Analysis Of Gas Transport Networks

Strategies For Simplifying And Solving Problems Related To Gas Transport Networks

MSc Applied Mathematics Thesis Project

by

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Preface

In front of you is the report for my master thesis project as part of the MSc programme Applied Mathematics at Delft University of Technology. Research for it started in June of 2018. The project finished at the start of May 2019 with a final presentation. This thesis project has been done with the assistance of Dr. J.W. van der Woude, who I had frequent meetings with during the research in order to stay up-to-date with the progress.

Before officially starting, we discussed a few possible options for topics that could be interesting to work with. In the end, it was decided that the best option would be to focus on the analysis of gas pipe networks by using methods similar to the ones known for electric networks. The idea originated from the desire to find better methods for computations within large networks. This would be done with the use of incidence matrices and other network properties, in a similar way as A. van der Schaft described in his article about the methods for resistive networks[1]. Therefore, the first few weeks were spent on the exploration of these incidence matrices and getting used to the general knowledge about graphs again.

A second work that was influential for this project was the MSc report by S. Maring which served as inspiration for which topics we could build upon[2]. Additionally, we applied a MatLab programme that we newly created to a network shown in that report.

It was followed by the first main research topic, which was to find a way to simplify parts of a gas transport network by reducing them to a single pipe. This is done similarly to how a configuration of multiple resistors within a resistive network can be reduced to fewer resistors. The resistance of the single resistor is then fully dependent on the resistances of the multiple resistors, as is the case for series or parallel connections for example. As such, the goal was to find a similar connection between the constant values in a configuration and the constant for the equivalent situation with fewer pipes.

The second part of the research consisted of translating a theorem in the aforementioned article of A. van der Schaft from the resistive network to gas pipe networks. A MatLab script would be created in order to apply said theorem for practical cases where a specific set of known values is used as input.

Lastly, the created MatLab script would be extended beyond just the gas pipes to also include valves and compressors, as these are also key parts in a gas transport network.

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1

Introduction

1

In the current times, gas remains an important source of energy for both industrial use and daily use in many homes. As a result, development and improvement of gas transport remains relevant as well. This research project aims to do precisely that with extensive use of the network structures.

This report will focus on three research questions, which are spread out over several chapters. Below is briefly described what kind of information each chapter contains and where these research questions are being covered.

First off, chapter 2 will give an introduction to the main definitions and network properties that have been applied for the research and are of importance for the terminology in the rest of this report. Additionally, this chapter discusses a set of particular network properties. These are Kirchoff's Circuit Laws as known for resistive networks[3] and will be translated to gas transport networks.

The following three chapters will discuss different components within a gas transport network: chapters 3, 4 and 5 respectively contain information about gas pipes, valves and compressors. These components will be related to similar parts within an electric network and have the relations between pressure loss and flow given as equations. Chapter 3 will discuss two potential equations for pipe segments, whereas chapters 4 and 5 respectively contain the details on two types of valves plus the mixing station, and two types of compressors. These three chapters combined contain the information for the first research question:

> *What are the best ways to describe the relations between pressure loss and volumetric flow for different types of components in a gas transport network?*

Next, chapter 6 discusses ways to reduce certain connections within a gas pipe network to a single pipe segment, those being series and parallel connections. The constant values of all pipes in the configuration will be related to the one in the single pipe segment in a similar way to the methods for resistive networks. Additionally, this chapter contains some words about the Δ/Y transforms in resistive networks and gas pipe networks. This chapter contains the information for the second research question:

> *What are the methods for finding the relation between the constant values in a configuration of multiple pipe segments and the constant values for a reduced and equivalent configuration?*

Chapter 7 will first describe a theorem from the article by A. van der Schaft[1]. It guarantees the uniqueness of all values in a resistive network when a specific set of values is given. A similar conclusion will be drawn for the gas pipe networks, although with different methods than the ones used in the article. Additionally, this chapter contains information about the algorithm that was created in order to apply the theorem to practical cases via numerical methods. Eventually, this will also be expanded beyond just the pipe segments and also include potential valves and compressors. The result of this is a toolbox that can be used to find all values in a gas transport network. This chapter contains the information for the third research question:

> *How can the uniqueness theorem be applied for practical gas transport cases with the help of numerical methods?*

Finally, chapter 8 discusses the conclusions that can be drawn from the research topics. Afterwards, chapter 9 has some words about the potential inaccuracies that could exist with the used methods and what could have been done better.

2

Network Properties & Definitions

In this chapter, some of the basics of networks will be discussed. This will start with some definitions for graphs and some of the important variables that will be used in the report. After that comes an explanation of incidence matrix, which is eventually followed by theory on Kirchoff's Circuit Laws.

2.1. Definitions

The first step is to give some recurring basic definitions and notations regarding networks. Let $G = (V, A)$ be the directed graph used to represent a network. V is the set of m vertices, whereas A defines a set of n arcs. Each arc serves as the connection between two vertices and will represent a particular component in the network. Vertices will be notated as v_i, whereas an arc in the direction from v_i to v_j will usually be written as $a_{i,j}$. An arc with reverse orientation is $a_{j,i}$. Meanwhile, a self-loop on vertex v_i is notated as $a_{i,i}$, but will see little to no use in this report. A figure of the representation of a connection is shown in figure 2.1.

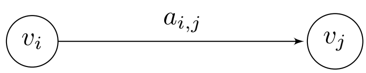

Figure 2.1: Arc within a graph.

A term that will be recurring in this report is the boundary vertex. Within a subgraph, such a vertex is one that has external influence from the rest of the full graph. The vector containing the labels of boundary vertices is notated as $\mathbf{b}$. The actual set of boundary sets uses the notation V_B and the remaining internal vertices uses V_I.

The vertices and arcs in the graph will also have some properties defined on them. In electric networks, the electric potentials are written as ψ_i, $i \in \{1, \ldots, m\}$ and will be defined on the corresponding vertices with the same i. The loss in potential is known as the voltage, which is defined on an arc from v_i to v_j with the notation $U_{i,j} = \psi_i - \psi_j$. Also defined on arcs are the internal currents, which are notated as $I_{i,j}$ on arc $a_{i,j}$. Another current that exists on some vertices within a subgraph is the external one, which is the current that is going into the network via boundary vertices. If vertex v_i has such a current, it will be notated as I_{Ei}. Note that only the i changes depending on the vertex. If any of the currents is negative, it means that it is going in the direction opposite of the arc orientation. Additionally, when vertices v_i and v_j are not connected via a single arc, the current $I_{i,j} = 0$. I_{Ei} is defined as equal to zero when the corresponding vector v_i has no external influence.

Lastly, the arcs usually also contain a certain constant. These have the property that they are the same positive number, regardless of the arc orientation. In other words, a constant with index i, j has the same value as the one with index j, i. In a resistor network for example, the resistance is defined on an arc $a_{i,j}$ and is notated as $R_{i,j} = R_{j,i}$. The situation is visualised in figure 2.2.

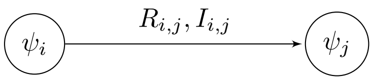

Figure 2.2: Arc within an electric network which contains a resistor component.

As for gas transport networks, pressures take the role of electric potentials. Pressure p_i is defined on the corresponding vertex v_i, $i \in \{1, \ldots, m\}$. The pressure loss is written as $\Delta p_{i,j} = p_i - p_j$ and is defined on the arc between v_i and v_j. The volumetric flow is also defined for arcs and is written as $Q_{i,j}$ on arc $a_{i,j}$. Similar to electric networks, there also exists an external flow aside from this internal flow. Like the external currents, the external flow is defined on boundary vertices and is dented by Q_{Ei}. Once again, only the i changes in this notation. Any negative volumetric flow implies that it is going in the direction opposite of the arc orientation.

Depending on the type of component that the arc represents, there is usually also a positive constant defined on it, just like the resistance in electric networks. The constants for the respective components will be discussed in detail in chapters 3, 4 and 5. The symmetry property also applies for these: a constant with index i, j is equal to one with index j, i. An arc from a gas transport network is visualised in figure 2.3. The constant that is used here as an example is $\alpha_{i,j}$.

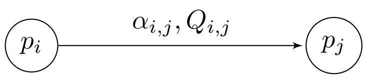

Figure 2.3: Arc within an electric network which contains a resistor component.

2.2. Graph Representations

In order to perform numerical analysis of graphs, there needs to be a way to implement it in the computer. Fortunately, there are several options for this. For this report though, two methods will be detailed. The first of these is by separating the source and terminal vertices of arcs. All the labels/indices of the source vertices are put in a vector $\mathbf{s}$, whereas the ones for the terminal vertices are in a vector $\mathbf{t}$. Naturally, it is important to be consistent with the arc order in these vectors. The j^{th} elements of both $\mathbf{s}$ and $\mathbf{t}$ need to be used for the same arc.

To show an example, suppose that the directed graph consists of vertices $V = \{v_1, v_2, v_3, v_4\}$ and arcs $A = \{a_{1,2}, a_{1,3}, a_{2,3}, a_{2,4}, a_{3,4}, a_{4,1}\}$. The situation is visualised in figure 2.4.

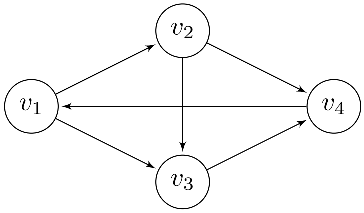

Figure 2.4: Example of a directed graph with 4 vertices and 6 arcs.

With this set of arcs, the resulting vectors for source and terminal vertices are $\mathbf{s} = (1, 1, 2, 2, 3, 4)^\top$ and $\mathbf{t} = (2, 3, 3, 4, 4, 1)^\top$. This will eventually the method used to define graphs in the computer.

The second method for describing graphs is with the use of incidence matrices in a similar way as done in the article by A. van der Schaft[1], usually notated as a matrix B. It is an $m \times n$ matrix constructed in such a way that each row represents a vertex, while each column represents an arc. Also suppose for this explanation that they are labelled $1, \ldots, m$ and $1, \ldots, n$ respectively. All matrix elements are either -1, 0 or 1, depending on the role of the vertices on the arcs:

- If vertex i is a source vertex for arc j, then element $B_{i,j} = -1$;
- If vertex i is not connected to arc j, then element $B_{i,j} = 0$;
- If vertex i is a terminal vertex for arc j, then element $B_{i,j} = 1$.

As an example, representing the graph in figure 2.4 this way gives matrix 2.1. The rows represent v_1, v_2, v_3 and v_4 in that order and the columns use the same arc order as the $\mathbf{s}$ and $\mathbf{t}$ vectors.

$$B = \begin{bmatrix} -1 & -1 & 0 & 0 & 0 & 1 \\ 1 & 0 & -1 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 1 & -1 \end{bmatrix} \tag{2.1}$$

The incidence matrix has a couple of convenient properties. First, because every arc is connected to only one source and one terminal vertex, each column of B has at most one -1 and one 1 element. This also means that the sum of all rows gives a row that exclusively contains 0 elements.

An application of the incidence matrix is that it can be used to create a vector with pressure losses. Suppose that $\mathbf{p}$ is a vector of length m containing the pressures on all vertices in the graph. The product $-B^\top \mathbf{p}$ then gives the desired vector with pressure losses. Note that the ordering of the pressures in $\mathbf{p}$ need to correspond with the row order.

This will once again be shown with the example in figure 2.4 and its incidence matrix B as defined above and pressure vector $\mathbf{p} = (p_1, p_2, p_3, p_4)^\top$. $-B^\top \mathbf{p}$ is expanded in equation 2.2. The last vector has all the pressure losses on each arc and will also be notated as $\Delta\mathbf{p}$.

$$-B^\top \mathbf{p} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ -1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \end{bmatrix} = \begin{bmatrix} p_1 - p_2 \\ p_1 - p_3 \\ p_2 - p_3 \\ p_2 - p_4 \\ p_3 - p_4 \\ p_4 - p_1 \end{bmatrix} = \begin{bmatrix} \Delta p_{1,2} \\ \Delta p_{1,3} \\ \Delta p_{2,3} \\ \Delta p_{2,4} \\ \Delta p_{3,4} \\ \Delta p_{4,1} \end{bmatrix} \tag{2.2}$$

Of course, the same applies for electric networks, where $-B^\top\boldsymbol{\psi}$ gives the vector $\mathbf{U}$ of voltages over each arc.

Another property of the incidence matrix is that each row shows how many arcs are oriented towards and away from the corresponding vertices:

- The amount of -1 entries in a single row shows how many arcs are oriented away from the vertex;
- The amount of 1 entries in a single row shows how many arcs are oriented towards the vertex.

Later on, it will be important to separate B into two parts. One part contains the rows of B that correspond with boundary vertices while the other takes the rows that correspond with internal vertices. The former will be called B_B and the latter B_I. For example, if v_1 and v_4 are boundary vertices in figure 2.4, then B_B and B_I are the matrices 2.3 and 2.4 respectively.

$$B_B = \begin{bmatrix} -1 & -1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & -1 \end{bmatrix} \tag{2.3}$$

$$B_I = \begin{bmatrix} 1 & 0 & -1 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 & -1 & 0 \end{bmatrix} \tag{2.4}$$

Now recall that $-B^\top\boldsymbol{\psi}$ and $-B^\top\mathbf{p}$ respectively give the vectors with voltages $\mathbf{U}$ and pressure losses $\Delta\mathbf{p}$. These equations can be separated into internal and external (boundary) parts, as shown in expressions 2.5 and 2.6.

$$\mathbf{U} = -B^\top\boldsymbol{\psi} = -B_I^\top\boldsymbol{\psi}_I - B_B^\top\boldsymbol{\psi}_B \tag{2.5}$$

$$\Delta\mathbf{p} = -B^\top\mathbf{p} = -B_I^\top\mathbf{p}_I - B_B^\top\mathbf{p}_B \tag{2.6}$$

Here, $\boldsymbol{\psi}_I$ and $\mathbf{p}_I$ are respectively the vectors with potentials and pressures on the internal vertices. Meanwhile, $\boldsymbol{\psi}_B$ and $\mathbf{p}_B$ represent the vectors with potentials and pressures on the boundary vertices.

For the earlier example with pressures, this is shown with expression 2.7

2

2

$$
\begin{aligned}
B_I^\top \mathbf{p}_I + B_B^\top \mathbf{p}_B &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 1 \\ -1 & 0 \\ 0 & -1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} p_2 \\ p_3 \end{bmatrix} + \begin{bmatrix} -1 & 0 \\ -1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} p_1 \\ p_4 \end{bmatrix} \\
&= \begin{bmatrix} p_2 \\ p_3 \\ p_3 - p_2 \\ -p_2 \\ -p_3 \\ 0 \end{bmatrix} + \begin{bmatrix} -p_1 \\ -p_1 \\ 0 \\ p_4 \\ p_4 \\ p_1 - p_4 \end{bmatrix} = \begin{bmatrix} p_2 - p_1 \\ p_3 - p_1 \\ p_3 - p_2 \\ p_4 - p_2 \\ p_4 - p_3 \\ p_1 - p_4 \end{bmatrix} \\
&= -\Delta \mathbf{p} = B^\top \mathbf{p}
\end{aligned}
\tag{2.7}
$$

2.3. Kirchoff's Circuit Laws

Kirchoff's Circuit Laws are two important theorems for electric networks. These are known as the Current Law and the Voltage Law[3].

First, for the Current Law, consider a vertex $v_j \in V$. The set A_j^+ contains all the arcs oriented towards v_j, whereas A_j^- contains all arcs oriented away from v_j. All arcs in A_j^+ are of the form $a_{i,j}$ and the ones in A_j^- are of the form $a_{j,i}$. Only the i is variable depending on the labelling used in the sets of arcs. Similarly, I_j^+ represents the total current going into v_j and I_j^- represents the total current exiting v_j. Naturally, I_j^+ corresponds to A_j^+ and I_j^- to A_j^-. The Current Law then states that $I_j^+ = I_j^-$ or equivalently $I_j^+ - I_j^- = 0$.

Now remember that a current exiting a vertex is the same as a negative current entering the vertex. Therefore, consider all currents to be their variant which enters v_j and let them be labelled as I_k, $k \in \{1, \ldots, \#A_j^+ + \#A_j^-\}$. The total current going into v_j (including negative currents) is the sum of all individual I_k values and is, according to the Current Law, equal to zero. This rule is shown in equation 2.8.

$$
\sum_{k=1}^{\#A_j^+ + \#A_j^-} I_k = 0. \tag{2.8}
$$

Since this rule says that the total ingoing current for any vertex is equal to zero, the two properties in system 2.9 will follow.

$$
\begin{aligned}
B_B \mathbf{I} &= -\mathbf{I}_E \\
B_I \mathbf{I} &= 0
\end{aligned}
\tag{2.9}
$$

Here, $\mathbf{I}$ is the vector of size n containing the currents on each arc. The same order as the one for the columns of B is used. Meanwhile, $\mathbf{I}_E$ is the vector of size m with its entries being the external ingoing flows on each boundary vertex.

Kirchoff's Current Law also applies to gas transport networks, in which the volumetric flows take the role of the currents. Therefore, let v_j, A_j^+ and A_j^- be the same as in the electric network. However, I_j^+, I_j^- and I_k now change to Q_j^+, Q_j^- and Q_k in this case, but the process remains the same. Eventually, when the rule is applied for the gas transport network, equation 2.10 follows.

$$\sum_{k=1}^{\#A_j^+ + \#A_j^-} Q_k = 0. \tag{2.10}$$

Translating property 2.9 to the gas transport case requires a vector $\mathbf{Q}$ containing the volumetric flows. System of equations 2.11 then follows. This time, $\mathbf{Q}_E$ is the vector with all external flows on the vertices.

$$\begin{aligned} B_B\mathbf{Q} &= -\mathbf{Q}_E \\ B_I\mathbf{Q} &= 0 \end{aligned} \tag{2.11}$$

Next, the Voltage Law states that the sum of all voltages in a cycle is equal to zero. It is important that all arcs are all oriented in a single direction (clockwise/counter-clockwise). Just like before, it is easily possible to reverse arcs if necessary. Therefore, suppose that the cycle contains vertices labelled as $\{v_1, \dots, v_k\}$ and arcs $\{a_{1,2}, a_{2,3}, \dots, a_{k-1,k}, a_{k,1}\}$. The voltages in this cycle are $\{V_{1,2}, V_{2,3}, \dots, V_{k-1,k}, V_{k,1}\}$ and the total sum of these is $V_{k,1} + \sum_{i=1}^{k-1} V_{i,i+1}$. Expanding the voltages in terms of electric potentials ($V_{i,j} = \psi_i - \psi_j$) gives equation 2.12.

$$V_{k,1} + \sum_{i=1}^{k-1} V_{i,i+1} = \psi_k - \psi_1 + \sum_{i=1}^{k-1} (\psi_i - \psi_{i+1}) \tag{2.12}$$

Expanding the sum results in expression 2.13.

$$(\psi_k - \psi_1) + (\psi_1 - \psi_2) + \dots + (\psi_{k-1} - \psi_k) \tag{2.13}$$

Here, the brackets can be moved in such a way that expression 2.14 remains. Also note that the first ψ_k term is moved to the back.

$$(-\psi_1 + \psi_1) + (-\psi_2 + \psi_2) + \dots + (-\psi_k + \psi_k) = \sum_{i=1}^{k} (-\psi_i + \psi_i) \tag{2.14}$$

Each part between brackets makes zero and thus the total sum is equal to zero. This shows the correctness of Kirchoff's Voltage Law.

Just like for the Current Law, the Voltage Law also has an application for gas transport networks. The only difference this time is that the pressures on the vertices take the role of the electric potentials. Aside from that, the method is exactly the same, with the eventual result shown in expression 2.15.

$$(p_k - p_1) + (p_1 - p_2) + \cdots + (p_{k-1} - p_k) = \sum_{i=1}^{k}(-p_i + p_i) = 0 \tag{2.15}$$

3

Pipe Segments

In this chapter, the most basic component of a gas pipe network shall be explored, namely the pipe segment. Its function is to simply move the substance or gas through the network. As a result of resistance on the inside of the pipe, the flow within it will usually decrease with increasing distance. Additionally, a pressure loss exists between the two end-points of the pipe. Some pipes are designed to be narrower or wider in order to allow less or more flow respectively. However, in this research it will be assumed that all pipes are uniform over the entire length.

3

3.1. Electric Analogy

In an electric network, the pipe segment is analogous to a resistor or a wire, the latter of which can be considered a resistor component with little or no resistance. The corresponding equation that relates voltage to current is the well-known Ohm's Law[4], shown in equation 3.1.

$$U = \psi_i - \psi_j = RI \tag{3.1}$$

The variables in the Ohm's Law equation are defined as follows, with the symbol between brackets being the unit of the variable:

- $U = \psi_i - \psi_j$ $[V]$ is the difference in potential between the two end points, more commonly known as the voltage;
- R $[\Omega]$ is the resistance of the resistor or wire, usually a constant in simple electricity networks;
- I $[A]$ is the current on the resistor or wire.

A similar equation of this form will be used for a pipe segment in a gas pipe network.

3.2. Equations

Depending on the source material, two very similar equations are generally used in order to describe the relation between the pressure loss and the volumetric flow on an arc. The first one is more often referred to in literature and is known as the Darcy-Weisbach equation[5]. The second equation is a variant of it that uses the difference between squared pressures instead of a regular pressure loss[2]. Both equations also use a number of variables that are considered to be constant for the sake of simplicity.

3.2.1. Darcy-Wesibach Equation

The Darcy-Weisbach equation[5] is usually used to describe the relation between the pressure loss and the flow velocity. This is shown below in equation 3.2.

$$\Delta p = \frac{fL\rho}{2d} V^2 \tag{3.2}$$

The variables in the Darcy-Weisbach equation are defined as follows:

- $\Delta p = p_i - p_j$ $[Pa = kg.m^{-1}.s^{-1}]$ is the total pressure loss on the pipe segment in the direction of arc $a_{i,j}$;
- f is the dimensionless Darcy friction factor;
- L $[m]$ is the length of the pipe segment;
- ρ $[kg.m^{-3}]$ is the density of the substance being moved through the pipe segment;
- d $[m]$ is the hydraulic diameter of the pipe;
- V $[m.s^{-1}]$ is the average flow velocity within the pipe.

3

However, in order to be able to properly apply Kirchoff's Law, the flow velocity needs to be converted to volumetric flow. This can be done by using the fact that the volumetric flow Q $[m^3.s^{-1}]$ is equal to the product of the area of the cross-section A $[m^2]$ and the average flow velocity, i.e. $Q = AV$ or alternatively $V = \frac{Q}{A}$. Furthermore, a perfectly cylindrical pipe segment will be assumed in every case, which means that the area can be expanded as $A = \pi r^2 = \frac{1}{4}\pi d^2$. Here, r $[m]$ is the radius and half of the diameter. Therefore, $V = \frac{4Q}{d^2\pi}$ and thus the Darcy-Weisbach can be converted to equation 3.3.

$$\Delta p = \frac{fL\rho}{2d}\left(\frac{4Q}{d^2\pi}\right)^2 = \frac{fL\rho}{2d}\frac{16Q^2}{d^4\pi^2} = \frac{8fL\rho}{d^5\pi^2}Q^2 \tag{3.3}$$

It is reasonable to assume that the Darcy friction factor, pipe length, density and diameter do not change over time, so the fraction $\frac{8fL\rho}{d^5\pi^2}$ will be noted as a positive constant α $[kg.m^{-7}]$. Hence, the result will be equation 3.4.

$$\Delta p = \alpha Q^2 \tag{3.4}$$

Lastly, it should be noted that the possibility of a negative pressure loss with resulting negative volumetric flow is ignored in this equation. In order to incorporate this possibility, the factor Q^2 will be replaced with $Q|Q|$. This way, if Q is negative, the right-hand side will be $-\alpha Q^2$, which corresponds with a negative pressure loss or a pressure increase in the direction of the arc.

In conclusion, the Darcy-Weisbach equation that will be used for the rest of this report will be equation 3.5.

$$\Delta p = p_i - p_j = \alpha Q|Q| \tag{3.5}$$

As an example, consider an arc $a_{i,j}$ with a pipe segment function. Let it have a constant $\alpha = 2$ and let the pressures be $p_i = 28$ and $p_j = 10$. Using equation 3.5 gives that $Q|Q| = \frac{28-10}{2} = 9$, so $Q = 3$.

3.2.2. Squared Pressures Variant

A slightly more complicated variant of the Darcy-Weisbach equation contains a difference between squared pressures on the left-hand side, instead of a regular pressure loss[2]. The form of this variant is shown in equation 3.6.

$$p_i^2 - p_j^2 = \hat{\alpha} Q|Q| \tag{3.6}$$

As can be seen, the constant in this equation is denoted by $\hat{\alpha}$. This is done to indicate that it is different from the constant used in the Darcy-Weisbach equation: Due to the different unit on the left hand side (Pa^2 instead of Pa), the one for $\hat{\alpha}$ needs to change as well. In order to find this unit, first write down the units of the known variables. $p_i^2 - p_j^2$ is expressed in $(kg.m^{-1}.s^{-1})^2$, whereas $Q|Q|$ is still expressed in $(m^3.s^{-1})^2$. The objective is to find the missing part in equation 3.7.

$$(kg.m^{-1}.s^{-1})^2 = \ldots \cdot (m^3.s^{-1})^2 \tag{3.7}$$

Now the solution is to give the right-hand side an extra factor of $(kg.m^{-4})^2 = kg^2.m^{-8}$ in order to make the equation valid, meaning that this would be the proper unit for the constant $\hat{\alpha}$.

$$(kg.m^{-1}.s^{-1})^2 = (kg.m^{-4})^2 \cdot (m^3.s^{-1})^2 \tag{3.8}$$

4

Valves

The next component to be observed is the valve. Two types will be discussed, namely the regular valve and the check valve. In the end of the chapter, the case of a mixing station will be converted to a construction that uses valves.

4.1. Regular Valves

First off, the function of the regular valve is to allow any positive or negative flow whenever it is opened or allow no flow if it is closed. In this research, the situation of a partially opened valve is ignored. In other words: this component is completely binary.

4

4.1.1. Electric Analogy

The regular valve has the same function in a gas pipe network as an on/off switch in an electric network. The latter also allows any current (flow) only when the switch is on (open) and nothing when it is off (closed).

4.1.2. Equations

Before finding an equation that properly describes the behaviour of regular valves, the individual open and closed cases need to be discussed.

Firstly, consider the case that the valve is closed. This simply means that no substance is being moved from one side to the other. In other words: the flow on the arc representing the valve is equal to zero. This is true for every possible pressure loss. Therefore, the equation for the opened valve becomes equation 4.1.

$$Q = 0 \tag{4.1}$$

Secondly, consider the case that the valve is opened. Now any amount of substance can go to the other side of the valve and be moved in the opposite direction. Therefore, $Q \in \mathbb{R}$ applies. Additionally, since it is assumed that the valve covers no significant distance in the network to cause a pressure loss, equation 4.2 is obtained for an opened valve.

$$\Delta p = 0 \tag{4.2}$$

In the next step, a binary constant $\beta \in \{0, 1\}$ is introduced. $\beta = 0$ will indicate a closed valve, while $\beta = 1$ indicates an opened one. Since an equation that follows either equation 4.1 or 4.2 is desired, the result will be equation 4.3.

$$\beta \Delta p + (1 - \beta) Q = 0 \tag{4.3}$$

Notice that if $\beta = 0$, the first $\beta \Delta p$ term becomes zero and all that remains is $(1 - 0) \cdot Q = Q = 0$. On the other hand, if $\beta = 1$, the second term is removed and $1 \cdot \Delta p = \Delta p = 0$ remains.

It should be noted that for a network containing regular valves with a β that remains constant, there exists an equivalent network in which all such valves are removed. If an arc $a_{1,2}$ in the graph contains a closed valve, simply delete the arc to remove all flow between the two vertices v_1 and v_2. Otherwise, if $a_{1,2}$ contains an opened valve, the solution is to delete the arc and merge vertices v_1 and v_2 into a single vertex with a single pressure. Therefore, no pressure loss exists.

4.2. Check Valves

The next type is the check valve[6]. Its function is to only allow flow in a single direction. It serves as a combination of the on and off situations.

4.2.1. Electric Analogy

In electric networks, the check valve can be considered to be analogous to a diode[7]. Just like a check valve, this component also works in such a way that current only goes one way on the arc in the defined direction of the diode. Current in the direction opposite of the arc orientation is not possible. Therefore, the current is equal to zero whenever the voltage is negative, i.e. the potential on the end of the arc is lower than the one at the start.

4.2.2. Equations

Going by the requirements for open and closed check valves described above, two sets can be defined. The closed case is active whenever the flow would go backwards in a regular pipe segment, i.e. when $p_i < p_j$ for an arc $a_{i,j}$ or $\Delta p < 0$. Q would then take a value 0. The set covering this case is shown in 4.4.

$$\{(\Delta p, Q) : \Delta p < 0, Q = 0\} \tag{4.4}$$

Meanwhile, the open case is active when there is no pressure loss due to the assumption that the valve covers no distance. The valve being opened also implies that any flow can move through it, so Δp is equal to zero, regardless of the value of Q. The set covering this case is shown in 4.5.

$$\{(\Delta p, Q) : \Delta p = 0, Q > 0\} \tag{4.5}$$

As for what happens when $\Delta p > 0$, this is a case without real application under the assumptions. The value of Q will not be defined, although it could perhaps be argued that due to the extra influence from an impossible pressure loss, the flow would be infinitely large.

All in all, the goal is to find an equation that closely covers the unity 4.6 of the described sets.

$$\{(\Delta p, Q) : \Delta p < 0, Q = 0\} \cup \{(\Delta p, Q) : \Delta p = 0, Q > 0\} \tag{4.6}$$

For diodes, the Shockley diode equation 4.7 is generally used[8]. It shows an exponential relation between current and voltage. In addition to the usual definitions, I_S represents the reverse bias saturation, r the ideality factor and U_T the thermal voltage.

$$I = I_S(\mathrm{e}^{\frac{U}{rU_T}} - 1) \tag{4.7}$$

Under the assumption that I_S, r and U_T are constant, equation 4.7 can be simplified to equation 4.8, with constants $C_1 = I_S$ and $C_2 = \mathrm{e}^{-rU_T}$.

$$I = C_1(C_2^U - 1) \tag{4.8}$$

Note that in this form, $I = 0$ only when $U = 0$ and not when $U \leq 0$. Additionally, I is defined for positive voltages, but it quickly gets large due to the exponential nature of the equation.

A similar equation has been sought for a check valve. However, it was eventually decided that euqation 4.9 would be used.

$$Q = (1 + \mathrm{sgn}(\Delta p))\beta\Delta p \tag{4.9}$$

It works in such a way that two simple linear parts exist, depending on whether Δp is positive or negative. If $\Delta p < 0$, then its sign $\mathrm{sgn}(\Delta p)$ is equal to -1. This results in the flow being equal to $Q = (1-1)\beta\Delta p = 0$, which means that set 4.4 is covered. Alternatively, when $\Delta p = 0$, the flow also becomes $Q = 0$. Lastly, if $\Delta p > 0$, then $\mathrm{sgn}(\Delta p) = 1$ and the flow is equal to $(1+1)\beta\Delta p = 2\beta\Delta p$.

Because of this, the constant β will be set to a very large number in order to approach the limit case, which covers set 4.5 as much as possible. It also incorporates a small pressure loss if a positive flow is being moved through the valve, adding to the more realistic imperfection.

Some more complicated equations have been sought that would perfectly cover the union of sets 4.4 and 4.5. Most of these involved very large expressions that use the sgn and sgn^2 functions a lot. Often, if these were turned into a function of variables Δp and Q by putting everything to one side and setting it equal to the function value, the use of sgn would cause the function to be discontinuous in some places. This caused problems when using them for the numerical analysis, so it was decided not to used these. However, the best candidate for such a function that would be fully continuous when put against a third variable is shown in equation 4.10.

$$f(\Delta p, Q) = (\Delta p)^2 Q^2(\mathrm{sgn}(Q) - \mathrm{sgn}(\Delta p)) + (\Delta p)^2(-\mathrm{sgn}(\Delta p) - 1) + Q^2(\mathrm{sgn}(Q) - 1) \tag{4.10}$$

The roots of this function are precisely the desired union of 4.4 and 4.5. It is constructed as the sum of two functions each with a special property. The first part, $(\Delta p)^2 Q^2(\mathrm{sgn}(Q) - \mathrm{sgn}(\Delta p))$ is equal to zero on every $(\Delta p, Q)$ where at least one of the two equals zero or both are either positive or negative at the same time, i.e. in quadrants I and III. Secondly, $(\Delta p)^2(-\mathrm{sgn}(\Delta p) - 1) + Q^2(\mathrm{sgn}(Q) - 1)$ is equal to zero only when $\Delta p \leq 0$ and $Q \geq 0$ positive. This is particularly important, as there is no point symmetry in the origin and neither is there symmetry on one of the axes. As a result, the function has no roots on the positive part of the Δp axis and the negative part of the Q axis. Additionally, there is a strong decline outside of quadrant II, meaning that the sum together with the first part remains negative inside quadrant IV and does not cause any new potential roots.

In the end though, it was decided that equation 4.9 would be used for further computations. This was because it gave slightly better results later on during the numerical analysis than using the roots of function 4.10. The latter also resulted in more iterations being required than when the former equation would be used.

4.3. Mixing Station

An often recurring network component is the mixing station. This is a point where substances from k separate pipes are mixed together into a new substance that will get transported further into the network. Additionally, it is important that the mixed substance can not be transported backwards into the two pipes. Therefore, consider $k+1$ arcs with a single vertex in the middle as shown in figure 4.1.

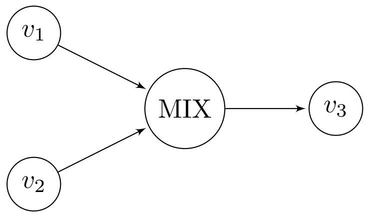

Figure 4.1: Mixing station setup.

Each arc represents a pipe segment, whereas the vertex is a mixing station. Since all of the network components in this approach are represented by arcs in a graph and thus not by vertices, a way needs to be found to convert this vertex to a construction suitable for the approach in this research. Luckily, this can be achieved with the use of the aforementioned check valves. As mentioned before, it is important that the mixed substance does not move backwards. Therefore, the arcs for separate pipe segments from which the original two substances came from are each split in two arcs. The first arc is identical to the original arc. The second arc however has the function of a check valve in the direction of the original arc in order to keep the substance from moving the opposite way. The arcs containing the check valves all come together in a single vertex. This construction is visualised in figure 4.2. The black arcs indicate the pipe segments and the red ones the check valves. The alternative setup follows all the requirements for a mixing station behaviour: multiple paths still come together in a single vertex and no backwards flow is possible from the middle vertex into those paths due to the valves. In conclusion, this setup allows for computations with mixing stations with the methods that only define segments on arcs and not vertices.

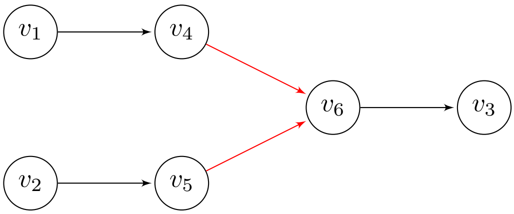

Figure 4.2: Alternative mixing station setup.

5

Compressors

The last type of component to be discussed is the compressor[9]. After having elaborated on the electric analogy, two types of compressor will be looked at. These are distinguished by the way the pressure is regulated. The first one has the pressure increased by a fixed amount, whereas the second one puts the pressure on the end point of the arc to a fixed value.

5.1. Electric Analogy

In electric networks, the analogy to a compressor is a battery. Similarly to what a compressor does with a pressure increase, the battery increases the electric potential. Generally speaking, a battery works in such a way that the difference in potential is constant instead of the endpoint having a fixed potential. Therefore, it is more closely related to the fixed pressure increase compressor.

As an example, if the arc is defined in the direction from v_1 to v_2 with potentials ψ_1 and ψ_2, then a $12V$ battery implies that $\psi_2 - \psi_1 = 12$.

5.2. Fixed Pressure Increase

A compressor with a fixed pressure increase works similarly to the usual battery in the electric network.

Once again, consider an arc with defined direction from v_1 to v_2. The pressures on these vertices are respectively p_1 and p_2. The difference between these pressures is considered to be a constant.

As a result, the corresponding equation becomes very simple, namely equation 5.1. The γ just represents the value used for the fixed increase.

$$p_2 - p_1 = -\Delta p = \gamma \tag{5.1}$$

5.3. Increase To Fixed Pressure

As mentioned, the alternative is a compressor working in such a way that it changes the pressure at the end point to a fixed value. In other words, if the same situation is used as before, p_2 will be equal to a constant, regardless of what the value of p_1 is.

The resulting equation 5.2 becomes once again very straight-forward. This time, the fixed pressure is represented by $\hat{\gamma}$.

$$p_2 = \hat{\gamma} \tag{5.2}$$

6

Network Reduction

In this chapter, the methods used to reduce a construction of a set of vertices and arcs will be discussed. In particular, it concerns the series and parallel connections. Each connection will first have the resistor case reduced, followed by a reduction of the gas pipe case, based on the former.

Of course, for resistor networks, it is assumed for this chapter that each arc $a_{i,i+1}, i \in \{1, \ldots, n-1\}$ contains a resistor with resistance $R_{i,i+1}$. On the vertices on each end of the arc, there exists a potential $\psi_j, j \in \{1, n\}$. The voltage $V_{i,i+1}$ is the difference in potential, i.e. $\psi_i - \psi_{i-1}$. The current $I_{i,i+1}$ on the arc can be calculated by using the aforementioned Ohm's Law (equation 3.1). The other important equation used here originates from Kirchoff's Law.

6.1. Series Connection

The first structure to be reduced is the series connection[10], which will be defined as follows. Consider a set of arcs $A = \{a_{1,2}, \ldots, a_{n-1,n}\}$ which respectively serve as the connections between the vertices $\{(v_1, v_2), \ldots, (v_{n-1}, v_n)\}$. Additionally, external influence on the structure is only allowed on the boundary vertices v_1 and v_n. Therefore, all internal vertices $v_2, \ldots, v_{n-1}$ have degree 2. An example of a series connection with two arcs is shown in figure 6.1.

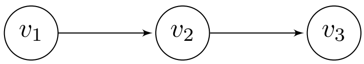

Figure 6.1: Series connection with two arcs.

Reducing the series connection to a single arc and two vertices in the electricity networks is done by following the method described below.

Given are the equations for Ohm's Law (system of equations 6.1) and Kirchoff's Law (equations 6.2).

$$\begin{aligned} \psi_1 - \psi_2 &= R_{1,2} I_{1,2} \\ \psi_2 - \psi_3 &= R_{2,3} I_{2,3} \\ &\vdots \\ \psi_{n-1} - \psi_n &= R_{n-1,n} I_{n-1,n} \end{aligned} \tag{6.1}$$

$$I_{in} = I_{1,2} = I_{2,3} = \ldots = I_{n-1,n} = I_{out} \tag{6.2}$$

The reduced series connection would have a corresponding Ohm's Law equation of the form shown in equation 6.3.

$$\psi_1 - \psi_n = R_{1,n} I_{1,n} \tag{6.3}$$

This can be achieved by taking the sum of all equations in system 6.1 followed by the use of Kirchoff's Law. First, taking the sum results in equation 6.4.

$$\sum_{i=1}^{n-1}(\psi_i - \psi_{i+1}) = \psi_1 - \psi_n = \sum_{i=1}^{n-1} R_{i,j} I_{i,j} \tag{6.4}$$

Kirchoff's Law states that $I_{in} = I_{1,n} = I_{out}$, which implies that every current mentioned in equations 6.2 is also equal to $I_{1,n}$. Applying this rule gives equation 6.5, which is in the form from equation 6.3.

$$\psi_1 - \psi_n = \left(\sum_{i=1}^{n-1} R_{i,j}\right) I_{1,n} \tag{6.5}$$

The result will be that the resistor of a reduced series connection has a resistance equal to the sum of all individual resistances (equation 6.6).

$$R_{1,n} = \sum_{i=1}^{n-1} R_{i,j} \tag{6.6}$$

In the context of gas pipe segments, an almost identical method is used to compute the constant value $\alpha_{1,n}$ for a Darcy-Weisbach equation corresponding to a series connection with constant values $\alpha_{1,2}, \alpha_{2,3}, \ldots, \alpha_{n-1,n}$.

First, the equations for Ohm's Law in system 6.1 are replaced with the Darcy-Weisbach equations for the $n-1$ arcs. This step is shown in system of equations 6.7.

6

$$\begin{aligned} p_1 - p_2 &= \alpha_{1,2} Q_{1,2} |Q_{1,2}| \\ p_2 - p_3 &= \alpha_{2,3} Q_{2,3} |Q_{2,3}| \\ &\vdots \\ p_{n-1} - p_n &= \alpha_{n-1,n} Q_{n-1,n} |Q_{n-1,n}| \end{aligned} \tag{6.7}$$

Additionally, Kirchoff's Law once again gives equation 6.8.

$$Q_{in} = Q_{1,2} = Q_{2,3} = \ldots = Q_{n-1,n} = Q_{out} \tag{6.8}$$

This time, an equation of form 6.9 is desired.

$$p_1 - p_n = \alpha_{1,n} Q_{1,n} |Q_{1,n}| \tag{6.9}$$

To obtain this, take the sum of all individual Darcy-Weisbach equations as shown in equation 6.10.

$$\sum_{i=1}^{n-1}(p_i - p_{i+1}) = p_1 - p_n = \sum_{i=1}^{n-1} \alpha_{i,j} Q_{i,j} |Q_{i,j}| \tag{6.10}$$

Reducing this then returns equation 6.11, which has the right form as described above with the sum on the place of $\alpha_{1,n}$.

$$p_1 - p_n = \left(\sum_{i=1}^{n-1} \alpha_{i,j} \right) Q_{1,n}|Q_{1,n}| \tag{6.11}$$

Therefore, the conclusion is that $C_{1,n}$ has the following relation to the individual α-values in the series connection.

$$\alpha_{1,n} = \sum_{i=1}^{n-1} \alpha_{i,j} \tag{6.12}$$

If, instead of the Darcy-Weisbach equation, the alternative equation with squared pressures is used, the only change will be that every $p_j, j \in \{1, \ldots, n\}$ is replaced with a $p_j^2, j \in \{1, \ldots, n\}$. This changes nothing to the result, since $\sum_{i=1}^{n-1}(p_i^2 - p_{i+1}^2)$ is still equal to $p_1^2 - p_n^2$, just like how $\sum_{i=1}^{n-1}(p_i - p_{i+1})$ equals $p_1 - p_n$.

6.1.1. Example

As an example with Darcy-Weisbach equation (the variant is almost identical), consider two pipe segments $a_{1,2}$ and $a_{2,3}$ placed in a series connection. Let the first one have a constant value $\alpha_{1,2} = 2$ and the second one with $\alpha_{2,3} = 1$. In the reduced series connection, the constant should be $\alpha_{1,3} = 3$. The situation is visualised in figure 6.2. To show this, suppose that the flow through the connection is $Q = 2$. The pressure loss from p_1 to p_2 should be $\alpha_{1,2}Q^2 = 2 \cdot 2^2 = 8$. Meanwhile, the pressure loss from p_2 to p_3 should be $\alpha_{2,3}Q^2 = 1 \cdot 2^2 = 4$. The result of $\alpha_{1,3} = 3$ would imply that the total pressure loss from p_1 to p_3 is $3 \cdot 2^2 = 12$. This is correct, because the sum of the two pressure losses gives the total loss and is indeed equal to $8 + 4 = 12$.

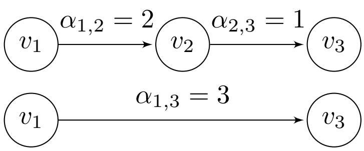

Figure 6.2: Example of a series connection.

6.2. Parallel Connection

Less straight-forward is the reduction of the parallel connection[11]. For this structure, let $A = \{a_1, \ldots, a_n\}$ be the set of parallel arcs that connect vertices v_1 and v_2. An example of a parallel connection with two arcs is shown in figure 6.3.

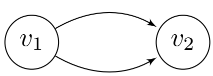

Figure 6.3: Parallel connection with two arcs.

For resistor networks, each $a_j \in A$ has a corresponding resistance R_j and potentials ψ_1 and ψ_2 together give a voltage $U = \psi_1 - \psi_2$. Naturally, the voltage is the same on all parallel arcs. By applying Ohm's Law, equations 6.13 are obtained for all $j \in \{1, \ldots, n\}$.

$$\psi_1 - \psi_2 = U = R_j I_j \Leftrightarrow I_j = \frac{U}{R_j} \tag{6.13}$$

The total current going from v_1 to v_2 is $I_{in} = I_1 + I_2 + \ldots + I_n = I_{out}$, as a result of Kirchoff's Law. This is precisely the current that will exist on the single arc from v_1 to v_2 in the reduced parallel connection. It will be defined as $I = I_1 + I_2 + \ldots + I_n$. The desired Ohm's Law equation needs to be of the form in equation 6.14.

$$U = RI \Leftrightarrow I = \frac{U}{R} \tag{6.14}$$

Using both the definition of I and Ohm's Law for the individual pipe segments gives the result in equation 6.15. This is an equation of the desired form, in which the sum is the resistance for the reduced parallel connection.

$$I = \sum_{j=1}^{n} I_j = \sum_{j=1}^{n} \frac{U}{R_j} = U \sum_{j=1}^{n} \frac{1}{R_j} \tag{6.15}$$

Therefore, R is related to the individual resistances $R_1, \ldots, R_n$ according to expression 6.16.

$$\frac{1}{R} = \sum_{j=1}^{n} \frac{1}{R_j} \tag{6.16}$$

Before going further into the methods for finding the result for the gas pipe networks, the $Q|Q|$ factor in the Darcy-Weisbach equation needs to be discussed. Just like in the resistor network case, the flow (current) Q needs to be expressed in terms of Δp, while only the opposite is known at the moment. Therefore, consider the equation $y = x|x|$. Expressing x in terms of y has to be done with $x = \operatorname{sgn}(y)\sqrt{|y|}$, as shown in expression 6.17. The function $\operatorname{sgn}(y)$ gives the sign of y. It is equal to -1 if y is negative, 1 if it is positive or 0 only if it is precisely equal to zero.

$$x|x| = \operatorname{sgn}(y)\sqrt{|y|}|\operatorname{sgn}(y)\sqrt{|y|}| = \operatorname{sgn}(y)\sqrt{|y|} \cdot 1 \cdot \sqrt{|y|} = \operatorname{sgn}(y)|y| = y \tag{6.17}$$

This is then applied to the Darcy-Weisbach equation with Q taking the role of x and $\frac{\Delta p}{\alpha}$ taking the role of y. The result is shown in expression 6.18. As per definition, α is positive, so $\operatorname{sgn}(\alpha) = 1$ and thus $\operatorname{sgn}(\frac{\Delta p}{\alpha_j}) = \operatorname{sgn}(\Delta p)$.

$$\begin{aligned} p_1 - p_2 = \Delta p = \alpha_j Q_j |Q_j| \Leftrightarrow Q_j|Q_j| = \frac{\Delta p}{\alpha_j} \Leftrightarrow \\ \Leftrightarrow Q_j = \operatorname{sgn}(\frac{\Delta p}{\alpha_j})\sqrt{\left|\frac{\Delta p}{\alpha_j}\right|} = \operatorname{sgn}(\Delta p)\sqrt{\frac{|\Delta p|}{\alpha_j}} \end{aligned} \tag{6.18}$$

6

This time, the desired equation has the form shown in equation 6.19.

$$p_1 - p_2 = \Delta p = \alpha Q|Q| \Leftrightarrow Q|Q| = \frac{\Delta p}{\alpha} \tag{6.19}$$

Next, equation 6.20 shows the application of Kirchoff's Law and substitution of the expression for the individual Q_j terms from equation 6.18.

$$Q = \sum_{j=1}^{n} Q_j = \sum_{j=1}^{n} \operatorname{sgn}(\Delta p)\sqrt{\frac{|\Delta p|}{\alpha_j}} = \operatorname{sgn}(\Delta p)\sqrt{|\Delta p|}\sum_{j=1}^{n}\frac{1}{\sqrt{\alpha_j}} \tag{6.20}$$

Substituting this last expression in $Q|Q|$ gives equation 6.21.

$$Q|Q| = \left(\operatorname{sgn}(\Delta p)\sqrt{|\Delta p|}\sum_{j=1}^{n}\frac{1}{\sqrt{\alpha_j}}\right)\left|\operatorname{sgn}(\Delta p)\sqrt{|\Delta p|}\sum_{j=1}^{n}\frac{1}{\sqrt{\alpha_j}}\right| \tag{6.21}$$

This is then further simplified in equation 6.22, the final result of which is of the desired form.

6

$$Q|Q| = \operatorname{sgn}(\Delta p)|\Delta p|\left(\sum_{j=1}^{n}\frac{1}{\sqrt{\alpha_j}}\right)^2 = \Delta p\left(\sum_{j=1}^{n}\frac{1}{\sqrt{\alpha_j}}\right)^2 \tag{6.22}$$

The squared part is completely in terms of α_j values and clearly in the place of $\frac{1}{\alpha}$. Thus, the conclusion for the relation between the α from the reduced parallel connection is that it relates to the individual α_j values according to equation 6.23.

$$\frac{1}{\alpha} = \left(\sum_{j=1}^{n}\frac{1}{\sqrt{\alpha_j}}\right)^2 \tag{6.23}$$

Alternatively, this relation can be written like equation 6.24.

$$\frac{1}{\sqrt{\alpha}} = \sum_{j=1}^{n}\frac{1}{\sqrt{\alpha_j}} \tag{6.24}$$

Once again, the method used to find the relation for the squared pressure variant is almost identical. The role of Δp remains the same throughout the process, but it is defined differently as $p_1^2 - p_2^2$.

6.2.1. Example

As an example, let $p_1 = 25$, $p_2 = 9$, $\alpha_1 = 1$, $\alpha_2 = 4$. Then the arcs have the corresponding system of Darcy Weisbach equations as shown in 6.25

$$\begin{aligned} 25 - 9 = 16 = 1 \cdot Q_1|Q_1| \\ 25 - 9 = 16 = 4 \cdot Q_2|Q_2| \end{aligned} \tag{6.25}$$

This gives resulting volumetric flows $Q_1 = 4$ and $Q_2 = 2$. Therefore, the total flow becomes $Q = 6$.

$$25 - 9 = 16 = C \cdot 6 \cdot |6| = 36\alpha \tag{6.26}$$

This gives $\alpha = \frac{4}{9}$. If equation 6.23 is correct, it should give the exact same value for α. This is indeed the case, going by the steps in 6.27: $\frac{1}{\alpha} = \frac{9}{4}$ implies that $\alpha = \frac{4}{9}$.

$$\left(\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{4}}\right)^2 = \left(\frac{3}{2}\right)^2 = \frac{9}{4} = \frac{1}{\alpha} \tag{6.27}$$

The situation is visualised in figure 6.4.

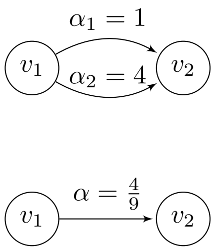

Figure 6.4: Example of a parallel connection.

6.3. Δ/Y TRANSFORMS

Lastly, some words need to be said about the Δ and Y constructions and the transform between them[12], as quite a bit of time was spent on this topic.

In this research, a Δ construction is a (sub)graph that consists of three vertices, all with an arc between each other. Every vertex also has external influence. Ignoring arc orientation, this construction is therefore a cycle of length 3 with external influence on every vertex in the cycle. A visualisation of the construction is shown in figure 6.5 Also note that if one vertex lacks external influence, the situation is simply a parallel connection with a series connection on one of the two paths. This can be reduced by combining the methods for series and parallel connections: first reduce the series part to a single arc and follow up with reduction of the parallel part.

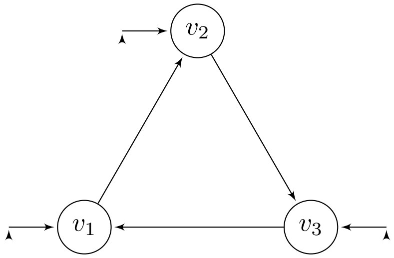

Figure 6.5: Δ configuration.

As for the Y construction, this is a (sub)graph consisting of four vertices. Three of these are only connected to the fourth one. Additionally, the three vertices have external influence, whereas the fourth one does not. This situation is shown in image 6.6.

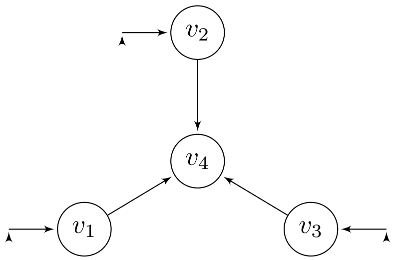

Figure 6.6: Y configuration.

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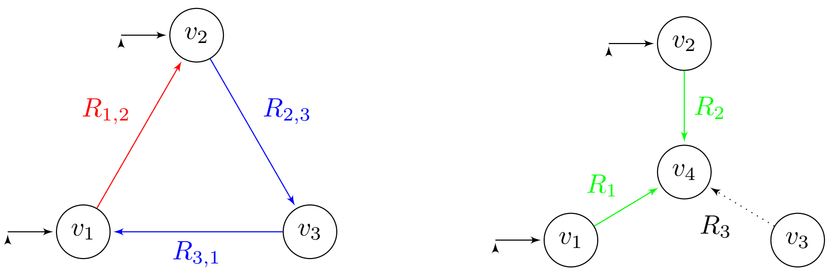
For resistor networks, it is possible to transform a Δ construction into an equivalent Y construction and vice versa[12]. This is done while exclusively using the constant resistance values on each arc. To achieve this, consider three cases. In each case, it is considered that one of the three vertices has no external input. The resulting setups will each be reduced to a single resistor.

The method will be shown for the case that v_3 has no external input



(a) Δ configuration. (b) Y configuration.

Figure 6.7: Figures for the case without external input on v_3.

Reducing both the parallel construction in 6.7a and the series connection in 6.7b to a single arc should give the same resistance.

Using the two methods described in sections 6.1 and 6.2, the reduction of the Δ situation results in an arc with resistance $R = \frac{R_{1,2}(R_{2,3}+R_{3,1})}{R_{1,2}+R_{2,3}+R_{3,1}}$. Meanwhile, the method for reducing series connections gives $R = R_1 + R_2$ as the resulting resistance in the Y construction and is set as equal to the previous equation. Repeating this process for the other two cases with no external input on v_1 and v_2 gives the system of three equations in 6.28.

$$\begin{aligned} \frac{R_{1,2}(R_{2,3}+R_{3,1})}{R_{1,2}+R_{2,3}+R_{3,1}} &= R_1 + R_2 \\ \frac{R_{2,3}(R_{3,1}+R_{1,2})}{R_{1,2}+R_{2,3}+R_{3,1}} &= R_2 + R_3 \\ \frac{R_{3,1}(R_{1,2}+R_{2,3})}{R_{1,2}+R_{2,3}+R_{3,1}} &= R_3 + R_1 \end{aligned} \tag{6.28}$$

This system of equations can conveniently be written by using vectors and a matrix as is shown in equation 6.29.

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} R_1 \\ R_2 \\ R_3 \end{bmatrix} = \begin{bmatrix} \frac{R_{1,2}(R_{2,3}+R_{3,1})}{R_{1,2}+R_{2,3}+R_{3,1}} \\ \frac{R_{2,3}(R_{3,1}+R_{1,2})}{R_{1,2}+R_{2,3}+R_{3,1}} \\ \frac{R_{3,1}(R_{1,2}+R_{2,3})}{R_{1,2}+R_{2,3}+R_{3,1}} \end{bmatrix} \tag{6.29}$$

In order to only have the vector $(R_1, R_2, R_3)^\top$ remaining on the left-hand side, both sides need to be multiplied from the left with the inverse of the matrix. This gives equation 6.30.

$$\begin{bmatrix} R_1 \\ R_2 \\ R_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} \frac{R_{1,2}(R_{2,3}+R_{3,1})}{R_{1,2}+R_{2,3}+R_{3,1}} \\ \frac{R_{2,3}(R_{3,1}+R_{1,2})}{R_{1,2}+R_{2,3}+R_{3,1}} \\ \frac{R_{3,1}(R_{1,2}+R_{2,3})}{R_{1,2}+R_{2,3}+R_{3,1}} \end{bmatrix} \tag{6.30}$$

The next step writes out the inverse of the matrix and takes the common factor of $\frac{1}{(R_{1,2}+R_{2,3}+R_{3,1})}$ out of the vector. Additionally, the individual elements from the vector on the right-hand side are expanded.

$$\begin{bmatrix} R_1 \\ R_2 \\ R_3 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ -1 & 1 & 1 \end{bmatrix} \frac{1}{(R_{1,2}+R_{2,3}+R_{3,1})} \begin{bmatrix} R_{1,2}R_{2,3} + R_{3,1}R_{1,2} \\ R_{2,3}R_{3,1} + R_{1,2}R_{2,3} \\ R_{3,1}R_{1,2} + R_{2,3}R_{3,1} \end{bmatrix} \tag{6.31}$$

The matrix and vector on the right-hand side can be multiplied in order to give the much simpler vector shown in equation 6.32.

$$\begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} R_{1,2}R_{2,3} + R_{3,1}R_{1,2} \\ R_{2,3}R_{3,1} + R_{1,2}R_{2,3} \\ R_{3,1}R_{1,2} + R_{2,3}R_{3,1} \end{bmatrix} = \begin{bmatrix} 2R_{3,1}R_{1,2} \\ 2R_{1,2}R_{2,3} \\ 2R_{2,3}R_{3,1} \end{bmatrix} = 2 \begin{bmatrix} R_{3,1}R_{1,2} \\ R_{1,2}R_{2,3} \\ R_{2,3}R_{3,1} \end{bmatrix} \tag{6.32}$$

Substituting this in equation 6.31 means that all of it can be reduced to equation 6.33. Note that the factors $\frac{1}{2}$ and 2 have cancelled each other out.

$$\begin{bmatrix} R_1 \\ R_2 \\ R_3 \end{bmatrix} = \frac{1}{R_{1,2} + R_{2,3} + R_{3,1}} \begin{bmatrix} R_{3,1}R_{1,2} \\ R_{1,2}R_{2,3} \\ R_{2,3}R_{3,1} \end{bmatrix} \tag{6.33}$$

Thus, the individual equations for R_1, R_2 and R_3 are obtained.

$$\begin{aligned} R_1 &= \frac{R_{3,1}R_{1,2}}{R_{1,2} + R_{2,3} + R_{3,1}} \\ R_2 &= \frac{R_{1,2}R_{2,3}}{R_{1,2} + R_{2,3} + R_{3,1}} \\ R_3 &= \frac{R_{2,3}R_{3,1}}{R_{1,2} + R_{2,3} + R_{3,1}} \end{aligned} \tag{6.34}$$

Conversely, to transform a Y construction into a Δ construction, start off by writing out R_1R_2, R_2R_3 and R_3R_1 by using the expressions in system 6.34. The result is shown in system 6.35.

$$\begin{aligned} R_1R_2 &= \frac{R_{1,2}^2R_{2,3}R_{3,1}}{(R_{1,2} + R_{2,3} + R_{3,1})^2} \\ R_2R_3 &= \frac{R_{1,2}R_{2,3}^2R_{3,1}}{(R_{1,2} + R_{2,3} + R_{3,1})^2} \\ R_1R_2 &= \frac{R_{1,2}R_{2,3}R_{3,1}^2}{(R_{1,2} + R_{2,3} + R_{3,1})^2} \end{aligned} \tag{6.35}$$

Once again, add these three equations in order to get equation 6.36.

$$R_1R_2 + R_2R_3 + R_3R_1 = \frac{R_{1,2}^2R_{2,3}R_{3,1} + R_{1,2}R_{2,3}^2R_{3,1} + R_{1,2}R_{2,3}R_{3,1}^2}{(R_{1,2} + R_{2,3} + R_{3,1})^2} \tag{6.36}$$

Isolate a factor $R_{1,2}R_{2,3}R_{3,1}$ in the numerator to get equation 6.37.

$$R_1R_2 + R_2R_3 + R_3R_1 = \frac{R_{1,2}R_{2,3}R_{3,1}(R_{1,2} + R_{2,3} + R_{3,1})}{(R_{1,2} + R_{2,3} + R_{3,1})^2} \tag{6.37}$$

Now a factor $R_{1,2}+R_{2,3}+R_{3,1}$ exists in both the numerator and the denominator, so one can be removed from the fraction.

$$R_1R_2 + R_2R_3 + R_3R_1 = \frac{R_{1,2}R_{2,3}R_{3,1}}{R_{1,2} + R_{2,3} + R_{3,1}} \tag{6.38}$$

Conveniently, the right hand side of equation 6.38 is exactly the same as either $R_1R_{2,3}$, $R_2R_{3,1}$ or $R_3R_{1,2}$. For this step, refer back to system 6.34. Therefore, equation 6.39 follows.

$$R_1R_2 + R_2R_3 + R_3R_1 = R_1R_{2,3} = R_2R_{3,1} = R_3R_{1,2} \tag{6.39}$$

From this, any of the three resistances in the Δ configuration can be obtained by dividing the correct equation by the resistance from the Y configuration, located on the right hand side. The final results, together with system 6.34, are shown in 6.40. Now the resistances in the Δ configuration are exclusively expressed in terms of resistances in the Y configuration and vice versa.

$$\begin{aligned} R_1 &= \frac{R_{3,1}R_{1,2}}{R_{1,2}+R_{2,3}+R_{3,1}} & R_{1,2} &= \frac{R_1R_2+R_2R_3+R_3R_1}{R_3} \\ R_2 &= \frac{R_{1,2}R_{2,3}}{R_{1,2}+R_{2,3}+R_{3,1}} & R_{2,3} &= \frac{R_1R_2+R_2R_3+R_3R_1}{R_1} \\ R_3 &= \frac{R_{2,3}R_{3,1}}{R_{1,2}+R_{2,3}+R_{3,1}} & R_{3,1} &= \frac{R_1R_2+R_2R_3+R_3R_1}{R_2} \end{aligned} \tag{6.40}$$

During the research, it was attempted to find similar expressions in a gas pipe network. Unfortunately though, this did not end up brining any useful results. It was first attempted to follow the same method under the assumption that it was allowed to consider the three similar cases. A result for the $\Delta \to$ Y transform was found and would be equation 6.41.

$$\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} \frac{\alpha_{1,2}(\alpha_{2,3}+\alpha_{3,1})}{\left(\sqrt{\alpha_{1,2}}+\sqrt{\alpha_{2,3}+\alpha_{3,1}}\right)^2} \\ \frac{\alpha_{2,3}(\alpha_{3,1}+\alpha_{1,2})}{\left(\sqrt{\alpha_{2,3}}+\sqrt{\alpha_{3,1}+\alpha_{1,2}}\right)^2} \\ \frac{\alpha_{3,1}(\alpha_{1,2}+\alpha_{2,3})}{\left(\sqrt{\alpha_{3,1}}+\sqrt{\alpha_{1,2}+\alpha_{2,3}}\right)^2} \end{bmatrix} \tag{6.41}$$

However the reverse transform resulted in expressions that were incredibly difficult to deal with. Additionally, the non-linearity of the relation between Δp and Q seemed to cause inconsistencies whenever the same α values, but different pressures on the vertices or different volumetric flows on the arcs would be used. This happened, even though the resulting α values in one configuration should be solely dependent on the α values in the other configuration and not of any pressures or flows.

7

Uniqueness Of Network Values & Numerical Applications

This chapter focuses on the solvability of practical cases. It will start by discussing a relevant theorem by A. van der Schaft[1] and translating it to the situation of a gas pipe network. Afterwards, a programme that was created based on this theorem will be detailed. It will also be explained how the algorithm can be expanded to include more components. Lastly, the programme will be applied to an example of large network.

7.1. Uniqueness Theorem

For the case of a resistive network, A. van der Schaft discussed theorem 1 the article mentioned before[1]. Note that the theorem as written here is worded differently than in the article in order to stay consistent with the terminology. However, the implications remain the same. Also note that this is only a part of the full theorem as written in the article.

Theorem 1 *Let $G = (V, A)$ be the graph representing a resistive network where each arc follows Ohm's Law and suppose that V_I and V_B are the subsets of V which respectively contain the internal and boundary vertices. Additionally, let R be a diagonal matrix containing the resistances on the arcs. Then for any vector $\boldsymbol{\psi}_B$ containing the electric potentials on the boundary vertices in V_B, there exist unique vectors $\boldsymbol{\psi}_I$, $\mathbf{I}$ and $\mathbf{I}_E$ such that all the equations for Kirchoff's Current Law and Ohm's Law corresponding to the network are satisfied.*

As mentioned before, the vector $\boldsymbol{\psi}_I$ contains the electric potentials on the internal vertices, $\mathbf{I}$ contains the internal currents and $\mathbf{I}_E$ contains the external currents. Also remember that the equations for Kirchoff's Current Law are the ones shown in system 7.1.

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$$\begin{aligned} B_B\mathbf{I} &= -\mathbf{I}_E \\ B_I\mathbf{I} &= 0 \end{aligned} \tag{7.1}$$

As for the system of equations of the form $\psi_i - \psi_j = R_{i,j}I_{i,j}$ according to Ohm's Law, those can be written with the use of the incidence matrix B, a diagonal matrix R and the vectors $\boldsymbol{\psi}$ and $\mathbf{I}$. The system is shown in this form in equation 7.2. Recall for this from section 2.2 that $B^\top\boldsymbol{\psi}$ creates a vector U containing the voltages over each arc.

$$-B^\top\boldsymbol{\psi} = -B_I^\top\boldsymbol{\psi}_I - B_B^\top\boldsymbol{\psi}_B = R\mathbf{I} \tag{7.2}$$

From systems 7.1 and 7.2, two properties can be obtained, those being equations 7.3 and 7.4. Observe that Ohm's Law is rewritten to $\mathbf{I} = R^{-1}\mathbf{U}$ and substituted in the equations for Kirchoff's Current Law.

$$B_I(R^{-1}B_I^\top\boldsymbol{\psi_I} + R^{-1}B_B^\top\boldsymbol{\psi_B}) = 0 \tag{7.3}$$

$$B_B(R^{-1}B_I^\top\boldsymbol{\psi_I} + R^{-1}B_B^\top\boldsymbol{\psi_B}) = -\mathbf{I}_E \tag{7.4}$$

The relation between $\mathbf{I}_E$ and $\boldsymbol{\psi}_B$ can be obtained by finding the expression for $\boldsymbol{\psi}_I$ from equation 7.3 and substituting it in equation 7.4. This process gives expression 7.5, followed by equation 7.6.

$$\boldsymbol{\psi}_I = -(B_I R^{-1} B_I^\top)^{-1} B_I R^{-1} B_B^\top \boldsymbol{\psi}_B \tag{7.5}$$

$$\mathbf{I}_E = B_B R^{-1}(B_B^\top (B_I R^{-1} B_I^\top)^{-1} B_I G B_B^\top + B_B^\top)\boldsymbol{\psi}_B \tag{7.6}$$

The matrix $L_B = B_B R^{-1}(B_B^\top (B_I R^{-1} B_I^\top)^{-1} B_I G B_B^\top + B_B^\top)$ then serves as a linear operator to relate the vectors $\mathbf{I}_E$ and $\boldsymbol{\psi}$ to each other.

The uniqueness of the variables follows from equations 7.5, 7.6 and 7.2[1]. The first two respectively give unique $\boldsymbol{\psi}_I$ and $\mathbf{I}_E$ vectors when B, R and $\boldsymbol{\psi}_B$ are known. Applying Ohm's Law then results in a unique vector $\mathbf{I}$ if it is rewritten to $\mathbf{I} = -R^{-1}(B_I^\top \boldsymbol{\psi}_I + B_B^\top \boldsymbol{\psi}_B)$ after $\boldsymbol{\psi}_B$ is found.

In the case of a gas pipe network, theorem 1 would be translated to theorem 2.

Theorem 2 *Let $G = (V, A)$ be the graph representing a gas pipe network where each arc follows a Darcy-Weisbach equation and suppose that V_I and V_B are the subsets of V which respectively contain the internal and boundary vertices. Additionally, let α be a diagonal matrix containing the constant values for each pipe on the arcs. Then for any vector $\mathbf{p}_B$ containing the pressures on the boundary vertices in V_B, there exist unique vectors $\mathbf{p}_I$, $\mathbf{Q}$ and $\mathbf{Q}_E$ such that all the Kirchoff and Darcy-Weisbach equations corresponding to the network are satisfied.*

Unfortunately though, some problems arise when translating the equations to their gas transport counterparts due to the non-linear nature of the Darcy-Weisbach equation. System 7.1 can still easily be converted to system 7.7.

$$\begin{aligned} B_B\mathbf{Q} &= -\mathbf{Q}_E \\ B_I\mathbf{Q} &= 0 \end{aligned} \tag{7.7}$$

Writing out the Darcy-Weisbach equation with the incidence matrix requires the use of the Hadamard product[13], due to the $Q|Q|$ factor. The Hadamard product is the operator used for element-wise multiplication of matrices or vectors of the same size and uses the symbol $\circ$ as shown in equation 7.8.

$$-B^\top \mathbf{p} = -B_I^\top \mathbf{p}_I - B_B^\top \mathbf{p}_B = \alpha\mathbf{Q} \circ |\mathbf{Q}| \tag{7.8}$$

This also means that isolating $\mathbf{Q}$ in a similar manner to $\mathbf{I} = R^{-1}\mathbf{U}$ becomes a problem. For individual Darcy-Weisbach equations, this is easily possible, as was done before in equation 6.18. Combining all equations of this type into an expression with matrices and vectors then requires taking the square root of individual elements in vectors. The expression would look something like equation 7.9, with the important property that the notation of the square root indicates an element-wise square root.

$$\mathbf{Q} = -\text{sgn}(B_I^\top \mathbf{p}_I + B_B^\top \mathbf{p}_B)\sqrt{\alpha^{-1}|B_I^\top \mathbf{p}_I + B_B^\top \mathbf{p}_B|} \tag{7.9}$$

Substituting this in the second equation from 7.7 would result in equation 7.10.

$$B_I(-\text{sgn}(B_I^\top \mathbf{p}_I + B_B^\top \mathbf{p}_B)\sqrt{\alpha^{-1}|B_I^\top \mathbf{p}_I + B_B^\top \mathbf{p}_B|}) = 0 \tag{7.10}$$

Following one of the later steps for the resistive network would mean that this equation is used to express $\mathbf{p}_I$ in terms of $\mathbf{p}_B$ in order to substitute it in an expression for $\mathbf{Q}_E$ based on the first equation of 7.7. Unfortunately though, it doesn't seem possible to isolate the $\mathbf{p}_I$ due to the added complexity of equation 7.10.

However, the fact remains that there are precisely as many equations as there are unknown variables. As for the number of equations, system 7.7 contains m equations, one for each of the m vertices in V and system 7.8 contains n equations for each of the n arcs in A. This means that the total amount of equations is $m+n$. Meanwhile, the unknown variables are split up between the following:

- $\mathbf{p}_I$ is a vector of size $\#V_I$, the amount of internal vertices;
- $\mathbf{Q}$ is a vector of size n;
- $\mathbf{Q}_E$ is a vector of size $\#V_B$, the amount of boundary vertices;

Of course, $\#V_I + \#V_B$ gives the total amount of vertices m, meaning that the total amount of unknown variables is $m + n$ and that there are exactly enough equations for the number of unknown variables. This means that it should be possible to find a solution for $\mathbf{p}_I$, $\mathbf{Q}$ and $\mathbf{Q}_E$, which will be done with the use of numerical methods. As for the uniqueness of the solution, this is expected, since the relation between Δp and Q remains bijective, despite the non-linear behaviour. Namely, for any arc with given properties, every Δp-Q combination is unique.

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7.2. NUMERICAL IMPLEMENTATION

In order to apply the theorem to practical cases of gas pipe networks, numerical methods will be used. The objective is to create an algorithm that shows the values for the volumetric flows and pressures, given the following input arguments:

- The network structure in the form of a graph $G = (V, A)$;
- The set of boundary vertices V_B in order to know what matrices B_I and B_B are;
- A vector $\boldsymbol{\alpha}$ containing the constant values for each pipe segment in the network;
- A vector $\mathbf{p}_B$ containing the pressures on each of the boundary vertices.

The output arguments need to be:

- A vector $\mathbf{Q}$ containing the internal flows going through the network;
- A vector $\mathbf{Q}_E$ containing the external flows going into the network;
- A vector $\mathbf{p}_I$ containing the pressures on the internal vertices with which one can find the pressure losses on each arc.

The output arguments are combined into a single vector $\mathbf{x} = (\mathbf{Q}_I^\top, \mathbf{Q}_E^\top, \mathbf{p}_I^\top)^\top$.

7.2.1. Multivariate Newton-Raphson Method

For this particular problem, finding the values via numerical methods is done with the use of the multivariate Newton-Raphson method[14]. For a function with a single variable, the widely known Newton-Raphson method iteratively approaches a root of the function. Similarly, the multivariate method does the same for a system of functions which use exactly as many unknown variables as there are functions. Therefore, all the relevant equations need to be converted to the form $\mathbf{f}(\mathbf{x}) = 0$. Additionally, the method requires initial values for $\mathbf{x}$. The vector containing these will be called $\mathbf{x}_0$.

After these preparations, the iterative steps will follow. For the regular Newton-Raphson methods with a single function of a single variable, iterative step 7.11 applies.

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)} \tag{7.11}$$

This is as a result from the desired property 7.12.

$$0 = f'(x_i)(x_{i+1} - x_i) + f(x_i) \tag{7.12}$$

For the multivariate Newton-Raphson method, the Jacobian will be used in the place of the regular derivative. Hence, equation 7.12 becomes equation 7.13.

$$0 = J(\mathbf{x}_i)(\mathbf{x}_{i+1} - \mathbf{x}_i) + \mathbf{f}(\mathbf{x}_i) \tag{7.13}$$

Isolating the vector $\mathbf{x}_{i+1}$ from this under the assumption that the Jacobian is invertible then results in iterative step 7.14 for the multivariate method.

$$\mathbf{x}_{i+1} = \mathbf{x}_i - J^{-1}(\mathbf{x}_i)\mathbf{f}(\mathbf{x}_i) \tag{7.14}$$

The iterations will then end if one of the following two conditions is satisfied:

- The norm of $\mathbf{x}_{i+1} - \mathbf{x}_i$ is below a certain small predetermined value $\varepsilon > 0$, meaning that the change between iterations sufficiently small;
- The predetermined amount of iterations is reached in order to not let the programme run for too long.

The uniqueness of the solution is guaranteed when the Jacobian is invertible in every iteration and when all vectors with initial values for the unknown variables lead towards a single outcome. This is the case under normal circumstances, though a definitive proof hasn't been worked on. The further details will focus more on finding the solution, rather than giving much value to the uniqueness of the solution.

7.2.2. Gas Pipe Network

Now this method will be implemented for the gas pipe network. The first step in the programme is to define the graph structure. The computations will require it to be in the form of an incidence matrix. Directly defining the incidence matrix will be a lot of work for large networks, so instead, the MatLab functions `digraph`, `incidence` and `full` will be used. Beforehand, all vertices are labelled with a number and the arcs will be defined via input of its source and terminal vertices, as was described before in section 2.2. To repeat, the `digraph` function requires two input vectors, `s` and `t`, which are respectively the vectors containing all source and terminal for the arcs. For example, let $\mathbf{s} = (1,2,3)^\top$ and $\mathbf{t} = (2,3,1)^\top$, then `G=digraph(s,t)` gives the directed graph G with arcs $\{a_{1,2}, a_{2,3}, a_{3,1}\}$. Applying the `incidence` function to this then gives the desired incidence matrix. However, this function only provides a sparse matrix, so `full` needs to be used in order to turn it into a full matrix B. The size of this matrix is $m \times n$, i.e. rows and columns equal to the number of vertices and arcs respectively. Also required is the set of boundary vertices, since these differ greatly in function from the internal vertices for this method. They will be stored with their labels in a vector $\mathbf{b}$ with k being its length. It is assumed that the labels are in increasing order. A custom function named `sortB(B,b)` then sorts the rows of the incidence matrix in such a way that the ones corresponding to boundary vertices are on the top. The rows for the internal vertices will be placed on the bottom. If $\mathbf{b} = (2,3)^\top$ and B has four rows for instance, then the rows of B will reordered as $(2,3,1,4)^\top$.

Next, the remaining required input arguments need to be defined. These are the diagonal matrix α and the vector $\mathbf{p}_B$. α stores the constant values used in the Darcy-Weisbach equation for each pipe segment on each diagonal element. Alternatively, the matrix could be defined by defining a vector $\boldsymbol{\alpha}$ containing the desired diagonal elements and applying the `diag` function to it. In fact, this is the method used in the eventual programme made for this project. Meanwhile, $\mathbf{p}_B$ stores the pressures on the boundary vertices. Of course, it is important that these values are stored in the correct order. The i^{th} component of $\boldsymbol{\alpha}$ needs to correspond to the i^{th} column of B. The same goes for the order of $\mathbf{p}_B$ in relation to $\mathbf{b}$.

As mentioned, the unknown variables will be stored in the vectors $\mathbf{Q}$, $\mathbf{Q}_E$ and $\mathbf{p}_I$. Since the Newton-Raphson method requires initial values for each of these, the vectors will have random entries between 0.5 and 1.5. The reason why a random entry between those values is chosen and they are not all the same number is to avoid potential cases where the difference between two values cannot be equal to zero in later extensions of the programme.

The vectors $\mathbf{p}_B$ and $\mathbf{p}_I$ will together be stored in a vector $\mathbf{p} = (\mathbf{p}_B^\top, \mathbf{p}_I^\top)^\top$.

Additionally, the single vector will be created containing all the unknown variables. The order used in this case is $\mathbf{x} = (\mathbf{Q}^\top, \mathbf{Q}_E^\top, \mathbf{p}_I^\top)^\top$ as mentioned in section 7.2.

The last step of preparation is to define the two values that would eventually end the algorithm and a vector used as reference in each iteration. The number $\varepsilon > 0$ is the error margin and is chosen very small. Going below this value means that the algorithm gave a sufficiently close result. Additionally, i is the largest amount of iterations for which the algorithm will run. It decreases by one after each iteration and shows the results at that point once it hits zero. $\mathbf{x}_{ref}$ is a vector containing the unknown values as computed in the previous iteration. This means that the entries will be replaced after each iteration. This is done in order to not have all the intermediate steps unnecessarily be stored in the memory.

Because the multivariate Newton-Raphson method specifically finds the roots of functons, the equations 7.7 for Kirchoff's Circuit Law will have to be changed to system 7.15.

$$\begin{aligned} \mathbf{Q}_E + B_B\mathbf{Q} &= 0 \\ B_I\mathbf{Q} &= 0 \end{aligned} \tag{7.15}$$

This could also be written as equation 7.16, for which $\hat{\mathbf{Q}}_E = (\mathbf{Q}_E^\top, 0^\top)^\top$. Note that the zero is a vector of size $\#V_I$ completely filled with zeros. In other words, this could be considered the external flow vector with the non-existent (zero) external influence on the internal vertices included. Additionally, B is the incidence matrix under the assumption that the `sortB` function has already been applied.

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$$\mathbf{f}_0(\mathbf{x}) = \hat{\mathbf{Q}}_E + B\mathbf{Q} = 0 \tag{7.16}$$

Meanwhile, the Darcy-Weisbach equations 7.8 will be converted to system 7.17

$$\begin{aligned} \mathbf{f}_1(\mathbf{x}) = B^\top\mathbf{p} + \alpha\mathbf{Q} \circ |\mathbf{Q}| = \\ B_I^\top\mathbf{p}_I + B_B^\top\mathbf{p}_B + \alpha\mathbf{Q} \circ |\mathbf{Q}| = 0 \end{aligned} \tag{7.17}$$

The general layout of the main programme is shown in the following algorithm.

Input: G, $\boldsymbol{\alpha}$, $\mathbf{p}_B$, $\mathbf{b}$
while $||\mathbf{x} - \mathbf{x}_{ref}|| > \varepsilon$ **and** $i > 0$ **do**
  $\mathbf{f}_0(\texttt{sortB}(B), (\mathbf{x}_{1+n}, \ldots, \mathbf{x}_{n+k})^\top, (\mathbf{x}_1, \ldots, \mathbf{x}_n)^\top)$
  $\mathbf{f}_1(\texttt{sortB}(B), \boldsymbol{\alpha}, \mathbf{p}_B, (\mathbf{x}_{1+n+k}, \ldots, \mathbf{x}_{m+n})^\top, (\mathbf{x}_1, \ldots, \mathbf{x}_n)^\top)$
  $\mathbf{f} = (\mathbf{f}_0^\top, \mathbf{f}_1^\top)^\top$
  $J = ((J_0(\texttt{sortB}(B), \mathbf{p}_B, (\mathbf{x}_{1+n+k}, \ldots, \mathbf{x}_{m+n})^\top))^\top,$
    $(\texttt{blkdiag}(J_{1B}(\boldsymbol{\alpha}, (\mathbf{x}_1, \ldots, \mathbf{x}_n)^\top)), J_{1P}(k, \texttt{sort}(B)))^\top)^\top$
  $\mathbf{x}_{ref} = \mathbf{x}$
  $\mathbf{x} = \mathbf{x}_{ref} - J^{-1}\mathbf{f}$
  $i = i - 1$
end while
Output: $\mathbf{Q}$, $\mathbf{Q}_E$, $\mathbf{p}_I$

To clarify, $(\mathbf{x}_1, \ldots, \mathbf{x}_n)^\top$ is the part of $\mathbf{x}$ used for $\mathbf{Q}_I$, $(\mathbf{x}_{1+n}, \ldots, \mathbf{x}_{n+k})^\top$ is the part used for $\mathbf{Q}_E$ and $(\mathbf{x}_{1+n+k}, \ldots, \mathbf{x}_{m+n})^\top$ is used for $\mathbf{p}_I$.

As can be seen, the algorithm refers to a number of functions, those being $\mathbf{f}_0$, $\mathbf{f}_1$, J_0, J_{1B}, J_{1P} in addition to the previously discussed `sortB` function.

- $\mathbf{f}_0$ uses input arguments B, $\mathbf{Q}_E$ and $\mathbf{Q}$ and returns $\mathbf{f}_0 = B\mathbf{Q} + \hat{\mathbf{Q}}_E$;
- $\mathbf{f}_1$ uses input arguments B, $\boldsymbol{\alpha}$, $\mathbf{p}_B$, $\mathbf{p}_I$ and $\mathbf{Q}$ and returns either $\mathbf{f}_1 = [\quad]$ (a vector of length zero) if B is zero-dimensional (this will be relevant later) or $\mathbf{f}_1 = \boldsymbol{\alpha} \circ \mathbf{Q} \circ |\mathbf{Q}| + B(\mathbf{p}_B^\top, \mathbf{p}_I^\top)^\top$

The Jacobian J is built from several blocks which for now are just J_0, J_{1B} and J_{1P}. J_0 is the block making the first m rows of J. The remaining n rows of the Jacobian will consist of a block diagonal matrix, for now only of a single block, $J_{1,B}$ on the left and a block J_{1P} on the right. In equation 7.18, a visualisation of the positioning of the blocks is shown.

$$J = \begin{bmatrix} \begin{bmatrix} & & \\ & J_0 & \\ & & \end{bmatrix} \\ \begin{bmatrix} & & \\ & J_{1B} & \\ & & \end{bmatrix} \begin{bmatrix} & & \\ & J_{1P} & \\ & & \end{bmatrix} \end{bmatrix} \tag{7.18}$$

J_0 is obtained when the partial derivatives of the Kirchoff equations to the unknown variables are taken. This results in matrix 7.19.

$$J_0 = \begin{bmatrix} \frac{\partial \mathbf{f}_0}{\partial Q_1} & \cdots & \frac{\partial \mathbf{f}_0}{\partial Q_n} & \frac{\partial \mathbf{f}_0}{\partial Q_{E1}} & \cdots & \frac{\partial \mathbf{f}_0}{\partial Q_{Ek}} & \frac{\partial \mathbf{f}_0}{\partial p_{I1}} & \cdots & \frac{\partial \mathbf{f}_0}{\partial p_{I(m-k)}} \end{bmatrix} \tag{7.19}$$

Conveniently, this can be converted to matrix 7.20. Differentiating the Kirchoff equations to $\mathbf{Q}$ leaves B, differentiating to $\mathbf{Q}_E$ leaves an identity matrix of size k only for the equations related to the boundary vertices and differentiating to $\mathbf{p}_I$ gives zeroes, since there are no pressures here. Therefore the parts where the functions are differentiated to $\mathbf{Q}_E$ and $\mathbf{p}_I$ will be implemented via a block diagonal matrix with blocks I_k and 0_{m-k}, where the latter is a square matrix of size $m-k$ filled with zeroes.

$$J_0 = \begin{bmatrix} B & \texttt{blkdiag}(I_k, 0_{m-k}) \end{bmatrix} \tag{7.20}$$

As for the other part of the Jacobian regarding the partial derivatives of the Darcy-Weisbach equations, this is split up in two blocks.

$$J_{1B} = \begin{bmatrix} \frac{\partial \mathbf{f}_1}{\partial Q_1} & \cdots & \frac{\partial \mathbf{f}_1}{\partial Q_n} \end{bmatrix} \tag{7.21}$$

Each Darcy-Weisbach equation is related to exactly one variable of Q, so assuming that B and $\mathbf{Q}$ use the same ordering with the arc labels, this results in a diagonal matrix with elements of the form $2\alpha|Q|$, since the derivative $\frac{d}{dx}(x|x|) = 2|x|$. This is shown in equation 7.22.

$$J_{1B} = 2 \cdot \texttt{diag}(\boldsymbol{\alpha}|\mathbf{Q}|) \tag{7.22}$$

The second block J_{1P} contains zeroes for the first k columns, since the Darcy-Weisbach equations are not influenced by external flows Q_E. The remaining $n-k$ columns are the same as the last $m-k$ columns of $B^\top$, again under the assumption that B is sorted properly.

$$J_{1P} = \begin{bmatrix} \frac{\partial \mathbf{f}_1}{\partial Q_{E1}} & \cdots & \frac{\partial \mathbf{f}_1}{\partial Q_{Ek}} & \frac{\partial \mathbf{f}_1}{\partial p_{I1}} & \cdots & \frac{\partial \mathbf{f}_1}{\partial p_{I(m-k)}} \end{bmatrix} \tag{7.23}$$

In the programme, this can be achieved by simply taking $B^\top$ and making the first k columns filled with zeroes with $J_{1P} = B^\top$, followed by $J_{1P}(:, 1:k) = 0$.

If, for example, the gas pipe network used is represented by the same graph as the one in figure 2.4, then the Jacobian would become matrix 7.24 under the assumption that the boundary vertices are v_1 and v_4. For this example, suppose that $q_{i,j} := \alpha_{i,j}|Q_{i,j}|$.

$$\begin{bmatrix}
-1 & -1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & -1 & 0 & 1 & 0 & 0 \\
1 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
2q_{1,2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 2q_{1,3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 2q_{2,3} & 0 & 0 & 0 & 0 & 0 & -1 & 1 \\
0 & 0 & 0 & 2q_{2,4} & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 2q_{3,4} & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 2q_{4,1} & 0 & 0 & 0 & 0
\end{bmatrix} \tag{7.24}$$

In the programme made for this project, all the eventual values of the known and unknown variables are compiled into two tables: one for all the vertices and one for all the arcs. The results for this example is shown in figure 7.1. Here, all values of $\boldsymbol{\alpha}$ are equal to 1 and the pressures on the boundary vertices v_1 and v_4 are respectively $p_1 = 25$ and $p_4 = 9$. `VType` indicates whether the vertex is an internal or boundary vertex, `EFlow` shows the external flow going into each vertex (is equal to 0 in case of no external influence) and `Iflow` shows the internal flows going over each arc. `AType` becomes relevant in the next subsection and is an indicator for the three different types of component.

```
Vertices =

  4×4 table

    Vertex    VType    Pressure          EFlow
    ______    _____    ________    _________________

      1         1         25        9.65685424949238
      2         0         17                       0
      3         0         17                       0
      4         1          9       -9.65685424949238

VType 0: Internal / VType 1: Boundary

Arcs =

  6×4 table

     Arc      AType    Constant          IFlow
    ______    _____    ________    ____________________

    1    2      1         1            2.82842712474619
    1    3      1         1            2.82842712474619
    2    3      1         1        3.59434997309081e-17
    2    4      1         1            2.82842712474619
    3    4      1         1            2.82842712474619
    4    1      1         1                          -4
```

Figure 7.1: Table with results for example network with α-values 1, $p_1 = 25$ and $p_4 = 9$.

7.2.3. EXPANSION WITH CHECK VALVES & COMPRESSORS

Now it will be explained how the toolbox can be expanded with the other two component types: the check valves and the compressors.

The first step is to split the graph G into three sub-graphs G_1, G_2 and G_3:

- G_1 is the sub-graph for which the n_1 arcs represent pipe segments;
- G_2 is the sub-graph for which the n_2 arcs represent check valves;
- G_3 is the sub-graph for which the n_3 arcs represent compressors.

Consequently, the incidence matrix B need to be split into three sub-matrices B_1, B_2 and B_3 as well which respectively are columns $1, \ldots, n_1$, $1 + n_1, \ldots, n_1 + n_2$ and $1 + n_1 + n_2, \ldots, n$. The dimensions of these sub-matrices are $m \times n_1$, $m \times n_2$ and $m \times n_3$. Therfore, every B in the previous steps exclusively for pipe segments is replaced by a B_1, like in equation 7.25. The equations corresponding to Kirchoff's Current Law still use the full matrix B, so these remain unchanged.

$$\begin{aligned} \mathbf{f}_1(\mathbf{x}) = B_1^\top \mathbf{p} + \alpha \mathbf{Q}_1 \circ |\mathbf{Q}_1| = \\ B_{1I}^\top \mathbf{p}_I + B_{1B}^\top \mathbf{p}_B + \alpha \mathbf{Q}_1 \circ |\mathbf{Q}_1| = 0 \end{aligned} \tag{7.25}$$

Additionally, the vector $\mathbf{Q}$ of internal volumetric flows is split up into three parts Q_1, Q_2 and Q_3 as well, each corresponding to one of the component types.

The equations for the check valves and compressors now need to be put in the correct forms $\mathbf{f}(\mathbf{x}) = 0$. For the check valves, equation 4.9 will first be converted to a form with vectors and matrices. This is done in equation 7.26. For simplicity, it is assumed that β is the same large constant for every equation.

$$\mathbf{Q}_2 = -\beta(1 - \operatorname{sgn}(B_2^\top \mathbf{p})) \circ (B_2^\top \mathbf{p}) \tag{7.26}$$

From this, vector function 7.27 can be obtained for which the root will be found.

$$\mathbf{f}_2(\mathbf{x}) = \mathbf{Q} + \beta(1 - \operatorname{sgn}(B_2^\top \mathbf{p})) \circ (B_2^\top \mathbf{p}) = 0 \tag{7.27}$$

As for the compressors, the version with the fixed pressure increase will be used. Equation 7.28 shows the system of compressor equations with vectors and matrices, where $\boldsymbol{\gamma}$ is a vector containing the fixed pressure increases on each arc. Naturally, it is important that it has the correct value for the cases where a compressor exists between two boundary vertices for which the pressures are known. Equation 7.29 is then the function for which the root will be found.

$$B_3 \mathbf{p} = \boldsymbol{\gamma} \tag{7.28}$$

$$\mathbf{f}_3 = B_3 \mathbf{p} - \boldsymbol{\gamma} = 0 \tag{7.29}$$

When including these extra component types, the Jacobian will now contain a large block in the bottom left which has the form of a block diagonal matrix which contains the partial derivatives to the volumetric flows. Each block is used for a single type of equation, i.e. block J_{1B} is for equation type 1 (Darcy-Weisbach equations), block J_{2B} is for type 2 (Check valves) and block J_{3B} is for type 3 (Compressors). All the blocks are also diagonal matrices, due to the equations for arcs only containing a single and unique volumetric flow. Only the Kirchoff equations are not represented in this block diagonal matrix. Those have their own blocks at the top.

In the bottom right are three vertically stacked blocks J_{1P}, J_{2P} and J_{3P} which contain the all derivatives of respectively $\mathbf{f}_1$, $\mathbf{f}_2$ and $\mathbf{f}_3$ to both the external flows and the internal pressures.

The new full Jacobian with blocks for the expanded situation is shown in equation 7.30.

$$J = \begin{bmatrix} \begin{bmatrix} & & J_0 & & \end{bmatrix} \\ \begin{bmatrix} \begin{bmatrix} J_{1B} \end{bmatrix} & & 0 \\ & \begin{bmatrix} J_{2B} \end{bmatrix} & \\ 0 & & \begin{bmatrix} J_{3B} \end{bmatrix} \end{bmatrix} & \begin{bmatrix} \begin{bmatrix} J_{1P} \end{bmatrix} \\ \begin{bmatrix} J_{2P} \end{bmatrix} \\ \begin{bmatrix} J_{3P} \end{bmatrix} \end{bmatrix} \end{bmatrix} \tag{7.30}$$

Blocks J_0, J_{1B} and J_{1P} remain the same as before. Deriving the check valve equations to each of the individual internal flows gives an identity matrix of size n_2 for the block, since every derivative becomes one.

$$J_{2B} = I_{n_2} \tag{7.31}$$

Because the compressor equations do not contain a variable of internal flow, all their derivatives to the flows become zero, thus the following applies.

$$J_{3B} = 0_{n_3} \tag{7.32}$$

The blocks J_{2P} will contain the partial derivatives of $\mathbf{f}_2$ to $\mathbf{Q}_{2E}$ and p_I. The ones to $\mathbf{Q}_{2E}$ all become zero, since none of these variables exist in the equation. Differentiating an individual check valve equation to the former pressure in Δp gives $\beta(1 + \text{sgn}(\Delta p))$, whereas the latter pressure in Δp gives the negative of it. Therefore, the part of the incidence matrix with respect to the internal vertices (i.e. B_{2I}) is important in order to indicate which of these two need to be applied. The part for the boundary vertices is of no importance, so the top rows of B_2 will be filled with zeroes in order to implement the partial derivatives to the external flows. This variant of B_2 will be notated as $\hat{B}_2$. Also note that $\mathbf{p}$ is the pressure vector with the pressures on the boundary vertices first and then the ones on the internal vertices, i.e. $\mathbf{p} = (\mathbf{p}_B^\top, \mathbf{p}_I^\top)^\top$. All in all, block J_{2P} will be built according to equation 7.33.

$$J_{2P} = \mathtt{diag}(1 - \mathrm{sgn}(B_2^\top \mathbf{p}))\beta \hat{B}_2^\top \tag{7.33}$$

Block J_{3P} becomes fairly simple. Again, there are no influences from external flows, so the derivatives to those become zero. The derivatives to the internal pressures are either -1 or $+1$, depending on whether the vertex has the role of a source vertex or a terminal vertex. Like with block J_{2P}, only the rows of B_3 that are related to the internal vertices (i.e. B_{3I}) are relevant, so the other rows will be filled with zeroes to accommodate for the partial derivatives to the external flows. Similarly to $\hat{B}_2$, this matrix will be called $\hat{B}_3$. Therefore, block J_{3P} will be built according to equation 7.34.

$$J_{3P} = \hat{B}_3^\top \tag{7.34}$$

In order to incorporate all the new blocks, the previous algorithm be modified as follows:

Input: G_1, G_2, G_3, $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, $\boldsymbol{\gamma}$, $\mathbf{p}_B$, $\mathbf{b}$
while $||\mathbf{x} - \mathbf{x}_{ref}|| > \varepsilon$ **and** $i > 0$ **do**
 $\mathbf{f}_0(\mathtt{sortB}(B), (\mathbf{x}_{1+n}, \ldots, \mathbf{x}_{n+k})^\top, (\mathbf{x}_1, \ldots, \mathbf{x}_n)^\top)$
 $\mathbf{f}_1(\mathtt{sortB}(B_1), \boldsymbol{\alpha}, \mathbf{p}_B, (\mathbf{x}_{1+n+k}, \ldots, \mathbf{x}_{m+n})^\top, (\mathbf{x}_1, \ldots, \mathbf{x}_{n_1})^\top)$
 $\mathbf{f}_2(\mathtt{sortB}(B_2), \boldsymbol{\beta}, \mathbf{p}_B, (\mathbf{x}_{1+n+k}, \ldots, \mathbf{x}_{m+n})^\top, (\mathbf{x}_{1+n_1}, \ldots, \mathbf{x}_{n_1+n_2})^\top)$
 $\mathbf{f}_3(\mathtt{sortB}(B_3), \boldsymbol{\gamma}, \mathbf{p}_B, (\mathbf{x}_{1+n+k}, \ldots, \mathbf{x}_{m+n})^\top, (\mathbf{x}_{1+n_1+n_2}, \ldots, \mathbf{x}_n)^\top)$
 $\mathbf{f} = (\mathbf{f}_0^\top, \mathbf{f}_1^\top, \mathbf{f}_2^\top, \mathbf{f}_3^\top)^\top$
 $J = (J_0^\top, (\mathtt{blkdiag}(J_{1B}, J_{2B}, J_{3B}), (J_{1P}^\top, J_{2P}^\top, J_{3P}^\top)^\top)^\top)^\top$
 $\mathbf{x}_{ref} = \mathbf{x}$
 $\mathbf{x} = \mathbf{x}_{ref} - J^{-1}\mathbf{f}$
 $i = i - 1$
end while
Output: $\mathbf{Q}$, $\mathbf{Q}_E$, $\mathbf{p}_I$

Note that the part of what was previously $\mathbf{Q}$ in the vector $\mathbf{x}$ is now split up between $\mathbf{Q}_1 = (\mathbf{x}_1, \ldots, \mathbf{x}_{n_1})^\top$, $\mathbf{Q}_2 = (\mathbf{x}_{1+n_1}, \ldots, \mathbf{x}_{n_1+n_2})^\top$ and $\mathbf{Q}_3 = (\mathbf{x}_{1+n_1+n_2}, \ldots, \mathbf{x}_n)^\top$.

Any potential further additions for components require extra lines for new functions $\mathbf{f}_4, \mathbf{f}_5, \ldots$, more square matrices $J_{4B}, J_{5B}, \ldots$ in the block diagonal matrix and more matrices $J_{5P}, J_{6P}, \ldots$ to add below J_{1P}, J_{2P} and J_{3P} in the Jacobian. For example, the other equations from chapters 3, 4 and 5, (e.g. pipe segment with squared pressures, regular valves or compressors with set pressure on the terminal vertex) can still be implemented in future research. Of course, all of these additions need to be properly made according to a correct equation corresponding to the behaviour of the component.

7.3. Gasunie Gas Transport Network Example

Lastly, the programme will be applied to an example of the Dutch gas tranport network used by the Gasunie[15]. An schematic image of it is shown in figure 7.2.

Figure 7.2: Gasunie gas transport network[15].

The example that will be looked at is the approximate structure of the main part of the high pressure grid that is used for the H-gas type, schematically visualised in figure 7.3[2].

Do note that this section will not go into much detail about the specifics of the network itself and how everything realistically works in practice. This is merely an application of the programme on a larger network. It will be assumed that every arc represents a pipe segment with constant value $\alpha = 1$. The set of boundary vertices is assumed to be $V_B = \{v_2, v_6, v_7, v_{12}, v_{18}, v_{19}, v_{20}, v_{22}, v_{23}\}$, which means that $\mathbf{b} = (2, 6, 7, 12, 18, 19, 20, 22, 23)^\top$. They are marked in blue in the visualisation of the graph. As for the pressures on these vertices, they are randomly generated. Starting with v_2, this will be set to $p_{B1} = p_2 = 1000$. Every pressure after this in the list takes a random value between 90% and 100% of the previous one in the list, so $0.9p_{B1} \leq p_{B2} = p_6 \leq p_{B1}$, $0.9p_{B2} \leq p_{B3} = p_7 \leq p_{B2}$, etcetera. In other words, the iterative step $p_{B(i+1)} = p_{Bi} \cdot (0.9 + 0.1 \cdot \texttt{rand()})$ is used for $i \in \{1, \ldots, k-1\}$. Here, the function `rand()` returns a random value between 0 and 1.

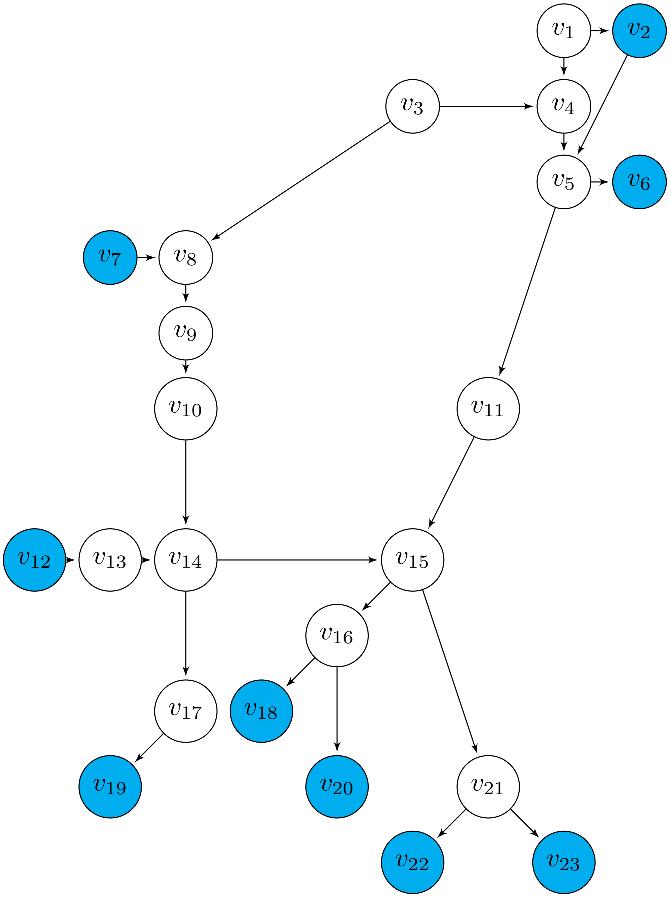

Figure 7.3: Graph of the general structure of the H-gas part of the network.

The result of one of these examples is shown in figure 7.4. Figure 7.4a shows the values of the variables related to the vertices, whereas figure 7.4b shows the ones related to the arcs.

Vertex	VType	Pressure	EFlow
1	0	959.359394476601	0
2	1	1000	15.4030096883861
3	0	883.885782779936	0
4	0	918.718788953203	0
5	0	918.495013484448	0
6	1	918.390778828242	-0.3228539239443
7	1	848.588719638137	-0.681217269696697
8	0	849.05277660667	0
9	0	821.796733114059	0
10	0	794.540689621448	0
11	0	834.255540565685	0
12	1	799.13866046447	3.99086546601317
13	0	783.211653296653	0
14	0	767.284646128837	0
15	0	750.016067646921	0
16	0	725.13235494314	0
17	0	741.721008227206	0
18	1	723.192871911687	-1.39265323446052
19	1	716.157370325576	-5.05604963401571
20	1	712.203264223853	-3.59570448164017
21	0	680.370423058008	0
22	1	675.942438696072	-2.10427763423363
23	1	641.418856980324	-6.24111897640827

(a) Table with vertex details.

Arc		AType	Constant	IFlow
1	2	1	1	-6.37499847242324
1	4	1	1	6.37499847242324
2	5	1	1	9.0280112159629
3	4	1	1	-5.90194935366836
3	8	1	1	5.90194935366836
4	5	1	1	0.473049118754875
5	6	1	1	0.3228539239443
5	11	1	1	9.17820641077348
7	8	1	1	-0.681217269696697
8	9	1	1	5.22073208397167
9	10	1	1	5.22073208397167
10	14	1	1	5.22073208397167
11	15	1	1	9.17820641077348
12	13	1	1	3.99086546601317
13	14	1	1	3.99086546601317
14	15	1	1	4.15554791596912
14	17	1	1	5.05604963401571
15	16	1	1	4.98835771610069
15	21	1	1	8.34539661064191
16	18	1	1	1.39265323446052
16	20	1	1	3.59570448164017
17	19	1	1	5.05604963401571
21	22	1	1	2.10427763423363
21	23	1	1	6.24111897640827

(b) Table with arc details.

Figure 7.4: Tables with variable details for the Gasunie network example.

8 Conclusion

The final conclusions of the research will be given as the answers to the three questions posed in the introduction in between a summary of what has been discussed.

The first thing done in the report was to repeat some of the known terminology and theory about graphs. Additionally, the basic notations used in the rest of the report are given at this point. This includes notations and definitions for both electric networks and gas transport networks. Afterwards, two methods for graph representation were detailed. The first of these was the incidence matrix B which is filled with -1, 0 or 1 on all the elements, depending on the role a vertex plays on an arc. The second method was done with the use of two vectors $\mathbf{s}$ and $\mathbf{t}$, these being the vectors respectively containing the source and terminal vertices for all the arcs. The incidence matrix would mostly be used for computations, whereas the method with the two vectors is used to define graphs for numerical implementations. Aside from discussing this familiar theory, Kirchoff's Circuit Laws are detailed. First was the Current Law and secondly the Voltage Law. Especially the former would be used a lot later on in the research.

Three types of components would be discussed in the next few chapters, these being the regular pipe segments, the valves and the compressors. They would be compared to their counterparts in the electric networks. This is related to the first research question that was posed:

> *What are the best ways to describe the relations between pressure loss and volumetric flow for different types of components in a gas transport network?*

To answer this question, the three component types are separately considered For the pipe segments, the counterpart is the wire or the resistor. They follow the Darcy-Weisbach equation $\Delta p = \alpha Q|Q|$, which has a form very similar to the equation for Ohm's Law. Alternatively, a variant with a difference between squared pressures can be used instead of the Δp, but this makes computations significantly more complicated. Secondly, two types of valves are considered. One is the regular valve which allows any flow when it is opened or no flow when it is closed. The other is the check valve which only allows gas to move in a single direction. Flow that would normally go in the opposite direction would therefore be halted, meaning that there exists no flow on that arc. The corresponding equations are $\beta\Delta p + (1-\beta)Q = 0$ for the regular valve and $Q = (1+\text{sgn}(\Delta p))\beta\Delta p$ for the check valves. The counterparts in the electric network are respectively the on/off switch and the diode. Additionally, the mixing station was discussed and it was explained how it is a combination of pipe segments and check valves. Lastly, the compressor was discussed with one type being a compressor where the terminal vertex on the arc has its pressure set to a fixed value and the other type being a compressor where the pressure is increased with a fixed value. The corresponding equation is either $p_j = \hat{\gamma}$ or $\Delta p = \gamma$ and its counterpart is the battery.

The next part of the report is fully dedicated to the second research question:

What are the methods for finding the relation between the constant values in a configuration of multiple pipe segments and the constant values for a reduced and equivalent configuration?

The first result for this was regarding the series connection, which goes in a similar way to the method in the electric network, i.e. the constant in the reduced configuration is the sum of the constants in the original configuration. Next was the parallel connection which is slightly different from the method in electric networks. There exists a relation of $\frac{1}{\sqrt{\alpha}}$ being equal to the sum of individual terms $\frac{1}{\sqrt{\alpha}_i}$. The third topic that was discussed here was the Δ/Y transform, although that eventually did not lead to a similar method for the gas pipe network due to what seemed to be inconsistencies with the values of the other variables in the network and the constants still being dependent on them.

The final chapter discussed the uniqueness theorem for resistive networks and how it translates to the gas pipe network. Even though the proof of the uniqueness theorem could not be replicated for the gas pipe network, it is still fairly safe to say that any solution found is a unique one. This lead to the answering of the third research question:

How can the uniqueness theorem be applied for practical gas transport cases with the help of numerical methods?

The uniqueness theorem can be applied with the use of the multivariate Newton-Raphson method. Every equation (Kirchoff, Darcy-Weisbach, Check Valve, Compressor) first needs to be put in the form $\mathbf{f}(\mathbf{x}) = 0$ Then the Jacobian is constructed with a number of blocks related to the equation type and the variable it is differentiated to. An iterative step is then repeated in order to approach the solution for the vector $\mathbf{x}$ containing the unknown variables. In the end, the method is applied in an example of a large gas transport network.

9
Discussion

Lastly, there will be a discussion about a number of things during the project that did not go as planned, lack accuracy or could be looked at in further research.

First of all, there might be better and more accurate options for the equations corresponding to the pipe segments or the check valves. For example, the equation with the squared pressures could potentially give more accurate results, but it is more difficult to do computations with. As for the check valves, there probably is a better equation, but the one used in the end turned out to work the best in the programme. It was attempted to create an equation which more accurately described the desired behaviour, but this caused the occasional failure when implemented in the programme. It might be possible to solve this problem, but this would have to be done during further research. In particular, it might be a good idea to look into finding an equation that is more closely related to the Shockley diode equation. In the end, the equation that was used was considered to be sufficiently good in order to obtain proper results.

Another issue came with the Δ/Y transform. The methods for the resistive network appeared to not be applicable to the gas pipe network due to inconsistencies with the values of the other variables. Unfortunately, this took up a lot of time without getting a satisfying conclusion. Perhaps there exists a solution for this problem, but it was not possible to find one during this research. It is also possible that some inaccuracies or errors in the calculations caused the inconsistencies, so it is strongly suggested to take another look at this at a later point.

Regarding the numerical implementations, there might be methods to reduce the work time, although the programme already seemed to work quite fast in its current state. For future research the main focus should be on implementing more equation types for different components in order to create a more complete toolbox. The only downside of this is that the programme will be using more and more separate functions, which means that working with it becomes a lot less convenient and more complicated.

A final improvement that could have been made was to look more into the details of the Dutch gas transport network in order to give a more informed and more realistic example. The current example is far from a legitimate case and right now the only purpose was to apply the programme to a large network and to see how well it would work for that. The programme still gave a very quick result, so in the end, at least this application turned out to be a success.

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