holds and take $|t-t'| \le \frac{1}{2}|t-s|$. If $t < s$, then also $t' < s$ and there is nothing to prove. Thus it suffices to consider the case $t, t' > s$. If $t' > t$, then

$$\begin{aligned}
\|K(t',s) - K(t,s)\|_{\mathcal{L}(X,Y)} &= \|K(t',t) - I\|_{\mathcal{L}(X_\alpha^t,Y)}\|K(t,s)\|_{\mathcal{L}(X,X_\alpha^t)} \\
&\le C(t'-t)^{\alpha-\frac{1}{2}}(t-s)^{-\alpha} \\
&= C\Big|\frac{t-t'}{t-s}\Big|^{\alpha-\frac{1}{2}}|t-s|^{-1/2},
\end{aligned} \tag{5.3.3}$$

where we used Lemma 5.3.2 and Proposition 5.3.1. In the case $t > t'$ the same estimate holds with t and t' interchanged. Since $t' - s \ge \frac{1}{2}(t-s)$, (4.3.6) also follows in this case. We can therefore conclude that K is an 2-standard kernel with $\epsilon = \alpha - \frac{1}{2}$, which finishes the proof. □

Remark 5.3.4. If X and $\widetilde{X}_{\frac{1}{2}}$ are Hilbert spaces, the assumption that $A \in \mathrm{SMR}(L^2(0,T;\widetilde{X}_{\frac{1}{2}}))$ in Theorem 5.3.3 can be checked by showing

$$\|t \mapsto S(t,s)x\mathbf{1}_{t>s}\|_{L^2(0,T;\widetilde{X}_{\frac{1}{2}})} \lesssim \|x\|_X, \qquad x \in X, \quad s \in [0,T],$$

using Proposition 4.2.10(i). By the proof of [Ver10, Theorem 4.3] it is therefore sufficient to check

$$\|t \mapsto A_\rho(s)^{\frac{1}{2}} e^{tA(s)}x\|_{L^2((0,T);X)} \lesssim \|x\|, \qquad s \in [0,T], \quad x \in X.$$

5.3.1. STOCHASTIC HEAT EQUATION ON A DOMAIN WITH TIME-DEPENDENT NEUMANN BOUNDARY CONDITION

As an application of the abstract extrapolation theory under Acquistapace-Terreni conditions, we deduce stochastic maximal L^p-regularity for an operator family which was previously considered in [Acq88, Sch04, Yag91] in the deterministic setting and in [SV03] and [Ver10, Example 8.2] in the stochastic setting. In particular, stochastic maximal $L^2(L^2)$-regularity was derived in the latter. Below we extend this to an $L^p(L^q)$-setting.

Example 5.3.5. Let $\epsilon \in (0, \frac{1}{2})$ and $T \in (0,\infty)$. On a smooth bounded domain $D \subseteq \mathbb{R}^d$ consider

$$\begin{cases} du + Au\,dt = G\,dW_H, & \text{on } [0,T]\times D, \\ Cu = 0 & \text{on } [0,T]\times \partial D, \\ u(0) = 0. \end{cases} \tag{5.3.4}$$

Here the differential operator A and boundary operator are given by

$$A(t,x)u = -\sum_{i,j=1}^{d} \partial_i a_{ij}(t,x)\partial_j u,$$
$$C(t,x)u = \sum_{ij=1}^{d} a_{ij}(t,x) n_i(x) \partial_j u,$$

where for $x \in \partial D$, $n(x) \in \mathbb{R}^d$ denotes the outer normal of D. Assume that the coefficients (a_{ij}) are real-valued, symmetric and suppose that there exists a $\kappa > 0$ such that

$$\sum_{i,j=1}^{n} a_{ij}(t,x)\xi_i\xi_j \geq \kappa|\xi|^2, \qquad x \in D,\ t \in [0,T],\ \xi \in \mathbb{R}^d.$$

(i) If D is a bounded C^2-domain and

$$\begin{aligned} a_{ij} &\in C^{1/2+\epsilon}([0,T]; C(\overline{D})), \\ a_{ij}(t,\cdot) &\in C^1(\overline{D}), && t \in [0,T] \\ \partial_k a_{ij} &\in C([0,T]\times\overline{D}) && t \in [0,T] \end{aligned}$$

for all $i,j,k \in \{1,\dots,d\}$, then for all $p \in (2,\infty)$ and $\nu \in A_{p/2}(0,T)$ (where $p=2$ and $\nu = 1$ is allowed as well) the mild solution u to (5.3.4) satisfies

$$\|u\|_{L^p(\Omega\times(0,T),\nu;W^{1,2}(D))} \leq C\|G\|_{L^p(\Omega\times(0,T),\nu;L^2(D;H))},$$

where C does not depend on G.

(ii) If D is a bounded $C^{3+\epsilon}$-domain and

$$a_{ij} \in C^{1+\frac{\epsilon}{2},2+\epsilon}([0,T]\times\overline{D}),$$

for all $i,j \in \{1,\dots,d\}$, then for all $p \in (2,\infty)$, $q,r \in (1,\infty)$, $\nu \in A_{p/2}(0,T)$ and $w \in A_q(D)$ the mild solution u to (5.3.4) satisfies

$$\begin{aligned} \|u\|_{L^r(\Omega;L^p((0,T),\nu;W^{1,q}(D,w)))} &\leq C\|G\|_{L^p(\Omega;L^p((0,T),\nu;L^q(D,w;H)))}, && q > 2 \\ \|u\|_{L^r(\Omega;W^{1,q}(D,w;L^p((0,T),\nu)))} &\leq C\|G\|_{L^r(\Omega;L^q(D,w;L^p((0,T),\nu;H)))}, \end{aligned}$$

where C does not depend on G.

Example 5.3.5(i) for $p = q = 2$ and $w \equiv 1$ has been shown in [Ver10, Example 8.2]. Using Theorem 5.3.3 we will extrapolate this to the case $p > 2$ and $w \in A_{p/2}$. In order to also treat the case $q \neq 2$ in Example 5.3.5(ii), we will to check the assumptions of Theorem 4.1.3. To check these assumptions we will use the kernel estimates in [EI70, Theorem 1.1], which requires more smoothness on the domain and the coefficients. This explains the difference in assumptions between (i) and (ii) in Example 5.3.5.

Proof of Example 5.3.5. For (i) note that in [Acq88, Sch04, Yag91] it is shown that the realization of $(A(t))_{t\in[0,T]}$ on $L^2(D)$ with domain

$$D(A(t)) := \{u \in W^{2,2}(D) : \mathrm{tr}_{\partial D}(C(t,\cdot)u) = 0\}$$

satisfies (AT). Let $\widetilde{X}_\beta = W^{2\beta,2}(D)$ for $\beta \in (0,1]$ and $\widetilde{X}_0 = L^2(D)$. Then $X^t_\beta \hookrightarrow \widetilde{X}_\beta$ for all $\beta \in [0,1]$ (see [Sch04, Example 2.8]) and we have $A \in \mathrm{SMR}(L^2(0,T;\widetilde{X}_{\frac{1}{2}}))$ by [Ver10, Example 8.2]. Therefore the result follows from Theorem 5.3.3.

For (ii) we will use Theorem 4.1.3. For this let Γ denote the Green kernel of the evolution family associated to the realization of A on $L^2(D)$, which exists by [EI70, Theorem 1.1]. Then the mild solution u to (5.3.4) is given by

$$u(t,x) = \int_0^t \int_D \Gamma(t,s,x,y) G(s,y)\,\mathrm{d}y\,\mathrm{d}W_H(s), \qquad (t,x) \in (0,T)\times D,$$

For $|\alpha| \leq 1$ define

$$K_\alpha((t,x),(s,y)) = \partial_x^\alpha \Gamma(t,s,x,y)\,\mathbf{1}_{t>s}, \qquad t,s \in (0,T), \quad x,y \in D.$$

By [Ver10, Example 8.2] we have that the operators

$$T_\alpha \colon L^2_{\mathscr{F}}(\Omega\times\mathbb{R}_+\times D;H)) \to L^2(\Omega\times\mathbb{R}_+\times D)$$

given by

$$T_\alpha G(t,x) := \int_0^t \int_D K_\alpha((t,x),(s,y)) G(s,y)\,\mathrm{d}y\,\mathrm{d}W_H(s), \quad (t,x) \in \mathbb{R}_+\times D,$$

are bounded for all $|\alpha| \leq 1$.

By [EI70, Theorem 1.1] we have for all $t > s$ and $x, y \in D$

$$\begin{aligned} \big|\partial_x^\beta K_\alpha((t,x),(s,y))\big| &\lesssim \frac{1}{(t-s)^{(|\alpha|+|\beta|+d)/2}} \exp\Big(-c\frac{|x-y|}{(t-s)^{1/2}}\Big), \qquad |\beta| = 1, \\ \big|\partial_t K_\alpha((t,x),(s,y))\big| &\lesssim \frac{1}{(t-s)^{(d+3)/2}} \exp\Big(-c\frac{|x-y|}{(t-s)^{1/2}}\Big). \end{aligned} \tag{5.3.5}$$

for some $c > 0$. Define $\boldsymbol{a} = (2,1,\dots,1)$ and let $|\cdot|_{\boldsymbol{a}}$ be the anisotropic distance on $\mathbb{R}\times\mathbb{R}^d$ as in Example 2.1.2. Using the uniform boundedness of re^{-r} for $r > 0$ and the boundedness of D in case $|\alpha| = 0$, we have for $t > s$ and $x, y \in D$

$$\big|\partial_x^\beta K_\alpha((t,x),(s,y))\big| \leq C\frac{1}{|(t,x)-(s,y)|_{\boldsymbol{a}}^{d+2}}, \qquad |\beta| = 1,$$

$$\big|\partial_t K_\alpha((t,x),(s,y))\big| \le C\,\frac{1}{|(t,x)-(s,y)|_{\boldsymbol{a}}^{d+3}}.$$

Take $|(t-t',x-x')|_{\boldsymbol{a}} \le \frac{1}{2}|(t-s,x-y)|_{\boldsymbol{a}}$ and first suppose that $t,t'>s$. Then, arguing as in the proof of Lemma 4.3.3 (see also Remark 4.3.4), we deduce that K_α satisfies

$$\big|K_\alpha((t,x),(s,y)) - K_\alpha((t',x'),(s,y))\big| \lesssim \frac{|(t-t',x-x')|_{\boldsymbol{a}}}{|(t-s,x-y)|_{\boldsymbol{a}}^{d+2}}$$

Next let us consider the case $t>s>t'$. Then $t-t'>t-s$ and thus also $\frac{1}{2}|x-y|^2 \ge t-t'$. Therefore using (5.3.5) with $\beta=0$ we have the estimate

$$\begin{aligned}\big|K_\alpha((t,x),(s,y)) - K_\alpha((t',x'),(s,y))\big| &\lesssim \frac{(t-s)^{1/2}}{(t-s)^{(d+2)/2}}\exp\Big(-c\frac{|x-y|}{(t-s)^{1/2}}\Big)\\ &\lesssim \frac{|(t-t',x-x')|_{\boldsymbol{a}}}{|t-s,x-y|_{\boldsymbol{a}}^{d+2}}.\end{aligned}$$

The case $t'>s>t$ follows analogously and the case $s>t,t'$ is trivial, so K_α satisfies (4.3.6) with $\omega(t)=Ct$. The smoothness assumption (4.3.5) follows similarly by considering the adjoint problem. Therefore K_α is a $(2,1)$-standard kernel on the product space of homogeneous type $(0,T)\times D$ with the metric $|\cdot-\cdot|_{\boldsymbol{a}}$ and the Lebesgue measure. We have thus checked the assumptions of Theorem 4.1.3 for T_α for all $|\alpha|\le 1$, which immediately implies the conclusion of (ii). □

5.4. VOLTERRA EQUATIONS

In [DL13] the results of [NVW12b] have been extended to the setting of integral equations:

$$U(t) + A\int_0^t \frac{1}{\Gamma(\alpha)}(t-s)^{\alpha-1}U(s)\,\mathrm{d}s = \int_0^t \frac{1}{\Gamma(\beta)}(t-s)^{\beta-1}U(s)\,\mathrm{d}W_H(s),$$

where $\alpha\in(0,2)$ and, $\beta\in(\frac{1}{2},2)$. The solution U is given by

$$U(t) = \int_0^t S_{\alpha\beta}(t-s)G(s)\,\mathrm{d}W_H(s), \qquad t\in\mathbb{R}_+,$$

where $S_{\alpha\beta}$ is the so-called resolvent associated with A, α and β. The maximal regularity result in [DL13, Theorem 3.1] gives L^p-estimates for $A^\theta\partial_t^\eta U$ in terms of G, where $\beta-\alpha\theta-\eta=\frac{1}{2}$ with $\theta\in[0,1)$ and $\eta\in(-1,1)$. In this case one has to estimate a stochastic convolution with kernel $k(t)=A^\theta\partial_t^\eta S_{\alpha\beta}(t)$. We will not go into details on Volterra equations further now, but restrict ourselves to checking that $K(t,s):=k(t-s)\mathbf{1}_{t<s}$ is a 2-standard kernel. Consequently our extrapolation theory can be applied to this setting as well.

If $\eta\in(-\frac{1}{2},1)$ we take $\epsilon\in(0,\frac{1}{2})$ such that $\eta+\epsilon\in(0,1)$. Then there is an $M>0$ such that (see [DL13, Remark 2.4])

$$\|\partial^\epsilon k(t)\| \le M\,t^{-\epsilon-\frac{1}{2}}, \qquad t\in\mathbb{R}_+.$$

If $\eta \in (-1, -\frac{1}{2})$, we let $\epsilon = -\eta$. Then $k(t) = \partial_t^{-\epsilon} A^\theta S_{\alpha\beta}(t)$ and there is an $M > 0$ such that (see [DL13, Remark 2.4])

$$\|\partial^\epsilon k(t)\| = \|A^\theta S_{\alpha\beta}(t)\| \le M t^{-\epsilon-\frac{1}{2}}, \qquad t \in \mathbb{R}_+.$$

Thus, writing $K(t,s) = \partial_t^{-\epsilon}\partial_t^\epsilon k(t-s)\,\mathbf{1}_{t>s}$, in both cases it follows from Lemma 5.4.1 below that K is a 2-standard kernel.

Lemma 5.4.1. *Let $\Phi\colon \mathbb{R}_+ \to \mathcal{L}(X,Y)$ be strongly measurable and suppose there exists a constant $M > 0$ and an $\epsilon \in (0, \frac{1}{2})$ such that*

$$\|\Phi(s)\| \le M s^{-\frac{1}{2}-\epsilon}, \qquad s > 0.$$

Let $k\colon \mathbb{R} \to \mathcal{L}(X,Y)$ be defined by

$$k(s) := \frac{1}{\Gamma(\epsilon)} \int_0^s (s-r)^{\epsilon-1}\Phi(r)\,\mathrm{d}r \qquad s \in \mathbb{R}_+$$

Then $K(t,s) := k(t-s)\,\mathbf{1}_{t>s}$ is an $(\epsilon,2)$-standard kernel.

Proof. Let $s > 0$ and assume $t \in [s, \frac{3}{2}s]$. By a change of variables it suffices to show

$$\|k(s) - k(t)\| \le M C_\epsilon \frac{(t-s)^\epsilon}{s^\epsilon} \frac{1}{s^{1/2}}.$$

To show this note that

$$\begin{aligned}
&\Gamma(\epsilon)\,\|k(s) - k(t)\| \\
&\quad\le \int_s^t (t-r)^{\epsilon-1}\|\Phi(r)\|\,\mathrm{d}r + \int_0^s \big|(s-r)^{\epsilon-1} - (t-r)^{\epsilon-1}\big|\,\|\Phi(r)\|\,\mathrm{d}r \\
&\quad\le M \underbrace{\int_s^t (t-r)^{\epsilon-1} r^{-\epsilon-\frac{1}{2}}\,\mathrm{d}r}_{\boxed{\text{A}}} + M \underbrace{\int_0^s \big|(s-r)^{\epsilon-1} - (t-r)^{\epsilon-1}\big| r^{-\epsilon-\frac{1}{2}}\,\mathrm{d}r}_{\boxed{\text{B}}}.
\end{aligned}$$

For $\boxed{\text{A}}$ note that

$$\boxed{\text{A}} \le s^{-\epsilon-\frac{1}{2}} \int_s^t (t-r)^{\epsilon-1}\,\mathrm{d}r = \epsilon^{-1}\frac{(t-s)^\epsilon}{s^\epsilon}\frac{1}{s^{1/2}}.$$

For $\boxed{\text{B}}$ we write $\boxed{\text{B}} = \boxed{\text{B1}} + \boxed{\text{B2}}$ where we have split the integral into parts over $(0, s/2)$ and $(s/2, s)$. For $\boxed{\text{B1}}$ we can write

$$\begin{aligned}
\boxed{\text{B1}} &= \frac{1}{1-\epsilon}\int_0^{s/2}\int_{s-r}^{t-r} x^{\epsilon-2}\,\mathrm{d}x\, r^{-\epsilon-\frac{1}{2}}\,\mathrm{d}r \\
&\le \frac{1}{1-\epsilon}\int_0^{s/2} (t-s)(s-r)^{\epsilon-2} r^{-\epsilon-\frac{1}{2}}\,\mathrm{d}r
\end{aligned}$$

$$\leq \frac{(t-s)(s/2)^{\epsilon-2}}{1-\epsilon}\int_0^{s/2} r^{-\epsilon-\frac{1}{2}}\,\mathrm{d}r$$
$$\leq \frac{(t-s)(s/2)^{\epsilon-2}}{(1-\epsilon)(\frac{1}{2}-\epsilon)}(s/2)^{\frac{1}{2}-\epsilon} = \frac{2\sqrt{2}}{(1-\epsilon)(\frac{1}{2}-\epsilon)}\frac{t-s}{s}\frac{1}{s^{1/2}}$$

where we used $\epsilon < \frac{1}{2}$. Finally, using $t \geq s$, we obtain

$$\boxed{\text{B2}} \leq (s/2)^{-\epsilon-\frac{1}{2}}\int_{s/2}^{s}\big((s-r)^{\epsilon-1}-(t-r)^{\epsilon-1}\big)\,\mathrm{d}r$$
$$= \epsilon^{-1}(s/2)^{-\epsilon-\frac{1}{2}}\big((t-s)^{\epsilon}+(s/2)^{\epsilon}-(t-s/2)^{\epsilon}\big)$$
$$\leq \epsilon^{-1}2^{\epsilon+\frac{1}{2}}\frac{(t-s)^{\epsilon}}{s^{\epsilon}}\frac{1}{s^{1/2}},$$

which implies the required estimate. □

5.5. p-INDEPENDENCE OF THE $\mathcal{R}$-BOUNDEDNESS OF STOCHASTIC CONVOLUTIONS

In this final section we prove the p-independence of a Banach space property which was introduced in [NVW15b]. Let X be a Banach space with type 2. For $\lambda \in \mathbb{C}$ with $\mathrm{Re}(\lambda) > 0$ let $k_\lambda : \mathbb{R}_+ \to \mathbb{C}$ be given by

$$k_\lambda(s) = \lambda^{1/2}e^{-\lambda s}, \qquad s \in \mathbb{R}_+,$$

and define $T_\lambda : L^p(\mathbb{R}_+;X) \to L^p(\mathbb{R}_+;\gamma(\mathbb{R}_+;X))$ by

$$T_\lambda f(s) = k_\lambda(s-\cdot)f(\cdot), \qquad s \in \mathbb{R}_+.$$

Then by Proposition 4.2.12

$$\|k\|_{\mathcal{K}_\gamma(L^p(\mathbb{R}_+))} \leq \tau_{2,X}\Big(\frac{|\lambda|}{2\,\mathrm{Re}(\lambda)}\Big)^{1/2}. \tag{5.5.1}$$

The following p-dependent condition was introduced in [NVW15b, NVW15c]:

(C_p) For each $\theta \in [0,\pi/2)$ the family $\mathcal{T} = \{T_\lambda : |\arg(\lambda)| \leq \theta\}$ is $\mathcal{R}$-bounded from $L^p(\mathbb{R}_+;X)$ into $L^p(\mathbb{R}_+;\gamma(\mathbb{R}_+;X))$.

Note that (5.5.1) implies that $\mathcal{T}$ is uniformly bounded. In [NVW15c] the condition (C_p) was combined with the boundedness of the H^∞-calculus in order to derive stochastic maximal L^p-regularity.

From [NVW15b, Theorems 4.7 and 7.1] it can be seen that in the following case the condition (C_p) holds for all $p \in (2,\infty)$:

- X is a 2-convex Banach function space and the dual of its concavification X^2 has the *Hardy–Littlewood property*, i.e. the lattice Hardy–Littlewood maximal operator is bounded on $L^p(\mathbb{R}^d;(X^2)^*)$ for some (all) $p \in (1,\infty)$. See Section 6.4 for an introduction to the Hardy–Littlewood property and lattice Hardy-Littlewood maximal operator.

In particular, UMD Banach function spaces have the Hardy–Littlewood property, but also e.g. L^∞. In particular, the space L^q satisfies (C_p) for any $q \in [2,\infty)$ and $p \in (2,\infty)$. In the case $q = 2$ one can additionally allow $p = 2$. On the other hand, L^q for $q > 2$ fails (C_2) (see [NVW12b, Theorem 6.1] and the proof of [NVW15c, Theorem 7.1]). A Banach function space with UMD and type 2 for which we do not know whether (C_p) holds for $p \in (2,\infty)$ is for instance $\ell^2(\ell^4)$. Some evidence against this can be found in [NVW15b, Theorem 8.2].

It was an open problem whether (C_p) is p-independent. Below we settle this issue. In the special case of Banach function spaces one could also derive this by rewriting (C_p) as a square function result (cf. [NVW15b, Theorem 7.1]) and using operator-valued Calderón–Zygmund theory (see Section 3.4).

Theorem 5.5.1. *Let X be Banach space with type 2 and let $p \in [2,\infty)$. If (C_p) holds, then for all $\theta \in [0,\pi/2)$, $q \in (2,\infty)$ and $w \in A_{q/2}(\mathbb{R}_+)$ the family*

$$\mathcal{T} = \{T_\lambda : |\arg(\lambda)| \le \theta\}$$

is $\mathcal{R}$-bounded from $L^q(\mathbb{R}_+, w; X)$ into $L^q(\mathbb{R}_+, w; \gamma(\mathbb{R}_+; X))$. In particular (C_q) holds for all $q \in (2,\infty)$.

Proof. Fix $n \in \mathbb{N}$. Let $\lambda_1, \dots, \lambda_n \in \Sigma_\theta$, $f_1, \dots, f_n \in L^q(\mathbb{R}_+, w; \gamma(\mathbb{R}_+; X))$. Let $\mathrm{Rad}_n(X)$ be the space X^n endowed with the norm

$$\|(x_j)_{j=1}^n\|_{\mathrm{Rad}_n(X)} := \Big\| \sum_{j=1}^n \varepsilon_j x_j \Big\|_{L^2(\Omega;X)},$$

where $(\varepsilon_j)_{j=1}^n$ is a Rademacher sequence. Replacing the $L^2(\Omega;X)$-norm by $L^r(\Omega;X)$ with $r \in [1,\infty)$ leads to an equivalent norm by the Kahane–Khintchine inequalities (see [HNVW17, Theorem 6.2.4]). Define a diagonal operator $k : \mathbb{R}_+ \to \mathcal{L}(\mathrm{Rad}_n(X))$ by

$$(k(s)x)_j = k_{\lambda_j}(s) x_j, \qquad j \in \{1,\dots,n\}, \quad x \in \mathrm{Rad}_n(X),$$

and set $K(s,t) := k(s-t)\mathbf{1}_{s>t}$. To prove the required $\mathcal{R}$-boundedness of $\mathcal{T}$, by the Kahane–Khintchine inequalities, Fubini's theorem and Proposition 2.8.6 it suffices to prove that $\|K\|_{\mathcal{K}_\gamma(L^q(\mathbb{R}_+,w))} \le C$, where C is independent of n. Now by (C_p) we know the latter is true for $w = 1$ and $q = p$. Therefore, by Theorem 4.4.11, it suffices to check that K satisfies the 2-standard kernel condition with constants only depending on θ. For this we check the condition of Lemma 4.3.3. Moreover, since K is of convolution type it suffices to check that $\|k'(s)\| \le Cs^{-3/2}$. Since $k'(s)$ is a diagonal operator we have for $x \in \mathrm{Rad}_n(X)$:

$$\|K'(s)x\|_{\mathrm{Rad}_n(X)} = \Big\| \sum_{j=1}^n \varepsilon_j k'_{\lambda_j}(s) x_j \Big\|_{L^2(\Omega;X)} \le Cs^{-3/2} \Big\| \sum_{j=1}^n \varepsilon_j x_j \Big\|_{L^2(\Omega;X)},$$

where we used the Kahane contraction principle and

$$|s^3 k'_{\lambda_j}(s)^2| \le \sup_{\lambda \in \Sigma_\theta} |\lambda|^3 e^{-2\mathrm{Re}(\lambda)} \le \frac{27}{8e^3 \cos^3(\theta)} := C^2.$$

This implies the required estimates for K and therefore finishes the proof. □

II

Banach function space-valued extensions of operators

6

BANACH FUNCTION SPACE-VALUED EXTENSIONS OF OPERATORS

This chapter is based on a selection of the results from

[3] E. Lorist and B. Nieraeth. Sparse domination implies vector-valued sparse domination. arXiv:2003.02233.

[4] N.J. Kalton, E. Lorist, and L. Weis. Euclidean structures and operator theory in Banach spaces. *To appear in Mem. Amer. Math. Soc.*, 2021.

[9] T.S. Hänninen and E. Lorist. Sparse domination for the lattice Hardy–Littlewood maximal operator. *Proc. Amer. Math. Soc.*, 147(1):271–284, 2019.

It is complemented by some results from

[11] A. Amenta, E. Lorist, and M.C. Veraar. Rescaled extrapolation for vector-valued functions. *Publ. Mat.*, 63(1):155–182, 2019.

The presentation of these results in this chapter is original. Multilinear versions of the main results in this chapter are contained in

[3] E. Lorist and B. Nieraeth. Sparse domination implies vector-valued sparse domination. arXiv:2003.02233, 2020.

[8] E. Lorist and B. Nieraeth. Vector-valued extensions of operators through multilinear limited range extrapolation. *J. Fourier Anal. Appl.*, 25(5):2608–2634, 2019.

Abstract. *In this chapter we develop two methods to extend a bounded operator on $L^p(\mathbb{R}^d)$ to a bounded operator on the Bochner space $L^p(\mathbb{R}^d;X)$, where X is a Banach function space. The first method is based on a factorization principle, which is a more flexible version of the factorization theory of Nikišin, Maurey and Rubio de Francia. The second method is based on sparse domination, which we extensively studied in Chapter 3. Using these extension theorems, we prove quantitative connections between Banach space properties like the (randomized)* UMD *property and the Hardy–Littlewood property.*

6.1. INTRODUCTION

For a bounded linear operator T on $L^p(\mathbb{R}^d)$ and a Banach space X we can define a linear operator $\widetilde{T}$ on $L^p(\mathbb{R}^d) \otimes X$ by setting

$$\widetilde{T}(f \otimes x) := Tf \otimes x, \qquad f \in L^p(\mathbb{R}^d),\, x \in X,$$

and extending by linearity. For $p \in [1,\infty)$ the space $L^p(\mathbb{R}^d) \otimes X$ is dense in the Bochner space $L^p(\mathbb{R}^d; X)$ and it thus makes sense to ask whether $\widetilde{T}$ extends to a bounded operator on $L^p(\mathbb{R}^d; X)$.

Such vector-valued extensions of operators prevalent in the theory of harmonic analysis have been actively studied in the past decades. A centerpoint of the theory is the result of Burkholder [Bur83] and Bourgain [Bou83] that the Hilbert transform on $L^p(\mathbb{R})$ extends to a bounded operator on $L^p(\mathbb{R}; X)$ if and only if the Banach space X has the UMD property (see Theorem 2.7.1). From this connection one can derive the boundedness of the vector-valued extension of many operators in harmonic analysis, like Fourier multipliers and Littlewood–Paley operators, as we have already seen in Chapter 3.

In case X is a Banach function space, very general extension theorems are known. These follow from a deep result of Bourgain [Bou84] and Rubio de Francia [Rub86] on the connection between the boundedness of the lattice Hardy–Littlewood maximal operator on $L^p(\mathbb{R}^d; X)$ and the UMD property of X. The boundedness of the lattice Hardy–Littlewood maximal operator allows one to use scalar-valued arguments to show the boundedness of the vector-valued extension of an operator. Moreover it connects the extension problem to the theory of Muckenhoupt weights. Combined this enabled Rubio de Francia to show an extension principle in [Rub86], yielding vector-valued extensions of operators on $L^p(\mathbb{T})$ satisfying weighted bounds. This result was subsequently extended by Amenta, Veraar and the author in [11], replacing $\mathbb{T}$ by $\mathbb{R}^d$ and adding weights in the conclusion.

As we saw in Chapter 3, weighted bounds for operators in harmonic analysis are nowadays often obtained through sparse domination. So to deduce the weighted boundedness of the vector-valued extension $\widetilde{T}$ of an operator T using [Rub86] and its generalization in [11], one typically goes through implications (1) and (3) in the following diagram

$$\begin{array}{ccc} \text{Sparse domination for } T & \xRightarrow{(1)} & \text{Weighted bounds for } T \\ \Big\Downarrow (2) & & (3) \Big\Downarrow \\ \text{Sparse domination for } \widetilde{T} & \xRightarrow{(4)} & \text{Weighted bounds for } \widetilde{T} \end{array}$$

In this diagram implications (1) and (4) are well-known and unrelated to the operator T, as we saw in Proposition 3.2.4. Another approach to obtain the weighted boundedness

of the vector-valued extension $\widetilde{T}$ of an operator T, through implications (2) and (4) in this diagram, was obtained by Culiuc, Di Plinio, and Ou in [CDO17] for $X = \ell^q$.

The advantage of the route through implications (2) and (4) over the route through implications (1) and (3) is that the Fubini-type techniques needed for implication (2) are a lot less technical than the ones needed for implication (3). Moreover implication (4) yields quantitative and in many cases sharp weighted estimates for $\widetilde{T}$, while the weight dependence in the arguments used for implication (3) is certainly not sharp. A downside of the approach through implications (2) and (4) is the fact that one needs sparse domination for T as a starting point, while one only needs weighted bounds in order to apply (3).

In this chapter we will provide an alternative, more flexible approach to implication (3) and extend implication (2) from ℓ^q to more general Banach function spaces X. Both implications have also been extended to the multilinear limited range setting by Nieraeth and the author in [3, 8]. To keep this dissertation at a reasonable length and to avoid the more involved notation of the multilinear limited range setting, we will stay in the linear setting in this chapter and refer to the dissertation of Nieraeth [Nie20, Part $\frac{1}{4}$] for a detailed treatment of this generalization to the multilinear limited range setting.

6.1.1. EXTENSION OF OPERATORS USING FACTORIZATION

Implication (3) for $X = \ell^q$ with $q \in (1,\infty)$ follows easily from Rubio de Francia extrapolation and Fubini's theorem. Indeed if T is a bounded linear operator on $L^p(\mathbb{R}^d, w)$ for all $w \in A_p$, we know by an application of Theorem 2.3.3 that T is a bounded linear operator on $L^q(\mathbb{R}^d, w)$ for all $w \in A_q$. Thus $\widetilde{T}$ is bounded on $L^q(\mathbb{R}^d, w; \ell^q)$ for all $w \in A_q$ by Fubini's theorem. Another application of Theorem 2.3.3 then yields boundedness of $\widetilde{T}$ on $L^p(\mathbb{R}^d, w; \ell^q)$ for all $w \in A_p$.

A generalization of this result, replacing ℓ^q by a UMD Banach function space, was first proven by Rubio de Francia in [Rub86, Theorem 5]. Extended in [11] by Amenta, Veraar and the author, this theorem reads as follows:

Theorem 6.1.1. *Let X be a* UMD *Banach function space over a measure space (Ω, μ), let $p \in (1,\infty)$ and $w \in A_p$. Let $f, g \in L^p(\mathbb{R}^d, w; X)$ and suppose that for some $p_0 \in (1,\infty)$ there is an increasing function $\phi\colon \mathbb{R}_+ \to \mathbb{R}_+$ such that for all $v \in A_{p_0}$ we have*

$$\|f(\cdot,\omega)\|_{L^{p_0}(\mathbb{R}^d,v)} \le \phi([v]_{A_{p_0}})\|g(\cdot,\omega)\|_{L^{p_0}(\mathbb{R}^d,v)}, \qquad \omega \in \Omega.$$

Then there exists an increasing $\psi\colon \mathbb{R}_+ \to \mathbb{R}_+$, depending on X, ϕ, p, p_0, d, such that

$$\|f\|_{L^p(\mathbb{R}^d,w;X)} \le \psi([w]_{A_p})\|g\|_{L^p(\mathbb{R}^d,w;X)}.$$

One obtains implication (3) as a direct corollary of Theorem 6.1.1 by taking $g \in L^p(\mathbb{R}^d, w) \otimes X$ and $f = \widetilde{T}g \in L^p(\mathbb{R}^d, w) \otimes X$. Indeed, for $g = \sum_{j=1}^m g_j \otimes x_j$ with $g_1, \cdots, g_m \in L^p(\mathbb{R}^d, w)$ and $x_1, \cdots, x_m \in X$ we have

$$\widetilde{T}g(t,\omega) = \sum_{j=1}^m Tg_m(t)x_m(\omega) = Tg(\cdot,\omega)(t), \qquad (t,\omega) \in \mathbb{R}^d \times \Omega. \tag{6.1.1}$$

Therefore we obtain by density:

Theorem 6.1.2. *Let X be a* UMD *Banach function space and let T be a bounded linear operator on $L^{p_0}(\mathbb{R}^d, v)$ for some $p_0 \in (1,\infty)$ and all $v \in A_{p_0}$. Suppose that there is an increasing function $\phi\colon \mathbb{R}_+ \to \mathbb{R}_+$ such that*

$$\|T\|_{L^{p_0}(\mathbb{R}^d,v)\to L^{p_0}(\mathbb{R}^d,v)} \le \phi([v]_{A_{p_0}}), \qquad v \in A_{p_0}.$$

Then $\widetilde{T}$ extends uniquely to a bounded linear operator on $L^p(\mathbb{R}^d, w; X)$ for all $p \in (1,\infty)$ and $w \in A_p$.

The right hand-side of (6.1.1) makes sense for any, not necessarily linear, operator T. However, to obtain an analog of Theorem 6.1.2 one needs to take care when considering $f(\cdot,\omega)$ and $\widetilde{T}f(\cdot,\omega)$ for $\omega \in \Omega$, as these are not necessarily functions in $L^{p_0}(\mathbb{R}^d, w)$ for all $w \in A_{p_0}$. This technicality can in applications be circumvented by using e.g. simple functions or smooth compactly supported functions and appealing to density. This density argument requires a little bit of structure of the operator. A sufficient condition is for example

$$|Tf - Tg| \le |T(f-g)|$$

for all $f, g\colon \mathbb{R}^d \to \mathbb{C}$ in the chosen dense subset of $L^p(\mathbb{R}^d, w)$.

Remark 6.1.3. In applications it is usually easily checked that a weighted estimate is dependent on the Muckenhoupt characteristic $[w]_{A_p}$, and not on any other information coming from w. However, checking that this dependence is nondecreasing can sometimes be tricky (see e.g. [12, Theorem 3.10]). Moreover, this monotonicity is not always explicitly stated in the literature. In Appendix 6.A we will show that the monotonicity condition in Theorem 6.1.2 redundant: an estimate depending on $[w]_{A_p}$ with no monotonicity assumption implies the same estimate with monotonicity.

The original proof of Theorem 6.1.1 is based on the factorization of ℓ^r-bounded families of operators on a r-convex Banach function space X through a weighted L^r-space. The classical approach for this factorization comes from the work of Nikišin [Nik70], Maurey [Mau73] and Rubio de Francia [Rub82, Rub86, Rub87] (see also [GR85]). This factorization is used to show that, on Banach function spaces, the UMD property is self-improving, i.e. there exists an $\epsilon > 0$ such that X^r has the UMD property for all $0 < r < 1+\epsilon$. Using a Rubio de Francia iteration argument with the lattice Hardy–Littlewood maximal operator (see Section 6.4 for an introduction) Theorem 6.1.1 then follows (see [Rub86, Theorem 5])

We will give a alternative approach to the proof of Theorem 6.1.1, which has various advantages over the original approach:

- Our approach yields quantitative bounds, allowing us to estimate the operator norm of $\widetilde{T}$ by a power of the UMD constant of X.

- The original approach relies upon the boundedness of the lattice Hardy-Littlewood maximal operator on $L^p(\mathbb{R}^d; X)$ whereas this will not be used in our approach. Therefore we will be able to use Theorem 6.1.1 to give a quantitative proof of the boundedness of the lattice Hardy-Littlewood maximal operator on UMD Banach function spaces (see Theorem 6.4.6).

- Instead of assuming X to have the UMD property, we will formulate a more abstract assumption in Theorem 6.3.1. This allows us to the deduce the UMD property of X from e.g. the ℓ^2-sectoriality of differentiation operators (see Theorem 6.3.5).

Our approach will be based on the factorization of an ℓ^2-bounded family of operators on a (not necessarily 2-convex!) Banach function space X through a weighted L^2-space. This factorization is a consequence of the abstract representation and factorization theory of Euclidean structures, which has been developed in [4] by Kalton, Weis and the author. We will not go into the details of this theory, as this could be a dissertation in itself. Instead, we will formulate the necessary results from [4] in the setting of the Euclidean structure ℓ^2, which gives rise to ℓ^2-boundedness. We will adapt the proofs of these results to this specific choice of an Euclidean structure in Section 6.2 and refer to [4, Chapter 1 and 2] for the general theory.

6.1.2. EXTENSION OF OPERATORS USING SPARSE DOMINATION

The proof of the sparse domination-based extension theorem depicted by implication (2) relies on the following two key ingredients:

- The equivalence between sparse forms and the L^1-norm of the bisublinear maximal function. This equivalence seems to have been used for the first time in [CDO17] by Culiuc, Di Plinio, and Ou.

- A sparse domination result for the bisublinear lattice maximal operator on UMD Banach function spaces.

For this second ingredient we will first study the lattice Hardy–Littlewood maximal operator in Section 6.4, which on an order-continuous Banach function space X is given by

$$M_{\mathrm{Lat}} f := \sup_B \langle |f| \rangle_{1,B} \mathbf{1}_B, \qquad f \in L^1_{\mathrm{loc}}(\mathbb{R}^d; X)$$

where the supremum is taken in the lattice sense over all balls $B \subseteq \mathbb{R}^d$. Using our abstract sparse domination theory from Chapter 3 and the previously discussed factorization-based extension principle, we will show sparse domination and sharp weighted estimates for this operator if X has the UMD property. In particular, this yields a quantitative version of the result of Bourgain [Bou84] and Rubio de Francia [Rub86] that M_{Lat} is bounded on $L^p(\mathbb{R}^d; X)$ if X has the UMD property. Moreover, the sharpness of our

sparse domination result allows us to compare the lattice Hardy–Littlewood maximal operator to the Rademacher maximal operator, which was introduced in Section 3.6.

We will introduce the bisublinear version of the lattice Hardy–Littlewood maximal operator, which we actually need to prove implication (2), in Section 6.5. This operator is part of a much more general, multilinear theory, for which we refer to [3] or the dissertation of Nieraeth [Nie20, Part $\frac{1}{4}$]. In Section 6.5 we will only treat the sparse domination result needed for the extension theorem.

Combining these two ingredients, we obtain the following theorem:

Theorem 6.1.4. *Let X be a* UMD *Banach function space over a measure space (Ω,μ) and let T be a linear operator such that for any $f,g\in L_c^\infty(\mathbb{R}^d)$ there exists a sparse collection of cubes $\mathcal{S}$ such that*

$$\int_{\mathbb{R}^d}|Tf|\cdot|g|\,\mathrm{d}t\le C_T\sum_{Q\in\mathcal{S}}\langle|f|\rangle_{1,Q}\langle|g|\rangle_{1,Q}|Q|.$$

Then for all simple functions $f\in L_c^\infty(\mathbb{R}^d,X)$ and $g\in L_c^\infty(\mathbb{R}^d)$ there exists a sparse collection of cubes $\mathcal{S}$ such that

$$\int_{\mathbb{R}^d}\|\widetilde{T}f\|_X\cdot|g|\,\mathrm{d}t\lesssim_{X,d} C_T\sum_{Q\in\mathcal{S}}\big\langle\|f\|_X\big\rangle_{1,Q}\big\langle|g|\big\rangle_{1,Q}|Q|.$$

In particular, $\widetilde{T}$ extends uniquely to a bounded linear operator on $L^p(\mathbb{R}^d,w;X)$ for all $p\in(1,\infty)$ and $w\in A_p$ with

$$\|\widetilde{T}\|_{L^p(\mathbb{R}^d,w;X)\to L^p(\mathbb{R}^d,w;X)}\lesssim_{X,p,d} C_T\,[w]^{\max\left\{\frac{1}{p-1},1\right\}}.$$

As in Theorem 6.1.2, we can allow for more general (not necessarily linear) operators T in Theorem 6.1.4 (see Theorem 6.6.1 and Corollary 6.6.2). Also note that the sparse form domination for T in the assumption of Theorem 6.1.4 is in particular satisfied if we have pointwise sparse domination for T as in Chapter 3 with $r=1$, which follows by integrating against a $g\in L_c^\infty(\mathbb{R}^d)$.

As was the case for our factorization-based extension theorem, the techniques used to prove Theorem 6.1.4 can also be used to deduce the UMD property of certain Banach spaces. In particular, in Theorem 6.6.3, we will obtain the following results:

- We will deduce that a Banach function space X has the UMD property if M_{Lat} is bounded on both $L^p(\mathbb{R}^d;X)$ and $L^{p'}(\mathbb{R}^d;X^*)$ for some $p\in(1,\infty)$. Moreover we obtain a quantitative estimate of the UMD constant $\beta_{p,X}$ in terms of the operator norm of M_{Lat} on $L^p(\mathbb{R}^d;X)$ and $L^{p'}(\mathbb{R}^d;X^*)$. Thus, combined with the previously discussed converse implication, we have an alternative proof of this equivalence by Bourgain [Bou84] and Rubio de Francia [Rub86].

- If X is a UMD Banach function space and Y is a UMD Banach space, we show that the Köthe–Bochner space $X(Y)$ has the UMD property with a quantitative estimate between the respective UMD constants. The qualitative part of this statement was first proven by Rubio de Francia [Rub86].

6.1.3. APPLICATIONS

Theorems 6.1.1 and 6.1.4 and their multilinear, limited range counterparts in [3, 8] have various interesting applications. They can for example be applied to obtain Banach function space-valued boundedness of:

- The bilinear Hilbert transform
- The variational Carleson operator
- Multilinear Calderón–Zygmund operators
- Bochner–Riesz multipliers
- Spherical maximal operators

For the details of these applications, we refer to [3, Section 6], [8, Section 5] and [11, Section 5]. In this dissertation we will focus on one specific, quite elaborate application. In Chapter 7 we will use Theorem 6.1.1 to deduce Banach function space-valued Littlewood–Paley–Rubio de Francia estimates, which we will enable us to prove Fourier multiplier theorems in Banach function spaces with UMD concavifications.

6.2. FACTORIZATION OF ℓ^r-BOUNDED FAMILIES OF OPERATORS

In this section we will prove that ℓ^r-bounded families of operators on a Banach function space X can be factored through a weighted L^r-space and specifically focus on the case $r = 2$. We will start with the factorization theory from the work of Nikišin [Nik70], Maurey [Mau73] and Rubio de Francia [Rub82, Rub86, Rub87]. We will not use this theorem in subsequent sections, but we find it instructive to first show how this classical, simpler factorization theorem works, before turning to our factorization theorem based on the abstract representation and factorization theory of Euclidean structures.

The following theorem was shown by Rubio de Francia in the following special cases:

- $X = L^p(\Omega)$ in [Rub82],
- $\Gamma = \{T\}$ for $T \in \mathcal{L}(X)$ in [Rub86, III Lemma 1],

see also [GR85]. An extensive description of the literature preceding the theorems of Rubio de Francia is given in the monograph of Gilbert [Gil79].

Theorem 6.2.1. *Take $r \in [1,\infty)$ and let X be an r-convex, order-continuous Banach function space over a measure space (Ω,μ) and let $\Gamma \subseteq \mathcal{L}(X)$ be a family of operators. The following are equivalent:*

(i) *Γ is ℓ^r-bounded.*

(ii) There exists a $C>0$ such that for every nonnegative $v\in (X^r)^*$, there is a $w\geq v$ with $\|w\|_{(X^r)^*}\leq 2\|v\|_{(X^r)^*}$ and

$$\|Tx\|_{L^r(\Omega,w)}\leq C\|x\|_{L^r(\Omega,w)}\qquad x\in X,\ T\in\Gamma. \tag{6.2.1}$$

Moreover $C>0$ can be chosen such that $2^{-1/r}C\leq\|\Gamma\|_{\ell^r}\leq 2^{1/r}C$.

Proof. We first prove (ii)⇒(i). Let $x_1,\cdots,x_n\in X$ and $T_1,\cdots,T_n\in\Gamma$. As $\sum_{k=1}^n|T_kx_k|^r\in X^r$, we can find a nonnegative $v\in(X^r)^*$ with $\|v\|_{(X^r)^*}=1$ such that

$$\Big\|\Big(\sum_{k=1}^n|T_kx_k|^r\Big)^{1/r}\Big\|_X^r=\Big\|\sum_{k=1}^n|T_kx_k|^r\Big\|_{X^r}=\int_\Omega\sum_{k=1}^n|T_kx_k|^r\,v\,\mathrm{d}\mu.$$

Then by assumption there exists a $w\geq v$ with $\|w\|_{(X^r)^*}\leq 2$ and

$$\Big(\int_\Omega\sum_{k=1}^n|T_kx_k|^r\,v\,\mathrm{d}\mu\Big)^{1/r}\leq C\Big(\int_\Omega\sum_{k=1}^n|x_k|^r\,w\,\mathrm{d}\mu\Big)^{1/r}\leq 2^{\frac{1}{r}}C\Big\|\Big(\sum_{k=1}^n|x_k|^r\Big)^{1/r}\Big\|_X,$$

so $\|\Gamma\|_{\ell^r}\leq 2^{1/r}C$.

Now for (i)⇒(ii) take a nonnegative $v\in(X^r)^*$. Without loss of generality we may assume that $\|v\|_{(X^r)^*}\leq 1$. Let $Y:=L^r(\Omega,v)$. Then $\|x\|_Y\leq\|x\|_X$ for all $x\in X$, i.e. $X\hookrightarrow Y$ contractively. We can therefore consider Γ as a family of operators from X to Y with

$$\Big(\sum_{k=1}^n\|T_kx_k\|_Y^r\Big)^{1/r}\leq\Big\|\Big(\sum_{k=1}^n|T_kx_k|^r\Big)^{1/r}\Big\|_X\leq\|\Gamma\|_{\ell^r}\Big\|\Big(\sum_{k=1}^n|x_k|^r\Big)^{1/r}\Big\|_X. \tag{6.2.2}$$

for all $x_1,\cdots x_n\in X$ and $T_1,\cdots,T_n\in\Gamma$. Define the sets

$$A:=\Big\{\Big(\sum_{k=1}^n|x_k|^r,\sum_{k=1}^n\|T_kx_k\|_Y^r\Big):x_k\in X,T_k\in\Gamma\Big\}\subseteq X^r\times\mathbb{R},$$
$$B:=\Big\{b\in(X^r)^*:\|b\|_{(X^r)^*}\leq 1\text{ and }b\geq 0\Big\}.$$

Then A and B are convex, and by the Banach-Alaoglu theorem B is weak*-compact.

Define $\Phi\colon A\times B\to\mathbb{R}$ by

$$\Phi(a,b):=\sum_{k=1}^n\|T_kx_k\|_Y^r-\|\Gamma\|_{\ell^r}^r\int_\Omega\sum_{k=1}^n|x_k|^r\,b\,\mathrm{d}\mu,\qquad a=\Big(\sum_{k=1}^n|x_k|^r,\sum_{k=1}^n\|T_kx_k\|_Y^r\Big).$$

Then Φ is linear in its first coordinate and affine in its second. Furthermore, by definition, $\Phi(a,\cdot)$ is weak*-continuous for all $a\in A$, and by (6.2.2) for any $a\in A$

$$\min_{b\in B}\Phi(a,b)=\sum_{k=1}^n\|T_kx_k\|_Y^r-\|\Gamma\|_{\ell^r}^r\Big\|\Big(\sum_{k=1}^n|x_k|^r\Big)^{1/r}\Big\|_X^r\leq 0.$$

Thus, by the Minimax lemma (see [Gra14a, Appendix H]), we have

$$\min_{b\in B}\sup_{a\in A}\Phi(a,b)=\sup_{a\in A}\min_{b\in B}\Phi(a,b)\leq 0,$$

so there exists $w_1 \in B$ such that $\Phi(a, w_1) \le 0$ for all $a \in A$. In particular, for any $x \in X$ and $T \in \Gamma$ we find that

$$\int_\Omega |Tx|^r v\, d\mu - \|\Gamma\|_{\ell^r}^r \int_\Omega |x|^r w_1 \, d\mu = \Phi\big((|x|^r, \|Tx\|_Y^r), w_1\big) \le 0.$$

Set $w_0 := v$. Iterating the argument with w_n in place of v yields a sequence $(w_n)_{n=0}^\infty$ satisfying

$$\Big(\int_\Omega |Tx|^r w_n \, d\mu\Big)^{1/r} \le \|\Gamma\|_{\ell^r} \Big(\int_\Omega |x|^r w_{n+1} \, d\mu\Big)^{1/r}, \qquad x \in X, \quad T \in \Gamma$$

for all $n \in \mathbb{N}$. Then the weight $w := \sum_{n=0}^\infty 2^{-n} w_n$ satisfies $w \ge v$, $\|w\|_{(X^r)^*} \le 2$ and (6.2.1) with $C = 2^{1/r}\|\Gamma\|_{\ell^r}$. □

In applications Theorem 6.2.1 is often applied as follows: for a fixed $y_1 \in X$ we take a positive $v \in (X^r)^*$ of norm one such that

$$\int_\Omega |y_1|^r v \, d\mu = \big\||y_1|^r\big\|_{X^r} = \|y_1\|_X^r.$$

Taking $w \ge v$ as in Theorem 6.2.1, we then have that Γ is uniformly bounded on $L^r(\Omega, w)$ and

$$\|x\|_{L^r(\Omega,w)} \le 2\|x\|_X, \qquad x \in X, \tag{6.2.3}$$

$$\|y_1\|_{L^r(\Omega,w)} \ge \|y_1\|_X. \tag{6.2.4}$$

Using (6.2.3) and (6.2.4), we can transfer the analysis of Γ back and forth between X and $L^r(\Omega, w)$. There is a notable difference between (6.2.3) and (6.2.4): we obtain (6.2.3) for all $x \in X$, whereas (6.2.4) only holds for one a prior fixed $y_1 \in X$. One can not expect both inequalities to hold for all $x \in X$ unless X is isomorphic to a weighted L^p-space.

Using the abstract representation and factorization theory of Euclidean structures, which was developed in [4] by Kalton, Weis and the author, we will now prove a version of Theorem 6.2.1 with $r = 2$ for Banach function spaces which are not necessarily 2-convex. A key observation for our theory to work is that one does only need (6.2.3) for one fixed $y_0 \in X$ in applications. This will allow us to build a 2-convex Banach function space Y based on y_0 and y_1, which is contractively embedded in X. The factorization theory from [4] applied to Y will then yield us a theorem in the spirit of Theorem 6.2.1 with (6.2.3) only for $x = y_0$, but for Banach function spaces X which are not necessarily 2-convex.

We will start with the technical heart of the proof, which is not yet specific to Banach function spaces and ℓ^2-boundedness, i.e. the following lemma holds more generally for any Euclidean structure α on a Banach space X, see [4, Lemma 2.5]. The proof of the lemma in the case $\Gamma = \varnothing$ is a variation of the proof of [AK16, Theorem 7.3.4], which is the key ingredient to prove the Maurey-Kwapień theorem on factorization of an operator $T : X \to Y$ through a Hilbert space (see [Kwa72a, Mau74]).

Lemma 6.2.2. *Let X be a Banach function space and let $Y \subseteq X$ be a subspace. Suppose that $F : X \to [0,\infty)$ and $G : Y \to [0,\infty)$ are positive homogeneous functions such that*

$$\Big(\sum_{k=1}^n F(x_k)^2\Big)^{1/2} \le \Big\|\Big(\sum_{k=1}^n |x_k|^2\Big)^{1/2}\Big\|_X, \qquad x_1,\dots,x_n \in X, \tag{6.2.5}$$

$$\Big\|\Big(\sum_{k=1}^n |y_k|^2\Big)^{1/2}\Big\|_X \le \Big(\sum_{k=1}^n G(y_k)^2\Big)^{1/2}, \qquad y_1,\dots,y_n \in Y. \tag{6.2.6}$$

Let $\Gamma \subseteq \mathcal{L}(X)$ be an ℓ^2-bounded family of operators. Then there exists a Γ-invariant subspace $Y \subseteq X_0 \subseteq X$ and a Hilbertian seminorm $\|\cdot\|_0$ on X_0 such that

$$\|Tx\|_0 \le 2\|\Gamma\|_{\ell^2}\|x\|_0 \qquad x \in X_0,\ T \in \Gamma, \tag{6.2.7}$$

$$\|x\|_0 \ge F(x) \qquad x \in X_0, \tag{6.2.8}$$

$$\|x\|_0 \le 4G(x) \qquad x \in Y. \tag{6.2.9}$$

Proof. Let X_0 be the smallest Γ-invariant subspace of X containing Y, i.e. set $Y_0 := Y$, define for $N \ge 1$

$$Y_N := \Big\{Tx : T \in \Gamma,\ x \in Y_{N-1}\Big\}.$$

and take $X_0 := \bigcup_{N\ge 0} Y_N$. We will prove the lemma in three steps.

Step 1: We will first show that G can be extended to a function G_0 on X_0, such that $2G_0$ satisfies (6.2.6) for all $y_1,\dots y_n \in X_0$. For this pick a sequence of real numbers $(a_N)_{N=1}^\infty$ such that $a_N > 1$ and $\prod_{N=1}^\infty a_N = 2$ and define $b_M := \prod_{N=1}^M a_N$ for $M \ge 1$. For $y \in Y$ we set $G_0(y) = G(y)$ and we will proceed by induction. Suppose that G_0 is defined on $\bigcup_{N=0}^M Y_N$ for some $M \in \mathbb{N}$ with

$$\Big\|\Big(\sum_{k=1}^n |y_k|^2\Big)^{1/2}\Big\|_X \le b_M\Big(\sum_{k=1}^n G_0(y_k)^2\Big)^{1/2} \tag{6.2.10}$$

for any $y_1,\dots,y_n \in \bigcup_{N=0}^M Y_N$.

For $y \in Y_{M+1} \setminus \bigcup_{N=0}^M Y_N$ pick a $T \in \Gamma$ and an $x \in Y_M$ such that $Tx = y$ and define

$$G_0(y) := \frac{\|\Gamma\|_{\ell^2}}{a_{M+1}-1} \cdot G_0(x).$$

For $y_1,\dots,y_n \in \bigcup_{N=0}^{M+1} Y_N$ we let $\mathcal{I} = \{k : y_k \in \bigcup_{N=0}^M Y_N\}$. For $k \notin \mathcal{I}$ we let T_k and x_k be as in the definition of G_0, i.e. $T_k x_k = y_k$. Then, by our definition of G_0, we have

$$\begin{aligned}\Big\|\Big(\sum_{k=1}^n |y_k|^2\Big)^{1/2}\Big\|_X &\le \Big\|\Big(\sum_{k\in\mathcal{I}} |y_k|^2\Big)^{1/2}\Big\|_X + \Big\|\Big(\sum_{k\notin\mathcal{I}} |y_k|^2\Big)^{1/2}\Big\|_X \\ &\le b_M\Big(\sum_{k\in\mathcal{I}} G_0(y_k)^2\Big)^{1/2} + b_M\|\Gamma\|_{\ell^2}\Big(\sum_{k\notin\mathcal{I}} G_0(x_k)^2\Big)^{1/2}\end{aligned}$$

$$\leq b_M\Big(\sum_{k=1}^n G_0(y_k)^2\Big)^{1/2} + b_M(a_{M+1}-1)\Big(\sum_{k=1}^n G_0(y_k)^2\Big)^{1/2}$$
$$= b_{M+1}\Big(\sum_{k=1}^n G_0(y_k)^2\Big)^{1/2}.$$

So G_0 satisfies (6.2.10) for $M+1$. Therefore, by induction, we can define G_0 on X_0, such that $2G_0$ satisfies (6.2.6) for all $y_1,\dots,y_n \in X_0$.

Step 2: For $x \in X$ define the function $\phi_x : X^* \to \mathbb{R}_+$ by $\phi_x(x^*) := |x^*(x)|^2$. We will construct a sublinear functional on the space

$$\mathbb{V} := \operatorname{span}\{\phi_x : x \in X_0\}.$$

For this note that every $\psi \in \mathbb{V}$ has a representation of the form

$$\psi = \sum_{k=1}^{n_u} \phi_{u_k} - \sum_{k=1}^{n_v} \phi_{v_k} + \sum_{k=1}^{n_x} \big(\phi_{T_k x_k} - \phi_{2\|\Gamma\|_{\ell^2} x_k}\big) \tag{6.2.11}$$

with $u_k \in X_0$, $v_k, x_k \in X$ and $T_k \in \Gamma$. Define $p : \mathbb{V} \to [-\infty,\infty)$ by

$$p(\psi) = \inf\left\{16\sum_{k=1}^{n_u} G_0(u_k)^2 - \sum_{k=1}^{n_v} F(v_k)^2\right\},$$

where the infimum is taken over all representations of ψ in the form of (6.2.11). This functional clearly has the following properties

$$p(a\psi) = ap(\psi), \qquad \psi \in \mathbb{V}, a > 0, \tag{6.2.12}$$
$$p(\psi_1+\psi_2) \leq p(\psi_1)+p(\psi_2), \qquad \psi_1,\psi_2 \in \mathbb{V}, \tag{6.2.13}$$
$$p(\phi_{Tx} - \phi_{2\|\Gamma\|_{\ell^2} x}) \leq 0, \qquad x \in X_0, T \in \Gamma, \tag{6.2.14}$$
$$p(-\phi_x) \leq -F(x)^2, \qquad x \in X_0, \tag{6.2.15}$$
$$p(\phi_x) \leq 16G_0(x)^2, \qquad x \in X_0. \tag{6.2.16}$$

We will check that $p(0) = 0$. It is clear that $p(0) \leq 0$. Let

$$0 = \sum_{k=1}^{n_u} \phi_{u_k} - \sum_{k=1}^{n_v} \phi_{v_k} + \sum_{k=1}^{n_x} \Big(\phi_{T_k x_k} - \phi_{2\|\Gamma\|_{\ell^2} x_k}\Big)$$

be a representation of the form of (6.2.11). So for any $x^* \in X^*$ we have

$$\sum_{k=1}^{n_u} |x^*(u_k)|^2 + \sum_{k=1}^{n_x} |x^*(T_k x_k)|^2 = \sum_{k=1}^{n_v} |x^*(v_k)|^2 + \sum_{k=1}^{n_x} |x^*(2\|\Gamma\|_{\ell^2} x_k)|^2. \tag{6.2.17}$$

Let

$$\mathbf{u} := (u_k)_{k=1}^{n_u}, \qquad \mathbf{v} := (v_k)_{k=1}^{n_v},$$

$$\mathbf{x} := (x_k)_{k=1}^{n_x}, \qquad \mathbf{y} := (T_k x_k)_{k=1}^{n_x},$$

be column vectors and define

$$\tilde{\mathbf{u}} = \begin{pmatrix} \mathbf{u} \\ \mathbf{y} \end{pmatrix}, \qquad \tilde{\mathbf{v}} = \begin{pmatrix} \mathbf{v} \\ 2\|\Gamma\|_{\ell^2}\mathbf{x} \end{pmatrix}.$$

Note that (6.2.17) implies, by the Hahn-Banach theorem, that

$$v_1, \cdots, v_{n_v}, x_1, \cdots, x_{n_x} \in \operatorname{span}\{u_1, \cdots, u_{n_u}, T_1 x_1, \cdots, T_{n_x} x_{n_x}\}.$$

Thus there exists a scalar matrix $\mathbf{A}$ with $\|\mathbf{A}\| = 1$ such that $\tilde{\mathbf{v}} = \mathbf{A}\tilde{\mathbf{u}}$. Therefore, applying the boundedness of $\mathbf{A}$ pointwise and using the ℓ^2-boundedness of Γ, we have

$$\begin{aligned}\Big\| \Big(\sum_{k=1}^{n_v} |v_k|^2 + 2\|\Gamma\|_{\ell^2} \sum_{k=1}^{n_x} |x_k|^2 \Big)^{1/2} \Big\|_X &\le \Big\| \Big(\sum_{k=1}^{n_u} |u_k|^2 + \sum_{k=1}^{n_x} |T_k x_k|^2 \Big)^{1/2} \Big\|_X \\ &\le \Big\| \Big(\sum_{k=1}^{n_u} |u_k|^2 \Big)^{1/2} \Big\|_X + \frac{1}{2} \Big\| 2\|\Gamma\|_{\ell^2} \Big(\sum_{k=1}^{n_x} |x_k|^2 \Big)^{1/2} \Big\|_X.\end{aligned}$$

In particular, by assumption (6.2.5) on F and (6.2.6) on $2G_0$, this implies

$$\Big(\sum_{k=1}^{n_v} F(v_k)^2 \Big)^{1/2} \le \Big\| \Big(\sum_{k=1}^{n_v} |v_k|^2 \Big)^{1/2} \Big\|_X \le 2 \Big\| \Big(\sum_{k=1}^{n_u} |u_k|^2 \Big)^{1/2} \Big\|_X \le 4 \Big(\sum_{k=1}^{n_u} G_0(u_k)^2 \Big)^{1/2}.$$

We conclude $p(0) \ge 0$ and thus $p(0) = 0$. Now with property (6.2.13) of p we have

$$p(\psi) + p(-\psi) \ge p(0) = 0,$$

so $p(\psi) > -\infty$ for all $\psi \in \mathbb{V}$. Combined with properties (6.2.12) and (6.2.13) this means that p is a sublinear functional.

Step 3. To complete the prove of the lemma, we construct a semi-inner product from our sublinear functional p using the Hahn–Banach theorem. Indeed, by applying the Hahn-Banach theorem (see [Rud91, Theorem 3.2]), we obtain a linear function f on $\mathbb{V}$ such that $f(\psi) \le p(\psi)$ for all $\psi \in \mathbb{V}$. By property (6.2.15) we know that $p(-\phi_x) \le 0$ and thus $f(\phi_x) \ge 0$ for all $x \in X_0$.

We take the complexification of $\mathbb{V}$

$$\mathbb{V}^{\mathbb{C}} = \{v_1 + i\, v_2 : v_1, v_2 \in \mathbb{V}\}$$

with addition and scalar multiplication defined as usual. We extend f to a complex linear functional on this space by $f(v_1 + i v_2) = f(v_1) + i f(v_2)$ and define a pseudo-inner product on X_0 by $\langle x, y\rangle = f(\rho_{x,y})$ with $\rho_{x,y} : X^* \to \mathbb{C}$ defined as $\rho_{x,y}(x^*) = x^*(x)\overline{x^*(y)}$ for all $x^* \in X^*$. This is well-defined since

$$\rho_{x,y} = \frac{1}{4}\big(\phi_{x+y} - \phi_{x-y} + i\phi_{x+iy} - i\phi_{x-iy}\big) \in \mathbb{V}^{\mathbb{C}}.$$

On X_0 we define $\|\cdot\|_0$ by the seminorm induced by this semi-inner product, i.e.

$$\|x\|_0 := \sqrt{\langle x, x\rangle} = \sqrt{f(\phi_x)}.$$

Then for $x \in X_0$ and $T \in \Gamma$ we have by property (6.2.14) of p

$$\|Tx\|_0^2 \le p(\phi_{Tx} - \phi_{2\|\Gamma\|_{\ell^2}x}) + f(\phi_{2\|\Gamma\|_{\ell^2}x}) \le 4\|\Gamma\|_{\ell^2}^2\|x\|_0^2.$$

By property (6.2.15) of p we have

$$\|x\|_0^2 = f(\phi_x) \ge -p(-\phi_x) \ge F(x)^2, \qquad x \in X_0,$$

and by property (6.2.16) of p we have

$$\|y\|_0^2 = f(\phi_y) \le p(\phi_y) \le 16G_0(y)^2 = 16G(y)^2, \qquad y \in Y.$$

So $\|\cdot\|_0$ satisfies (6.2.7)-(6.2.9). □

We want use the lattice structure of X to make Lemma 6.2.2 more concrete. Let us first note the following property of a Hilbertian seminorm on a function space.

Lemma 6.2.3. *Let $X \subseteq L^0(\Omega)$ be a vector space with a Hilbertian seminorm $\|\cdot\|_0$. Suppose that there is a $C > 0$ such that for $x_1 \in L^0(S)$ and $x_2 \in X$*

$$|x_1| \le |x_2| \Rightarrow x_1 \in X \text{ and } \|x_1\|_0 \le C\|x_2\|_0.$$

Then there exists a seminorm $\|\cdot\|_1$ on X such that

$$\begin{aligned} \tfrac{1}{C}\|x\|_0 \le \|x\|_1 \le C\|x\|_0 \qquad & x \in X, \\ \|x_1 + x_2\|_1^2 = \|x_1\|_1^2 + \|x_2\|_1^2, \qquad & x_1, x_2 \in X : x_1 \wedge x_2 = 0. \end{aligned}$$

Proof. Let Π be the collection of all finite measurable partitions of Ω, partially ordered by refinement. We define

$$\|x\|_1 = \inf_{\pi\in\Pi}\sup_{\pi'\ge\pi}\Big(\sum_{E\in\pi'}\|x\mathbf{1}_E\|_0^2\Big)^{1/2}, \qquad x \in X,$$

which is clearly a seminorm. For a $\pi \in \Pi$, write $\pi = \{E_1, \cdots, E_n\}$ and let $(\varepsilon_k)_{k=1}^n$ be a Rademacher sequence. Then we have for all $x \in X$ that

$$\sum_{k=1}^n \|x\mathbf{1}_{E_k}\|_0^2 = \mathbb{E}\sum_{j=1}^n\sum_{k=1}^n \varepsilon_j\varepsilon_k\langle x\mathbf{1}_{E_j}, x\mathbf{1}_{E_k}\rangle = \mathbb{E}\Big\|\sum_{k=1}^n \varepsilon_k \cdot x\mathbf{1}_{E_k}\Big\|_0^2 \le C^2\|x\|_0^2$$

and, since $\bigcup_{k=1}^n E_k = \Omega$, we deduce in the same fashion

$$\|x\|_0^2 \le C^2\,\mathbb{E}\Big\|\sum_{k=1}^n \varepsilon_k \cdot x\mathbf{1}_{E_k}\Big\|_0^2 = C^2\sum_{k=1}^n \|x\mathbf{1}_{E_k}\|_0^2 = C^2\sum_{k=1}^n \|x\mathbf{1}_{E_k}\|_0^2.$$

Therefore we have $\frac{1}{C}\|x\|_0 \le \|x\|_1 \le C\|x\|_0$ for all $x \in X$ and $\pi \in \Pi$. Furthermore if $x, y \in X$ with $x \wedge y = 0$, then for $\pi \ge \{\operatorname{supp} x, \Omega \setminus \operatorname{supp} x\}$ we have

$$\sum_{E\in\pi} \|(x+y)\mathbf{1}_E\|_0^2 = \sum_{E\in\pi} \|x\mathbf{1}_E\|_0^2 + \sum_{E\in\pi} \|y\mathbf{1}_E\|_0^2.$$

So we also get $\|x+y\|_1^2 = \|x\|_1^2 + \|y\|_1^2$, which proves the lemma. □

With this lemma at our disposal, we are now ready to reformulate Lemma 6.2.2 in a way that resembles Theorem 6.2.1. Note that if X is 2-convex and order-continuous, we can take $Y = X$ and $F(x) = \|x\|_{L^2(\Omega,v)}$ for $v \in (X^2)^*$ of norm 1 in the following lemma. This yields the difficult implication of Theorem 6.2.1 for $r = 2$, since in this case $X \subseteq L^2(\Omega, w)$ and (6.2.19) implies $w \gtrsim v$.

Lemma 6.2.4. *Let X and Y be Banach function spaces over a measure space (Ω, μ). Suppose that Y is 2-convex, order-continuous and $Y \hookrightarrow X$ contractively. Let $F\colon X \to [0,\infty)$ be a positive homogeneous function such that*

$$\Big(\sum_{k=1}^n F(x_k)^2\Big)^{1/2} \le \Big\|\Big(\sum_{k=1}^n |x_k|^2\Big)^{1/2}\Big\|_X, \qquad x_1,\dots,x_n \in X.$$

Let $\Gamma \subseteq \mathcal{L}(X)$ be an ℓ^2-bounded family of operators. Then there exists a $w \in (Y^2)^*$ with $\|w\|_{(Y^2)^*} \lesssim 1$ such that

$$\|Tx\|_{L^2(\Omega,w)} \lesssim \|\Gamma\|_{\ell^2}\|x\|_{L^2(\Omega,w)} \qquad x \in X \cap L^2(\Omega,w),\ T \in \Gamma. \tag{6.2.18}$$

$$\|x\|_{L^2(\Omega,w)} \gtrsim F(x), \qquad x \in Y. \tag{6.2.19}$$

Moreover, the implicit constants are absolute.

Proof. Define $G: Y \to [0,\infty)$ by $G(x) = \|x\|_Y$, for which (6.2.6) follows from the contractive embedding $Y \hookrightarrow X$ and the 2-convexity of Y. For $m \in L^\infty(\Omega)$ let T_m be the pointwise multiplication operator given by $T_m x = m \cdot x$ for $x \in X$ and set

$$\mathcal{M} = \{T_m : m \in L^\infty(\Omega), \|m\|_{L^\infty(\Omega)} \le 1\} \subseteq \mathcal{L}(X). \tag{6.2.20}$$

Note that $\mathcal{M}$ is ℓ^2-bounded with $\|\mathcal{M}\|_{\ell^2} = 1$, so if we define

$$\Gamma_0 := \big(\tfrac{1}{2\|\Gamma\|_{\ell^2}} \cdot \Gamma\big) \cup \big(\tfrac{1}{2} \cdot \mathcal{M}\big),$$

then Γ_0 is ℓ^2-bounded with $\|\Gamma_0\|_{\ell^2} \le 1$. Applying Lemma 6.2.2 to Γ_0, we obtain a Γ- and $\mathcal{M}$-invariant subspace $Y \subseteq X_0 \subseteq X$ and a Hilbertian seminorm $\|\cdot\|_0$ on X_0 satisfying

$$\|Tx\|_0 \le 4\|\Gamma\|_{\ell^2}\|x\|_0, \qquad x \in X_0,\ T \in \Gamma, \tag{6.2.21}$$

$$\|Tx\|_0 \le 4\|x\|_0, \qquad x \in X_0,\ T \in \mathcal{M}, \tag{6.2.22}$$

$$\|x\|_0 \ge F(x) \qquad x \in X_0, \tag{6.2.23}$$

$$\|x\|_0 \le 4\|x\|_Y, \qquad x \in Y. \tag{6.2.24}$$

Then (6.2.22) implies that if $x_1 \in L^0(\Omega)$, $x_2 \in X_0$ and $|x_1| \le |x_2|$, we have $x_1 \in X_0$ with $\|x_1\| \le 4\|x_2\|$. Thus we may, at the the loss of an absolute constant in (6.2.21)-(6.2.24), furthermore assume

$$\|x_1 + x_2\|_0^2 = \|x_1\|_0^2 + \|x_2\|_0^2, \qquad x_1, x_2 \in X : x_1 \wedge x_2 = 0 \tag{6.2.25}$$

by Lemma 6.2.3.

Let $u \in Y$ such that $u > 0$ a.e. and define a measure

$$\lambda(E) = \|u\mathbf{1}_E\|_0^2, \qquad E \in \Sigma.$$

Using (6.2.25), the σ-additivity of this measure follows from

$$\lambda\Big(\bigcup_{k=1}^{\infty} E_k\Big) = \Big\|\sum_{k=1}^{\infty} u\mathbf{1}_{E_k}\Big\|_0^2 = \lim_{n\to\infty}\Big\|\sum_{k=1}^{n} u\mathbf{1}_{E_k}\Big\|_0^2 = \sum_{k=1}^{\infty}\lambda(E_k)$$

for $E_1, E_2, \cdots \in \Sigma$ pairwise disjoint, where the second step is justified by the order-continuity of Y and (6.2.24). Moreover, again using (6.2.24), we have for any $E \in \Sigma$ with $\mu(E) = 0$

$$\lambda(E) = \|u\mathbf{1}_E\|_0^2 \lesssim \|u\mathbf{1}_E\|_Y = \|\mathbf{1}_\varnothing\|_Y = 0$$

so λ is absolutely continuous with respect to μ. Thus, by the Radon-Nikodym theorem, we can find a $f \in L^1(\Omega)$ such that

$$\|u\mathbf{1}_E\|_0^2 = \lambda(E) = \int_E f\,\mathrm{d}\mu$$

for all $E \in \Sigma$. Define $w = u^{-2}f$, which is a weight since $u, f \ge 0$ a.e.

Take $x \in Y$ and let $(v_n)_{n=1}^{\infty}$ be a sequence of functions of the form

$$v_n = u\sum_{j=1}^{m_n} a_j^n \mathbf{1}_{E_j^n}, \qquad a_j^n \in \mathbb{C}, \quad E_j^n \in \Sigma,$$

such that $|v_n| \uparrow |x|$. Then $\lim_{n\to\infty}\|v_n - x\|_0 = 0$ by the order- continuity of Y and (6.2.24). Therefore we have, by (6.2.25) and the monotone convergence theorem, that

$$\|x\|_0^2 = \lim_{n\to\infty}\sum_{j=1}^{m_n}|a_j^n|^2\|u\mathbf{1}_{E_j^n}\|_0^2 = \lim_{n\to\infty}\sum_{j=1}^{m_n}\int_{E_j^n}|a_j^n|^2u^2w\,\mathrm{d}\mu = \int_\Omega |x|^2 w\,\mathrm{d}\mu.$$

In particular, (6.2.19) now follows from (6.2.23) and by (6.2.24) we have

$$\|w\|_{(Y^2)^*} \le \sup_{\|x\|_Y\le 1}\int_\Omega |x|^2 w\,\mathrm{d}\mu \lesssim 1.$$

For $T \in \Gamma$ and $x \in Y$ define $m_n = \min(1, nu\cdot|Tx|^{-1})$ for $n \in \mathbb{N}$. Then $m_n \cdot Tx \in Y$ and $|m_n \cdot Tx| \uparrow |Tx|$. So, by the monotone convergence theorem, (6.2.21) and (6.2.22), we have

$$\begin{aligned}\|Tx\|_{L^2(\Omega,w)} &= \lim_{n\to\infty}\Big(\int_\Omega |m_n\cdot Tx|^2 w\,\mathrm{d}\mu\Big)^{\frac12}\\ &= \lim_{n\to\infty}\|m_n\cdot Tx\|_0\\ &\lesssim \|\Gamma\|_{\ell^2}\|x\|_{L^2(\Omega,w)}.\end{aligned}$$

To conclude, note that Y is dense in $X \cap L^2(\Omega, w)$ by order-continuity. Therefore, since T is bounded on X as well, this estimate extends to all $x \in X \cap L^2(\Omega, w)$, i.e. (6.2.18) holds □

Lemma 6.2.4 does not only cover Theorem 6.2.1 in the case $r = 2$, but it also allows for $Y \neq X$. This enables us to deduce a factorization theorem for Banach function spaces X that are not 2-convex. For the special case $X = L^p(S)$ the following result can be found in the work of Le Merdy and Simard [LS02, Theorem 2.1]. See also Johnson and Jones [JJ78] and Simard [Sim99].

Theorem 6.2.5. *Let X be an order-continuous Banach function space over a measure space (Ω, μ) and let $\Gamma \subseteq \mathcal{L}(X)$. Then Γ is ℓ^2-bounded if and only if there exists a constant $C > 0$ such that for all $y_0, y_1 \in X$ there is a weight w such that*

$$\|Tx\|_{L^2(\Omega,w)} \leq C\|x\|_{L^2(\Omega,w)}, \quad x \in X \cap L^2(\Omega, w),\ T \in \Gamma \tag{6.2.26}$$

$$\|y_0\|_{L^2(\Omega,w)} \lesssim \|y_0\|_X, \tag{6.2.27}$$

$$\|y_1\|_{L^2(\Omega,w)} \gtrsim \|y_1\|_X. \tag{6.2.28}$$

Moreover $C > 0$ can be chosen such that $\|\Gamma\|_{\ell^2} \simeq C$ and the implicit constants are absolute.

Proof. We will first prove the 'if' statement, which is very similar to the 'if' statement of Theorem 6.2.1. Let $x_1, \ldots, x_n \in X$ and $T_1, \ldots, T_n \in \Gamma$. Define

$$y_0 = \Big(\sum_{k=1}^n |x_k|^2\Big)^{\frac{1}{2}}, \qquad y_1 = \Big(\sum_{k=1}^n |T_k x_k|^2\Big)^{\frac{1}{2}}.$$

Then we have, by applying (6.2.26)-(6.2.28), that

$$\|y_1\|_X^2 \lesssim \sum_{k=1}^n \int_\Omega |T_k x_k|^2 w \,\mathrm{d}\mu \leq C^2 \sum_{k=1}^n \int_\Omega |x_k|^2 w \,\mathrm{d}\mu \lesssim C^2 \|y_0\|_X^2,$$

so $\|\Gamma\|_{\ell^2} \lesssim C$.

Now for the converse take $y_0, y_1, \tilde{u} \in X$ with $\|y_0\|_X = \|y_1\|_X = \|\tilde{u}\|_X = 1$ and $\tilde{u} > 0$ a.e. Define

$$u := \frac{1}{3}\big(|y_0| \vee |y_1| \vee \tilde{u}\big),$$

then $\|u\|_X \leq 1$ and $\|y_j u^{-1}\|_{L^\infty(\Omega)} \leq 3$ for $j = 0, 1$. Let

$$Y = \{x \in X : x^2 u^{-1} \in X\}$$

with norm $\|x\|_Y := \|x^2 u^{-1}\|_X^{1/2}$. Then Y is an order-continuous Banach function space and for $x_1, \cdots, x_n \in Y$ we have

$$\Big\|\Big(\sum_{k=1}^n |x_k|^2\Big)^{1/2}\Big\|_Y = \Big\|\sum_{k=1}^n |x_k|^2 u^{-1}\Big\|_X^{1/2} \leq \Big(\sum_{k=1}^n \big\||x_k|^2 u^{-1}\big\|_X\Big)^{1/2} = \Big(\sum_{k=1}^n \|x_k\|_Y^2\Big)^{1/2},$$

i.e. Y is 2-convex. Moreover by Hölders inequality for Banach function spaces ([LT79, Proposition 1.d.2(i)]), we have

$$\|x\|_X \le \big\|x^2 u^{-1}\big\|_X^{1/2} \|u\|_X^{1/2} \le \|x\|_Y, \qquad x \in Y,$$

so Y is contractively embedded in X. Conversely we have for $j = 0, 1$

$$\|y_j\|_Y \le \|y_j\|_X \big\|y_j u^{-1}\big\|_{L^\infty(\Omega)} \le 3\|y_j\|_X. \tag{6.2.29}$$

Now define

$$F(x) = \begin{cases} \|x\|_X, & \text{if } x \in \operatorname{span}\{y_1\}, \\ 0, & \text{otherwise} \end{cases}$$

and for $x_1, \dots, x_n \in X$ let $a_1, \dots, a_n \in \mathbb{C}$ be such that $x_k = a_k y_1$ if $x_k \in \operatorname{span}\{y_1\}$ and $a_k = 0$ otherwise. Then we have

$$\Big(\sum_{k=1}^n F(x_k)^2\Big)^{1/2} = \Big(\sum_{k=1}^n |a_k|^2\Big)^{1/2} \le \Big\|\Big(\sum_{k=1}^n |x_k|^2\Big)^{1/2}\Big\|_X.$$

Therefore, applying Lemma 6.2.4, there is a weight $w \in (Y^2)^*$ with $\|w\|_{(Y^2)^*} \lesssim 1$ such that (6.2.18) and (6.2.19) hold. In particular, using (6.2.29), this implies

$$\|y_0\|_{L^2(\Omega,w)} \le \big\||y_0|^2\big\|_{Y^2}^{1/2} \|w\|_{(Y^2)^*}^{1/2} \lesssim \|y_0\|_Y \le 3\|y_0\|_X,$$
$$\|y_1\|_{L^2(\Omega,w)} \gtrsim \|y_1\|_X,$$

proving the theorem. □

6.3. EXTENSIONS OF OPERATORS I: FACTORIZATION

In this section we will apply Theorem 6.2.5 to obtain a more general version of Theorem 6.1.1. We will apply this factorization-based extension theorem to deduce the following results:

- We will show that the dyadic UMD$^+$ property is equivalent to the UMD property on Banach function spaces.
- We will show that the UMD property is necessary for the ℓ^2-sectoriality of certain differentiation operators on $L^p(\mathbb{R}^d; X)$, where X is a Banach function space.

Moreover, in the next section we will use it to prove the boundedness of the lattice Hardy–Littlewood maximal operator on UMD Banach function spaces.

Let $p \in [1, \infty)$ and let w be a weight on $\mathbb{R}^d$. For a bounded linear operator T on $L^p(\mathbb{R}^d, w)$, we define the linear operator $\widetilde{T}$ on $L^p(\mathbb{R}^d, w) \otimes X$ by setting

$$\widetilde{T}(f \otimes x) := Tf \otimes x, \qquad f \in L^p(\mathbb{R}^d, w),\ x \in X,$$

and extending by linearity. If $\widetilde{T}$ extends to a bounded operator on $L^p(\mathbb{R}^d, w; X)$ we denote this operator again by $\widetilde{T}$. For a family of bounded operators $\Gamma \subseteq \mathcal{L}(L^p(\mathbb{R}^d, w))$ we denote $\widetilde{\Gamma} := \{\widetilde{T} : T \in \Gamma\}$.

Theorem 6.3.1. *Let X be an order-continuous Banach function space over a measure space (Ω, Σ, μ), let $p \in (1,\infty)$ and $w \in A_p$. Assume that there is a family of operators $\Gamma \subseteq \mathcal{L}(L^p(\mathbb{R}^d, w))$ and an increasing function $\phi\colon \mathbb{R}_+ \to \mathbb{R}_+$ such that*

- *For all weights $v\colon \mathbb{R}^d \to (0,\infty)$ we have*

$$[v]_{A_2} \le \phi\Big(\sup_{T\in\Gamma} \|T\|_{L^2(\mathbb{R}^d,v)\to L^2(\mathbb{R}^d,v)}\Big).$$

- *$\widetilde{\Gamma}$ is ℓ^2-bounded on $L^p(\mathbb{R}^d, w; X)$.*

Let $f, g \in L^p(\mathbb{R}^d, w; X)$ and suppose that there is an increasing function $\psi\colon \mathbb{R}_+ \to \mathbb{R}_+$ such that for all $v \in A_2$ we have

$$\|f(\cdot,\omega)\|_{L^2(\mathbb{R}^d,v)} \le \psi([v]_{A_2})\|g(\cdot,\omega)\|_{L^2(\mathbb{R}^d,v)}, \qquad \omega \in \Omega.$$

Then there is an absolute constant $c > 0$ such that

$$\|f\|_{L^p(\mathbb{R}^d,w;X)} \le c \cdot \psi \circ \phi\big(c\|\widetilde{\Gamma}\|_{\ell^2}\big)\|g\|_{L^p(\mathbb{R}^d,w;X)}.$$

Proof. Let $u \in L^p(\mathbb{R}^d, w)$ be such that there is a $c_K > 0$ with $u \ge c_K \mathbf{1}_K$ for every compact $K \subseteq \mathbb{R}^d$. Let $x \in X$ be such that $x > 0$ a.e. and

$$\|u \otimes x\|_{L^p(\mathbb{R}^d,w;X)} \le \|g\|_{L^p(\mathbb{R}^d,w;X)}.$$

Since X is order-continuous, $L^p(\mathbb{R}^d, w; X)$ is an order-continuous Banach function space over the measure space

$$(\mathbb{R}^d \times \Omega, w\,\mathrm{d}t\,\mathrm{d}\mu).$$

So, by Theorem 6.2.5, we can find a weight v on $\mathbb{R}^d \times \Omega$ and a numerical constant $c > 0$ such that

$$\|Th\|_{L^2(\mathbb{R}^d\times\Omega, v\cdot w)} \le c\|\widetilde{\Gamma}\|_{\ell^2}\|h\|_{L^2(\mathbb{R}^d\times\Omega, v\cdot w)} \tag{6.3.1}$$

for all $T \in \widetilde{\Gamma}$ and $h \in L^p(\mathbb{R}^d, w; X) \cap L^2(\mathbb{R}^d \times \Omega, v \cdot w)$,

$$\big\||g| + u \otimes x\big\|_{L^2(\mathbb{R}^d\times\Omega, v\cdot w)} \le c\big\||g| + u \otimes x\big\|_{L^p(\mathbb{R}^d,w;X)}, \tag{6.3.2}$$

$$\|f\|_{L^2(\mathbb{R}^d\times\Omega, v\cdot w)} \ge \frac{1}{c}\|f\|_{L^p(\mathbb{R}^d,w;X)}. \tag{6.3.3}$$

Note that (6.3.2) and the definition of x imply

$$\|g\|_{L^2(\mathbb{R}^d\times\Omega, v\cdot w)} \le 2c\|g\|_{L^p(\mathbb{R}^d,w;X)}. \tag{6.3.4}$$

Moreover (6.3.2) implies that $u \in L^2(\mathbb{R}^d, v(\cdot,\omega)\cdot w)$ for μ-a.e. $\omega \in \Omega$. Therefore by the definition of u we know that $v(\cdot,\omega)\cdot w$ is locally integrable on $\mathbb{R}^d$. Let $\mathcal{A}$ be the $\mathbb{Q}$-linear

span of indicator functions of rectangles with rational corners, which is a a countable, dense subset of both $L^p(\mathbb{R}^d, w)$ and $L^2(\mathbb{R}^d, w(\cdot,\omega))$ for μ-a.e. $\omega \in \Omega$. Define

$$\mathcal{B} = \{\psi \otimes (x\,\mathbf{1}_E) : \psi \in \mathcal{A},\, E \in \Sigma\} \subseteq L^p(\mathbb{R}^d, w; X) \cap L^2(\mathbb{R}^d \times \Omega, v \cdot w),$$

where the inclusion follows from $u \otimes x \in L^2(\mathbb{R}^d \times \Omega, v \cdot w)$. Testing (6.3.1) on all $h \in \mathcal{B}$ we find that for all $T \in \Gamma$ and $\psi \in \mathcal{A}$

$$\|T\psi\|_{L^2(\mathbb{R}^d, v(\cdot,\omega)\cdot w)} \le c\,\|\widetilde{\Gamma}\|_{\ell^2}\,\|\psi\|_{L^2(\mathbb{R}^d, v(\cdot,\omega)\cdot w)}, \qquad \omega \in \Omega.$$

Since $\mathcal{A}$ is countable and dense in $L^2(\mathbb{R}^d, w(\cdot,s))$, we have by assumption that $v(\cdot,\omega)w \in A_2$ with $[v(\cdot,\omega)\cdot w]_{A_2} \le \phi(c\,\|\widetilde{\Gamma}\|_{\ell^2})$ for μ-a.e. $\omega \in \Omega$. Therefore, using Fubini's theorem, our assumption, (6.3.3) and (6.3.4), we obtain

$$\begin{aligned}
\|f\|_{L^p(\mathbb{R}^d, w; X)} &\le c\Big(\int_\Omega \int_{\mathbb{R}^d} |f|^2 v\cdot w\,\mathrm{d}t\,\mathrm{d}\mu\Big)^{1/2} \\
&\le c\cdot\psi\circ\phi\big(c\,\|\widetilde{\Gamma}\|_{\ell^2}\big)\Big(\int_\Omega \int_{\mathbb{R}^d} |g|^2 v\cdot w\,\mathrm{d}t\,\mathrm{d}\mu\Big)^{1/2} \\
&\le 2c^2\cdot\psi\circ\phi\big(c\,\|\widetilde{\Gamma}\|_{\ell^2}\big)\|g\|_{L^p(\mathbb{R}^d, w; X)},
\end{aligned}$$

proving the statement. □

Let us point out some choices of $\Gamma \subseteq \mathcal{L}(L^p(\mathbb{R}^d, w))$ that satisfy the assumptions Theorem 6.3.1 when X has UMD property:

- $\Gamma = \{H\}$, where H is the Hilbert transform.
- $\Gamma = \{R_k : k = 1, \dots, d\}$ where R_k is the k-th Riesz projection.
- $\Gamma := \{T_B : B \text{ a ball in } \mathbb{R}^d\}$, where $T_B\colon L^p(\mathbb{R}^d, w) \to L^p(\mathbb{R}^d, w)$ is the averaging operator
$$T_B f(t) := \langle f\rangle_{1,B}\,\mathbf{1}_B(t), \qquad t \in \mathbb{R}^d.$$

In each of these cases one obtains Theorem 6.1.1 as a corollary:

Proof of Theorem 6.1.1. For $j = 1, \cdots, d$ denote the k-th Riesz projection on $L^p(\mathbb{R}^d, w)$ by R_k and set $\Gamma = \{R_k : k = 1, \dots, d\}$. Then we have for any weight v on $\mathbb{R}^d$

$$[v]_{A_2} \lesssim_d \Big(\sup_{T\in\Gamma} \|T\|_{L^2(\mathbb{R}^d, v)\to L^2(\mathbb{R}^d, v)}\Big)^4.$$

by [Gra14a, Theorem 7.4.7]. Moreover by the triangle inequality, the fact that we can test ℓ^2-boundedness on distinct operators, Theorem 2.7.1 and Theorem 3.4.1 we have

$$\|\widetilde{\Gamma}\|_{\ell^2} \le \sum_{k=1}^d \|R_k\|_{L^p(\mathbb{R}^d, w; X)\to L^p(\mathbb{R}^d, w; X)} \lesssim_{X,p,d} [w]_{A_p}^{\max\{\frac{1}{p-1}, 1\}}$$

Thus Γ satisfies the assumptions of Theorem 6.3.1. Now let $f, g \in L^p(\mathbb{R}^d, w; X)$ and suppose that for some $p_0 \in (1,\infty)$ there is an increasing function $\phi\colon \mathbb{R}_+ \to \mathbb{R}_+$ such that for all $v \in A_{p_0}$ we have

$$\|f(\cdot,\omega)\|_{L^{p_0}(\mathbb{R}^d,v)} \le \phi([v]_{A_{p_0}})\|g(\cdot,\omega)\|_{L^{p_0}(\mathbb{R}^d,v)}, \qquad \omega \in \Omega.$$

Then by Rubio de Francia extrapolation (Theorem 2.3.3) there is an increasing function $\psi\colon \mathbb{R}_+ \to \mathbb{R}_+$, depending on ϕ, p, p_0, d, such that for all $v \in A_2$ we have

$$\|f(\cdot,\omega)\|_{L^2(\mathbb{R}^d,v)} \le \psi([v]_{A_2})\|g(\cdot,\omega)\|_{L^2(\mathbb{R}^d,v)}, \qquad \omega \in \Omega.$$

Therefore by Theorem 6.3.1 we obtain

$$\|f\|_{L^p(\mathbb{R}^d,w;X)} \le c \cdot \psi\big(C_{X,p,d} \cdot [w]_{A_p}^{\max\{\frac{4}{p-1},4\}}\big)\|g\|_{L^p(\mathbb{R}^d,w;X)}$$

which implies the conclusion of Theorem 6.1.1. □

6.3.1. RANDOMIZED UMD PROPERTIES

As a first application of Theorem 6.3.1, we will prove the equivalence of the UMD property and the dyadic UMD$^+$ property, introduced in Subsection 2.4.2. Two natural questions regarding these randomized UMD properties are the following:

- Does either the UMD$^-$ property or the UMD$^+$ property imply the UMD property? For the UMD$^-$ property it turns out that this is not the case, as any L^1-space has it, see [Gar90]. For the UMD$^+$ property this is an open problem. For general Banach spaces it is known that one cannot expect a better than quadratic bound relating $\beta_{p,X}$ and $\beta^+_{p,X}$ (see [Gei99, Corollary 5]).
- The dyadic UMD property implies its non-dyadic counterpart. Does the same hold for the dyadic UMD$^+$ and UMD$^-$ properties? For the UMD$^-$ property it is known that the constants $\beta^-_{p,X}$ and $\beta^{\Delta,-}_{p,X}$ are not the same in general, as explained in [CV11]. The relation between the norm of the Hilbert transform on $L^p(\mathbb{T};X)$ and $\beta^{\Delta,+}_{p,X}$ and $\beta^{\Delta,+}_{p,X}$ has recently been investigated in [OY19].

Using Theorem 6.3.1, we will show that on Banach function spaces the dyadic UMD$^+$ property implies the UMD property (and thus also the UMD$^+$ property), with a quadratic estimate of the respective constants. The equivalence of the UMD$^+$ property and the UMD property on Banach function spaces has previously been shown in unpublished work of T.P. Hytönen, using Stein's inequality to deduce the ℓ^2-boundedness of the Poisson semigroup on $L^p(\mathbb{R}^d;X)$, from which the boundedness of the Hilbert transform on $L^p(\mathbb{R}^d;X)$ was concluded using Theorem 6.2.5.

Theorem 6.3.2. *Let X be a Banach function space over a measure space (Ω,μ). Assume that X has the dyadic* UMD$^+$ *property and cotype $q \in (1,\infty)$. Then X has the* UMD *property with for $p \in (1,\infty)$*

$$\beta_{p,X} \lesssim_p q\big(c_{q,X}\beta^{\Delta,+}_{p,X}\big)^2.$$

Proof. Denote the standard dyadic system on $[0,1)$ by $\mathscr{D}$, i.e.

$$\mathscr{D} := \bigcup_{k\in\mathbb{N}} \mathscr{D}_k, \qquad \mathscr{D}_k := \{2^{-k}([0,1)+j) : j = 0,\dots,2^k-1\}.$$

Viewing $[0,1)$ as the torus $\mathbb{T}$, set $\mathscr{D}_k^\alpha := \mathscr{D}_k + \alpha$ and $\mathscr{D}^\alpha := \mathscr{D} + \alpha$ for $\alpha \in \{0, \frac{1}{3}, \frac{2}{3}\}$. Then $(\mathscr{D}_k^\alpha)_{k=1}^n$ is a Paley-Walsh filtration on $[0,1)$ for all $n \in \mathbb{N}$ and $\alpha \in \{0, \frac{1}{3}, \frac{2}{3}\}$. Let $p \in (1,\infty)$ and define

$$\Gamma := \Big\{\mathbb{E}(\cdot|\mathscr{D}_k^\alpha) : k \in \mathbb{N}, \alpha \in \Big\{0, \tfrac{1}{3}, \tfrac{2}{3}\Big\}\Big\} \subseteq \mathcal{L}(L^p([0,1))).$$

By a dyadic version of Stein's inequality, which can be proven analogously to [HNVW16, Theorem 4.2.23], we have for $f_1, \cdots, f_n \in L^p([0,1);X)$

$$\Big\| \sum_{\alpha\in\{0,\frac{1}{3},\frac{2}{3}\}} \sum_{k=1}^n \varepsilon_k \mathbb{E}(f_k|\mathscr{D}_k)\Big\|_{L^p([0,1)\times\Omega';X)} \le 3\beta_{p,X}^{\Delta,+} \Big\| \sum_{k=1}^n \varepsilon_k f_k\Big\|_{L^p([0,1)\times\Omega';X)},$$

where $(\varepsilon_k)_{k=1}^n$ is a Rademacher sequence on a probability space $(\Omega',\mathbb{P})$. So, by Proposition 2.4.1, Proposition 2.6.3 and the fact that we can test ℓ^2-boundedness on distinct operators, we know that $\widetilde{\Gamma}$ is ℓ^2-bounded with

$$\|\widetilde{\Gamma}\|_{\ell^2} \lesssim_p \sqrt{q} c_{q,X} \beta_{p,X}^{\Delta,+}. \tag{6.3.5}$$

Let $w : [0,1) \to (0,\infty)$ and set $C := \sup_{T\in\Gamma} \|T\|_{L^2([0,1),w)\to L^2([0,1),w)}$. Let $I \subseteq [0,1)$ be an interval. Then there exists an $I' \in \bigcup_{\alpha\in\{0,\frac{1}{3},\frac{2}{3}\}} \mathscr{D}^\alpha$ such that $I \subseteq I'$ and $|I'| \le 3|I|$. Fix $\alpha \in \{0, \frac{1}{3}, \frac{2}{3}\}$ and $k \in \mathbb{N}$ such that $I' \in \mathscr{D}_k^\alpha$. Applying $\mathbb{E}(\cdot|\mathscr{D}_k^\alpha)$ to the function $f = (w+\varepsilon)^{-1}\mathbf{1}_{I'}$ for some $\varepsilon > 0$ we obtain

$$\int_{I'} \Big(\frac{1}{|I'|}\int_{I'} (w(t)+\varepsilon)^{-1}\,\mathrm{d}t\Big)^2 w(s)\,\mathrm{d}s \le C^2 \int_{I'} \frac{w(t)}{(w(t)+\varepsilon)^2}\,\mathrm{d}t$$

which implies

$$\langle w\rangle_{1,I}\big\langle (w+\varepsilon)^{-1}\big\rangle_{1,I} \le 9\langle w\rangle_{1,I'}\big\langle (w+\varepsilon)^{-1}\big\rangle_{1,I'} \le 9C^2$$

So by letting $\varepsilon \to 0$ with the monotone convergence theorem we obtain $w \in A_2$ with $[w]_{A_2} \lesssim C^2$.

For $I \in \mathscr{D}$ let D_I be the Haar projection as defined in (3.8.1). Let $\mathcal{A}$ be the set of all $f \in L^p([0,1);X)$ such that $D_I f \neq 0$ for only finitely many $I \in \mathscr{D}$. Then for all $f \in \mathcal{A}$, $w \in A_2$ and $\epsilon_I \in \{-1,1\}$ we have

$$\Big\| \sum_{I\in\mathscr{D}} \epsilon_I D_I f(\cdot,\omega)\Big\|_{L^2([0,1),w)} \lesssim [w]_{A_2} \|f(\cdot,\omega)\|_{L^2([0,1),w)}, \qquad \omega \in \Omega$$

by Theorem 3.8.1. Now note that Theorem 6.3.1 also holds with $[0,1)$ in place of $\mathbb{R}^d$ with the exact same proof. Thus, applying this adapted version of Theorem 6.3.1, we obtain

$$\Big\| \sum_{I\in\mathscr{D}} \epsilon_I D_I f\Big\|_{L^p([0,1);X)} \lesssim_p q\big(c_{q,X}\beta_{p,X}^{\Delta,+}\big)^2 \|f\|_{L^p([0,1);X)} \tag{6.3.6}$$

for all $f \in \mathcal{A}$ and $\epsilon_I \in \{-1,1\}$. This extends to all $f \in L^p([0,1);X)$ by density (see [HNVW16, Lemma 4.2.12]), so

$$\beta_{p,X} \lesssim_p q \left(c_{q,X}\beta_{p,X}^{\Delta,+}\right)^2$$

as (6.3.6) characterizes the UMD property of X by [HNVW16, Theorem 4.2.13]. □

Remark 6.3.3. The assumption that X has finite cotype may be omitted in Theorem 6.3.2, since the dyadic UMD$^+$ property implies that there exists a constant $C_p > 0$ such that X has cotype $C_p\beta_{p,X}^{+,\Delta}$ with constant less than C_p (see [HLN16, Lemma 32]). This would yield the bound $\beta_{p,X} \lesssim_p \left(\beta_{p,X}^{\Delta,+}\right)^3$ for all $p \in (1,\infty)$ in the conclusion of Theorem 6.3.2.

6.3.2. ℓ^2-SECTORIALITY AND THE UMD PROPERTY

Recall the definition of a sectorial operator A from Section 5.2. We say that a sectorial operator A on a Banach function space X is ℓ^2-sectorial (respectively $\mathcal{R}$-sectorial) if the resolvent set

$$\{\lambda R(\lambda, A) : \lambda \neq 0, |\arg \lambda| > \sigma\}$$

is ℓ^2-bounded (respectively $\mathcal{R}$-bounded) for some $\sigma \in (0,\pi)$.

It is well-known that both the differentiation operator $Df := f'$ with domain $W^{1,p}(\mathbb{R};X)$ and the Laplacian $-\Delta$ with domain $W^{2,p}(\mathbb{R}^d;X)$ are $\mathcal{R}$-sectorial if X has the UMD property (see [KW04, Example 10.2] and [HNVW17, Theorem 10.3.4]). Since any UMD Banach space has finite cotype, it follows from Proposition 2.6.3 that these differentiation operators are also ℓ^2-sectorial. Using Theorem 6.3.1 we can turn this into an 'if and only if' statement for order-continuous Banach function spaces. We start with a lemma to check the weight assumption in Theorem 6.3.1.

Lemma 6.3.4. *Let $0 \neq \varphi \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ be real-valued and let $w\colon \mathbb{R}^d \to (0,\infty)$ be a weight. Suppose that there is a $C > 0$ such that for all $f \in L^2(\mathbb{R}^d, w)$ and $\lambda \in \mathbb{R}$ we have*

$$\|\varphi_\lambda * f\|_{L^2(\mathbb{R}^d,w)} \leq C\|f\|_{L^2(\mathbb{R}^d,w)}$$

where $\varphi_\lambda(t) := |\lambda|^d\varphi(\lambda t)$ for $t \in \mathbb{R}^d$. Then $w \in A_2$ and $[w]_{A_2} \lesssim_{\phi,d} C^4$.

Proof. Let $\psi = \varphi_{-1} * \varphi$. Then $\psi(-t) = \psi(t)$ for all $t \in \mathbb{R}^d$ and $\psi(0) = \|\varphi\|^2_{L^2(\mathbb{R}^d)} > 0$. Moreover

$$\|\psi\|_{L^\infty(\mathbb{R}^d)} \leq \|\varphi\|^2_{L^2(\mathbb{R}^d)},$$

so ψ is continuous by the density of $C_c(\mathbb{R}^d)$ in $L^2(\mathbb{R}^d)$. Therefore we can find a $\delta > 0$ such that $\psi(t) > \delta$ for all $|t| < \delta$. Define $\psi_\lambda(t) := \lambda^d\psi(\lambda t)$ for $\lambda > 0$. Then we have for all $f \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d, w)$ that

$$\|\psi_\lambda * f\|_{L^2(\mathbb{R}^d,w)} = \|\varphi_{-\lambda} * \varphi_\lambda * f\|_{L^2(\mathbb{R}^d,w)} \leq C^2\|f\|_{L^2(\mathbb{R}^d,w)}$$

Now let B be a ball in $\mathbb{R}^d$ of radius $r>0$ and let $f \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d, w)$ be nonnegative and supported on Q. Take $\lambda = \frac{\delta}{2r}$, then for $t \in B$

$$\psi_\lambda * f(t) = \lambda^d \int_B \psi\big(\lambda(t-s)\big) f(s)\,\mathrm{d}s \geq_d \frac{\delta^{d+1}}{|B|} \int_B f(s)\,\mathrm{d}s. \tag{6.3.7}$$

Now let $w\colon \mathbb{R}^d \to (0,\infty)$ be a weight. Applying (6.3.7) and the assumption to the function $f = (w+\varepsilon)^{-1}\mathbf{1}_B$ for some $\varepsilon>0$ we obtain

$$\int_B \Big(\frac{1}{|B|}\int_B (w(t)+\varepsilon)^{-1}\,\mathrm{d}t\Big)^2 w(s)\,\mathrm{d}s \lesssim_{\phi,d} C^4 \int_B \frac{w(t)}{(w(t)+\varepsilon)^2}\,\mathrm{d}t$$

which implies

$$\langle w\rangle_{1,B}\big\langle (w+\varepsilon)^{-1}\big\rangle_{1,B}\,\mathrm{d}t \lesssim_{\phi,d} C^4$$

So, by letting $\varepsilon \to 0$ with the monotone convergence theorem, we obtain $w \in A_2$ with $[w]_{A_2} \lesssim_{\phi,d} C^4$. □

Using Lemma 6.3.4 to check the weight condition of Theorem 6.3.1, the announced theorem follows readily.

Theorem 6.3.5. *Let X be an order-continuous Banach function space and let $p \in (1,\infty)$. The following are equivalent:*

(i) X has the UMD *property.*

(ii) The differentiation operator D on $L^p(\mathbb{R};X)$ is ℓ^2-sectorial.

(iii) The Laplacian $-\Delta$ on $L^p(\mathbb{R}^d;X)$ is ℓ^2-sectorial.

Proof. We have already discussed the implications (i) ⇒ (ii) and (i) ⇒ (iii). We will prove (iii) ⇒ (i), the proof of (ii) ⇒ (i) is similar. Take $\lambda \in \mathbb{R}$ and define the operators

$$T_\lambda := -\lambda^2\Delta(1-\lambda^2\Delta)^{-2} = -\Delta R\Big(-\frac{1}{\lambda^2},-\Delta\Big)\cdot\frac{1}{\lambda^2}R\Big(-\frac{1}{\lambda^2},-\Delta\Big).$$

Since $-\Delta$ is ℓ^2-sectorial on $L^p(\mathbb{R}^d;X)$, we know that the family of operators

$$\widetilde{\Gamma} = \{\widetilde{T}_\lambda : \lambda \in \mathbb{R}\}$$

is ℓ^2-bounded on $L^p(\mathbb{R}^d;X)$. Furthermore we have for $f \in L^2(\mathbb{R}^d)$ that $T_1 f = \varphi * f$ with $\varphi \in L^1(\mathbb{R}^d)\cap L^2(\mathbb{R}^d)$ such that

$$\hat{\varphi}(\xi) = \frac{(2\pi|\xi|)^2}{\big(1+(2\pi|\xi|)^2\big)^2}, \qquad \xi \in \mathbb{R}^d.$$

Moreover $T_\lambda f = \varphi_\lambda * f$ for $\varphi_\lambda(x) = |\lambda|^d\varphi(\lambda x)$ and $\lambda \in \mathbb{R}$. Using Lemma 6.3.4 this implies that the assumptions of Theorem 6.3.1 are satisfied.

Since the Riesz projections R_k for $k = 1, \dots, d$ are bounded on $L^2(\mathbb{R}^d, w)$ for all $w \in A_2$ by Theorem 3.4.1, applying Theorem 6.3.1 we find that for all $f \in C_c^\infty(\mathbb{R}^d; X)$

$$\|R_k f\|_{L^p(\mathbb{R}^d;X)} \lesssim_d \|\widetilde{\Gamma}\|_{\ell^2}^4 \|f\|_{L^p(\mathbb{R}^d;X)}, \qquad k = 1, \cdots, d.$$

So, by the density of $C_c^\infty(\mathbb{R}^d; X)$ in $L^p(\mathbb{R}^d; X)$, the Riesz projections R_k are bounded on $L^p(\mathbb{R}^d; X)$, which means that X has the UMD property by Theorem 2.7.1. □

The proof scheme of Theorem 6.3.5 can be adapted to various other operators. We mention two examples:

- In [7] it was shown that the UMD property is sufficient for the ℓ^2-boundedness of a quite large family of convolution operators on $L^p(\mathbb{R}^d; X)$. Using a similar proof as the one presented in Theorem 6.3.5, one can show that the UMD property of the Banach function space X is necessary for the ℓ^2-boundedness of this family of operators.

- On a general Banach space X we know by a result of Coulhon and Lamberton [CL86] (recently quantified by Hytönen [Hyt15]), that the maximal L^p-regularity of $(-\Delta)^{1/2}$ implies that X has the UMD property. Maximal L^p-regularity implies the $\mathcal{R}$-sectoriality of $(-\Delta)^{1/2}$ on $L^p(\mathbb{R}^d; X)$ by a result of Clément and Prüss [CP01] and the converse holds if X has the UMD property by [Wei01b]. It is therefore a natural question to ask whether the $\mathcal{R}$-sectoriality of $(-\Delta)^{1/2}$ on $L^p(\mathbb{R}^d; X)$ also implies that X has the UMD property. By the equivalence of $\mathcal{R}$-sectoriality and ℓ^2-sectoriality on Banach lattices with finite cotype, we can show that this is indeed the case for Banach function spaces with finite cotype, using a similar proof as in the proof of Theorem 6.3.5. The question for general Banach spaces remains open. This is also the case for the question whether the $\mathcal{R}$-sectoriality of $-\Delta$ on $L^p(\mathbb{R}^d; X)$ implies that X has the UMD property, see [HNVW17, Problem 7].

6.4. THE LATTICE HARDY–LITTLEWOOD MAXIMAL OPERATOR

We now turn our attention to the lattice Hardy–Littlewood maximal operator, which will play an important role in our sparse domination-based extension theorem. We will study this operator on a space of homogeneous type, although we will restrict ourselves to $\mathbb{R}^d$ for the extension theorem. We will start by introducing the Hardy–Littlewood property of a Banach function space X and study some of its properties. Afterwards we will be in a position to define the lattice Hardy–Littlewood maximal operator and deduce sharp weighted bounds using sparse domination. We will end this section with a comparison between the lattice Hardy–Littlewood maximal operator and the Rademacher maximal operator introduced in Section 3.6.

6.4.1. THE HARDY–LITTLEWOOD PROPERTY

Let X be a Banach function space, let (S,d,μ) be a space of homogeneous type with dyadic system $\mathscr{D}$ and let $\mathcal{D}\subseteq\mathscr{D}$ be a finite collection of dyadic cubes. For $f\in L^1_{\mathrm{loc}}(S;X)$ we define

$$M^{\mathcal{D}}_{\mathrm{Lat}}f := \sup_{Q\in\mathcal{D}} \langle |f|\rangle_{1,Q}\,\mathbf{1}_Q,$$

where the supremum is taken in the lattice sense. We say that X has the *Hardy–Littlewood property* and write $X\in \mathrm{HL}$ if, for some $p\in(1,\infty)$, we have

$$\mu_{p,X} := \sup_{\mathcal{D}} \|M^{\mathcal{D}}_{\mathrm{Lat}}\|_{L^p([0,1);X)\to L^p([0,1);X)} < \infty,$$

where the supremum is taken over all finite collections of dyadic cubes $\mathcal{D}$ in $[0,1)$. We took the unit interval $[0,1)$ in this definition, since it was shown by Deleaval, Kriegler and Kemppainen in [DKK18, Lemma 3.4] that for any finite collection of dyadic cubes $\mathcal{D}\subseteq\mathscr{D}$ one has

$$\|M^{\mathcal{D}}_{\mathrm{Lat}}\|_{L^p(S;X)\to L^p(S;X)} \le \mu_{p,X}. \tag{6.4.1}$$

The Hardy–Littlewood property is independent of $p\in(1,\infty)$, which was shown by García–Cuerva, Macias and Torrea in [GMT93]. Our first goal will be to prove sparse domination for $M^{\mathcal{D}}_{\mathrm{Lat}}$, which also implies this p-independence. We start with a weak L^1-estimate.

Lemma 6.4.1. *Let X be a Banach function space and let (S,d,μ) be a space of homogeneous type with a dyadic system $\mathscr{D}$. If $X\in\mathrm{HL}$, then we have for any finite collection of dyadic cubes $\mathcal{D}\subseteq\mathscr{D}$ and $p\in(1,\infty)$*

$$\|M^{\mathcal{D}}_{\mathrm{Lat}}\|_{L^1(S;X)\to L^{1,\infty}(S;X)} \lesssim_{S,\mathscr{D},p} \mu_{p,X}.$$

Proof. Fix $\mathcal{D}\subseteq\mathscr{D}$ finite and take $f\in L^1(S;X)$ with norm 1. For $\lambda>0$ define

$$\mathcal{S} := \{Q\in\mathscr{D} : Q \text{ maximal (w.r.t inclusion) such that } \langle\|f\|_X\rangle_{1,Q} > \lambda\}$$

and set

$$O := \bigcup_{Q\in\mathcal{S}} Q = \{M^{\mathscr{D}}(\|f\|_X) > \lambda\}.$$

For a fixed $P\in\mathcal{D}$ note that if $P\setminus O\neq\varnothing$, then

$$\begin{aligned}\langle|f|\rangle_{1,P}\,\mathbf{1}_P &= \Big\langle |f|\,\mathbf{1}_{S\setminus O} + \sum_{\substack{Q\in\mathcal{S}:\\ Q\subseteq P}} |f|\,\mathbf{1}_Q\Big\rangle_{1,P}\mathbf{1}_P\\ &= \Big\langle |f|\,\mathbf{1}_{S\setminus O} + \sum_{Q\in\mathcal{S}} \langle|f|\rangle_{1,Q}\,\mathbf{1}_Q\Big\rangle_{1,P}\mathbf{1}_P\end{aligned}$$

using the disjointness of the cubes in $\mathcal{S}$ and

$$\big\langle\langle f\rangle_{1,Q}\,\mathbf{1}_Q\big\rangle_{1,P} = \langle f\,\mathbf{1}_Q\rangle_{1,P}, \qquad Q\subseteq P$$

in the second equality. Taking the supremum over $P \in \mathcal{D}$ we can estimate

$$\begin{aligned} M_{\mathrm{Lat}}^{\mathcal{D}} f &\le \sup_{P\in\mathcal{D}} \Big\langle |f|\,\mathbf{1}_{S\setminus O} + \sum_{Q\in\mathcal{S}} \langle |f|\rangle_{1,Q}\,\mathbf{1}_Q \Big\rangle_{1,P} \mathbf{1}_P + \langle |f|\rangle_{1,P}\,\mathbf{1}_O \\ &\le M_{\mathrm{Lat}}^{\mathcal{D}} g + b, \end{aligned}$$

where $b = \langle |f|\rangle_{1,P}\,\mathbf{1}_O$ and

$$g := g_1 + g_2 := |f|\,\mathbf{1}_{S\setminus O} + \sum_{Q\in\mathcal{S}} \langle |f|\rangle_{1,Q}\,\mathbf{1}_Q .$$

By the disjointness of the cubes in $\mathcal{S}$, we have $\|g\|_{L^1(S;X)} = \|f\|_{L^r(S;X)} = 1$. Moreover, since

$$\operatorname{supp} b \subseteq O = \{M^{\mathscr{D}}(\|f\|_X) > \lambda\}$$

and $M^{\mathscr{D}}$ is weak L^1-bounded by Proposition 2.2.1, we have

$$\big|\{\|b\|_X > \lambda\}\big| \le \big|\{M^{\mathscr{D}}\big(\|f\|_X\big) > \lambda\}\big| \le \frac{1}{\lambda}.$$

Next we estimate the L^∞-norm of g. We have, by the Lebesgue differentiation theorem, that

$$\|g_1\|_X = \|f\|_X\,\mathbf{1}_{S\setminus O} \le M^{\mathscr{D}}\big(\|f\|_X\big)\,\mathbf{1}_{S\setminus O} \le \lambda$$

and, using the maximality of the cubes in $\mathcal{S}$, we have

$$\|g_2\|_X = \Big\| \sum_{Q\in\mathcal{S}} \langle |f|\rangle_{1,Q}\,\mathbf{1}_Q \Big\|_X \lesssim_{S,\mathscr{D}} \sum_{Q\in\mathcal{S}} \langle \|f\|_X\rangle_{1,\hat{Q}}\,\mathbf{1}_Q \le \lambda,$$

where $\hat{Q}$ is the dyadic parent of $Q \in \mathcal{S}$. Thus we have $\|g\|_{L^\infty(S;X)} \lesssim_{S,\mathscr{D}} \lambda$.

Combining the estimates for g and b with (6.4.1), we obtain for $p \in (1,\infty)$

$$\begin{aligned} \Big|\Big\{\big\|M_{\mathrm{Lat}}^{\mathcal{D}} f\big\|_X > 2\lambda\Big\}\Big| &\le \Big|\Big\{\big\|M_{\mathrm{Lat}}^{\mathcal{D}} g\big\|_X > \lambda\Big\}\Big| + \big|\{\|b\|_X > \lambda\}\big| \\ &\le \mu_{p,X}\,\frac{\|g\|_{L^p(S;X)}^p}{\lambda^p} + \frac{1}{\lambda} \\ &\lesssim_{S,\mathscr{D},p} \mu_{p,X}\,\frac{\|g\|_{L^1(S;X)}\cdot\lambda^{p-1}}{\lambda^p} + \frac{1}{\lambda} \le \mu_{p,X}\,\frac{2}{\lambda}. \end{aligned}$$

Taking the supremum over $f \in L^1(S;X)$ with norm 1 yields the conclusion. □

Using Lemma 6.4.1 we can prove sparse domination for $M_{\mathrm{Lat}}^{\mathcal{D}}$, which was shown for an arbitrary locally finite (not necessarily doubling) Borel measure on $\mathbb{R}^d$ by Hänninen and the author in [9]. The argument presented in [9] was tailor-made for $M_{\mathrm{Lat}}^{\mathcal{D}}$, whereas here we prefer to employ the abstract sparse domination principle in Theorem 3.2.2 once more. We will revisit the argument from [9] in Section 6.5 in the context of the bisublinear lattice Hardy–Littlewood maximal operator.

Proposition 6.4.2. *Let X be a Banach function space, let $\mathscr{D}$ be a dyadic system in S and $p \in (1,\infty)$. Suppose that X is r-convex for $r \in [1,\infty)$ and $X \in \mathrm{HL}$. Then for any finite collection of cubes $\mathcal{D} \subseteq \mathscr{D}$ and $f \in L^1(S;X)$ there exists a $\frac{1}{2}$-sparse collection of cubes $\mathcal{S} \subseteq \mathscr{D}$ such that*

$$\big\| M^{\mathcal{D}}_{\mathrm{Lat}} f(s) \big\|_X \lesssim_{S,\mathscr{D},p,r} \mu_{p,X} \cdot \Big(\sum_{Q \in \mathcal{S}} \langle \|f\|_X \rangle^r_{1,Q} \mathbf{1}_Q(s) \Big)^{1/r}, \qquad s \in S.$$

Proof. We will check the assumptions of Theorem 3.2.2 for $M^{\mathcal{D}}_{\mathrm{Lat}}$. By Lemma 6.4.1 we know that we can view $M^{\mathcal{D}}_{\mathrm{Lat}}$ as a bounded operator

$$M^{\mathcal{D}}_{\mathrm{Lat}} \colon L^1(S;X) \to L^{1,\infty}\big(S; X(\ell^\infty(\mathcal{D}))\big)$$

given by

$$M^{\mathcal{D}}_{\mathrm{Lat}} f = \big(\langle |f| \rangle_{1,Q} \mathbf{1}_Q \big)_{Q \in \mathcal{D}}.$$

For any collection of cubes $\mathcal{D}' \subseteq \mathcal{D}$ we interpret $M^{\mathcal{D}'}_{\mathrm{Lat}}$ similarly. For $Q \in \mathscr{D}$ set

$$\mathcal{D}(Q) := \{P \in \mathcal{D} : P \subseteq Q\}$$

and define $T_Q := M^{\mathcal{D}(Q)}_{\mathrm{Lat}}$. Then $\{T_Q\}_{Q \in \mathscr{D}}$ is a 1-localization family for $M^{\mathcal{D}}_{\mathrm{Lat}}$. Furthermore we have for $f \in L^1(S;X)$ and $s \in Q \in \mathscr{D}$ that

$$\mathcal{M}^{\#}_{M^{\mathcal{D}}_{\mathrm{Lat}},Q} f(s) = \sup_{\substack{Q' \in \mathscr{D}(Q)\colon\, s',s'' \in Q' \\ s \in Q'}} \operatorname{ess\,sup} \big\| T_{Q \setminus Q'} f(s') - T_{Q \setminus Q'} f(s'') \big\|_{X(\ell^\infty(\mathcal{D}))} = 0,$$

where the last step follows from the fact that $T_{Q \setminus Q'} f = M^{\mathcal{D}(Q) \setminus \mathcal{D}(Q')}_{\mathrm{Lat}} f$ is constant on Q'. So $\mathcal{M}^{\#}_{M^{\mathcal{D}}_{\mathrm{Lat}},Q}$ is trivially bounded from $L^1(S;X)$ to $L^{1,\infty}(S)$.

To check the localized ℓ^r-estimate for $M^{\mathcal{D}}_{\mathrm{Lat}}$ take $Q_1, \cdots, Q_n \in \mathcal{D}$ with $Q_n \subseteq \cdots \subseteq Q_1$. Then for $s \in Q_n$ and $f \in L^1(S;X)$ we have

$$\begin{aligned}
\big\| T_{Q_1} f(s) \big\|_{X(\ell^\infty(\mathcal{D}))} &= \Big\| \sup\Big\{ \big| T_{Q_n} f(s) \big|, \big| T_{Q_{n-1} \setminus Q_n} f(s) \big|, \dots, \big| T_{Q_1 \setminus Q_2} f(s) \big| \Big\} \Big\|_{X(\ell^\infty(\mathcal{D}))} \\
&\le \Big\| \Big(\big| T_{Q_n} f(s) \big|^r + \sum_{k=1}^{n-1} \big| T_{Q_k \setminus Q_{k+1}} f(s) \big|^r \Big)^{1/r} \Big\|_{X(\ell^\infty(\mathcal{D}))} \\
&\le \Big(\big\| T_{Q_n} f(s) \big\|^r_{X(\ell^\infty(\mathcal{D}))} + \sum_{k=1}^{n-1} \big\| T_{Q_k \setminus Q_{k+1}} f(s) \big\|^r_{X(\ell^\infty(\mathcal{D}))} \Big)^{1/r},
\end{aligned}$$

using the r-convexity of X in the last step. Having checked all assumptions of Theorem 3.2.2 for $M^{\mathcal{D}}_{\mathrm{Lat}}$, it follows that for any $Q \in \mathcal{D}$ there is a $\frac{1}{2}$-sparse collection of cubes $\mathcal{S}_Q \subseteq \mathscr{D}(Q)$ such that

$$\big\| T_Q(s) \big\|_Y \lesssim_{S,\mathscr{D},p,r} \mu_{p,X} \Big(\sum_{P \in \mathcal{S}} \langle \|f\|_X \rangle^r_{1,P} \mathbf{1}_P(s) \Big)^{1/r}, \qquad s \in Q.$$

Let $\mathcal{D}'$ be the maximal cubes (with respect to set inclusion) of $\mathcal{D}$, which are pairwise disjoint. Then $\mathcal{S} := \bigcup_{Q \in \mathcal{D}'} \mathcal{S}_Q$ is a $\frac{1}{2}$-sparse collection of cubes that satisfies the claimed sparse domination as $T_Q(s) = M^{\mathcal{D}}_{\mathrm{Lat}} f(s)$ for any $Q \in \mathcal{D}'$ and $s \in Q$ and $M^{\mathcal{D}}_{\mathrm{Lat}} f$ is zero outside $\bigcup_{Q \in \mathcal{D}'} Q$. □

As a direct corollary of Proposition 6.4.2 and Proposition 3.2.4, we now see that the Hardy–Littlewood property is independent of $p \in (1,\infty)$. Moreover $M^{\mathcal{D}}_{\mathrm{Lat}}$ satisfies weighted bounds, which we will discuss more generally in Subsection 6.4.2.

Corollary 6.4.3. *Let X be a Banach function space with $X \in \mathrm{HL}$. For $p, q \in (1,\infty)$ we have*

$$\mu_{p,X} \eqsim_{p,q} \mu_{q,X}.$$

The sparse domination result in Proposition 6.4.2 is sharp. In fact, we can show that the exponent

$$r^* := \sup\{r \in (1,\infty) : X \text{ is } r\text{-convex}\}$$

is critical: The sparse domination in Proposition 6.4.2 holds for all $r < r^*$ and fails for all $r > r^*$. Moreover it holds for $r = r^*$ if X is r^*-convex, but we do not settle the case $r = r^*$ if X is not r-convex, which happens for example if $X = L^{p,r}(\mathbb{R})$ with $p \in (1,r)$.

Proposition 6.4.4. *Let X be a Banach function space and let $\mathscr{D}$ be the standard dyadic system in $[0,1)$. Take $r \in (1,\infty)$ and assume that for each finite collection $\mathcal{D} \subseteq \mathscr{D}$ of dyadic intervals and $f \in L^1([0,1);X)$ there exists a $\frac{1}{2}$-sparse collection of intervals $\mathcal{S} \subseteq \mathscr{D}$ such that*

$$\|M^{\mathcal{D}}_{\mathrm{Lat}} f(t)\|_X \lesssim_{X,r} \Big(\sum_{Q\in\mathcal{S}} \langle \|f\|_X\rangle^r_{1,Q} \mathbf{1}_Q(t)\Big)^{1/r}, \qquad t \in [0,1).$$

Then X is q-convex for all $q \in [1,r)$.

Proof. Fix $n \in \mathbb{N}$ and define $Q_k = [0, 2^{-(n-k)})$ for $k = 0, \cdots, n$. Let $x_1, \cdots, x_n \in X$ be pairwise disjointly supported and assume without loss of generality that $\|x_1\|_X \le \cdots \le \|x_n\|_X$. Define $\mathcal{D} = \bigcup_{k=0}^n Q_k$ and $f = \sum_{k=1}^n \mathbf{1}_{Q_k \setminus Q_{k-1}} x_k$. Let $\mathcal{S} \subseteq \mathscr{D}$ be $\frac{1}{2}$-sparse such that

$$\|M^{\mathcal{D}}_{\mathrm{Lat}} f(t)\|_X \lesssim_{X,r} \Big(\sum_{Q\in\mathcal{S}} \langle \|f\|_X\rangle^r_{1,Q} \mathbf{1}_Q(t)\Big)^{1/r}, \qquad t \in [0,1). \tag{6.4.2}$$

Note that

$$\langle |f|\rangle_{1,Q_k} \ge \frac{\mu(Q_k \setminus Q_{k-1})}{\mu(Q_k)} |x_k| \ge \Big(1 - \frac{1}{2}\Big)|x_k| = \frac{1}{2}|x_k|.$$

Since the x_k's are disjointly supported, we have $\big|\sum_{k=1}^n x_k\big| = \sup_{1\le k\le n} |x_k|$. Therefore we have

$$\Big\|\sum_{k=1}^n x_k\Big\|_X = \big\|\sup_{1\le k\le n} |x_k|\big\|_X \le 2\,\|M^{\mathcal{D}}_{\mathrm{Lat}} f(t)\|_X, \qquad t \in Q_0. \tag{6.4.3}$$

Moreover, since $\|x_1\|_X \le \cdots \le \|x_n\|_X$, we have that

$$\langle \|f\|_X\rangle_{1,Q_k} = \frac{1}{\mu(Q_k)} \sum_{j=1}^k \mu(Q_j \setminus Q_{j-1}) \|x_j\|_X \le \|x_k\|_X.$$

Since $f \equiv 0$ on Q_0, this yields

$$\Big(\sum_{Q\in\mathcal{S}} \langle \|f\|_X\rangle^r_{1,Q} \mathbf{1}_Q(t)\Big)^{1/r} \le \Big(\sum_{k=1}^n \big(\langle \|f\|\rangle_{1,Q_k}\big)^r\Big)^{1/r} \le \Big(\sum_{k=1}^n \|x_k\|_X^r\Big)^{\frac{1}{r}}, \qquad t \in Q_0 \tag{6.4.4}$$

Combining (6.4.2), (6.4.3) and (6.4.4), we deduce that

$$\Big\|\sum_{k=1}^n x_k\Big\|_X \lesssim_{X,r} \Big(\sum_{k=1}^n \|x_k\|_X^r\Big)^{\frac{1}{r}},$$

for all pairwise disjoint vectors $x_1,\cdots,x_n \in X$. This is called an *upper r-estimate* for X. By [LT79, Theorem 1.f.7], this implies that X is q-convex for all $q \in [1,r)$. □

Remark 6.4.5. The proof of Proposition 6.4.4 can be extended to any space of homogeneous type (S,d,μ) with a dyadic system $\mathscr{D}$ such that for any $n \in \mathbb{N}$ there are dyadic cubes $Q_0 \subseteq \cdots \subseteq Q_n$ with $\mu(Q_{k-1}) \le \frac{1}{2}\mu(Q_k)$ for $k = 1,\dots,n$.

It was proven by Bourgain [Bou84] and Rubio de Francia [Rub86] that a sufficient condition for X to have the Hardy–Littlewood property is that X has the UMD property. We will recover this result using Theorem 6.3.1 and obtain an explicit estimate of $\mu_{p,X}$ in terms of the UMD constant $\beta_{p,X}$. Tracking this dependence in the proof of Bourgain and Rubio de Francia would be hard, as it involves the weight characteristic dependence of the inequality [Rub86, (a.5)].

Theorem 6.4.6. *Let X be Banach function space with cotype $q \in (1,\infty)$. If $X \in$ UMD, then $X \in$ HL and for $p \in (1,\infty)$ we have*

$$\mu_{p,X} \lesssim_p q\big(c_{q,X}\beta_{p,X}\big)^2.$$

Proof. Let $p \in (1,\infty)$ and $f \in L^p(\mathbb{R})$. Define for any interval $I \subseteq \mathbb{R}$ the averaging operator

$$T_I f(t) := \langle f\rangle_{1,I}\mathbf{1}_I(t), \qquad t \in \mathbb{R}$$

and set $\Gamma := \{T_I : I \text{ an interval in } \mathbb{R}\}$. Then we know that $\widetilde{\Gamma}$ is ℓ^2-bounded on $L^p(\mathbb{R};X)$ with

$$\|\widetilde{\Gamma}\|_{\ell^2} \lesssim_p \sqrt{q}c_{q,X}\beta_{p,X}$$

by [HNVW17, Proposition 8.1.13] and Proposition 2.6.3.

Let $w\colon \mathbb{R} \to (0,\infty)$ and set $C := \sup_{T\in\Gamma}\|T\|_{L^2(\mathbb{R},w)\to L^2(\mathbb{R},w)}$. Fix an interval $I \subseteq \mathbb{R}$. Applying T_I to the function $(w+\varepsilon)^{-1}\mathbf{1}_I$ for some $\varepsilon > 0$ we obtain

$$\int_I\Big(\frac{1}{|I|}\int_I (w(t)+\varepsilon)^{-1}\,\mathrm{d}t\Big)^2 w(s)\,\mathrm{d}s \le C^2\int_I \frac{w(t)}{(w(t)+\varepsilon)^2}\,\mathrm{d}t$$

which implies

$$\Big(\frac{1}{|I|}\int_Q w(t)\,\mathrm{d}t\Big)\Big(\frac{1}{|I|}\int_B (w(t)+\varepsilon)^{-1}\,\mathrm{d}t\Big) \le C^2$$

So by letting $\varepsilon \to 0$ with the monotone convergence theorem, we obtain $w \in A_2$ with $[w]_{A_2} \le C^2$. Therefore Γ satisfies the assumptions of Theorem 6.3.1 with $\phi(t) = t^2$.

Fix a finite collection of dyadic intervals $\mathcal{D} \subseteq \mathscr{D}$ in $\mathbb{R}$. For any simple function $f \in L^p(\mathbb{R};X)$ we have

$$M_{\mathrm{Lat}}^{\mathcal{D}} f(t,\omega) \le M^{\mathscr{D}}\big(f(\cdot,\omega)(t)\big), \qquad t \in \mathbb{R},\ \omega \in \Omega.$$

So by Theorem 6.3.1, using the weighted boundedness of $M^{\mathscr{D}}$ from Proposition 2.3.2(v), we know that for any simple function $f \in L^p(\mathbb{R};X)$ we have

$$\|M^{\mathcal{D}}_{\mathrm{Lat}} f\|_{L^p(\mathbb{R};X)} \lesssim_p q\big(c_{q,X}\beta_{p,X}\big)^2 \|f\|_{L^p(\mathbb{R};X)}.$$

Thus, by the density of the simple functions in $L^p(\mathbb{R};X)$ and restricting to the unit interval $[0,1)$, we obtain

$$\|M^{\mathcal{D}}_{\mathrm{Lat}}\|_{L^p([0,1);X)\to L^p([0,1);X)} \le \|M^{\mathcal{D}}_{\mathrm{Lat}}\|_{L^p(\mathbb{R};X)\to L^p(\mathbb{R};X)} \lesssim_p q\big(c_{q,X}\beta_{p,X}\big)^2.$$

Taking the supremum over all finite collections of dyadic intervals, the theorem follows. □

Remark 6.4.7.

- As in the proof of Theorem 6.1.1, we could also use $\Gamma = \{H\}$ or $\Gamma = \{R_k : k = 1,\cdots,d\}$ in the proof of Theorem 6.4.6, where H is the Hilbert transform and R_k is the k-th Riesz projection. This yields a bound on $\mu_{p,X}$ in terms of the norm of the Hilbert transform or Riesz projections.

- As already discussed in Remark 6.3.3, the assumption that X has finite cotype may be omitted in Theorem 6.4.6. This yields the bound $\mu_{p,X} \lesssim \beta^3_{p,X}$ in the conclusion of Theorem 6.4.6.

- The converse of Theorem 6.4.6 holds if we impose the Hardy–Littlewood property on both X and X^* (see [Bou84] and [Rub86]). We will provide a quantitative proof of this fact in Section 6.6.

6.4.2. THE HARDY–LITTLEWOOD MAXIMAL OPERATOR

Let X be an order-continuous Banach function space and let (S,d,μ) be a space of homogeneous type with dyadic system $\mathscr{D}$. Take $p \in (1,\infty)$ and $w \in A_p$. If $X \in \mathrm{HL}$ we define the *(dyadic) lattice Hardy–Littlewood maximal operator* for $f \in L^p(S;X)$ by

$$M^{\mathscr{D}}_{\mathrm{Lat}} f := \sup_{Q\in\mathscr{D}: s\in Q} \langle |f| \rangle_{1,Q} \mathbf{1}_Q,$$
$$M_{\mathrm{Lat}} f := \sup_{B\ni s} \langle |f| \rangle_{1,B} \mathbf{1}_B,$$

where the suprema are taken in the lattice sense and the second supremum is taken over all balls $B \subseteq S$ containing s.

Our main result in this section is that both $M^{\mathscr{D}}_{\mathrm{Lat}}$ and M_{Lat} are bounded operators on $L^p(S,w;X)$. This is a direct consequence of the sparse domination result in Proposition 6.4.2, the weighted estimates in Proposition 3.2.4 and the existence of adjacent dyadic systems as in Proposition 2.1.1.

Theorem 6.4.8. *Let X be an order-continuous Banach function space and let (S,d,μ) be a space of homogeneous type with dyadic system $\mathscr{D}$. Suppose that X is r-convex for $r\in[1,\infty)$ and $X\in \mathrm{HL}$. Both $M_{\mathrm{Lat}}^{\mathscr{D}}$ and M_{Lat} are bounded operators on $L^p(\mathbb{R}^d,w;X)$ for all $p\in(1,\infty)$ and $w\in A_p$ with*

$$\|M_{\mathrm{Lat}}^{\mathscr{D}}\|_{L^p(S,w;X)\to L^p(S,w;X)} \lesssim_{S,\mathscr{D},p,r} \mu_{p,X}\,[w]_{A_p}^{\max\left\{\frac{1}{p-1},\frac{1}{r}\right\}}$$
$$\|M_{\mathrm{Lat}}\|_{L^p(S,w;X)\to L^p(S,w;X)} \lesssim_{S,p,r} \mu_{p,X}\,[w]_{A_p}^{\max\left\{\frac{1}{p-1},\frac{1}{r}\right\}}$$

Proof. We start with the claim for $M_{\mathrm{Lat}}^{\mathscr{D}}$. Let $f\in L^p(S,w;X)$ and let $\mathcal{D}_k$ be a finite collection of cubes for each $k\in\mathbb{N}$ such that $\mathcal{D}_k\subseteq\mathcal{D}_{k+1}$ and $\bigcup_{k\in\mathbb{N}}\mathcal{D}_k=\mathscr{D}$. By Proposition 6.4.2 and Proposition 3.2.4 we have

$$\sup_{k\in\mathbb{N}}\|M_{\mathrm{Lat}}^{\mathcal{D}_k}f\|_{L^p(S,w;X)} \lesssim_{S,\mathscr{D},p,r} \mu_{p,X}\,[w]_{A_p}^{\max\left\{\frac{1}{p-1},\frac{1}{r}\right\}}.$$

Thus, using the Fatou property of X, it follows that $M_{\mathrm{Lat}}^{\mathscr{D}}f(s)\in X$ for a.e. $s\in S$. Moreover, since X is order-continuous, $\left(M_{\mathrm{Lat}}^{\mathcal{D}_n}f(s)\right)_{k\in\mathbb{N}}$ converges to $M_{\mathrm{Lat}}^{\mathscr{D}}f(s)$ for a.e. $s\in S$. As $M_{\mathrm{Lat}}^{\mathcal{D}_k}f$ is a simple function for each $k\in\mathbb{N}$, we can conclude that $M_{\mathrm{Lat}}^{\mathscr{D}}f$ is strongly measurable, i.e. $M_{\mathrm{Lat}}^{\mathscr{D}}f\in L^0(S;X)$. Furthermore, using the Fatou property of $L^p(S,w;X)$, we have that $M_{\mathrm{Lat}}^{\mathscr{D}}$ is a bounded operator on $L^p(S,w;X)$ for $p\in(1,\infty)$ with

$$\|M_{\mathrm{Lat}}^{\mathscr{D}}\|_{L^p(S;X)\to L^p(S;X)} \lesssim_{S,\mathscr{D},p,r} \mu_{p,X}\,[w]_{A_p}^{\max\left\{\frac{1}{p-1},\frac{1}{r}\right\}}. \tag{6.4.5}$$

To see that $M_{\mathrm{Lat}}f: S\to X$ is well-defined, we note that by Proposition 2.1.1, there exist dyadic systems $\mathscr{D}^1,\cdots,\mathscr{D}^m$ such that for any ball $B\subseteq S$

$$\langle|f|\rangle_{1,B}\,\mathbf{1}_B \lesssim_S \sum_{j=1}^m M_{\mathrm{Lat}}^{\mathscr{D}^j}f\in L^p(S,w;X).$$

Since X is order-continuous, we know that $L^p(S,w;X)$ is an order-complete Banach function space (see [LT79, Theorem 1.a.8]) and therefore $M_{\mathrm{Lat}}f\in L^p(S,w;X)$. Moreover we have

$$\left\|M_{\mathrm{Lat}}f\right\|_{L^p(S,w;X)} \lesssim_S \sum_{j=1}^m \|M_{\mathrm{Lat}}^{\mathscr{D}^j}\|_{L^p(S;X)\to L^p(S;X)} \lesssim_{S,p,r} \mu_{p,X}\,[w]_{A_p}^{\max\left\{\frac{1}{p-1},\frac{1}{r}\right\}},$$

which finishes the proof. □

Remark 6.4.9.

- On a UMD Banach function space X, the boundedness of a centered version of M_{Lat} on $L^p(\mathbb{R}^d;X)$ with $\|M_{\mathrm{Lat}}\|_{L^p(\mathbb{R}^d;X)\to L^p(\mathbb{R}^d;X)}\le C$ for a constant $C>0$ independent of d has been shown by Kriegler and Deleaval in [DK19a].

- In the particular case $X = \ell^r$, the dependence on the A_p-characteristic in Theorem 6.4.8 is sharp. This can be shown by a similar argument as in Proposition 3.6.2 (see [CMP12]). In the general case that X is a Banach function space that is r-convex for some $r \in (1,\infty)$, the exponent

$$r^* := \sup\{r \in (1,\infty) : X \text{ is } r\text{-convex}\}$$

is again critical: The weighted estimate with the dependence $[w]_{A_p}^{\max\{\frac{1}{p-1},\frac{1}{r}\}}$ holds for all $r < r^*$ and fails for all $r > r^*$. This follows from embedding a copy of ℓ_n^r with $r < r^*$ into X for a large enough n (by applying [LT79, Theorem 1.f.12]) and using the sharpness in the case ℓ_n^r.

6.4.3. COMPARISON WITH THE RADEMACHER MAXIMAL OPERATOR

To finish our study of the lattice Hardy–Littlewood maximal operator, we will compare it to the Rademacher maximal operator, introduced in Section 3.6. Let X be a Banach function space with finite cotype and let $\mathscr{D}$ be the standard dyadic system on $[0,1)$. If $X \in \mathrm{HL}$, then by the Khintchine–Maurey inequalities (see Proposition 2.5.1) we have

$$M_{\mathrm{Rad}}^{\mathscr{D}} f(s) \lesssim_X M_{\mathrm{Lat}}^{\mathscr{D}} f(s), \qquad s \in [0,1),$$

so in particular X has the RMF property. Thus we know by Theorem 6.4.6 that any UMD Banach function space has the RMF property.

Comparing the sparse domination result for $M_{\mathrm{Rad}}^{\mathcal{D}}$ in Theorem 3.6.1 with the sparse domination result for $M_{\mathrm{Lat}}^{\mathcal{D}}$ in Proposition 6.4.2, we see that the sparse operator for $M_{\mathrm{Rad}}^{\mathcal{D}}$ is smaller than the sparse operator for $M_{\mathrm{Lat}}^{\mathcal{D}}$. Since the sparse domination for $M_{\mathrm{Lat}}^{\mathcal{D}}$ on $[0,1)$ is sharp by Proposition 6.4.4, it follows that the operators $M_{\mathrm{Rad}}^{\mathscr{D}}$ and $M_{\mathrm{Lat}}^{\mathscr{D}}$ are incomparable on any RMF Banach function space that is not ∞-convex, , i.e. the dyadic lattice Hardy–Littlewood maximal operator is strictly larger than the Rademacher maximal operator. As the only ∞-convex RMF Banach function spaces are the finite dimensional ones, we have the following corollary.

Corollary 6.4.10. *Let X be an infinite dimensional Banach function space with* RMF *and* HL . *Then there does not exist a $C > 0$ such that for all $f \in L^p([0,1);X)$*

$$M_{\mathrm{Lat}}^{\mathscr{D}} f(s) \le C\, M_{\mathrm{Rad}}^{\mathscr{D}} f(s), \qquad s \in [0,1).$$

6.5. THE BISUBLINEAR (LATTICE) HARDY–LITTLEWOOD MAXIMAL OPERATOR

As mentioned in the introduction, our second approach to extend a bounded operator T on $L^p(\mathbb{R}^d)$ to a bounded operator on $L^p(\mathbb{R}^d;X)$ is based on sparse domination for a bisublinear version of the lattice Hardy–Littlewood maximal operator, which we will introduce in this section. The results presented here are part of a much more general,

multilinear theory. However, to keep our results accessible and in the spirit of the rest of this dissertation, we will only discuss a special case of the bilinear results and refer to [3] or the dissertation of Nieraeth [Nie20, Part $\frac{1}{4}$] for further results. For notational simplicity we will also restrict to $\mathbb{R}^d$ in this section.

Let us start by introducing the scalar version of the bisublinear Hardy-Littlewood maximal operator. For $f, g \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$ we define

$$M(f,g) := \sup_B \langle |f| \rangle_{1,B} \langle |g| \rangle_{1,B} \mathbf{1}_B,$$

where the supremum is taken over all balls B in $\mathbb{R}^d$. Note that we trivially have

$$M(f,g)(t) \leq Mf(t) \cdot Mg(t), \qquad t \in \mathbb{R}^d.$$

Thus, by Proposition 2.2.1 and Hölder's inequality, we immediately obtain that for $p_1, p_2 \in (1,\infty)$ and $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$ we have that

$$M \colon L^{p_1}(\mathbb{R}^d) \times L^{p_2}(\mathbb{R}^d) \to L^p(\mathbb{R}^d)$$

is a bounded operator. Moreover, Proposition 2.3.2 yields (non-sharp) weighted estimates for the bisublinear Hardy–Littlewood maximal operator.

Our main reason for introducing the bisublinear Hardy-Littlewood maximal operator is its intimate connection with the sparse forms appearing in Theorem 6.1.4.

Proposition 6.5.1. *Let $f, g \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$. There exists an η-sparse collection of cubes $\mathcal{S}$ in $\mathbb{R}^d$ such that*

$$\|M(f,g)\|_{L^1(\mathbb{R}^d)} \lesssim_d \sum_{Q \in \mathcal{S}} \langle |f| \rangle_{1,Q} \langle |g| \rangle_{1,Q} |Q|.$$

Conversely, for any η-sparse collection of cubes $\mathcal{S}$ in $\mathbb{R}^d$ we have

$$\sum_{Q \in \mathcal{S}} \langle |f| \rangle_{1,Q} \langle |g| \rangle_{1,Q} |Q| \lesssim_d \eta \|M(f,g)\|_{L^1(\mathbb{R}^d)}. \tag{6.5.1}$$

Proof. The first claim follows from [Nie19, Lemma 2.9] and Proposition 2.1.1. For the second claim we have

$$\|M(f,g)\|_{L^1(\mathbb{R}^d)} \leq \eta \sum_{Q \in \mathcal{S}} \int_{E_Q} \langle |f| \rangle_{1,Q} \langle |g| \rangle_{1,Q} \, \mathrm{d}t \lesssim_d \eta \sum_{Q \in \mathcal{S}} \langle |f| \rangle_{1,Q} \langle |g| \rangle_{1,Q} |Q|. \qquad \square$$

Proposition 6.5.1 will allow us to rewrite the sparse forms in Theorem 6.1.4 in terms of the bisublinear maximal operator, which is an essential step in its proof. It also allows us to deduce sharp weighted estimates for the bisublinear Hardy–Littlewood maximal operator. For our purposes the case $p_2 = p_1'$ with dual weights suffices, for further weighted estimates we refer to [Nie19, Proposition 2.7].

Proposition 6.5.2. *Let $p \in (1,\infty)$, $w \in A_p$ and set $w' := w^{-\frac{1}{p-1}}$. Then we have for $f \in L^p(\mathbb{R}^d, w)$ and $g \in L^{p'}(\mathbb{R}^d, w')$*

$$\|M(f,g)\|_{L^1(\mathbb{R}^d)} \lesssim_{p,d} [w]_{A_p}^{\max\{\frac{1}{p-1},1\}} \|f\|_{L^p(\mathbb{R}^d,w)} \|g\|_{L^{p'}(\mathbb{R}^d,w')}.$$

Proof. This follows from Proposition 6.5.1 and the proof of Proposition 3.2.4, in particular from (3.2.9) with $p_0 = r = 1$. □

6.5.1. THE BISUBLINEAR LATTICE HARDY–LITTLEWOOD MAXIMAL OPERATOR

We can also define a lattice version of the bisublinear Hardy–Littlewood maximal operator. Let X be an order-continuous Banach function space over a measure space (Ω,μ) and $p_1,p_2\in(1,\infty)$. Since X is order-continuous, X^* is a Banach function space as well and

$$x\in X,\, x^*\in X^*\Rightarrow x\cdot x^*\in L^1(\Omega).$$

Suppose that $X,X^*\in \mathrm{HL}$, which by Theorem 6.4.6 is the case if $X\in \mathrm{UMD}$. We define the bisublinear analog of M_{Lat} for $f\in L^{p_1}(\mathbb{R}^d;X)$ and $g\in L^{p_2}(\mathbb{R}^d;X^*)$ as

$$M_{\mathrm{Lat}}(f,g):=\sup_B \langle |f|\rangle_{1,B}\cdot\langle |g|\rangle_{1,B}\mathbf{1}_B,$$

where the supremum is taken in the lattice sense in $L^1(\Omega)$ over all balls B in $\mathbb{R}^d$. Note that we have

$$M_{\mathrm{Lat}}(f,g)(t)\le M_{\mathrm{Lat}}f(t)\cdot M_{\mathrm{Lat}}g(t),\qquad t\in\mathbb{R}^d,$$

which means that, by the order-completeness of $L^p(\mathbb{R}^d;L^1(\Omega))$, Theorem 6.4.8 and Hölder's inequality, we have for $p_1,p_2\in(1,\infty)$ and $\frac{1}{p}=\frac{1}{p_1}+\frac{1}{p_2}$ that

$$M_{\mathrm{Lat}}\colon L^{p_1}(\mathbb{R}^d;X)\times L^{p_2}(\mathbb{R}^d;X^*)\to L^p(\mathbb{R}^d;L^1(\Omega))$$

is a well-defined, bounded operator. We can also deduce (non-sharp) weighted estimates for the bisublinear lattice Hardy–Littlewood maximal operator from Theorem 6.4.8. To obtain sharp weighted estimates we will deduce a sparse domination result for a dyadic version of the bisublinear lattice Hardy–Littlewood maximal operator. This time, rather than extending the sparse domination principle from Chapter 3 to the bisublinear case, we will extend the argument from Hänninen and the author in [9] to the bisublinear setting.

Let $\mathcal{D}\subseteq\mathscr{D}$ be a finite collection of dyadic cubes in $\mathbb{R}^d$. Define the bisublinear analog of $M^{\mathcal{D}}_{\mathrm{Lat}}$ for $f\in L^1_{\mathrm{loc}}(\mathbb{R}^d;X)$ and $g\in L^1_{\mathrm{loc}}(\mathbb{R}^d;X^*)$ as

$$M^{\mathcal{D}}_{\mathrm{Lat}}(f,g):=\sup_{Q\in\mathcal{D}}\langle |f|\rangle_{1,Q}\cdot\langle |g|\rangle_{1,Q}\mathbf{1}_Q,$$

where the supremum is taken in the lattice sense in $L^1(\Omega)$.

Proposition 6.5.3. *Let X be an order-continuous Banach function space over a measure space (Ω,μ) and $p\in(1,\infty)$. Suppose that $X,X^*\in\mathrm{HL}$. Then for any finite collection of dyadic cubes $\mathcal{D}\subseteq\mathscr{D}$, $f\in L^1_{\mathrm{loc}}(\mathbb{R}^d;X)$ and $g\in L^1_{\mathrm{loc}}(\mathbb{R}^d;X^*)$ there exists a $\frac{1}{2}$-sparse collection of cubes $\mathcal{S}\subseteq\mathcal{D}$ such that*

$$\big\|M^{\mathcal{D}}_{\mathrm{Lat}}(f,g)(t)\big\|_{L^1(\Omega)}\lesssim_{\mathscr{D},p,d}\mu_{p,X}\mu_{p',X^*}\sum_{Q\in\mathcal{S}}\big\langle\|f\|_X\big\rangle_{1,Q}\big\langle\|g\|_{X^*}\big\rangle_{1,Q}\mathbf{1}_Q(t),\qquad t\in\mathbb{R}^d.$$

Proof. Let $f \in L^1_{\mathrm{loc}}(\mathbb{R}^d; X)$, $g \in L^1_{\mathrm{loc}}(\mathbb{R}^d; X^*)$ and fix a finite collection of dyadic cubes $\mathcal{D} \subseteq \mathscr{D}$. Note that we have

$$M^{\mathcal{D}}_{\mathrm{Lat}}(f,g)(t) \le M^{\mathcal{D}}_{\mathrm{Lat}} f(t) \cdot M^{\mathcal{D}}_{\mathrm{Lat}} g(t), \qquad t \in \mathbb{R}^d,$$

so by Hölder's inequality for weak L^p-spaces and Lemma 6.4.1 we have

$$A_0 := \sup_{\mathcal{D} \subseteq \mathscr{D} \text{ finite}} \big\| M^{\mathcal{D}}_{\mathrm{Lat}} \big\|_{L^1(\mathbb{R}^d;X)\times L^1(\mathbb{R}^d;X^*) \to L^{1/2,\infty}(\mathbb{R}^d;L^1(\Omega))} \lesssim_{\mathscr{D},p,d} \mu_{p,X}\mu_{p',X^*}.$$

For a cube $Q \in \mathcal{D}$, we define its stopping children $\mathrm{ch}_{\mathcal{D}}(Q)$ to be the collection of maximal cubes $Q' \in \mathcal{D}$ such that $Q' \subsetneq Q$ and

$$\Big\| \sup_{\substack{P \in \mathcal{D} \\ Q' \subseteq P \subseteq Q}} \langle |f| \rangle_{1,P} \langle |g| \rangle_{1,P} \Big\|_{L^1(\Omega)} > 4A_0 \big\langle \|f\|_X \big\rangle_{1,Q} \big\langle \|g\|_{X^*} \big\rangle_{1,Q}. \tag{6.5.2}$$

Let $\mathcal{S}^1$ be the maximal cubes in $\mathcal{D}$, define recursively $\mathcal{S}^{k+1} := \bigcup_{Q \in \mathcal{S}^k} \mathrm{ch}_{\mathcal{D}}(Q)$ and set $\mathcal{S} := \bigcup_{k=1}^{\infty} \mathcal{S}^k$.

Fix $Q \in \mathcal{S}$ and let $E_Q := Q \setminus \bigcup_{Q' \in \mathrm{ch}_{\mathcal{D}}(Q)} Q'$. Define the set

$$Q^* := \Big\{ t \in \mathbb{R}^d : \big\| M^{\mathcal{D}}_{\mathrm{Lat}}(f\mathbf{1}_Q, g\mathbf{1}_Q)(t) \big\|_{L^1(\Omega)} > 4A_0 \big\langle \|f\|_X \big\rangle_{1,Q} \big\langle \|g\|_{X^*} \big\rangle_{1,Q} \Big\}.$$

Then by the definition of A_0 we have

$$|Q^*|^2 \le \frac{1}{4} \frac{\|f\mathbf{1}_Q\|_{L^1(\mathbb{R}^d;X)} \|g\mathbf{1}_Q\|_{L^1(\mathbb{R}^d;X^*)}}{\big\langle \|f\|_X \big\rangle_{1,Q} \big\langle \|g\|_{X^*} \big\rangle_{1,Q}} = \frac{1}{4}|Q|^2. \tag{6.5.3}$$

Moreover, for $Q' \in \mathrm{ch}_{\mathcal{S}}(Q)$ and $t \in Q'$, we have by (6.5.2)

$$\big\| M^{\mathcal{D}}_{\mathrm{Lat}}(f\mathbf{1}_Q, g\mathbf{1}_Q)(t) \big\|_{L^1(\Omega)} \ge \Big\| \sup_{\substack{P \in \mathcal{D}: \\ Q' \subseteq P \subseteq Q}} \langle |f| \rangle_{1,P} \langle |g| \rangle_{1,P} \Big\|_{L^1(\Omega)} > 4A_0 \big\langle \|f\|_X \big\rangle_{1,Q} \big\langle \|g\|_{X^*} \big\rangle_{1,Q},$$

so $t \in Q^*$ and thus $Q' \subseteq Q^*$. Using the disjointness of the cubes in $\mathrm{ch}_{\mathcal{D}}(Q)$ and (6.5.3), we get

$$\sum_{Q' \in \mathrm{ch}_{\mathcal{S}}(Q)} |Q'| \le |Q^*| \le \frac{1}{2}|Q|.$$

So $|E_Q| \ge \frac{1}{2}|Q|$, which means that $\mathcal{S}$ is a $\frac{1}{2}$-sparse collection of dyadic cubes.

Next, we check that $M^{\mathcal{D}}_{\mathrm{Lat}}(f,g)$ is pointwise dominated by the sparse operator associated to $\mathcal{S}$. For each $P \in \mathcal{D}$ we define

$$\pi_{\mathcal{S}}(P) := \{Q \in \mathcal{S} : Q \text{ minimal such that } P \subseteq Q\},$$

which allows us to partition $\mathcal{D}$ as

$$\mathcal{D} = \bigcup_{Q \in \mathcal{S}} \{P \in \mathcal{D} : \pi_{\mathcal{S}}(P) = Q\}.$$

Fix $Q \in \mathcal{S}$, $t \in Q$ and let $Q' \in \mathcal{D}$ be the minimal cube such that $t \in Q'$ and $\pi_{\mathcal{S}}(Q') = Q$. If $Q' \subsetneq Q$ we have by the definition of Q' that

$$\Big\| \sup_{\substack{P \in \mathcal{D}: \\ \pi_{\mathcal{S}}(P)=Q}} \langle |f| \rangle_{1,P} \langle |g| \rangle_{1,P} \mathbf{1}_P(t) \Big\|_{L^1(\Omega)} = \Big\| \sup_{\substack{P \in \mathcal{D}: \\ Q' \subseteq P \subseteq Q}} \prod_{j=1}^{m} \langle f_j \rangle_{r_j,P} \Big\|_X$$
$$\leq 4 A_0 \big\langle \|f\|_X \big\rangle_{1,Q} \big\langle \|g\|_{X^*} \big\rangle_{1,Q} \mathbf{1}_Q(t).$$

If $Q' = Q$ the same estimate follows directly from the triangle inequality in X and X^*. We can conclude for any $t \in \mathbb{R}^d$

$$\begin{aligned} \big\| M^{\mathcal{D}}_{\mathrm{Lat}}(f,g)(t) \big\|_{L^1(\Omega)} &= \Big\| \sup_{Q \in \mathcal{S}} \sup_{\substack{P \in \mathcal{D}: \\ \pi_{\mathcal{S}}(P)=Q}} \langle |f| \rangle_{1,P} \langle |g| \rangle_{1,P} \mathbf{1}_P(t) \Big\|_{L^1(\Omega)} \\ &\leq \sum_{Q \in \mathcal{S}} \Big\| \sup_{\substack{P \in \mathcal{D}: \\ \pi_{\mathcal{S}}(P)=Q}} \langle |f| \rangle_{1,P} \langle |g| \rangle_{1,P} \mathbf{1}_P(t) \Big\|_{L^1(\Omega)} \\ &\leq 4 A_0 \sum_{Q \in \mathcal{S}} \big\langle \|f\|_X \big\rangle_{1,Q} \big\langle \|g\|_{X^*} \big\rangle_{1,Q} \mathbf{1}_Q(t), \end{aligned}$$

which proves the claim. □

We obtain the following, important corollary from Proposition 6.5.1 and Proposition 6.5.3. It is exactly this statement that will be the key to prove sparse domination for $\widetilde{T}$ from sparse domination for T.

Corollary 6.5.4. *Let X be an order-continuous Banach function space over a measure space (Ω, μ) and $p \in (1, \infty)$. Suppose that $X, X^* \in \mathrm{HL}$. For $f \in L^p(\mathbb{R}^d; X)$ and $g \in L^{p'}(\mathbb{R}^d; X^*)$ we have*

$$\| M_{\mathrm{Lat}}(f,g) \|_{L^1(\mathbb{R}^d \times \Omega)} \lesssim_{p,d} \mu_{p,X} \cdot \mu_{p',X^*} \cdot \big\| M(\|f\|_X, \|g\|_{X^*}) \big\|_{L^1(\mathbb{R}^d)}.$$

Proof. Using Proposition 2.1.1, the monotone convergence theorem, Proposition 6.5.3 and Proposition 6.5.1, we can find $\mathscr{D}^1, \dots, \mathscr{D}^m$ such that for a.e. $t \in \mathbb{R}^d$

$$\begin{aligned} \big\| M_{\mathrm{Lat}}(f,g)(t) \big\|_{L^1(\Omega)} &\lesssim_d \sum_{j=1}^{m} \sup_{\mathcal{D} \subseteq \mathscr{D}^j \text{ finite}} \big\| M^{\mathcal{D}}_{\mathrm{Lat}}(f,g)(t) \big\|_{L^1(\Omega)} \\ &\lesssim_{p,d} \mu_{p,X} \mu_{p',X^*} \cdot M(\|f\|_X, \|g\|_{X^*})(t), \end{aligned}$$

Taking $L^1(\mathbb{R}^d)$-norms yields the desired conclusion. □

Corollary 6.5.4 combined with Proposition 6.5.2 yields the announced sharp weighted estimates for the bisublinear lattice Hardy–Littlewood maximal operator. Again we only state the case $p_2 = p_1'$ with dual weights and refer to [3, Corollary 3.6] for further weighted estimates.

Theorem 6.5.5. *Let X be an order-continuous Banach function space over (Ω,μ), let $p\in(1,\infty)$, $w\in A_p$ and set $w':=w^{-\frac{1}{p-1}}$. Suppose that $X,X^*\in \mathrm{HL}$. Then we have*

$$\|M_{\mathrm{Lat}}(f,g)\|_{L^p(\mathbb{R}^d,w;X)\times L^{p'}(\mathbb{R}^d,w';X^*)\to L^1(\mathbb{R}^d\times\Omega)}\lesssim_{p,d}\mu_{p,X}\cdot\mu_{p',X^*}\cdot[w]_{A_p}^{\max\{\frac{1}{p-1},1\}}.$$

Remark 6.5.6. If we would redo the proof of Lemma 6.4.1 in the bisublinear case (see [3, Lemma 3.3], we could replace $\mu_{p,X}\cdot\mu_{p',X^*}$ by

$$\|M_{\mathrm{Lat}}\|_{L^p(\mathbb{R}^d;X)\times L^{p'}(\mathbb{R}^d;X^*)\to L^1(\mathbb{R}^d\times\Omega)}$$

in the estimates in Proposition 6.5.3, Corollary 6.5.4 and Theorem 6.5.5. By [3, Proposition 4.3] we know that

$$\max\{\mu_{p,X},\mu_{p',X^*}\}\lesssim\|M_{\mathrm{Lat}}\|_{L^p(\mathbb{R}^d;X)\times L^{p'}(\mathbb{R}^d;X^*)\to L^1(\mathbb{R}^d\times\Omega)}\le\mu_{p,X}\cdot\mu_{p',X^*},$$

so this would yield slightly sharper estimates.

6.6. EXTENSIONS OF OPERATORS II: SPARSE DOMINATION

The sparse domination-based extension theorem in the introduction relies on the following two ingredients:

- The equivalence between sparse forms and the L^1-norm of the bisublinear maximal function in Proposition 6.5.1.
- The sparse domination result for the bisublinear lattice maximal operator in Corollary 6.5.4.

Having discussed both in the previous section, we are therefore ready to prove this extension theorem. Since $X,X^*\in\mathrm{HL}$ if X has the UMD property by Theorem 6.4.6, the sparse domination claim in Theorem 6.1.4 is a direct consequence of the following result in the case $Y=\mathbb{C}$ and Proposition 6.5.1. Recall that $X(Y)$ is a Köthe–Bochner space as introduced in Section 2.5.

Theorem 6.6.1. *Let Y be a Banach space and let T be an operator such that for any $f\in L^\infty_c(\mathbb{R}^d;Y)$ and $g\in L^\infty_c(\mathbb{R}^d)$*

$$\big\|\|Tf\|_Y\cdot g\big\|_{L^1(\mathbb{R}^d)}\le C_T\big\|M(\|f\|_Y,g)\big\|_{L^1(\mathbb{R}^d)}.$$

Let X be a Banach function space over a measure space (Ω,μ) and assume $X,X^*\in\mathrm{HL}$. Furthermore suppose that for all simple $f\in L^\infty_c(\mathbb{R}^d;X(Y))$ the function $\widetilde{T}f\colon\mathbb{R}^d\to X(Y)$ given by

$$\widetilde{T}f(t,\omega):=T(f(\cdot,\omega))(t),\qquad(t,\omega)\in\mathbb{R}^d\times\Omega$$

is well-defined and strongly measurable. Then for all simple functions $f\in L^\infty_c(\mathbb{R}^d;X(Y))$ and $g\in L^\infty_c(\mathbb{R}^d)$ we have

$$\big\|\|\widetilde{T}f\|_{X(Y)}\cdot g\big\|_{L^1(\mathbb{R}^d)}\lesssim_{p,d}\mu_{p,X}\cdot\mu_{p',X^*}\cdot C_T\big\|M(\|f\|_{X(Y)},g)\big\|_{L^1(\mathbb{R}^d)}.$$

Proof. We first note that X is q-concave for some $q \in (1,\infty)$, and thus order-continuous, by [GMT93, Theorem 2.8]. Let $f \in L^\infty_c(\mathbb{R}^d; X(Y))$ and $g_0 \in L^\infty_c(\mathbb{R}^d; X^*)$ be simple functions. Then we have that $f(\cdot,\omega) \in L^\infty_c(\mathbb{R}^d; Y)$ and $g_0(\cdot,\omega) \in L^\infty_c(\mathbb{R}^d)$ for a.e. $\omega \in \Omega$. Using Fubini's Theorem, the assumption on T and Corollary 6.5.4, we have

$$\begin{aligned}
\big\| \|\widetilde{T} f\|_Y \cdot g_0 \big\|_{L^1(\mathbb{R}^d\times\Omega)} &= \Big\| \omega \mapsto \big\| \|Tf(\cdot,\omega)\|_Y, g_0(\cdot,\omega)\big\|_{L^1(\mathbb{R}^d)} \Big\|_{L^1(\Omega)} \\
&\lesssim C_T \Big\| \omega \mapsto \|M(\|f(\cdot,\omega)\|_Y, g_0(\cdot,\omega))\|_{L^1(\mathbb{R}^d)} \Big\|_{L^1(\Omega)} \\
&= C_T \big\| M_{\mathrm{Lat}}(\|f\|_Y, g_0)\big\|_{L^1(\mathbb{R}^d\times\Omega)} \\
&\lesssim_{p,d} \mu_{p,X}\cdot\mu_{p',X^*}\cdot C_T \big\| M(\|f\|_{X(Y)}, \|g_0\|_{X^*})\big\|_{L^1(\mathbb{R}^d)}.
\end{aligned}$$

Now by duality (see e.g. [HNVW16, Proposition 1.3.1]) we have for any $g \in L^\infty_c(\mathbb{R}^d)$

$$\begin{aligned}
\big\| \|\widetilde{T} f\|_{X(Y)} \cdot g\big\|_{L^1(\mathbb{R}^d)} &= \sup_{\|h\|_{L^\infty(\mathbb{R}^d;X^*)}=1} \big\| \|\widetilde{T} f\|_Y \cdot g\cdot h\big\|_{L^1(\mathbb{R}^d\times\Omega)} \\
&\lesssim_{p,d} \mu_{p,X}\cdot\mu_{p',X^*}\cdot C_T \sup_{\|h\|_{L^\infty(\mathbb{R}^d;X^*)}=1} \big\| M(\|f\|_{X(Y)}, \|gh\|_{X^*})\big\|_{L^1(\mathbb{R}^d)} \\
&= \mu_{p,X}\cdot\mu_{p',X^*}\cdot C_T \big\| M(\|f\|_{X(Y)}, g)\big\|_{L^1(\mathbb{R}^d)},
\end{aligned}$$

proving the theorem. □

If T is linear in Theorem 6.6.1, we have for simple functions $f_1,\dots,f_m \in L^p(\mathbb{R}^d, w; Y)$ and $x_1,\dots,x_m \in X$ that

$$\widetilde{T}\Big(\sum_{j=1}^m g_j \otimes x_j\Big)(t,\omega) = \sum_{j=1}^m Tg_j(t)\otimes x_j(\omega), \qquad (t,\omega)\in\mathbb{R}^d\times\Omega.$$

Thus, in this case $\widetilde{T}$ coincides with the tensor extension of T.

Our sparse domination-based extension theorem, the second part of Theorem 6.1.4, is now an easy consequence of Theorem 6.6.1 and the weighted estimates in Proposition 6.5.2. We once again formulate a more general version using Köthe–Bochner spaces, bisublinear maximal operators and HL-assumptions, from which the second part of Theorem 6.1.4 follows by taking $Y=\mathbb{C}$ and using Proposition 6.5.1 and Theorem 6.4.6

Corollary 6.6.2. *Assume the conditions of Theorem 6.6.1 and additionally suppose that for all simple functions $f, g \in L^\infty_c(\mathbb{R}^d; Y)$ we have*

$$\|Tf - Tg\|_Y \le \|T(f-g)\|_Y.$$

Then for all $p\in(1,\infty)$ and all $w\in A_p$ we have

$$\|\widetilde{T}\|_{L^p(\mathbb{R}^d,w;X(Y))\to L^p(\mathbb{R}^d,w;X(Y))} \lesssim_{p,d} \mu_{p,X}\cdot\mu_{p',X^*}\cdot C_T\cdot [w]^{\max\left\{\frac{1}{p-1},1\right\}}.$$

Proof. By Theorem 6.6.1 we have for all simple functions $f\in L^\infty_c(\mathbb{R}^d;X(Y))$ and $g\in L^\infty_c(\mathbb{R}^d)$

$$\big\| \|\widetilde{T} f\|_{X(Y)}\cdot g\big\|_{L^1(\mathbb{R}^d)} \lesssim_{p,d} \mu_{p,X}\cdot\mu_{p',X^*}\cdot C_T \big\| M(\|f\|_{X(Y)}, g)\big\|_{L^1(\mathbb{R}^d)}.$$

Thus by Proposition 6.5.2 we obtain

$$\big\| \|\widetilde{T} f\|_{X(Y)} \cdot g \big\|_{L^1(\mathbb{R}^d)} \lesssim_{p,d} \mu_{p,X} \cdot \mu_{p',X^*} \cdot C_T \cdot [w]_{A_p}^{\max\{\frac{1}{p-1},1\}} \|f\|_{L^p(\mathbb{R}^d,w;X(Y))} \|g\|_{L^{p'}(\mathbb{R}^d,w')},$$

which by duality implies

$$\|\widetilde{T} f\|_{L^p(\mathbb{R}^d,w;X(Y))} \lesssim_{p,d} \mu_{p,X} \cdot \mu_{p',X^*} \cdot C_T \cdot [w]_{A_p}^{\max\{\frac{1}{p-1},1\}} \|f\|_{L^p(\mathbb{R}^d,w;X(Y))}.$$

So, by the additional assumption on T, we have for simple functions $f_1, f_2 \in L^\infty_c(\mathbb{R}^d; X(Y))$

$$\|\widetilde{T} f_1 - \widetilde{T} f_2\|_{L^p(\mathbb{R}^d,w;X(Y))} \le \|\widetilde{T}(f_1 - f_2)\|_{L^p(\mathbb{R}^d,w;X(Y))} \lesssim \big\| f_1 - f_2 \big\|_{L^p(\mathbb{R}^d,w;X(Y))}.$$

It follows that $\widetilde{T}$ is Lipschitz continuous. Therefore, by density, $\widetilde{T}$ extends uniquely to a bounded operator on $L^p(\mathbb{R}^d, w; X(Y))$ with the claimed bound. □

As a consequence of Corollary 6.6.2 and the sparse domination result for the Haar projections in Theorem 3.8.1, we can simultaneously give a proof of the following two results of Bourgain [Bou84] and Rubio de Francia [Rub86]:

- If X is a Banach function space with $X, X^* \in \mathrm{HL}$, then $X \in \mathrm{UMD}$.
- If X is a UMD Banach function space and Y is a UMD Banach space, then the Köthe–Bochner space $X(Y)$ also has the UMD property

The first statement follows from the following theorem taking $Y = \mathbb{C}$, whereas the second follows by using Theorem 6.4.6 to obtain $X, X^* \in \mathrm{HL}$ and then using the following theorem to deduce $X(Y) \in \mathrm{UMD}$. Note that the quantitative information we obtain in the second statement is also sharper than the bound obtained by the arguments of Rubio de Francia [Rub86].

Theorem 6.6.3. *Let X be a Banach function space and let Y be a Banach space. Suppose that $X, X^* \in \mathrm{HL}$ and $Y \in \mathrm{UMD}$, then $X(Y) \in \mathrm{UMD}$ with for any $p \in (1,\infty)$*

$$\beta_{p,X(Y)} \lesssim_p \mu_{p,X} \cdot \mu_{p',X^*} \cdot \beta_{p,Y}$$

Proof. Let $\mathscr{D}$ be the standard dyadic system in $\mathbb{R}$ and for $I \in \mathscr{D}$ let D_I be the Haar projection on $L^p(\mathbb{R}; Y)$ as in (3.8.1). Fix $\epsilon_I \in \{-1, 1\}$ for all $I \in \mathscr{D}$ and for $f \in L^p(\mathbb{R}; Y)$ define the operator

$$Tf(t) := \sum_{I \in \mathscr{D}} \epsilon_I D_I f(t), \qquad t \in \mathbb{R}.$$

Then, by Theorem 3.8.1 and Proposition 6.5.1, we know that T satisfies for $f \in L^\infty_c(\mathbb{R}; Y)$ and $g \in L^\infty_c(\mathbb{R})$

$$\big\| \|Tf\|_Y \cdot g \big\|_{L^1(\mathbb{R})} \lesssim \beta_{p,Y} \big\| M(\|f\|_Y, g) \big\|_{L^1(\mathbb{R})}.$$

Therefore, by Corollary 6.6.2, we know that $\widetilde{T}$ is bounded on $L^p(\mathbb{R}; X(Y))$ with

$$\|\widetilde{T}\|_{L^p(\mathbb{R};X(Y)) \to L^p(\mathbb{R};X(Y))} \lesssim_p \mu_{p,X} \cdot \mu_{p',X^*} \cdot \beta_{p,Y}.$$

Thus, denoting the Haar projection associated to $I \in \mathscr{D}$ on $L^p(\mathbb{R}; X)$ once again by D_I, we have for all $f \in L^p(\mathbb{R}; X)$ and any choice of $\epsilon_I \in \{-1, 1\}$ that

$$\Big\| \sum_{I \in \mathscr{D}} \epsilon_I D_I f \Big\|_{L^p(\mathbb{R}; X(Y))} \lesssim_p \mu_{p,X} \cdot \mu_{p',X^*} \cdot \beta_{p,Y} \cdot \|f\|_{L^p(\mathbb{R}; X(Y))}.$$

As this inequality characterizes the UMD constant of $X(Y)$ by [HNVW16, Theorem 4.2.13], this proves the theorem. □

Combining Theorem 6.4.6 and Theorem 6.6.3 we have shown that for any Banach function space X we have

$$X, X^* \in \mathrm{HL} \Leftrightarrow X \in \mathrm{UMD}.$$

Moreover, by Remark 6.5.6 these two conditions are equivalent to the boundedness of the bisublinear lattice Hardy–Littlewood maximal operator. Combined with Theorem 2.7.1 we therefore have:

Theorem 6.6.4. *Let X be a Banach function space over a measure space (Ω, μ). The following are equivalent:*

(i) $X \in \mathrm{UMD}$.

(ii) $X, X^* \in \mathrm{HL}$.

(iii) *The Hilbert transform H is bounded on $L^p(\mathbb{R}; X)$ for some (all) $p \in (1, \infty)$.*

(iv) *The Riesz projections R_k for $k = 1, \dots, d$ are bounded on $L^p(\mathbb{R}^d; X)$ for some (all) $p \in (1, \infty)$.*

(v) *The bisublinear Hardy–Littlewood maximal operator is bounded from $L^p(\mathbb{R}^d; X) \times L^{p'}(\mathbb{R}^d; X^*)$ to $L^1(\mathbb{R}^d \times \Omega)$ for some (all) $p \in (1, \infty)$.*

Remark 6.6.5. Using the sparse domination for the scalar-valued variants of the operators in (i)-(iv) and applying Remark 6.5.6, we can deduce from Corollary 6.6.2 that the involved constants in (i)-(iv) can all be estimated linearly by

$$\|M_{\mathrm{Lat}}\|_{L^p(\mathbb{R}^d; X) \times L^{p'}(\mathbb{R}^d; X^*) \to L^1(\mathbb{R}^d \times \Omega)}.$$

Conversely, by Theorem 6.4.6 we have

$$\|M_{\mathrm{Lat}}\|_{L^p(\mathbb{R}^d; X) \times L^{p'}(\mathbb{R}^d; X^*) \to L^1(\mathbb{R}^d \times \Omega)} \lesssim \mu_{p,X} \cdot \mu_{p',X^*} \lesssim \beta_{p,X}^4.$$

It would be interesting to see whether this estimate can be improved. This would require a different proof of (a bisublinear version of) Theorem 6.4.6, perhaps in the spirit of the proof of Theorem 6.6.3.

6.A. MONOTONE DEPENDENCE ON THE MUCKENHOUPT CHARACTERISTIC

For Rubio de Francia extrapolation as in Theorem 2.3.3 and our factorization-based extension theorem in Theorem 6.3.1 one needs an estimate of the form

$$\|f\|_{L^p(\mathbb{R}^d,w)} \le \phi([w]_{A_p})\|g\|_{L^p(\mathbb{R}^d,w)} \tag{6.A.1}$$

for all $w \in A_p$, where $\phi\colon [1,\infty) \to [1,\infty)$ is a nondecreasing function independent of w; this is often overlooked in the literature. In applications it is often easily checked that a weighted estimate is dependent on the Muckenhoupt characteristic $[w]_{A_p}$, and not on any other information coming from w. However, checking that this dependence is nondecreasing in $[w]_{A_p}$ can be tricky (see for example [12, Theorem 3.10]). Moreover, this monotonicity is usually not explicitly stated in the literature.

In this appendix we show that the monotonicity condition in (6.A.1) is redundant when working with pairs of nonnegative functions: an estimate depending on $[w]_{A_p}$ with no monotonicity assumption implies the estimate (6.A.1).

Theorem 6.A.1. *Fix $p \in (1,\infty)$, let $f, g \in L^0(\mathbb{R}^d)$ and suppose that there exists a function $C\colon [1,\infty) \to [1,\infty)$ such that for all $w \in A_p$ we have*

$$\|f\|_{L^p(\mathbb{R}^d,w)} \le C([w]_{A_p})\|g\|_{L^p(\mathbb{R}^d,w)}.$$

Then

$$\phi(t) := \sup\left\{\frac{\|f\|_{L^p(\mathbb{R}^d,w)}}{\|g\|_{L^p(\mathbb{R}^d,w)}} : w \in A_p, [w]_{A_p} = t\right\}$$

is nondecreasing, $\phi(t) \le C(t)$ for all $t \in [1,\infty)$ and for all $w \in A_p$ we have

$$\|f\|_{L^p(\mathbb{R}^d,w)} \le \phi([w]_{A_p})\|g\|_{L^p(\mathbb{R}^d,w)}. \tag{6.A.2}$$

Proof. Without loss of generality we may assume $f, g \in L^p(\mathbb{R}^d, w)$ for all $w \in A_p$. It is clear that $\phi(t) \le C(t)$ for all $t \in [1,\infty)$, and (6.A.2) holds. We will show that ϕ is nondecreasing. Let $1 \le t < s < \infty$ and $\varepsilon > 0$. Fix $w \in A_p$ with $[w]_{A_p} = t$ such that

$$\|f\|_{L^p(\mathbb{R}^d,w)} \ge \big(\phi([w]_{A_p}) - \varepsilon\big)\|g\|_{L^p(\mathbb{R}^d,w)},$$

and fix a ball $B_0 \subseteq \mathbb{R}^d$ such that

$$\|f\mathbf{1}_{B_0}\|_{L^p(\mathbb{R}^d,w)} \le \varepsilon\|g\|_{L^p(\mathbb{R}^d,w)} \quad \text{and} \quad \|g\mathbf{1}_{B_0}\|_{L^p(\mathbb{R}^d,w)} \le \frac{\varepsilon}{2s^{\frac{1}{p}}}\|g\|_{L^p(\mathbb{R}^d,w)}. \tag{6.A.3}$$

Divide B_0 into two sets B_0^+ and B_0^- such that $|B_0^+| = |B_0^-| = |B_0|/2$ and $w(x) > w(y)$ for all $x \in B_0^+$ and $y \in B_0^-$. For any $\sigma \in [1,\infty)$ we define a weight

$$w_\sigma(x) := \begin{cases} \sigma \cdot w(x) & \text{if } x \in B_0^+ \\ w(x) & \text{if } x \in B_0^-, \end{cases}$$

and for $B \subseteq \mathbb{R}^d$ define a function $f_B\colon [1,\infty) \to [1,\infty)$ by

$$f_B(\sigma) := \langle w_\sigma \rangle_{1,B} \cdot \langle w^{-1} \rangle_{\frac{1}{p-1},B}$$

Then f_B is of the form

$$f_B(\sigma) = (\alpha_0 + \alpha_+ \cdot \sigma)\Big(\beta_0 + \beta_+ \cdot \sigma^{-\frac{1}{p-1}}\Big)^{p-1}$$

with $\alpha_-, \alpha_+, \beta_-, \beta_+$ constants depending on B which satisfy

$$\alpha_- < \alpha_+, \qquad \beta_- > \beta_+, \qquad (\alpha_- + \alpha_+)(\beta_- + \beta_+)^{p-1} \le [w]_{A_p}.$$

So if we restrict to $[1, 2^p s]$ we know that $f_B \in C^1([1, 2^p s])$ with norm independent of B.

For each $n \in \mathbb{N}$ define a function

$$f_n := \sup_{B \in \mathcal{B}_n} f_B$$

on $[1, 2^p t]$, where each $\mathcal{B}_n$ is a finite collection of balls in $\mathbb{R}^d$, such that $\mathcal{B}_n \subseteq \mathcal{B}_{n+1}$ and $\bigcup_{n=1}^\infty \mathcal{B}_n$ contains all balls in $\mathbb{R}^d$ with rational center and radius. Then the sequence $(f_n)_{n=1}^\infty$ is nondecreasing and bounded, so it converges pointwise to some function f. Restricting to $[1, 2^p s]$, we also have that the sequence $(f_n)_{n=1}^\infty$ is equicontinuous, so by the Arzelà–Ascoli theorem we know that f is continuous on $[1, 2^p s]$. By a density argument we get that

$$f(\sigma) = \sup_{\substack{B \subseteq \mathbb{R}^d \\ B \text{ rational}}} f_B(\sigma) = \sup_{B \subseteq \mathbb{R}^d} f_B(\sigma) = [w_\sigma]_{A_p}.$$

Since $f(1) = [w]_{A_p} = t$ and

$$f(2^p s) \ge \frac{1}{|B_0|} \int_{B_0^+} 2^p s w(x)\, \mathrm{d}x \Big(\frac{1}{|B_0|} \int_{B_0^-} w(x)^{-\frac{1}{p-1}}\, \mathrm{d}x \Big)^{p-1} \ge \frac{f_{B_0}(1)}{2^p} 2^p s \ge s,$$

there exists $\sigma \in [1, 2^p s]$ such that $s = f(\sigma) = [w_\sigma]_{A_p}$.

Now by construction and (6.A.3) we have

$$\|g \mathbf{1}_{B_0}\|_{L^p(\mathbb{R}^d, w_\sigma)} \le \sigma^{1/p} \|g \mathbf{1}_{B_0}\|_{L^p(\mathbb{R}^d, w)} \le \varepsilon \|g\|_{L^p(\mathbb{R}^d, w)}.$$

Combining this with (6.A.3) and the triangle inequality yields

$$\begin{aligned}
\|f\|_{L^p(w_s)} &\ge \|f \mathbf{1}_{B_0^c}\|_{L^p(\mathbb{R}^d, w)} + \|f \mathbf{1}_{B_0}\|_{L^p(\mathbb{R}^d, w)} - \|f \mathbf{1}_{B_0}\|_{L^p(\mathbb{R}^d, w)} \\
&\ge \|f\|_{L^p(\mathbb{R}^d, w)} - \|f \mathbf{1}_{B_0}\|_{L^p(\mathbb{R}^d, w)} \\
&\ge (\phi(t) - 2\varepsilon) \|g\|_{L^p(\mathbb{R}^d, w)} \\
&\ge (\phi(t) - 2\varepsilon) \Big(\|g\|_{L^p(\mathbb{R}^d, w_s)} - \|g \mathbf{1}_{B_0}\|_{L^p(\mathbb{R}^d, w_s)} \Big) \\
&\ge (\phi(t) - 2\varepsilon)(1 - \varepsilon) \|g\|_{L^p(\mathbb{R}^d, w_s)}.
\end{aligned}$$

Thus $\phi(s) \ge (\phi(t) - 2\varepsilon)(1 - \varepsilon)$, and since $\varepsilon > 0$ was arbitrary this implies $\phi(s) \ge \phi(t)$, so ϕ is nondecreasing. □

7

FOURIER MULTIPLIERS IN BANACH FUNCTION SPACES

This chapter is based on the paper

[10] A. Amenta, E. Lorist, and M.C. Veraar. Fourier multipliers in Banach function spaces with UMD concavifications. *Trans. Amer. Math. Soc.*, 371(7):4837–4868, 2019.

It is complemented by a section on Littlewood–Paley–Rubio de Francia estimates from

[11] A. Amenta, E. Lorist, and M.C. Veraar. Rescaled extrapolation for vector-valued functions. *Publ. Mat.*, 63(1):155–182, 2019.

Abstract. *Using the factorization-based extension theorem of Chapter 6, we prove Banach function space-valued Littlewood–Paley–Rubio de Francia-type estimates . These Littlewood–Paley–Rubio de Francia-type estimates enable us to prove various operator-valued Fourier multipliers on Banach function spaces, which are extensions of the Coifman–Rubio de Francia–Semmes multiplier theorem. Our results involve a new boundedness condition on sets of operators, which we call $\ell^r(\ell^s)$-boundedness and which implies ℓ^2- and $\mathcal{R}$-boundedness in many cases.*

7.1. INTRODUCTION

For an interval $I \subseteq \mathbb{R}$, let S_I denote the Fourier projection onto I, defined by $S_I f := \mathcal{F}^{-1}(\mathbf{1}_I \widehat{f})$ for Schwartz functions $f \in \mathcal{S}(\mathbb{R})$. For every collection $\mathcal{I}$ of pairwise disjoint intervals and every $q \in (0,\infty]$ we consider the operator

$$\mathcal{S}_{\mathcal{I},q}(f) := \big(\sum_{I\in\mathcal{I}} |S_I f|^q\big)^{1/q},$$

interpreted as a supremum when $q = \infty$. If $\mathcal{I}$ is a dyadic decomposition of $\mathbb{R}$, then the classical Littlewood–Paley inequality states that for $p \in (1,\infty)$

$$\big\|\mathcal{S}_{\mathcal{I},2} f\big\|_{L^p(\mathbb{R})} \eqsim_p \|f\|_{L^p(\mathbb{R})}, \qquad f \in \mathcal{S}(\mathbb{R}).$$

A surprising extension of this classical Littlewood–Paley square function estimate was shown by Rubio de Francia in [Rub85]: for all $q \in [2,\infty)$ and $p \in (q',\infty)$ and *any* collection $\mathcal{I}$ of mutually disjoint intervals in $\mathbb{R}$ we have

$$\big\|\mathcal{S}_{\mathcal{I},q} f\big\|_{L^p(\mathbb{R})} \lesssim_{p,q} \|f\|_{L^p(\mathbb{R})}, \qquad f \in \mathcal{S}(\mathbb{R}). \tag{7.1.1}$$

This result (particularly the $q = 2$ case) is now known as the Littlewood–Paley–Rubio de Francia theorem. As a consequence, Coifman, Rubio de Francia and Semmes [CRdFS88] showed that if $p \in (1,\infty)$ and $\frac{1}{s} > \big|\frac{1}{p} - \frac{1}{2}\big|$, then every $m\colon \mathbb{R} \to \mathbb{C}$ of bounded s-variation uniformly on dyadic intervals induces a bounded Fourier multiplier T_m on $L^p(\mathbb{R})$. This is analogous to the situation for the Marcinkiewicz multiplier theorem (the $s = 1$ case of the Coifman–Rubio de Francia–Semmes theorem), which follows from the classical Littlewood–Paley theorem. We refer to [Lac07] for a survey of these results.

In this chapter we are interested in analogues of the results above in the vector-valued setting, i.e. estimates like (7.1.1) for functions in $\mathcal{S}(\mathbb{R};X)$ and multiplier theorems for operator-valued Fourier multipliers $m\colon \mathbb{R} \to \mathcal{L}(X)$, where X is a Banach (function) space.

7.1.1. LITTLEWOOD–PALEY–RUBIO DE FRANCIA ESTIMATES

Let X be a Banach space. The definition of S_I extends directly to the X-valued Schwartz functions $f \in \mathcal{S}(\mathbb{R};X)$. Vector-valued extensions of the Littlewood–Paley–Rubio de Francia theorem for the case $q = 2$ case are studied in [BGT03, GT04, HP06, HTY09, PSX12] via a reformulation in terms of random sums,

$$\mathbb{E}\Big\|\sum_{I\in\mathcal{I}} \varepsilon_I S_I f\Big\|_{L^p(\mathbb{R};X)} \lesssim \|f\|_{L^p(\mathbb{R};X)}, \qquad f \in \mathcal{S}(\mathbb{R};X),$$

where $(\varepsilon_I)_{I\in\mathcal{I}}$ is a Rademacher sequence. If this estimate holds then we say that X has the LPR_p property. When X is a Banach function space with finite cotype, this is equivalent to the boundedness of $\mathcal{S}_{\mathcal{I},2}$ on $L^p(\mathbb{R};X)$ by the Khintchine-Maurey inequalities (see Proposition 2.5.1). When $q \neq 2$, no analogue of the boundedness of $\mathcal{S}_{\mathcal{I},q}$ for general Banach spaces is known.

The LPR_p property is quite mysterious. In [HTY09, Theorem 1.2] it was shown that if a Banach space X has LPR_p property for some $p \geq 2$, then X has the UMD property and type 2. However, the converse is only known to hold when the collection $\mathcal{I}$ consists of intervals of equal length. The most general sufficient condition currently known is in [PSX12, Theorem 3]: if X is a 2-convex Banach function space and the 2-concavification X^2 has the UMD property, then X has the LPR_p property for all $p > 2$. This result is proved by an extension of Rubio de Francia's argument for the scalar-valued case. Every Banach space X that is known to have the LPR_p property is either of this form, or is isomorphic to a Hilbert space (and hence has the LPR_p property for all $p \in [2,\infty)$ by Rubio de Francia's original proof).

We prove the following theorem, a more precise version of which appears as Theorem 7.2.3.

Theorem 7.1.1. *Let $q \in [2,\infty)$, and suppose X is a q'-convex Banach function space such that $X^{q'}$ has the* UMD *property. Then there exists a increasing function $\phi\colon \mathbb{R}_+ \to \mathbb{R}_+$, depending on X, p, q, such that for $p \in (q',\infty)$, and $w \in A_{p/q'}$*

$$\|\mathcal{S}_{\mathcal{I},q} f\|_{L^p(\mathbb{R},w;X)} \leq \phi([w]_{A_{p/q'}}) \|f\|_{L^p(\mathbb{R},w;X)}, \qquad f \in \mathcal{S}(\mathbb{R};X).$$

We deduce this result, which includes [PSX12, Theorem 3] as a special case, directly from the scalar case $X = \mathbb{C}$ and the factorization-based extension theorem in Chapter 6, see Section 7.2 for further details. We do not obtain sharp dependence on Muckenhoupt characteristics in Theorem 7.1.1 and consequently we also do not obtain sharp dependence on Muckenhoupt characteristics in the Fourier multiplier theorems that we will deduce from Theorem 7.1.1. If we would apply our sparse domination-based extension theorem instead, we could obtain sharp weighted estimates in the case $q = 2$, see also Remark 7.2.4.

7.1.2. FOURIER MULTIPLIER THEOREMS

An operator-valued analogue of the Coifman–Rubio de Francia–Semmes theorem was obtained in [HP06], where the Banach space X was assumed to satisfy the LPR_p property. Naturally, $\mathcal{R}$-boundedness assumptions play an important role in the results of [HP06]. The main goal of this chapter is to prove a wider range of Coifman–Rubio de Francia–Semmes type results in case X is a Banach function space. We will use Theorem 7.1.1 to prove such results under a UMD assumption on a q-concavification of X. This naturally leads to an '$\ell^2(\ell^{q'})$-boundedness' condition, where one would usually expect an $\mathcal{R}$-boundedness condition. This new condition turns out to imply $\mathcal{R}$-boundedness. We investigate the more general notion of $\ell^r(\ell^s)$-boundedness in Section 7.3.

The following multiplier theorem is the fundamental result of this chapter. Let

$$\Delta = \{\pm[2^k, 2^{k+1}), k \in \mathbb{Z}\}$$

denote the standard dyadic partition of $\mathbb{R}$. Let X and Y be Banach function spaces and, for a set of bounded linear operators $\Gamma \subseteq \mathcal{L}(X,Y)$, let $V^s(\Delta;\Gamma)$ denote the space of func-

tions $m\colon \mathbb{R} \to \operatorname{span}(\Gamma)$ with bounded s-variation uniformly on dyadic intervals $J \in \Delta$, measured with respect to the Minkowski norm on $\operatorname{span}(\Gamma)$.

Theorem 7.1.2. *Let $q \in (1,2]$, $p \in (q,\infty)$, $s \in [1,q)$, and let $w \in A_{p/q}$. Let X and Y be Banach function spaces such that X^q and Y have the* UMD *property. Let $\Gamma \subseteq \mathcal{L}(X,Y)$ be absolutely convex and $\ell^2(\ell^{q'})$-bounded, and suppose that $m \in V^s(\Delta;\Gamma)$. Then the Fourier multiplier T_m is bounded from $L^p(\mathbb{R},w;X)$ to $L^p(\mathbb{R},w;Y)$.*

The case $q = 2$ and $w = 1$ of Theorem 7.1.2 was considered in [HP06, Theorem 2.3] for Banach spaces $X = Y$ with the LPR_p property. Our approach only works for Banach function spaces (and closed subspaces thereof), but as discussed before these are currently the only known examples of Banach spaces with LPR_p. As the parameter q decreases we assume less of X, but more of Γ and m. In Section 7.5 we prove Theorem 7.1.2, along with various other extensions and modifications of this result. In particular we obtain the following reformulation of Theorem 7.1.2 for Lebesgue spaces.

Theorem 7.1.3. *Let $s \in [2,\infty)$. Suppose that $m\colon \mathbb{R} \to \mathcal{L}(L^r(\mathbb{R}^d,w))$ for some $r \in (1,\infty)$ and all $w \in A_r(\mathbb{R}^d)$. Furthermore suppose that there is a increasing function $\phi\colon \mathbb{R}_+ \to \mathbb{R}_+$ such that for $w \in A_r(\mathbb{R}^d)$*

$$\sup_{t\in\mathbb{R}} \|m(t)\|_{\mathcal{L}(L^r(\mathbb{R}^d,w))} + \sup_{J\in\Delta} |J|^{\frac{1}{s}} [m]_{C^{1/s}(J;\mathcal{L}(L^r(\mathbb{R}^d,w)))} \leq \phi([w]_{A_r}).$$

Then the Fourier multiplier T_m is bounded on $L^p(\mathbb{R};L^r(\mathbb{R}^d))$ in each of the following cases:

(i) $r \in [2,\infty)$ *and* $\frac{1}{s} > \max\{\frac{1}{2} - \frac{1}{p}, \frac{1}{2} - \frac{1}{r}, \frac{1}{p} - \frac{1}{r}\}$,

(ii) $r \in (1,2]$ *and* $\frac{1}{s} > \max\{\frac{1}{p} - \frac{1}{2}, \frac{1}{r} - \frac{1}{2}, \frac{1}{r} - \frac{1}{p}\}$.

The result follows from the combination of Proposition 7.5.9 and Example 7.5.14. The condition on s becomes less restrictive as the numbers p, r, and 2 get closer. Taking $p = r$ or $r = 2$ is particularly illustrative: the condition on s is then $\frac{1}{s} > \left|\frac{1}{p} - \frac{1}{2}\right|$, as in the Coifman–Rubio de Francia–Semmes theorem. However, even if $p = r$, the operator-valued nature of the symbol m prevents us from deducing the boundedness of T_m from the scalar-valued case by a Fubini argument. Using the same techniques, one could also deduce versions of Theorem 7.1.3 with Muckenhoupt weights in the $\mathbb{R}$- and $\mathbb{R}^d$-variables.

In Section 7.5.4 we will present some new Coifman–Rubio de Francia–Semmes-type theorems on UMD Banach spaces (not just Banach function spaces) which are complex interpolation spaces between a Hilbert space and a UMD space. Typical examples which are not Banach function spaces include the space of Schatten class operators, and more generally non-commutative L^p-spaces. Our results in this context are weaker than those that we obtain for Banach function spaces, but nonetheless they seem to be new even for scalar multipliers.

7.1.3. NOTATION

Throughout this chapter we write $\phi_{a,b,\dots}$ to denote an increasing function on $\mathbb{R}_+$ which depends only on the parameters $a, b, \dots$, and which may change from line to line. Increasing dependence on the Muckenhoupt characteristic of weights is used in applications of extrapolation theorems. As we saw in Appendix 6.A, monotone dependence on the Muckenhoupt characteristic can be deduced from a more general estimate in terms of the Muckenhoupt characteristic.

7.2. LITTLEWOOD–PALEY–RUBIO DE FRANCIA ESTIMATES

In this section we apply Theorem 6.1.1 to the operators $\mathcal{S}_{\mathcal{I},q}$, which will result in Banach function space-valued Littlewood–Paley–Rubio de Francia estimates. As a warm-up we consider the operator $\mathcal{S}_{\Delta,2}$, where $\Delta := \{\pm[2^k, 2^{k+1}), k \in \mathbb{Z}\}$ is the standard dyadic partition of $\mathbb{R}$. Theorem 6.1.1 yields a direct proof of the classical Littlewood–Paley estimate in UMD Banach function spaces.

Proposition 7.2.1. *Let X be a* UMD *Banach function space, $p \in (1,\infty)$, and $w \in A_p$. Then for all $f \in L^p(\mathbb{R}, w; X)$,*

$$\phi_{X,p}([w]_{A_p})^{-1}\|f\|_{L^p(\mathbb{R},w;X)} \le \|\mathcal{S}_{\Delta,2}(f)\|_{L^p(\mathbb{R},w;X)} \le \phi_{X,p}([w]_{A_p})\|f\|_{L^p(\mathbb{R},w;X)}.$$

Proof. In the scalar case the result was obtained in [Kur80, Theorem 1], using Theorem 6.A.1 for the increasing dependence on $[w]_{A_p}$. Therefore the estimate

$$\|\mathcal{S}_{\Delta,2}(f)\|_{L^p(\mathbb{R},w;X)} \le \phi_{X,p}([w]_{A_p})\|f\|_{L^p(\mathbb{R},w;X)}$$

follows by applying Theorem 6.1.1 on simple functions $g \in L^p(\mathbb{R}, w; X)$ and $f = \mathcal{S}_{\Delta,2}(g)$. The converse estimate may be proved using a duality argument or another application of Theorem 6.1.1 with a simple function $f \in L^p(\mathbb{R}, w; X)$ and $g = \mathcal{S}_{\Delta,2}(f)$. □

Remark 7.2.2. Theorem 7.2.1 actually holds for all UMD Banach spaces, where the ℓ^2-sum in $\|\mathcal{S}_{\Delta,2}(f)\|_{L^p(\mathbb{R},w;X)}$ must be replaced by a suitable Rademacher sum. It was proved in [Bou86, Zim89] in the unweighted case and in [FHL20] in the weighted case. As noted in Section 3.8, this result can also be obtained using Theorem 3.1.1.

Next we establish weighted Littlewood–Paley–Rubio de Francia estimates for Banach function spaces with UMD concavifications (Theorem 7.1.1 in the introduction). The unweighted case with $q = 2$ was first proved in [PSX12], but we do not use this result in our proof.

Theorem 7.2.3. *Let $q \in [2,\infty)$ and let X be a Banach function space with $X^{q'} \in$ UMD. Then for all collections $\mathcal{I}$ of mutually disjoint intervals in $\mathbb{R}$, all $p \in (q',\infty)$, $w \in A_{p/q'}$, and $f \in L^p(\mathbb{R}, w; X)$,*

$$\|\mathcal{S}_{\mathcal{I},q}(f)\|_{L^p(\mathbb{R},w;X)} \le \phi_{X,p,q}([w]_{A_{p/q'}})\|f\|_{L^p(\mathbb{R},w;X)}.$$

Proof. Let (Ω,μ) be the measure space over which X is defined and let $f \in L^p(\mathbb{R},w;X)$ be simple. The scalar case of the result is proved in [Rub85, Theorem 6.1] for $q=2$, and [Kró14, Theorem B] for $q>2$. Monotonicity in $[w]_{A_{p/q'}}$ is contained in [Kró14] for $q>2$, and can be deduced from Theorem 6.A.1 when $q=2$. Thus for all $v \in A_{p/q'}$ and a.e. $\omega \in \Omega$ we have

$$\begin{aligned}\big\|\mathcal{S}_{\mathcal{I},q}(f)(\cdot,\omega)^{q'}\big\|_{L^{p/q'}(\mathbb{R},v;X^{q'})} &= \big\|\mathcal{S}_{\mathcal{I},q}(f)(\cdot,\omega)\big\|^{q'}_{L^p(\mathbb{R},v;X)}\\ &\le \phi_{p,q}([v]_{A_{p/q'}})\|f(\cdot,\omega)\|^{q'}_{L^p(\mathbb{R},v;X)}\\ &= \phi_{p,q}([v]_{A_{p/q'}})\big\||f(\cdot,\omega)|^{q'}\big\|_{L^{p/q'}(\mathbb{R},v;X^{q'})}.\end{aligned}$$

Therefore, since $X^{q'} \in$ UMD, applying Theorem 6.1.1 with $p_0 = p/q'$, $f = \mathcal{S}_{\mathcal{I},q}(f)^{q'}$ and $g = f^{q'}$ yields

$$\big\|(\mathcal{S}_{\mathcal{I},q}f)^{q'}\big\|_{L^r(\mathbb{R},w;X^{q'})} \le \phi_{X,p,q}([w]_{A_r})\big\||f|^{q'}\big\|_{L^r(\mathbb{R},w;X^{q'})}.$$

for all $r \in (1,\infty)$. Taking $r = p/q'$, rescaling and appealing to density yields the result. □

Remark 7.2.4.

- In the scalar case of Theorem 7.2.3 there is also a weak-type estimate for $p = q'$ and $w \in A_1$. The strong-type estimate seems to remain an open problem (see [Rub85, (6.4)]).

- In the scalar case sparse domination and (sharp) weighted estimates for $\mathcal{S}_{\mathcal{I},2}$ were shown by Garg, Roncal and Shrivastava [GRS21] using time-frequency analysis. Alternatively one can check the weak L^2-boundedness of our sharp grand maximal truncation operator $\mathcal{M}^{\#}_{\widetilde{\mathcal{S}}_{\mathcal{I},2},\alpha}$ using [PSX12, Lemma 4.5], where $\widetilde{\mathcal{S}}_{\mathcal{I},2}$ is a smooth version of $\mathcal{S}_{\mathcal{I},2}$. Combined with the trivial $L^2(\mathbb{R})$ estimate this also yields sparse domination and (non-sharp!) weighted estimates by Theorem 3.1.1 and Proposition 3.2.4. This method can be extended to $\mathcal{S}_{\mathcal{I},q}$ for $q \in (2,\infty)$.

- In the case $q=2$ we could also use the sparse domination-based extension theorem, rather than the factorization-based extension theorem, to prove Theorem 7.2.3. This would yield sharp weighted estimates in Theorem 7.2.3. Moreover, as noted in the previous bullet, with some additional work this could be extended to $q \in (2,\infty)$.

When $q=2$, the estimate in Theorem 7.2.3 can be used to obtain extensions of the Marcinkiewicz multiplier theorem. This is done in [HP06, Theorem 2.3]. For $q>2$ a slight variation will be needed to make this work. The following estimate, which combines Proposition 7.2.1 and Theorem 7.2.3, is a key ingredient in the Fourier multiplier theory that we will develop in Section 7.5. Recall that we set $\Delta = \{\pm[2^k, 2^{k+1}), k \in \mathbb{Z}\}$.

Theorem 7.2.5. *Let $q \in [2,\infty)$ and let X be a Banach function space such that $X^{q'} \in$ UMD. Let $\mathcal{I}$ be a collection of mutually disjoint intervals in $\mathbb{R}$, and for all $J \in \Delta$ let*

$$\mathcal{I}^J := \{I \in \mathcal{I} : I \subseteq J\}.$$

Then for all $p \in (q',\infty)$, $w \in A_{p/q'}$ and $f \in L^p(\mathbb{R}, w; X)$,

$$\Big\| \Big(\sum_{J\in\Delta} |\mathcal{S}_{\mathcal{I}^J,q}(f)|^2 \Big)^{1/2} \Big\|_{L^p(\mathbb{R},w;X)} \leq \phi_{X,p,q}([w]_{A_{p/q'}}) \|f\|_{L^p(\mathbb{R},w;X)}.$$

Proof. If $q = 2$ this follows from Theorem 7.2.3, so we need only consider $q > 2$. By Theorem 2.3.3 it suffices to take $p = 2$. Using Theorem 7.2.3 and Proposition 7.2.1 we estimate

$$\begin{aligned}
\Big\| \Big(\sum_{J\in\Delta} |\mathcal{S}_{\mathcal{I}^J,q}(f)|^2 \Big)^{1/2} \Big\|_{L^2(\mathbb{R},w;X)} &= \Big(\sum_{J\in\Delta} \Big\| \Big(\sum_{I\in\mathcal{I}^J} |S_I S_J f|^q \Big)^{1/q} \Big\|^2_{L^2(\mathbb{R},w;X)} \Big)^{1/2} \\
&\leq \phi_{X,q}([w]_{A_{2/q'}}) \Big(\sum_{J\in\Delta} \|S_J f\|^2_{L^2(\mathbb{R},w;X)} \Big)^{1/2} \\
&\leq \phi_{X,q}([w]_{A_{2/q'}}) \|\mathcal{S}_{\Delta,2} f\|_{L^2(\mathbb{R},w;X)} \\
&\leq \phi_{X,q}([w]_{A_{2/q'}}) \|f\|_{L^2(\mathbb{R},w;X)},
\end{aligned}$$

proving the theorem. □

If X is a Hilbert space, then one cannot apply Theorem 7.2.3 with $q = 2$. Instead, the following modification of Theorem 7.2.3 holds.

Proposition 7.2.6. *Let X be a Hilbert space, and let $\mathcal{I}$ be a collection of mutually disjoint intervals in $\mathbb{R}$. Then for all $p \in (2,\infty)$, $w \in A_{p/2}$, and $f \in L^p(\mathbb{R}, w; X)$,*

$$\Big\| \Big(\sum_{I\in\mathcal{I}} \|S_I f\|^2_X \Big)^{1/2} \Big\|_{L^p(\mathbb{R},w)} \leq \phi_p([w]_{A_{p/2}}) \|f\|_{L^p(\mathbb{R},w;X)}.$$

Proof. To prove this it suffices to consider $X = \ell^2$ (by restriction to a separable Hilbert space, see [HNVW16, Theorem 1.1.20]). Now the result will follow from Fubini's theorem, the result in the scalar-valued case, and a randomisation argument.

Let $(\varepsilon_I)_{I\in\mathcal{I}}$ and $(r_n)_{n\geq 1}$ be a Rademacher sequences on probability spaces Ω_ε and Ω_r respectively. Then writing

$$F = \sum_{n\geq 1} r_n f_n \in L^p(\mathbb{R}, w; L^p(\Omega_r)),$$

where $f = (f_n)_{n\geq 1} \in L^p(\mathbb{R}, w; \ell^2)$, it follows from Fubini's theorem and Khintchine's inequality (see Proposition 2.5.1 with $X = \mathbb{C}$) that

$$\Big\| \Big(\sum_{I\in\mathcal{I}} \|S_I f\|^2_{\ell^2} \Big)^{1/2} \Big\|_{L^p(\mathbb{R},w)} \eqsim_p \Big\| \sum_{I\in\mathcal{I}} \varepsilon_I S_I F \Big\|_{L^p(\Omega_r; L^p(\mathbb{R},w; L^p(\Omega_\varepsilon)))}.$$

Now we can argue pointwise in Ω_r. By Khintchine's inequality and the scalar case of the Littlewood–Paley–Rubio de Francia theorem [Rub85, Theorem 6.1], we obtain

$$\Big\| \sum_{I\in\mathcal{I}} \varepsilon_I S_I F \Big\|_{L^p(\mathbb{R},w;L^p(\Omega_\varepsilon))} \eqsim_p \Big\| \Big(\sum_{I\in\mathcal{I}} |S_I F|_X^2 \Big)^{1/2} \Big\|_{L^p(\mathbb{R},w)} \leq \phi([w]_{A_{p/2}}) \|F\|_{L^p(\mathbb{R},w)}.$$

The result now follows by taking $L^p(\Omega_r)$-norms and applying Khintchine's inequality once more. □

Remark 7.2.7. If X is a Hilbert space, $\mathcal{I}$ a collection of mutually disjoint intervals in $\mathbb{R}$ and $q \in (2,\infty)$, then for all $p \in (q',\infty)$, $w \in A_{p/q'}$ and $f \in L^p(\mathbb{R},w;X)$, we have

$$\Big\| \Big(\sum_{I\in\mathcal{I}} \|S_I f\|_X^q \Big)^{1/q} \Big\|_{L^p(\mathbb{R},w)} \leq \phi_{p,q}([w]_{A_{p/q'}}) \|f\|_{L^p(\mathbb{R},w;X)}$$
$$\Big\| \Big(\sum_{J\in\Delta} \Big(\sum_{I\in\mathcal{I}^J} \|S_I f\|_X^q \Big)^{2/q} \Big)^{1/2} \Big\|_{L^p(\mathbb{R},w)} \leq \phi_{p,q}([w]_{A_{p/q'}}) \|f\|_{L^p(\mathbb{R},w;X)}.$$

These estimates are weaker than Theorem 7.2.3 and Theorem 7.2.5. To prove the first estimate it is enough to consider $X = \ell^2$. In this case

$$\Big\| \Big(\sum_{I\in\mathcal{I}} \|S_I f\|_{\ell^2}^q \Big)^{1/q} \Big\|_{L^p(\mathbb{R},w)} \leq \|\mathcal{S}_{\mathcal{I},q} f\|_{L^p(\mathbb{R},w;\ell^2)}$$

by Minkowski's inequality, so the result follows from Theorem 7.2.3. The second estimate is proved similarly.

7.3. $\ell^r(\ell^s)$-BOUNDEDNESS

Our operator-valued multiplier theorems involve a new condition on sets of bounded operators $\Gamma \subseteq \mathcal{L}(X,Y)$, which we call $\ell^r(\ell^s)$-*boundedness*. This generalises the more familiar notions of $\mathcal{R}$-boundedness and ℓ^r-boundedness introduced in Section 2.6. In this section we introduce and explore the concept.

7.3.1. DEFINITION AND BASIC PROPERTIES

We start with the definition of $\ell^r(\ell^s)$-boundedness. As ℓ^r-boundedness, it can only be defined for families of operators on Banach *function* spaces.

Definition 7.3.1. Let X and Y be Banach function spaces, $\Gamma \subseteq \mathcal{L}(X,Y)$ and $r,s \in [1,\infty]$. We say that Γ is $\ell^r(\ell^s)$-*bounded* if for all finite doubly-indexed sequences $(T_{j,k})_{j,k=1}^{n,m}$ in Γ and $(x_{j,k})_{j,k=1}^{n,m}$ in X,

$$\Big\| \Big(\sum_{j=1}^n \Big(\sum_{k=1}^m |T_{j,k} x_{j,k}|^s \Big)^{r/s} \Big)^{1/r} \Big\|_Y \lesssim \Big\| \Big(\sum_{j=1}^n \Big(\sum_{k=1}^m |x_{j,k}|^s \Big)^{r/s} \Big)^{1/r} \Big\|_X.$$

The least admissible implicit constant is called the $\ell^r(\ell^s)$-bound of Γ, and denoted $\|\Gamma\|_{\ell^r(\ell^s)}$.

As discussed in Section 2.6, for $\mathcal{R}$- and ℓ^2-boundedness it suffices to consider *subsets* of Γ in the defining inequality. Just as for ℓ^r-boundedness, this is not the case for $\ell^r(\ell^s)$-boundedness with $r, s \neq 2$: one must consider *sequences*, allowing for repeated elements. We say that an operator $T \in \mathcal{L}(X,Y)$ is $\ell^r(\ell^s)$-bounded if the singleton $\{T\}$ is.

If a set $\Gamma \subseteq \mathcal{L}(X,Y)$ is $\ell^r(\ell^s)$-bounded, then so is its closure in the strong operator topology, and likewise its absolutely convex hull absco(Γ). This can be proven analogously to the proof of the statement for ℓ^r-boundedness in [KU14]. Moreover we once again have that if $\Gamma_1, \Gamma_2 \subseteq \mathcal{L}(X)$ are $\ell^r(\ell^s)$-bounded, then $\Gamma_1 \cup \Gamma_2$ is $\ell^r(\ell^s)$-bounded as well.

It is immediate from the definition that ℓ^r-boundedness and $\ell^r(\ell^r)$-boundedness are equivalent. The following proposition encapsulates a few other connections between ℓ^r-, and $\ell^r(\ell^s)$-boundedness. The following proposition shows in particular that if Γ is $\ell^2(\ell^s)$- or $\ell^s(\ell^2)$-bounded for some $s \in [1,\infty]$, then Γ is ℓ^2-bounded, and hence $\mathcal{R}$-bounded if Y has finite cotype.

Proposition 7.3.2. *Let X and Y be Banach function spaces and $\Gamma \subseteq \mathcal{L}(X,Y)$.*

(i) Let $r, s \in [1,\infty]$. If Γ is $\ell^r(\ell^s)$-bounded, then Γ is ℓ^r- and ℓ^s-bounded with $\|\Gamma\|_{\ell^r} \leq \|\Gamma\|_{\ell^r(\ell^s)}$ and $\|\Gamma\|_{\ell^s} \leq \|\Gamma\|_{\ell^r(\ell^s)}$.

(ii) Let $p, s \in [1,\infty]$. If X is p-concave, Y is p-convex, and Γ is ℓ^s-bounded, then Γ is $\ell^p(\ell^s)$-bounded with $\|\Gamma\|_{\ell^p(\ell^s)} \leq \|\Gamma\|_{\ell^s}$.

Proof. (i) follows by taking one index to be a singleton. For (ii), consider doubly-indexed finite sequences $(T_{j,k})_{j,k=1}^{m,n}$ in Γ and $(x_{j,k})_{j,k=1}^{m,n}$ in X. Then we have

$$\begin{aligned}
\Big\| \Big(\sum_{j=1}^m \Big(\sum_{k=1}^n |T_{j,k} x_{j,k}|^s \Big)^{p/s} \Big)^{1/p} \Big\|_Y &\leq \Big(\sum_{j=1}^m \Big\| \Big(\sum_{k=1}^n |T_{j,k} x_{j,k}|^s \Big)^{1/s} \Big\|_X^p \Big)^{1/p} \\
&\leq \|\Gamma\|_{\ell^s} \Big(\sum_{j=1}^m \Big\| \Big(\sum_{k=1}^n |x_{j,k}|^s \Big)^{1/s} \Big\|_Y^p \Big)^{1/p} \\
&\leq \|\Gamma\|_{\ell^s} \Big\| \Big(\sum_{j=1}^m \Big(\sum_{k=1}^n |x_{j,k}|^s \Big)^{p/s} \Big)^{1/p} \Big\|_X,
\end{aligned}$$

so $\|\Gamma\|_{\ell^p(\ell^s)} \leq \|\Gamma\|_{\ell^s}$. □

Duality and interpolation may be used to establish $\ell^r(\ell^s)$-boundedness, as shown in the following two propositions.

Proposition 7.3.3. *Let X, Y be Banach function spaces, and let $\Gamma \subseteq \mathcal{L}(X,Y)$. Let $r, s \in [1,\infty]$. If Γ is $\ell^r(\ell^s)$-bounded, then the adjoint family*

$$\Gamma^* = \{T^* : T \in \Gamma\} \subseteq \mathcal{L}(Y^*, X^*)$$

is $\ell^{r'}(\ell^{s'})$-bounded with $\|\mathcal{T}^*\|_{\ell^{r'}(\ell^{s'})} = \|\Gamma\|_{\ell^r(\ell^s)}$.

Proof. This follows from the duality relation $X(\ell^r_m(\ell^s_n))^* = X^*(\ell^{r'}_m(\ell^{s'}_n))$ (see [LT79, Section 1.d]). □

To exploit interpolation we must assume order-continuity, which holds automatically for reflexive spaces and thus in particular for UMD spaces.

Proposition 7.3.4. *Let X and Y be order continuous Banach function spaces and $\Gamma \subseteq \mathcal{L}(X,Y)$. Let $r_k, s_k \in [1,\infty]$ for $k = 0,1$. If Γ is $\ell^{r_k}(\ell^{s_k})$-bounded for $k = 0,1$, then Γ is $\ell^{r_\theta}(\ell^{s_\theta})$-bounded for all $\theta \in (0,1)$, where $r_\theta := [r_0, r_1]_\theta$ and $s_\theta := [s_0, s_1]_\theta$. Moreover we have the estimate*

$$\|\Gamma\|_{\ell^{r_\theta}(\ell^{s_\theta})} \le \|\Gamma\|^{\theta}_{\ell^{r_0}(\ell^{s_0})} \|\Gamma\|^{1-\theta}_{\ell^{r_1}(\ell^{s_1})} \le \max\{\|\Gamma\|_{\ell^{r_0}(\ell^{s_0})}, \|\Gamma\|_{\ell^{r_1}(\ell^{s_1})}\}.$$

Proof. This follows from Calderón's theory of complex interpolation for order continuous vector-valued function spaces [Cal64]. □

Combining Proposition 7.3.2(i) with Proposition 7.3.4 we deduce the following.

Corollary 7.3.5. *Let X and Y be order continuous Banach function spaces and $\Gamma \subset \mathcal{L}_b(X,Y)$. Fix $r, s \in [1,\infty]$ and suppose that Γ is $\ell^r(\ell^s)$-bounded. If*

$$r \le u \le v \le s \quad \text{or} \quad s \le v \le u \le r,$$

then Γ is $\ell^u(\ell^v)$-bounded with $\|\Gamma\|_{\ell^u(\ell^v)} \le \|\Gamma\|_{\ell^r(\ell^s)}$.

If we use Theorem 6.1.2 to extend a family of bounded operators on $L^p(\mathbb{R}^d, w)$ to a family of bounded operators on $L^p(\mathbb{R}^d, w)$, then this family of extensions is automatically $\ell^r(\ell^s)$-bounded. This observation is a convenient source of $\ell^r(\ell^s)$-bounded families.

Proposition 7.3.6. *Fix $p_0 \in (1,\infty)$, and suppose that $\Gamma \subseteq \mathcal{L}(L^p(w))$ for some $p \in (p_0,\infty)$ and $w \in A_{p/p_0}$. In addition suppose that there is an increasing function $\phi\colon \mathbb{R}_+ \to \mathbb{R}_+$ such that*

$$\|T\|_{L^p(\mathbb{R}^d,w)\to L^p(\mathbb{R}^d,w)} \le \phi([w]_{A_{p/p_0}}), \qquad T \in \Gamma.$$

Let X be a Banach function space with $X^{p_0} \in \mathrm{UMD}$, and let $\widetilde{\Gamma}$ be the set of tensor extensions of operators in Γ. Then for all $p, r, s \in (p_0,\infty)$ and all $w \in A_{p/p_0}$, $\widetilde{\Gamma}$ is $\ell^r(\ell^s)$-bounded on $L^p(\mathbb{R}^d, w; X)$ with

$$\|\widetilde{\Gamma}\|_{\ell^r(\ell^s)} \le \phi_{X,p_0,p,r,s,d}([w]_{A_{p/p_0}}).$$

Proof. Consider doubly-indexed finite sequences $(T_{j,k})_{j,k=1}^{m,n}$ in Γ and let $(g_{j,k})_{j,k=1}^{m,n}$ be a boundedly indexed sequence of simple functions in $L^p(\mathbb{R}^d, w; X)$. Let (Ω,μ) be the underlying measure space of X, and define

$$F, G\colon \mathbb{R}^d \times \Omega \times \{1,\dots,m\} \times \{1,\dots,n\} \to \mathbb{R}_+$$

by

$$F(\cdot,\omega,j,k) = |T_{j,k}g_{j,k}(\cdot,\omega)|^{p_0} \quad \text{and} \quad G(\cdot,\omega,j,k) = |g_{j,k}(\cdot,\omega)|^{p_0}.$$

Then, from the assumption on Γ, we see that for some $p \in (p_0,\infty)$ and all $w \in A_{p/p_0}$,

$$\|F(\cdot,\omega,j,k)\|_{L^{p/p_0}(\mathbb{R}^d,w)} \leq \phi([w]_{A_{p/p_0}})^{p_0}\|G(\cdot,\omega,j,k)\|_{L^{p/p_0}(\mathbb{R}^d,w)}.$$

Letting $Y := X(\ell^r_m(\ell^s_n))$, it follows from Theorem 6.6.3 that $Y = X^{p_0}(\ell^{r/p_0}_m(\ell^{s/p_0}_n)) \in \mathrm{UMD}$, with UMD constants independent of $m,n \in \mathbb{N}$. Hence Theorem 6.1.1 implies that for all $p \in (p_0,\infty)$ and $w \in A_{p/p_0}$,

$$\|F\|_{L^{p/p_0}(\mathbb{R}^d,w;Y^{p_0})} \leq \phi_{X,p_0,p,r,s,d}([w]_{A_{p/p_0}})\|G\|_{L^{p/p_0}(\mathbb{R}^d,w;Y^{p_0})}.$$

Rescaling and a density argument now proves the claim. □

Taking X to be the scalar field $\mathbb{C}$, so that $X^{p_0} = X$ for any p_0, we obtain the following special case. Note that in this case a more direct proof may be given as in [12, Theorem 2.3].

Proposition 7.3.7. *Fix $p_0 \in (1,\infty)$, and suppose that $\Gamma \subseteq \mathcal{L}(L^p(\mathbb{R}^d,w))$ for some $p \in (p_0,\infty)$ and $w \in A_{p/p_0}$. In addition suppose that there is an increasing function $\phi\colon \mathbb{R}_+ \to \mathbb{R}_+$ such that*

$$\|T\|_{L^p(\mathbb{R}^d,w)\to L^p(\mathbb{R}^d,w)} \leq \phi([w]_{A_{p/p_0}}), \qquad T \in \Gamma.$$

Then for all $p,r,s \in (p_0,\infty)$ and all $w \in A_{p/p_0}$, Γ is $\ell^r(\ell^s)$-bounded on $L^p(\mathbb{R}^d,w)$ with

$$\|\Gamma\|_{\ell^r(\ell^s)} \leq \phi_{p_0,p,r,s,d}([w]_{A_{p/p_0}}).$$

To end this section we present a technical lemma on the $\ell^r(\ell^s)$-boundedness of the closure of a family of operators on spaces other than that in which the closure was taken. It is used in our multiplier result for intermediate spaces, where several Lebesgue spaces are used simultaneously. A similar result can be proved with general order-continuous Banach function spaces in place of Lebesgue spaces.

Lemma 7.3.8. *Let (Ω,d,μ) be a metric measure space and assume μ is locally finite. Let $p \in (1,\infty)$ and let Γ be a family of operators such that $\Gamma \subseteq \mathcal{L}(L^p(\Omega))$ is uniformly bounded and absolutely convex. Let $\overline{\Gamma}$ denote the closure of Γ in $\mathcal{L}(L^p(\Omega))$. Suppose $q \in (1,\infty)$, let w be a locally integrable weight on Ω and assume that $\Gamma \subseteq \mathcal{L}(L^q(\Omega,w))$ is $\ell^r(\ell^s)$-bounded for some $r,s \in [1,\infty]$. Then $\overline{\Gamma}$ is $\ell^r(\ell^s)$-bounded on $L^q(\Omega,w)$ with $\|\overline{\Gamma}\|_{\ell^r(\ell^s)} = \|\Gamma\|_{\ell^r(\ell^s)}$.*

Proof. Fix $(T_{m,n})_{m=1,n=1}^{M,N}$ in $\overline{T}$ and $(f_{m,n})_{m=1,n=1}^{M,N}$ in $L^q(\Omega,w)$. By a density argument we may assume each for each m,n that $f_{m,n}$ is bounded and supported on a bounded subset of Ω, which implies $f_{m,n} \in L^p(\Omega)$. For each m,n choose $(T^{(k)}_{m,n})_{k\geq 1}$ in Γ such that

$T^{(k)}_{m,n} \to T_{m,n}$ in $\mathcal{L}(L^p(\Omega))$. Then also $T^{(k)}_{m,n} f_{m,n} \to T_{m,n} f_{m,n}$ in $L^p(\Omega)$. By passing to subsequences we may suppose that for all m, n we have $T^{(k)}_{m,n} f_{m,n} \to T_{m,n} f_{m,n}$, μ-a.e. Therefore, by Fatou's lemma,

$$\begin{aligned}\Big\|\Big(\sum_{m=1}^M\Big(\sum_{n=1}^N |T_{m,n} f_{m,n}|^s\Big)^{r/s}\Big)^{1/r}\Big\|_{L^q(\Omega,w)} &\le \liminf_{k\to\infty}\Big\|\Big(\sum_{m=1}^M\Big(\sum_{n=1}^N |T^{(k)}_{m,n} f_{m,n}|^s\Big)^{r/s}\Big)^{1/r}\Big\|_{L^q(\Omega,w)}\\ &\le \|\Gamma\|_{\ell^r(\ell^s)}\Big\|\Big(\sum_{m=1}^M\Big(\sum_{n=1}^N |f_{m,n}|^s\Big)^{r/s}\Big)^{1/r}\Big\|_{L^q(\Omega,w)},\end{aligned}$$

with the appropriate adjustment if $r = \infty$ or $s = \infty$. So $\overline{\Gamma}$ is indeed $\ell^r(\ell^s)$-bounded on $L^q(w)$. □

7.3.2. $\ell^r(\ell^s)$-BOUNDEDNESS OF SINGLE OPERATORS

As noted before, a single operator $T \in \mathcal{L}(X,Y)$ can fail to be $\ell^r(\ell^s)$-bounded. For positive operators we have the following result, which is an adaptation of [MS96, Lemma 4].

Proposition 7.3.9. *Let X and Y be Banach function spaces and let $P \in \mathcal{L}(X,Y)$ be a positive operator. Then P is $\ell^r(\ell^s)$-bounded for all $r,s \in [1,\infty]$, and we have $\|\{P\}\|_{\ell^r(\ell^s)} \le \|P\|_{\mathcal{L}(X,Y)}$.*

Proof. Let $(x_{j,k})_{j,k=1}^{m,n}$ be a doubly-indexed sequence in X, and note that by positivity of P we may take the elements of the sequence to be positive. By positivity of P we can estimate

$$\begin{aligned}\Big\|\Big(\sum_{j=1}^m\Big(\sum_{k=1}^n |Px_{j,k}|^s\Big)^{r/s}\Big)^{1/r}\Big\|_Y &= \Big\|\sup_{\|(b_j)\|_{\ell^{r'}_m}\le 1}\sum_{j=1}^m b_j \sup_{\|(a^j_k)\|_{\ell^{s'}_n}\le 1}\sum_{k=1}^n a^j_k P x_{j,k}\Big\|_Y\\ &\le \Big\|P\Big(\sup_{\|(b_j)\|_{\ell^{r'}_m}\le 1}\sum_{j=1}^m b_j \sup_{\|(a^j_k)\|_{\ell^{s'}_n}\le 1}\sum_{k=1}^n a^j_k x_{j,k}\Big)\Big\|_Y\\ &\le \|P\|_{\mathcal{L}(X,Y)}\Big\|\Big(\sum_{j=1}^m\Big(\sum_{k=1}^n |x_{j,k}|^s\Big)^{r/s}\Big)^{1/r}\Big\|_X,\end{aligned}$$

so $\|\{P\}\|_{\ell^r(\ell^s)} \le \|P\|_{\mathcal{L}(X,Y)}$. □

For an ℓ^1-bounded operator on a Lebesgue space one has $\ell^r(\ell^s)$-boundedness for all $r,s \in [1,\infty]$ (see [HNVW16, Theorem 2.7.2]). The result below actually holds with $L^p(\Omega)$ replaced by any Banach lattice X with a Levi norm (see [Buh75] and [Lin16, Fact 2.5]). A duality argument implies a similar result for ℓ^∞-boundedness.

Proposition 7.3.10. *Let $p \in [1,\infty)$ and $T \in \mathcal{L}(L^p(\Omega))$. If T is ℓ^1-bounded, then $\{T\}$ is $\ell^r(\ell^s)$-bounded for all $r,s \in [1,\infty]$.*

Remark 7.3.11. Even on L^p it can be quite hard to establish the $\ell^r(\ell^s)$-boundedness of a single operator. By using i.i.d. s-stable random variables $\xi_1,\dots,\xi_n\colon \Omega \to \mathbb{R}$ (see [LT91,

Section 5]), for $p \in (0,s)$ one can linearise the estimate by writing

$$\Big(\sum_{j=1}^n |Tx_j|^s\Big)^{1/s} = C_{p,s}\Big\|T\sum_{j=1}^n \xi_j x_j\Big\|_{L^p(\Omega)}.$$

By using Fubini's theorem and Minkowski's inequality, one can deduce that any $T \in \mathcal{L}(L^p)$ is $\ell^r(\ell^s)$-bounded if $p \le r \le s \le 2$ or $2 \le s \le r \le p$. Most of the remaining cases seem to be open (see [Kwa72b, Problem 2] and [DLOT17, Corollary 1.44]).

7.3.3. NON-EXAMPLES

We end this section with two examples to demonstrate that $\ell^r(\ell^s)$-boundedness is not just the conjunction of ℓ^r- and ℓ^s-boundedness. Consider the class of kernels

$$\mathcal{K} = \{k \in L^1(\mathbb{R}) : |k * f| \le Mf \text{ a.e. for all simple } f\colon \mathbb{R} \to \mathbb{R}\},$$

where M is the Hardy–Littlewood maximal operator. For $k \in \mathcal{K}$ and $f \in L^p(\mathbb{R})$ with $p \in (1,\infty)$ define the operator T_k by

$$T_k f(t) = \int_{\mathbb{R}} k(t-s)f(s)\,\mathrm{d}s, \qquad t \in \mathbb{R}$$

and set $\Gamma = \{T_k : k \in \mathcal{K}\}$.

Example 7.3.12. Let $p \in (1,\infty)$. The family of operators $\Gamma \subseteq \mathcal{L}(L^p(\mathbb{R}))$ defined above is ℓ^r-bounded for all $r \in [1,\infty]$, but not $\ell^1(\ell^s)$- or $\ell^\infty(\ell^s)$- bounded for any $s \in (1,\infty)$.

Proof. The ℓ^r-boundedness of Γ for $r \in [1,\infty]$ is proved in [NVW15b, Theorem 4.7]. Since $\Gamma = \Gamma^*$, Proposition 7.3.3 says that $\ell^1(\ell^s)$-boundedness of Γ on $L^p(\mathbb{R})$ implies $\ell^\infty(\ell^{s'})$-boundedness on $L^{p'}(\mathbb{R})$, so it suffices to show that Γ is not $\ell^\infty(\ell^s)$-bounded on $L^p(\mathbb{R})$ for any $s \in (1,\infty)$. We follow the proof of [NVW15b, Proposition 8.1].

Fix $n \in \mathbb{N}$ and for $i,j \in \mathbb{N}$ define $f_{i,j} \in L^p(\mathbb{R})$ by

$$f_{i,j}(t) = \mathbf{1}_{(0,1]}(t)\,\mathbf{1}_{(2^{-j},2^{-j+1}]}(t-(i-1)2^{-n}), \qquad t \in \mathbb{R},$$

so that

$$\Big\|\sup_{1\le i\le 2^n}\Big(\sum_{j=1}^n |f_{i,j}(t)|^s\Big)^{1/s}\Big\|_{L^p(\mathbb{R})} \le \Big\|\sup_{1\le i\le 2^n}\mathbf{1}_{(0,1]}\Big\|_{L^p(\mathbb{R})} = 1. \tag{7.3.1}$$

Next, for $i,j \in \mathbb{N}$ define

$$k_{i,j}(t) = \frac{1}{2^{-j+2}}\,\mathbf{1}_{(-2^{-j+1},2^{-j+1})}(t), \qquad t \in \mathbb{R},$$

and $T_{i,j} = T_{k_{i,j}}$. Then $T_{i,j} \in \Gamma$, as for any simple function f and $t \in \mathbb{R}$ we have

$$\begin{aligned}|T_{i,j}f(t)| = |k_{i,j} * f(t)| &= \frac{1}{2^{-j+2}}\Big|\int_{\mathbb{R}} \mathbf{1}_{(-2^{-j+1},2^{-j+1})}(t-\tau)f(\tau)\,d\tau\Big| \\ &= \frac{1}{2^{-j+2}}\Big|\int_{t-2^{-j+1}}^{t+2^{-j+1}} f(\tau)\,d\tau\Big| \le Mf(t).\end{aligned}$$

Furthermore, for any $1 \le j \le n$, $t \in (0,1]$ and $1 \le i \le 2^n$ with $t \in ((i-1)2^{-n}, i2^{-n}]$,

$$\begin{aligned}|T_{i,j} f_{i,j}(t)| &= \frac{1}{2^{-j+2}} \int_{t-2^{-j+1}-(i-1)2^{-n}}^{t+2^{-j+1}-(i-1)2^{-n}} \mathbf{1}_{(2^{-j},2^{-j+1}]}(\tau)\, d\tau \\ &\ge \frac{1}{2^{-j+2}} \int_{2^{-j}}^{2^{-j+1}} \mathbf{1}_{(2^{-j},2^{-j+1}]}(\tau)\, d\tau = \frac{2^{-j}}{2^{-j+2}} = \frac{1}{4}.\end{aligned}$$

Therefore

$$\Big\| \sup_{1\le i\le 2^n} \Big(\sum_{j=1}^n |T_{i,j} f_{i,j}(t)|^s \Big)^{1/s} \Big\|_{L^p(\mathbb{R})} \ge \Big\| \Big(\frac{n}{4^s}\Big)^{1/s} \mathbf{1}_{(0,1]} \Big\|_{L^p(\mathbb{R})} = \frac{n^{1/s}}{4}$$

which tends to ∞ as $n \to \infty$. Combining this with (7.3.1) it follows that Γ is not $\ell^\infty(\ell^s)$-bounded on $L^p(\mathbb{R})$. □

The previous example can be modified to construct examples of operator families which are not $\ell^2(\ell^s)$-bounded, by using stochastic integral operators introduced in Chapter 4. For $k \in \mathcal{K}$ and $f \in L^p(\mathbb{R}_+)$ with $p \in (2,\infty)$, define

$$S_k f(t) := \int_0^t |k(t-s)|^{\frac{1}{2}} f(s)\, \mathrm{d}W(s),$$

where W is a standard Brownian motion on a probability space $(\Omega, \mathbb{P})$. Then S_k is bounded from $L^p(\mathbb{R}_+)$ to $L^p(\mathbb{R}_+ \times \Omega)$ by Proposition 4.2.12 and Proposition 4.2.3. Define

$$\mathcal{S} := \{S_k : k \in \mathcal{K}\}.$$

Example 7.3.13. Let $p \in (2,\infty)$. The family of operators $\mathcal{S}$ from $L^p(\mathbb{R}_+)$ to $L^p(\mathbb{R}_+ \times \Omega)$ is ℓ^r-bounded for all $r \in [2,\infty)$, but not $\ell^2(\ell^r)$-bounded for any $r \in (2,\infty)$.

Proof. Let $r \in [2,\infty)$ and $X = \ell^r$. Take $f \in L^p(\mathbb{R}_+; X)$ and $k \in L^1(\mathbb{R}_+; X)$ such that $k_j \in \mathcal{K}$ for all $j \in \mathbb{N}$. By Theorem 2.9.1 we know that

$$\Big(\mathbb{E} \Big\| \int_0^t |k(t-s)|^{\frac{1}{2}} |f(s)|\, \mathrm{d}W(s) \Big\|_X^p \Big)^{1/p} \simeq \Big\| \Big(\int_0^t |k(t-s)||f(s)|^2\, \mathrm{d}s \Big)^{\frac{1}{2}} \Big\|_X, \qquad t \in \mathbb{R}_+.$$

This implies that $\mathcal{S}$ is ℓ^r-bounded from $L^p(\mathbb{R}_+)$ to $L^p(\mathbb{R}_+ \times \Omega)$ if and only if Γ restricted to $\mathbb{R}_+$ is $\ell^{r/2}$-bounded on $L^{p/2}(\mathbb{R}_+)$, so $\mathcal{S}$ is ℓ^r-bounded for all $r \in [2,\infty)$ by Example 7.3.12. Repeating the argument with $X = \ell^2(\ell^r)$, we also get from Example 7.3.12 that $\mathcal{S}$ is not $\ell^2(\ell^r)$-bounded for any $r \in (2,\infty)$. □

7.4. THE FUNCTION SPACES $V^s(\mathcal{J}; Y)$ AND $R^s(\mathcal{J}; Y)$

The multipliers that we will consider are members of the space of functions of bounded s-variation, which we denote by $V^s(\mathcal{J}, Y)$ for $s \ge 1$. This space contains the class of $1/s$-Hölder continuous functions. In our arguments we will also use the atomic function space $R^s(\mathcal{J}, Y)$, which was introduced in the scalar case in [CRdFS88].

Definition 7.4.1.

(i) Let Y be a Banach space, $J=[J_-,J_+]\subseteq\mathbb{R}$ a bounded interval and $s\in[1,\infty)$. A function $f\colon\mathbb{R}\to Y$ is said to be of *bounded s-variation on J*, or $f\in V^s(J;Y)$, if

$$\|f\|_{V^s(J;Y)}:=\|f\|_{L^\infty(J;Y)}+[f]_{V_s(J;Y)}<\infty,$$

where

$$[f]_{V_s(J;Y)}:=\sup_{J_-=t_0<\cdots<t_N=J_+}\Big(\sum_{i=1}^N\|f(t_{i-1})-f(t_i)\|_Y^s\Big)^{1/s}.$$

Furthermore we define $V^\infty(J;Y)=L^\infty(J;Y)$.

(ii) When $\mathcal{J}$ is a collection of mutually disjoint bounded intervals in $\mathbb{R}$, the space $V^s(\mathcal{J};Y)\subseteq L^\infty(\mathbb{R};Y)$ consists of all $f\in L^\infty(\mathbb{R};Y)$ such that

$$\|f\|_{V^s(\mathcal{J};Y)}:=\sup_{J\in\mathcal{J}}\|f|_J\|_{V^s(J;Y)}<\infty.$$

If $\mathcal{J}=(J_k)_{k\in\mathbb{N}}$ is ordered, we define $V_0^s(\mathcal{J};Y)\subseteq V^s(\mathcal{J};Y)$ to be the closed subspace consisting of $f\in V^s(\mathcal{J};Y)$ with $\lim_{k\to\infty}\|f|_{J_k}\|_{V^s(J;Y)}=0$.

Clearly $V^s(\mathcal{J};Y)\hookrightarrow V^t(\mathcal{J};Y)$ contractively when $1\le s\le t\le\infty$, and $V^s(\mathcal{J};Y)$ is complete when Y is complete.

In our applications the space Y is usually the span of a bounded and absolutely convex subset B of a normed space Z (i.e. a *disc* in Z), equipped with the Minkowski norm

$$\|x\|_B:=\inf\{\lambda>0:\tfrac{x}{\lambda}\in B\},$$

and we write $V^s(\mathcal{J};B):=V^s(\mathcal{J};\operatorname{span}B)$. Clearly $\|x\|_Z\lesssim_B\|x\|_B$ for $x\in Y$. If the Minkowski norm on $\operatorname{span}B$ is complete, then B is called a *Banach disc*. If Z is a Banach space and B is closed, then B is a Banach disc [PB87, Proposition 5.1.6], but this is not a necessary condition [PB87, Proposition 3.2.21].

Definition 7.4.2.

(i) Let Y be a normed space, $J\subseteq\mathbb{R}$ a bounded interval, and $s\in[1,\infty)$. Say that a function $a\colon J\to Y$ is an *$R^s(J;Y)$-atom*, written $a\in R^s_{\mathrm{at}}(J;Y)$, if there exists a set $\mathcal{I}$ of mutually disjoint subintervals of J and a set of vectors $(c_I)_{I\in\mathcal{I}}\subseteq Y$ such that

$$a=\sum_{I\in\mathcal{I}}c_I\mathbf{1}_I\quad\text{and}\quad\Big(\sum_{I\in\mathcal{I}}\|c_I\|_Y^s\Big)^{1/s}\le 1.$$

Define $R^s(J;Y)\subseteq L^\infty(J;Y)$ by

$$R^s(J;Y):=\Big\{f\in L^\infty(J;Y):f=\sum_{k=1}^\infty\lambda_k a_k,(\lambda_k)\in\ell^1,(a_k)\subseteq R^s_{\mathrm{at}}(J;Y)\Big\},$$

where the series $f = \sum_{k=1}^{\infty} \lambda_k a_k$ converges in $L^\infty(J;Y)$. Define a norm on $R^s(J;Y)$ by

$$\|f\|_{R^s(J;Y)} := \inf\Big\{\|\lambda_k\|_{\ell^1} : f = \sum_{k=1}^{\infty} \lambda_k a_k \text{ as above}\Big\}.$$

Furthermore we define $R^\infty(J;Y) := L^\infty(J;Y)$.

(ii) When $\mathcal{J}$ is a collection of mutually disjoint bounded intervals in $\mathbb{R}$, the space $R^s(\mathcal{J};Y) \subseteq L^\infty(\mathbb{R};Y)$ consists of all $f \in L^\infty(\mathbb{R};Y)$ such that

$$\|f\|_{R^s(\mathcal{J};Y)} := \sup_{J\in\mathcal{J}} \|f|_J\|_{R^s(J;Y)} < \infty.$$

If $\mathcal{J} = (J_k)_{k\in\mathbb{N}}$ is ordered, we define $R^s_0(\mathcal{J};Y) \subseteq R^s(\mathcal{J};Y)$ to be the closed subspace consisting of $f \in R^s(\mathcal{J};Y)$ with $\lim_{k\to\infty} \|f|_{J_k}\|_{R^s(J_k;Y)} = 0$.

Clearly $R^s(\mathcal{J};Y) \hookrightarrow R^t(\mathcal{J};Y)$ contractively when $1 \le s \le t \le \infty$, and $R^s(\mathcal{J};Y)$ is complete when Y is complete. As with the classes V^s, when B is a disc in a normed space Z, we put the Minkowski norm on the linear span of B and write $R^s(\mathcal{J};B) := R^s(\mathcal{J};\operatorname{span} B)$.

For $\alpha \in (0,1]$ and an interval $J \subseteq \mathbb{R}$ we let $C^\alpha(J;Y)$ denote the space of α-Hölder continuous functions with $\|f\|_{C^\alpha(J;Y)} = \max\big\{\|f\|_\infty, [f]_{C^\alpha(J;Y)}\big\}$, where

$$[f]_{C^\alpha(J;Y)} := \sup_{x,y\in J} \frac{\|f(x)-f(y)\|_Y}{|x-y|^\alpha}.$$

Lemma 7.4.3. *Let $s \in [1,\infty)$, let Y be a Banach space and fix a bounded interval $J \subseteq \mathbb{R}$.*

(i) If $q \in (s,\infty)$, then $R^s(J;Y) \subseteq V^s(J;Y) \subseteq R^q(J;Y)$ and for all $f \in L^\infty(J;Y)$ we have

$$\|f\|_{R^q(J;Y)} \lesssim_{q,s} \|f\|_{V^s(J;Y)} \lesssim \|f\|_{R^s(J;Y)}.$$

(ii) We have $C^{1/s}(J;Y) \subseteq V^s(J;Y)$, and for all $f \in V^s(J;Y)$,

$$\|f\|_{V^s(J;Y)} \le \|f\|_\infty + |J|^{1/s}[f]_{C^{1/s}(J;Y)}.$$

Proof. For (i) we note that both $R^s(J;Y) \subseteq V^s(J;Y)$ and the second norm estimate follow directly from the fact that for any atom $a \in R^s_{\mathrm{at}}(J;Y)$ with

$$a = \sum_{I\in\mathcal{I}} c_I \mathbf{1}_I$$

we have by Minkowski's inequality that

$$\|a\|_{V^s(J;Y)} \le \sup_{I\in\mathcal{I}} \|c_I\|_Y + \Big(\sum_{\substack{I,J\in\mathcal{I}\\ I\neq J}} \|c_I - c_J\|^s\Big)^{1/s} \le 1 + 2\Big(\sum_{I\in\mathcal{I}} \|c_I\|^s\Big)^{1/s} \le 3.$$

The embedding $V^s(J;Y) \subseteq R^q(J;Y)$ with the first norm estimate is shown in [CRdFS88, Lemme 2] for scalar functions, and the argument extends to the general case. Part (ii) is straightforward to check. □

We end this section with complex interpolation containments for the V^s- and R^s-classes. It is an open problem whether complex interpolation of the V^s-classes as below can be proved with $\varepsilon = 0$ (see [Pis16, Chapter 12]). It is also not clear whether converse inclusions hold, but since we don't need them we leave the question open.

Theorem 7.4.4. *Suppose $1 \le q_0 \le q_1 \le \infty$, $\theta \in (0,1)$, $\varepsilon > 0$ and let Y be a Banach space. Then for all bounded intervals $J \subseteq \mathbb{R}$ we have continuous inclusions*

$$V^{[q_0,q_1]_\theta - \varepsilon}(J;Y) \hookrightarrow [V^{q_0}(J;Y), V^{q_1}(J;Y)]_\theta, \tag{7.4.1}$$

$$R^{[q_0,q_1]_\theta}(J;Y) \hookrightarrow [R^{q_0}(J;Y), R^{q_1}(J;Y)]_\theta, \qquad q_1 \neq \infty. \tag{7.4.2}$$

Furthermore, if $\mathcal{J} = (J_k)_{k\in\mathbb{N}}$ is an ordered collection of mutually disjoint bounded intervals in $\mathbb{R}$, then we have continuous inclusions

$$V_0^{[q_0,q_1]_\theta - \varepsilon}(\mathcal{J};Y) \hookrightarrow [V_0^{q_0}(\mathcal{J};Y), V_0^{q_1}(\mathcal{J};Y)]_\theta \tag{7.4.3}$$

$$R_0^{[q_0,q_1]_\theta}(\mathcal{J};Y) \hookrightarrow [R_0^{q_0}(\mathcal{J};Y), R_0^{q_1}(\mathcal{J};Y)]_\theta, \qquad q_1 \neq \infty. \tag{7.4.4}$$

Proof. For $q_0 = 1$ and $q_1 = \infty$ we have (7.4.1) by applying subsequently [Pis16, Lemma 12.11], [BL76, Theorem 3.4.1], and [BL76, Theorem 4.7.1],

$$\begin{aligned} V^{[q_0,q_1]_\theta - \varepsilon}(J;Y) &\hookrightarrow \big(V^1(J;Y), L^\infty(J;Y)\big)_{\theta_\varepsilon,\infty} \\ &\hookrightarrow \big(V^1(J;Y), L^\infty(J;Y)\big)_{\theta,1} \\ &\hookrightarrow \big[V^1(J;Y), L^\infty(J;Y)\big]_\theta \end{aligned}$$

with

$$\theta_\varepsilon = 1 - \frac{1}{\frac{1}{1-\theta} - \varepsilon} < \theta.$$

The intermediate cases follow from the reiteration theorem for complex interpolation [BL76, Theorem 4.6.1].

In the remainder of the proof we will need the following notation: when $\mathcal{I}_k$ is a collection of intervals for each $k \in \mathbb{N}$ and $I \in \mathcal{I}_k$, let $\pi_{I,k}$ denote the canonical projection $\ell^\infty(\mathcal{I}_k;Y) \to Y$. We abbreviate Banach couples (X_0, X_1) by $X_\bullet$, and use this shorthand for expressions like

$$[\ell^{p_\bullet}(\mathbb{N};X)]_\theta = [\ell^{p_0}(\mathbb{N};X), \ell^{p_1}(\mathbb{N};X)]_\theta.$$

Define the open strip $\mathbb{S} := \{z \in \mathbb{C} : \operatorname{Re} z \in (0,1)\}$. We let $\mathcal{F}(X_\bullet)$ denote the space of bounded continuous functions from the closed strip $\overline{\mathbb{S}}$ to the sum $X_0 + X_1$ whose restrictions to $\mathbb{S}$ is analytic and whose restrictions to the sets $\{z \in \mathbb{C} : \operatorname{Re} z = 0\}$ and $\{z \in \mathbb{C} : \operatorname{Re} z = 1\}$ map continuously into X_0 and X_1 respectively, equipped with the norm

$$\|F\|_{\mathcal{F}(X_\bullet)} := \max\Big(\sup_{t\in\mathbb{R}} \|F(it)\|_{X_0}, \sup_{t\in\mathbb{R}} \|F(1+it)\|_{X_1}\Big).$$

For (7.4.2) let $1 \le q_0 \le q_1 \le \infty$ and write $q_\theta := [q_0, q_1]_\theta$ for brevity. Suppose $f \in R^{q_\theta}(J;Y)$, with atomic decomposition

$$f = \sum_{k=1}^{\infty} \lambda_k a_k = \sum_{k=1}^{\infty} \lambda_k \sum_{I \in \mathcal{I}_k} \mathbf{1}_I \pi_{I,k}(c_k),$$

where $c_k \in \ell^{q_\theta}(\mathcal{I}_k;Y)$ for each $k \in \mathbb{N}$.

Let $\varepsilon > 0$. For each $k \in \mathbb{N}$ we have $\ell^{q_\theta}(\mathcal{I}_k;Y) = [\ell^{q_\bullet}(\mathcal{I}_k;Y)]_\theta$ with equal norms [Tri78, Theorem 1.18.1], hence there exists a function $C_k \in \mathcal{F}(\ell^{q_\bullet}(\mathcal{I}_k;Y))$ with $C_k(\theta) = c_k$ and $\|C_k\|_{\mathcal{F}(\ell^{q_\bullet}(\mathcal{I}_k;Y))} \le (1+\varepsilon)\|c_k\|_{\ell^{q_\theta}(\mathcal{I}_k;Y)} \le 1+\varepsilon$. For all $z \in \overline{\mathbb{S}}$ and $t \in J$, define

$$A_k(z)(t) := \sum_{I \in \mathcal{I}_k} \mathbf{1}_I(t) \pi_{I,k}(C_k(z)),$$

noting that for each t there is at most one non-zero term in the sum. It follows from $\|C_k\|_{\mathcal{F}(\ell^{q_\bullet}(\mathcal{I}_j;Y))} \le 1+\varepsilon$ that $\|A_k\|_{\mathcal{F}(R^{q_\bullet}(J;Y))} \le 1+\varepsilon$ for all $z \in \overline{\mathbb{S}}$.

We will show that each $A_k \colon S \to R^{q_0}(J;Y) + R^{q_1}(J;Y)$ is analytic on $\mathbb{S}$, using that $R^{q_0}(J;Y) + R^{q_1}(J;Y) = R^{q_1}(J;Y)$ and $\ell^{q_0}(\mathcal{I}_k;Y) + \ell^{q_1}(\mathcal{I}_k;Y) = \ell^{q_1}(\mathcal{I}_k;Y)$. Fix $z_0 \in \mathbb{S}$. Since C_k is analytic with values in $\ell^{q_1}(\mathcal{I}_k;Y)$, there exists a Taylor expansion

$$C_k(z) = \sum_{n=0}^{\infty} (z-z_0)^n \beta_{k,n}$$

for z in a neighbourhood of z_0, where $(\beta_{k,n})_{n=0}^{\infty} \subseteq \ell^{q_1}(\mathcal{I}_k;Y)$. Thus for such z we have

$$A_k(z) = \sum_{I \in \mathcal{I}_k} \mathbf{1}_I \pi_{I,k}(C_k(z)) = \sum_{n=0}^{\infty} (z-z_0)^n \sum_{I \in \mathcal{I}_k} \mathbf{1}_I \pi_{I,k}(\beta_{k,n}) =: \sum_{n=0}^{\infty} (z-z_0)^n \gamma_{k,n}$$

using the mutual disjointness of $\mathcal{I}_k$ to interchange the sums. The functions $\gamma_{k,n}$ are in $R^{q_1}(J;Y)$ as we can write

$$\|\gamma_{k,n}\|_{R^{q_1}(J;Y)} = \Big\| \sum_{I \in \mathcal{I}_k} \mathbf{1}_I \pi_{I,k}(\beta_{k,n}) \Big\|_{R^{q_1}(J;Y)} \le \|\beta_{k,n}\|_{\ell^{q_1}(\mathcal{I}_k;Y)} < \infty.$$

Similarly we can show that each $A_k \colon \overline{\mathbb{S}} \to R^{q_1}(J;Y)$ is continuous.

Now for $z \in \overline{\mathbb{S}}$ define

$$F(z) := \sum_{k=1}^{\infty} \lambda_k A_k(z).$$

Since the functions $A_k \colon \mathbb{S} \to R^{q_0}(J;Y) + R^{q_1}(J;Y)$ are bounded uniformly in k, continuous on $\overline{\mathbb{S}}$, and analytic on $\mathbb{S}$, and since $\lambda \in \ell^1$, and each A_k maps into $R^{q_0}(J;Y) + R^{q_1}(J;Y)$, we find that $F \in \mathcal{F}(R^{q_\bullet}(J;Y))$. Furthermore we have

$$F(\theta) = \sum_{k=1}^{\infty} \lambda_k A_k(\theta) = \sum_{k=1}^{\infty} \lambda_k \sum_{I \in \mathcal{I}_k} \mathbf{1}_I \pi_{I,k}(C_k(\theta)) = f$$

and

$$\|F\|_{\mathcal{F}(R^{q_\bullet}(J;Y))} \le \|\lambda_k\|_{\ell^1} \sup_{k \in \mathbb{N}} \|A_k\|_{\mathcal{F}(R^{q_\bullet}(J;Y))} \le (1+\varepsilon)\|\lambda_k\|_{\ell^1}.$$

Since $\varepsilon > 0$ was arbitrary, taking the infimum over all atomic decompositions of f and all $F \in \mathcal{F}(R^{q_\bullet}(J;Y))$ with $F(\theta) = f$ completes the proof.

Now consider a collection $\mathcal{J}$ of mutually disjoint bounded intervals in $\mathbb{R}$. We will only prove (7.4.3), as the proof of (7.4.4) is similar. We introduce the following notation: if $J = [J_-, J_+) \subseteq \mathbb{R}$ is a bounded interval and $f \in L^0(J;Y)$, we let $f_J \in L^0([0,1);Y)$ be the function

$$f_J(x) := f((J_+ - J_-)x + J_+) \qquad x \in [0,1).$$

Then for each $s \in [1,\infty]$ the map $\tau_J \colon V^s(J;Y) \to V^s([0,1);Y)$ defined by $\tau_J(f) := f_J$ is an isometry. Consequently we can write

$$\|f\|_{V^s(\mathcal{J};Y)} = \sup_{J\in\mathcal{J}} \|f|_J\|_{V^s(J;Y)} = \sup_{J\in\mathcal{J}} \|\tau_J(f|_J)\|_{V^s([0,1);Y)},$$

and therefore the map $\Phi \colon V_0^s(\mathcal{J};Y) \to c_0(\mathcal{J}; V^s([0,1);Y))$ defined by

$$\Phi(f) := (\tau_J(f|_J))_{J\in\mathcal{J}}$$

is an isometry. Since the intervals in $\mathcal{J}$ are mutually disjoint, Φ is an isometric isomorphism. Thus Φ^{-1} induces an isometric isomorphism

$$\Phi^{-1} \colon c_0\big(\mathcal{J}; [V^{q_\bullet}([0,1);Y)]_\theta\big) = \big[c_0(\mathcal{J}; V^{q_\bullet}([0,1);Y))\big]_\theta \to [V_0^{q_\bullet}(\mathcal{J};Y)]_\theta,$$

using [Tri78, Remark 3, §1.18.1]. By (7.4.1) we have

$$V^{[q_0,q_1]_\theta-\varepsilon}([0,1);Y) \hookrightarrow [V^{q_\bullet}([0,1);Y)]_\theta,$$

so that Φ^{-1} yields an embedding

$$c_0(\mathcal{J}; V^{[q_0,q_1]_\theta-\varepsilon}([0,1);Y)) \hookrightarrow [V_0^{q_\bullet}(\mathcal{J};Y)]_\theta.$$

Precomposing with Φ gives the bounded inclusion

$$V_0^{[q_0,q_1]_\theta-\varepsilon}(\mathcal{J};Y) \hookrightarrow [V_0^{q_\bullet}(\mathcal{J};Y)]_\theta$$

and completes the proof. □

7.5. FOURIER MULTIPLIER THEOREMS

Let X, Y be Banach function spaces and let $m \colon \mathbb{R} \to \mathcal{L}(X,Y)$. In this section we will develop sufficient conditions on X and Y which imply that the Fourier multiplier operator

$$T_m \colon \mathcal{S}(\mathbb{R}^d;X) \to \mathcal{S}'(\mathbb{R}^d;Y), \qquad T_m f = (m\widehat{f})^\vee.$$

extends to a bounded operator from $L^p(\mathbb{R},w;X)$ to $L^p(\mathbb{R},w;Y)$. In particular we will prove operator-valued variants of the multiplier theory of Coifmann-Rubio de Francia–Semmes, i.e. we will show the boundedness of T_m for $m \in V^s(\Delta;\mathcal{L}(X,Y))$ and w in

a suitable Muckenhoupt class. We will only consider multipliers m defined on $\mathbb{R}$; extensions to multipliers defined on $\mathbb{R}^d$ can be obtained by an induction argument as in [Kró14, Section 4], [Lac07] and [Xu96], and extensions to multipliers on the torus $\mathbb{T}$ can be obtained by transference, see [11, Proposition 4.1]. In this case one must consider multipliers defined on $\hat{\mathbb{T}} = \mathbb{Z}$, where bounded s-variation for a function on $\mathbb{Z}$ is defined analogously to Definition 7.4.1.

We start with a result that is well-known in the unweighted setting (see [HHN02, ŠW07]). It will be used in the proof of Theorem 7.5.16. Recall that $\Delta = \{\pm[2^k, 2^{k+1}), k \in \mathbb{Z}\}$ is the standard dyadic partition of $\mathbb{R}$.

Theorem 7.5.1 (Vector-valued Marcinkiewicz multiplier theorem)**.** *Let X and Y be UMD Banach spaces, and suppose $\Gamma \subseteq \mathcal{L}(X,Y)$ is absolutely convex and $\mathcal{R}$-bounded. Suppose $m \in V^1(\Delta;\Gamma)$. Then for all $p \in (1,\infty)$ and $w \in A_p$,*

$$\|T_m\|_{L^p(\mathbb{R},w;X)\to L^p(\mathbb{R},w;Y)} \leq \phi_{X,Y,p}([w]_{A_p})\|\Gamma\|_{\mathcal{R}}\|m\|_{V^1(\Delta;\Gamma)}.$$

Proof. To prove the result one can repeat the argument in [HHN02, Theorem 4.3] using weighted Littlewood–Paley inequalities with sharp cut-off functions, which can be found for instance in [FHL20]. □

Our starting point for multiplier theorems for $m \in V^s$ with $s > 1$ will be the Littlewood–Paley–Rubio de Francia estimates developed in Section 7.2

7.5.1. MULTIPLIERS IN HILBERT SPACES

The first part of the following theorem is an analogue of [Kró14, Theorem A(i)], and the second part is an unweighted analogue of [Kró14, Theorem A(ii)]. The second part is also proved in [HP06, Proposition 3.3]. The exponents (p,s) for which each part of the theorem applies are pictured in Figure 7.1.

Theorem 7.5.2. *Let X and Y be Hilbert spaces, $p,s \in (1,\infty)$, and consider a multiplier $m \in V^s(\Delta;\mathcal{L}(X,Y))$.*

(i) If $s \leq 2$ and $p \geq s$, then for all $w \in A_{p/s}$ we have

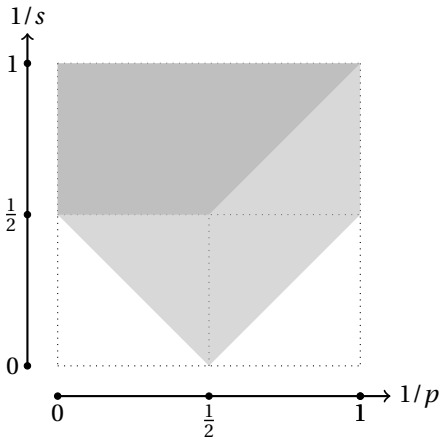
$$\|T_m\|_{L^p(\mathbb{R},w;X)\to L^p(\mathbb{R},w;Y)} \leq \phi_{p,s}([w]_{A_{p/s}})\|m\|_{V^s(\Delta;\mathcal{L}(X,Y))}.$$

(ii) If $\frac{1}{s} > \left|\frac{1}{p} - \frac{1}{2}\right|$ we have

$$\|T_m\|_{L^p(\mathbb{R};X)\to L^p(\mathbb{R};Y)} \lesssim_{p,s} \|m\|_{V^s(\Delta;\mathcal{L}(X,Y))}.$$

To prove Theorem 7.5.2 we use the following proposition, which is a version of Theorem 7.5.2(i) for R-class multipliers. The techniques used to prove this proposition are strongly related to those used in the proof of our main result for UMD Banach function spaces, Theorem 7.5.6.

Figure 7.1: Allowable exponents for Theorem 7.5.2: the weighted case (i) dark shaded, the unweighted case (ii) light shaded.



Proposition 7.5.3. *Let X and Y be Hilbert spaces, $s \in (1,2]$, and consider a multiplier $m \in R^s(\Delta; \mathcal{L}(X,Y))$. Then for all $p > s$ and $w \in A_{p/s}$ we have*

$$\|T_m\|_{L^p(\mathbb{R},w;X)\to L^p(\mathbb{R},w;Y)} \le \phi_{p,s}([w]_{A_{p/s}})\|m\|_{R^s(\Delta;\mathcal{L}(X,Y))}.$$

Proof. We only consider the case $s < 2$. The case $s = 2$ is similar, but simpler. Fix $\varepsilon > 0$ and let $f \in L^p(\mathbb{R}, w; X)$. By approximation we may assume that the dyadic Littlewood–Paley decomposition of f has finitely many nonzero terms and set $\Delta_f = \{J \in \Delta : S_J f \neq 0\}$. For each $J \in \Delta_f$ let

$$m|_J = \sum_{k=1}^N \lambda_k a_k^J, \qquad a_k^J = \sum_{I\in\mathcal{J}_k^J} c_I^{J,k}\mathbf{1}_I$$

be an $R^s(J;\mathcal{L}(X,Y))$-atomic decomposition of the restriction $m|_J$ with λ_k independent of J and

$$\sum_{k=1}^N |\lambda_k| \le (1+\varepsilon)\|m\|_{R^s(\Delta;\mathcal{L}(X,Y))}$$

as in [HP06, Theorem 2.3].

Note that $S_J T_m = T_m S_J$, where we abuse notation by letting S_J denote either the X- or Y-valued Fourier projection. By the Littlewood–Paley estimate (see [FHL20, Theorem 3.4]), Hölder's inequality, Remark 7.2.7, and $w \in A_{p/s} \subseteq A_p$, we have

$$\begin{aligned}\|T_m f\|_{L^p(\mathbb{R},w;Y)} &\le \phi_p([w]_{A_p})\Big\|\Big(\sum_{J\in\Delta_f}\|T_m S_J f\|_Y^2\Big)^{1/2}\Big\|_{L^p(\mathbb{R},w)}\\ &\le \phi_p([w]_{A_p})\Big\|\Big(\sum_{J\in\Delta_f}\Big(\sum_{k=1}^N|\lambda_k|\sum_{I\in\mathcal{J}_k^J}\|c_I^{J,k}S_I f\|_Y\Big)^2\Big)^{1/2}\Big\|_{L^p(\mathbb{R},w)}\end{aligned}$$

$$\leq \phi_p([w]_{A_p}) \sum_{k=1}^N |\lambda_k| \Big\| \Big(\sum_{J\in\Delta_f} \Big(\sum_{I\in\mathcal{J}_k^J} \|c_I^{J,k}\|^s \Big)^{\frac{2}{s}} \Big(\sum_{I\in\mathcal{J}_k^J} \|S_I f\|_X^{s'} \Big)^{\frac{2}{s'}} \Big)^{\frac{1}{2}} \Big\|_{L^p(\mathbb{R},w)}$$

$$\leq \phi_p([w]_{A_p}) \sum_{k=1}^N |\lambda_k| \Big\| \Big(\sum_{J\in\Delta_f} \Big(\sum_{I\in\mathcal{J}_k^J} \|S_I f\|_X^{s'} \Big)^{2/s'} \Big)^{1/2} \Big\|_{L^p(\mathbb{R},w)}$$

$$\leq \phi_{p,s}([w]_{A_{p/s}}) \sum_{k=1}^N |\lambda_k| \, \|f\|_{L^p(\mathbb{R},w;X)}.$$

Since $\varepsilon > 0$ was arbitrary this implies

$$\|T_m f\|_{L^p(\mathbb{R},w;Y)} \leq \phi_{p,s}([w]_{A_{p/s}}) \|m\|_{R^s(\Delta;\mathcal{L}(X,Y))} \|f\|_{L^p(\mathbb{R},w;X)}$$

for all $w \in A_{p/s}$ and $f \in L^p(\mathbb{R},w;X)$. □

Proof of Theorem 7.5.2. **Part (i):** We first consider the case $p < s$ and $s < 2$. Let $w \in A_{p/s}$ and take $\sigma \in (s,2]$ such that $w \in A_{p/\sigma}$, which is possible by Proposition 2.3.2(iii). By Lemma 7.4.3 we know that $m \in R^\sigma(\Delta;\mathcal{L}(X,Y))$ with

$$\|m\|_{R^\sigma(\Delta;\mathcal{L}(X,Y))} \lesssim_{s,\sigma} \|m\|_{V^s(\Delta;\mathcal{L}(X,Y))},$$

so by Proposition 7.5.3 we obtain

$$\|T_m\|_{\mathcal{L}(L^p(\mathbb{R},w;X),L^p(\mathbb{R},w;Y))} \leq \phi_{p,s}([w]_{A_{p/s}}) \|m\|_{V^s(\Delta;\mathcal{L}(X,Y))}.$$

Next we consider the case $p > s = 2$. Observe that by [HNVW16, Proposition 5.3.16] it suffices to prove the result for the truncated multipliers

$$m_N := \mathbf{1}_{\bigcup_{n=1}^N J_n} m,$$

where $\Delta = (J_n)_{n=1}^\infty$ is an arbitrary ordering of Δ. Since $m_N \in V_0^s(\Delta;\mathcal{L}(X,Y))$ uniformly, without loss of generality we may work with an arbitrary multiplier $m \in V_0^s(\Delta;\mathcal{L}(X,Y))$. Fix $w \in A_{p/2}$. Then by Proposition 2.3.2(iv) there exists a $\delta > 0$ such that $w^{1+\delta} \in A_{p/2}$. Take

$$\theta = \frac{2}{p}\Big(1 - \frac{1}{1+\delta}\Big), \qquad p_0 = (1+\delta)(1-\theta)p, \quad \text{and} \quad \sigma = 2-\theta.$$

Then $\theta \in (0,1)$, $\sigma \in (1,2)$ and $p_0 = p + (p-2)\delta > p$, so by the first case we have

$$\|T_m\|_{L^{p_0}(\mathbb{R},w;X)\to L^{p_0}(\mathbb{R},w;Y)} \leq \phi_{p_0,\sigma}([w]_{A_{p/2}}) \|m\|_{V_0^\sigma(\Delta;\mathcal{L}(X,Y))}.$$

Moreover by Plancherel's theorem (which is valid since X and Y are Hilbert spaces) we know that

$$\|T_m\|_{\mathcal{L}(L^2(\mathbb{R};X),L^2(\mathbb{R};Y))} \leq \|m\|_{L^\infty(\mathbb{R};\mathcal{L}(X,Y))}. \tag{7.5.1}$$

Since

$$\frac{1}{[p_0,2]_\theta} = \frac{1}{p(1+\delta)} + \frac{1}{p} - \frac{1}{p(1+\delta)} = \frac{1}{p},$$

we know by [Tri78, Theorem 1.18.5] that $L^p(\mathbb{R}, w; X) = [L^{p_0}(w^{1+\delta}, X), L^2(\mathbb{R}; X)]_\theta$, and likewise with X replaced by Y. Moreover since $[\sigma, \infty]_\theta = \frac{2-\theta}{1-\theta} > 2$ we have the continuous inclusions

$$\begin{aligned} V^2(\Delta; \mathcal{L}(X,Y)) &\hookrightarrow [V_0^\sigma(\Delta; \mathcal{L}(X,Y)), V_0^\infty(\Delta; \mathcal{L}(X,Y))]_\theta \\ &\hookrightarrow [V_0^\sigma(\Delta; \mathcal{L}(X,Y)), L^\infty(\mathbb{R}; \mathcal{L}(X,Y))]_\theta \end{aligned}$$

by Theorem 7.4.4. By bilinear complex interpolation [BL76, §4.4] applied to the bilinear map $(m, f) \mapsto T_m f$ we have boundedness of $T_m \colon L^p(\mathbb{R}, w; X) \to L^p(\mathbb{R}, w; Y)$ with the required norm estimate.

Finally we consider the case $p = s \geq 2$; we will use another interpolation argument. Fix $w \in A_1$. Then by Proposition 2.3.2((iv)) there exists a $\delta > 0$ such that $w^{1+\delta} \in A_1$. Fix $p_1 \in (s, s + (s-1)\delta)$. By the argument of the previous cases we have

$$\|T_m\|_{L^{p_1}(w^{1+\delta}; X) \to L^{p_1}(w^{1+\delta}; Y)} \leq \phi_{p_1, s}([w]_{A_1}) \|m\|_{V^s(\Delta; \mathcal{L}(X,Y))}.$$

Let $\theta \in (0,1)$ be such that $\theta(1+\delta)s = p_1$. Such a θ exists since $p_1 < s + (s-1)\delta$. Choose $p_0 \in (1, s)$ such that $[p_0, p_1]_\theta = s$. Such a p_0 exists since $p_1 > s$ and $[1, p_1]_\theta < s$. Indeed, the latter follows from

$$\frac{s}{[1, p_1]_\theta} = s(1-\theta) + s\frac{\theta}{p_1} = s - \frac{p_1}{1+\delta} + \frac{1}{1+\delta} > 1.$$

Since $p_0 < s \leq 2$ we have by duality with the previous cases (taking $w = 1$) that

$$\|T_m\|_{L^{p_0}(\mathbb{R}; X) \to L^{p_0}(\mathbb{R}; Y)} \lesssim_{p_0, s} \|m\|_{V^s(\Delta; \mathcal{L}(X,Y))}.$$

As before our choice of θ yields $L^s(\mathbb{R}, w; X) = [L^{p_0}(\mathbb{R}, X), L^{p_1}(w^{1+\delta}; X)]_\theta$, and likewise with X replaced by Y. Therefore by complex interpolation we have boundedness of $T_m \colon L^s(\mathbb{R}, w; X) \to L^s(\mathbb{R}, w; Y)$ with the required norm estimate.

Part (ii): The case $p = 2$ is clear from (7.5.1) and the embedding of the V^s-classes in L^∞. For $p > 2$ we may assume without loss of generality that $m \in V_0^s(\Delta; \mathcal{L}(X,Y))$ as in Part (i). Moreover, by embedding of the V^s-classes, we may assume that $s > 2$.

Let $\sigma \in \left(s, \left(\frac{1}{2} - \frac{1}{p}\right)^{-1}\right)$ and fix $t \in (2, \infty)$ such that $[2, t]_{\frac{\sigma}{2}} = p$. Such a t exists since $p > 2$ and

$$\frac{1}{p} = \frac{1}{[2, t]_{\frac{\sigma}{2}}} = \frac{1}{2} - \frac{1}{\sigma} + \frac{2}{\sigma}\frac{1}{t},$$

which implies that

$$\frac{1}{t} = \frac{2}{s}\left(\frac{1}{p} + \frac{1}{\sigma} - \frac{1}{2}\right) > 0.$$

Using the boundedness properties

$$\begin{aligned} V_0^\infty(\Delta; \mathcal{L}(X,Y)) \times L^2(\mathbb{R}; X) &\to L^2(\mathbb{R}; Y) \quad \text{and} \\ V_0^2(\Delta; \mathcal{L}(X,Y)) \times L^t(\mathbb{R}; X) &\to L^t(\mathbb{R}; Y) \end{aligned}$$

of the bilinear map $(m,f)\mapsto T_m f$, which follow from (7.5.1) and Part (i) respectively, we have boundedness of $T_m\colon L^p(\mathbb{R},w;X)\to L^p(\mathbb{R},w;Y)$ with the required norm estimate by bilinear complex interpolation [BL76, §4.4]. Here we use [Tri78, Theorem 1.18.4] and Theorem 7.4.4 to identify the interpolation spaces as before. The case $p<2$ follows by a duality argument. □

Remark 7.5.4.

1. If the multiplier is scalar-valued and $X=Y$, then Theorem 7.5.2 follows simply from the scalar case and a standard Hilbert space tensor extension argument (see [HNVW16, Theorem 2.1.9]).

2. As in [Kró14, Theorem A], a weighted version of Theorem 7.5.2(ii) can be proved, but we omit it to avoid limited range Muckenhoupt weight classes.

7.5.2. MULTIPLIERS IN UMD BANACH FUNCTION SPACES

We now turn to our main result (Theorem 7.5.6). Its proof is inspired by that of [HP06, Theorem 2.3], which is a generalisation of the Hilbert space result in Theorem 7.5.2. Besides the regularity assumption on the multiplier as in the Hilbert space case, we will need an $\ell^2(\ell^q)$-boundedness assumption. We first prove a result for R-class multipliers, analogous to Proposition 7.5.3.

Proposition 7.5.5. *Let $q\in(1,2]$, $p\in(q,\infty)$, and $w\in A_{p/q}$. Let X and Y be Banach function spaces with $X^q\in$ UMD and $Y\in$ UMD. Let $\Gamma\subseteq\mathcal{L}(X,Y)$ be absolutely convex and $\ell^2(\ell^{q'})$-bounded, and suppose $m\in R^q(\Delta;\Gamma)$. Then*

$$\|T_m\|_{\mathcal{L}(L^p(\mathbb{R},w;X),L^p(\mathbb{R},w;Y))}\le\phi_{X,Y,p,q}([w]_{A_{p/q}})\|\Gamma\|_{\ell^2(\ell^{q'})}\|m\|_{R^q(\Delta;\Gamma)}.$$

Proof. Fix $\varepsilon>0$ and let $f\in L^p(\mathbb{R},w;X)$. We begin as in the proof of Proposition 7.5.3: we assume that the dyadic Littlewood–Paley decomposition of f has finitely many nonzero terms and set $\Delta_f=\{J\in\Delta: S_J f\neq 0\}$. For each $J\in\Delta_f$ let

$$m|_J=\sum_{k=1}^N \lambda_k a_k^J,\qquad a_k^J=\sum_{I\in\mathcal{J}_k^J} c_I^{J,k}\mathbf{1}_I$$

be a $R^q(J;\Gamma)$-atomic decomposition of the restriction $m|_J$ with λ_k independent of J, with each $\mathcal{J}_k^J$ finite, and with

$$\sum_{k=1}^N|\lambda_k|\le(1+\varepsilon)\|m\|_{R^q(\Delta;\mathcal{L}(X,Y))}.$$

As before, $S_J T_m=T_m S_J$. By the Littlewood–Paley theorem for UMD Banach function spaces (Proposition 7.2.1), using that $Y\in$ UMD and $w\in A_{p/q}\subseteq A_p$, we have

$$\|T_m f\|_{L^p(\mathbb{R},w;Y)}\le\phi_{Y,p}([w]_{A_p})\Big\|\Big(\sum_{J\in\Delta_f}|T_m S_J f|^2\Big)^{1/2}\Big\|_{L^p(\mathbb{R},w;Y)}$$

$$= \phi_{Y,p}([w]_{A_p}) \Big\| \Big(\sum_{J\in\Delta_f} \Big| \sum_{k=1}^N \lambda_k \sum_{I\in\mathcal{J}_k^J} c_I^{J,k} S_I f \Big|^2 \Big)^{1/2} \Big\|_{L^p(\mathbb{R},w;Y)}$$
$$\leq \phi_{Y,p}([w]_{A_p}) \sum_{k=1}^N |\lambda_k| \Big\| \Big(\sum_{J\in\Delta_f} \Big| \sum_{I\in\mathcal{J}_k^J} c_I^{J,k} S_I f \Big|^2 \Big)^{1/2} \Big\|_{L^p(\mathbb{R},w;Y)}.$$

We estimate the right hand-side by

$$\sum_{k=1}^N |\lambda_k| \Big\| \Big(\sum_{J\in\Delta_f} \Big| \sum_{I\in\mathcal{J}_k^J} c_I^{J,k} S_I f \Big|^2 \Big)^{1/2} \Big\|_{L^p(\mathbb{R},w;Y)}$$
$$\leq \sum_{k=1}^N |\lambda_k| \Big\| \Big(\sum_{J\in\Delta_f} \Big(\Big(\sum_{I\in\mathcal{J}_k^J} \|c_I^{J,k}\|_\Gamma^q \Big)^{1/q} \Big(\sum_{I\in\mathcal{J}_k^J} \Big| \frac{c_I^{J,k} S_I f}{\|c_I^{J,k}\|_\Gamma} \Big|^{q'} \Big)^{1/q'} \Big)^2 \Big)^{1/2} \Big\|_{L^p(\mathbb{R},w;Y)}$$
$$\leq \sum_{k=1}^N |\lambda_k| \Big\| \Big(\sum_{J\in\Delta_f} \Big(\sum_{I\in\mathcal{J}_k^J} \Big| \frac{c_I^{J,k}}{\|c_I^{J,k}\|_\Gamma} S_I f \Big|^{q'} \Big)^{2/q'} \Big)^{1/2} \Big\|_{L^p(\mathbb{R},w;Y)}.$$

By the definition of the Minkowski norm, the operators $c_I^{J,k}/\|c_I^{J,k}\|_\Gamma$ all lie in Γ, so by $\ell^2(\ell^{q'})$-boundedness of Γ we have

$$\|T_m f\|_{L^p(\mathbb{R},w;Y)} \leq \phi_{Y,p}([w]_{A_p}) \|\Gamma\|_{\ell^2(\ell^{q'})} \sum_{k=1}^N |\lambda_k| \Big\| \Big(\sum_{J\in\Delta_f} \Big(\sum_{I\in\mathcal{J}_k^J} |S_I f|^{q'} \Big)^{2/q'} \Big)^{1/2} \Big\|_{L^p(\mathbb{R},w;X)}.$$

By Theorem 7.2.5, we obtain

$$\Big\| \Big(\sum_{J\in\Delta_f} \Big(\sum_{I\in\mathcal{J}_k^J} |S_I f|^{q'} \Big)^{2/q'} \Big)^{1/2} \Big\|_{L^p(\mathbb{R},w;X)} \leq \phi_{X,p,q}([w]_{A_{p/q}}) \|f\|_{L^p(\mathbb{R},w;X)}.$$

Since $\sum_{k=1}^N |\lambda_k| \leq (1+\varepsilon)\|m\|_{R^q(\Delta;\Gamma)}$ and $\varepsilon > 0$ was arbitrary, this finishes the proof. □

Our main multiplier theorem now follows directly from Proposition 7.5.5 and Lemma 7.4.3.

Theorem 7.5.6. *Let X and Y be Banach function spaces, and let $\Gamma \subseteq \mathcal{L}(X,Y)$ be absolutely convex. Let $q \in (1,2]$, $s \in [1,q)$ and $m \in V^s(\Delta;\Gamma)$.*

(i) Suppose that $X^q \in$ UMD, $Y \in$ UMD, and Γ is $\ell^2(\ell^{q'})$-bounded. Then for all $p \in (q,\infty)$ and $w \in A_{p/q}$ we have

$$\|T_m\|_{L^p(\mathbb{R},w;X)\to L^p(\mathbb{R},w;Y)} \leq \phi_{X,Y,p,q}([w]_{A_{p/q}}) \|\Gamma\|_{\ell^2(\ell^{q'})} \|m\|_{V^s(\Delta;\Gamma)}.$$

(ii) Suppose that $X \in$ UMD, $(Y^*)^q \in$ UMD, Γ is $\ell^2(\ell^q)$-bounded, and $m \in V^s(\Delta;\Gamma)$. Then for all $p \in (1,q')$ we have

$$\|T_m\|_{L^p(\mathbb{R};X)\to L^p(\mathbb{R};Y)} \lesssim_{X,Y,p,q} \|\Gamma\|_{\ell^2(\ell^q)} \|m\|_{V^s(\Delta;\Gamma)}.$$

Proof. The first part follows directly from Proposition 7.5.5 and Lemma 7.4.3. For the second part a standard duality argument shows that

$$\|T_m\|_{L^p(\mathbb{R};X)\to L^p(\mathbb{R};Y)} \le \|T_{m^*}\|_{\mathcal{L}(L^{p'}(\mathbb{R};Y^*),L^{p'}(\mathbb{R};X^*))},$$

with $m^*\colon \mathbb{R}\to \operatorname{span}(\Gamma^*)$ defined by $m^*(t) = m(t)^*$ for all $t\in\mathbb{R}$. Applying the first part to m^*, using Proposition 7.3.3 to show that $\mathcal{T}^*$ is $\ell^2(\ell^{q'})$-bounded and noting that $m^*\in V^q(\Delta;\Gamma^*)$, completes the proof. □

If $q=2$ and $w=1$ in Theorem 7.5.6, we recover [HP06, Corollary 2.5] for Banach function spaces, except for the endpoint $p=2$, which is missing since we work in the weighted setting. Of course Theorem 7.5.6(ii) could also be stated with weights. However, to formulate for which weights it holds we would have to introduce limited range Muckenhoupt weight classes.

Remark 7.5.7. The $\ell^2(\ell^{q'})$-boundedness assumption in Theorem 7.5.6 arises naturally from the proof. It is known that boundedness of T_m implies $\mathcal{R}$-boundedness—and thus ℓ^2-boundedness if X has finite cotype—of the image of the Lebesgue points of m (see [CP01] or [HNVW16, Theorem 5.3.15]). However, $\ell^2(\ell^{q'})$-boundedness is not necessary, as may be seen by considering $m = nS$ where $n\in R^q(\Delta)$ is a scalar multiplier and $S\colon X\to Y$ is a bounded linear operator. In this case T_m will be bounded, but $\{S\}$ need not be $\ell^2(\ell^{q'})$-bounded for $q\neq 2$ (see Example 7.3.13 and [KU14, Example 2.16]).

Using complex interpolation, the reverse Hölder inequality, and the openness of the UMD property, we can obtain a result for the endpoint $p=q=s$ in Theorem 7.5.6.

Proposition 7.5.8. *Let X and Y be Banach function spaces. Let $q,r\in(1,2)$ and suppose that $X^q\in$ UMD and $(Y^*)^r\in$ UMD. Let $\Gamma\subseteq\mathcal{L}(X,Y)$ be absolutely convex and both $\ell^2(\ell^{q'})$- and $\ell^2(\ell^r)$-bounded. Let $s=\min\{q,r\}$ and suppose that $m\in V^s(\Delta;\Gamma)$. Then for all $w\in A_1$,*

$$\|T_m\|_{L^q(\mathbb{R},w;X)\to L^q(\mathbb{R},w;Y)} \le \phi_{X,Y,q,r}([w]_{A_1})\max\bigl\{\|\Gamma\|_{\ell^2(\ell^{q'})},\|\Gamma\|_{\ell^2(\ell^r)}\bigr\}\|m\|_{V^s(\Delta;\Gamma)}.$$

Proof. Fix $w\in A_1$. By Proposition 2.3.2(iv) there exists an $\delta>0$ such that $w^{1+\delta}\in A_1$. By the openness of the UMD property of Banach function spaces (see [Rub86, Theorem 4]) we know that there exist

$$q_0\in\bigl(q,\max\{2,q+(q-1)\delta\}\bigr),\qquad r_0\in(r,2)$$

such that $X^{q_0},(Y^*)^{r_0}\in$ UMD. By Corollary 7.3.5 we know that Γ is $\ell^2(\ell^{q_0'})$- and Γ is $\ell^2(\ell^{r_0})$-bounded with

$$\|\Gamma\|_{\ell^2(\ell^{q_0'})}\le\|\Gamma\|_{\ell^2(\ell^{q'})}\quad\text{and}\quad\|\Gamma\|_{\ell^2(\ell^{r_0})}\le\|\Gamma\|_{\ell^2(\ell^r)}. \tag{7.5.2}$$

Fix $p_1\in(q_0,q+(q-1)\delta)$. By Theorem 7.5.6 and (7.5.2) we know that

$$\|T_m\|_{\mathcal{L}(L^{p_1}(w^{1+\delta};X),L^{p_1}(w^{1+\delta};Y))}\le\phi_{X,Y,p_1,q_0}([w]_{A_1})\|\Gamma\|_{\ell^2(\ell^{q'})}\|m\|_{V^s(\Delta;\Gamma)}.$$

Let $\theta \in (0,1)$ be such that $\theta(1+\delta)q = p_1$, and fix $p_0 \in (1,q)$ such that $[p_0,p_1]_\theta = q$. These parameters exist by the same argument as in Theorem 7.5.2(i). Since $p_0 < r_0'$, we know by Theorem 7.5.6((ii)) and (7.5.2) that

$$\|T_m\|_{\mathcal{L}(L^{p_0}(\mathbb{R};X),L^{p_0}(\mathbb{R};Y))} \lesssim_{X,Y,p_0,r_0} \|\Gamma\|_{\ell^2(\ell^r)}\|m\|_{V^s(\Delta;\Gamma)}.$$

Therefore by complex interpolation as in Theorem 7.5.2(i) we have boundedness of $T_m\colon L^q(\mathbb{R},w;X) \to L^q(\mathbb{R},w;Y)$ with the required norm estimate. □

When dealing with operator-valued multipliers m, to check the hypotheses of our results, one needs an $\ell^2(\ell^{q'})$-bounded subset $\Gamma \subseteq \mathcal{L}(X,Y)$ whose span contains $m(\mathbb{R})$, such that m has the appropriate regularity *when measured with respect to the Minkowski norm induced by* Γ. An obvious naïve choice is to assume that $m(\mathbb{R})$ is $\ell^2(\ell^{q'})$-bounded and to take $\Gamma = \overline{m(\mathbb{R})}$, but m may not be sufficiently regular with respect to the Γ-Minkowski norm. By making Γ larger m becomes more regular in the Γ-Minkowski norm, but enlarging Γ may violate $\ell^2(\ell^{q'})$-boundedness. Constructing such a set Γ given a general multiplier m is quite subtle (except of course in the scalar case, where the Minkowski norm on the one-dimensional span of m is equivalent to the absolute value on $\mathbb{C}$). Below we give an example where these problems may be surmounted using extrapolation techniques.

Proposition 7.5.9. *Let $\alpha \in (0,1]$. Suppose that for some $p_0 \in (1,\infty)$ and all $w \in A_{p_0}(\mathbb{R}^d)$ we have that $m\colon \mathbb{R} \to \mathcal{L}(L^{p_0}(\mathbb{R}^d,w))$ satisfies the following Hölder-type condition:*

$$\sup_{t\in\mathbb{R}} \|m(t)\|_{\mathcal{L}(L^{p_0}(\mathbb{R}^d,w))} + \sup_{J\in\Delta} |J|^\alpha [m]_{C^\alpha(J;\mathcal{L}(L^{p_0}(\mathbb{R}^d,w)))} \le \phi([w]_{A_{p_0}}). \tag{7.5.3}$$

Then there exists a family of operators Γ such that $m \in V^{1/\alpha}(\Delta;\Gamma)$ and Γ is $\ell^u(\ell^v)$-bounded on $L^p(\mathbb{R}^d,w)$ for all $p,u,v \in (1,\infty)$ and $w \in A_p(\mathbb{R}^d)$, with

$$\|\Gamma\|_{\ell^u(\ell^v)} \le \phi_{p,u,v}([w]_{A_p}).$$

Proof. For each $J \in \Delta$ define

$$\Gamma(J) := m(J) \cup \Big\{ \frac{m(x)-m(y)}{|x-y|^\alpha} |J|^\alpha : x \ne y \in J \Big\},$$

and set $\Gamma := \bigcup_{J\in\Delta} \Gamma(J)$. Note that $m(\mathbb{R}) \subseteq \Gamma$. We will show that Γ has the desired properties.

Since $m(x) \in \Gamma$ and $\frac{m(x)-m(y)}{|x-y|^\alpha}|J|^\alpha \in \Gamma$ for all $J \in \Delta$ and all $x \ne y \in J$, by the definition of the Minkowski and Hölder norms, we have $\|m(x)\|_\Gamma \le 1$ and $|J|^\alpha [m]_{C^\alpha(J;\Gamma)} \le 1$, from which it follows directly that $m \in V^{1/\alpha}(\Delta;\Gamma)$.

By (7.5.3) we have

$$\|Tf\|_{L^p(\mathbb{R}^d,w)\to L^p(\mathbb{R}^d,w)} \le \phi_p([w]_{A_p})$$

for some $p \in (1,\infty)$, all $w \in A_p(\mathbb{R}^d)$ and all $T \in \Gamma$. Thus the $\ell^u(\ell^v)$-boundedness result follows directly from Proposition 7.3.7. □

In the next example we specialise to the case $X = Y = L^r$ and $s \in (1,2)$. Results for $s \in [2,\infty)$ will be presented in Example 7.5.14. Note that the ℓ^2-boundedness or $\ell^2(\ell^s)$-boundedness assumptions can be deduced for instance from weight-uniform Hölder estimates as in Proposition 7.5.9.

Example 7.5.10. Let (Ω,μ) be a σ-finite measure space. Let $p,r \in (1,\infty)$ and let $\Gamma \subseteq \mathcal{L}(L^r(\Omega))$ be absolutely convex. Let $s \in (1,2)$ and $m \in V^s(\Delta;\Gamma)$. Then T_m is bounded on $L^p(\mathbb{R},w;L^r(\Omega))$ in each of the following cases:

(i) If $r = 2$,

  (a) $p \in [s,\infty)$ and $w \in A_{p/s}$.

  (b) $p \in (1,s']$ and $w \equiv 1$.

(ii) If $r \in (2,\infty)$,

  (a) $p \in (2,\infty)$, $w \in A_{p/2}$ and Γ is ℓ^2-bounded.

  (b) $p \in (1,r)$, $s \in (1,r')$, $w \equiv 1$ and Γ is $\ell^2(\ell^s)$-bounded.

(iii) If $r \in (1,2)$,

  (a) $p \in (1,2)$, $w \equiv 1$ and Γ is ℓ^2-bounded.

  (b) $p \in (r,\infty)$, $s \in (1,r)$, $w \in A_{p/s}$ and Γ is $\ell^2(\ell^{s'})$-bounded.

Proof. The case (i)(a) follows from Theorem 7.5.2 and the case (i)(b) from a duality argument. The cases (ii)(a) and (iii)(a) follow from Theorem 7.5.6(i) and (ii) with $q = 2$. For (iii)(b) choose $q \in (s,r)$ such that $w \in A_{p/q}$. By Corollary 7.3.5, Γ is $\ell^2(\ell^{q'})$-bounded, and therefore Theorem 7.5.6(i) applies. Similarly, (ii)(b) follows from Theorem 7.5.6((ii)). □

There is some overlap between the cases in Example 7.5.10. For $X = L^r(\Omega)$, we can exploit that we always have either $X^2 \in$ UMD or $(X^*)^2 \in$ UMD. This is not possible for general UMD Banach function spaces, which restricts the class of multipliers that can be handled by our results, as shown in the following example.

Example 7.5.11. Let (Ω,μ) be a σ-finite measure space. Let $p \in (1,\infty)$, $r \in (1,2)$, and let $\Gamma \subseteq \mathcal{L}(L^r(\Omega) \oplus L^{r'}(\Omega))$ be absolutely convex. Let $s \in (1,r)$ and $m \in V^s(\Delta;\Gamma)$. Then T_m is bounded on $L^p(\mathbb{R},w;L^r(\Omega) \oplus L^{r'}(\Omega))$ in each of the following cases:

(i) $p \in (r,\infty)$, $w \in A_{p/s}$ and Γ is $\ell^2(\ell^{s'})$-bounded.

(ii) $p \in (1,r')$, $w \equiv 1$ and Γ is $\ell^2(\ell^s)$-bounded.

The result follows from Theorem 7.5.6 in the same way as in Example 7.5.10.

7.5.3. MULTIPLIERS IN INTERMEDIATE UMD BANACH FUNCTION SPACES

We can prove stronger results, allowing for multipliers of lower regularity, if we consider 'intermediate' spaces $X = [Y, H]_\theta$ where $Y^q \in$ UMD for some $q \in (1,2]$ and H is a Hilbert space. For example, when $r \in (2,\infty)$, we have $L^r = [L^{r_0}, L^2]_\theta$ for some $r_0 \in (r,\infty)$ and $\theta \in (0,1)$. In order to use interpolation methods we will need that $\mathrm{span}(\mathcal{T})$ with the Minkowski norm is a Banach space, i.e. that $\mathcal{T}$ is a Banach disc (see below Definition 7.4.1).

Theorem 7.5.12. *Let $p \in (1,\infty)$, $q \in (1,2]$ and $\theta \in (0,1)$. Let Y and H be Banach function spaces over the same measure space, with $Y^q \in$ UMD, H a Hilbert space, and $Y \cap H$ dense in both Y and H. Let $X = [Y,H]_\theta$. Suppose $\Gamma \subseteq \mathcal{L}(Y \cap H)$ is a Banach disc which is $\ell^2(\ell^{q'})$-bounded on Y and uniformly bounded on H. Let $s \in (1,\infty)$ and suppose that $m \in V^s(\Delta;\Gamma)$.*

(i) If $s < \min\{p, [q,2]_\theta\}$ and $s \geq [q,1]_\theta$, then

$$\|T_m\|_{L^p(\mathbb{R},w;X)\to L^p(\mathbb{R},w;X)} \leq \phi_{Y,p,q,s,\theta}([w]_{A_{p/s}})\|m\|_{V^s(\Delta;\Gamma)}\|\mathcal{T}\|_{\ell^2(\ell^{q'})}$$

for all $w \in A_{p/s}$.

(ii) If

$$\frac{1}{s} > \max\Big\{\frac{1}{[q,2]_\theta} - \frac{1}{p}, \frac{1-\theta}{q}, \frac{1}{p} - \frac{\theta}{2}\Big\}$$

and $p > [q,1]_\theta$, then

$$\|T_m\|_{L^p(\mathbb{R};X)\to L^p(\mathbb{R};X)} \lesssim_{Y,p,q,s,\theta} \|m\|_{V^s(\Delta;\Gamma)}\|\mathcal{T}\|_{\ell^2(\ell^{q'})}.$$

The allowable exponents (p,s) in Theorem 7.5.12 are shown in Figure 7.2. The symmetry in Figure 7.2 is due to the equalities

$$\frac{\theta}{2} = \frac{1}{[\infty,2]_\theta} - 0 = \frac{1}{[q,1]_\theta} - \frac{1}{[q,2]_\theta} = \frac{1}{[q,2]_\theta} - \frac{1}{[q,\infty]_\theta}$$

and

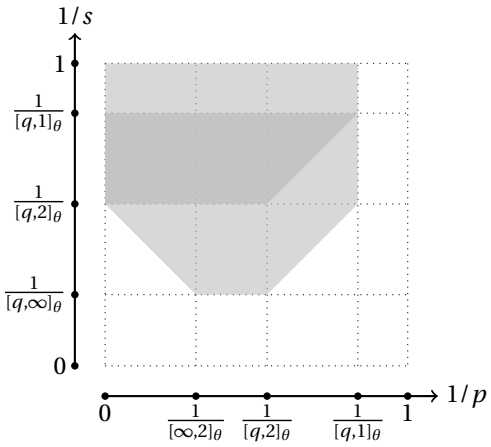
$$\frac{1-\theta}{q} = \frac{1}{[q,\infty]_\theta} - 0 = \frac{1}{[q,2]_\theta} - \frac{1}{[\infty,2]_\theta}.$$

Proof. As in the proof of Theorem 7.5.2, it suffices to consider decaying multipliers $m \in V_0^s(\Delta;\Gamma)$. Moreover, by Lemma 7.4.3, Proposition 2.3.2((iii)) and the openness of the assumptions on s, it suffices to consider $m \in R_0^s(\Delta;\Gamma)$. Throughout the proof we let $r_{s,\theta,q} \in [1,\infty)$ be the unique number such that

$$[q, r_{s,\theta,q}]_\theta = s,$$

which exists if $[q,1]_\theta \leq s < [q,\infty]_\theta$.

Figure 7.2: Allowable exponents for Theorem 7.5.12: the weighted case (i) dark shaded, the unweighted case (ii) light shaded.



Part (i): First assume $s \neq [q,1]_\theta$, so that $r_{s,\theta,q} > 1$. Fix a weight $w \in A_1$. Take $t > q$ and define $\sigma = [t, r_{s,\theta,q}]_\theta > s$. By Proposition 7.5.5 we have boundedness of the bilinear map

$$R_0^q(\Delta;\Gamma) \times L^t(\mathbb{R}, w; Y) \to L^t(\mathbb{R}, w; Y), \qquad (m, f) \mapsto T_m f$$

using that Γ is $\ell^2(\ell^{q'})$-bounded on Y. Moreover, since $s \leq [q,2]_\theta$, we know that $r_{s,\theta,q} \leq 2$, so we have by Theorem 7.5.2((i)) and Lemma 7.4.3 that the bilinear map

$$R_0^{r_{s,\theta,q}}(\Delta;\Gamma) \times L^{r_{s,\theta,q}}(\mathbb{R}, w; H) \to L^{r_{s,\theta,q}}(\mathbb{R}, w; H), \qquad (m, f) \mapsto T_m f$$

is bounded, using

$$\|m\|_{R^s(\Delta;\mathcal{L}(H))} \lesssim \|m\|_{R^s(\Delta;\Gamma)} \tag{7.5.4}$$

by the uniform boundedness of Γ on H.

We define a bilinear map

$$\begin{aligned}\big(R_0^s(\Delta;\Gamma) \cap R_0^{r_{s,\theta,q}}(\Delta;\Gamma)\big) \times \big(L^t(\mathbb{R}, w; Y) \cap L^{r_{s,\theta,q}}(\mathbb{R}, w; H)\big)& \\ \to L^r(\mathbb{R}, w; Y) \cap L^{r_{s,\theta,q}}(\mathbb{R}, w; H), \qquad (m, f) \mapsto T_m f.&\end{aligned}$$

This is well-defined as it is the extension of the map $(m, f) \mapsto T_m f$ defined for $m \in R_0^{s \wedge r_{s,\theta,q}}(\Delta;\Gamma)$ and $f \in \mathcal{S}(\mathbb{R}; Y \cap H)$. Here we use that $Y \cap H$ is dense in both Y and H. By bilinear complex interpolation [BL76, §4.4] we have boundedness of

$$\begin{aligned}[R_0^q(\Delta;\Gamma), R_0^{r_{s,\theta,q}}(\Delta;\Gamma)]_\theta \times [L^t(\mathbb{R}, w; Y), L^{r_{r,\theta,q}}(\mathbb{R}, w; H)]_\theta& \\ \to [L^t(\mathbb{R}, w; Y), L^{r_{s,\theta,q}}(\mathbb{R}, w; H)]_\theta, \qquad (m, f) \mapsto T_m f.&\end{aligned}$$

Here we use that the Minkowski norm on the linear span of Γ is complete, i.e. that $\Gamma \subseteq \mathcal{L}(Y \cap H)$ is a Banach disc.

By Theorem 7.4.4 we have

$$R_0^{[q,r_{s,\theta,q}]_\theta}(\Delta;\Gamma) \hookrightarrow [R_0^q(\Delta;\Gamma), R_0^{r_{s,\theta,q}}(\Delta;\Gamma)]_\theta.$$

Using this embedding and complex interpolation of weighted Bochner spaces (see [Tri78, Theorem 1.18.5]; note that the proof simply extends to the case $X_0 \neq X_1$), we get boundedness of

$$R_0^s(\Delta;\Gamma) \times L^\sigma(\mathbb{R}, w; X) \to L^\sigma(\mathbb{R}, w; X), \qquad (m, f) \mapsto T_m f$$

with norm estimate

$$\big\| T_m f \big\|_{L^\sigma(\mathbb{R},w;X)} \leq \phi_{Y,q,s,t,\sigma,\theta}([w]_{A_1}) \|m\|_{R^s(\Delta;\Gamma)} \|\mathcal{T}\|_{\ell^2(\ell^{q'})} \big\| f \big\|_{L^\sigma(\mathbb{R},w;X)}$$

for all $w \in A_1$ and all simple functions $f\colon \mathbb{R} \to X$. By Theorem 2.3.3 and density of the simple functions we deduce

$$\| T_m f \|_{L^p(\mathbb{R},w;X)} \leq \phi_{Y,p,q,s,t,\sigma,\theta}([w]_{A_{p/\sigma}}) \|m\|_{R^s(\Delta;\Gamma)} \|\mathcal{T}\|_{\ell^2(\ell^{q'})} \| f \|_{L^p(\mathbb{R},w;X)}$$

for all $p \in [\sigma, \infty)$ and all $w \in A_{p/\sigma}$. Taking t arbitrarily close to q and using Proposition 2.3.2(iii) proves the case $[q,1]_\theta \neq s$.

Next if $[q,1]_\theta = s$ and $w \in A_{p/s}$, then by Proposition 2.3.2(iii) we can choose $t \in (s, [q,2]_\theta)$ such that $w \in A_{p/t}$. By the previous case T_m is bounded on $L^p(\mathbb{R}, w; X)$ for all $m \in R^t(\Delta;\Gamma)$ and hence also for $m \in R^s(\Delta;\Gamma)$, which completes the proof.

Part (ii): By embedding of the R^s-spaces and the fact that

$$\frac{1}{[q,2]_\theta} > \max\Big\{\frac{1}{[q,2]_\theta} - \frac{1}{p}, \frac{1-\theta}{q}, \frac{1}{p} - \frac{\theta}{2}\Big\}$$

for $p > [q,1]_\theta$, we may assume that $s > [q,2]_\theta$ without loss of generality. Note that this implies that $r_{s,\theta,q} > 2$. We will consider three cases:

Case 1: $p \geq [\infty, 2]_\theta$. Since

$$\frac{1}{p\theta} > \frac{1}{\theta}\Big(\frac{\theta}{2} + \frac{1-\theta}{q} - \frac{1}{s}\Big) = \frac{1}{2} - \frac{1}{r_{s,\theta,q}}$$

we can find a $p_1 > p\theta \geq 2$ such that $p_1 < p$ and $p_1 < (\frac{1}{2} - \frac{1}{r_{s,\theta,q}})^{-1}$. Therefore we know by Theorem 7.5.2((ii)), using (7.5.4), that the bilinear map

$$R_0^{r_{s,\theta,q}}(\Delta;\Gamma) \times L^{p_1}(\mathbb{R}; H) \to L^{p_1}(\mathbb{R}; H), \qquad (m, f) \mapsto T_m f$$

is bounded. Since $p < [\infty, p_1]_\theta$ we can find a $p_0 \in (p, \infty)$ such that $p = [p_0, p_1]_\theta$. By Proposition 7.5.5 we have boundedness of the bilinear map

$$R_0^q(\Delta;\Gamma) \times L^{p_0}(\mathbb{R}; Y) \to L^{p_0}(\mathbb{R}; Y), \qquad (m, f) \mapsto T_m f,$$

using that Γ is $\ell^2(\ell^{q'})$-bounded on Y. We can now finish the proof using bilinear complex interpolation, Theorem 7.4.4 and complex interpolation of Bochner spaces as in the first part.

Case 2: $[q,2]_\theta < p < [\infty,2]$. Note that $R_0^{r_{s,\theta,q}}(\Delta;\Gamma) \hookrightarrow L^\infty(\mathbb{R};\Gamma)$. Therefore by Plancherel's theorem and (7.5.4) the bilinear map

$$R_0^{r_{s,\theta,q}}(\Delta;\Gamma) \times L^2(\mathbb{R};H) \to L^2(\mathbb{R};H), \qquad (m,f) \mapsto T_m f$$

is bounded. Since $[q,2]_\theta < p < [\infty,2]_\theta$ we can find a $p_0 \in (q,\infty)$ such that $p = [p_0,2]_\theta$. By Proposition 7.5.5 we have boundedness of the bilinear map

$$R_0^{q}(\Delta;\Gamma) \times L^{p_0}(\mathbb{R};Y) \to L^{p_0}(\mathbb{R};Y), \qquad (m,f) \mapsto T_m f,$$

using that Γ is $\ell^2(\ell^{q'})$-bounded on Y. The proof can now be finished as before.

Case 3: $[q,1]_\theta < p \le [q,2]$. Let $\tilde{p} \in (1,2]$ be such that $p = [q,\tilde{p}]_\theta$. Then since

$$\frac{1}{\tilde{p}} < \frac{1}{\theta}\Big(\frac{\theta}{2} + \frac{1}{s} - \frac{1-\theta}{q}\Big) = \frac{1}{2} + \frac{1}{r_{s,\theta,q}},$$

we can find a $1 < p_1 < \tilde{p}$ such that $p_1 > (\frac{1}{2} + \frac{1}{r_{s,\theta,q}})^{-1}$. Therefore we know by Theorem 7.5.2(ii), using (7.5.4), that the bilinear map

$$R_0^{r_{s,\theta,q}}(\Delta;\Gamma) \times L^{p_1}(\mathbb{R};H) \to L^{p_1}(\mathbb{R};H), \qquad (m,f) \mapsto T_m f$$

is bounded. Since $p_1 < \tilde{p}$, we can find a $p_0 \in (q,\infty)$ such that $p = [p_0,p_1]_\theta$. By Proposition 7.5.5 we have boundedness of the bilinear map

$$R_0^{q}(\Delta;\Gamma) \times L^{p_0}(\mathbb{R};Y) \to L^{p_0}(\mathbb{R};Y), \qquad (m,f) \mapsto T_m f,$$

again using that Γ is $\ell^2(\ell^{q'})$-bounded on Y. The proof can again be finished as before. □

The conditions on m in Theorem 7.5.12((ii)) with $q = 2$ are less restrictive than the conditions of [HP06, Theorem 3.6], which allows for Banach spaces with the LPR_p property. The proof of Theorem 7.5.12(ii) can also be used to improve the conditions of [HP06, Theorem 3.6]

Remark 7.5.13. A weighted variant of part (ii) of Theorem 7.5.12 holds for an appropriate class of weights, by using a weighted variant of Theorem 7.5.2(ii) (see [Kró14, Theorem A(ii)]) and limited range extrapolation (see [CMP11, Theorem 3.31]). However, as this involves limited range Muckenhoupt weight classes, the technical details are left to the interested reader.

We continue with an application to $X = L^r$ for $s \in [2,\infty)$. Results for $s \in (1,2)$ have been previously covered by Example 7.5.10.

Example 7.5.14. Let (Ω,μ) be a σ-finite measure space and let $p,r\in(1,\infty)$. Let Γ be an absolutely convex and ℓ^2-bounded family of operators on $L^t(\Omega)$ for all $t\in(1,\infty)$. Let $s\in[2,\infty)$ and assume $m\in V^s(\Delta;\Gamma)$. Then T_m is bounded on $L^p(\mathbb{R};L^r(\Omega))$ in each of the following cases:

(i) $r\in[2,\infty)$ and $\frac{1}{s}>\max\{\frac{1}{2}-\frac{1}{p},\frac{1}{2}-\frac{1}{r},\frac{1}{p}-\frac{1}{r}\}$.

(ii) $r\in(1,2]$ and $\frac{1}{s}>\max\{\frac{1}{p}-\frac{1}{2},\frac{1}{r}-\frac{1}{2},\frac{1}{r}-\frac{1}{p}\}$.

Proof. It suffices to prove (i), as (ii) follows from a duality argument. Let $\overline{\Gamma}$ be the closure of Γ in $\mathcal{L}(L^2(\Omega))$. Then $\overline{\Gamma}$ is a Banach disc. Moreover, by Lemma 7.3.8 we know that $\overline{\Gamma}\subseteq\mathcal{L}(L^t(\Omega))$ is ℓ^2-bounded for all $t\in(1,\infty)$. We will check the conditions of Theorem 7.5.12(ii) with $\overline{\Gamma}$, $q=2$, $Y=L^t(\Omega)$ for an appropriate $t>r$ and $H=L^2(\Omega)$. Choose $\theta\in(0,\frac{2}{r})$ such that

$$\frac{1}{s}>\max\Big\{\frac{1}{2}-\frac{1}{p},\frac{1-\theta}{2},\frac{1}{p}-\frac{\theta}{2}\Big\}.$$

Since $s\geq 2$ it follows that $p>[2,1]_\theta$. Now the result follows by choosing $t>r$ such that $r=[t,2]_\theta$. □

In a similar way we obtain the following from Theorem 7.5.12(i) and duality. This partly improves Example 7.5.10.

Example 7.5.15. Let (Ω,μ) be a σ-finite measure space and let $p,r\in(1,\infty)$. Let Γ be an absolutely convex and ℓ^2-bounded family of operators on $L^t(\Omega)$ for all $t\in[2,\infty)$. Let $s\in(1,2)$ and assume $m\in V^s(\Delta;\Gamma)$. Then T_m is bounded on $L^p(\mathbb{R},w;L^r(\Omega))$ if $\frac{1}{p}<\frac{1}{s}\leq\frac{1}{r}+\frac{1}{2}$ and $w\in A_{p/s}$.

7.5.4. MULTIPLIERS IN INTERMEDIATE UMD BANACH SPACES

In this final subsection we consider general UMD Banach spaces (not just Banach function spaces) and use interpolation to improve the conditions of Theorem 7.5.1 considerably, assuming X is an interpolation space between a UMD space and a Hilbert space, and using the same interpolation scheme as in Theorem 7.5.12. This result is new even for scalar-valued multipliers, and it implies sufficient conditions for Fourier multipliers on the space of Schatten class operators.

Theorem 7.5.16. *Let $p\in(1,\infty)$ and $\theta\in(0,1)$. Let Y and H be an interpolation couple, with $Y\in$ UMD, H a Hilbert space, and $Y\cap H$ dense in both Y and H. Let $X=[Y,H]_\theta$. Suppose $\Gamma\subseteq\mathcal{L}(Y\cap H)$ is a Banach disc which is $\mathcal{R}$-bounded on Y and uniformly bounded on H. Let $s\in(1,\infty)$ and suppose that $m\in V^s(\Delta;\Gamma)$.*

(i) If $1/s>\min\{1/p,1-(\theta/2)\}$, then

$$\|T_m\|_{L^p(\mathbb{R},w;X)\to L^p(\mathbb{R},w;X)}\leq\phi_{Y,p,s,\theta}([w]_{A_{p/s}})\|m\|_{V^s(\Delta;\Gamma)}\|\Gamma\|_{\mathcal{R}}$$

for all $w\in A_{p/s}$.

(ii) If

$$\frac{1}{s} > \max\Big\{1-\frac{\theta}{2}-\frac{1}{p}, 1-\theta, \frac{1}{p}-\frac{\theta}{2}\Big\},$$

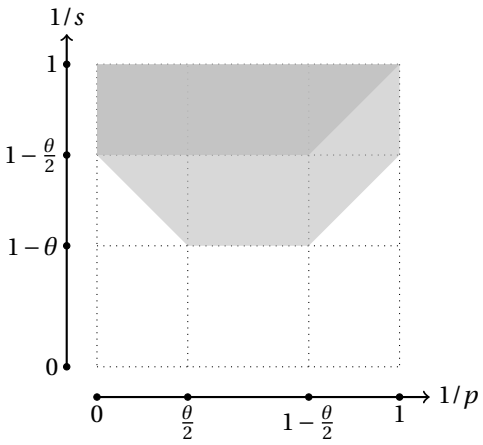
then

$$\|T_m\|_{L^p(\mathbb{R};X)\to L^p(\mathbb{R};X)} \lesssim_{Y,p,s,\theta} \|m\|_{V^s(\Delta;\Gamma)}\|\Gamma\|_{\mathcal{R}}.$$

The allowable exponents (p,s) above are shown in Figure 7.3.

Proof. To prove the result one can argue as in Theorem 7.5.12 with $q=1$ and using Theorem 7.5.1 instead of Proposition 7.5.5. □

Figure 7.3: Allowable exponents for Theorem 7.5.16: the weighted case (i) dark shaded, the unweighted case (ii) light shaded.



In the next example we apply Theorem 7.5.16 to operator-valued multipliers on the Schatten class operators $\mathscr{S}^r \subseteq \mathcal{L}(\ell^2)$ for $r \in [1,\infty]$. This is potentially useful for Schur multipliers (see [HNVW16, Theorem 5.4.3] and [PS11, Theorem 4]). For $r \in (1,\infty)$ these spaces have the UMD property, and for $p,q \in [1,\infty]$ one has $\mathscr{S}^{[p,q]_\theta} = [\mathscr{S}^p, \mathscr{S}^q]_\theta$ (see [HNVW16, Propositions 5.4.2 and D.3.1]).

Example 7.5.17. Let $X = \mathscr{S}^r$ with $p,r \in (1,\infty)$ and $\Gamma \subseteq \mathcal{L}(\mathscr{S}^t)$ be absolutely convex and $\mathcal{R}$-bounded for all $t \in (1,\infty)$. Let $s \in (1,\infty)$ and assume $m \in V^s(\Delta;\Gamma)$. Then T_m is bounded on $L^p(\mathbb{R};\mathscr{S}^r)$ in each of the following cases:

(i) $r \in [2,\infty)$ and $\frac{1}{s} > \max\Big\{\frac{1}{p'}-\frac{1}{r}, \big|\frac{1}{r}-\frac{1}{r'}\big|, \frac{1}{p}-\frac{1}{r}\Big\}$.

(ii) $r \in (1,2]$ and $\frac{1}{s} > \max\Big\{\frac{1}{r}-\frac{1}{p'}, \big|\frac{1}{r}-\frac{1}{r'}\big|, \frac{1}{r}-\frac{1}{p}\Big\}$.

In particular, if $p \in [r \wedge r', r \vee r']$ then T_m is bounded on $L^p(\mathbb{R}; \mathscr{S}^r)$ if $r \in (1,\infty)$ and $\frac{1}{s} > |\frac{1}{r} - \frac{1}{r'}|$.

Proof. The result follows from Theorem 7.5.16(ii) by arguing as in Example 7.5.14. A similar result can be derived on $L^p(\mathbb{R}, w; \mathscr{S}^r)$ by Theorem 7.5.16(i). □

REFERENCES

[1] E. Lorist. Space-time weighted maximal regularity for parabolic (S)PDE. In preparation.

[2] N. Lindemulder and E. Lorist. A discrete framework for the interpolation of Banach spaces. In preparation.

[3] E. Lorist and B. Nieraeth. Sparse domination implies vector-valued sparse domination. arXiv:2003.02233.

[4] N.J. Kalton, E. Lorist, and L. Weis. Euclidean structures and operator theory in Banach spaces. *To appear in Mem. Amer. Math. Soc.*, 2021.

[5] E. Lorist. On pointwise ℓ^r-sparse domination in a space of homogeneous type. *J. Geom. Anal.*, 2020. Online first.

[6] E. Lorist and M.C. Veraar. Singular stochastic integral operators. *To appear in Anal. PDE*, 2020.

[7] E. Lorist. The ℓ^s-boundedness of a family of integral operators on UMD Banach function spaces. In *Positivity and Noncommutative Analysis: Festschrift in Honour of Ben de Pagter on the Occasion of his 65th Birthday*, pages 365–379, Cham, 2019. Springer International Publishing.

[8] E. Lorist and B. Nieraeth. Vector-valued extensions of operators through multilinear limited range extrapolation. *J. Fourier Anal. Appl.*, 25(5):2608–2634, 2019.

[9] T.S. Hänninen and E. Lorist. Sparse domination for the lattice Hardy–Littlewood maximal operator. *Proc. Amer. Math. Soc.*, 147(1):271–284, 2019.

[10] A. Amenta, E. Lorist, and M.C. Veraar. Fourier multipliers in Banach function spaces with UMD concavifications. *Trans. Amer. Math. Soc.*, 371(7):4837–4868, 2019.

[11] A. Amenta, E. Lorist, and M.C. Veraar. Rescaled extrapolation for vector-valued functions. *Publ. Mat.*, 63(1):155–182, 2019.

[12] C. Gallarati, E. Lorist, and M.C. Veraar. On the ℓ^s-boundedness of a family of integral operators. *Rev. Mat. Iberoam.*, 32(4):1277–1294, 2016.

[AB02] W. Arendt and S. Bu. The operator-valued Marcinkiewicz multiplier theorem and maximal regularity. *Math. Z.*, 240(2):311–343, 2002.

[Acq88] P. Acquistapace. Evolution operators and strong solutions of abstract linear parabolic equations. *Differential Integral Equations*, 1(4):433–457, 1988.

[Agr18] A. Agresti. A quasilinear approach to fully nonlinear parabolic (S)PDEs on $\mathbb{R}^d$. arXiv:1802.06395, 2018.

[AIS01] K. Astala, T. Iwaniec, and E. Saksman. Beltrami operators in the plane. *Duke Math. J.*, 107(1):27–56, 2001.

[AK16] F. Albiac and N.J. Kalton. *Topics in Banach space theory*, volume 233 of *Graduate Texts in Mathematics*. Springer, second edition, 2016.

[AKMP12] P. Auscher, C. Kriegler, S. Monniaux, and P. Portal. Singular integral operators on tent spaces. *J. Evol. Equ.*, 12(4):741–765, 2012.

[AM15] R. Alvarado and M. Mitrea. *Hardy spaces on Ahlfors-regular quasi metric spaces*, volume 2142 of *Lecture Notes in Mathematics*. Springer, Cham, 2015.

[Ama95] H. Amann. *Linear and Quasilinear Parabolic Problems. Vol. I, Abstract Linear Theory*, volume 89 of *Monographs in Mathematics*. Birkhäuser Boston Inc., Boston, MA, 1995.

[And15] T.C. Anderson. *A Framework for Calderón-Zygmund Singular Integral Operators on Spaces of Homogeneous Type*. PhD thesis, Brown University, 2015.

[ANP14] P. Auscher, J. van Neerven, and P. Portal. Conical stochastic maximal L^p-regularity for $1 \leqslant p < \infty$. *Math. Ann.*, 359(3-4):863–889, 2014.

[Ant17] M. Antoni. *Regular Random Field Solutions for Stochastic Evolution Equations*. PhD thesis, Karlsruhe Institute of Technology, 2017.

[AT87] P. Acquistapace and B. Terreni. A unified approach to abstract linear nonautonomous parabolic equations. *Rend. Sem. Mat. Univ. Padova*, 78:47–107, 1987.

[AT92] P. Acquistapace and B. Terreni. Regularity properties of the evolution operator for abstract linear parabolic equations. *Differential Integral Equations*, 5(5):1151–1184, 1992.

[AV14] T.C. Anderson and A. Vagharshakyan. A simple proof of the sharp weighted estimate for Calderón-Zygmund operators on homogeneous spaces. *J. Geom. Anal.*, 24(3):1276–1297, 2014.

[AV20a] A. Agresti and M.C. Veraar. Nonlinear parabolic stochastic evolution equations in critical spaces Part I. Stochastic maximal regularity and local existence. *To appear in Nonlinearity.* arXiv:2001.00512, 2020.

[AV20b] A. Agresti and M.C. Veraar. Nonlinear parabolic stochastic evolution equations in critical spaces Part II. Blow-up criteria and instantaneous regularization. arXiv:2012.04448, 2020.

[AV20c] A. Agresti and M.C. Veraar. Stability properties of stochastic maximal L^p-regularity. *J. Math. Anal. Appl.*, 482(2):123553, 35, 2020.

[BBR20] J. Bailey, G. Brocchi, and M.C. Reguera. Quadratic sparse domination and weighted estimates for non-integral square functions. arXiv:2007.15928, 2020.

[BCN12] J.J. Betancor, A.J. Castro, and A. Nowak. Calderón–Zygmund operators in the Bessel setting. *Monatsh. Math.*, 167(3-4):375–403, 2012.

[BCP62] A. Benedek, A.-P. Calderón, and R. Panzone. Convolution operators on Banach space valued functions. *Proc. Nat. Acad. Sci. U.S.A.*, 48:356–365, 1962.

[BD20] T.A. Bui and X.T. Duong. Sharp weighted estimates for square functions associated to operators on spaces of homogeneous type. *J. Geom. Anal.*, 30(1):874–900, 2020.

[BFP16] F. Bernicot, D. Frey, and S. Petermichl. Sharp weighted norm estimates beyond Calderón-Zygmund theory. *Anal. PDE*, 9(5):1079–1113, 2016.

[BGT03] E. Berkson, T. A. Gillespie, and J. L. Torrea. Vector valued transference. In *Functional Space Theory and its applications (Wuhan, 2003)*, pages 1–27, 2003.

[BH09] Z. Brzeźniak and E. Hausenblas. Maximal regularity for stochastic convolutions driven by Lévy processes. *Probab. Theory Related Fields*, 145(3-4):615–637, 2009.

[BK03] S. Blunck and P.C. Kunstmann. Calderón-Zygmund theory for non-integral operators and the H^∞ functional calculus. *Rev. Mat. Iberoamericana*, 19(3):919–942, 2003.

[BL76] J. Bergh and J. Löfström. *Interpolation spaces. An introduction.* Springer-Verlag, Berlin-New York, 1976. Grundlehren der Mathematischen Wissenschaften, No. 223.

[BMT07] D. Buraczewski, T. Martinez, and J. L. Torrea. Calderón-Zygmund operators associated to ultraspherical expansions. *Canad. J. Math.*, 59(6):1223–1244, 2007.

[Bou83] J. Bourgain. Some remarks on Banach spaces in which martingale difference sequences are unconditional. *Ark. Mat.*, 21(2):163–168, 1983.

[Bou84] J. Bourgain. Extension of a result of Benedek, Calderón and Panzone. *Ark. Mat.*, 22(1):91–95, 1984.

[Bou86] J. Bourgain. Vector-valued singular integrals and the H^1-BMO duality. In *Probability theory and harmonic analysis (Cleveland, Ohio, 1983)*, volume 98 of *Monogr. Textbooks Pure Appl. Math.*, pages 1–19. Dekker, New York, 1986.

[Brz95] Z. Brzeźniak. Stochastic partial differential equations in M-type 2 Banach spaces. *Potential Anal.*, 4(1):1–45, 1995.

[Brz97] Z. Brzeźniak. On stochastic convolution in Banach spaces and applications. *Stochastics Stochastics Rep.*, 61(3-4):245–295, 1997.

[BS88] C. Bennett and R. Sharpley. *Interpolation of operators*, volume 129 of *Pure and Applied Mathematics*. Academic Press, Inc., Boston, MA, 1988.

[Buc93] S. Buckley. Estimates for operator norms on weighted spaces and reverse Jensen inequalities. *Trans. Amer. Math. Soc.*, 340(1):253–272, 1993.

[Buh75] A. V. Buhvalov. The duality of functors that are generated by spaces of vector-valued functions. *Izv. Akad. Nauk SSSR Ser. Mat.*, 39(6):1284–1309, 1438, 1975.

[Bur83] D.L. Burkholder. A geometric condition that implies the existence of certain singular integrals of Banach-space-valued functions. In *Conference on harmonic analysis in honor of Antoni Zygmund, Vol. I, II (Chicago, Ill., 1981)*, Wadsworth Math. Ser., pages 270–286. Wadsworth, Belmont, CA, 1983.

[Cal64] A.P. Calderón. Intermediate spaces and interpolation, the complex method. *Studia Math.*, 24:113–190, 1964.

[CCV18] P.A. Cioica-Licht, S.G. Cox, and M.C. Veraar. Stochastic integration in quasi-Banach spaces. *To appear in Studia Mathematica*. arXiv:1804.08947, 2018.

[CDO17] A. Culiuc, Di Plinio, F., and Y. Ou. A sparse estimate for multisublinear forms involving vector-valued maximal functions. In *Bruno Pini Mathematical Analysis Seminar 2017*, volume 8 of *Bruno Pini Math. Anal. Semin.*, pages 168–184. Univ. Bologna, Alma Mater Stud., Bologna, 2017.

[CF14] R. Chill and A. Fiorenza. Singular integral operators with operator-valued kernels, and extrapolation of maximal regularity into rearrangement invariant Banach function spaces. *J. Evol. Equ.*, 14(4-5):795–828, 2014.

[CGR+17] Ó. Ciaurri, T.A. Gillespie, L. Roncal, J.L. Torrea, and J.L. Varona. Harmonic analysis associated with a discrete Laplacian. *J. Anal. Math.*, 132:109–131, 2017.

[Chr90] M. Christ. *Lectures on singular integral operators*, volume 77 of *CBMS Regional Conference Series in Mathematics*. American Mathematical Society, Providence, RI, 1990.

[Cio20] P.A. Cioica-Licht. An L^p-theory for the stochastic heat equation on angular domains in $\mathbb{R}^2$ with mixed weights. arXiv:2003.03782, 2020.

[CK18] R. Chill and S. Król. Weighted inequalities for singular integral operators on the half-line. *Studia Math.*, 243(2):171–206, 2018.

[CKL19] P.A. Cioica-Licht, K.-H. Kim, and K. Lee. On the regularity of the stochastic heat equation on polygonal domains in $\mathbb{R}^2$. *J. Differential Equations*, 267(11):6447–6479, 2019.

[CKLL18] P.A. Cioica-Licht, K.-H. Kim, K. Lee, and F. Lindner. An L_p-estimate for the stochastic heat equation on an angular domain in $\mathbb{R}^2$. *Stochastics and Partial Differential Equations: Analysis and Computations*, 6(1):45–72, 2018.

[CL86] T. Coulhon and D. Lamberton. Régularité L^p pour les équations d'évolution. In *Séminaire d'Analyse Fonctionelle 1984/1985*, volume 26 of *Publ. Math. Univ. Paris VII*, pages 155–165. Univ. Paris VII, Paris, 1986.

[CM13a] D. Cruz-Uribe and K. Moen. A fractional Muckenhoupt-Wheeden theorem and its consequences. *Integral Equations Operator Theory*, 76(3):421–446, 2013.

[CM13b] D. Cruz-Uribe and K. Moen. One and two weight norm inequalities for Riesz potentials. *Illinois J. Math.*, 57(1):295–323, 2013.

[CMP11] D.V. Cruz-Uribe, J.M. Martell, and C. Pérez. *Weights, extrapolation and the theory of Rubio de Francia*, volume 215 of *Operator Theory: Advances and Applications*. Birkhäuser/Springer Basel AG, Basel, 2011.

[CMP12] D.V. Cruz-Uribe, J.M. Martell, and C. Pérez. Sharp weighted estimates for classical operators. *Adv. Math.*, 229(1):408–441, 2012.

[Cob83] F. Cobos. On the type of interpolation spaces and $S_{p,q}$. *Math. Nachr.*, 113:59–64, 1983.

[CP01] P. Clément and J. Prüss. An operator-valued transference principle and maximal regularity on vector-valued L_p-spaces. In *Evolution equations and their applications in physical and life sciences (Bad Herrenalb, 1998)*, volume 215 of *Lecture Notes in Pure and Appl. Math.*, pages 67–87. Dekker, New York, 2001.

[CPSW00] P. Clément, B. de Pagter, F.A. Sukochev, and H. Witvliet. Schauder decompositions and multiplier theorems. *Studia Math.*, 138(2):135–163, 2000.

[CR16] J.M. Conde-Alonso and G. Rey. A pointwise estimate for positive dyadic shifts and some applications. *Math. Ann.*, 365(3-4):1111–1135, 2016.

[CRdFS88] R. Coifman, J. L. Rubio de Francia, and S. Semmes. Multiplicateurs de Fourier de $L^p(\mathbf{R})$ et estimations quadratiques. *C. R. Acad. Sci. Paris Sér. I Math.*, 306(8):351–354, 1988.

[Cru17] D. Cruz-Uribe. Elementary proofs of one weight norm inequalities for fractional integral operators and commutators. In *Harmonic analysis, partial differential equations, Banach spaces, and operator theory. Vol. 2*, volume 5 of *Assoc. Women Math. Ser.*, pages 183–198. Springer, Cham, 2017.

[CV11] S. Cox and M.C. Veraar. Vector-valued decoupling and the Burkholder-Davis-Gundy inequality. *Illinois J. Math.*, 55(1):343–375 (2012), 2011.

[CW71] R.R. Coifman and G. Weiss. *Analyse harmonique non-commutative sur certains espaces homogènes.* Lecture Notes in Mathematics, Vol. 242. Springer-Verlag, Berlin-New York, 1971.

[Cwi74] M. Cwikel. On $(L^{p_0}(A_0), L^{p_1}(A_1))_{\theta,q}$. *Proc. Amer. Math. Soc.*, 44:286–292, 1974.

[DGPP05] O. Dragičević, L. Grafakos, M.C. Pereyra, and S. Petermichl. Extrapolation and sharp norm estimates for classical operators on weighted Lebesgue spaces. *Publ. Mat.*, 49(1):73–91, 2005.

[DHP03] R. Denk, M. Hieber, and J. Prüss. $\mathcal{R}$-boundedness, Fourier multipliers and problems of elliptic and parabolic type. *Mem. Amer. Math. Soc.*, 166(788):viii+114, 2003.

[DK18] H. Dong and D. Kim. On L_p-estimates for elliptic and parabolic equations with A_p weights. *Trans. Amer. Math. Soc.*, 370(7):5081–5130, 2018.

[DK19a] L. Deleaval and C. Kriegler. Dimension free bounds for the vector-valued Hardy-Littlewood maximal operator. *Rev. Mat. Iberoam.*, 35(1):101–123, 2019.

[DK19b] H. Dong and N.V. Krylov. Fully nonlinear elliptic and parabolic equations in weighted and mixed-norm Sobolev spaces. *Calc. Var. Partial Differential Equations*, 58(4):Art. 145, 32, 2019.

[DKK18] L. Deleaval, M. Kemppainen, and C. Kriegler. Hörmander functional calculus on UMD lattice valued L^p spaces under generalised Gaussian estimates. *To appear in J. Anal. Math.* arXiv:1806.03128, 2018.

[DL98] G. Da Prato and A. Lunardi. Maximal regularity for stochastic convolutions in L^p spaces. *Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei (9) Mat. Appl.*, 9(1):25–29, 1998.

[DL13] G. Desch and S.-O. Londen. Maximal regularity for stochastic integral equations. *J. Appl. Anal.*, 19(1):125–140, 2013.

[DLOT17] H.G. Dales, N.J. Laustsen, T. Oikhberg, and V.G. Troitsky. Multi-norms and Banach lattices. *Dissertationes Math.*, 524:115, 2017.

[DM99] X.T. Duong and A. McIntosh. Singular integral operators with non-smooth kernels on irregular domains. *Rev. Mat. Iberoamericana*, 15(2):233–265, 1999.

[DO18] F. Di Plinio and Y. Ou. Banach-valued multilinear singular integrals. *Indiana Univ. Math. J.*, 67(5):1711–1763, 2018.

[Dor00] G. Dore. Maximal regularity in L^p spaces for an abstract Cauchy problem. *Adv. Differential Equations*, 5(1-3):293–322, 2000.

[DR96] X.T. Duong and D.W. Robinson. Semigroup kernels, Poisson bounds, and holomorphic functional calculus. *J. Funct. Anal.*, 142(1):89–128, 1996.

[Du20] K. Du. $W^{2,p}$-solutions of parabolic SPDEs in general domains. *Stochastic Process. Appl.*, 130(1):1–19, 2020.

[DZ14] G. Da Prato and J. Zabczyk. *Stochastic equations in infinite dimensions*, volume 152 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, second edition, 2014.

[EI70] S.D. Èĭdel′man and S.D. Ivasišen. Investigation of the Green's matrix of a homogeneous parabolic boundary value problem. *Trudy Moskov. Mat. Obšč.*, 23:179–234, 1970.

[Emm03] H. Emmerich. *The diffuse interface approach in materials science: Thermodynamic concepts and applications of phase-field models*, volume 73. Springer Science & Business Media, 2003.

[EN00] K.-J. Engel and R. Nagel. *One-parameter semigroups for linear evolution equations*, volume 194 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 2000.

[Fef70] C. Fefferman. Inequalities for strongly singular convolution operators. *Acta Math.*, 124:9–36, 1970.

[FH18] S. Fackler and T.P. Hytönen. Off-diagonal sharp two-weight estimates for sparse operators. *New York J. Math.*, 24:21–42, 2018.

[FHL20] S. Fackler, T.P. Hytönen, and N. Lindemulder. Weighted estimates for operator-valued Fourier multipliers. *Collect. Math.*, 71(3):511–548, 2020.

[Fig90] T. Figiel. Singular integral operators: a martingale approach. In *Geometry of Banach spaces (Strobl, 1989)*, volume 158 of *London Math. Soc. Lecture Note Ser.*, pages 95–110. Cambridge Univ. Press, Cambridge, 1990.

[FN19] D. Frey and B. Nieraeth. Weak and Strong Type $A_1 - A_\infty$ Estimates for Sparsely Dominated Operators. *J. Geom. Anal.*, 29(1):247–282, 2019.

[Gar90] D.J.H. Garling. Random martingale transform inequalities. In *Probability in Banach spaces 6 (Sandbjerg, 1986)*, volume 20 of *Progr. Probab.*, pages 101–119. Birkhäuser Boston, Boston, MA, 1990.

[Gar07] D.J.H. Garling. *Inequalities: a journey into linear analysis*. Cambridge University Press, Cambridge, 2007.

[Gei99] S. Geiss. A counterexample concerning the relation between decoupling constants and UMD-constants. *Trans. Amer. Math. Soc.*, 351(4):1355–1375, 1999.

[Gil79] J.E. Gilbert. Nikišin-Stein theory and factorization with applications. In *Harmonic analysis in Euclidean spaces (Proc. Sympos. Pure Math., Williams Coll., Williamstown, Mass., 1978), Part 2*, Proc. Sympos. Pure Math., XXXV, Part, pages 233–267. Amer. Math. Soc., Providence, R.I., 1979.

[GMT93] J. García-Cuerva, R. Macías, and J.L. Torrea. The Hardy-Littlewood property of Banach lattices. *Israel J. Math.*, 83(1-2):177–201, 1993.

[GR85] J. García-Cuerva and J.L. Rubio de Francia. *Weighted norm inequalities and related topics*, volume 116 of *North-Holland Mathematics Studies*. North-Holland Publishing Co., Amsterdam, 1985.

[Gra14a] L. Grafakos. *Classical Fourier analysis*, volume 249 of *Graduate Texts in Mathematics*. Springer, New York, third edition, 2014.

[Gra14b] L. Grafakos. *Modern Fourier analysis*, volume 250 of *Graduate Texts in Mathematics*. Springer, New York, third edition, 2014.

[GRS21] R. Garg, L. Roncal, and S. Shrivastava. Quantitative weighted estimates for Rubio de Francia's Littlewood–Paley square function. *J. Geom. Anal.*, 31(1):748–771, 2021.

[GT04] T.A. Gillespie and J.L. Torrea. Transference of a Littlewood-Paley-Rubio inequality and dimension free estimates. *Rev. Un. Mat. Argentina*, 45(1):1–6 (2005), 2004.

[GV17a] C. Gallarati and M.C. Veraar. Evolution families and maximal regularity for systems of parabolic equations. *Adv. Differential Equations*, 22(3-4):169–190, 2017.

[GV17b] C. Gallarati and M.C. Veraar. Maximal regularity for non-autonomous equations with measurable dependence on time. *Potential Anal.*, 46(3):527–567, 2017.

[GW03] M. Girardi and L. Weis. Operator-valued Fourier multiplier theorems on $L_p(X)$ and geometry of Banach spaces. *J. Funct. Anal.*, 204(2):320–354, 2003.

[Haa06] M.H.A. Haase. *The functional calculus for sectorial operators*, volume 169 of *Operator Theory: Advances and Applications*. Birkhäuser Verlag, Basel, 2006.

[Heb90] W. Hebisch. A multiplier theorem for Schrödinger operators. *Colloq. Math.*, 60/61(2):659–664, 1990.

[HH14] T.S. Hänninen and T.P. Hytönen. The A_2 theorem and the local oscillation decomposition for Banach space valued functions. *J. Operator Theory*, 72(1):193–218, 2014.

[HH16] T.S. Hänninen and T.P. Hytönen. Operator-valued dyadic shifts and the $T(1)$ theorem. *Monatsh. Math.*, 180(2):213–253, 2016.

[HHN02] R. Haller, H. Heck, and A. Noll. Mikhlin's theorem for operator-valued Fourier multipliers in n variables. *Math. Nachr.*, 244:110–130, 2002.

[HK12] T.P. Hytönen and A. Kairema. Systems of dyadic cubes in a doubling metric space. *Colloq. Math.*, 126(1):1–33, 2012.

[HL18] T.P. Hytönen and K. Li. Weak and strong A_p-A_∞ estimates for square functions and related operators. *Proc. Amer. Math. Soc.*, 146(6):2497–2507, 2018.

[HL19] F. Hummel and N. Lindemulder. Elliptic and parabolic boundary value problems in weighted function spaces. arXiv:1911.04884, 2019.

[HLN16] T.P. Hytönen, S. Li, and A. Naor. Quantitative affine approximation for UMD targets. *Discrete Anal.*, 37(6), 2016.

[HLP13] T.P. Hytönen, M.T. Lacey, and C. Pérez. Sharp weighted bounds for the q-variation of singular integrals. *Bull. Lond. Math. Soc.*, 45(3):529–540, 2013.

[HMP08] T.P. Hytönen, A. McIntosh, and P. Portal. Kato's square root problem in Banach spaces. *J. Funct. Anal.*, 254(3):675–726, 2008.

[HNVW16] T.P. Hytönen, J.M.A.M. van Neerven, M.C. Veraar, and L. Weis. *Analysis in Banach Spaces. Volume I: Martingales and Littlewood-Paley Theory*, volume 63 of *Ergebnisse der Mathematik und ihrer Grenzgebiete*. Springer, 2016.

[HNVW17] T.P. Hytönen, J.M.A.M. van Neerven, M.C. Veraar, and L. Weis. *Analysis in Banach spaces. Volume II: Probabilistic methods and operator theory*, volume 67 of *Ergebnisse der Mathematik und ihrer Grenzgebiete*. Springer, 2017.

[HNVW2x] T.P. Hytönen, J.M.A.M. van Neerven, M.C. Veraar, and L. Weis. *Analysis in Banach spaces. Volume III: Harmonic Analysis and Spectral Theory*. Ergebnisse der Mathematik und ihrer Grenzgebiete. Springer, 202x. In Preparation.

[Hör60] L. Hörmander. Estimates for translation invariant operators in L^p spaces. *Acta Math.*, 104:93–140, 1960.

[Hor19] L. Hornung. Quasilinear parabolic stochastic evolution equations via maximal L^p-regularity. *Potential Anal.*, 50(2):279–326, 2019.

[HP06] T.P. Hytönen and D. Potapov. Vector-valued multiplier theorems of Coifman-Rubio de Francia-Semmes type. *Arch. Math. (Basel)*, 87(3):245–254, 2006.

[HP13] T.P. Hytönen and C. Pérez. Sharp weighted bounds involving A_∞. *Anal. PDE*, 6(4):777–818, 2013.

[HPR12] T.P. Hytönen, C. Pérez, and E. Rela. Sharp reverse Hölder property for A_∞ weights on spaces of homogeneous type. *J. Funct. Anal.*, 263(12):3883–3899, 2012.

[HRT17] T.P. Hytönen, L. Roncal, and O. Tapiola. Quantitative weighted estimates for rough homogeneous singular integrals. *Israel J. Math.*, 218(1):133–164, 2017.

[HTY09] T.P. Hytönen, J.L. Torrea, and D.V. Yakubovich. The Littlewood-Paley-Rubio de Francia property of a Banach space for the case of equal intervals. *Proc. Roy. Soc. Edinburgh Sect. A*, 139(4):819–832, 2009.

[HV15] T.P. Hytönen and A.V. Vähäkangas. The local non-homogeneous Tb theorem for vector-valued functions. *Glasg. Math. J.*, 57(1):17–82, 2015.

[HW06] T.P. Hytönen and L. Weis. A $T1$ theorem for integral transformations with operator-valued kernel. *J. Reine Angew. Math.*, 599:155–200, 2006.

[Hyt04] T.P. Hytönen. Fourier embeddings and Mihlin-type multiplier theorems. *Math. Nachr.*, 274/275:74–103, 2004.

[Hyt06] T.P. Hytönen. An operator-valued Tb theorem. *J. Funct. Anal.*, 234(2):420–463, 2006.

[Hyt07] T.P. Hytönen. Anisotropic Fourier multipliers and singular integrals for vector-valued functions. *Ann. Mat. Pura Appl. (4)*, 186(3):455–468, 2007.

[Hyt12] T.P. Hytönen. The sharp weighted bound for general Calderón-Zygmund operators. *Ann. of Math.*, 175(3):1473–1506, 2012.

[Hyt14] T.P. Hytönen. The vector-valued nonhomogeneous Tb theorem. *Int. Math. Res. Not. IMRN*, (2):451–511, 2014.

[Hyt15] T.P. Hytönen. A quantitative Coulhon-Lamberton theorem. In *Operator semigroups meet complex analysis, harmonic analysis and mathematical physics*, volume 250 of *Oper. Theory Adv. Appl.*, pages 273–279. Birkhäuser/Springer, Cham, 2015.

[Hyt20] T.P. Hytönen. An operator-valued $T(1)$ theorem for symmetric singular integrals in UMD spaces. arXiv:2008.04143, 2020.

[IRV18] G.H. Ibañez-Firnkorn, M.S. Riveros, and R.E. Vidal. Sharp bounds for fractional operator with $L^{\alpha,r'}$-Hörmander contidtions. arXiv:1804.096318, 2018.

[JJ78] W.B. Johnson and L. Jones. Every L_p operator is an L_2 operator. *Proc. Amer. Math. Soc.*, 72(2):309–312, 1978.

[JZ04] B. Jacob and H. Zwart. Counterexamples concerning observation operators for C_0-semigroups. *SIAM J. Control Optim.*, 43(1):137–153, 2004.

[Kal78] N.J. Kalton. The endomorphisms of $L_p(0 \leq p \leq 1)$. *Indiana Univ. Math. J.*, 27(3):353–381, 1978.

[Kal02] O. Kallenberg. *Foundations of modern probability*. Probability and its Applications (New York). Springer-Verlag, New York, second edition, 2002.

[Kar19] G.A. Karagulyan. An abstract theory of singular operators. *Trans. Amer. Math. Soc.*, 372(7):4761–4803, 2019.

[Kem11] M. Kemppainen. On the Rademacher maximal function. *Studia Math.*, 203(1):1–31, 2011.

[Kem13] M. Kemppainen. Some remarks on the dyadic Rademacher maximal function. *Colloq. Math.*, 131(1):113–128, 2013.

[Kim04] K.-H. Kim. On stochastic partial differential equations with variable coefficients in C^1 domains. *Stochastic Process. Appl.*, 112(2):261–283, 2004.

[Kim05] K.-H. Kim. L_p estimates for SPDE with discontinuous coefficients in domains. *Electron. J. Probab.*, 10:no. 1, 1–20, 2005.

[Kim15] I. Kim. A BMO estimate for stochastic singular integral operators and its application to SPDEs. *J. Funct. Anal.*, 269(5):1289–1309, 2015.

[Kim20] I. Kim. An L^p-maximal regularty estimate of moments of solutions to second-order stochastic partial differential equations. arXiv:2011.11028, 2020.

[KK04] K.-H. Kim and N. V. Krylov. On the Sobolev space theory of parabolic and elliptic equations in C^1 domains. *SIAM J. Math. Anal.*, 36(2):618–642, 2004.

[KK18] I. Kim and K.-H. Kim. A regularity theory for quasi-linear stochastic PDEs in weighted Sobolev spaces. *Stochastic Process. Appl.*, 128(2):622–643, 2018.

[KK20] I. Kim and K.-H. Kim. On the L_p-boundedness of the stochastic singular integral operators and its application to L_p-regularity theory of stochastic partial differential equations. *Trans. Amer. Math. Soc.*, 373(8):5653–5684, 2020.

[KKL15] I. Kim, K.-H. Kim, and S. Lim. Parabolic BMO estimates for pseudo-differential operators of arbitrary order. *J. Math. Anal. Appl.*, 427(2):557–580, 2015.

[KKL16] I. Kim, K.-H. Kim, and S. Lim. An $L_q(L_p)$-theory for parabolic pseudo-differential equations: Calderón-Zygmund approach. *Potential Anal.*, 45(3):463–483, 2016.

[KL99a] N. V. Krylov and S. V. Lototsky. A Sobolev space theory of SPDEs with constant coefficients in a half space. *SIAM J. Math. Anal.*, 31(1):19–33, 1999.

[KL99b] N. V. Krylov and S. V. Lototsky. A Sobolev space theory of SPDEs with constant coefficients on a half line. *SIAM J. Math. Anal.*, 30(2):298–325, 1999.

[KN14] V. Kozlov and A. Nazarov. The Dirichlet problem for non-divergence parabolic equations with discontinuous in time coefficients in a wedge. *Math. Nachr.*, 287(10):1142–1165, 2014.

[KPW10] M. Köhne, J. Prüss, and M. Wilke. On quasilinear parabolic evolution equations in weighted L_p-spaces. *J. Evol. Equ.*, 10(2):443–463, 2010.

[Kró14] S. Król. Fourier multipliers on weighted L^p spaces. *Math. Res. Lett.*, 21(4):807–830, 2014.

[Kry94a] N. V. Krylov. A W_2^n-theory of the Dirichlet problem for SPDEs in general smooth domains. *Probab. Theory Related Fields*, 98(3):389–421, 1994.

[Kry94b] N.V. Krylov. A generalization of the Littlewood-Paley inequality and some other results related to stochastic partial differential equations. *Ulam Quart.*, 2(4):16 ff., approx. 11 pp. (electronic), 1994.

[Kry99] N.V. Krylov. An analytic approach to SPDEs. In *Stochastic partial differential equations: six perspectives*, volume 64 of *Math. Surveys Monogr.*, pages 185–242. Amer. Math. Soc., Providence, RI, 1999.

[Kry00] N.V. Krylov. SPDEs in $L_q((0,\tau],L_p)$ spaces. *Electron. J. Probab.*, 5:Paper no. 13, 29 pp. (electronic), 2000.

[Kry08] N.V. Krylov. A brief overview of the L_p-theory of SPDEs. *Theory Stoch. Process.*, 14(2):71–78, 2008.

[Kry09] N.V. Krylov. On divergence form SPDEs with VMO coefficients. *SIAM J. Math. Anal.*, 40(6):2262–2285, 2009.

[KU14] P. Kunstmann and A. Ullmann. $\mathcal{R}_s$-sectorial operators and generalized Triebel-Lizorkin spaces. *J. Fourier Anal. Appl.*, 20(1):135–185, 2014.

[Kun15] P.C. Kunstmann. A new interpolation approach to spaces of Triebel-Lizorkin type. *Illinois J. Math.*, 59(1):1–19, 2015.

[Kur80] D. S. Kurtz. Littlewood-Paley and multiplier theorems on weighted L^p spaces. *Trans. Amer. Math. Soc.*, 259(1):235–254, 1980.

[KVW16] S. Kwapień, M.C. Veraar, and L. Weis. R-boundedness versus γ-boundedness. *Ark. Mat.*, 54(1):125–145, 2016.

[KW04] P.C. Kunstmann and L. Weis. Maximal L_p-regularity for parabolic equations, Fourier multiplier theorems and H^∞-functional calculus. In *Functional analytic methods for evolution equations*, volume 1855 of *Lecture Notes in Math.*, pages 65–311. Springer, Berlin, 2004.

[Kwa72a] S. Kwapień. Isomorphic characterizations of inner product spaces by orthogonal series with vector valued coefficients. *Studia Math.*, 44:583–595, 1972.

[Kwa72b] S. Kwapień. On operators factorizable through L_p space. pages 215–225. Bull. Soc. Math. France, Mém. No. 31–32, 1972.

[Lac07] M.T. Lacey. *Issues related to Rubio de Francia's Littlewood-Paley inequality*, volume 2 of *New York Journal of Mathematics. NYJM Monographs.* State University of New York, University at Albany, Albany, NY, 2007.

[Lac17] M.T. Lacey. An elementary proof of the A_2 bound. *Israel J. Math.*, 217(1):181–195, 2017.

[Ler08] A.K. Lerner. On some weighted norm inequalities for Littlewood-Paley operators. *Illinois J. Math.*, 52(2):653–666, 2008.

[Ler11] A.K. Lerner. Sharp weighted norm inequalities for Littlewood-Paley operators and singular integrals. *Adv. Math.*, 226(5):3912–3926, 2011.

[Ler13] A.K. Lerner. A simple proof of the A_2 conjecture. *Int. Math. Res. Not.*, (14):3159–3170, 2013.

[Ler16] A.K. Lerner. On pointwise estimates involving sparse operators. *New York J. Math.*, 22:341–349, 2016.

[Ler19] A.K. Lerner. A weak type estimate for rough singular integrals. *Rev. Mat. Iberoam.*, 35(5):1583–1602, 2019.

[Li18] K. Li. Sparse domination theorem for multilinear singular integral operators with L^r-Hörmander condition. *Michigan Math. J.*, 67(2):253–265, 2018.

[Lin14a] N. Lindemulder. Parabolic initial-boundary value problems with inhomogeneous data: a maximal weighted L^q-L^p-regularity approach. Master's thesis, Delft University of Technology, Delft, the Netherlands, 2014.

[Lin14b] F. Lindner. Singular behavior of the solution to the stochastic heat equation on a polygonal domain. *Stoch. Partial Differ. Equ. Anal. Comput.*, 2(2):146–195, 2014.

[Lin16] N. Lindemulder. Vector-valued extensions of linear operators on L^∞. In *Ordered Structures and Applications (Positivity VII, Zaanen Centennial Conference)*. Birkhäuser Verlag, 2016.

[Lin18] N. Lindemulder. Second order operators subject to Dirichlet boundary conditions in weighted Triebel-Lizorkin spaces: Parabolic problems. arXiv:1812.05462, 2018.

[Lin20] N. Lindemulder. Maximal regularity with weights for parabolic problems with inhomogeneous boundary conditions. *J. Evol. Equ.*, 20(1):59–108, 2020.

[Lio69] J.-L. Lions. *Quelques méthodes de résolution des problèmes aux limites non linéaires*. Dunod; Gauthier-Villars, Paris, 1969.

[LL16] M.T. Lacey and K. Li. On A_p–A_∞ type estimates for square functions. *Math. Z.*, 284(3-4):1211–1222, 2016.

[LM03] C. Le Merdy. The Weiss conjecture for bounded analytic semigroups. *J. London Math. Soc. (2)*, 67(3):715–738, 2003.

[LN18] A.K. Lerner and F. Nazarov. Intuitive dyadic calculus: The basics. *Expositiones Mathematicae*, 2018.

[LO20] A.K. Lerner and S. Ombrosi. Some remarks on the pointwise sparse domination. *J. Geom. Anal.*, 30(1):1011–1027, 2020.

[LOR17] A.K. Lerner, S. Ombrosi, and I.P. Rivera-Ríos. On pointwise and weighted estimates for commutators of Calderón-Zygmund operators. *Adv. Math.*, 319:153–181, 2017.

[LPR15] T. Luque, C. Pérez, and E. Rela. Optimal exponents in weighted estimates without examples. *Math. Res. Lett.*, 22(1):183–201, 2015.

[LR15] W. Liu and M. Röckner. *Stochastic partial differential equations: an introduction*. Universitext. Springer, Cham, 2015.

[LS02] C. Le Merdy and A. Simard. A factorization property of R-bounded sets of operators on L^p-spaces. *Math. Nachr.*, 243:146–155, 2002.

[LT79] J. Lindenstrauss and L. Tzafriri. *Classical Banach spaces. II*, volume 97 of *Ergebnisse der Mathematik und ihrer Grenzgebiete*. Springer-Verlag, Berlin-New York, 1979.

[LT91] M. Ledoux and M. Talagrand. *Probability in Banach spaces*, volume 23 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (3)*. Springer-Verlag, Berlin, 1991. Isoperimetry and processes.

[Lun95] A. Lunardi. *Analytic semigroups and optimal regularity in parabolic problems*, volume 16 of *Progress in Nonlinear Differential Equations and their Applications*. Birkhäuser Verlag, Basel, 1995.

[LV20] N. Lindemulder and M.C. Veraar. The heat equation with rough boundary conditions and holomorphic functional calculus. *J. Differential Equations*, 269(7):5832–5899, 2020.

[Mar04] J.M. Martell. Sharp maximal functions associated with approximations of the identity in spaces of homogeneous type and applications. *Studia Math.*, 161(2):113–145, 2004.

[Mau73] B. Maurey. Théorèmes de factorisation pour les opérateurs linéaires à valeurs dans un espace $L^p(U,\ \mu)$, $0 < p \leq +\infty$. In *Séminaire Maurey-Schwartz Année 1972–1973: Espaces L^p et applications radonifiantes, Exp. No. 15*, page 8. Centre de Math., École Polytech., Paris, 1973.

[Mau74] B. Maurey. Un théorème de prolongement. *C. R. Acad. Sci. Paris Sér. A*, 279:329–332, 1974.

[McC84] T.R. McConnell. On Fourier multiplier transformations of Banach-valued functions. *Trans. Amer. Math. Soc.*, 285(2):739–757, 1984.

[Mey91] P. Meyer-Nieberg. *Banach lattices*. Universitext. Springer-Verlag, Berlin, 1991.

[MS65] B. Muckenhoupt and E. M. Stein. Classical expansions and their relation to conjugate harmonic functions. *Trans. Amer. Math. Soc.*, 118:17–92, 1965.

[MS96] S. Montgomery-Smith. Stability and dichotomy of positive semigroups on L_p. *Proc. Amer. Math. Soc.*, 124(8):2433–2437, 1996.

[MTX15] T. Ma, J.L. Torrea, and Q. Xu. Weighted variation inequalities for differential operators and singular integrals. *J. Funct. Anal.*, 268(2):376–416, 2015.

[MTX17] T. Ma, J.L. Torrea, and Q. Xu. Weighted variation inequalities for differential operators and singular integrals in higher dimensions. *Sci. China Math.*, 60(8):1419–1442, 2017.

[MV12] M. Meyries and M.C. Veraar. Sharp embedding results for spaces of smooth functions with power weights. *Studia Math.*, 208(3):257–293, 2012.

[Naz01] A.I. Nazarov. L_p-estimates for a solution to the Dirichlet problem and to the Neumann problem for the heat equation in a wedge with edge of arbitrary codimension. *J. Math. Sci. (New York)*, 106(3):2989–3014, 2001.

[Nie19] B. Nieraeth. Quantitative estimates and extrapolation for multilinear weight classes. *Math. Ann.*, 375(1-2):453–507, 2019.

[Nie20] Z. Nieraeth. *Sharp estimates and extrapolation for multilinear weight classes*. PhD thesis, Delft University of Technology, 2020.

[Nik70] E.M. Nikišin. Resonance theorems and superlinear operators. *Uspehi Mat. Nauk*, 25(6(156)):129–191, 1970.

[NRV13] F. Nazarov, A. Reznikov, and A. Volberg. The proof of A_2 conjecture in a geometrically doubling metric space. *Indiana Univ. Math. J.*, 62(5):1503–1533, 2013.

[NS07] A. Nowak and K. Stempak. Riesz transforms for multi-dimensional Laguerre function expansions. *Adv. Math.*, 215(2):642–678, 2007.

[NS12] A. Nowak and P. Sjögren. Calderón-Zygmund operators related to Jacobi expansions. *J. Fourier Anal. Appl.*, 18(4):717–749, 2012.

[NVW07] J.M.A.M. van Neerven, M.C. Veraar, and L. Weis. Stochastic integration in UMD Banach spaces. *Ann. Probab.*, 35(4):1438–1478, 2007.

[NVW08] J.M.A.M. van Neerven, M.C. Veraar, and L. Weis. Stochastic evolution equations in UMD Banach spaces. *J. Funct. Anal.*, 255(4):940–993, 2008.

[NVW12a] J.M.A.M. van Neerven, M.C. Veraar, and L. Weis. Maximal L^p-regularity for stochastic evolution equations. *SIAM J. Math. Anal.*, 44(3):1372–1414, 2012.

[NVW12b] J.M.A.M. van Neerven, M.C. Veraar, and L. Weis. Stochastic maximal L^p-regularity. *Ann. Probab.*, 40(2):788–812, 2012.

[NVW15a] J.M.A.M. van Neerven, M.C. Veraar, and L. Weis. Maximal γ-regularity. *J. Evol. Equ.*, 15(2):361–402, 2015.

[NVW15b] J.M.A.M. van Neerven, M.C. Veraar, and L. Weis. On the R-boundedness of stochastic convolution operators. *Positivity*, 19(2):355–384, 2015.

[NVW15c] J.M.A.M. van Neerven, M.C. Veraar, and L. Weis. Stochastic integration in Banach spaces—a survey. In *Stochastic analysis: a series of lectures*, volume 68 of *Progr. Probab.*, pages 297–332. Birkhäuser/Springer, Basel, 2015.

[Osę17] A. Osękowski. Inequalities for Hilbert operator and its extensions: the probabilistic approach. *Ann. Probab.*, 45(1):535–563, 2017.

[OY19] A. Osękowski and I.S. Yaroslavtsev. The Hilbert transform and orthogonal martingales in Banach spaces. *Int. Math. Res. Not. IMRN*, 2019. Online first.

[Paz83] A. Pazy. *Semigroups of linear operators and applications to partial differential equations*, volume 44 of *Applied Mathematical Sciences*. Springer-Verlag, New York, 1983.

[PB87] P. Pérez Carreras and J. Bonet. *Barrelled locally convex spaces*, volume 131 of *North-Holland Mathematics Studies*. North-Holland Publishing Co., Amsterdam, 1987. Notas de Matemática [Mathematical Notes], 113.

[Pis16] G. Pisier. *Martingales in Banach spaces*, volume 155 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2016.

[PS04] J. Prüss and G. Simonett. Maximal regularity for evolution equations in weighted L_p-spaces. *Arch. Math. (Basel)*, 82(5):415–431, 2004.

[PS07] J. Prüss and G. Simonett. H^∞-calculus for the sum of non-commuting operators. *Trans. Amer. Math. Soc.*, 359(8):3549–3565, 2007.

[PS11] D. Potapov and F. Sukochev. Operator-Lipschitz functions in Schatten-von Neumann classes. *Acta Math.*, 207(2):375–389, 2011.

[PS16] J. Prüss and G. Simonett. *Moving interfaces and quasilinear parabolic evolution equations*, volume 105 of *Monographs in Mathematics*. Birkhhäuser/Springer, [Cham], 2016.

[PSW18] J. Prüss, G. Simonett, and M. Wilke. Critical spaces for quasilinear parabolic evolution equations and applications. *J. Differential Equations*, 264(3):2028–2074, 2018.

[PSX12] D. Potapov, F. Sukochev, and Q. Xu. On the vector-valued Littlewood-Paley-Rubio de Francia inequality. *Rev. Mat. Iberoam.*, 28(3):839–856, 2012.

[PV02] S. Petermichl and A. Volberg. Heating of the Ahlfors-Beurling operator: weakly quasiregular maps on the plane are quasiregular. *Duke Math. J.*, 112(2):281–305, 2002.

[PV19] P. Portal and M.C. Veraar. Stochastic maximal regularity for rough time-dependent problems. *Stoch. Partial Differ. Equ. Anal. Comput.*, 7(4):541–597, 2019.

[Roz90] B.L. Rozovskiĭ. *Stochastic evolution systems*, volume 35 of *Mathematics and its Applications (Soviet Series)*. Kluwer Academic Publishers Group, Dordrecht, 1990.

[RRT86] J.L. Rubio de Francia, F. J. Ruiz, and J.L. Torrea. Calderón-Zygmund theory for operator-valued kernels. *Adv. in Math.*, 62(1):7–48, 1986.

[RT88] F.J. Ruiz and J.L. Torrea. Vector-valued Calderón-Zygmund theory and Carleson measures on spaces of homogeneous nature. *Studia Math.*, 88(3):221–243, 1988.

[Rub82] J.L. Rubio de Francia. Weighted norm inequalities and vector valued inequalities. In *Harmonic analysis (Minneapolis, Minn., 1981)*, volume 908 of *Lecture Notes in Math.*, pages 86–101. Springer, Berlin-New York, 1982.

[Rub85] J.L. Rubio de Francia. A Littlewood-Paley inequality for arbitrary intervals. *Rev. Mat. Iberoam.*, 1(2):1–14, 1985.

[Rub86] J.L. Rubio de Francia. Martingale and integral transforms of Banach space valued functions. In *Probability and Banach spaces (Zaragoza, 1985)*, volume 1221 of *Lecture Notes in Math.*, pages 195–222. Springer, Berlin, 1986.

[Rub87] J.L. Rubio de Francia. Linear operators in Banach lattices and weighted L^2 inequalities. *Math. Nachr.*, 133:197–209, 1987.

[Rud91] W. Rudin. *Functional analysis*. International Series in Pure and Applied Mathematics. McGraw-Hill, Inc., New York, second edition, 1991.

[RV17] J. Rozendaal and M.C. Veraar. Fourier multiplier theorems on Besov spaces under type and cotype conditions. *Banach J. Math. Anal.*, 11(4):713–743, 2017.

[Sch04] R. Schnaubelt. Asymptotic behaviour of parabolic nonautonomous evolution equations. In *Functional analytic methods for evolution equations*, volume 1855 of *Lecture Notes in Math.*, pages 401–472. Springer, Berlin, 2004.

[Sim99] A. Simard. Factorization of sectorial operators with bounded H^∞-functional calculus. *Houston J. Math.*, 25(2):351–370, 1999.

[Sol01] V.A. Solonnikov. L_p-estimates for solutions of the heat equation in a dihedral angle. *Rend. Mat. Appl. (7)*, 21(1-4):1–15, 2001.

[Ste93] E.M. Stein. *Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals*, volume 43 of *Princeton Mathematical Series*. Princeton University Press, Princeton, NJ, 1993.

[Ste15] K. Stempak. On some structural properties of spaces of homogeneous type. *Taiwanese J. Math.*, 19(2):603–613, 2015.

[SV03] M. Sanz-Solé and P.-A. Vuillermot. Equivalence and Hölder-Sobolev regularity of solutions for a class of non-autonomous stochastic partial differential equations. *Ann. Inst. H. Poincaré Probab. Statist.*, 39(4):703–742, 2003.

[ŠW07] Ž. Štrkalj and L. Weis. On operator-valued Fourier multiplier theorems. *Trans. Amer. Math. Soc.*, 359(8):3529–3547, 2007.

[Tan79] H. Tanabe. *Equations of evolution*, volume 6 of *Monographs and Studies in Mathematics*. Pitman (Advanced Publishing Program), Boston, Mass., 1979.

[Tan97] H. Tanabe. *Functional analytic methods for partial differential equations*, volume 204 of *Monographs and Textbooks in Pure and Applied Mathematics*. Marcel Dekker Inc., New York, 1997.

[Tri78] H. Triebel. *Interpolation theory, function spaces, differential operators*, volume 18 of *North-Holland Mathematical Library*. North-Holland Publishing Co., Amsterdam-New York, 1978.

[Tri83] H. Triebel. *Theory of function spaces*, volume 78 of *Monographs in Mathematics*. Birkhäuser Verlag, Basel, 1983.

[Ver10] M.C. Veraar. Non-autonomous stochastic evolution equations and applications to stochastic partial differential equations. *J. Evol. Equ.*, 10(1):85–127, 2010.

[Wei01a] L. Weis. A new approach to maximal L_p-regularity. In *Evolution equations and their applications in physical and life sciences (Bad Herrenalb, 1998)*, volume 215 of *Lecture Notes in Pure and Appl. Math.*, pages 195–214. Dekker, New York, 2001.

[Wei01b] L. Weis. Operator-valued Fourier multiplier theorems and maximal L_p-regularity. *Math. Ann.*, 319(4):735–758, 2001.

[Wil07] M. Wilson. The intrinsic square function. *Rev. Mat. Iberoam.*, 23(3):771–791, 2007.

[Wil08] M. Wilson. *Weighted Littlewood-Paley theory and exponential-square integrability*, volume 1924 of *Lecture Notes in Mathematics*. Springer, Berlin, 2008.

[Xu96] Q. Xu. Fourier multipliers for $L_p(\mathbf{R}^n)$ via q-variation. *Pacific J. Math.*, 176(1):287–296, 1996.

[Yag91] A. Yagi. Abstract quasilinear evolution equations of parabolic type in Banach spaces. *Boll. Un. Mat. Ital. B (7)*, 5(2):341–368, 1991.

[Yag10] A. Yagi. *Abstract parabolic evolution equations and their applications*. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2010.

[Zaa67] A.C. Zaanen. *Integration*. North-Holland Publishing Co., Amsterdam; Interscience Publishers John Wiley & Sons, Inc., New York, 1967.

[Zim89] F. Zimmermann. On vector-valued Fourier multiplier theorems. *Studia Math.*, 93(3):201–222, 1989.

[Zor20] P. Zorin-Kranich. Variational truncations of singular integrals on spaces of homogeneous type. arXiv:2009.04541, 2020.

SUMMARY

In the study of partial differential equations from a functional analytic viewpoint, harmonic analysis methods have been developed hand in hand with regularity theory for such equations in the past decades. In contrast, harmonic analysis has not yet fully made its entrance in the study of the stochastic counterparts of these partial differential equations. In this dissertation we will develop new methods in vector-valued harmonic analysis to treat stochastic partial differential equations from a functional analytic viewpoint.

In Part I of this dissertation we will develop harmonic analysis methods to treat singular stochastic integral operators of the form

$$S_K G(t) := \int_0^\infty K(t,s)G(s)\,\mathrm{d}W_H(s), \qquad t \in \mathbb{R}_+,$$

where X and Y are Banach spaces, G is an adapted stochastic process taking values in X, W_H is a cylindrical Brownian motion and K is a given operator-valued kernel $K\colon \mathbb{R}_+ \times \mathbb{R}_+ \to \mathcal{L}(X,Y)$ with a singularity in $t = s$. The L^p-boundedness of such operators plays in important role in the analysis of SPDEs from a functional analytic viewpoint.

As a preparation, we prove a general sparse domination theorem in Chapter 3, in which a vector-valued operator is controlled pointwise by a positive, local expression. This local expression is called a sparse operator and is of the form

$$\Big(\sum_{Q\in\mathcal{S}}\Big(\frac{1}{|Q|}\int_Q \|f(t)\|_X^{p_0}\,\mathrm{d}t\Big)^{r/p_0}\mathbf{1}_Q\Big)^{1/r}, \qquad f \in L^{p_0}_{\mathrm{loc}}(\mathbb{R}^d;X)$$

for a sparse collection of cubes $\mathcal{S}$ in $\mathbb{R}^d$ and $p_0 \in [1,\infty)$. We use the structure of the operator to allow for $r \in [1,\infty)$, rather than the more thoroughly studied case $r = 1$. This sparse domination theorem is applicable to various operators from both harmonic analysis and (S)PDE. Indeed, starting with applications in harmonic analysis, we prove the A_2-theorem for vector-valued Calderón–Zygmund operators in a space of homogeneous type, from which we deduce an anisotropic, mixed-norm Mihlin multiplier theorem. Furthermore, we show quantitative weighted norm inequalities for Littlewood–Paley operators and the Rademacher maximal operator.

In Chapter 4 we develop extrapolation theory for singular stochastic integral operators. In particular, we prove L^p-extrapolation results under a Hörmander condition on the kernel. Sparse domination and sharp weighted bounds using the sparse domination result from Chapter 3 are obtained under a Dini condition on the kernel, leading to a stochastic version of the solution to the A_2-conjecture. We also discuss the closely

related γ-Fourier multiplier operators and develop an extrapolation theory for singular stochastic-deterministic integral operators.

In Chapter 5 we apply the results of Chapter 4 to obtain p-independence and weighted bounds for stochastic maximal L^p-regularity both in the complex and real interpolation scale. As a consequence, we obtain several new regularity results for the stochastic heat equation and its time-dependent variants on $\mathbb{R}^d$ and on smooth and angular domains. We also treat stochastic Volterra equations and show the p-independence of the $\mathcal{R}$-boundedness of stochastic convolution operators.

In Part II of this dissertation, motivated by the use of the tensor extension of various classical operators prevalent in harmonic analysis in the study of (S)PDEs, we will develop two general sufficient conditions for a bounded operator T on $L^p(\mathbb{R}^d)$ to have a bounded tensor extension $\widetilde{T}$ on $L^p(\mathbb{R}^d; X)$ when X is a Banach function space.

In Chapter 6 we prove implications (2) and (3) in the following diagram

$$\begin{array}{ccc} \text{Sparse domination for } T & \xRightarrow[(1)]{} & \text{Weighted bounds for } T \\ \Big\Downarrow (2) & & (3) \Big\Downarrow \\ \text{Sparse domination for } \widetilde{T} & \xRightarrow[(4)]{} & \text{Weighted bounds for } \widetilde{T} \end{array}$$

whereas implications (1) and (4) are well-known and unrelated to the operator T. Both implication (3) and the combination of implications (2) and (4) represent a Banach function space-valued extension theorem. Implication (3) is based on a factorization principle, which resembles the factorization theory of Nikišin, Maurey and Rubio de Francia, but is more flexible. Implication (2) is based on sparse domination for the lattice Hardy–Littlewood maximal operator. Using these extension theorems, we provide quantitative connections between Banach space properties like the (randomized) UMD property and the Hardy–Littlewood property.

Using implication (3), we prove Banach function space-valued Littlewood–Paley–Rubio de Francia-type estimates in Chapter 7. These Littlewood–Paley–Rubio de Francia-type estimates enable us to prove various operator-valued Fourier multiplier theorems on Banach function spaces, which are extensions of the Coifman–Rubio de Francia–Semmes multiplier theorem. Our results involve a new boundedness condition on sets of operators, which we call $\ell^r(\ell^s)$-boundedness and which implies $\mathcal{R}$-boundedness in many cases.

SAMENVATTING

Harmonische analyse methodes voor het bestuderen van partiële differentiaalvergelijkingen vanuit een functionaalanalyse oogpunt zijn afgelopen decennia hand in hand ontwikkeld met regulariteitstheorie voor zulke vergelijkingen. Harmonische analyse heeft echter nog niet volledig haar intrede gemaakt in de analyse van de stochastische varianten van deze partiële differentiaalvergelijkingen. In deze dissertatie zullen we nieuwe vectorwaardige harmonische analyse methodes ontwikkelen om stochastische partiële differentiaalvergelijkingen te bestuderen.

In Deel I van deze dissertatie zullen we harmonische analyse methodes ontwikkelen om singuliere stochastische integraaloperatoren van de vorm

$$S_K G(t) := \int_0^\infty K(t,s)G(s)\,\mathrm{d}W_H(s), \qquad t \in \mathbb{R}_+,$$

te bestuderen. Hier zijn X en Y Banach ruimtes, G is een aangepast stochastisch proces met waardes in X, W_H is een cylindrische Brownse beweging en K is een gegeven operatorwaardige kern $K\colon \mathbb{R}_+ \times \mathbb{R}_+ \to \mathcal{L}(X,Y)$ met een singulariteit in $t = s$. De L^p-begrensdheid van zulke operatoren speelt een belangrijke rol in de analyse van SPDV's vanuit een functionaalanalyse oogpunt.

Als voorbereiding bewijzen we een algemene schaarse dominantie stelling in Hoofdstuk 3, waarin een vectorwaardige operator puntsgewijs wordt gedomineerd door een positieve, lokale uitdrukking. Deze lokale uitdrukking heet een schaarse operator en is van de vorm

$$\Big(\sum_{Q\in\mathcal{S}}\Big(\frac{1}{|Q|}\int_Q \|f(t)\|_X^{p_0}\,\mathrm{d}t\Big)^{r/p_0}\mathbf{1}_Q\Big)^{1/r}, \qquad f \in L^{p_0}_{\mathrm{loc}}(\mathbb{R}^d;X)$$

voor een schaarse verzameling kubussen $\mathcal{S}$ in $\mathbb{R}^d$ en $p_0 \in [1,\infty)$. We gebruiken de structuur van de operator om $r \in [1,\infty)$ toe te staan, in plaats van het grondiger bestudeerde geval $r = 1$. Deze schaarse dominantie stelling is van toepassing op verschillende operatoren uit zowel de harmonische analyse als uit (S)PDV. Om te beginnen met toepassingen in de harmonische analyse bewijzen we de A_2-stelling voor vectorwaardige Calderón–Zygmund-operatoren in een ruimte van homogeen type, waaruit we een anisotrope, gemengde norm Mihlin mutliplicatorstelling afleiden. Verder laten we kwantitatieve gewogen normongelijkheden zien voor Littlewood-Paley operatoren en de Rademacher maximaaloperator.

In Hoofdstuk 4 ontwikkelen we extrapolatietheorie voor singuliere stochastische integraaloperatoren. We bewijzen L^p-extrapolatieresultaten onder een Hörmander voorwaarde op de kern. We verkrijgen schaarse dominantie en scherpe gewogen grenzen

met behulp van het schaarse dominantie resultaat uit Hoofdstuk 3 onder een Dini voorwaarde op de kern, wat leidt tot een stochastische versie van de oplossing van het A_2-vermoeden. We bespreken ook de verwante γ-Fourier multiplicatoroperatoren en ontwikkelen een extrapolatietheorie voor singuliere stochastisch-deterministische integraaloperatoren.

In Hoofdstuk 5 passen we de resultaten van Hoofdstuk 4 toe om p-onafhankelijkheid en gewogen grenzen te verkrijgen voor stochastische maximale L^p-regulariteit, zowel in de complexe als in de reële interpolatieschaal. Als gevolg hiervan verkrijgen we verschillende nieuwe regulariteitsresultaten voor de stochastische warmtevergelijking en tijdsafhankelijke varianten in $\mathbb{R}^d$ en in gladde en hoekige domeinen. We behandelen ook stochastische Volterra vergelijkingen en tonen de p-onafhankelijkheid van de $\mathcal{R}$-begrensdheid van stochastische convolutieoperatoren aan.

In Deel II van deze dissertatie, gemotiveerd door het gebruik van de begrensdheid van de tensor-extensie van verscheidene klassieke harmonische analyse operatoren in de studie van (S)PDV's, zullen we twee algemene voldoende voorwaarden ontwikkelen voor een begrensde operator T op $L^p(\mathbb{R}^d)$ om een begrensde tensorextensie $\widetilde{T}$ op $L^p(\mathbb{R}^d; X)$ te hebben als X een Banachfunctieruimte is.

In Hoofdstuk 6 bewijzen we implicaties (2) en (3) in het volgende diagram

$$\begin{array}{ccc} \text{Schaarse dominantie voor } T & \xRightarrow{(1)} & \text{Gewogen afschattingen voor } T \\ \Big\Downarrow (2) & & (3) \Big\Downarrow \\ \text{Schaarse dominantie voor } \widetilde{T} & \xRightarrow{(4)} & \text{Gewogen afschattingen voor } \widetilde{T} \end{array}$$

In dit diagram zijn implicaties (1) en (4) alom bekend en hebben geen verband met de operator T. Zowel implicatie (3) als de combinatie van de implicaties (2) en (4) vertegenwoordigen een Banachfunctieruimte-waardige extensiestelling. Implicatie (3) is gebaseerd op een factorisatieprincipe, dat lijkt op de factorisatietheorie van Nikišin, Maurey en Rubio de Francia, maar dat flexibeler is. Implicatie (2) is gebaseerd op schaarse dominantie voor de rooster Hardy–Littlewood maximaaaloperator. Met behulp van deze extensiestellingen bewijzen we kwantitatieve connecties tussen Banach ruimte eigenschappen zoals de (gerandomiseerde) UMD-eigenschap en de Hardy–Littlewood eigenschap.

Met behulp van implicatie (3) bewijzen we Banachfunctieruimte-waardige afschattingen van Littlewood–Paley–Rubio de Francia-type in Hoofdstuk 7. Deze afschattingen stellen ons in staat om verscheidene operatorwaardige Fourier multiplicatorstellingen op Banach functieruimtes te bewijzen. Deze stellingen zijn uitbreidingen van de Coifman–Rubio de Francia–Semmes multiplicatorstelling. Onze resultaten gebruiken een nieuwe voorwaarde voor de begrensdheid van verzamelingen operatoren, die we $\ell^r(\ell^s)$-begrensdheid noemen en die in veel gevallen $\mathcal{R}$-begrensdheid impliceert.

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CURRICULUM VITÆ

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Emiel Lorist was born on January 22, 1992 in Utrecht, the Netherlands. He completed his secondary education in 2010 in a combined programme between Christelijk Gymnasium Utrecht and Junior College, an institute of Utrecht University. In the same year he started his studies in both Applied Mathematics and Applied Physics at Delft University of Technology. He obtained his propaedeutic degree in both Applied Mathematics and Applied Physics at the TU Delft in 2011. Afterwards he only continued his studies in Applied Mathematics, obtaining both his BSc degree and MSc cum laude in Applied Mathematics in 2013 and 2016 respectively. He wrote his master thesis titled *'Maximal functions, factorization, and the $\mathcal{R}$-boundedness of integral operators'* under the supervision of prof. dr. ir. M.C. Veraar.

During his master studies, he paused his academic activities for a year to become a member of the Nuon Solar Team, a team consisting of 15 students with the aim to build a solar powered car to compete in the World Solar Challenge. He became world champion solar car racing in 2015 with this team.

In 2016 he started his PhD research under the supervision of prof. dr. ir. M.C. Veraar and prof. dr. J.M.A.M. van Neerven at Delft University of Technology. Part of this research was carried out during multiple week-long visits to prof. dr. L. Weis at Karlsruhe Institute of Technology, Germany and a one month visit to prof. dr. T.P. Hytönen at University of Helsinki, Finland. In October 2020 he won the Dutch Royal Mathematical Society (KWG) PhD prize for a presentation about his research on singular stochastic integral operators.

LIST OF PUBLICATIONS

PREPRINTS

[1] E. Lorist. Space-time weighted maximal regularity for parabolic (S)PDE. In preparation.

[2] N. Lindemulder and E. Lorist. A discrete framework for the interpolation of Banach spaces. In preparation.

[3] E. Lorist and B. Nieraeth. Sparse domination implies vector-valued sparse domination. arXiv:2003.02233.

JOURNAL PAPERS / BOOKS / BOOK CHAPTERS

[4] N.J. Kalton, E. Lorist, and L. Weis. Euclidean structures and operator theory in Banach spaces. *To appear in Mem. Amer. Math. Soc.*, 2021.

[5] E. Lorist. On pointwise ℓ^r-sparse domination in a space of homogeneous type. *J. Geom. Anal.*, 2020. Online first.

[6] E. Lorist and M.C. Veraar. Singular stochastic integral operators. *To appear in Anal. PDE*, 2020.

[7] E. Lorist. The ℓ^s-boundedness of a family of integral operators on UMD Banach function spaces. In *Positivity and Noncommutative Analysis: Festschrift in Honour of Ben de Pagter on the Occasion of his 65th Birthday*, pages 365–379, Cham, 2019. Springer International Publishing.

[8] E. Lorist and B. Nieraeth. Vector-valued extensions of operators through multilinear limited range extrapolation. *J. Fourier Anal. Appl.*, 25(5):2608–2634, 2019.

[9] T.S. Hänninen and E. Lorist. Sparse domination for the lattice Hardy–Littlewood maximal operator. *Proc. Amer. Math. Soc.*, 147(1):271–284, 2019.

[10] A. Amenta, E. Lorist, and M.C. Veraar. Fourier multipliers in Banach function spaces with UMD concavifications. *Trans. Amer. Math. Soc.*, 371(7):4837–4868, 2019.

[11] A. Amenta, E. Lorist, and M.C. Veraar. Rescaled extrapolation for vector-valued functions. *Publ. Mat.*, 63(1):155–182, 2019.

[12] C. Gallarati, E. Lorist, and M.C. Veraar. On the ℓ^s-boundedness of a family of integral operators. *Rev. Mat. Iberoam.*, 32(4):1277–1294, 2016.