General formulation of equivalent moment factor for elastic lateral torsional buckling of slender rectangular sections and I-sections

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Abstract

In the past decades, great progress has been made in analyzing lateral torsional buckling of slender beams. The phenomena has been accurately described by differential equations, closed form solutions are available for specific cases and the solution for any load and any boundary condition can be obtained by finite element analysis. Timber and steel design standards provide a procedure based on equivalent moment factors. With this procedure, beams can be designed straightforwardly. However, modern designers continue to push the envelope and more irregular load patterns are found, on which the design standards do not provide solutions. Consequently, designers are forced to determine the equivalent moment factors based on case-specific literature and/or conservative assumptions. Unfortunately, this makes many challenging modern designs uneconomical. Furthermore, significant inconsistencies between the different design procedures are found. For that purpose, this paper proposes a solution in the form of a general formulation to determine equivalent moment factors for any loading on a single-span beam for both free and restrained lateral bending and/or warping at the supports, for both I-sections and rectangular slender sections loaded in the shear center. It is shown that the obtained moment factors are accurate and in good agreement with design standards and literature, and a wide range of irregular load patterns is considered.

Keywords: Elastic laterial torsional buckling, moment equivalent factor, moment gradient factor, steel beams, timber beams, analytic energy approach

1. Introduction

The occurrence of elastic lateral torsional buckling (abbreviated as LTB) means a loss of structural stability and therefore, it is considered as an ultimate limit state failure mode that has to be addressed during the design of especially slender long-span laterally unsupported beams. LTB is the rotation and translation of transversely loaded beams out of plane as a result of buckling of the compressive part of the cross-section due to bending. According to Lindner [1], the load level for which lateral torsional buckling takes place, called the critical load, depends on beam geometry, boundary conditions (rotational and translational restraints at the supports), distribution of the loading over the length of the beam, location of the loading within the cross-section, material properties and, as discussed by Pi and Trahair [2], pre-buckling deformations. This paper focuses on the influence of the load distribution and boundary conditions on the elastic critical load via the so-called *equivalent moment factor*, also called the moment gradient correction factor, for symmetrical beams loaded at the shear centre by uniform loads, point loads and bending moments at the supports (e.g. as shown in Figure 4) and supported by fork supports, with the possibility to prevent warping as an additional feature. The equivalent moment factor is used to transfer any loading situation into an equivalent simply supported beam that is loaded by a uniform bending moment, for which the critical load can be easily found. I-sections and rectangular slender sections, which are often made from steel and timber respectively, are considered.

During the last few decades, research on lateral torsional buckling concentrated on finite element formulations of advanced problems, due to the increasing complexity of solving the governing differential equations, as described by Timoshenko and Gere [3], for non-standard loading cases. As a result of the increasing application of computer programs in the design of structures, a clear understanding of buckling phenomena is becoming more relevant to make optimal use of the structural capacity. In this study, an analytic approach based on the principle of conservation of energy is followed, leading to analytic expressions which can be used by the designer to obtain more insight in the structural behaviour regarding lateral torsional buckling. Although analytic expressions might not be as accurate as numerical analyses, the simplicity of these expressions offers the possibility to quickly converge to the optimal design parameters of the considered structure, which is especially relevant in early design stages. However, the current design codes (such as EN 1993-1-1 [4], EN 1995-

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1-1 [5], AS-4100 [6] and ANSI/AISC 360-16 [7]) only offer solutions for basic load cases (e.g. mid-span point loads, uniformly distributed loads, clamped beams) and simplified general formulations which might lead to unnoticed significant errors or over-conservative results, hence creating the need for a general accurate analytic approach, considering all load combinations within a predefined framework, to evaluate lateral torsional buckling. At the moment, engineers are often referred to case-specific literature and are making use of rules of thumb (e.g. taking the effective buckling length equal to the distance between zero-moment-points, also known as points of contraflexure). This paper describes a general closed-form analytic approach to obtain the equivalent moment factor, which is one of the parameters influencing the total buckling behaviour, without over-simplifying the problem. This closed-form expression can be directly used by the designer or implemented in structural analysis software to incorporate design check procedures for lateral torsional buckling.

First, the current design codes are summarized to show the design procedures, followed by a review of the relevant literature. Significant inconsistencies between between the different design codes and approaches in literature are found. Subsequently, the theoretical framework and the general theory are presented and applied to some specific cases. Finally, the general formulation is compared with literature, showing good agreement, and the limitations of the approach are discussed.

2. Design code procedures

The critical moment regarding lateral torsional buckling for double symmetric beams loaded in the shear centre is found by Timoshenko and Gere [3]:

$$M_{cr} = C_1 \frac{\pi^2 EI_z}{(kL)^2} \sqrt{\left(\frac{k}{k_w}\right)^2 \frac{I_w}{I_z} + \frac{(kL)^2 GI_t}{\pi^2 EI_z}} \tag{1}$$

in which coefficient C_1 is the equivalent moment factor accounting for the non-uniform moment distribution. The influence of the support conditions on the length over which the beam buckles are considered by the lateral bending coefficient k and the warping coefficient k_w. For free lateral bending and/or warping at the supports, the values are equal to 1 and for prevented lateral bending and/or warping at the supports, the values become 0.5 (equivalent to a column that is clamped on both sides). It is noted that in practice, complete prevention of warping at the supports is unrealistic and therefore, prevented warping is merely interpreted as a theoretical limit case. The fork support as is shown in Figure 2b offers both free lateral bending and warping. For I-sections, the moment resistance is a combination of warping and uniform (Saint-Venant) torsion. For rectangular slender sections, the effects of warping tend to be negligible, as discussed by Trahair [8] and Chajes [9], which reduces Eq. 1 to

$$M_{cr} = C_1 \frac{\pi}{kL} \sqrt{EI_z GI_t} \tag{2}$$

Table 1: Equivalent moment factors according to EC3 [4]

Load case	C_1
Uniform moment ($\psi = 1$)	1.00
Linear moment from M to 0 ($\psi = 0$)	1.75
Linear moment from M to -M ($\psi = -1$)	2.30
Mid-span point load	1.35
Point loads on 0.25L and 0.75L	1.04
Uniformly distributed load	1.13

Eq. 1 is used in steel standards and Eq. 2 is used in timber standards, as most of the applied timber sections are rectangular and slender. In fact, lateral torsional buckling of a slender rectangular section follows the same expressions as an I-section with neglected warping. Hence, this study treats rectangular slender sections as if they are I-sections to generalize the approach. In this section, the calculation of the critical moment for lateral torsional buckling according to several design standards is elaborated.

2.1. Eurocode 3 for steel

Lateral torsional buckling of double symmetric steel beams is considered in Eurocode EN 1993-1-1 [4] for steel structures as an ultimate limit state. For a beam loaded in the shear centre, EC3 gives for the critical bending moment

$$M_{cr} = C_1 \frac{\pi}{L_b} \sqrt{EI_z GI_t} \sqrt{1 + \frac{\pi^2}{L_b^2} \frac{EI_w}{GI_t}} \tag{3}$$

Eqs. 1 and 3 are identical when k and k_w have the same value and the buckling length L_b is set to kL. Using the approach of EC3, it is not possible to have different values for the lateral bending and warping coefficients. The equivalent moment factor C_1 is prescribed for basic load cases based on the work of Gardner and Nethercot [10] and Trahair [8], as shown by Table 1. For a simply supported beam loaded at the supports by a moment M and ψM , with $-1 \le \psi \le 1$, EC3 gives an analytic expression: $C_1 = 1.75 - 1.05\psi + 0.3\psi^2 \le 2.3$. Furthermore, EC3 gives design diagrams for obtaining C_1 for the loading cases with support moments combined with uniformly distributed loads or mid-span point loads, based on the work of Bijlaard and Steenbergen [11].

2.2. Eurocode 5 for timber

The Eurocode EN 1995-1-1 [5] for timber structures provides guidelines for transversely loaded rectangular slender sections, for which warping is neglected. The critical moment is given by

$$M_{cr} = \frac{\pi}{L_{eff}} \sqrt{EI_z GI_t} \tag{4}$$

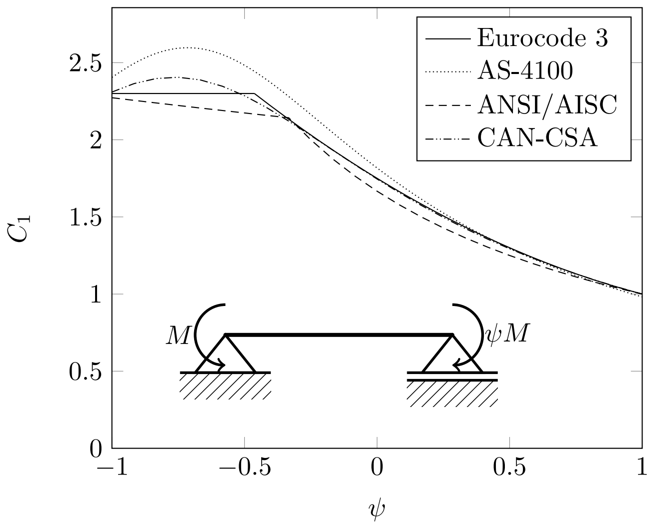


Figure 1: C_1 as function of β for the different standards

in which L_{eff} is the effective buckling length. For a lateral bending and warping free beam, such that $k = k_w = 1$, the effective length is equal to $1/C_1$ and hence, Eq. 4 is equal to Eq. 2. The current version of EC5 only gives the effective length for 3 cases: for uniform moments 1.00, for uniformly distributed loads 0.90 and for mid-span point loads 0.80. For other loadcases, the designer is referred to literature.

2.3. Australian Steel Standards

The Australian Steel Standards AS-4100 [6] make use of Eq. 1. Instead of equivalent moment factors for specific load-cases, the AS-4100 provides a general expression for beams that are laterally bending and warping free:

$$C_1 = \frac{1.7M_{max}}{\sqrt{M_a^2 + M_b^2 + M_c^2}} \tag{5}$$

where M_{max} is the maximum absolute moment within the span and M_a, M_b and M_c are the absolute values of the moments at respectively $L/4$, $L/2$ and $3L/4$ of the span.

2.4. American Standards

Both the American Steel Standard (ANSI/AISC 360-16 [7]) and the Timber Standard (AFPA-TR14 [12]) give the same closed-form expression to calculate the equivalent moment factor for a given moment distribution, based on the work of Kirby and Nethercot [13].

$$C_1 = \frac{12.5M_{max}}{2.5M_{max} + 3M_a + 4M_b + 3M_c} \tag{6}$$

2.5. Canadian Standards

The Canadian Steel Standard CAN-CSA S16-14 [14] provides the following closed-form expression for C_1:

$$C_1 = \frac{4M_{max}}{\sqrt{M_{max}^2 + 4M_a^2 + 7M_b^2 + 4M_c^2}} \leq 2.5 \tag{7}$$

Figure 1 compares the expressions of the considered design standards for a linear moment gradient on a simply supported beam. The standards are in correspondence with each other, although differences can be significant. According to Sahraei et al. [15], the Australian standard delivers, with respect to finite element analysis, the most accurate solution for basic load cases.

3. Literature review

Lateral torsional buckling of beams can be studied in two ways, as mentioned by Pi and Trahair [16]. Firstly, by considering the nonlinear differential equations, as posed by Timoshenko and Gere [3], which can be solved in closed form for a few simplistic cases. For more general cases finite integrals, series solution, finite differences, numerical integration and finite element methods can be used. Secondly, an energy-conservation based approach can be followed: the work done by the load during buckling must be equal to the increment in strain energy. Using approximate deformation shapes, the buckling load can be determined. The accuracy of energy-based methods is highly depending on the applied deformation shape.

Using an energy-based approach, Chajes [9] derived expressions for lateral-torsional buckling of simply or warping restraining supported I-sections for two load-cases: uniform bending and mid-span point loads. The books of Kirby and Nethercot [13] and Trahair [8] comprehend a more elaborate collection of load cases based on both energy approaches and finite element analyses. Trahair described amongst others simply supported and restrained beams loaded by one point load at arbitrary location, beams loaded by mid-span point loads combined with support bending moments and beams loaded by a uniformly distributed load combined with support bending moments. Although this set of load-cases is quite elaborate, only basic load-cases are considered and there is no general expression fulfilling all the cases. This study aims to find an expression that is approximately valid for all load-cases, without over-simplifying the problem.

During the years, lots of experimental data have been obtained regarding the equivalent moment factor (for example by Xiao et al. [17] and Burow et al. [18]). However, besides verification of theories, the use of experimental data is rather limited, since information is only obtained for specific cases. Both [17, 18] show that the experimental and theoretical values for the critical buckling moments are close to each other for the considered cases and hence, this study will not further consider experimental values.

Using both the Bubnov-Galerkin method and the finite element method, Lim et al. [19] found an expression to capture the equivalent moment factor for linear moment gradients, as defined in Figure 1, for elastic lateral torsional buckling of I-sections:

$$C_{1,Lim} = \frac{2}{\sqrt{S_1}\sqrt{(1+\psi)^2 + S_2(1-\psi)^2}} \tag{8}$$

Table 2: S_1 and S_2 values from Lim et al. [19]

k	k_w	S_1	S_2
1.0	1.0	1.00	0.16
0.5	0.5	1.00	0.18
1.0	0.5	0.80	0.10

in which S_1 and S_2 are coefficients depending on the support conditions and are given in Table 2 for different lateral bending and warping restraints k and k_w respectively.

Serna et al. [20] studied C_1 more in depth with the aid of a finite difference approach. Tables and graphs are given for a wide variety of cases (linear moments and uniformly distributed load or a concentrated load together with one or two support moments), loaded in the shear centre. They found that coefficient C_1 is slightly dependent on the length of span, following the observations of Nethercot and Rockey [21]. Based on this elaborate analysis, Serna et al. proposed an improved expression for $k = k_w = 1$ which governs the studied load cases accurately and conservatively, without taking into account the length dependency:

$$C_{1,Serna} = \sqrt{\frac{35M_{max}^2}{M_{max}^2 + 9M_a^2 + 16M_b^2 + 9M_c^2}} \tag{9}$$

Also following the finite difference approach, Suryoatmono and Ho [22] found equivalent moment factors for beams under uniform loading and one or two support moments. Furthermore, they showed that significant differences occur when using the general formulas from design standards, leading to over-conservative designs. Sahraei et al. [15] considered elastic lateral torsional buckling of rectangular slender wooden beams and presented a simplified expression to find the equivalent moment factor for a wide variety of load cases:

$$M_{cr,Sahraei} = C_r C_b C_L C_p \sqrt{EI_z GI_t} \tag{10}$$

in which

- C_r accounts for partial twist restraint at beam-ends, meaning that the rotation out of plane is not fully fixed at the supports.
- C_b is the equivalent moment factor accounting for non-uniform load distributions, defined as C_1 in the Eurocode 3 for steel [4].
- C_L considers the influence of the load-location within the cross-section with reference to the shear centre.
- C_p is a coefficient that accounts for pre-buckling deformations (the so-called second order effects).

In their study, Sahraei et al. derived detailed expressions for C_r, C_L and C_p. For C_b, the equivalent moment factor, the expression of the Australian Standard (Eq. 5) is adopted, as this expression compared best to finite element analyses of standard load cases. However, it is not known if Eq. 5 performs sufficiently accurate for more irregular loading cases. Furthermore, the strength of a simplistic expression becomes questionable when more complex cases are studied, since these expressions are most likely over-simplifications of the physical problem and might not be accurate. The aim of this study is to find a more generally applicable closed-form expression for the equivalent moment factor, which subsequently can be implemented in the study of Sahraei et al. to further improve the accuracy of Eq. 10. Furthermore, this closed-form expression can be used in the design to consider irregular load patterns, without the need to perform an elaborate finite element analysis or study case-specific literature.

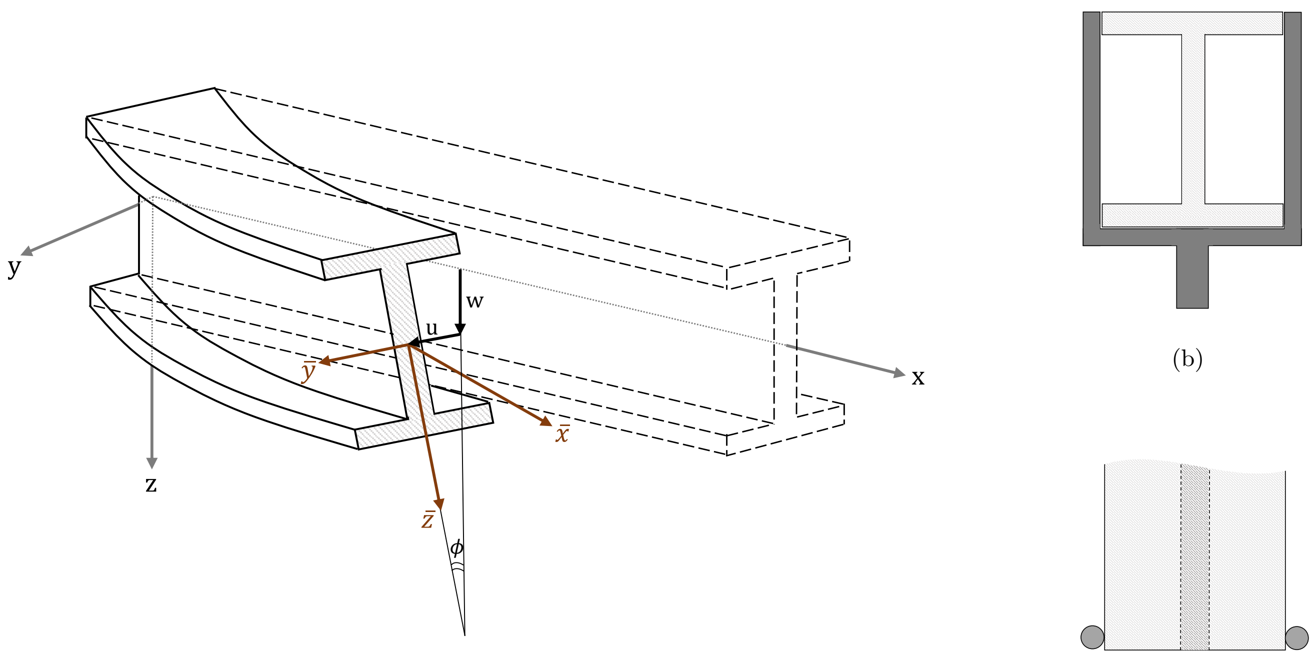
4. Theoretical framework

The beam is considered in two configurations: the undeformed configuration (x, y and z) and the deformed configuration ($\bar{x}$, $\bar{y}$ and $\bar{z}$) just after the occurrence of lateral torsional buckling with displacements u, w and ϕ. Both of the configurations are defined in Figure 2a. For the purpose of this paper, the following key assumptions are made:

1. The beam is prismatic with an I-section. The theory also holds for slender rectangular sections, by implementing $I_w \approx 0$ in Eq. 1. Because only double symmetric sections are studied, bending moments around the different axes are uncoupled.
2. The Euler-Bernoulli beam theory is applied, meaning that plane sections remain plane and shear deformations are not considered.
3. The loading acts transversely in the shear centre of the beam, not causing any additional torsional loading on the beam. No lateral loading is included.
4. Displacements are assumed to be small such that first order approximations of the deformed configuration are valid and pre-buckling deformations can be neglected.
5. Elastic material behaviour is assumed, meaning that no energy dissipation can take place.
6. The considered beam is supported on both sides by fork supports, as shown in Figures 2b and 2c. The fork supports prevent lateral displacements u and twisting ϕ and allow for lateral bending. In this study, the possibility to prevent warping ($\phi' = 0$) at the supports is also included. However, both of the supports should either have warping restrained or allowed. In the span, the beam is laterally unsupported.
7. The beam is not loaded in axial direction.

5. Derivation of general formulation

The derivation of a general formulation for the equivalent moment factor C_1 is based on the energy approach of



(a) (b) (c)

Figure 2: (a) Definitions of the coordinate-systems in undeformed (x, y and z) and deformed configuration (post-buckling, $\bar{x}$, $\bar{y}$ and $\bar{z}$) with displacements u, w and ϕ; (b) Side view of fork support; (c) Top view of fork support

the American Wood Council [12]. C_1 describes the ratio between the critical load of a beam loaded by a uniform bending moment (Figure 3) and a beam loaded by arbitrary loading and hence, both beams must be considered. For the purpose of simplicity, the arbitrary loading is first simplified to a single point load and two end-moments (Figure 4). Later on, arbitrary loading is considered as a superposition of multiple point loads and a general theory is formulated. In this way, any loading can be simulated by representing it as a collection of point loads.

5.1. General considerations

In this derivation, an energy-based approach is followed. The energy prior to buckling is equaled with the energy just after buckling, following assumption 5. Since the total amount of energy in the system stays constant, the increase in energy

$$\Delta E = \Delta U + \Delta V = 0 \tag{11}$$

with U the strain energy and V the potential energy. For the considered I-section and following the key assumptions, strain energy can be the result of bending (around both y- and z-axes), torsion and warping. For a beam loaded in bending around a certain axis, the strain energy U_b taken by the beam is found by integrating the internal work over the complete volume.

$$U_b = \frac{1}{2} \int_V \sigma_b \epsilon_b \, dV \tag{12}$$

Combining Eq. 12 with Hooke's law ($\sigma_b = E\epsilon_b$), the kinematic relation $\epsilon_b = \kappa \cdot z/EI$ and the moment of inertia $I_y = \int_A z^2 dA$, leads to

$$U_b = \frac{1}{2} \int_0^L \frac{M_b^2}{EI} \, dx \tag{13}$$

In the same manner the strain energy taken by warping U_w and torion U_t are derived (see [23, 9] for complete derivations).

$$U_w = \frac{1}{2} \int_0^L \frac{M_w^2}{EI_w} \, dx \tag{14}$$

$$U_t = \frac{1}{2} \int_0^L \frac{M_t^2}{GI_t} \, dx \tag{15}$$

The potential energy is obtained by the position of the loading with respect to a certain reference state. For the considered problem, the change in potential energy ΔV is equal but opposite to the work done by the loading in the considered interval. Since the deformations are assumed to be small (assumption 4), the bending moments in the undeformed and deformed configuration can be related by $M_{\bar{y}} \approx M_y$ and $M_{\bar{z}} \approx \phi M_y$.

5.2. Uniform moment over beam

A simply supported beam loaded by a uniform bending moment M_0 is shown in Figure 3. Prior to lateral torsional buckling, only bending around the y-axis takes place.

$$U_I = U_{b,y} = \frac{1}{2}\int_0^L \frac{M_y^2}{EI_y}\,dx \tag{16}$$

Just after lateral torsional buckling, bending around both $\bar{y}$ and $\bar{z}$-axes, warping and torsion are considered in the deformed configuration, leading to

$$U_{II} = U_{b,\bar{y}} + U_{b,\bar{z}} + U_{t,\bar{x}} + U_{\bar{w}} = \frac{1}{2}\int_0^L \left\{ \frac{M_{\bar{y}}^2}{EI_y} + \frac{M_{\bar{z}}^2}{EI_z} + \frac{M_{\bar{x}}^2}{GI_t} + \frac{M_{\bar{w}}^2}{EI_w} \right\} dx \tag{17}$$

Using the relations between the deformed and undeformed configurations and assuming small deformations do no influence warping and torsional behaviour, Eq. 17 can be elaborated to

$$U_{II} = \frac{1}{2}\int_0^L \left\{ \frac{M_y^2}{EI_y} + \frac{(\phi M_y)^2}{EI_z} + \frac{M_x^2}{GI_t} + \frac{M_w^2}{EI_w} \right\} dx \tag{18}$$

Next, the change in strain energy is found by $\Delta U = U_{II} - U_I$, giving with Eqs. 16 and 18

$$\Delta U = \frac{1}{2}\int_0^L \left\{ \frac{(\phi M_y)^2}{EI_z} + \frac{M_x^2}{GI_t} + \frac{M_w^2}{EI_w} \right\} dx \tag{19}$$

The change in potential energy ΔV is equal and opposite to the work done by the loading. For the considered beam, the only work is done by the end moments, giving

$$\Delta V = -(\theta_{LTB,A} + \theta_{LTB,B}) \cdot M_0 = -2\theta_{LTB,A} \cdot M_0 \tag{20}$$

in which $\theta_{LTB,A}$ and $\theta_{LTB,A}$ are defined in Figure 3. The transverse displacement w_{LTB} due to only lateral torsional buckling is related to the lateral displacement u_{LTB} via $w_{LTB} = \phi\, u_{LTB}$. Furthermore, u_{LTB} is found with the moment-area theorem, whilst applying $M_{\bar{z}} \approx \phi M_y$, leading to

$$w_{LTB}(x_p) = x_p\theta_{LTB,A} - \int_0^{x_p} \phi \cdot \frac{\phi M_y}{EI_z} \cdot (x_p - x)\,dx \tag{21}$$

From Eq. 21, $\theta_{LTB,A}$ can be solved by considering the transverse displacement $w_{LTB}(x_p = L) = 0$, giving

$$\theta_{LTB,A} = \int_0^L \frac{\phi^2 M_y}{EI_z}\frac{(L-x)}{L}\,dx \tag{22}$$

Following the same approach, an expression for $\theta_{LTB,B}$ is found:

$$\theta_{LTB,B} = \int_0^L \frac{\phi^2 M_y}{EI_z}\frac{x}{L}\,dx \tag{23}$$

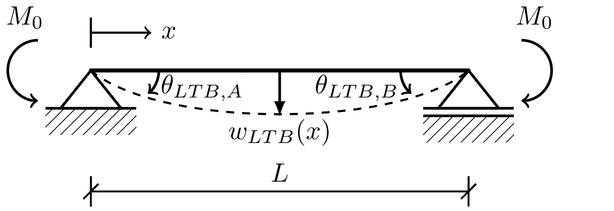


Figure 3: Simply supported beam loaded by a uniform bending moment M_0 with deformations due to LTB in plane of loading

For a uniform bending moment $M_y = M_0$ and assuming a buckling shape ϕ that is symmetric around $x = L/2$, Eq. 22 further reduces to

$$\theta_{LTB,A} = \frac{M_0}{2EI_z}\int_0^L \phi^2\,dx \tag{24}$$

Substitution of $\theta_{LTB,A}$ in Eq. 20 results in

$$\Delta V = -\frac{M_0^2}{EI_z}\int_0^L \phi^2\,dx \tag{25}$$

Inserting Eqs. 19 and 25 in Eq. 11 leads to

$$\Delta E = \frac{1}{2}\int_0^L \left\{ -\frac{(\phi M_0)^2}{EI_z} + \frac{M_x^2}{GI_t} + \frac{M_w^2}{EI_w} \right\} dx = 0 \tag{26}$$

5.3. Point load and support-moments on beam

A simply supported beam loaded by a point load F at γL (with $0 < \gamma < 1$) and two end moments $M_A = \alpha FL$ and $M_B = \beta FL$ is shown in Figure 4 together with the corresponding bending moment line. Since the point load acts in the same plane as the support-moments, the change in strain energy ΔU is found by Eq. 19. Furthermore, the change in potential energy ΔV is found in a similar way as Eq. 20, adding the change in potential of the point load:

$$\Delta V = -Fw_{LTB}(\gamma L) - \theta_{LTB,A}M_A - \theta_{LTB,B}M_B \tag{27}$$

in which w_{LTB}, $\theta_{LTB,A}$ and $\theta_{LTB,B}$ are given by Eqs. 21, 22 and 23 respectively. The total energy change in the system is now found by

$$\frac{1}{2}\int_0^L \left\{ \frac{(\phi M_y)^2}{EI_z} + \frac{M_x^2}{GI_t} + \frac{M_w^2}{EI_w} \right\} dx + \Delta V = 0 \tag{28}$$

Considering the constitutive relations $M_x = GI_t(d\phi/dx)$ and $M_w = -EI_w(d^2\phi/dx^2)$, and applying the same buckling shape $\phi(x)$ for the uniform loading and the considered case, the torsion and warping terms in Eqs. 26 and 28 work out to be equal and hence, the other terms in these expressions should be equal as well. The accuracy of this approach depends on the similarity of the buckling shapes

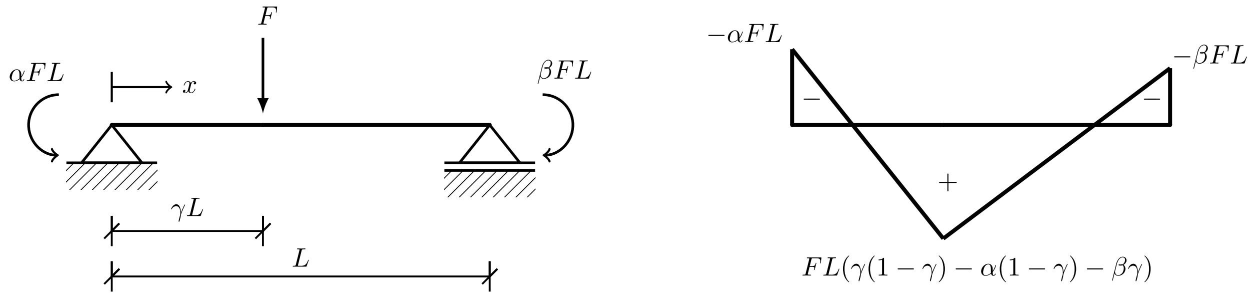


Figure 4: Simply supported beam loaded by a point load and end-moments with the corresponding bending moment line M_y

of the considered and uniform load cases, as will be discussed later. Elaborating this gives

$$\frac{1}{2}\int_0^L \left\{ \frac{(\phi M_y)^2}{EI_z} + \frac{(\phi M_{0,eq})^2}{EI_z} \right\} dx - F w_{LTB}(\gamma L) - \theta_{LTB,A} M_A - \theta_{LTB,B} M_B = 0 \tag{29}$$

which can be solved to obtain the equivalent uniform moment $M_{0,eq}$, once a certain buckling shape $\phi(x)$ is specified. The equivalent moment factor C_1 is now found by

$$C_1 = \frac{M_{max}}{M_{0,eq}} \tag{30}$$

5.4. Arbitrary point loading on a beam

In this study, arbitrary loading is defined as a random collection of N point loads $\delta_i F$, in which F is the unit load and δ_i the collection of multipliers with $i = 1..N$, on a beam with support-moments αFL and βFL on both sides. The loads are applied on locations γ_i. Intermediate moment loads are not considered but can be implemented by the reader by considering the loss of potential energy by this moment load during buckling in the same manner as the support-moments. Taking multiple point loads into account, Eq. 29 changes to

$$\frac{1}{2}\int_0^L \left\{ \frac{(\phi M_y)^2}{EI_z} + \frac{(\phi M_{0,eq})^2}{EI_z} \right\} dx - \sum_{i=1}^{N} \delta_i F\, w_{LTB}(\gamma_i L) - \theta_{LTB,A} M_A - \theta_{LTB,B} M_B = 0 \tag{31}$$

and the moment M_y is found by

$$M_y(x) = -\alpha F(L-x) - \beta F x + \sum_{i=1}^{N} \begin{cases} \delta_i F x(1-\gamma_i) & 0 \le x \le \gamma_i L \\ \delta_i F \gamma_i (L-x) & \gamma_i L \le x \le L \end{cases} \tag{32}$$

Different sizes of point loads can be included in two ways: either by the multipliers δ_i or by manipulating the location collection γ_i such that multiple unit loads F on the same location represent a certain relative size. In the remainder of this paper, the latter approach is used in order to reduce the number of different input parameters. By implementing Eqs. 31 and 32 in Eq. 30, a general expression has been derived to determine the equivalent moment factor C_1 once a certain buckling shape $\phi(x)$ is assumed. For a fork support that allows for warping ($k = k_w = 1$), many authors (e.g. [3, 8, 9]) assume a sinusoidal buckling shape according to

$$\phi(x) = a \sin\frac{\pi x}{L} \tag{33}$$

with a being an unknown scaling variable. Eq. 33 fulfills the boundary conditions: twisting is prevented on both sides, so $\phi(0) = \phi(L) = 0$, and warping is allowed meaning $M_w(0) = M_w(L) = 0$ and hence $\phi''(0) = \phi''(L) = 0$. Substitution of the assumed buckling shape $\phi(x)$ in Eq. 31, solving to $M_{0,eq}$ using the moment distribution as defined in Eq. 32 and inserting the obtained $M_{0,eq}$ in Eq. 30 results in a general expression for the equivalent moment factor C_1:

$$C_1 = \frac{M_{max}}{FL\sqrt{D_1(\alpha^2+\beta^2) + D_2\alpha\beta + D_3\alpha + D_4\beta + D_5}} \tag{34}$$

In which coefficients D_1 to D_5, given in Appendix A, are fully defined by the collection of locations of point loads γ_i. Appendix A also gives an expression to calculate the maximum bending moment in the beam. For example, when a beam is loaded by point loads of 15 kN at $L/2$ and 10 kN at $3L/4$, $\gamma_i = [0.5, 0.5, 0.5, 0.75, 0.75]$ with a unit point load F of 5 kN. In this manner, it is possible to include different sizes of point loads, although it is not possible to include point loads with different directions in this formulation. These can be included by elaborating Eq. 31 with an adjusted moment distribution. The coefficients D_1 to D_5 have a physical meaning.

- D_1 and D_2 are constant values, respectively 0.2827 and 0.4347, taking account of the influence of the support-moments on the equivalent moment factor. These factors are constant as only the magnitude of the support-moments is susceptible to change.

- D_5 considers the influence of the point loads on C_1 by the moment distribution and the work done by the point loads. For $\alpha = \beta = 0$, Eq. 34 reduces to $M_{max}/(FL\sqrt{D_5})$.

- D_3 and D_4 are interaction factors for the point loads and the support-moments. The vertical displacements $w_{LTB}(\gamma_i L)$ are influenced by the size of the support-moments and the support rotations θ_{LTB} are influenced by the point loads, hence creating cross-terms.

Using the same formulations, it is possible to prevent warping at the supports while allowing for lateral bending ($k = 1$ and $k_w = 0.5$) by assuming a different buckling shape for both the uniform moment load case and the arbitrary loading

$$\phi(x) = a\left(1 - \cos\frac{2\pi x}{L}\right) \tag{35}$$

which fulfills the boundary conditions: $\phi(0) = \phi(L) = 0$ and warping is prevented meaning $\phi'(0) = \phi'(L) = 0$. As discussed by Trahair [8], the influence of warping-restraints at the supports on the critical moment depends on the torsional stiffness GI_t, the warping stiffness EI_w and the length L of the beam. However, these effects are completely neglected by assuming that Eq. 35 holds for both the uniform moment and the arbitrary loading. Restrained warping at the supports has a favourable influence on lateral torsional buckling, leading to larger C_1 values. Tables 3 and 4 give coefficients D_3, D_4 and D_5 for a single point load and multiple point loads respectively for both free and restrained warping at the supports. For prevented warping, coefficients D_1 and D_2 are constants: 0.2700 and 0.4600 respectively.

5.5. Uniformly distributed loading

Uniformly distributed loads q are handled in the same manner. The force-related term in the potential energy change ΔV is replaced by an integral over the complete length of the beam:

$$\sum_{i=1}^{N} \delta_i F\, w_{LTB}(\gamma_i L) = \int_0^L q \cdot w_{LTB}(\gamma L)\, d\gamma \tag{36}$$

and $M_A = \alpha q L^2$ and $M_B = \beta q L^2$ are implemented in Eq. 31, leading to

$$C_1 = \frac{M_{max}}{qL^2\sqrt{D_1(\alpha^2+\beta^2) + D_2\alpha\beta + D_3\alpha + D_4\beta + D_5}} \tag{37}$$

The coefficients D_1 and D_2 remain unchanged and coefficients D_3, D_4 and D_5 are given in Table 4. For the combination of point loads and uniformly distributed loads, one should represent the UDL as a collection of equivalent equally-spaced point loads, such that the expressions in Appendix A together with Eq. 34 can be applied. This approach is further elaborated in section 7.

6. Validation

In this section, the derived general formulations for point loads (Eq. 34) and uniformly distributed loads (Eq.

Table 3: Coefficients D_3, D_4 and D_5 for 1 point load at γL with and without warping allowed at the supports

γ	0.1	0.2	0.25	0.33	0.4	0.5	0.6	0.67	0.75	0.8	0.9
$k_w = 1$											
D_3	-0.0562	-0.1087	-0.1312	-0.1592	-0.1735	-0.1757	-0.1581	-0.1369	-0.1069	-0.0864	-0.0434
D_4	-0.0434	-0.0864	-0.1069	-0.1369	-0.1581	-0.1757	-0.1735	-0.1592	-0.1312	-0.1087	-0.0562
D_5	0.0028	0.0106	0.0158	0.0241	0.0300	0.0335	0.0300	0.0241	0.0158	0.0106	0.0028
$k_w = 0.5$											
D_3	-0.0540	-0.1072	-0.1322	-0.1664	-0.1863	-0.1925	-0.1730	-0.1483	-0.1144	-0.0919	-0.0460
D_4	-0.0460	-0.0919	-0.1144	-0.1483	-0.1730	-0.1925	-0.1863	-0.1664	-0.1322	-0.1072	-0.0540
D_5	0.0027	0.0107	0.0163	0.0263	0.0340	0.0388	0.0340	0.0263	0.0163	0.0107	0.0027

Table 4: Coefficients D_3, D_4 and D_5 multiple point loads at $\gamma_i L$ with and without warping allowed at the supports

γ_i	UDL	[0.25, 0.75]	[0.33, 0.67]	[0.25, 0.50]	[0.25, 0.50, 0.75]	[0.33, 0.50, 0.67]	[0.33, 0.33, 0.67]
$k_w = 1$							
D_3	-0.1086	-0.2382	-0.2962	-0.3069	-0.4138	-0.4718	-0.4554
D_4	-0.1086	-0.2382	-0.2962	-0.2826	-0.4138	-0.4718	-0.4331
D_5	0.0122	0.0578	0.0912	0.0923	0.1774	0.2341	0.2066
$k_w = 0.5$							
D_3	-0.1150	-0.2466	-0.3147	-0.3247	-0.4733	-0.5073	-0.4811
D_4	-0.1150	-0.2466	-0.3147	-0.3069	-0.4733	-0.5073	-0.4630
D_5	0.0134	0.0611	0.1004	0.1029	0.2279	0.2628	0.2270

Load case	Bresser (1)	EC3 (2)	AS-4100 (3)	ANSI (4)	Trahair (5)	Serna (6)	(1)/(2)	(1)/(3)	(1)/(4)	(1)/(5)	(1)/(6)
$\psi = 1$	1.000	1.000	0.982	1.000	1.000	1.000	1.00	1.02	1.00	1.00	1.00
$\psi = 0.75$	1.141	1.131	1.120	1.111	1.131	1.137	1.01	1.02	1.03	1.01	1.00
$\psi = 0.5$	1.324	1.300	1.297	1.250	1.300	1.310	1.02	1.02	1.06	1.02	1.01
$\psi = 0.25$	1.564	1.506	1.525	1.429	1.506	1.532	1.04	1.03	1.09	1.04	1.02
$\psi = 0$	1.881	1.750	1.818	1.667	1.750	1.815	1.07	1.03	1.13	1.07	1.04
$\psi = -0.25$	2.284	2.031	2.164	2.000	2.031	2.155	1.12	1.06	1.14	1.12	1.06
$\psi = -0.5$	2.712	2.300	2.483	2.174	2.350	2.488	1.18	1.09	1.25	1.15	1.09
$\psi = -0.75$	2.940	2.300	2.594	2.222	2.500	2.652	1.28	1.13	1.32	1.18	1.11
$\psi = -1$	2.766	2.300	2.404	2.273	2.500	2.523	1.20	1.15	1.22	1.11	1.10
UDL	1.133	1.130	1.166	1.136	1.130	1.136	1.00	0.97	1.00	1.00	1.00
Mid PL	1.366	1.350	1.388	1.316	1.350	1.276	1.01	0.98	1.04	1.01	1.07
Quarter PL	1.494	1.450	1.363	1.364	1.450	1.390	1.03	1.10	1.10	1.03	1.07
2 PL (0.25, 0.75)	1.040	1.040	0.982	1.000	1.088	1.000	1.00	1.06	1.04	0.96	1.04
Mid PL SC	1.830	1.700	1.817	1.705	1.800	1.596	1.08	1.01	1.07	1.02	1.15
Mid PL DC	1.732	1.640	1.700	1.923	1.710	1.435	1.06	1.02	0.90	1.01	1.21
UDL SC	2.276	2.200	2.404	2.083	2.250	2.197	1.03	0.95	1.09	1.01	1.04
UDL DC	2.613	2.300	2.500	2.381	2.420	2.574	1.14	1.05	1.10	1.08	1.02
Average							1.07	1.04	1.09	1.05	1.06
Standard dev.							0.08	0.05	0.10	0.06	0.06

Table 5: Comparison of the equivalent moment factors C_1 with the general formulation of this study for several standard load cases and free warping ($k = k_w = 1$)

37) are compared with the available literature. In Table 5, equivalent moment factors are given for practical load-cases like a moment gradient from M to ψM, uniformly distributed loads (UDL) and point loads (PL) for simply supported, single clamped (SC) and double clamped (DC) beams based on Eqs. 34 and 37, Eurocode 3 [4], the Australian Steel Standard AS-4100 [6], the American Steel Standard ANSI/AISC 360-17 [7] and the general formulation from Serna et al. [20] as given by Eq. 9. Only supports with $k = k_w = 1$ are considered in Table 5 since most design standards do not offer expressions to include support restraints. The average ratio between the found general formulations and literature lies between 1.04 and 1.09, with the lowest ratio for AS-4100, meaning that this study results in a non-conservative equivalent moment factor with respect to literature, which could be the result of the conservative nature of general simplified design expressions. For $\psi < 0$, the ratio increases significantly as a result of the assumed symmetric sinusoidal buckling shape not being accurate for very asymmetric loading. This discrepancy for asymmetric load patterns is shown by Figure 5, in which the finite difference results of Serna et al. for an IPE500 steel beam with length $L = 8m$ are elaborated, with parameters chosen to represent practical usual cases (although differences with other lengths and profiles are small).

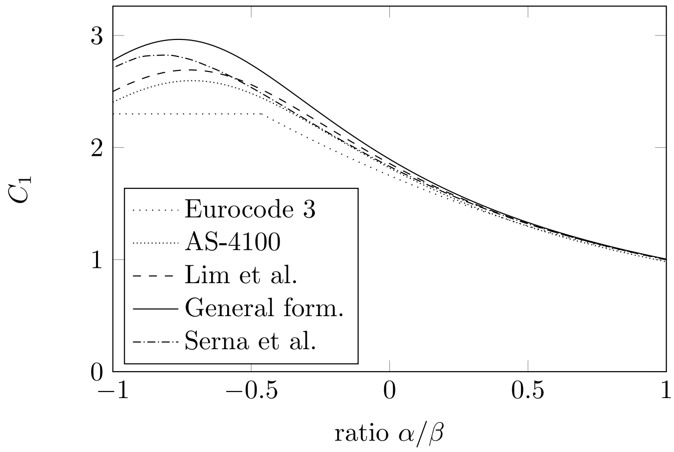
For a beam loaded by two equal end-moments or one end-moment and a concentrated mid-span load (Figures 6 and 7), the general formulation shows strong similarity with the finite difference results of Serna et al. for both free and restrained warping. For the case with one end-moment differences up to 13.8% are obtained, following from the introduced asymmetry of the loading for large α, causing deviations from the assumed symmetric buckling shapes. Figures 8 and 9 compare the general formulation with literature for an uniformly distributed load with one or two (equal) end-moments and it is found that, for this load cases, both the Australian Standard AS-4100 and Eq. 37 show good correspondence with the finite difference results of Serna et al. [20] and Suryoatmono and Ho [22] with differences up to 6.7% and 12.7% respectively. The available design standards and literature are captured with reasonably accuracy with the aid of Eqs. 34 and 37, depending on the complexity and asymmetry of the considered loading pattern. Hence, when making use of the general formulation, it is important to verify whether the assumed buckling shapes (Eqs. 33 and 35) are sufficiently accurate for the purpose of the analysis. Instead of case-specific (possibly more accurate) analysis, this study offers a general formulation that can be used to consider all loading cases within the formulated theoretical framework with no significant loss of accuracy.

7. Application

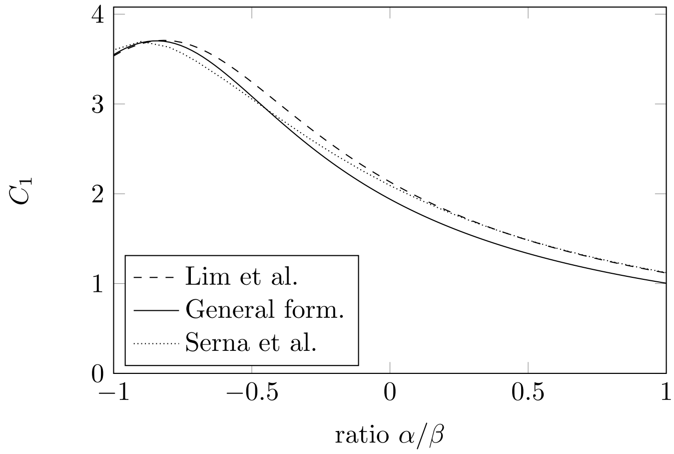
With the aid of the general formulations for point loads (Eq. 34) and uniformly distributed loads (Eq. 37), some specific cases are analyzed and compared with the design standards to show the application of the theory.

7.1. Elaboration of example

A simply supported beam with length $L = 8m$ without warping restraints ($k = k_w = 1$), loaded at 1/2 of the span by a point load $F = 12.5kN$ is considered. The beam

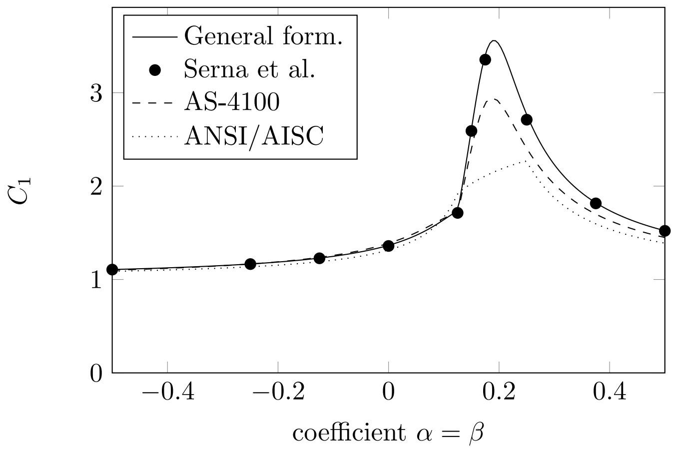


(a)

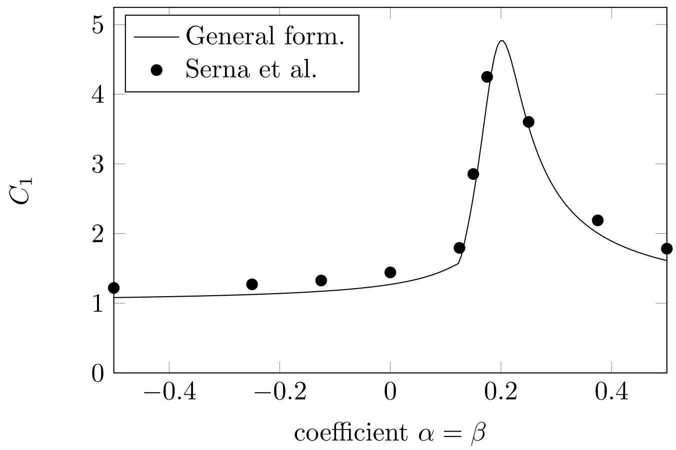


(b)

Figure 5: C_1 as function of the ratio α/β (or coefficient ψ in Eurocode 3) for (a) unrestrained warping with $k_w = 1$ and (b) restrained warping with $k_w = 0.5$

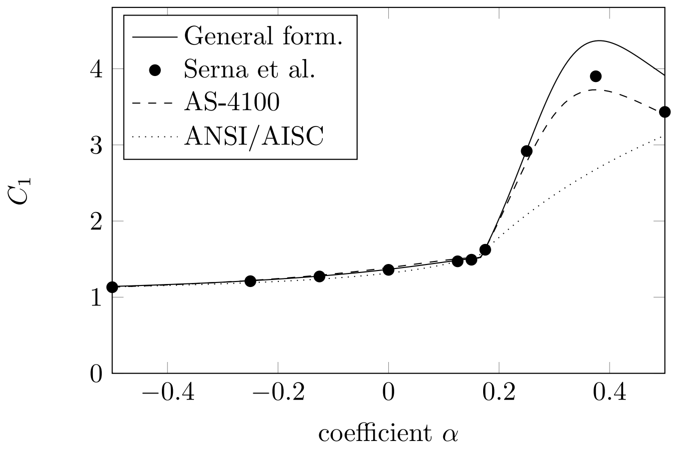


(a)

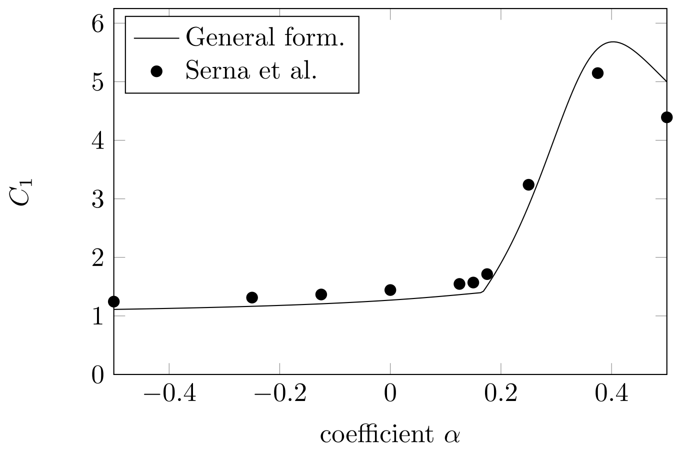


(b)

Figure 6: C_1 for a beam loaded by two equal end-moments and a concentrated mid-span load for (a) unrestrained warping with $k_w = 1$ and (b) restrained warping with $k_w = 0.5$

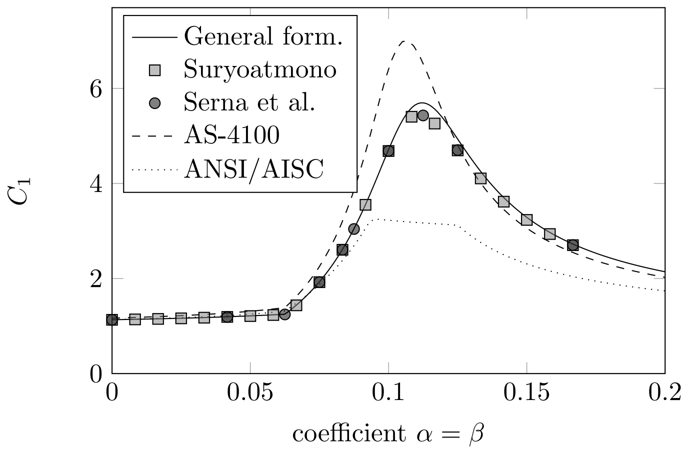


(a)

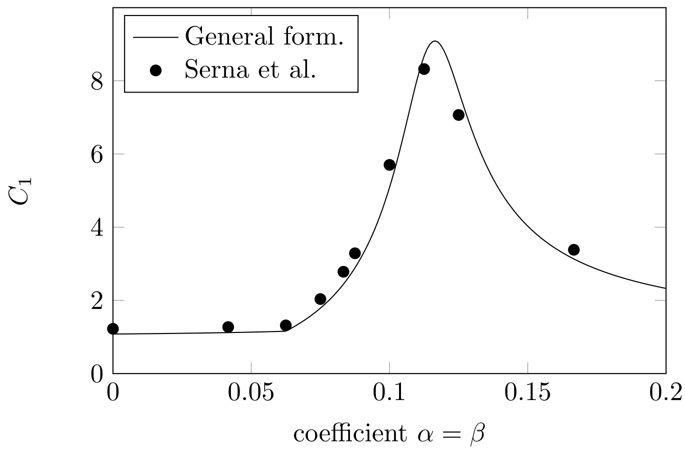


(b)

Figure 7: C_1 for a beam loaded by one end-moment ($\beta = 0$) and a concentrated mid-span load for (a) unrestrained warping with $k_w = 1$ and (b) restrained warping with $k_w = 0.5$

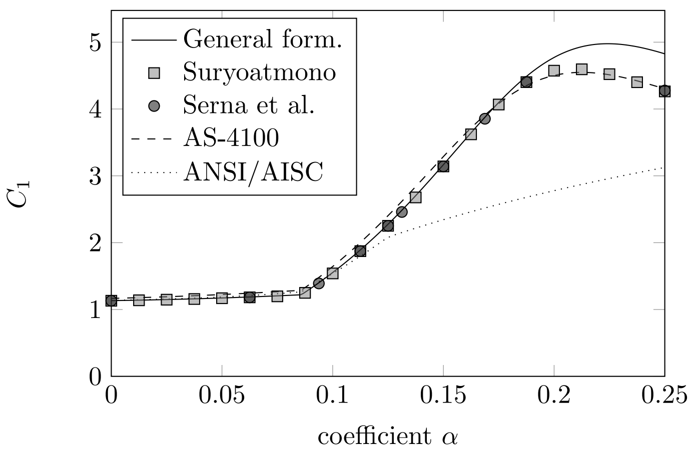


(a)

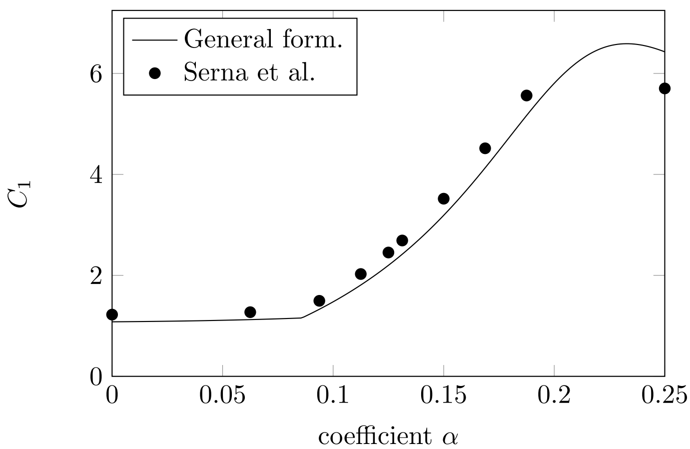


(b)

Figure 8: C_1 for a beam loaded by two end-moments and a uniformly distributed load for (a) unrestrained warping with $k_w = 1$ and (b) restrained warping with $k_w = 0.5$



(a)



(b)

Figure 9: C_1 for a beam loaded by one end-moment ($\beta = 0$) and a uniformly distributed load for (a) unrestrained warping with $k_w = 1$ and (b) restrained warping with $k_w = 0.5$

is loaded by support-moments $M_A = M_B = 17.5kNm$, leading to $\alpha = \beta = M_A/FL = 0.175$. For a beam loaded at 1/2 of the span, Table 3 gives $D_3 = -0.1757$, $D_4 = -0.1757$ and $D_5 = 0.0335$ and from Appendix A follows $D_1 = 0.2827$ and $D_2 = 0.4347$. The maximum moment $M_{max} = 17.5kNm$ is found at the supports. From Eq. 34, the equivalent moment factor $C_1 = 3.405$. From the diagrams of Eurocode 3 for concentrated loading and support-moments follows $C_{1,EC3} = 2.30$. Furthermore, $C_{1,ANSI} = 2.886$ and $C_{1,AS-4100} = 2.108$ are found based on Eqs. 6 and 5. Using the numerical finite difference approach, Serna et al. found $C_{1,Serna} = 3.354$ for this case, which is assumed to be (close to) the exact value. Hence, when using the design standards, the structure is over-conservative by a factor 1.18 for ANSI and even 1.62 for AS-4100, leading to inefficient use of the cross-section.

When considering the same beam loaded at $L/3$ with $\alpha = 0.175$ and $\beta = 0.035$, the general formulation for point loads results in $C_1 = 3.083$. For non-midspan loadings, Eurocode 3 does not provide a solution. Assuming that a mid-span load is a proper representation of the considered case, Eurocode 3 provides $C_{1,EC3} = 1.60$. The American Steel Standard gives $C_{1,ANSI} = 2.720$ and the Australian Steel Standard provides $C_{1,AS-4100} = 4.344$, which is 1.6 times larger than $C_{1,ANSI}$. The inconsistency between different design standards is the result of the non-physical nature of simplified expressions created by curve-fitting, causing specific cases, with for example $M_a = M_c = 0$, to deviate significantly from the physical reality. The general formulation of Eq. 34 is based on physical considerations and is therefore less likely to show these inconsistencies.

7.2. Combination of PL and UDL

General formulations are given for cases with only point loads or UDL. However, it is also possible to consider the combination of PL and UDL by representing the UDL by a set of n equivalent equally-spaced point loads and add these locations to γ_i to make use of Eq. 37 (although it is also possible to solve Eq. 31 including the work done by the UDL and the corresponding M_y instead of using the general formulation). Consider a simply supported beam loaded by a mid-span point load F and an UDL $q = \theta F/L$. The UDL is represented by n equivalent unit loads $F_i = qL/n$, where it is advised to use $n > 10$ to obtain a proper representation of the UDL. The concentrated load is represented by n/θ unit point loads F_i. The collection of locations γ_i is now found by

$$\gamma_i = \left[\frac{2j-1}{2n}L;\ \gamma_{F,k}\right] \text{ for } j = 1...n \tag{38}$$

with $k = n/\theta$ and $\gamma_{F,k} = \gamma_F$, where n is chosen such that k is an integer. Following this approach, it has been found that the equivalent moment factor for the combination of a mid-span point load and an UDL can be accurately described by

$$C_1 = \begin{cases} 1.27 \cdot \theta^{-0.034} & \text{for } \theta \leq 25 \\ 1.13 & \text{for } \theta > 25 \end{cases} \tag{39}$$

Using the general formulations of this study, design figures and diagrams can be straightforwardly created. For this purpose, it is advised to program the general formulations and expressions of Appendix A. Furthermore, it is also possible to consider multiple uniformly distributed loads over parts of a span by changing the distance between the equivalent point loads for each UDL.

7.3. Intermediate torsional/lateral supports

This study considers single-span beams without intermediate supports. To increase the resistance to lateral torsional buckling, intermediate torsional and/or lateral supports can be added. A first indication of the equivalent moment factor can be achieved by assuming that each segment, defined as the part between two supports, behaves as an independent individual segment. This assumption holds for similar segments that have similar resistances. For non-similar segments however, the assumption of no interaction does not hold anymore since the stronger segments restrain the weaker ones and more elaborate analysis should be performed. The same effects are obtained for multi-span beams over multiple supports. These effects are not further elaborated in this study.

7.4. Approach within context of design procedure

With the aid of the derived general formulation, the equivalent moment factor for single span beams without intermediate restraints is determined for both I-section (often steel) and rectangular slender sections (often timber). The determination of the critical theoretical (Euler) buckling load is a general physical problem, which does not depend on the considered material. In the regular design procedures, the theoretical buckling load, as determined with the equivalent moment factor (Eqs. 1 and 2), is reduced in order to incorporate effects of material imperfections, geometric imperfections, internal stresses etc. The reduction of the theoretical buckling load is a material dependent procedure and is therefore not further considered in this study, but is mentioned here to capture the complete design procedure. Furthermore, for general application of the derived general formulation, the influence of the loading location within the cross-section on the buckling load should be studied in more detail.

8. Conclusions

Lateral torsional buckling is considered as an ultimate limit state failure mode during the design of beam structures. This study focused on the influence of the moment distribution via the equivalent moment factor C_1 on the critical elastic moment for both I-sections (often steel) and slender rectangular sections (often timber). The considered design standards offer different simplified general solutions and some practical case-specific solutions. The advantage of simplicity comes with the disadvantage of loss of accuracy. Furthermore, the goal of design standards is to be conservative rather than to be accurate. On the other side, literature offers many case-specific studies that go in depth on a certain load case, mostly leading to an accurate, but complex, solution.

This paper has presented a general formulation that is in between the simplicity of the design standards and the specific complex solution procedures as can be found in literature. This general formulation is derived based on the principle of conservation of energy and can be used for support-moments, point loads (Eq. 34), uniformly distributed loads (Eq. 37) and a combination of those, for free warping ($k = k_w = 1$) and restrained warping ($k = 1$ and $k_w = 0.5$) at the supports. For the case with free warping, coefficients D_1 to D_5 are defined in Appendix A. The results of this paper have been compared with design standards (EC3[4], AS-4100[6], ANSI/AISC 360-17[7]) as well as literature (Trahair[8], Serna et al.[20], Lim et al. [19] and Suryoatmono and Ho [22]) and it is found that this paper is in good correspondence with design standards and literature. On average, the ratio between the results of this study and literature lies between 1.04 and 1.09. The expected accuracy of the general formulation decreases for asymmetric loading as a result of the assumed symmetric buckling shapes (Eqs. 33 and 35) not representing the physical reality. The influence of length, warping and torsion stiffness on C_1 for restrained warping ($k_w = 0.5$) is neglected in this study, inducing a potential loss of accuracy.

The application of the general formulations has been shown with the aid of practical examples, pointing out possible drawbacks of using design standards. By manipulating the collection of locations γ_i, it is possible to represent complex load-cases of multiple UDL and point loads combined. Instead of using the general formulation, it is also possible to make use of Eq. 31. The general formulation presented in this paper is limited to elastic material behaviour, in plane loading, loading in the shear center and double-symmetric cross-sections and can not consider the influence of axial loading. In order to work with the general formulation, one must be aware of these limitations.

In a more general picture, this paper contributes to the solution of Sahraei et al. [15], which takes account of partial twist restraints, load height, moment distribution and pre-buckling deformations. In their study, Sahraei et al. made use of the Australian Steel Standards to consider the influence of the moment distribution. With the contribution of this paper, the theory of Sahraei et al. is improved and more physical background is obtained.

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AppendixA. Expressions D_1 to D_5 for $k = k_w = 1$

$$D_1 = \frac{1}{3} - \frac{1}{2\pi^2} \approx 0.2827 \tag{A.1}$$

$$D_2 = \frac{1}{3} + \frac{1}{\pi^2} \approx 0.4347 \tag{A.2}$$

$$D_3 = \sum_{i=1}^{N} \left\{ -(\gamma_i - 1)P_i + Q_i + R_i + \frac{\gamma_i - 1}{\pi^2} \right\} \tag{A.3}$$

$$D_4 = \sum_{i=1}^{N} \left\{ \gamma_i P_i - Q_i + \frac{\gamma_i^3 - \gamma_1}{3} - \frac{\gamma_i}{\pi^2} \right\} \tag{A.4}$$

$$\begin{aligned} D_5 = \sum_{i=1}^{N} \Bigg\{ & P_i S_i - Q_i \left[\sum_{j=1}^{N} \gamma_j - T_i \right] \\ & + \frac{1}{3} \left(\gamma_i^4 + (U_i - [n + 2 - i])\gamma_i^3 + \gamma_i^2 \right) \\ & - \sum_{j=i+1 \leq N}^{N} R_j \cdot \gamma_i + \frac{1}{\pi^2} \left(-\frac{1}{2}\gamma_i^2 + (T_i - U_i)\gamma_i \right) \Bigg\} \end{aligned} \tag{A.5}$$

with

$$\begin{aligned} P_i &= \frac{1}{\pi^2} \cos^2 \pi\gamma_i \\ Q_i &= \frac{1}{\pi^3} \sin \pi\gamma_i \cos \pi\gamma_i \\ R_i &= \frac{-\gamma_i^3 + 3\gamma_i^2 - 2\gamma_i}{3} \\ S_i &= \sum_{j=1}^{i-1} [\gamma_j(\gamma_i - 1)] + \sum_{j=i}^{N} [\gamma_i(\gamma_j - 1)] \\ T_i &= \frac{1 + 2(N - i)}{2} \\ U_i &= \sum_{j=i+1 \leq N}^{N} \gamma_j \\ & 0 \leq \gamma_i \leq 1 \end{aligned} \tag{A.6}$$

and the maximum moment

$$M_{max} = max(|M_{y,\gamma_i}|) \tag{A.7}$$

with

$$\begin{aligned} M_{y,\gamma_i} =& FL \left[\sum_{j=1}^{i-1} \gamma_j + \gamma_i \cdot \left(N - i + 1 - \sum_{j=1}^{N} \gamma_j \right) \right. \\ & \left. - \alpha(1 - \gamma_i) - \beta\gamma_i \right] \end{aligned} \tag{A.8}$$