Fact Checking Fibonacci

by

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to obtain the degree of Bachelor of Science
at the Delft University of Technology,
to be defended publicly on Wednesday August 9, 2023 at 11:00 AM (GMT +2).

Student number:	5381657	
Project duration:	April 25, 2023 – July, 2023	
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Contents

1 Introduction

John Horton Conway (1937-2020) was a renowned mathematician who made significant contributions to various areas of mathematics, including group theory, number theory, and combinatorial game theory. He was known for his remarkable intuition and ability to come up with elegant solutions to difficult problems. However, he was also famously informal in his approach to mathematics, often relying on his intuition and geometric insight to guide his reasoning rather than following traditional methods of proof.

Conway was known for being precise in his writing, but he did not always provide complete proofs. Sometimes, he would skip steps or leave out details that he felt were ”elementary” and could be easily filled in by the reader. This approach often led to criticism from more traditional mathematicians, who argued that Conway’s methods were not rigorous enough.

However, despite this criticism, Conway’s work has stood the test of time and has had a profound impact on the field of mathematics. Many of his ideas, such as the discovery of the Conway groups and the invention of surreal numbers, have led to new areas of research and have inspired generations of mathematicians.

Ultimately, Conway’s legacy lies not only in his groundbreaking research but also in his unconventional approach to mathematics, which challenged the status quo and encouraged others to think outside the box. This makes understanding his work particularly hard, because he is not an ordinary thinker and sometimes sees things in mathematics that no one else have ever seen it before. This thesis is devoted to one of his recent articles that is interesting to read, but where not everything is immediately clear for the reader.

In this thesis, we will have a careful look at one of his works, namely the paper called ”The Extra Fibonacci Series and the Empire State Building” written by John Conway and Alex Ryba [1]. This article is published in 2016 where he wrote down some properties and **Facts** about the so-called Extra Fibonacci Series. He then connects those Series of integers with the Empire State Building. He orders all those series in a Some Facts in this article, more likely to be called Lemmas, are not always provided with complete proofs. Again this is typical Conway. He mostly provides some small hints in the text, which gives us a direction of his thought process. But sometimes, he writes down a Fact and does not give us a hint, because Conway apparently sees it immediately. This makes it difficult for an average reader to understand why it is true. Therefore, the main goal of this thesis is to provide a clear explanation or proofs to the Facts stated in the article.

2 Summary

In his paper titled ”The Extra Fibonacci Series and the Empire State Building,” Conway begins by introducing the well-known Fibonacci and Lucas numbers. He then presents a new series called the ”extraFib series,” which consists of infinite sequences of integers that follow Fibonacci’s rule and have a positive sequence to the right. There are several facts about the extraFib series that require proof as they may not be immediately apparent.

Conway proceeds to introduce the Zeckendorf notation, a method of representing any integer as the sum of non-consecutive distinct Fibonacci numbers. Prior to establishing the connection between the extraFib series and the Empire State Building, he introduces the concept of the Garden State. The Garden State serves as a display area for all the extraFib series, which are listed vertically. The term ”Garden State” is a play on words by Conway, who spent the latter part of his life in New Jersey, often referred to as the Garden State. The array’s wall, consisting of terms between vertical lines, represents the garden wall, and the garden expands beyond the wall.

Conway then introduces the concept of ”reversal” in the extraFib series to obtain the ”left wall term” and determine the central term, referred to as the ”pillar” of the Empire State Building. He organizes all the extraFib series from the Garden State in a way that aligns their central terms in the same column. Some extraFib series share the same ”inner width” between the left and right terms, resulting in them being grouped together in a so-called ”block.” The resulting infinite array takes on a structure resembling the Empire State Building, where the inner width increases by 1 as you move to the block below. This array exhibits various intriguing structures that will become apparent to readers as they progress through the thesis.

Not only does Conway discover a connection between the extraFib series and the Empire State Building, but he also identifies a relationship between the Fibonacci and Lucas numbers and the standard trigonometric formulas. He coins the term ”Fibonometry” for this novel concept, which will be explained in the thesis. It offers a fascinating perspective on how seemingly unrelated mathematical concepts can be interconnected.

3
Fact Checking

The Facts stated in the paper of John Conway and Alex Ryba will be proven by first stating definitions and providing background information before we actually give a proof to a Fact.

Fibonacci numbers and Lucas numbers are two related sequences of numbers that have many interesting properties and connections.

Fibonacci numbers are a series of numbers where each number is the sum of the two preceding numbers. The sequence begins with 0 and 1, resulting in the initial values of 0, 1, 1, 2, 3, 5, 8, 13, 21, and so on. These numbers are named after Leonardo of Pisa, also known as Fibonacci, an Italian mathematician who introduced them to the Western world through his book ”Liber Abaci” published in 1202. However, the sequence had already been known in Indian mathematics long before Fibonacci’s time. Fibonacci encountered this sequence while investigating the growth of a rabbit population. He observed that the number of rabbit pairs in each generation equaled the sum of the pairs from the previous generation and the newly born pairs in the current one. This observation led to the recursive relationship that defines the Fibonacci sequence. After Fibonacci’s book was published, the sequence gained popularity in Europe and has since found applications in various fields of mathematics and science. It showcases captivating properties and connections, such as the golden ratio, which represents the limit of the ratio between consecutive Fibonacci numbers.

Lucas numbers are a similar sequence of numbers, but they start with 2 and 1 instead of 0 and 1. So the Lucas sequence begins as 2, 1, 3, 4, 7, 11, 18, 29, and so on. The Lucas sequence is named after the French mathematician Édouard Lucas, who introduced it in the late 19^{th} century. Like the Fibonacci sequence, Lucas numbers also exhibit interesting properties and relationships. They have a connection to the golden ratio as well, similar to the Fibonacci numbers. The ratio between consecutive Lucas numbers also approaches the golden ratio as the terms increase.

A sequence is an enumerated collection of numbers where each number has a fixed position, i.e. has an index. Usually the term ”series” is used for an infinite sequence of numbers that is to be added, but in this thesis we will use the term series to refer to a doubly infinite sequence of integers where a position of a term is not fixed.

Definition 1. *Let (x_i) be an indexed sequence of integers. If i runs over $\mathbb{N}$, we say it is a sequence. If i runs over $\mathbb{Z}$ and each term does not have a fixed position, we say it is a series.*

We can continue the Fibonacci and Lucas sequences backwards with the Fibonacci’s rule, then we obtain the Fibonacci and Lucas series, which are both palindromic except for the signs.

... -8 5 -3 2 -1 1 0 1 1 2 3 5 8 ...

... 18 -11 7 -4 3 -1 2 1 3 4 7 11 18 ...

We say that these series "end in positive integers" because F_n and L_n are positive for $n \geq 0$. In this report, we want to review the "extra Fibonacci series"—briefly, "extraFib" series—that have these properties: each term is the sum of the previous two, and they end in positive integers, meaning that after some term in the series, that the terms after that are all positive integers.
The extraFib series form the rows of the "extraFib array." Because its rows, being series, do not have well defined starting points, the array has many "states" that differ only by sliding individual rows left or right.

Definition 2. *Let (x_i) be a series that satisfies the recurrence $x_i + x_{i+1} = x_{i+2}$. We say that (x_i) is ExtraFib if there exists an N such that $x_i > 0$ if $i > N$.*

0	1	1	2	3	5	8	13	21	34	...
1	3	4	7	11	18	29	47	76	123	...
2	4	6	10	16	26	42	68	110	178	...
3	6	9	15	24	39	63	102	165	267	...
4	8	12	20	32	52	84	136	220	356	...
5	9	14	23	37	60	97	157	254	411	...
6	11	17	28	45	73	118	191	309	500	...
7	12	19	31	50	81	131	212	343	555	...
8	14	22	36	58	94	152	246	398	644	...
9	16	25	41	66	107	173	280	453	733	...
10	17	27	44	71	115	186	301	487	788	...

Figure 3.1: Garden State (Conway 2016)

This extraFib array is described in its Garden State (Figure 3.1). In the Garden State, you see the "tail" of the ExtraFib series, where all terms are positive in a row. We now wonder if we can find all ExtraFib series in the Garden State. We will later find out that this is the case.

We define the seed terms to be the terms in the first column, which is just a sequence $0, 1, 2, 3, \cdots$. The wall terms are between the lines in the second column. The sequence of the wall terms are harder to understand and this will be clear later in Fact 4. After we have defined those two terms, everything after follows by the recurrence. We call the terms after the bold line the *garden terms*. Seed terms can be seen as the index of a row. The wall column turns out to be the so-called the lower Wythoff sequence, a well known series of numbers, **1 3 4 6 8 9 11 12 14 16 17 ....** This sequence can be found online as well and has different ways to describe them [2]. For instance one description is that this sequence has numbers that can be written as a sum of different Fibonacci numbers including the Fibonacci number 1. After knowing the seed column and the wall column, we can generate the garden terms by using the Fibonacci's rule.

We say that n has a Fibonacci representation if it can be written as a sum of descending Fibonacci numbers, $n = F_a + F_b + \cdots + F_k$ with $k \geq 2$ (since $F_2 = F_1 = 1$). We define $out(n) = F_{a+1} + F_{b+1} + \cdots + F_{k+1}$. So the $out(n)$ is the shifted version of the Fibonacci numbers that build up n. In our Garden State, the seed terms s for $s = 0, 1, 2, \cdots$ are followed by the wall terms $out(s) + 1$. It will become clear in Fact 1 that function $out(n)$ actually does not depend on the choice of the Fibonacci numbers where n is made out from.

We say that $n = F_a + F_b + \cdots + F_k$ where $a > b > \cdots$ is a *Zeckendorf representation* of n if the indices differ by more than one. It is not immediately clear that each number admits a Zeckendorf representation. We will show in Fact 2 that this expansion is uniquely determined for any number n.

The Zeckendorf notation is analogous to the usual binary notation. In binary notation, each digit represents a power of 2. In a canonical Zeckendorf notation, for instance 11 does not exist, because the requirement is that no adjacent Fibonacci numbers are used in the representation.

The Zeckendorf notation is where the bit string $...\gamma\beta\alpha$ represents $... + \gamma F_4 + \beta F_3 + \alpha F_2$ rather than $... + 4\gamma + 2\beta + \alpha$. We also use this for sums of distinct Fibonacci numbers that may contain two adjacent ones ("noncanonical Zeckendorf expansions"). If $***$ is possibly a noncanonical Zeckendorf notation, then $|***|$ represents its canonicalization. Thus $|1011| = 10000$ is the canonical Zeckendorf notation for 8, since $F_5 + F_3 + F_2 = 5 + 2 + 1 = 8 = F_6$. Note that $out(n)$ is equivalent to putting a 0 after the binary notation of n.

Fact 1 *The function $out(n)$ is well-defined.*

Proof. $out(n)$ is well-defined when the outcome is the same for a different Fibonacci representation of n. First we need to know if there is a Fibonacci representation for any n. This follows from Fact 2. Fact 1 will be established once we prove Fact 2.

Suppose that P is a Zeckendorf representation of n and Q be another representation of n which is not Zeckendorf. Note that the $out(n)$ will put a 0 after the bit string of n. So $out(Q)$ will put a 0 after the bit string of Q. Since Q is not a Zeckendorf notation, from the left we can find 11 in its bit string. we find 11 either in the beginning of the bit string, then we can replace that with 100 or find it later and we can replace 011 with 100. In both cases we can replace Q by Q', where Q' is the new representation of n. Then $out(Q) = out(Q')$ holds because 011 is still the same as 100 after shifting. We can repeat this process for Q'. Every time we replace 011 in the bit string by 100, the number of 1's decrease and eventually we arrive at the Zeckendorf representation. □

Fact 2 *The Zeckendorf expansion of n is unique*

Proof. This Fact states two things. We first give a proof to that there exists a Zeckendorf expansion for every $n \in \mathbb{N}$. There is an algorithm to find it. Choose F_p to be the largest Fibonacci number that is less than or equal to n (when n is equal to a Fibonacci number, then we already have the Zeckendorf expansion and we are done, so we may suppose that $n > F_p$).
Find a Fibonacci number that is less than or equal to $n - F_p$. This cannot be F_{p-1} because $n < F_p + F_{p-1} = F_{p+1}$ holds. since we assumed that F_p is the largest Fibonacci number that is smaller than n. It follows that the next Fibonacci number we can use for the expansion is less or equal to F_{p-2}. Eventually, we will get that the remainder is equal to the next Fibonacci number, since 1 is a Fibonacci number. The algorithm halts.

To prove uniqueness, suppose that the sets P and Q are distinct Zeckendorf representations for n. So $n = \sum P = \sum Q$. Consider then $P' = P \setminus Q$ and $Q' = Q \setminus P$, which are sets where common elements are removed. Then still $\sum P' = \sum Q'$ holds, so either both are empty sets and we are done, or both are non-empty. Let F_p the largest element of P' and F_q the largest element of Q'. Since these sets are disjoint, $F_p \neq F_q$. WLOG let $F_p < F_q$. We will need the following fact. If a Zeckendorf expansion of n begins with F_p, then it can be at most $F_p + F_{p-2} + \cdots + (F_3 \ or \ F_2)$. It becomes at most F_{p+1} when we again add 1. This follows from the fact that the sum of adjacent Fibonacci numbers can be written as the next Fibonacci number of the largest of the two, i.e. $F_n + F_{n+1} = F_{n+2}$ for any n.

Using this information we can conclude that $\sum P' < F_{p+1}$ and also $\sum P' < F_q$ by assumption. We assumed that $\sum P' = \sum Q'$, but the largest term $F_q \in Q'$ is already larger than $\sum P'$, which gives a contradiction. So P' and Q' must be empty sets. Therefore $P = Q$ and we have proved the uniqueness. □

Let $\tau = \frac{1+\sqrt{5}}{2}$ and $\sigma = \frac{1-\sqrt{5}}{2}$. These are the roots of the equation $x^2 = x + 1$ and have nice properties such as $\tau + \sigma = 1$, $\tau - \sigma = \sqrt{5}$ and $\tau\sigma = -1$. Those two numbers are the roots of the equation $x^{n+1} = x^n + x^{n-1}$. So their power sequence satisfy Fibonacci's rule. We will write out those starting with 1.

$$1, \sigma, \sigma^2, \sigma^3, \cdots$$

$$1, \tau, \tau^2, \tau^3, \cdots$$

Then if we subtract the second sequence from the first, we obtain the following sequence:

$$0, \sqrt{5}, \sqrt{5}, 2\sqrt{5}, \cdots$$

and after dividing every term by $\sqrt{5}$, we get the Fibonacci sequence. We can write a series out of those two sequences by continuing them backwards. Then every extraFib series can be described from these two power series by $a\sigma^n + b\tau^n$ where $n \in \mathbb{Z}$. For an extraFib, $b \geq 0$ holds. The extraFib series is palindromic when $a = b$.

Therefore we have a formula for the n-th Fibonacci which is known as the *Binet formula*. This formula will be used multiple times in this thesis.

$$F_n = \frac{\tau^n - \sigma^n}{\sqrt{5}}$$

Also we derive a formula for the n-th Lucas number, which is validated by the recurrence relation [3]:

$$L_n = \tau^n + \sigma^n$$

It is handy to establish Fact 3, because once we have that, we can calculate $out(n)$ directly without having to find the Zeckendorf representation for n. Multiply n with τ and then subtract σ^2. Find the first integer that is greater that that number. Then we have found our $out(n)$.

Fact 3 *The unique integer in the open unit interval* $(\tau n - \sigma^2, \tau n - \sigma)$ *is* $out(n)$

Proof. First note that $1-\sigma^2 = -\sigma$. We will show that $out(n) - \tau n$ is in the open unit interval $(-\sigma^2, 1-\sigma^2)$, which is just the shifted version of Fact 3.

We know that $\tau = \frac{1+\sqrt{5}}{2} > 0$ and $\sigma = \frac{1-\sqrt{5}}{2} < 0$. We will first show that $F_{r+1} - \tau F_r = \sigma^r$.

$$\begin{aligned}
F_{r+1} - \tau F_r &= \frac{\tau^{r+1} - \sigma^{r+1}}{\sqrt{5}} - \frac{\tau(\tau^r - \sigma^r)}{\sqrt{5}} \\
&= \frac{\tau^{r+1} - \sigma^{r+1}}{\sqrt{5}} - \frac{\tau^{r+1} - \tau\sigma^r}{\sqrt{5}} \\
&= \frac{-\sigma^{r+1} + \tau\sigma^r}{\sqrt{5}} \\
&= \frac{\sigma^r(\tau - \sigma)}{\sqrt{5}} = \sigma^r F_1 = \sigma^r
\end{aligned}$$

Note that we can write $n = F_a + F_b + \cdots$ for any n. So we have

$$\begin{aligned}
out(n) - \tau n &= (F_{a+1} + F_{b+1} + \cdots) - \tau(F_a + F_b + \cdots) \\
&= (F_{a+1} - \tau F_a) + (F_{b+1} - \tau F_b) + \cdots \\
&= \sigma^a + \sigma^b + \cdots
\end{aligned} \tag{3.1}$$

by the identity shown above. We claim that $\sigma^a + \sigma^b + \cdots$ is smaller than $1 - \sigma^2$ and greater than $-\sigma^2$. With a simple calculation we see that $1 - \sigma^2 = -\sigma = \frac{\sigma^2}{1-\sigma^2}$. Then by using the geometric series we see that $\frac{\sigma^2}{1-\sigma^2} = \sigma^2 \sum_{i=0}^{\infty} (\sigma^2)^i = \sigma^2 + \sigma^4 + \cdots$ since $\sigma^2 < 1$. Since $\sigma < 0$, even powers of σ are positive and odd powers are negative. Therefore this sum is the upper bound for $out(n) - \tau n$. Similarly, $-\sigma^2 = \frac{\sigma^3}{1-\sigma^2} = \sigma^3 \sum_{i=0}^{\infty} (\sigma^2)^i = \sigma^3 + \sigma^5 + \cdots$. Then this sum is the lower bound for $out(n) - \tau n$ and we have proved Fact 3. □

Fact 4 *The canonical Zeckendorf notation for the typical row of the Garden State is* $***, |***1|, ***01, ***010, ***0100, \ldots$ *where that for the seed is* $***$.

Proof. Fact 4 is a definition of the rows in Table 1 via the wall term. From the second term after the wall term n, the row grows with $out(n)$, because $out(n)$ puts a 0 after the bit string of n. The seed terms can be seen as the index of a row. We will show that the row $***, |***1|, ***01, ***010, ***0100, \ldots$ satisfies Fibonacci's rule. Let $n = F_a + F_b + F_c + \cdots$ be a seed term $***$. Then the wall term that follows is $out(n) + 1 = F_{a+1} + F_{b+1} + F_{c+1} + \cdots + F_2$, which is equal to $|***1|$ by the definition. By adding those two terms we get, $F_{a+2} + F_{b+2} + F_{c+2} + \cdots + F_2$ which is equal to $***01$, which is the first garden term given in the definition. If the seed term's bit string was $***1$, then the wall term is $|***11| = ***100$. Adding those two terms will indeed give a term in the form $***01$. If the seed term's bit string was $***0$ then the wall term is $***01$ and adding those two terms will again give a term in the form $***01$.

So the first 3 terms satisfy the Fibonacci's rule. From here, it is clear that the next terms also satisfy Fibonacci's rule: all the terms satisfy the equation $n + out(n) = out(out(n))$ for any $n \in \mathbb{N}$. □

Fact 5 is important because only the garden terms are followed by $out(n)$, so we can locate where the seed term and the wall term is for any extraFib series. For this, we need to know if every "tail" of any extraFib is in the Garden State. Fact 7 guarantees this. Note that if we know two consecutive terms, then we know which extraFib series it is by the Fibonacci's rule. Thus when term n is followed by $out(n)$, then they are in the garden and we can find the extraFib series in which those terms appear.

Fact 5
(a) A garden term n is followed by $out(n)$.
(b) A seed term s is followed by $out(s) + 1$.
(c) A wall term w is followed by $out(w) - 1$.

Proof. This Fact is analogous to Fact 4. Nevertheless, we will show that this Fact 5 is true to be complete. (a) That the garden term n is followed by $out(n)$, is given by the definition. (b) By the definition of a row in Fact 4, a seed term $***$ is followed by $|***1|$ which is $***0 + 1 = out(***) + 1$. We do not care about the representation whether it is canonical or not, because the out function is well-defined. (c) By the definition of a row in Fact 4, a wall term $|***1|$ is followed by $***01$. We know that $out(|***1|) = ***10 = ***01+1$, so indeed, a wall term $|***1|$ is followed by $out(|***1|)-1$. □

Fact 6 *Every positive integer appears exactly once in the garden and once as a seed, and zero also appears just once as a seed.*

Proof. Every positive integer appears exactly once as a seed, since the seed is just the index that starts at 0 and increased with 1. So zero also appears just once as a seed. It is left to show that every positive integer appears exactly once in the garden. If we take any $n \in \mathbb{N}$ and find the unique index (seed term) of the row where n is in, then we have proved this Fact. This is because one term cannot appear twice in its extraFib series due to Fibonacci's rule except for the original Fibonacci series. We can find the unique index of an arbitrary n as follows: First remove all the zeros from the right in its Zeckendorf notation until you come across a one. Then you have a number with Zeckendorf representation $***01$. Now take the sequence with seed $***$. By Fact 4 it is given by $***, ***1, ***01$. It is obvious that this is the unique row that contains n. □

When a series is represented in the Garden State, it means that it appears as a row in the Garden State. The Garden State consists of sequences and not series. We need to know where the wall term is in the series, then we can determine where a series is represented in the Garden State. Fact 5 provides the answer. We compute the outs of a series x_n, then for every consecutive terms x_t, x_{t+1} in the series, we look at the outs. If $out(x_t)+1 = x_{t+1}$ it follows from Fact 5 that x_{t+1} is a wall term.

Fact 7 *Every series that satisfies Fibonacci's rule and ends with positive integers is represented in the Garden State.*

Proof. From Fact 6, we know that every positive integer appears exactly once in the Garden. If some term n in a series is followed by $out(n)$, this series be seen in the Garden by Fact 5. This series also satisfies the Fibonacci's rule. We want to show that for any extraFib series, there exist a term x_k s.t. $x_{k+1} = out(x_k)$. We will show that x_{k+1} lies in the open interval $(\tau x_k - \sigma^2, \tau x_k - \sigma)$, then the Fact follows from Fact 3.

If we write $x_k = a\sigma^k + b\tau^k$, then $x_{k+1} = a\sigma^{k+1} + b\tau^{k+1}$. We have to show that $-a\sigma^{k-1} + b\tau^{k+1} - \sigma^2 < a\sigma^{k+1} + b\tau^{k+1} < -a\sigma^{k-1} + b\tau^{k+1} - \sigma$. Since $\sigma < 1$, for k large enough, the terms $a\sigma^{k-1}$ and $a\sigma^{k+1}$ are close to 0, so we can disregard those terms in the inequality. Therefore we have $b\tau^{k+1} - \sigma^2 < b\tau^{k+1} < b\tau^{k+1} - \sigma$ which obviously holds because $-\sigma^2 < 0$ and $-\sigma > 0$.

Fact 7 is sort of a corollary from the previous Facts. □

Fact 8 *If X_n is any extraFib series, so too is any positive multiple mX_n.*

Proof. Let X_n to be an extraFib series. First it is clear that when you multiply a series that ends in positive integers with a positive number m, that the multiplied series also ends in positive integer. Now let x_a, x_b, x_c three consecutive arbitrary terms of X_n. Then for mX_n, those terms become mx_a, mx_b, mx_c. Then $mx_a + mx_b = mx_c$ holds from the definition of an extraFib series. □

Fact 9 *The multiples of any extraFib series appear in order in the extraFib array.*

Proof. Fact 9 states that if we have $a, b \in \mathbb{N}$ where we assume $a > b$ and X_n is any extraFib series, then the seed term of aX_n is greater than the seed term of bX_n.
To see this, suppose for a fixed t, that bX_t is the seed term of the multiplied extraFib series bX_n. Then by Fact 5b we have $bX_{t+1} = out(bX_t) + 1$. By Fact 3, we know that $out(bX_t)$ is the smallest integer greater than $\tau bX_t - \sigma^2$. Therefore it follows that $out(bX_t) + 1 > \tau bX_t + (1 - \sigma^2)$. So we have that $bX_{t+1} > \tau bX_t + (1 - \sigma^2)$

If we have $a > b$, by multiplying both sides on the last inequality by $\frac{a}{b}$, we have $aX_{t+1} > \tau aX_t + \frac{a}{b}(1 - \sigma^2) > \tau aX_t + (1 - \sigma^2) > out(aX_t)$ by Fact 3 and since $\frac{a}{b} > 1$. This inequality says that aX_{t+1} is greater than $out(aX_t)$. So aX_t cannot be a garden term, nor a wall term by Fact 3. Therefore aX_t is either the seed or a term that is located to the left of the seed. If aX_t is the seed, then since $a > b$, this index is larger than bX_t, so the series aX_n appear later than bX_n in the extraFib array. If aX_t is located to the left of the seed, then the actual seed of this extraFib series is greater than aX_t, meaning that the argument in the first case still holds and therefore the series aX_n still appear later than bX_n in the extraFib array. □

We will show that in general, the signs of an extraFib series has the form $\cdots .-, +, -, +, +, +, +, \cdots$. First we know that an extraFib series has to end in positive integers, so eventually will only have positive signs on the "right" side. We will show that it will have alternating signs on the "left" side. We assume that 0 is a positive number. Since extraFib series satisfy the Fibonacci's rule, $E_a + E_{a+1} = E_{a+2}$ holds for every term in the extraFib series. Assuming that those terms are greater, we deduce that $E_{a+1} < E_{a+2}$. In case one term is 0, then it is a multiple of the original Fibonacci series, so we know that the signs alternate. Since we have deduced that the terms in any extraFib series decrease as it goes backwards, there must be a negative term in the series. Also from Fibonacci's rule, the first negative term we found has to be followed by a positive term, so the term before the negative term must be positive and larger in absolute value. Using this logic, we find that any extraFib series has the form $\cdots .-, +, -, +, +, +, +, \cdots$.

We will define a *reverse* for an extraFib series. A reverse is obtained by taking its terms in the reverse order around the pivot term. The pivot term is the first term after the last negative term. After reversing the terms, every term after the pivot becomes positive and the terms before the pivot will alternate in sign, starting with - from the first term before the pivot. We can define the reverse series as follows: $R_a = (-1)^a X_{-a}$, with X_0 as the pivot.

Fact 10 *The reversal of an extraFib series is also an extraFib series.*

Proof. This Fact follows immediately from the definition of the reversal of a series and Fibonacci's rule reverses to itself:

$$R_a + R_{a+1} = (-1)^a F_{-a} + (-1)^{a+1} F_{-a-1} = (-1)^a (F_{-a} - F_{-a-1}) = (-1)^a F_{-a-2} = R_{a+2}$$

. □

If we continue the Fibonacci sequence backwards, we obtain the series

$$..., 13, -8, 5, -3, 2, -1, 1, 0, 1, 1, 2, 3, 5, 8, 13, ...$$

We see that this series is its own reverse with the pivot being the term 0, ignoring the sign.

Not every extraFib series is its own reverse. For instance the series

$$... - 19, 12, -7, 5, -2, 3, 1, 4, 5, 9, 14, ...$$

Its reverse is namely

$$... - 14, 9, -5, 4, -1, 3, 2, 5, 7, 12, 19...$$

We see that the pivot term here is 3.

For our series above, we will find the right wall term. We find that $out(5) = out(3 + 2) = 5 + 3 = 8$ but the term after is 9, and $out(9) = out(8 + 1) = 13 + 2 = 15$ but the term after 9 is 14. Thus 9 is our right wall term. The left wall term is found similarly but after reversing the series first. We find that

$out(7) = out(5+2) = 8+3 = 11$ but the term after is 12, and $out(12) = out(8+3+1) = 13+5+2 = 20$ but the term after is 19. The left wall term is 12. We define the left seed as the term that comes right before the wall term when we reverse the series. We can indicate both wall terms in the original series by | |, so we have:

$$..., 31, -19, |12|, -7, 5, -2, 3, 1, 4, 5, |9|, 14, 23, ...$$

The central term is the middle term in a series that lies between the left and the right wall terms. This definition is only valid when there are uneven number of terms between the two wall term. We will show in Fact 12 that this is indeed the case. The "Empire State of the extraFib array" (also called the "Empire State Building") is obtained by sliding the rows of the array so as to align these central terms (indicated by double lines) to form a central "pillar". The pillar is not always the same as the pivot term. The situation can be illustrated as follows:

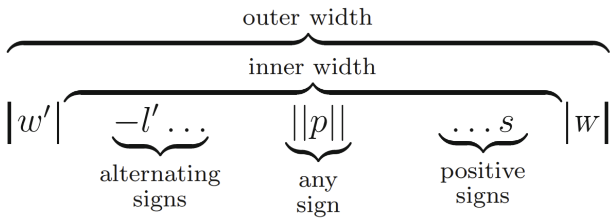

Figure 3.2: (Conway 2016)

Block r of the Empire State Building consists of those series with an inner width of $2r + 1$. See Appendix for the Empire State Building.

Fact 11 *Both wall terms are positive.*

Proof. We define a series x_t with $x_0 = out(l) + 1, x_1 = -l$ with $l \geq 0$. We show that this series is an extraFib series indexed so that x_0 is the left wall and x_1 the left seed. We want to show that the right wall is also positive. We fix r such that $F_{2r-1} \leq l < F_{2r+1}$, where r will give the block of the series (Fact 13). This bound will have further applications. By Fact 3, we find that $out(l) + 1$ is equal to $\tau l - \sigma + 1$ rounded down which we can write as $\tau(l + 1) - \theta$ where $0 < \theta < 1$. We have $x_t = F_{t-1}x_0 + F_t x_1$. It is with ease verified that this relation holds. For $t = 0$ and $t = 1$, one can see that it works. It follows from recurrence that it works for every next terms. This series satisfy Fibonacci's rule since it is a linear combination of two series that satisfy Fibonacci's rule. Thus by using properties of τ and σ and the definitions of x_0 and x_1 we have,

$$\begin{aligned}
\sqrt{5}x_t =& \sqrt{5}(F_{t-1}x_0 + F_t x_1) = (\tau^{t-1} - \sigma^{t-1})(\tau(l+1) - \theta) - (\tau^t - \sigma^t)l \\
=& \tau^t l + \tau^t - \theta\tau^{t-1} - \sigma^{t-1}\tau l - \sigma^{t-1}\tau + \theta\sigma^{t-1} - \tau^t l + \sigma^t l \\
=& \tau^t - \theta\tau^{t-1} - \sigma^{t-1}\tau l - \sigma^{t-2} + \theta\sigma^{t-1} + \sigma^t l \\
=& \tau^t - \theta\tau^{t-1} - (\sigma^{t-1}\tau l - \sigma^t l) - \sigma^{t-2} + \theta\sigma^{t-1} \\
=& \tau^t - \theta\tau^{t-1} - (\tau - \sigma)\sigma^{t-1} - \sigma^{t-2} + \theta\sigma^{t-1} \\
=& \tau^t - \theta\tau^{t-1} - \sqrt{5}l\sigma^{t-1} + \sigma^{t-2} + \theta\sigma^{t-1}.
\end{aligned} \tag{3.2}$$

We deduce that $\sqrt{5}x_t > \tau^t - \tau^{t-1} - O(|\sigma|^{t-2}) = \tau^{t-2} - O(|\sigma|^{t-2})$ since $0 < \theta < 1$ and σ^n tends to 0 as n grows. In $O(|\sigma|^{t-2})$ are all the terms that includes the powers of σ which all tend to 0 as the power increases. Eventually this term is positive because $\tau > 1$ and $|\sigma| < 1$. We conclude that our series ends in positive integers and therefore x_t is an extraFib series and is the reverse of the series with seed l. Then the right wall term that is after the seed l is positive. So both wall terms are positive. □

Fact 12 *The two wall terms are separated by an odd number $2r + 1$ of intermediate terms.*

Proof. We work in the same setting as in the proof of Fact 11, so $x_t = F_{t-1}x_0 + F_t x_1$. We will prove Fact 12 by showing that $x_{2r+2} = out(x_{2r+1}) + 1$, then we know that x_{2r+2} is the wall term by Fact 5. There cannot be another index after $2r + 1$ that satisfies this rule because every extraFib appears in the Garden State and after the wall term, the terms grow with out. Since x_0 was our left wall term, $2r+3$ is the outer width of the series and the wall terms are separated by $2r+1$ intermediate terms. Using equation 3.2 and the fact that $\sigma - \tau = -\sqrt{5}$:

$$\begin{aligned} x_{t+1} - \tau x_t &= \frac{1}{\sqrt{5}}(\tau^{t+1} - \theta\tau^t - \sqrt{5}l\sigma^t + \sigma^{t-1} + \theta\sigma^t) - \frac{\tau}{\sqrt{5}}(\tau^t - \theta\tau^{t-1} - \sqrt{5}l\sigma^{t-1} + \sigma^{t-2} + \theta\sigma^{t-1}) \\ &= \frac{1}{\sqrt{5}}(-\sqrt{5}l\sigma^{t-1} + \sigma^{t-2} + \theta\sigma^{t-2})(\sigma - \tau) \\ &= \sqrt{5}l\sigma^{t-1} - \sigma^{t-2} - \theta\sigma^{t-1} \end{aligned} \tag{3.3}$$

.

We first set $\theta = 1$, use the equation 3.3, $F_{2r-1} \leq l < F_{2r+1}$ and the properties of σ and τ and to obtain the lower bound:

$$\begin{aligned} x_{2r+2} - \tau x_{2r+1} &> \sqrt{5}l\sigma^{2r} - (\sigma^{2r-1} + \sigma^{2r}) \\ &= \sqrt{5}l\sigma^{2r} - \sigma^{2r+1} \\ &= \sigma^{2r}(\sqrt{5}l - \sigma) \\ &> \sigma^{2r}\sqrt{5}F_{2r-1} \\ &= \sigma^{2r}(\tau^{2r-1} - \sigma^{2r-1}) = -\sigma - \sigma^{4r-1} > -\sigma \end{aligned}$$

And for the upper bound we first set $\theta = 0$:

$$\begin{aligned} x_{2r+2} - \tau x_{2r+1} &< \sqrt{5}l\sigma^{2r} - \sigma^{2r-1} \\ &\leq \sigma^{2r}\sqrt{5}(F_{2r+1} - 1) - \sigma^{2r-1} \\ &= \sigma^{2r}(\tau^{2r+1} - \sigma^{2r+1} - \tau + \sigma) - \sigma^{2r-1} \\ &= \tau - \sigma^{4r+1} + \sigma^{2r-1} - \sigma^{2r-1} < \tau \end{aligned}$$

Combining those bounds we find $-\sigma < x_{2r+2} - \tau x_{2r+1} < \tau$. Rearranging gives $\tau x_{2r+1} - \sigma < x_{2r+2} < \tau x_{2r+1} + \tau$. This is equal to $\tau x_{2r+1} - \sigma^2 + 1 < x_{2r+2} < \tau x_{2r+1} + 2 - \sigma^2$. From this inequality, it follows from Fact 3 that $x_{2r+2} = out(x_{2r+1}) + 1$

□

Fact 13 *There are F_{2r} series with inner width $2r+1$ (so outer width $2r+3$). These form a block of the Empire State Building that we call "block r". They are the series with seeds in the half-open interval $[F_{2r-1}, F_{2r+1})$ defined by adjacent odd rank Fibonacci numbers.*

Proof. Again we work in the same setting as in the proof of Fact 11, where r is fixed such that $F_{2r-1} \leq l < F_{2r+1}$. Block r of the Empire State Building consists of those series with an inner width of $2r+1$. There are F_{2r} series in Block r, because $F_{2r+1} - F_{2r-1} = (F_{2r} + F_{2r-1}) - F_{2r-1} = F_{2r}$. They are the series with seeds in the half-open interval $[F_{2r-1}, F_{2r+1})$ because we have assumed that in the proof of Fact 11. □

Fact 14 *The palindromic extraFib series are either multiples of the Fibonacci series or multiples of the Lucas series.*

Proof. The palindromic extraFib series are extraFib series that is its own reverse. Any palindromic extraFib series can be written in two ways:

$$\cdots, -x_{-3}, x_{-2}, -x_{-1}, x_0, x_1, x_2, x_3, \cdots, \text{if } x_0 > x_1$$

or

$$\cdots, x_{-3}, -x_{-2}, x_{-1}, x_0, x_1, x_2, x_3, \cdots, \text{if } x_0 \leq x_1$$

Where x_0 is our pivot but also the pillar, because of the palindromic property. The last negative term is at x_{-1} or x_{-2} because we know that the terms before the pivot must alternate in sign and there are only 2 possible alternating patterns. For the first case it follows from the palindromic property that the following systems of equations must hold:

$$-x_1 + x_0 = x_1$$
$$x_0 + x_1 = x_2$$

So it follows that:

$$x_0 = 2x_1$$
$$x_2 = 3x_1$$

This gives exactly all multiples of the Lucas series since it has to satisfy the Fibonacci's rule.($x_0 = 0$ gives the series with only zeros.). Similarly for the second case:

$$x_1 + x_0 = x_1$$
$$x_0 + x_1 = x_2$$

So it follows that:

$$x_0 = 0$$
$$x_1 = x_2$$

which is exactly all multiples of the Fibonacci series since it has to satisfy the Fibonacci's rule. □

Fact 15 *The palindromic series in block r are either multiples of the Fibonacci series (r even) or multiples of the Lucas series (r odd).*

Proof. The wall terms are both positive by Fact 11 and they have odd indices in an even-numbered block r. This is because the outer width is given by $2r + 3$, so the index of the right wall term is found by $\frac{2r+3-1}{2} = r + 1$. For the left wall, the index is then $-(r + 1)$. In case r is even, the indices of the wall terms are odd. So it follows that the negative terms that lie between the left wall and the pivot of a row in such a block must have even indices. This is exactly the second case in the proof of Fact 14. Therefore the palindromic series in such a block are multiples $n \times F$ of the Fibonacci series F. Similarly, in an odd-numbered block the palindromic rows are multiples $n \times L$ of the Lucas series L. □

We now have cut the Building into Blocks. We see palindromic series in block r are either multiples of the Fibonacci series (r odd) or multiples of the Lucas series (r even). We call odd blocks Fi-blocks and even blocks Lu-blocks. We can cut the blocks even further into slices by underlining the palindromic series they contain. Each slice has its height which is equal to the number of rows that it contains. The underlines are marked fi and lu in the two cases. So in case when r is odd, we talk of $Fifi$ blocks and in case when r is even, we talk of $Lulu$ blocks.

Fact 16 *In a Fifi block r (r even), there are a "lot" (L_r) of "fine" (height F_r) slices. In a Lulu block r (r odd), there are a "few" (F_r) "large" (height L_r) ones.*

Proof. Let F be the Fibonacci series and L be the Lucas series. It is enough to show that the last series in Fifi block r is $L_{r+1}F$ and the last series in Lulu block r is $F_{r+1}L$. The last series in block r has the seed $F_{2r+1} - 1$ because of Fact 13 and Fact 9. The wall term is then $out(F_{2r+1} - 1) + 1 = F_{2r}$. It follows from Fact 17 that these are the terms numbered r and $r + 1$ in L_rF or F_rL.

Since every slice in Fifi block ends with a palindromic series that is a multiple of the Fibonacci series, we know how many slices are inside the Fifi block r: $L_{r+1} - L_{r-1} = L_r$. For now we claim that $F_{2r} = L_rF_r$ and this will be proven in Fact 17. We know from Fact 13 that there are F_{2r} series in a Fifi block r and now we know that there are L_r slices in the same Fifi block. Therefore the height of the slices are $\frac{F_{2r}}{L_r} = F_r$. Similarly In a Lulu block r, there are $F_{r+1} - F_{r-1} = F_r$ slices. Using $F_{2r} = L_rF_r$ and the fact that there are F_{2r} series in a Lulu block r, the height of the slices are $\frac{F_{2r}}{F_r} = L_r$. □

Fact 17 *$F_{2r} = L_rF_r$ and $F_{2r-1} - 1$ is L_rF_{r-1} or F_rL_{r-1} according as r is odd or even.*

Proof. We will give a direct proof of $F_{2r} = L_rF_r$ by using the Binet formula. We know that $F_n = \frac{\tau^n-\sigma^n}{\sqrt{5}}$ and $L_n = \tau^n + \sigma^n$. By using the identity $a^2 - b^2 = (a+b)(a-b)$ we obtain that $L_nF_n = \frac{\tau^{2n}-\sigma^{2n}}{\sqrt{5}}$ which is equal to F_{2n}.

Let r be odd. Using the Binet formula again, we find that $F_{2r-1} - 1 = \frac{\tau^{2r-1}-\sigma^{2r-1}}{\sqrt{5}} - 1$ and $L_rF_{r-1} = \frac{(\tau^r+\sigma^r)(\tau^{r-1}-\sigma^{r-1})}{\sqrt{5}}$. We can write this out and we obtain $\frac{\tau^{2r-1}-\sigma^{2r-1}}{\sqrt{5}} - \frac{\tau^r\sigma^{r-1}-\tau^{r-1}\sigma^r}{\sqrt{5}}$. It is left to show that $\frac{\tau^r\sigma^{r-1}-\tau^{r-1}\sigma^r}{\sqrt{5}} = 1$ to establish the equation $F_{2r-1} - 1 = L_rF_{r-1}$. Since r odd, we can write it as $r = 2n+1$ for some n:

$$\begin{aligned}\frac{\tau^{2n+1}\sigma^{2n} - \tau^{2n}\sigma^{2n+1}}{\sqrt{5}} &= \frac{\tau(\tau\sigma)^{2n} - \sigma(\tau\sigma)^{2n}}{\sqrt{5}}\\ &= \frac{\tau(-1)^{2n} - \sigma(-1)^{2n}}{\sqrt{5}}\\ &= \frac{\tau-\sigma}{\sqrt{5}}\\ &= F_1\\ &= 1\end{aligned}$$

Let r be even. $F_rL_{r-1} = \frac{(\tau^r-\sigma^r)(\tau^{r-1}+\sigma^{r-1})}{\sqrt{5}}$. We write this out and we get $\frac{\tau^{2r-1}-\sigma^{2r-1}}{\sqrt{5}} - \frac{\tau^{r-1}\sigma^r-\tau^r\sigma^{r-1}}{\sqrt{5}}$. It is left to show that $\frac{\tau^{r-1}\sigma^r-\tau^r\sigma^{r-1}}{\sqrt{5}} = 1$ to establish the equation $F_{2r-1} - 1 = F_rL_{r-1}$. Since r even, we can write it as $r = 2n$ for some n:

$$\begin{aligned}\frac{\tau^{2n-1}\sigma^{2n} - \tau^{2n}\sigma^{2n-1}}{\sqrt{5}} &= \frac{\sigma(\tau\sigma)^{2n-1} - \tau(\tau\sigma)^{2n-1}}{\sqrt{5}}\\ &= \frac{\sigma(-1)^{2n-1} - \tau(-1)^{2n-1}}{\sqrt{5}}\\ &= \frac{\tau-\sigma}{\sqrt{5}}\\ &= F_1\\ &= 1\end{aligned}$$

□

Fact 18 *Entries to the right of the central pillar are positive. Entries to the left of it alternate in sign. In a Lulu block, entries in the central pillar are positive whereas in a Fifi block they can be positive, negative, or zero.*

Proof. We work in the same setting as in the proof of Fact 11 and use the equation 3.2. We need to show that for every $t \geq r+2$, $x_t > 0$, as these are the entries to the right of the central pillar. Note that we have $x_1 = -l$ which is a negative number. Since the signs alternate, negative terms might only occur for odd indices, so we only consider t odd. If r is odd (recall $F_{2r-1} \leq l < F_{2r+1}$),

$$\begin{aligned}\sqrt{5}x_t &= \tau^t - \theta\tau^{t-1} - \sqrt{5}l\sigma^{t-1} + \sigma^{t-2} + \theta\sigma^{t-1}\\ &> \tau^{r+2} - \tau^{r+1} - \sqrt{5}(F_{2r+1} - 1)\sigma^{r+1} + \sigma^r\\ &= \tau^{r+2} - \tau^{r+1} - \sqrt{5}(\frac{\tau^{2r+1} - \sigma^{2r+1}}{\sqrt{5}} - 1)\sigma^{r+1} + \sigma^r\\ &= \tau^{r+2} - \tau^{r+1} - \tau^{2r+1}\sigma^{r+1} + \sigma^{2r+1}\sigma^{r+1} + \sqrt{5}\sigma^{r+1} + \sigma^r\\ &= \tau^{r+2} - \tau^{r+1} - (\tau\sigma)^{r+1}\tau^r + \sigma^{3r+2} + (\tau-\sigma)\sigma^{r+1} + \sigma^r\\ &= \tau^{r+2} - \tau^{r+1} - \tau^r + \sigma^{3r+2} + \tau\sigma^{r+1} - \sigma^{2r+1} + \sigma^r\\ &= \tau^{r+2} - \tau^{r+2} + \sigma^{3r+2} - \sigma^r - \sigma^{r+2} + \sigma^r\\ &= \sigma^{3r+2} - \sigma^{r+2} > 0\end{aligned}$$

If r is even, $t \geq r+3$ and we have,

$$\begin{aligned}
\sqrt{5}x_t =&\tau^t - \theta\tau^{t-1} - \sqrt{5}l\sigma^{t-1} + \sigma^{t-2} + \theta\sigma^{t-1} \\
>&\tau^{r+3} - \tau^{r+2} - \sqrt{5}F_{2r+1}\sigma^{r+3} + \sigma^{r+1} \\
=&\tau^{r+3} - \tau^{r+2} - \sqrt{5}(\frac{\tau^{2r+1} - \sigma^{2r+1}}{\sqrt{5}})\sigma^{r+3} + \sigma^{r+1} \\
=&\tau^{r+3} - \tau^{r+2} - \tau^{2r+1}\sigma^{r+3} + \sigma^{2r+1}\sigma^{r+3} + \sigma^{r+1} \\
=&\tau^{r+3} - \tau^{r+2} - (\tau\sigma)^{r+3}\tau^{r-2} + \sigma^{3r+4} + \sigma^{r+1} \\
=&\tau^{r+3} - \tau^{r+2} + \tau^{r-2} + \sigma^{3r+4} + \sigma^{r+1} \\
=&\tau^{r+1} + \tau^{r-2} + \sigma^{3r+4} + \sigma^{r+1} > 0
\end{aligned} \tag{3.4}$$

Under reversal, this shows that entries to the left of the pillar must alternate in sign. In a Lulu block, r is odd, meaning that the the central entries have an even index. Since negative terms only occur in odd indices on the left side of the central entry and the fact that the signs alternate, the entries in the central pillar are positive. In case of a Fifi block, r even, entries in the central pillar can be positive, negative, or zero. □

Conway introduces a so called Fibonometry: For each standard trigonometry formula expressed as a linear relationship between products of sines and cosines, there exists a corresponding relationship between Fibonacci and Lucas numbers. In this relationship, Fibonacci numbers replace sines, Lucas numbers replace cosines, and only the coefficients are altered.

The precise rule for replacing is that an angle $\theta = p\alpha + q\beta + r\gamma + \cdots$ become a subscript $n = pa + qb + rc + \cdots$. We then replace $sin(\theta)$ by $\frac{i^n}{2}F_n$ and $cos(\theta)$ by $\frac{i^n}{2}L_n$ and insert a factor of (-5) for each successive pair of sines in a term (and so $(-5)^k$ if the term contains $2k$ or $2k+1$ sines in all). For instance $sin^4(\theta)$ is replaced by $(-5)^2(\frac{i^n}{2}F_n)^4$. The simplest case is $F_nL_n = F_{2n}$ which corresponds to $2sin(\theta)cos(\theta) = sin(2\theta)$. By following the rule for replacing, we have:

$$\begin{aligned}
2sin(\theta)cos(\theta) =&sin(2\theta) \\
2 * \frac{i^n}{2}F_n\frac{i^n}{2}L_n =&\frac{i^{2n}}{2}F_{2n} \\
\frac{i^{2n}}{2}F_nL_n =&\frac{i^{2n}}{2}F_{2n} \\
F_nL_n =&F_{2n}
\end{aligned}$$

Note that this identity appeared in Fact 17 and was proven with the Binet formula. Hence we can easily validate any equation of Fibonacci and Lucas numbers. We will give another example. From the equation $cos(3\theta) = 4cos^3(\theta) - 3cos(\theta)$ we have:

$$\begin{aligned}
cos(3\theta) =&4cos^3(\theta) - 3cos(\theta) \\
\frac{i^{(3n)}}{2}L_{3n} =&4(\frac{i^n}{2}L_n)^3 - 3\frac{i^n}{2}L_n \\
\frac{(-i)^n}{2}L_{3n} =&\frac{(-i)^n}{2}L_n^3 - \frac{3i^n}{2}L_n \\
(-i)^nL_{3n} =&(-i)^nL_n^3 - 3i^nL_n \\
L_{3n} =&L_n^3 - (-1)^n3L_n
\end{aligned}$$

This formula is easily checked by using the identity $L_n = \tau^n + \sigma^n$, $\tau\sigma = -1$:

$$\begin{aligned}
L_n^3 - (-1)^n3L_n =&(\tau^n + \sigma^n)^3 - (-1)^n3(\tau^n + \sigma^n) \\
=&\tau^{3n} + \sigma^{3n} + (-1)^n3\tau + (-1)^n3\sigma - (-1)^n3\tau - (-1)^n3\sigma \\
=&\tau^{3n} + \sigma^{3n} \\
=&L_{3n}
\end{aligned}$$

Fact 19 *In a series with seed s that lies in block r, the central entry is congruent to $(-1)^r sF_{r-1}(mod F_r)$.*

Proof. Let c be the central entry of a series x_n with seed s that lies in block r. We need to show that $c \cong (-1)^r sF_{r-1}(mod F_r)$, i.e. $c = kF_r + (-1)^r sF_{r-1}$ for some k constant integer. Let $s = F_a + F_b + \cdots + F_z$ to be the extended Zeckendorf representation of the seed s. It is called the extended Zeckendorf because we added $F_z = F_0 = 0$ on the unique Zeckendorf representation; This does not change the value of s. By *shifting* this expansion we obtain $w = F_{a+1} + F_{b+1} + \cdots + F_{z+1}$, a sum with value $out(s) + F_{z+1} = out(s) + F_1 = out(s) + 1$, which is the wall term. Now it follows from the Fibonacci's rule that the term with index $t \in \mathbb{Z}$ is the shifted expansion $x_t = F_{a+t} + F_{b+t} + \cdots + F_{z+t}$. The central term, which lies r terms away from the seed term, is then $c = F_{a-r} + F_{b-r} + \cdots + F_{z-r}$. We then apply the following Fibonometric identity to complete the proof: $F_{n-r} = (-1)^r F_n F_{r-1} + (-1)^{r+n} F_{1-n} F_r$ (we can validate this identity easily by applying the Binet formula).

$$
\begin{aligned}
c =&F_{a-r} + F_{b-r} + \cdots + F_{z-r}\\
=&((-1)^r F_a F_{r-1} + (-1)^{r+a} F_{1-a} F_r) + ((-1)^r F_b F_{r-1} + (-1)^{r+b} F_{1-b} F_r) +\\
&\cdots + ((-1)^r F_z F_{r-1} + (-1)^{r+z} F_{1-z} F_r)\\
=&kF_r + (-1)^r (F_a + F_b + \cdots + F_z) F_{r-1}\\
=&kF_r + (-1)^r sF_{r-1}
\end{aligned}
$$

Where $k = (-1)^{r+a} F_{1-a} + (-1)^{r+b} F_{1-b} + \cdots + (-1)^{r+c} F_{1-c}$ is a constant integer. □

Recall Fact 16, where it showed how the Fifi and Lulu blocks were sliced. For the following Facts about the central pillar, we define the alternative *segmentation* as follows. In a Fifi block, instead of a Fifi block is sliced into F_r slices of height L_r, it is sliced into F_r slices of height L_r. In a Lulu block (r odd), the segmentation of the central pillar is the same as the slicing. Thus in every block r is the central pillar cut into F_r segments of length L_r.

Fact 20 *In a Fifi block r, the central entry is congruent to sF_{r-1} modulo F_r, and in the m^{th} segment it lies in the range $[m - F_r, m)$.*

Proof. That the central entry is congruent to sF_{r-1} modulo F_r follows from the proof of Fact 19, because r is even in a Fifi block. To prove the second statement, we need to show that the central entry c lies in the right range. For the proof, we work in the same setting as in Fact 11, where x_t is defined with $x_0 = out(l) + 1$, $x_1 = -l$ with $l \geq 0$. In this case, x_1 is our left seed, hence $c = x_{r+1}$. We write $l = F_{2r-1} + mL_r - n$, where $1 \leq m \leq F_r$ and $1 \leq n \leq L_r$. From equation 3.2 we have:

$$
\begin{aligned}
x_{r+1} &= (\tau^{r+1} - \theta(\tau^r - \sigma^r) - \sqrt{5}(F_{2r-1} + mL_r - n)\sigma^r + \sigma^{r-1})/\sqrt{5}\\
&= [F_{r+1} - \theta F_r - (-1)^r (F_{r-1} + m)] + [(n-1)\sigma^r - m\sigma^{2r} + (\sigma^{3r-1} - (-1)^r \sigma^{r-1})/\sqrt{5}]
\end{aligned}
\tag{3.5}
$$

Here, the first summand is the *major* term and the second one is a *small correction*. The major term belongs to an open interval of length F_r, which is either $(-m, F_r - m)$ or $(2F_{r-1} + m, L_r + m)$ according as r is even or odd. Since r is even in a Fifi block, we have that the major term equals $F_{r+1} - \theta F_r - F_{r-1} - m$. For the lower bound, we have

$$F_{r+1} - \theta F_r > F_{r-1} \ (0 < \theta < 1)$$

So

$$F_{r+1} - \theta F_r - F_{r-1} - m > F_{r-1} - F_{r-1} - m = -m$$

For the upper bound, note that $F_{r+1} - F_{r-1} = F_r$. It follows that

$$F_{r+1} - \theta F_r - F_{r-1} - m = F_r - \theta F_r - m > F_r - m$$

For r even, we will show that the small correction is positive and less than 1, showing that if the reverse has seed l, then its central entry in a Fifi block, and for the series with seed l, the central entry lies in the interval $(-m, F_r - m]$.

$$\begin{aligned}(n-1)\sigma^r - m\sigma^{2r} + (\sigma^{3r-1} - \sigma^{r-1})/\sqrt{5} &> -m\sigma^{2r} + \frac{\sigma^{3r-1}}{\sqrt{5}}\\ &> F_r\sigma^{2r} + \frac{\sigma^{3r-1}}{\sqrt{5}}\\ &> \frac{\tau^{2r}\sigma^{2r} + \sigma^{4r}}{\sqrt{5}} + \frac{\sigma^{3r-1}}{\sqrt{5}}\\ &= \frac{1 + \sigma^{4r} + \sigma^{3r-1}}{\sqrt{5}} > 0\end{aligned}$$

And we also have

$$\begin{aligned}(n-1)\sigma^r - m\sigma^{2r} + (\sigma^{3r-1} - \sigma^{r-1})/\sqrt{5} &< n\sigma^r - \frac{\sigma^{r-1}}{\sqrt{5}}\\ &< L_r\sigma^r - \sigma^{r-1}\\ &= \tau^r\sigma^r + \sigma^{2r} - \sigma^{r-1}\\ &< 1 + \sigma^0 - \sigma^0 = 1\end{aligned}$$

However, reversal changes the sign of the central entry in a Fifi block, and for the series with seed l, the central entry lies in the interval $[m - F_r, m)$. □

Fact 21 *In a Lulu block, say block r (r odd), the central entry is congruent to sF_{r-2} modulo F_r and in the l^{th} slice it lies in the range $[2F_{r-1} + l, L_r + l)$. However, the last central entry in a Lulu block is aberrant and takes the value $L_r + l$ rather than L_r.*

Proof. First note that from Fact 19 for a seed s if r odd, the central entry c is congruent to $-sF_{r-1}(mod F_r)$. This is equivalent as saying that $c = kF_r - sF_{r-1}$ for some integer k. We use $F_{r-1} = F_r - F_{r-2}$ to obtain $c = kF_r - s(F_r - F_{r-2}) = (k - s)F_r + sF_{r-2}$ where $k - s$ is again an integer. It follows that the central entry is congruent to sF_{r-2} modulo F_r.
We will complete the proof using the proof of Fact 20. We use the letter m (like in the proof of Fact 20) for the l^{th} slice, because in a Lulu block (r odd), the segmentation of the central pillar is the same as the slicing. We will first show that the small correction in equation 3.5 is still bounded in magnitude by 1, but this time it can take both positive and negative signs.

$$\begin{aligned}(n-1)\sigma^r - m\sigma^{2r} + (\sigma^{3r-1} + \sigma^{r-1})/\sqrt{5} &> L_r\sigma^r - F_r\sigma^{2r} + \frac{\sigma^{3r-1}}{\sqrt{5}}\\ &= \tau^r\sigma^r + \sigma^{2r} - \frac{\sigma^r - \sigma^{3r}}{\sqrt{5}} + \frac{\sigma^{3r-1}}{\sqrt{5}}\\ &> -1 + \frac{\sigma^3 + \sigma^2}{\sqrt{5}} > -1\end{aligned}$$

And

$$\begin{aligned}(n-1)\sigma^r - m\sigma^{2r} + (\sigma^{3r-1} + \sigma^{r-1})/\sqrt{5} &< (\sigma^{3r-1} + \sigma^{r-1})/\sqrt{5}\\ &< \sigma^{3r-1} + \sigma^{r-1} < 1\end{aligned}$$

The major term now becomes $F_{r+1} - \theta F_r + F_{r-1} + m$. The lower bound is found by using the inequality $F_{r-1} < F_{r+1} - \theta F_r$ since $0 < \theta < 1$. So

$$F_{r+1} - \theta F_r + F_{r-1} + m > F_{r-1} + F_{r-1} + m = 2F_{r-1} + m$$

For the upper bound, we use the identity $L_n = F_{n-1} + F_{n+1}$. We have

$$F_{r+1} - \theta F_r + F_{r-1} + m = L_r - \theta F_r + m < L_r + m$$

This places the central entry in the interval $[2F_{r-1}+m, L_r+m]$. However, the only positive term in the small correction is the last one: $(\sigma^{3r-1}+\sigma^{r-1})/\sqrt{5} = \sigma^{r-1}(1+\sigma^{2r})/\sqrt{5}$. If this is to outweigh the negative contribution of $-\theta F_r$, we have that $\sigma^{r-1}(1+\sigma^{2r})/\sqrt{5} > \theta F_r$. We have

$$\begin{aligned}\theta <& \frac{\sigma^{r-1}(1+\sigma^{2r})/\sqrt{5}}{F_r}\\ =& \frac{\sigma^{r-1}(1+\sigma^{2r})/\sqrt{5}}{(\tau^r-\sigma^r)/\sqrt{5}}\\ =& \frac{\sigma^{r-1}(1+\sigma^{2r})}{(\tau^r-\sigma^r)}\\ =& -\sigma^{2r-1}\end{aligned}$$

Since $(\tau^r-\sigma^r) * -\sigma^{2r-1} = \sigma^{r-1}+\sigma^{3r-1} = \sigma^{r-1}(1+\sigma^{2r})$.

Note that we can write $\theta = -\sigma+\tau l - out(l)$. Then from the proof of Fact 3, we know that $out(l)-\tau l = \sigma^a+\sigma^b+\cdots$ where $a, b, \cdots$ are the indices of the Fibonacci numbers of the Zeckendorf notation of l. Then we have for block r,

$$\begin{aligned}\sigma^a+\sigma^b+\cdots \leq& \sigma^{2r}+\sigma^{2r-2}+\cdots+\sigma^2\\ =& \sigma^2(\sigma^{2r-2}+\cdots+1)\\ =& \sigma^2\frac{1-\sigma^{2r}}{1-\sigma^2}\\ =& \sigma^2\frac{1-\sigma^{2r}}{-\sigma}\\ =& -\sigma+\sigma^{2r+1}\end{aligned}$$

So we have the inequality $\theta \leq -\sigma+\sigma-\sigma^{2r+1} = \sigma^{2r+1}$. It follows that the only seed in block r that could give such a small value of θ must have the Zeckendorf expansion $F_{2r}+F_{2r-2}+\cdots+F_2$. We will show that the series grown from this seed is the Lucas multiple $F_{r+1}\times L$. So this gives the last row in a Lulu block r and the last central entry in a Lulu block is indeed aberrant and takes the value L_r+l. We first show that $F_{2r}+F_{2r-2}+\cdots+F_2 = F_{r+1}\times L_r$, which shows that this sum indeed is the seed of the Lucas multiple $F_{r+1}\times L$. By using Fact 17 we have:

$$\begin{aligned}F_{2r}+F_{2r-2}+\cdots+F_2 &= F_{r+1}\times L_r\\ &= (F_{r-1}+F_r)\times L_r\\ &= F_{r-1}L_r+F_rL_r\\ &= F_{2r-1}-1+F_{2r}\end{aligned}$$

The term F_{2r} cancels out and it is left to show that $F_{2r-2}+F_{2r-4}+\cdots+F_2 = F_{2r-1}-1$. By using that $F_n-F_{n+1} = -F_{n-1}$ we have:

$$\begin{aligned}F_{2r-2}+F_{2r-4}+\cdots+F_2 &= F_{2r-1}-1\\ F_{2r-2}+F_{2r-4}+\cdots+F_2 &= F_{2r-1}+F_{2r-3}-F_{2r-3}+F_{2r-5}-F_{2r-5}+\cdots-1\\ F_{2r-2}-F_{2r-1}+F_{2r-4}-F_{2r-3}+\cdots+F_2-F_3 &= -F_{2r-3}-F_{2r-5}-\cdots-1\\ -F_{2r-3}-F_{2r-5}-\cdots-1 &= -F_{2r-3}-F_{2r-5}-\cdots-1\end{aligned}$$

Secondly, we show that $F_{2r+1}+F_{2r-1}+\cdots+F_3+1 = F_{r+1}\times L_{r+1}$. Then the sum on the left side of the equation is the wall of the Lucas multiple $F_{r+1}\times L$. From this, we can conclude that the series grown from the seed is the Lucas multiple $F_{r+1} \times L$. By using Fact 17 we have:

$$\begin{aligned} F_{2r+1} + F_{2r-1} + \cdots + F_3 + 1 &= F_{r+1} \times L_{r+1} \\ F_{2r+1} + F_{2r-1} + \cdots + F_3 + 1 &= F_{2r+2} \\ F_{2r-1} + \cdots + F_3 + 1 &= F_{2r+2} - F_{2r+1} \\ F_{2r-1} + \cdots + F_3 + 1 &= F_{2r} \\ F_{2r-3} + \cdots + F_3 + 1 &= F_{2r} - F_{2r-1} \\ F_{2r-3} + \cdots + F_3 + 1 &= F_{2r-2} \\ &\cdots \\ 1 &= 1 \end{aligned}$$

□

Fact 22 *The central entry of a series in a Lulu block is adjacent to the two terms of the series that have the smallest absolute values*

Proof. If a series belongs to a Lulu block, its central entry is positive (Fact 18) and is larger than its right-hand neighbor (because its left neighbor is negative). But we know that the central entry is smaller than all other entries to its right by the Fibonacci's rule, since the terms on the right of the central entry are positive (Fact 18). Hence if we consider the reversal, Fact 22 immediately follows. □

Fact 23 *The center is the term with smallest absolute value in any series in a Fifi block.*

Proof. In a Fifi block, a similar argument as for Fact 22 shows that the term with the smallest absolute value that lies at or to the right of the central column is either the central entry or the term two places to its right. From Fact 20, the central entry is bounded by $F_r - 1$ in absolute value. We will show that the entry two places to the right of the central entry is larger than F_r in a Fifi block r even. Recall inequality 3.4 for $t \geq r + 3$:

$$\sqrt{5}x_t > \tau^{r+1} - \tau^{r-2} + \sigma^{3r+4} + \sigma^{r+1}$$

For the entry two places to the right of the central entry (x_{r+3}) we have to show that,

$$\sqrt{5}x_{r+3} > \tau^{r+1} - \tau^{r-2} + \sigma^{3r+4} + \sigma^{r+1} > \tau^r - \sigma^r \tag{3.6}$$

Then after dividing both sides by $\sqrt{5}$ and using the Binet formula our Fact follows. The second inequality is verified as follows:

$$\begin{aligned} \tau^{r+1} - \tau^{r-2} + \sigma^{3r+4} + \sigma^{r+1} &> \tau^r - \sigma^r \\ \tau^{r+1} - \tau^r - \tau^{r-2} &> -\sigma^{3r+4} - \sigma^{r+1} - \sigma^r \\ \tau^{r-1} - \tau^{r-2} &> -\sigma^{3r+4} - \sigma^{r+2} \\ \tau^{r-3} &> -\sigma^{3r+4} - \sigma^{r+2} \end{aligned}$$

The second inequality follows immediately because the left side is positive and the right side is negative since r is even.

□

Fact 24 *A nonpalindromic series has two centers, its "fi center" X_f and its "lu center" X_l, where f and l differ by 1. Its multiples mX_n will be centered at either mX_f or mX_l according as they lie in Fifi or Lulu blocks.*

Proof. Let X_n be a nonpalindromic series. If X_n is in a Lulu block with the "lu center" X_l, the central term is adjacent to the two terms of the series that have the smallest absolute values (Fact 22). If the multiples mX_n is in a Lulu block again, then mX_l is again the central term, since adjacent terms are still the smallest in absolute value. If the multiple mX_n is in a Fifi block, then the central term is either mX_{l-1} or mX_{l+1} by Fact 23. Those two terms are the candidates for the "fi center" and only differ by 1 from the "lu center" X_l.
On the other hand, If the series X_n is in a Fifi block with the "fi center" X_f, then it should be the smallest term in absolute value (Fact 23). If the multiples mX_n is in a Fifi block again, then the central entry is

mX_f. If the multiple mX_n is in a Lulu block. Then by Fact 22, the central entry cannot be mX_f and it is either mX_{f-1} or mX_{f+1}. Those two terms are the candidates for the ”lu center” and only differ by 1 from the ”fi center” X_f. □

A

Appendix

block								0	1	2	3	4	5	6	7	8	9	10
0 (	*fi*						1	0	1	1	2	3	5	8	13	21	34	55
1 (	*lu*					3	-1	2	1	3	4	7	11	18	29	47	76	123
2	*fi*				4	-2	2	0	2	2	4	6	10	16	26	42	68	
	fi				6	-3	3	0	3	3	6	9	15	24	39	63	102	
	fi				8	-4	4	0	4	4	8	12	20	32	52	84	136	
3				12	-7	5	-2	3	1	4	5	9	14	23	37	60	97	
				17	-10	7	-3	4	1	5	6	11	17	28	45	73	118	
				9	-5	4	-1	3	2	5	7	12	19	31	50	81	131	
	lu			14	-8	6	-2	4	2	6	8	14	22	36	58	94	152	
				19	-11	8	-3	5	2	7	9	16	25	41	66	107	173	
				11	-6	5	-1	4	3	7	10	17	27	44	71	115	186	
				16	-9	7	-2	5	3	8	11	19	30	49	79	128	207	
	lu			21	-12	9	-3	6	3	9	12	21	33	54	87	141	228	
4			33	-20	13	-7	6	-1	5	4	9	13	22	35	57	92		
			46	-28	18	-10	8	-2	6	4	10	14	24	38	62	100		
	fi		25	-15	10	-5	5	0	5	5	10	15	25	40	65	105		
			38	-23	15	-8	7	-1	6	5	11	16	27	43	70	113		
			51	-31	20	-11	9	-2	7	5	12	17	29	46	75	121		
	fi		30	-18	12	-6	6	0	6	6	12	18	30	48	78	126		
			43	-26	17	-9	8	-1	7	6	13	19	32	51	83	134		
			22	-13	9	-4	5	1	6	7	13	20	33	53	86	139		
	fi		35	-21	14	-7	7	0	7	7	14	21	35	56	91	147		
			48	-29	19	-10	9	-1	8	7	15	22	37	59	96	155		
			27	-16	11	-5	6	1	7	8	15	23	38	61	99	160		
	fi		40	-24	16	-8	8	0	8	8	16	24	40	64	104	168		
			53	-32	21	-11	10	-1	9	8	17	25	42	67	109	176		
			32	-19	13	-6	7	1	8	9	17	26	43	69	112	181		
	fi		45	-27	18	-9	9	0	9	9	18	27	45	72	117	189		
			24	-14	10	-4	6	2	8	10	18	28	46	74	120	194		
			37	-22	15	-7	8	1	9	10	19	29	48	77	125	202		
	fi		50	-30	20	-10	10	0	10	10	20	30	50	80	130	210		
			29	-17	12	-5	7	2	9	11	20	31	51	82	133	215		
			42	-25	17	-8	9	1	10	11	21	32	53	85	138	223		
	fi		55	-33	22	-11	11	0	11	11	22	33	55	88	143	231		
5		88	-54	34	-20	14	-6	8	2	10	12	22	34	56	90	146		
		122	-75	47	-28	19	-9	10	1	11	12	23	35	58	93	151		
		67	-41	26	-15	11	-4	7	3	10	13	23	36	59	95	154		
		101	-62	39	-23	16	-7	9	2	11	13	24	37	61	98	159		
		135	-83	52	-31	21	-10	11	1	12	13	25	38	63	101	164		
		80	-49	31	-18	13	-5	8	3	11	14	25	39	64	103	167		
		114	-70	44	-26	18	-8	10	2	12	14	26	40	66	106	172		
		59	-36	23	-13	10	-3	7	4	11	15	26	41	67	108	175		
		93	-57	36	-21	15	-6	9	3	12	15	27	42	69	111	180		
		127	-78	49	-29	20	-9	11	2	13	15	28	43	71	114	185		
	lu	72	-44	28	-16	12	-4	8	4	12	16	28	44	72	116	188		
		106	-65	41	-24	17	-7	10	3	13	16	29	45	74	119	193		

Figure A.1: The Empire State Building: blocks 0 to 4 and part of block 5 (Conway, 2016)

	lu	72	-44	28	-16	12	-4	8	4	12	16	28	44	72	
		106	-65	41	-24	17	-7	10	3	13	16	29	45	74	
		140	-86	54	-32	22	-10	12	2	14	16	30	46	76	
		85	-52	33	-19	14	-5	9	4	13	17	30	47	77	
		119	-73	46	-27	19	-8	11	3	14	17	31	48	79	
		64	-39	25	-14	11	-3	8	5	13	18	31	49	80	
		98	-60	38	-22	16	-6	10	4	14	18	32	50	82	
		132	-81	51	-30	21	-9	12	3	15	18	33	51	84	
		77	-47	30	-17	13	-4	9	5	14	19	33	52	85	
		111	-68	43	-25	18	-7	11	4	15	19	34	53	87	
		56	-34	22	-12	10	-2	8	6	14	20	34	54	88	
	lu	90	-55	35	-20	15	-5	10	5	15	20	35	55	90	
		124	-76	48	-28	20	-8	12	4	16	20	36	56	92	
		69	-42	27	-15	12	-3	9	6	15	21	36	57	93	
		103	-63	40	-23	17	-6	11	5	16	21	37	58	95	
		137	-84	53	-31	22	-9	13	4	17	21	38	59	97	
		82	-50	32	-18	14	-4	10	6	16	22	38	60	98	
		116	-71	45	-26	19	-7	12	5	17	22	39	61	100	
		61	-37	24	-13	11	-2	9	7	16	23	39	62	101	
		95	-58	37	-21	16	-5	11	6	17	23	40	63	103	
		129	-79	50	-29	21	-8	13	5	18	23	41	64	105	
		74	-45	29	-16	13	-3	10	7	17	24	41	65	106	
5	*lu*	108	-66	42	-24	18	-6	12	6	18	24	42	66	108	
		142	-87	55	-32	23	-9	14	5	19	24	43	67	110	
		87	-53	34	-19	15	-4	11	7	18	25	43	68	111	
		121	-74	47	-27	20	-7	13	6	19	25	44	69	113	
		66	-40	26	-14	12	-2	10	8	18	26	44	70	114	
		100	-61	39	-22	17	-5	12	7	19	26	45	71	116	
		134	-82	52	-30	22	-8	14	6	20	26	46	72	118	
		79	-48	31	-17	14	-3	11	8	19	27	46	73	119	
		113	-69	44	-25	19	-6	13	7	20	27	47	74	121	
		58	-35	23	-12	11	-1	10	9	19	28	47	75	122	
		92	-56	36	-20	16	-4	12	8	20	28	48	76	124	
	lu	126	-77	49	-28	21	-7	14	7	21	28	49	77	126	
		71	-43	28	-15	13	-2	11	9	20	29	49	78	127	
		105	-64	41	-23	18	-5	13	8	21	29	50	79	129	
		139	-85	54	-31	23	-8	15	7	22	29	51	80	131	
		84	-51	33	-18	15	-3	12	9	21	30	51	81	132	
		118	-72	46	-26	20	-6	14	8	22	30	52	82	134	
		63	-38	25	-13	12	-1	11	10	21	31	52	83	135	
		97	-59	38	-21	17	-4	13	9	22	31	53	84	137	
		131	-80	51	-29	22	-7	15	8	23	31	54	85	139	
		76	-46	30	-16	14	-2	12	10	22	32	54	86	140	
		110	-67	43	-24	19	-5	14	9	23	32	55	87	142	
	lu	144	-88	56	-32	24	-8	16	8	24	32	56	88	144	
6 (	232	-143	89	-54	35	-19	16	-3	13	10	23	33	56	89	145

Figure A.2: The Empire State Building: the rest of block 5 (Conway, 2016)

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