

# ALGEBRAIC MULTILEVEL PRECONDITIONING FOR HELMHOLTZ EQUATION

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**Abstract.** *We propose efficient algebraic multilevel preconditioning for the Helmholtz equation with high wave numbers. Our algebraic method is mainly based on using new multilevel incomplete  $LDL^T$  techniques for symmetric indefinite systems.*

## 1 INTRODUCTION

We discuss the numerical solution of Helmholtz equations

$$-\Delta u - k^2 u = f \in \Omega \subset \mathbb{R}^2$$

on a bounded domain  $\Omega$ . This kind of problem arises e.g. in modeling of acoustic scattering. Often one is interested in using high wave numbers  $k$  which leads to a symmetric but highly indefinite system.

When  $\Omega$  corresponds to the truncation of an originally infinite region, we typically have to impose on part of  $\partial\Omega$  a Sommerfeld boundary condition,

$$\frac{\partial u}{\partial n} = iku,$$

to avoid spurious reflection from the boundary. In this case the system even is complex-valued. We consider the discretization of this system using finite differences and investigate algebraic multilevel preconditioning for this type of problem. Note that in order to resolve the wave numbers properly, the grid size has to be proportional to  $\frac{1}{k}$  which means that the size of the system is increasing as the wave number increases. Our approach to solve these systems of equations with increasing wave number consists in a new algebraic multilevel method designed for symmetric indefinite systems. The method uses:

1. symmetric maximum weight matchings to improve the block diagonal dominance of the system on every level,
2. an inverse-based pivoting strategy to compute  $A \approx LDL^T$  while at the same time keeping  $\|L^{-1}\| \leq \kappa$  below a bound  $\kappa$ . This requires to postpone several rows and columns to the end, which are then recursively treated within a multilevel framework.

## 2 SYMMETRIC WEIGHTED MATCHINGS

Symmetric weighted matchings [5, 7, 15] can be viewed as a preprocessing step that rescales and permutes the original matrix such that the block diagonal dominance of the system is improved. As a consequence of this strategy, all entries are at most one in modulus and in addition the diagonal blocks are either  $1 \times 1$  scalars  $a_{ii}$ , such that  $|a_{ii}| = 1$  (in exceptional cases we will have  $a_{ii} = 0$ ), or they are  $2 \times 2$  blocks

$$\begin{pmatrix} a_{ii} & a_{i,i+1} \\ a_{i+1,i} & a_{i+1,i+1} \end{pmatrix} \text{ such that } |a_{ii}|, |a_{i+1,i+1}| = 1 \text{ and } |a_{i+1,i}| = |a_{i,i+1}| = 1. \quad (1)$$

Numerical observations [7, 15, 11, 14] indicate that symmetric maximum weight matchings typically waive dynamic symmetric pivoting strategies like [3].

We will briefly describe the idea of symmetric maximum weight matchings. Basically, symmetric maximum weight matchings consist of two steps:

In a first step, a nonsymmetric maximum weight matching [6, 13] is applied that yields  $n \times n$  positive diagonal matrices  $D_r, D_c$  and a permutation matrix  $P_{\mathcal{M}}$  such that

$$A_{\mathcal{M}} := P_{\mathcal{M}}^T D_r A D_c$$

satisfies

$$|a_{\mathcal{M},ij}| \leq 1, \quad |a_{\mathcal{M},ii}| = 1, \quad \text{for all } i, j = 1, \dots, n.$$

In a second step, the permutation from the first step is decomposed into a product of cycles. Each cycle itself is broken up into a product of 2-cycles which gives a new permutation matrix  $P_S$ . The scaling procedure is symmetrized by taking  $D_s = (D_r D_c)^{1/2}$ . The associated matrix is given by

$$A_S = P_S^T D_s A D_s P_S.$$

If  $|A| = |A|^T$ , then  $|a_{S,ij}| \leq 1$  for all  $i, j = 1, \dots, n$  and  $A_S$  can be written as  $D + E$  where  $D$  is a block diagonal part of  $A_S$  with blocks of size  $1 \times 1$  and  $2 \times 2$  such that any  $2 \times 2$  block satisfies (1), while the  $1 \times 1$  blocks are either 1 or zero in modulus. For details we refer to [7, 15].

In order to construct the incomplete, algebraic, multilevel factorization efficiently, care has to be taken that not too much fill-in is introduced during the elimination process. To do we compress the graph of  $A_S$  and apply the fill-in reducing reordering to the compressed system. Here we will use a nested dissection reordering from METIS [12]. Finally, we extend the permutation to the original system  $A_S$  to get a suitable reordering.

### 3 ALGEBRAIC MULTILEVEL PRECONDITIONING

Next we give an overview over a symmetric approximate multilevel factorization that is based on three parts which are repeated in a multilevel framework. The components consist of (i) reordering of the system, (ii) approximate factorization using inverse-based pivoting and, (iii) recursive application to the system of postponed updates.

Suppose that initially the system is reordered and rescaled such that

$$P^T D A D P = \hat{A}, \tag{2}$$

where  $D, P \in R^{n,n}$ ,  $D$  is a diagonal matrix and  $P$  is a permutation matrix, both obtained from symmetric maximum weight matchings followed by the reordering. We expect  $\hat{A}$  to have many diagonal blocks of size  $1 \times 1$  or  $2 \times 2$  that are well-conditioned.

Given  $\hat{A}$  we compute an incomplete factorization  $LDL^T = \hat{A} + E$  of  $\hat{A}$ . Suppose that at step  $k$  of the algorithm we have

$$\hat{A} = \begin{pmatrix} B & F^T \\ F & C \end{pmatrix} = \begin{pmatrix} L_B & 0 \\ L_F & I \end{pmatrix} \begin{pmatrix} D_B & 0 \\ 0 & S_C \end{pmatrix} \begin{pmatrix} L_B^T & L_F^T \\ 0 & I \end{pmatrix}, \tag{3}$$

where  $L_B \in R^{k,k}$  is lower triangular with unit diagonal and  $D_B \in R^{k,k}$  is block diagonal with diagonal blocks of size  $1 \times 1$  and  $2 \times 2$ . Also,  $S_C = C - L_F D_B L_F^T$  denotes the approximate Schur complement. Following [1] we can easily estimate

$$\left\| \begin{pmatrix} L_B & 0 \\ L_F & I \end{pmatrix}^{-1} \right\| \leq \kappa \tag{4}$$

in every step for a prescribed bound  $\kappa$  based on a sparse adaption of the method presented in [4]. If at step  $k$  the approximate factorization fails to satisfy (4), then row and column

$k$  are permuted to the end. Otherwise we proceed with the approximate factorization where essentially any  $|l_{ik}| \leq \varepsilon$  is dropped. Here  $\varepsilon$  is a given drop tolerance.

When the approximate  $LDL^T$  decomposition has finally traversed  $\hat{A}$  we are faced with a system of the form

$$Q^T \hat{A} Q = \begin{pmatrix} L_{11} & 0 \\ L_{21} & I \end{pmatrix} \begin{pmatrix} D_{11} & 0 \\ 0 & S_{22} \end{pmatrix} \begin{pmatrix} L_{11}^T & L_{21}^T \\ 0 & I \end{pmatrix}. \quad (5)$$

We then apply the algorithm recursively to the remaining Schur-complement system  $S_{22}$  that consists of all postponed updates. Hence,  $S_{22}$  is now explicitly computed and the strategies for reordering, scaling and factorization are recursively applied to  $S_{22}$  finally leading to a multilevel factorization.

For the iterative solution of linear systems we use the simplified QMR method [10, 9] in order to fully exploit the symmetry of the original system  $A$  and the computed preconditioner.

#### 4 NUMERICAL EXPERIMENTS

We now present numerical experiments that show that the previously outlined AMG preconditioners can be successfully applied to the Helmholtz equations. All large scale numerical experiments were computed in MATLAB using the ILUPACK toolbox [2]. We use a Linux AMD Dual Opteron 1.6 GHz with 16GB memory. The toolbox was compiled using *gcc* and *g77* with `-O` flag. The ILUPACK toolbox uses PARDISO's maximum weighting interface [16].

As test example we consider the model problem

$$-\Delta u - k^2 u = 0 \text{ in } \Omega = [0, 1]^2$$

with boundary conditions

$$\begin{aligned} \frac{\partial u}{\partial \nu} &= 0 && \text{north and south boundaries} \\ \frac{\partial u}{\partial \nu} &= e^{-\frac{1}{2}(y-\frac{1}{2})^2} && \text{west boundary} \\ \frac{\partial u}{\partial \nu} &= -iku && \text{east boundary} \end{aligned}$$

As discretization we use second order finite differences for the Laplacian operator and first order discretization for the boundary conditions. Thus, the discretized system  $A_h$  will be complex symmetric. It can be decomposed as

$$A_h = K_h + S_h - k^2 M_h,$$

where  $K_h$  denotes the stiffness matrix,  $M_h$  denotes the mass matrix and  $S_h$  denotes the contribution from the Sommerfeld boundary condition  $\frac{\partial u}{\partial \nu} = -iku$ . Since we are interested

	Iteration steps (SQMR), backward error $10^{-12}$					
	10	20	40	60	80	100
complete $LDL^T$ for						
$K_h + k^2 M_h$	44	132	543	1556	> 2000	—
$K_h + S_h - ik^2 M_h$	35	126	679	> 2000	—	—
Algebraic multilevel preconditioning for						
$K_k + k^2 M_h$	65	223	967	2435	3713	—
$K_k + S_h - ik^2 M_h$	58	212	1058	2999	—	—
$A_h$	13	17	21	28	34	72

Table 1: Number of SQMR steps:  $LDL^T$  versus algebraic multilevel preconditioning

	Computation time [sec]					
	10	20	40	60	80	100
Algebraic multilevel preconditioning for						
$K_k + k^2 M_h$	0.02	0.20	2.90	18.0	55.0	—
$K_k + S_h - ik^2 M_h$	0.04	0.20	3.30	24.0	—	—
$A_h$	0.07	0.20	0.53	1.60	3.09	8.60

Table 2: Computational time for the algebraic multilevel preconditioning method

in varying high wave numbers  $k = 10, 20, \dots, 100$ , the mesh size  $hk = 2\pi/20$  needs to be adapted with respect to the wave number. In our experiments we used approximately 20 grid points per wavelength. For the algebraic multilevel preconditioning parameters we used  $\varepsilon = 10^{-2}$  and  $\kappa = 5$ .

First, we compare the performance of the AMG applied to different parts of the original system. Here we focus on  $K_h + k^2 M_h$ ,  $K_h + S - h - ik^2 M_h$  and finally on  $A_h$  itself. The first matrix  $K_h + k^2 M_h$  is real symmetric positive definite and with respect to the spectral properties better than using  $K_h$  as approximation to  $A_h$  (see [8]). Numerical methods will also compare preconditioners applied to  $K_h + S_h - ik^2 M_h$  and  $A_h$  itself. The results are summarized in Tables 1 and 2.

The numerical example indicates that our algebraic multilevel method can be successfully applied to the numerical solution of Helmholtz equations with high wave numbers. Currently filtering techniques are being developed to improve the preconditioner even further.

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