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# NON-TYCHONOFF $e$-COMPACTIFIABLE SPACES

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ABSTRACT. We construct a non-Tychonoff space $X$ which is $e$-compactifiable, thus answering a question of S. Hechler. We also answer a question of R. M. Stephenson: whether there exists a Tychonoff space, the largest $e$-compactification of which has a noncompact semiregularization.

**1. Introduction.** All spaces are Hausdorff. In **[He]** S. Hechler introduced the class of $e$-compactifiable spaces, i.e. spaces which admit an $e$-compactification. He posed the question whether there exist non-Tychonoff $e$-compactifiable spaces. We show that such spaces exist. In **[St]** R. M. Stephenson observed that an $e$-compactifiable space has a largest $e$-compactification $eX$, and he asked whether the space $(eX)_S$—the semiregularization of $eX$—is always compact. We show that this need not be the case, even if the space $X$ is assumed to be Tychonoff. The example of the space we present is based on an example of J. Chaber.

**2. Preliminary definitions and theorems.**

DEFINITION 2.1 **[He]**. Let $D$ be a dense subspace of $X$. $X$ is said to be *e-compact with respect to $D$* if each open cover of $X$ contains a finite subcollection that covers $D$. If so, $X$ is called an *e-compactification* of $D$ and $D$ is called *e-compactifiable*. $\square$

Observe that within this terminology the expression "let $X$ be an $e$-compact space" is meaningless. From this definition it readily follows that an $e$-compactification of a space $X$ is an $H$-closed extension. The following theorem shows that the converse need not be true.

THEOREM 2.2 **[He]**. *Let $pX$ be an extension of $X$. Then the following statements are equivalent*:

(i) *$pX$ is an e-compactification of $X$.*

(ii) *Every ultrafilter on $X$ has an accumulation point in $pX$.*

(iii) *$pX$ is $H$-closed and $X \cup \{q\}$ is regular, for all $q \in pX$.* $\square$

It follows that an $e$-compactifiable space is regular. The converse is not the case. From 2.2(iii) we can conclude that each noncompact $\mathfrak{R}$-closed space (i.e. a regular space which is closed in every regular space in which it is embedded, see **[BS]**) is an example of a regular non-$e$-compactifiable space. It is clear that every Tychonoff space is $e$-compactifiable, and in **[He]** the question appeared whether the converse

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holds. In the next section we show that this is not the case. We were unable to characterize the class of $e$-compactifiable spaces in terms of some separation property.

The following properties of $e$-compactifiable spaces are known.

THEOREM 2.3 **[He]**. (i) *Let $pX$ be an $e$-compactification of $X$. Then* $\mathrm{cl}_{pX} Y$ *is an $e$-compactification of $Y$, for each $Y \subset X$.*

(ii) *Let $p_\iota X_\iota$ be an $e$-compactification of $X_\iota$ ($\iota \in I$). Then $\prod p_\iota X_\iota$ is an $e$-compactification of $\prod X_\iota$.* □

Recall that a subset $U \subset X$ is regular-closed if $\mathrm{cl}\,\mathrm{int}\, U = U$. The collection of regular-closed subsets of $X$ is a closed base for some topology on $X$. $X$ supplied with this topology is called the semiregularization of $X$, to be denoted by $X_S$. $X$ is called semiregular if $X$ is homeomorphic to $X_S$.

In **[St]** R. M. Stephenson observed that Theorem 2.3 implies that each $e$-compactifiable space $X$ has a largest $e$-compactification $eX$, i.e. if $\alpha X$ is an $e$-compactification of $X$ then the map id: $X \to \alpha X$ has a continuous extension over $eX$.

THEOREM 2.4. (i) **[St]** *Let $X$ be an $e$-compactifiable space. Then $X$ is an open subspace of $eX$ and $eX - X$ is a closed discrete subspace of $eX$.*

(ii) *Let $f$: $X \to Y$ be a continuous map and assume that both $X$ and $Y$ are $e$-compactifiable. Then there is a continuous extension $ef$: $eX \to eY$ of $f$.*

PROOF. (ii) According to 2.3(ii) we have that $eX \times eY$ is an $e$-compactification of $X \times Y$. Define $\tilde{X} = \{(x, f(x)): x \in X\} \subset X \times Y$. $\tilde{X}$ is a closed subset of $X \times Y$ and $\prod_X \restriction \tilde{X}$: $\tilde{X} \to X$ is a homeomorphism. Since $\mathrm{cl}_{eX\times eY} \tilde{X}$ is an $e$-compactification of $\tilde{X}$, the map $(\prod_X \restriction \tilde{X})^{-1}$: $X \to \tilde{X}$ has an extension $e(\prod_X \restriction \tilde{X})^{-1}$: $eX \to \mathrm{cl}_{eX\times eY} \tilde{X}$. Define $ef = \prod_{eY} \circ e(\prod_X \restriction \tilde{X})^{-1}$. □

As a method to answer the question of S. Hechler, R. M. Stephenson asked the following question.

"Let $X$ be an $e$-compactifiable space. Is the space $(eX)_S$ always compact?"

Our example of a non-Tychonoff $e$-compactifiable space provides a negative answer to this question. A partial positive answer to Stephenson's question is the following

THEOREM 2.5 **[St]**. *Let $X$ be a regular space. If disjoint regular closed sets are contained in disjoint open subsets (in particular, if $X$ is normal), then $X$ is Tychonoff (hence $e$-compactifiable) and $(eX)_S$ is compact.* □

Our second example shows that the answer is negative if $X$ is only assumed to be Tychonoff. The following simple lemma is one of the keys to the construction.

LEMMA 2.6. *Let $X$ be a Tychonoff space. Then $(eX)_S$ is compact iff the map* $e(\mathrm{id})$: *$eX \to \beta X$ is injective.*

PROOF. Observe that $X$ is a subspace of $(eX)_S$ and that the map $e(\mathrm{id})$: $(eX)_S \to \beta X$ is also continuous. Then we have "$\to$", since $(eX)_S$ is a compactification of $X$ and "$\leftarrow$" holds because $(eX)_S$ is minimal Hausdorff and the topology of $\beta X$ is weaker than that of $(eX)_S$. □

**3. The results.** The following theorem is the key to our construction of a non-Tychonoff $e$-compactifiable space.

THEOREM 3.1. *Perfect preimages of e-compactifiable spaces are e-compactifiable.*

PROOF. Let $X$ be an $e$-compactifiable space and let $f\colon Y \to X$ be a perfect map. We construct an $e$-compactification $\alpha Y$ of $Y$ in the following way. The underlying set of $\alpha Y$ is $Y \oplus (eX - X)$ and a topology is defined by

(i) $Y$ is open in $\alpha Y$;

(ii) For $p \in \alpha Y - Y = eX - X$ the collection $\mathcal{U}_p = \{\{p\} \cup f^{-1}(X \cap U)\colon U$ open in $eX$ & $p \in U\}$ is taken as a local base in $p \in \alpha Y$.

One readily sees that $\alpha Y$ is a Hausdorff extension of $Y$. To see that $\alpha Y$ is an $e$-compactification of $Y$, consider an ultrafilter $\mathcal{F}$ on $Y$. Then $f(\mathcal{F}) = \{f(F)\colon F \in \mathcal{F}\}$ is an ultrafilter on $X$; hence $f(\mathcal{F})$ has an accumulation point $q$ in $eX$. If $q \in X$ then, since $f$ is perfect, $\mathcal{F}$ has an accumulation point in $f^{-1}(q)$. If $q \in eX - X$, then $f(F) \cap U_q \neq \varnothing$ for each open neighborhood $U_q$ of $q$ in $eX$ and $F \in \mathcal{F}$. Since $f(F) \subset X$ it follows that $F \cap f^{-1}(U_q \cap X) \neq \varnothing$, i.e. $q$—considered as an element of $\alpha Y$—is an accumulation point of $\mathcal{F}$. $\square$

In **[Ch]** J. Chaber constructed examples of non-Tychonoff perfect preimages of Tychonoff spaces, and so these examples establish the existence of non-Tychonoff $e$-compactifiable spaces. From 2.3(i) it follows that subspaces of perfect preimages of Tychonoff spaces are $e$-compactifiable. We were not able to construct $e$-compactifiable spaces outside this particular class. Observe that a space $X$ in this class (with $|X| > 1$) admits nonconstant real-valued continuous functions.

*Question* 3.2. Do there exist $e$-compactifiable spaces on which every real-valued continuous function is constant?

Let us now answer the question of R. M. Stephenson, whether there exist Tychonoff spaces $X$ for which $(eX)_S$ is not compact. Our strategy is as follows. We construct a Tychonoff space $X$, a point $p \in \beta X - X$ and an extension $\alpha X$ of $X$ such that $|\alpha X - X| > 1$ and such that the map $f\colon \alpha X \to X \cup \{p\}$ $(\subset \beta X)$ defined by $f(x) = x$ $(x \in X)$ and $f(\alpha X - X) = p$ is perfect. It then follows that $\alpha X$ is $e$-compactifiable, and since $e\alpha X$ can be considered as an $e$-compactification of $X$, we can conclude from the diagram below that the map $e(\mathrm{id})\colon eX \to \beta X$ is not injective. ($e_1$ is the extension of id: $X \to \alpha X \subset e\alpha X$ to $eX$ (see 2.4(iii)).) ($e_2$ is the extension of id: $X \cup \{p\} \to \beta(X \cup \{p\})$ to $e(X \cup \{p\}$.) Indeed, the diagram shows that $e(\mathrm{id}) = e_2 \circ ef \circ e_1$; hence $e(\mathrm{id})^{-1}(p) > 1$.

$$\begin{tikzcd}
 & eX \arrow[drr, "e_1"] & & \\
 & \alpha X \arrow[d, "f"] & & e\alpha X \arrow[ll] \arrow[d, "ef"] \\
X \arrow[uur] \arrow[ur] \arrow[r] \arrow[dr] & X \cup \{p\} \arrow[d] & & e(X \cup \{p\}) \arrow[ll] \arrow[dll, "e_2"] \\
 & \beta X = \beta(X \cup \{p\}) & &
\end{tikzcd}$$

The example we present is almost identical to the one constructed by J. Chaber. The only difference lies in the fact that we want the point $p$ to lie in the Čech-Stone remainder of $X$. For the reader's convenience we give the construction in detail.

EXAMPLE 3.3. Put $T = (\omega_1 + 1) \times (\omega_1 + 1) - \{(\omega_1, \omega_1)\}$. The set of pairs of the form $(\alpha, \omega_1) \in T$ will be called the left edge of $T$. The set of pairs of the form $(\omega_1, \alpha) \in T$ will be called the right edge of $T$. Define the space $T^n$, for $n \in \mathbf{N}$, as the space obtained by identification in the sum $\oplus_{i=1}^{n} T(i)$ where $T(i) = T \times \{i\}$, of the right edge of $T(i)$ with the left edge of $T(i+1)$. Let $\varphi_n$: $\oplus_{i=1}^{n} T(i) \to T^n$ be the corresponding identification-map. For each $0 \leqslant k \leqslant n$ we define an open subset $U_k^n \subset T^n$, by

$$U_k^n = \begin{cases} \operatorname{int} \varphi_n(T(1)) & (k = 0), \\ \operatorname{int} \varphi_n(T(k) \cup T(k+1)) & (k = 1, \dots, n-1), \\ \operatorname{int} \varphi_n(T(n)) & (k = n). \end{cases}$$

Finally we define $X = \oplus_{n=1}^{\infty} T^n$.

It is well known that $|\beta T^n - T^n| = 1$, for each $n \in \mathbf{N}$. For $\alpha < \omega_1$ put $Z_\alpha = [\alpha, \omega_1] \times [\alpha, \omega_1] - \{(\omega_1, \omega_1)\}$. Then $\{Z_\alpha: \alpha < \omega_1\}$ is a base for the unique nonfixed $z$-ultrafilter on $T$. If we define, for $n \in \mathbf{N}$, $Z_\alpha^n = \varphi_n(\oplus_{i=1}^{n}(Z_\alpha \times (i)))$ then $\{Z_\alpha^n: \alpha < \omega_1\}$ is a base for the unique nonfixed $z$-ultrafilter $\mathbf{Z}^n$ on $T^n$.

Next we define a point $p \in \beta X - X$. Let $\mathcal{G}$ be a nonfixed ultrafilter on $\mathbf{N}$. For $G \in \mathcal{G}$ and $\alpha < \omega_1$ put $Z(G, \alpha) = \bigcup\{Z_\alpha^n: n \in G\}$. It is easy to verify that the collection $\{Z(G, \alpha): G \in \mathcal{G}, \alpha < \omega_1\}$ is a base for a nonfixed $z$-ultrafilter $\mathcal{F}$ on $X$. Let $p \in \beta X - X$ be the point in $\beta X$ corresponding to $\mathcal{F}$, i.e. $\{p\} = \bigcap\{\mathrm{cl}_{\beta X} F: F \in \mathcal{F}\}$. In the space $X \cup \{p\}$ we have the following: If $U$ is open in $X$ then $U \cup \{p\}$ is a neighborhood of $p$ in $X \cup \{p\}$ iff $\exists G \in \mathcal{G} \exists \alpha < \omega_1$ such that $Z(G, \alpha) \subset U$. (*)

(This is not completely trivial, since $X$ is not normal. However, it follows easily by considering the space $\tilde{X} = \oplus_{n=1}^{\infty} \mathrm{cl}_{\beta X} T^n \subset \beta X$, which is $\sigma$-compact (hence normal). We omit the details.)

Let us now introduce a topology on the set $X \cup [0, 1]$ ($[0, 1]$ is the unit interval) in the following way. For $t \in [0, 1]$ let $\{V_l(t)\}_{l=1}^{\infty}$ be a countable local base at $t$. For $t \in [0, 1]$, $l \in \mathbf{N}$, $G \in \mathcal{G}$, $\alpha < \omega_1$ define

$$U(t, l, G, \alpha) = \bigcup_{n \in G} \bigcup_{s \in V_l(t)} \left(U_{[n.s]}^n \cap Z_\alpha^n\right) \cup V_l(t).$$

(Here $[n.s]$ denotes the greatest integer not greater than $n.s$.) And next we put:

$X$ is open in $X \cup [0, 1]$.

For $t \in [0, 1]$ the collection $\{U(t, l, G, \alpha): l \in \mathbf{N}, G \in \mathcal{G}, \alpha < \omega_1\}$ is defined to be a local base of $t$ in $X \cup [0, 1]$.

Observe that $[0, 1]$ is embedded in $X \cup [0, 1]$. It is easy to check that $X \cup [0, 1]$ is a Hausdorff space. In fact our topology has more open sets than Chaber's.

*Claim.* Let $U$ be a subset of $X \cup [0, 1]$ which contains $[0, 1]$. Then $U$ is neighborhood of $[0, 1]$ in $X \cup [0, 1]$ iff $\exists G \in \mathcal{G} \exists \alpha < \omega_1$ such that $Z(G, \alpha) \subset U$.

PROOF. Assume $Z(G, \alpha) \subset U$. Then, for $t \in [0, 1]$, $t \in U(t, l, G, \alpha) \subset Z(G, \alpha)$. Hence $[0, 1] \subset \operatorname{int} U$. On the other hand, assume $[0, 1] \subset \operatorname{int} U$. Then, $\forall t \in [0, 1]$ $\exists l(t) \in \mathbf{N}\, \exists G(t) \in \mathcal{G}\, \exists \alpha(t) < \omega_1$ such that

$$t \in U(t, l(t), G(t), \alpha(t)) \subset U.$$

Since $[0, 1]$ is compact, $[0, 1]$ can be covered by finitely many of these sets. Say $[0, 1] \subset \bigcup_{i=1}^{k} U(t_i, l(t_i), G(t_i), \alpha(t_i))$ $(\subset U)$. Put $G = \bigcap_{i=1}^{k} G(t_i)$ $(\in \mathcal{G})$ and $\alpha = \sup\{\alpha(t_i)\colon i \leqslant k\}$ $(< \omega_1)$. We claim that $Z(G, \alpha) \subset \bigcup_{i=1}^{k} U(t_i, l(t_i), G(t_i), \alpha(t_i))$ $(\subset U)$. Choose $p \in Z(G, \alpha)$, say $p \in Z_\alpha^n$ for some $n \in G$. Since $T^n = \bigcup_{k=0}^{n} U_k^n$, there exists $k \leqslant n$ such that $p \in U_k^n$. Choose $s \in [0, 1]$ such that $[n.s] = k$. If $s \in U(t_i, l(t_i), G(t_i), \alpha(t_i))$ then, since $G \subset G(t_i)$ and $Z_\alpha \subset Z_{\alpha(t_i)}$, we conclude that $p \in Z(\alpha(t_i)) \cap U_{[n.s]}^n$ for "some" $n \in G(t_i)$, i.e. $p \in U(t_i, l(t_i), G(t_i), \alpha(t_i))$. The claim follows.

From the claim and from $(*)$ we conclude that the space obtained from $X \cup [0, 1]$ by identifying $[0, 1]$ to a point is homeomorphic to $X \cup \{p\}$. Obviously the map $f$: $X \cup [0, 1] \to X \cup \{p\}$ defined by $f(x) = x$ $(x \in X)$ and $f[0, 1] = p$ is a perfect map. Hence, all the required properties are satisfied. $\square$

REMARK 3.4. It is well known that each space $T^n$, as defined in 3.3, has a unique (nontrivial) regular extension, namely $\beta T^n$. It follows that $\mathrm{cl}_{eX} T^n \simeq \beta T^n$, for all $n \in \mathbf{N}$. Consider the space $\tilde{X} = \bigoplus_{n=1}^{\infty} \beta T^n$. Then $X \subset \tilde{X} \subset eX$. $\tilde{X}$ is a $\sigma$-compact, hence normal, and according to 2.5 this implies that $(e\tilde{X})_S = \beta\tilde{X} = \beta X$. Since $(eX)_S \neq \beta X$, we conclude that the map id: $\tilde{X} \to eX$ cannot be extended continuously to $e\tilde{X}$. At first glance this may seem a contradiction, but it is not. One cannot use 2.4(ii) to ensure that such an extension should exist since $eX$ is not $e$-compactifiable ($eX$ is not even semiregular), nor the fact that $e\tilde{X}$ is the largest $e$-compactification, since $eX$ is not an $e$-compactification of $\tilde{X}$. (A nonfixed ultrafilter on $\tilde{X} - X$ does not have an accumulation point in $eX$.)

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