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The Issue of Transients in Leakage-based Model Reference Adaptive Control of Switched Linear Systems

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Abstract

The literature has proven that attaining good transient behavior in leakage-based robust adaptive control of uncertain switched systems is intrinsically challenging. In fact, because the gains of the inactive subsystems must exponentially vanish during inactive times as an effect of leakage action, new learning transients will repeatedly arise at each switching instant. In this paper, a new leakage-based mechanism is designed for robust adaptive control of uncertain switched systems: in contrast to the available designs, the key innovation of the proposed one is that the adaptive gains of the inactive subsystems can be kept constant to their switched-off values, thus preventing vanishing gains. Bounded stability of the closed-loop switched system is guaranteed thanks to the introduction of an auxiliary gain playing the role of leakage. A benchmark example commonly adopted in adaptive switched literature shows that the proposed strategy can consistently improve the transient behavior under various families of switching signals.

Keywords:

Adaptive switched systems, learning transient, dwell-time switching, leakage.

1. Introduction

With a wide range of application in several fields, such as networked control systems [1, 2], circuit and power systems [3, 4], multi-agent systems [5], fault-tolerant control [6, 7] and many more, switched systems have drawn enormous interest over the last decades. Switched systems are a special type of hybrid dynamic systems, constituted of continuous-time subsystems, also called modes, and a switching law determining the activation of the subsystems. Switched systems not only find application in several technological areas, but also bring several theoretical challenges, spanning from stability to control.

Stability is the most basic desirable property of a switched system [8–10]. The state of the art has shown that stability under arbitrary switching cannot in general be achieved unless a common

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Lyapunov function to all subsystems exists [11–14]: therefore, many researchers have concentrated on several classes of slowly-switching signals for which stability can be derived. Dwell-time (DT) switching requires the time interval between consecutive switching to be no less than a sufficiently large constant. Average dwell time (ADT) was put forward in [15], as the extension of DT: in ADT the dwell-time is defined on an average sense, i.e. fast switching is allowed, provided it is compensated by slow switching later on [16]. Some conservativeness of DT and ADT has been later relaxed by the concepts of mode-dependent dwell time (MDDT) and mode-dependent average dwell time (MDADT), where each mode has its own dwell time or average dwell time [17].

From the control point of view, adaptive control of switched system is a quite recent research field aiming at controlling switched systems with parametric uncertainties [18–22]. In adaptive control of uncertain switched systems the challenge is to design not only an adaptation mechanism for each subsystem, but also a slowly-switching law for which stability can be derived (typically in the framework of dwell time and its extensions as mentioned above). If adaptive control of uncertain switched systems presents some challenges, the literature has shown that *robust adaptive control* of switched systems is even more challenging [19, 21, 23, 24]. Notably, in line with the issues highlighted in the famous Rohr’s counterexample [25] or in the books [26, 27], *adaptive control designs are generally not robust*. Lack of robustness refers to the fact that, in the presence of disturbances and/or unmodelled dynamics, not only asymptotic convergence of the tracking error is lost, but also the adaptive closed-loop might become unstable, because the integral action of the adaptive law would integrate the "bad" disturbances and/or unmodelled dynamics [26, 27]. Solutions to the robustness issue in adaptive non-switched systems involve modifying the adaptive law, such as adding projection, dead-zone or leakage as a way to reduce the integral action of the adaptive law [26, Chap. 8].

Lack of robustness of adaptive design transfers to switched systems as well: some designs have been recently proposed aiming at robust adaptive control of switched systems [19, 23, 28–31]. Such designs can be classified in two families. In the first family, one adopts a sliding mode perspective, in which uncertainties can be compensated by sufficiently high robustification terms [28–30]: unfortunately, this approach requires to monotonically increase the control gains, which might lead to unpractically high control inputs. In the second family, one tries to extend the adaptive law modifications proposed for non-switched systems (projection/dead-zone/leakage) in the switched framework [19, 23, 31]. The leakage modification is quite interesting in view of the fact that it does not require any a priori knowledge on the uncertainty [26, Chap. 8], so it can potentially handle larger parametric uncertainty than projection-based robustification, as illustrated in [24, 31, 32]. Unfortunately, while avoiding high gains, the leakage-based methodology has serious performance issues, which can be highlighted by referring to the representative work [31]. In contrast with (non-robust) adaptive control of switched systems, where the control gains of the inactive subsystems can be kept constant at their switched-off values [19, 20, 22], in leakage-based robust adaptive control of switched systems one requires the control gains of the inactive subsystems to *decrease exponentially during inactive interval*. This is necessary in order to prove (bounded) stability, because such exponential decrease is a stabilizing effect of the leakage action. Of course, *such a vanishing-gain mechanism would lead to a new learning transient whenever an inactive system is activated again*. This is up to now, the biggest challenge in robust adaptive control of switched systems.

This work is motivated by the following research question: how to design a robust adaptive controller for uncertain switched systems that does not require the control gains to vanish during

inactive times?

A positive answer to this question is provided here. A new leakage-based framework is proposed, whose main contribution is to allow the control gains of the inactive subsystems to stay constant at their switched-off value, while guaranteeing stability of the closed-loop switched system. This is achieved via a new auxiliary gain that provides a suitable leakage action during inactive time intervals. A benchmark example commonly adopted in adaptive and robust adaptive literature shows that the proposed strategy can consistently improve the transient of the closed-loop system under various families of slowly-switching signals (in the framework of dwell time and its extensions).

The paper is organized as follows. The problem formulation and definition are presented in Section 2. The proposed robust adaptive mechanism is presented in Section 3, while its stability analysis is provided in Section 4. In Section 5, the effectiveness of the proposed controller is extensively studied using the benchmark example. Section 6 presents concluding remarks.

The following notations are used in this paper: $\lambda_{\min}(\cdot)$, $\lambda_{\max}(\cdot)$ and $\|(\cdot)\|$ denote the minimum eigenvalue, maximum eigenvalue and Euclidean norm of (\cdot) respectively; $D > 0$ denotes a positive definite matrix D ; $\text{diag}\{\cdot\}$ denotes a diagonal matrix with diagonal elements defined in $\{\cdot\}$.

2. Problem Formulation and Definition

In the following we recall the main concepts of model reference adaptive control, the most studied framework for adaptive control of switched systems [18–20, 22]. Consider the following switched linear system:

$$\dot{x}(t) = A_{\sigma(t)}x(t) + B_{\sigma(t)}u_{\sigma(t)}(t) + d(t), \quad \sigma(t) \in \Omega \quad (1)$$

where $x \in \mathbb{R}^n$ is the state vector; $u_{\sigma} \in \mathbb{R}^m$ is the (switched) control input; $d \in \mathbb{R}^n$ is an external bounded disturbance with unknown bound, and $\sigma(\cdot)$ is the piecewise constant switching signal (to be defined later) taking values in $\Omega := \{1, 2, \dots, N\}$, with switching instants denoted by t_l, t_{l+1}, \dots , and with intervals in between instants denoted by $[t_l, t_{l+1})$, $l = 1, 2, \dots$, and being N the number of subsystems. The switched linear system (1) is uncertain when the entries of the matrices $A_p \in \mathbb{R}^{n \times n}$ and $B_p \in \mathbb{R}^{n \times m}$, $p \in \Omega$ are unknown.

A switched reference model representing the desired behavior for each subsystem is given as:

$$\dot{x}_m(t) = A_{m\sigma(t)}x_m(t) + B_{m\sigma(t)}r(t), \quad \sigma(t) \in \Omega \quad (2)$$

where $x_m \in \mathbb{R}^n$ is the desired state vector, and $r \in \mathbb{R}^m$ is a bounded user-defined signal. The matrices $A_{mp} \in \mathbb{R}^{n \times n}$ and $B_{mp} \in \mathbb{R}^{n \times m}$ are known and $A_{mp}, p \in \Omega$, are Hurwitz matrices (so that the desired behavior of each subsystem is stable). It is known from literature [18–20, 22] that the state-feedback mode-dependent control that makes (1) behave like (2) is

$$u_{\sigma(t)}^*(t) = K_{\sigma(t)}^{*T}x(t) + L_{\sigma(t)}^*r(t)$$

where $K_p^* \in \mathbb{R}^{n \times m}$ and $L_p^* \in \mathbb{R}^{m \times m}$, $p \in \Omega$, are nominal parameters satisfying the following matching conditions:

$$A_p + B_p K_p^{*T} = A_{mp}, \quad B_p L_p^* = B_{mp}. \quad (3)$$

As the matrices (A_p, B_p) are unknown, the gain K_p^* and L_p^* in (3) are unknown. Define K_p and L_p as the (time-dependent) estimates of the ideal parameters K_p^* and L_p^* , respectively. Thus, the following controller is introduced:

$$u_{\sigma(t)}(t) = K_{\sigma(t)}^T(t)x(t) + (L_{\sigma(t)}(t) + \Gamma_{\sigma(t)}(t))r(t), \quad (4)$$

where $K_p \in \mathbb{R}^{n \times m}$, $L_p \in \mathbb{R}^{m \times m}$ and $\Gamma_p = \text{diag}\{\gamma_{ip}\}$, $p \in \Omega$, $i = 1, \dots, m$ are to be updated from appropriately designed adaptive laws. As compared to (3), the introduction of Γ_p in (4) is for robustness reasons, as it will be clear from Section 3.

Let $e(t) = x(t) - x_m(t)$ be the tracking error. After substituting (4) into (1) and subtracting (2), we obtain the dynamics of the tracking error as follows:

$$\dot{e}(t) = A_{m\sigma(t)}e(t) + B_{\sigma(t)}(\tilde{K}_{\sigma(t)}^T(t)x(t) + (\tilde{L}_{\sigma(t)}(t) + \Gamma_{\sigma(t)}(t))r(t)) + d(t) \quad (5)$$

where $\tilde{K}_{\sigma} = K_{\sigma} - K_{\sigma}^*$ and $\tilde{L}_{\sigma} = L_{\sigma} - L_{\sigma}^*$ are the parameter estimation errors.

The following class of switching signals $\sigma(\cdot)$ is considered for the switched system (1):

Definition 1. (Mode-dependent Average Dwell Time) [17]. Consider two time instants $t_2 \geq t_1 \geq 0$. Let $N_p(t_1, t_2)$ be the number of times subsystem p is activated over the interval $[t_1, t_2]$, and let $T_p(t_1, t_2)$ denote the total running time of subsystem p over the interval $[t_1, t_2]$, $\forall p \in \Omega$. We say that the switching signal $\sigma(\cdot)$ has mode-dependent average dwell time (MDADT) τ_p if there exist $N_{0p} \geq 1$ and $\tau_p > 0$ such that

$$N_p(t_1, t_2) \leq N_{0p} + \frac{T_p(t_1, t_2)}{\tau_p}, \quad \forall t_2 \geq t_1 \geq 0 \quad (6)$$

where N_{0p} is termed as mode-dependent chatter bound.

Remark 1. The MDADT class is adopted in this work because it comprises various families of switching laws considered in literature [9], such as dwell time (DT) switching ($N_{0p} = 1$, $\tau_p = \tau$, $\forall p \in \Omega$), mode-dependent dwell time (MDDT) switching ($N_{0p} = 1$, $\tau_p = \tau_p$, $\forall p \in \Omega$), and average dwell-time (ADT) switching ($N_{0p} = N_0$, $\tau_p = \tau$, $\forall p \in \Omega$).

After appropriate minor modifications, one can easily extend **Definition 1** so as to include the class of mode-mode-dependent dwell time (MMDDT) switching signals, introduced in [31] to handle the case in which the next subsystem to be switched on is known. When the switching sequence is known, we indicate the fact that the next mode to be switched on after p is q with $q \in \mathcal{N}(p)$. Then, we present the following definition.

Definition 2. Mode-mode-dependent Average Dwell Time. Let $N_{pq}(t_1, t_2)$ be the number of times subsystems p is activated over the interval $[t_1, t_2]$ with $q \in \mathcal{N}(p)$, and let $T_{pq}(t_1, t_2)$ denote the total running time of subsystem p switching to q over the interval $[t_1, t_2]$, $\forall p, q \in \Omega$. We say that $\sigma(\cdot)$ has a mode-mode-dependent average dwell time (MMDADT) τ_{pq} if there exist positive numbers $N_{0pq} \geq 1$ and τ_{pq} such that

$$N_{pq}(t_1, t_2) \leq N_{0pq} + \frac{T_{pq}(t_1, t_2)}{\tau_{pq}}, \quad \forall t_2 \geq t_1 \geq 0 \quad (7)$$

where N_{0pq} is termed as mode-mode-dependent chatter bound.

Remark 2. The MMDADT switching signal is an extension in an average sense of the MMDDT proposed in [31]. It is introduced for consistency, in order to be able to run fair comparisons with many switching families introduced in literature.

We now introduce some standard stability concepts. In robust adaptive control of uncertain switched systems, uniform boundedness of the tracking error and of the closed-loop signal is what one can aim at [23, 24, 28–31]. This concept is formalized in the following two definitions:

Definition 3. (Uniform Ultimate Boundedness (UUB)) [26]. *The uncertain switched system (1) under switching signal $\sigma(\cdot)$ is uniformly ultimately bounded if there exists a convex and compact set \mathcal{C} such that for every initial condition $x(0) = x_0$, there exists a finite time $T(x_0)$ such that $x(t) \in \mathcal{C}$ for all $t \geq T(x_0)$.*

Definition 4. (Ultimate Bound). *A signal $\Phi(\cdot)$ is said to have ultimate bound b if there exists a positive constant b , and for any $a \geq 0$, there exists $T = T(a, b)$, where b and T are independent of t_0 , such that $\|\Phi(t_0)\| \leq a \Rightarrow \|\Phi(t)\| \leq b, \forall t \geq t_0 + T$.*

The following assumption is standard in literature (see Sect. 4.1 in survey paper [33] for details on this assumption), in order to handle adaptive control of multi-input linear subsystems in (1).

Assumption 1. *The matching conditions (3) hold for some unknown K_p^* and L_p^* , and there exists a family of known matrices $S_p \in \mathbb{R}^{m \times m}, p \in \Omega$, such that $M_p = L_p^* S_p = (L_p^* S_p)^T = S_p^T L_p^{*T} > 0, \forall p \in \Omega$.*

The problem formulation can be finally given as:

Problem 1. *Under Assumption 1, develop an adaptive law for the control parameters in (4) and a switching law based on MDADT (or MMDADT if the switching sequence is known) such that, without requiring knowledge of the nominal values of A_p and $B_p, \forall p \in \Omega$, uniform ultimate boundedness of all closed-loop signals is guaranteed, including the tracking error in (5).*

3. Controller Design

In this section, novel adaptive laws for the gains in (4) are proposed to solve **Problem 1**. Correspondingly, stabilizing switching laws are given in the framework of MDADT switching (or MMDADT if the switching sequence is known).

3.1. Adaptive control

For compactness, let us denote with p the index corresponding to the active subsystem at time t (e.g. in the interval $t \in [t_l, t_{l+1})$). If p is an active system, we use $\bar{p} \in \mathcal{I}(p)$ to indicate the set of inactive subsystems with respect to p . Let $P_p > 0$ be the solution to

$$A_{mp}^T P_p + P_p A_{mp} + (1 + \kappa_p) P_p \leq 0, \quad (8)$$

where κ_p is a user-defined scalar.

Then, consider the leakage-based adaptive laws

$$\dot{K}_p^T(t) = -S_p^T B_{mp}^T P_p e(t) x^T(t) - \delta_p K_p^T(t), \quad \dot{K}_{\bar{p}}^T(t) = 0, \quad (9a)$$

$$\dot{L}_p(t) = -S_p^T B_{mp}^T P_p e(t) r^T(t) - \delta_p L_p(t), \quad \dot{L}_{\bar{p}}(t) = 0, \quad (9b)$$

$$\dot{\gamma}_{ip}(t) = 0,$$

$$\dot{\gamma}_{i\bar{p}}(t) = -\left(\beta_{i\bar{p}} + \delta_{\bar{p}}(\{K_{\bar{p}}(t)K_{\bar{p}}^T(t)\}_{ii} + \{L_{\bar{p}}^T(t)L_{\bar{p}}(t)\}_{ii})\right)\gamma_{i\bar{p}}(t) + \beta_{i\bar{p}}\epsilon_{i\bar{p}}, \quad (9c)$$

$$\text{with } \delta_\sigma \geq \max_{\sigma \in \Omega} \left(\frac{\lambda_{\max}(M_\sigma^{-1}) + \kappa_\sigma}{2}, 2\kappa_\sigma \lambda_{\max}(M_\sigma^{-1}) \right) > 0, \quad (9d)$$

$$\text{and } \gamma_{ip}(t_0), \gamma_{i\bar{p}}(t_0) > \epsilon_{i\bar{p}}, \quad (9e)$$

where the notation $\{K_{\bar{p}}K_{\bar{p}}^T\}_{ii}$ and $\{L_{\bar{p}}^T L_{\bar{p}}\}_{ii}$ is used to indicate diagonal elements along the corresponding matrices; $\beta_{i\bar{p}}, \epsilon_{i\bar{p}} \in \mathbb{R}^+$ $i = 1, \dots, m$ are static design scalars and t_0 is the initial time. The leakage bound in condition (9d) stems from Assumption 1, which implies that an upper bound on the perturbation in matrix B_σ is known. Such consideration is not only standard in literature (cf. the aforementioned survey paper [33]), but also valid in practical systems (cf. [34]).

Remark 3. *The inequality (8) is equivalent to the standard Lyapunov inequality*

$$(A_{mp}^T + (1 + \kappa_p)/2I)P_p + P_p(A_{mp}^T + (1 + \kappa_p)/2I) \leq 0 \quad (10)$$

which highlights how A_{mp} of the reference models should be chosen in such a way to have their eigenvalues which sufficiently large real part (implying sufficiently high exponential decay). This is not restrictive, since it is a standard requirement for stability of switched systems. Stability of switched systems is typically achieved by requiring that the possibly destabilizing effect of switching are compensated by the exponential decrease of the Lyapunov function in between switching instants [9–11].

Remark 4. *For comparison purposes, let us explicitly recall the robust adaptive law in [31]*

$$\dot{K}_p^T(t) = -S_p^T B_{mp}^T P_p e(t) x^T(t) - \delta_p M_p K_p^T(t), \quad (11a)$$

$$\dot{L}_p(t) = -S_p^T B_{mp}^T P_p e(t) r^T(t) - \delta_p M_p L_p(t), \quad (11b)$$

$$\dot{K}_{\bar{p}}^T(t) = -\delta_{\bar{p}} M_{\bar{p}} K_{\bar{p}}^T(t), \quad (11c)$$

$$\dot{L}_{\bar{p}}(t) = -\delta_{\bar{p}} M_{\bar{p}} L_{\bar{p}}(t), \quad (11d)$$

where the leakage rates δ_p must satisfy: $\delta_p \geq \lambda_{\max}(M_p^{-1}) \geq 0$. Albeit the slightly different leakage action (see the simulation section for more details on this point), it can be seen that the adaptive laws (11) are designed such that the gains for the inactive subsystems vanish exponentially during the inactive times, as an effect of leakage. This is required in order to prove UUB [31]. Unfortunately, this mechanism implies that the gains will drop to zero if a subsystem remains inactive for sufficiently long time. This will lead to a new learning transient every time the subsystem is switched-on again. This undesirable scenario is avoided by (9a) and (9b) where the adaptive gains are kept constant after the subsystem is switched-off.

3.2. Switching Laws:

In this section, a stabilizing switching law is given in terms of MDADT switching (or MMDADT if the switching sequence is known).

We define $\varrho_{M\sigma} \triangleq \lambda_{\max}(\mathbf{P}_\sigma)$, $\varrho_{m\sigma} \triangleq \lambda_{\min}(\mathbf{P}_\sigma)$, $\bar{\varrho}_M \triangleq \max_{\sigma \in \Omega}(\varrho_{M\sigma})$ and $\underline{\varrho}_m \triangleq \min_{\sigma \in \Omega}(\varrho_{m\sigma})$. Following **Definition 1** of MDADT, the switching law is proposed via:

$$\tau_p > \tau_p^* = \ln \mu_p / \chi_p, \quad \forall p \in \Omega \quad (12)$$

and any $N_{0p} \geq 1$, where $\mu_p \triangleq \varrho_{Mp} / \varrho_{mp}$; χ_p is a user-defined scalar satisfying $0 < \chi_p < \kappa_p, \forall p \in \Omega$.

According to **Remark 1**, the switching law (12) includes DT, MDDT, and ADT as special cases. For the scenario when the next subsystem q to be switched after subsystem p is known, we propose a MMDADT switching law in line with **Definition 2** via

$$\tau_{pq} > \tau_{pq}^* = \frac{1}{\chi_p} \ln \mu_{pq}, \quad \forall p \in \Omega, \quad q = \mathcal{N}(p), \quad (13)$$

where $\mu_{pq} = \varrho_{Mq} / \varrho_{mp}$. The MMDADT law is proposed for subsequent comparisons with [31].

Remark 5. *It is important to notice that, when selecting the same κ_p as [31] (thus obtaining the same P_p and μ_p in (8) and (12), respectively), one will obtain exactly the same τ_p as [31] (since the design parameter ζ_p in [31] plays exactly the same role as χ_p here). Therefore, the proposed adaptation mechanism does not introduce any restriction in τ_p as compared to the state of the art. This allows a fair comparison of the proposed method with the method in [31], i.e. the methods can be compared for the same switching signals.*

4. Stability Analysis of the Proposed Adaptation Framework

The following lemma is useful for stability analysis:

Lemma 1. [35] *Let $\Phi \in \mathbb{R}^g, \varphi \in \mathbb{R}_s$ be vector-valued signals, and let $W \in \mathbb{R}^{g \times g}, G \in \mathbb{R}^{g \times s}$ be constant matrices. Then, the following inequality holds:*

$$\pm 2\Phi^T W G \varphi \leq \Phi^T W W^T \Phi + \varphi^T G^T G \varphi$$

The stability properties of the proposed adaptation framework can now be stated:

Theorem 1. *Under **Assumption 1**, the closed-loop switched system formed by the switched system (1), the reference model (2), the controller (4), the adaptive laws (9), and the switching law (12), is Uniformly Ultimately Bounded (UUB) and an ultimate bound b on the tracking error e can be found as*

$$b \in \left[0, \sqrt{\frac{\bar{\varrho}_M}{\underline{\varrho}_m^2} \mathcal{B} \prod_{p=1}^N \mu_p^{N_{0p}}} \right], \quad (14)$$

$$\mathcal{B} \triangleq \max_{p \in \Omega} \left(\frac{\zeta_1}{\sqrt{\underline{\varrho}_m} (\kappa_p - \chi_p)} + \sqrt{\frac{\zeta_1^2}{\underline{\varrho}_m (\kappa_p - \chi_p)^2} + \frac{\zeta_2}{(\kappa_p - \chi_p)}} \right)^2,$$

where the scalars $\zeta_1, \zeta_2 \in \mathbb{R}^+$ are defined during the proof.

Proof. Stability relies on the Lyapunov candidate:

$$V = e^T(t)P_{\sigma(t)}e(t) + \sum_{s=1}^N \text{tr}[\tilde{K}_s(t)M_s^{-1}\tilde{K}_s^T(t)] + \sum_{s=1}^N \text{tr}[\tilde{L}_s^T(t)M_s^{-1}\tilde{L}_s(t)] + \sum_{s=1}^N \text{tr}[\underline{\Gamma}_s\Gamma_s(t)], \quad (15)$$

where $\underline{\Gamma}_\sigma = \text{diag}\{1/\underline{\gamma}_{i\sigma}\}$. In fact, from (9c) and the initial conditions (9e), it can be verified that $\exists \underline{\gamma}_{i\sigma}, \bar{\gamma}_{i\sigma} \in \mathbb{R}^+$ such that

$$\underline{\gamma}_{i\sigma} \leq \gamma_{i\sigma}(t) \leq \bar{\gamma}_{i\sigma}, \quad \forall t \geq t_0. \quad (16)$$

Analysis of (15) at the switching instants is required, since P_p is different for different subsystems generally (i.e. $V(\cdot)$ might be discontinuous at switching instants). Let subsystem $\sigma(t_{l+1}^-)$ be active when $t \in [t_l, t_{l+1})$ and subsystem $\sigma(t_{l+1})$ be active when $t \in [t_{l+1}, t_{l+2})$. Without the loss of generality, the behavior of $V(\cdot)$ is studied at the switching instant $t_{l+1}, l \in \mathbb{N}^+$.

At the switching instant t_{l+1} , we have before switching

$$\begin{aligned} V(t_{l+1}^-) &= e^T(t_{l+1}^-)P_{\sigma(t_{l+1}^-)}e(t_{l+1}^-) + \sum_{s=1}^N \text{tr}[\underline{\Gamma}_s(t_{l+1}^-)\Gamma_s(t_{l+1}^-)] \\ &\quad + \sum_{s=1}^N \text{tr}[\tilde{K}_s(t_{l+1}^-)M_s^{-1}\tilde{K}_s^T(t_{l+1}^-)] + \sum_{s=1}^N \text{tr}[\tilde{L}_s^T(t_{l+1}^-)M_s^{-1}\tilde{L}_s(t_{l+1}^-)] \end{aligned}$$

and after switching

$$\begin{aligned} V(t_{l+1}) &= e^T(t_{l+1})P_{\sigma(t_{l+1})}e(t_{l+1}) + \sum_{s=1}^N \text{tr}[\underline{\Gamma}_s(t_{l+1}^-)\Gamma_s(t_{l+1})] \\ &\quad + \sum_{s=1}^N \text{tr}[\tilde{K}_s(t_{l+1})M_s^{-1}\tilde{K}_s^T(t_{l+1})] + \sum_{s=1}^N \text{tr}[\tilde{L}_s^T(t_{l+1})M_s^{-1}\tilde{L}_s(t_{l+1})] \end{aligned}$$

According to the continuity of the tracking error $e(\cdot)$ in (5) and the continuity of the parameter estimates updated via (9), we have $e(t_{l+1}^-) = e(t_{l+1}), \tilde{K}_s(t_{l+1}^-) = \tilde{K}_s(t_{l+1}), \tilde{L}_s(t_{l+1}^-) = \tilde{L}_s(t_{l+1})$, and $\Gamma_s(t_{l+1}^-) = \Gamma_s(t_{l+1})$ for any switching law. Due to $e^T(t)P_p e(t) \leq \varrho_{Mp} e^T(t)e(t)$, and $e^T(t)P_p e(t) \geq \varrho_{mp} e^T(t)e(t)$ we have

$$\begin{aligned} V(t_{l+1}) - V(t_{l+1}^-) &= e^T(t_{l+1}^-)(P_{\sigma(t_{l+1})} - P_{\sigma(t_{l+1}^-)})e(t_{l+1}^-) \\ &\leq \frac{\varrho_{Mp} - \varrho_{mp}}{\varrho_{mp}} e^T(t_{l+1}^-)P_{\sigma(t_{l+1}^-)}e(t_{l+1}^-) \\ &\leq \frac{\varrho_{Mp} - \varrho_{mp}}{\varrho_{mp}} V(t_{l+1}^-) \end{aligned}$$

Then, we obtain the following inequality for $V(\cdot)$ at the switching instant t_{l+1} :

$$V(t_{l+1}) \leq \mu_p V(t_{l+1}^-) \quad (17)$$

with $\mu_p = \varrho_{Mp}/\varrho_{mp} \geq 1$.

Next, the behavior of $V(t)$ is studied between two consecutive switching instants, i.e., when $t \in [t_l, t_{l+1})$. In the following, let $\sigma(t) = p$ denote an active subsystem and an inactive is denoted as \bar{p} when $t \in [t_{l+1}, t_{l+2})$. Let us also use the notation $\mathcal{I}(p)$ to indicate all inactive subsystems when subsystem p is active. Then using (8), (5) and (9a)-(9c) we have

$$\begin{aligned} \dot{V} &\leq -e^T(1 + \kappa_p)P_p e + 2e^T P_p B_p (\tilde{K}_p x + (\tilde{L}_p + \Gamma_p)r) + 2e^T P_p d \\ &\quad + 2 \sum_{s=1}^N \text{tr}[\tilde{K}_s M_s^{-1} \dot{K}_s^T] + 2 \sum_{s=1}^N \text{tr}[\tilde{L}_s^T M_s^{-1} \dot{L}_s] + 2 \sum_{s=1}^N \text{tr}[\underline{\Gamma}_s \dot{\Gamma}_s] \\ &\leq -\kappa_p e^T P_p e + 2e^T P_p B_p \Gamma_p r + d^T P_p d + \sum_{\bar{p} \in \mathcal{I}(p)} \text{tr}[\underline{\Gamma}_{\bar{p}} \dot{\Gamma}_{\bar{p}}] \\ &\quad - 2 \text{tr}[\tilde{K}_p \delta_p M_p^{-1} K_p^T] - 2 \text{tr}[\tilde{L}_p^T \delta_p M_p^{-1} L_p]. \end{aligned} \quad (18)$$

The following simplification can be made using **Lemma 1**:

$$-2 \text{tr}[\tilde{K}_p \delta_p M_p^{-1} K_p^T] < -\text{tr}[\tilde{K}_p M_p^{-1} (2\delta_p I - M_p^{-1}) \tilde{K}_p^T] + \text{tr}[K_p^* \delta_p^2 K_p^{*T}], \quad (19)$$

$$-2 \text{tr}[\tilde{L}_p^T \delta_p M_p^{-1} L_p] < -\text{tr}[\tilde{L}_p^T M_p^{-1} (2\delta_p I - M_p^{-1}) \tilde{L}_p] + \text{tr}[L_p^{*T} \delta_p^2 L_p^*]. \quad (20)$$

Further, noting $\underline{\Gamma}_{\bar{p}} \dot{\Gamma}_{\bar{p}} = \text{diag}\{\dot{\gamma}_{i\bar{p}}/\underline{\gamma}_{i\bar{p}}\}$, $i = 1, \dots, m$, the following can be deduced from (9c) and (16)

$$\begin{aligned} \frac{\dot{\gamma}_{i\bar{p}}}{\underline{\gamma}_{i\bar{p}}} &= \frac{-\left(\beta_{i\bar{p}} + \delta_{\bar{p}} \left(\{K_{\bar{p}} K_{\bar{p}}^T\}_{ii} + \{L_{\bar{p}}^T L_{\bar{p}}\}_{ii}\right)\right) \gamma_{i\bar{p}} + \beta_{i\bar{p}} \epsilon_{i\bar{p}}}{\underline{\gamma}_{i\bar{p}}} \\ &\leq -\delta_{\bar{p}} \left(\{K_{\bar{p}} K_{\bar{p}}^T\}_{ii} + \{L_{\bar{p}}^T L_{\bar{p}}\}_{ii}\right) + (\beta_{i\bar{p}} \epsilon_{i\bar{p}})/\underline{\gamma}_{i\bar{p}}. \end{aligned} \quad (21)$$

Moreover, using the relations $\tilde{K}_\sigma = K_\sigma - K_\sigma^*$, $\tilde{L}_\sigma = L_\sigma - L_\sigma^*$ and **Lemma 1** we have

$$\begin{aligned} \text{tr}[\tilde{K}_\sigma^T \tilde{K}_\sigma] &= \text{tr}[K_\sigma^T K_\sigma - 2K_\sigma^T K_\sigma^* + K_\sigma^{*T} K_\sigma^*] \\ &\leq 2 \text{tr}[K_\sigma^T K_\sigma + K_\sigma^{*T} K_\sigma^*], \end{aligned} \quad (22)$$

$$\begin{aligned} \text{tr}[\tilde{L}_\sigma^T \tilde{L}_\sigma] &= \text{tr}[L_\sigma^T L_\sigma - 2L_\sigma^T L_\sigma^* + L_\sigma^{*T} L_\sigma^*] \\ &\leq 2 \text{tr}[L_\sigma^T L_\sigma + L_\sigma^{*T} L_\sigma^*]. \end{aligned} \quad (23)$$

Using (19)-(23), (18) is simplified as

$$\begin{aligned} \dot{V} &\leq -\kappa_p V + 2e^T P_p B_p \Gamma_p r + d^T P_p d \\ &\quad - \text{tr}[\tilde{K}_p M_p^{-1} (2\delta_p I - (M_p^{-1} + \kappa_p I)) \tilde{K}_p^T] + \text{tr}[K_p^{*T} \delta_p^2 K_p^*] - \text{tr}[\tilde{L}_p M_p^{-1} (2\delta_p I - (M_p^{-1} + \kappa_p I)) \tilde{L}_p^T] \\ &\quad + \sum_{\bar{p} \in \mathcal{I}(p)} \left[\text{tr}[\tilde{K}_{\bar{p}}^T \tilde{K}_{\bar{p}} (\kappa_{\bar{p}} \lambda_{\max}(M_{\bar{p}}^{-1}) - \frac{1}{2} \delta_{\bar{p}})] + \frac{1}{2} \text{tr}[K_{\bar{p}}^{*T} \delta_{\bar{p}} K_{\bar{p}}^*] \right] + \sum_{s=1}^N \text{tr}[\kappa_s \underline{\Gamma}_s \Gamma_s] + \text{tr}[L_p^{*T} \delta_p^2 L_p^*] \\ &\quad + \sum_{\bar{p} \in \mathcal{I}(p)} \left[\text{tr}[\tilde{L}_{\bar{p}}^T \tilde{L}_{\bar{p}} (\kappa_{\bar{p}} \lambda_{\max}(M_{\bar{p}}^{-1}) - \frac{1}{2} \delta_{\bar{p}})] + \frac{1}{2} \text{tr}[L_{\bar{p}}^{*T} \delta_{\bar{p}} L_{\bar{p}}^*] \right] + \sum_{\bar{p} \in \mathcal{I}(p)} \sum_{i=1}^m (\beta_{i\bar{p}} \epsilon_{i\bar{p}})/\underline{\gamma}_{i\bar{p}} \\ &\leq -\kappa_p V + 2\|e\| \|P_p B_p \Gamma_p r\| + \varrho_{M_p} \|d\|^2 + \text{tr}[K_p^{*T} \delta_p^2 K_p^*] + \text{tr}[L_p^{*T} \delta_p^2 L_p^*] \\ &\quad + \sum_{s=1}^N \text{tr}[\kappa_s \underline{\Gamma}_s \Gamma_s] + \sum_{\bar{p} \in \mathcal{I}(p)} \left(\frac{1}{2} \text{tr}[K_{\bar{p}}^{*T} \delta_{\bar{p}} K_{\bar{p}}^*] + \frac{1}{2} \text{tr}[L_{\bar{p}}^{*T} \delta_{\bar{p}} L_{\bar{p}}^*] + \sum_{i=1}^m (\beta_{i\bar{p}} \epsilon_{i\bar{p}})/\underline{\gamma}_{i\bar{p}} \right). \end{aligned} \quad (24)$$

By definition $r(t) \in \mathcal{L}_\infty$ and by design $\Gamma_s \in \mathcal{L}_\infty$ from (16). Therefore, $\exists \zeta_1 \in \mathbb{R}^+$ such that $\|P_p B_p \Gamma_p r\| \leq \zeta_1 \forall p \in \Omega$. Further we define a scalar ζ_2 as

$$\begin{aligned} \zeta_2 \triangleq & \bar{\varrho}_M \|d\|^2 + \max_{p \in \Omega} (\text{tr}[K_p^{*T} \delta_p^2 K_p^*] + \text{tr}[L_p^{*T} \delta_p^2 L_p^*]) + \sum_{s=1}^N \text{tr}[\kappa_s \underline{\Gamma}_s \Gamma_s] \\ & + \sum_{\bar{p} \in \mathcal{I}(p)} \left(\frac{1}{2} \text{tr}[K_{\bar{p}}^{*T} \delta_{\bar{p}} K_{\bar{p}}^*] + \frac{1}{2} \text{tr}[L_{\bar{p}}^{*T} \delta_{\bar{p}} L_{\bar{p}}^*] + \sum_{i=1}^m (\beta_{i\bar{p}} \epsilon_{i\bar{p}}) / \gamma_{i\bar{p}} \right). \end{aligned} \quad (25)$$

Again, the definition of the Lyapunov function (15) yields

$$V \geq \lambda_{\min}(P_p) \|e\|^2 \geq \underline{\varrho}_m \|e\|^2. \quad (26)$$

We had defined earlier $0 < \chi_p < \kappa_p$. Hence, using (25)-(26), (24) is simplified as

$$\dot{V} \leq -\chi_p V - (\kappa_p - \chi_p)V + 2\zeta_1 \sqrt{V/\underline{\varrho}_m} + \zeta_2. \quad (27)$$

Thus, $\dot{V} \leq -\epsilon_p V$ is established when

$$V \geq \max_{p \in \Omega} \left(\frac{\zeta_1}{\sqrt{\underline{\varrho}_m}(\kappa_p - \chi_p)} + \sqrt{\frac{\zeta_1^2}{\underline{\varrho}_m(\kappa_p - \chi_p)^2} + \frac{\zeta_2}{(\kappa_p - \chi_p)}} \right)^2.$$

So we obtain that a positive \mathcal{B} as

$$\mathcal{B} = \max_{p \in \Omega} \left(\frac{\zeta_1}{\sqrt{\underline{\varrho}_m}(\kappa_p - \chi_p)} + \sqrt{\frac{\zeta_1^2}{\underline{\varrho}_m(\kappa_p - \chi_p)^2} + \frac{\zeta_2}{(\kappa_p - \chi_p)}} \right)^2. \quad (28)$$

In light of this, further analysis is needed to observe the behavior of $V(t)$ between the two consecutive switching instants, i.e. $t \in [t_l \ t_{l+1})$, for two possible cases:

- (i) when $V(t) \geq \mathcal{B}$, we have $\dot{V}(t) \leq -\chi_p V(t)$ from (27) implying exponential decrease of $V(t)$;
- (ii) when $V(t) < \mathcal{B}$, no exponential decrease can be derived.

Behavior of $V(t)$ is discussed below individually for these two cases.

Case (i): There exists a time, call it T_1 , when $V(t)$ enters into the bound \mathcal{B} and $\bar{N}(t)$ denotes the number of all switching intervals for $t \in [t_0 \ t_0 + T_1)$. Accordingly, for $t \in [t_0 \ t_0 + T_1)$, using

(17), (27) and from the **Definition 1** we have

$$\begin{aligned}
V(t) &\leq \exp\left(-\chi_{\sigma(\bar{N}(t)-1)}(t_{\bar{N}(t)} - t_{\bar{N}(t)-1})\right) V(t_{\bar{N}(t)-1}) \\
&\leq \mu_{\sigma(\bar{N}(t)-1)} \exp\left(-\chi_{\sigma(\bar{N}(t)-1)}(t_{\bar{N}(t)} - t_{\bar{N}(t)-1})\right) V(t_{\bar{N}(t)-1}^-) \\
&\leq \mu_{\sigma(\bar{N}(t)-1)} \exp\left(-\chi_{\sigma(\bar{N}(t)-1)}(t_{\bar{N}(t)} - t_{\bar{N}(t)-1})\right) \cdot \\
&\mu_{\sigma(\bar{N}(t)-2)} \exp\left(-\chi_{\sigma(\bar{N}(t)-2)}(t_{\bar{N}(t)-1} - t_{\bar{N}(t)-2})\right) V(t_{\bar{N}(t)-2}^-) \\
&\quad \vdots \\
&\leq \mu_{\sigma(\bar{N}(t)-1)} \exp\left(-\chi_{\sigma(\bar{N}(t)-1)}(t_{\bar{N}(t)} - t_{\bar{N}(t)-1})\right) \cdot \\
&\mu_{\sigma(\bar{N}(t)-2)} \exp\left(-\chi_{\sigma(\bar{N}(t)-2)}(t_{\bar{N}(t)-1} - t_{\bar{N}(t)-2})\right) \cdot \\
&\quad \cdots \mu_{\sigma(t_0)} \exp(-\chi_{\sigma t_0}(t_1 - t_0)) V(t_0) \\
&= \prod_{p=1}^N \mu_p^{N_p} \exp\left(\sum_{p=1}^N \chi_p T_p(t_0, t_0 + T_1)\right) V(t_0) \\
&= c \exp\left(\sum_{p=1}^m \left(\frac{\ln \mu_p}{\tau_p} - \chi_p\right) T_p(t_0, t_0 + T_1)\right) V(t_0), \tag{29}
\end{aligned}$$

where $c \triangleq \exp\left(\sum_{p=1}^m N_{0p} \ln \mu_p\right)$ is a constant. Substituting the MDADT condition $\tau_p > \ln \mu_p / \chi_p$ to (29) yields $V(t) < cV(t_0)$ for $t \in [t_0, t_0 + T_1)$. Moreover, as $V(t_0 + T_1) < \mathcal{B}$, one has $V(t_{\bar{N}(t)+1}^-) < \mu_{\sigma(t_{\bar{N}(t)+1}^-)} \mathcal{B}$ from (17) at the next switching instant $t_{\bar{N}(t)+1}$ after $t_0 + T_1$. This implies that $V(t)$ may be larger than \mathcal{B} from the instant $t_{\bar{N}(t)+1}$. This necessitates further analysis.

We assume $V(t) \geq \mathcal{B}$ for $t \in [t_{\bar{N}(t)+1}, t_0 + T_2)$, where T_2 denotes the time before next switching. Let $\bar{N}(t)$ represent the number of all switching intervals for $t \in [t_{\bar{N}(t)+1}, t_0 + T_2)$. Then, substituting $V(t_0)$ with $V(t_{\bar{N}(t)+1})$ in (29) and following the similar procedure for analysis as (29), we have $V(t) \leq cV(t_{\bar{N}(t)+1}) < c\mu_{\sigma(t_{\bar{N}(t)+1}^-)} \mathcal{B}$ for $t \in [t_{\bar{N}(t)+1}, t_0 + T_2)$. Since $V(t_0 + T_2) < \mathcal{B}$, we have $V(t_{\bar{N}(t)+\bar{N}(t)+2}^-) < c\mu_{\sigma(t_{\bar{N}(t)+\bar{N}(t)+2}^-)} \mathcal{B}$ at the next switching instant $t_{\bar{N}(t)+\bar{N}(t)+2}$ after $t_0 + T_2$. If we follow similar lines of proof recursively, we can come to the conclusion that $V(t) < c \max_{p \in \Omega} \{\varrho_{Mp} / \varrho_{mp}\} \mathcal{B} < c\mu \mathcal{B}$ where $\mu = \bar{\varrho}_M / \underline{\varrho}_m$ for $t \in [t_0 + T_1, \infty)$. This confirms that once $V(t)$ enters the interval $[0, \mathcal{B}]$, it cannot exceed the bound $c\mu \mathcal{B}$ any time later with the ADT switching law (12).

Case (ii): It can be easily verified that the same argument below (29) also holds for Case (ii).

Next, we study the dynamics of the tracking error: based on the aforementioned analysis about UUB, it can be obtained that

$$V(t) \leq \max\{V(t_0), c\mu \mathcal{B}\}, \forall t \in t_0. \tag{30}$$

Then, it follows from that the tracking error is upper bounded in the following form:

$$\|e(t)\|^2 \leq \frac{1}{\underline{\varrho}_m} \max \left\{ V(t_0), c\mu\mathcal{B} \right\}. \quad (31)$$

Substituting $c \triangleq \exp\left(\sum_{p=1}^m N_{0p} \ln \mu_p\right)$ and $\mu = \bar{\varrho}_M / \underline{\varrho}_m$ into (31), thus the tracking error is UUB with an ultimate bound b with

$$b \in \left[0, \sqrt{\frac{\bar{\varrho}_M}{\underline{\varrho}_m^2} \mathcal{B} \prod_{p=1}^N \mu_p^{N_{0p}}} \right] \quad (32)$$

Thus, observing the stability arguments of the Cases (i) and (ii), it can be concluded that the closed-loop system remains UUB. \square

Theorem 1 reveals that stability of the ideal model reference closed loop (i.e. the switched linear closed-loop system arising from (1), (2) and the ideal control law before (3)) can be proven, along the arguments of [9–11], via the first quadratic term of the Lyapunov function in (15). Other remarks to compare **Theorem 1** with the state of the art follow:

Remark 6. *Because we keep the control gains constant during inactive times, one has to introduce a new mechanism to achieve stability. The proposed new mechanism is the auxiliary gain $\Gamma_{\sigma(t)}$ in (4), together with its adaptation law (9c). This gain plays the role of a leakage action for all inactive subsystems. Note that the second and the third term in the Lyapunov function (15) are summations over all (active and inactive) subsystems. In order to achieve exponential decrease of the Lyapunov function far enough from the origin (i.e. (24)), the items regarding active and inactive subsystems of $\dot{V}(t)$ should be offset by the corresponding items of $V(t)$, respectively. Since the derivative of the adaptive laws $K_{\bar{p}}, L_{\bar{p}}$ for inactive subsystems \bar{p} equals to zero, only the relative items regarding the active subsystem remain in $\dot{V}(t)$. That's why $\Gamma_{\bar{p}}$ is put forward: to compensate the missing part of inactive systems in $\dot{V}(t)$ such that (24) can be attained. Therefore, the crucial difference between [31] and the proposed scheme is the use the of auxiliary gains $\Gamma_{\sigma(t)}$ which avoids exponentially vanishing gains $K_{\bar{p}}, L_{\bar{p}}$ for the inactive subsystems \bar{p} . It is worth noticing that, with $\gamma_{i\sigma}$ being lower bounded by a positive value, the Lyapunov function V in (15) does not reach zero. However, the origin of tracking error and parametric estimation errors is not excluded; V may not reach origin, but the tracking error e and parameters estimation errors $\tilde{K}_\sigma, \tilde{L}_\sigma$ can still be zeros even if $\gamma_{i\sigma} \neq 0$. Eventually, the ultimate bound (32) on the tracking error e is still around the origin.*

Remark 7. *It has to be noted that, for a certain subsystem p , Γ_p might be different at switched-off and switched-on times, due to the evolution of $\gamma_{\bar{p}}$ in (9c) during inactive time intervals. This might lead to some transient at switched-on instant. However, there are clear evidences for such transient to be smaller than the one in [31]. The first evidence is that any possible transient in (4) is contributed only by $\Gamma_{\sigma(t)}$ which enters as a feedforward term: feedforward terms have less effect on learning transients than feedback terms. In the proposed design, the feedback gains $K_{\sigma(t)}$ do not contribute any transient, whereas the transients in [31] arise from both feedback terms $K_{\sigma(t)}$ and feedforward terms $L_{\sigma(t)}$. The second evidence is that the effects of transients in $\Gamma_{\sigma(t)}$ can be reduced by properly tuning the design parameters: for example, selecting $\gamma_{i\bar{p}}(t_0)$ and $\beta_{i\bar{p}}, \epsilon_{i\bar{p}}$ in (9c) very close to each other, with relatively high $\beta_{i\bar{p}}$, will induce a fast decrease of $\gamma_{i\bar{p}}$ to its lower bound. Therefore, $\gamma_{i\bar{p}}$ will be almost the same at switched-on and switched-off times.*

In other words, the intuition behind (4) and (9) is that it will sensibly reduce learning transients at switched-on instants. Of course, improved transient cannot be formally proven because any bound on transient performance of adaptive closed-loop systems is in general very conservative [27]. Nevertheless, one can verify the improved transient performance in simulations, as done in Section 5.

For a proper comparison with [31], **Theorem 1** is now modified to account for MMDADT in **Definition 2**.

Corollary 1. *Under **Assumption 1**, the closed-loop switched system formed by system (1), the reference model (2), the controller (4), the adaptive laws (9), and the switching law (13), is Uniformly Ultimately Bounded (UUB) and an ultimate bound b on the tracking error e can be found as*

$$b \in \left[0, \sqrt{\frac{1}{\underline{\varrho}_m} \max_{\substack{p,q \in \Omega \\ q \in \mathcal{N}(p)}} \{\mu_{pq}\} \mathcal{B} \prod_{p=1}^N \mu_{pq}^{N_{0pq}}} \right], \quad (33)$$

where the scalar \mathcal{B} is the same positive constant as in **Theorem 1**.

Proof. The proof follows the same steps as Theorem 1 with same Lyapunov function (15) being adopted. The main difference arises from the relationship of the values between the Lyapunov function at switching instant t_{l+1} , which can be expressed as:

$$V(t_{l+1}) \leq \frac{\varrho_{M\sigma(t_{l+1})}}{\varrho_{m\sigma(t_{l+1}^-)}} V(t_{l+1}^-) = \max_{\substack{p,q \in \Omega \\ q \in \mathcal{N}(p)}} \{\mu_{\sigma(t_{l+1})\sigma(t_{l+1}^-)}\} V(t_{l+1}^-). \quad (34)$$

Here, we define $\mu_{pq} = \mu_{\sigma(t_{l+1})\sigma(t_{l+1}^-)}$, $p, q \in \Omega, q \in \mathcal{N}(p)$. The analysis of the Lyapunov function during the switching intervals is identical with (18)-(28). Since the switching sequence is known, the maximum increase of the Lyapunov function at the switching instants is $\max_{\substack{p,q \in \Omega \\ q \in \mathcal{N}(p)}} \{\mu_{pq}\}$ instead of $\bar{\varrho}_M/\underline{\varrho}_m$ as in the MDADT case. The rest of the proof follows the lines from (24)-(32) after substituting $\mu_{\sigma(t_{l+1}^-)}$ with $\mu_{\sigma(t_{l+1})\sigma(t_{l+1}^-)}$ and $c \triangleq \exp(\sum_{p=1}^m N_{0pq} \ln \mu_{pq})$. We conclude that the adaptive law (9) and the switching law with MMDADT (13) lead to UUB stability with bounds (33). \square

5. Simulation Results

A benchmark example commonly adopted in switched adaptive literature [31, 36–40] is considered to show how the proposed strategy compares to the state of the art, i.e. the approach in [31]. The example is a simplified model of a Highly Maneuverable Aircraft Technology (HiMAT) with the following three subsystems:

$$A_1 = \begin{bmatrix} -0.8435 & 0.97505 & -0.0048 \\ 8.7072 & -1.1643 & 0.0026 \\ 0 & 1 & 0 \end{bmatrix}, \quad B_1 = \begin{bmatrix} -0.1299 & -0.092 & -0.0107 & -0.0827 \\ -7.6833 & -4.7974 & 4.8178 & -5.7416 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

$$A_2 = \begin{bmatrix} -1.8997 & 0.98312 & -0.00073 \\ 11.720 & -2.6316 & 0.00088 \\ 0 & 1 & 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} -0.2436 & -0.1708 & -0.00497 & -0.1997 \\ -46.206 & -31.604 & 22.396 & -31.179 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Table 1: Parameters for the six switching families under consideration (note that DT/ADT, MDDT/MDADT, and MMDDT/MMDADT have the same τ^* , τ_p^* , and τ_{pq}^* , as they only differ in terms of chattering bound).

Switching strategies	DT/ADT	MDDT/MDADT	MMDDT/MMDADT
Switching sequences	Unknown	Unknown	Known in advance
Parameters	$\tau^* = 23.7$ $\mu = 278.3$ $\kappa = 0.25$	$\tau_1^* = 16.3, \tau_2^* = 11.8, \tau_3^* = 13.2$ $\mu_1 = 48.6, \mu_2 = 278.3, \mu_3 = 154.1$ $\kappa_1 = 0.25, \kappa_2 = 0.5, \kappa_3 = 0.4$	$\tau_{13}^* = 16.3, \tau_{32}^* = 12.6$ $\tau_{21}^* = 11.8, \tau_{23}^* = 10$ $\mu_{13} = 48.6, \mu_{32} = 120.3, \mu_{21} = 272.4$ $\kappa_1 = 0.25, \kappa_2 = 0.5, \kappa_3 = 0.4$

$$A_3 = \begin{bmatrix} -1.2206 & 0.99411 & -0.00084 \\ -64.071 & -1.8876 & 0.00046 \\ 0 & 1 & 0 \end{bmatrix}, B_3 = \begin{bmatrix} -0.0662 & -0.0315 & -0.0141 & -0.0749 \\ -27.333 & -13.163 & 11.058 & -26.878 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

5.1. Design of reference model

Three ideal controllers and reference models arise from the same design of [31], whose parameters are given below for completeness:

$$K_1^* = \begin{bmatrix} 0.6219 & 0.7469 & 1.4508 \\ 0.3969 & 0.4671 & 0.9013 \\ -0.3174 & -0.4621 & -0.9483 \\ 0.4534 & 0.5572 & 1.0902 \end{bmatrix}, L_1^* = I_{4 \times 4}, A_{m1} = \begin{bmatrix} -0.9949 & 0.7939 & -0.3562 \\ -2.1076 & -14.5691 & -26.2966 \\ 0 & 1 & 0 \end{bmatrix}.$$

$$K_2^* = \begin{bmatrix} 0.1984 & 0.6793 & 1.5202 \\ 0.1368 & 0.4646 & 1.0392 \\ -0.0642 & -0.3289 & -0.7527 \\ 0.1431 & 0.4585 & 1.0212 \end{bmatrix}, L_2^* = I_{4 \times 4}, A_{m2} = \begin{bmatrix} -1.9997 & 0.6484 & -0.7487 \\ -7.6710 & -70.3615 & -151.7803 \\ 0 & 1 & 0 \end{bmatrix}.$$

$$K_3^* = \begin{bmatrix} -0.6674 & 0.6397 & 1.4517 \\ -0.3220 & 0.3081 & 0.6995 \\ 0.3287 & -0.2599 & -0.6292 \\ -0.6423 & 0.6288 & 1.4175 \end{bmatrix}, L_3^* = I_{4 \times 4}, A_{m3} = \begin{bmatrix} -1.1228 & 0.8986 & -0.2163 \\ -20.6916 & -43.2036 & -93.9421 \\ 0 & 1 & 0 \end{bmatrix}.$$

5.2. Comparison

For a fair comparison purposes with [31], the leakage action in the adaptive laws in (9) is slightly modified as

$$\dot{K}_p^T(t) = -S_p^T B_{mp}^T P_p e(t) x^T(t) - \delta_p M_p K_p^T(t), \quad \dot{K}_{\bar{p}}^T(t) = 0, \quad (35a)$$

$$\dot{L}_p(t) = -S_p^T B_{mp}^T P_p e(t) r^T(t) - \delta_p M_p L_p(t), \quad \dot{L}_{\bar{p}}(t) = 0, \quad (35b)$$

$$\dot{\gamma}_{ip}(t) = 0,$$

$$\dot{\gamma}_{i\bar{p}}(t) = -\left(\beta_{i\bar{p}} + \delta_{\bar{p}}(\{K_{\bar{p}}(t)K_{\bar{p}}^T(t)\}_{ii} + \{L_{\bar{p}}^T(t)L_{\bar{p}}(t)\}_{ii})\right)\gamma_{i\bar{p}}(t) + \beta_{i\bar{p}}\epsilon_{i\bar{p}}, \quad (35c)$$

$$\text{with } \delta_p \geq \lambda_{\max}(M_p^{-1}) \geq 0, \quad (35d)$$

$$\text{and } \gamma_{ip}(t_0), \gamma_{i\bar{p}}(t_0) > \epsilon_{i\bar{p}}, \quad (35e)$$

which allows us a perfect comparisons with [31] under the same choice of design parameters. Please note that the only difference between (11) and (9) is the special choice of the leakage gain, which requires some knowledge of L_p^* .

Let $\kappa_1 = 0.25, \kappa_2 = 0.5, \kappa_3 = 0.4$ as in [31]. By solving (8), we get the following positive definite matrices:

$$P_1 = \begin{bmatrix} 0.7337 & -0.0162 & -0.3781 \\ -0.0162 & 0.0549 & 0.0800 \\ -0.3781 & 0.0800 & 2.3960 \end{bmatrix}, P_2 = \begin{bmatrix} 0.5225 & -0.0028 & -0.0517 \\ -0.0028 & 0.0092 & 0.0132 \\ -0.0517 & 0.0132 & 1.9764 \end{bmatrix},$$

$$P_3 = \begin{bmatrix} 0.7942 & -0.0063 & -0.3177 \\ -0.0063 & 0.0167 & 0.0241 \\ -0.3177 & 0.0241 & 2.4767 \end{bmatrix}.$$

As explained in **Remark 1**, this implies that the same parameters for DT, MDDT, MMDDT can be obtained as in [31]. Table 1 shows such parameters, whereas Fig. 1 shows three switching signals satisfying the DT, MDDT and MMDDT requirements (such signals are the same as [31]). We also provide three additional switching families which satisfy the ADT, MDADT and MDDADT requirements: such signals have the same τ^* , τ_p^* , and τ_{pq}^* as DT, MDDT, MMDDT, and they only differ in terms of chattering bound. The chattering bound allows fast switching, compensate by slow switching later on: this can be seen from the three switching signals depicted in Fig. 2. Then,

Table 2: Total RMS and transient RMS errors for the six switching laws (the transient RMS error is calculated for one second after each switching).

	DT	ADT	MDDT	MDADT	MMDDT	MMDADT
Total RMS error						
Method in [31]	0.1295	0.1257	0.1172	0.1116	0.1369	0.1353
Proposed method	0.1153	0.1123	0.0895	0.0876	0.1195	0.1176
Improvement	11.0%	10.7%	23.6%	21.5%	12.7%	13.1%
Transient RMS error						
Method in [31]	0.1904	0.1893	0.1930	0.1938	0.2009	0.1520
Proposed method	0.1085	0.0785	0.1398	0.1382	0.1293	0.0820
Improvement	43.0%	58.6%	27.6%	28.7%	35.7%	46.1%

we consider for the proposed adaptation laws (9) the same design parameters as [31], i.e. the adaptive gains $S_1 = S_2 = S_3 = 10I_{4 \times 4}$, and the leakage rates $\delta_1 = \delta_2 = \delta_3 = 0.05$. What is left to design in (9) are the parameters for (9c) which are taken $\varepsilon_{i\bar{p}} = 0.1, \beta_{i\bar{p}} = 2$ with $i = 1, 2, 3, 4$. The initial conditions are $x(0) = [0 \ 0 \ 0]^T, x_m = [2 \ 2 \ 1]^T, K_p(0) = 0.8K_p^*, L_p(0) = 0.8L_p^*$, the disturbance is $d(t) = [0.2 \sin(10t) \ 0.15e^{-t} \ 0.1 \cos(\pi t)]$, and the reference input is $r(t) = [2 \sin(t) \ \cos(t) \ 0.5 \sin(0.5t) \ 0]^T$.

The comparison in terms of tracking errors are depicted in Figs. 3-5, for all six switching signals of Figs. 1-2 (upper plots are the tracking errors for the approach in [31], lower plots are the tracking errors for the proposed approach). From the lower plots of each figure, it is noticeable that the learning transients of the proposed methods are considerably reduced, in contrast with the method of [31]. This confirms that the bad effects of vanishing gains are alleviated. On the other hand,

it has to be acknowledged that the learning transients are not completely removed because the adaptive gain Γ_σ evolve during inactive times. However, the intuition of **Remark 7** is confirmed, i.e. the feedforward term Γ_σ has less effect on transient performance than the feedback gain K_σ in [31].

The performance improvements are quantified in Table 2 and visualized in Fig. 9, which show that not only the total Root Mean Square (RMS) error is reduced, but especially the transient RMS error is sensibly reduced. The transient RMS error is calculated for one second after each switching, as a way to measure the learning transients. The table shows that the improvement in terms of transient is much more pronounced than the improvement over the whole simulation: notice how the transient improvements range in 27-58%, depending on the switching signal.

5.3. Additional simulations

To further elaborate on consistency of the proposed result, we test a different leakage action, i.e. we test the proposed adaptive laws (9) against the state-of-the-art adaptive laws (11), where the terms $\delta_p M_p$ and $\delta_{\bar{p}} M_{\bar{p}}$ are replaced with δ_p and $\delta_{\bar{p}}$, respectively. This leakage action represents the case when M_p is unknown and thus cannot be used for control design. All the other parameters are left unchanged. The results are summarized in Table 3 and visualized in Fig. 10 (the plots are not shown for compactness): again, consistent improvements can be noticed. The improvements in terms of total RMS error are sometimes smaller than before, while transient improvements range in 36-82%, depending on the switching signal.

Table 3: Total RMS and transient RMS errors for the six switching laws with alternative leakage term (the transient RMS error is calculated for one second after each switching).

	DT	ADT	MDDT	MDADT	MMDDT	MMDADT
Total RMS error						
Method in [31]	0.0997	0.0990	0.0796	0.0776	0.1018	0.0998
Proposed method	0.0952	0.0961	0.0605	0.0620	0.0971	0.0960
Improvement	4.5%	2.9%	24.0%	20.0%	4.6%	3.8%
Transient RMS error						
Method in [31]	0.1738	0.1564	0.1364	0.1320	0.1437	0.0871
Proposed method	0.0794	0.0285	0.0787	0.0793	0.0907	0.0556
Improvement	54.3%	81.8%	42.3%	39.9%	36.9%	36.2%

6. Conclusions

This paper proposed a new adaptive framework based on leakage mechanism for the robust control of uncertain switched linear systems. Owing to introduction of an auxiliary gain, the proposed framework allows the adaptive gains of the inactive subsystems to keep the same values as switched-off: this is in clear contrast with the state of the art where the control gains of the inactive subsystems should vanish during inactive times. This innovation sensibly reduces the learning transient at switched-on instants for various families of dwell-time based switching signals (and their extensions).

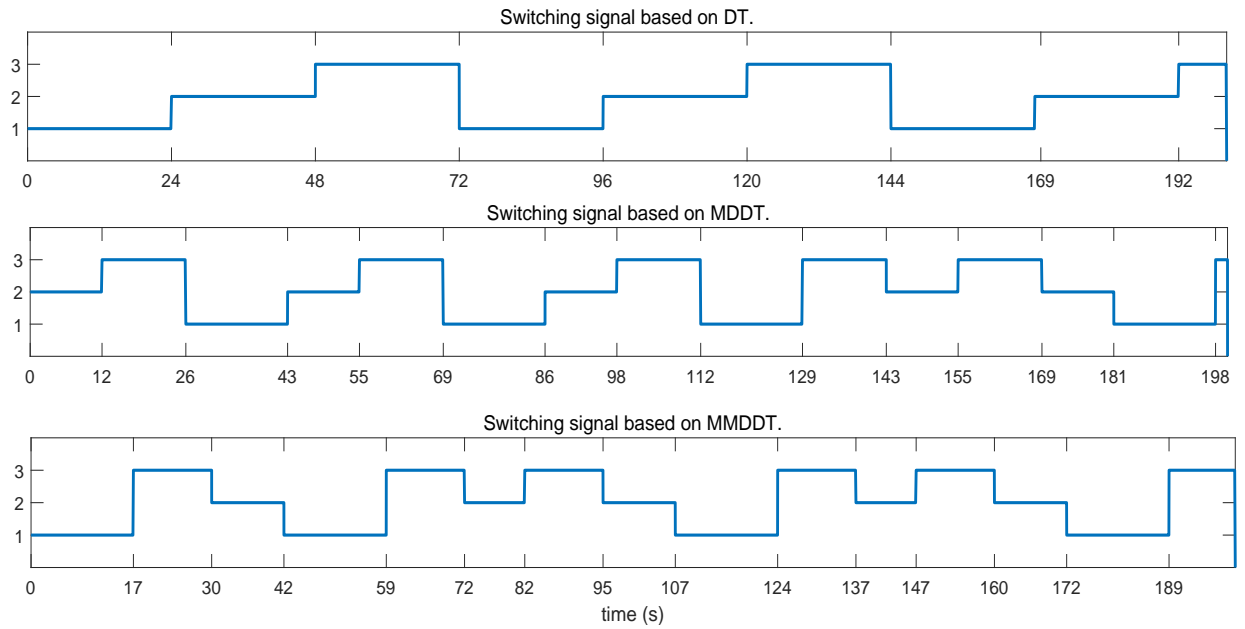


Figure 1: Switching signal based on DT, MDDT, MDDDT

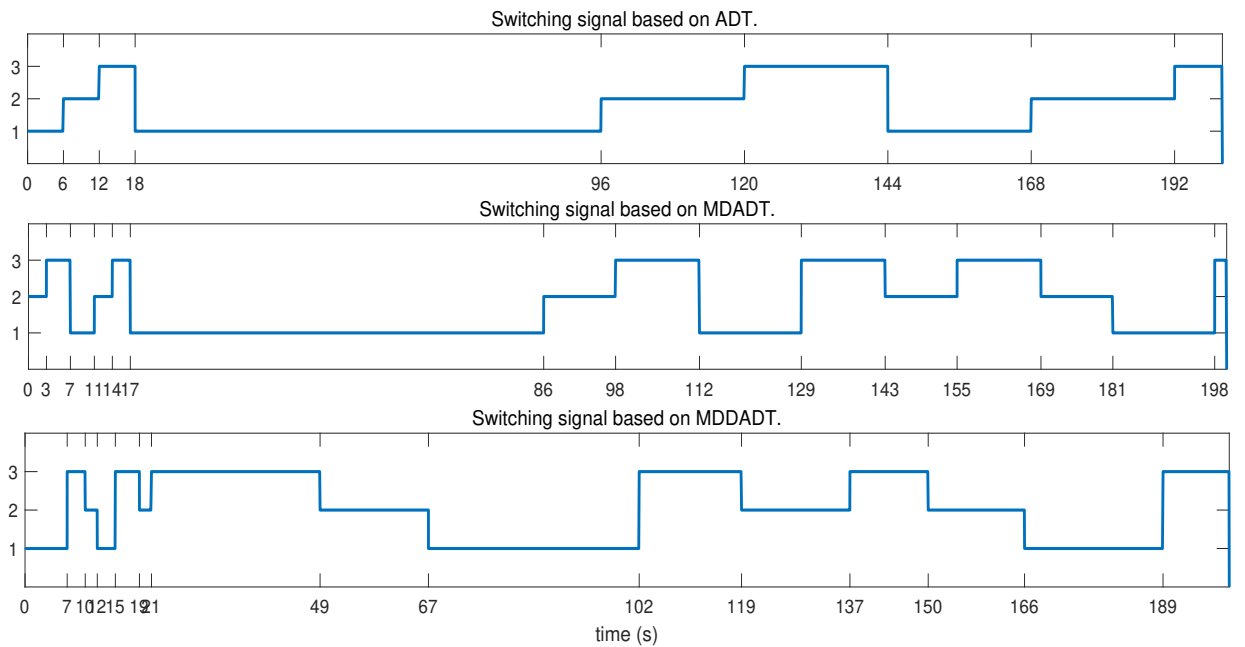


Figure 2: Switching signal based on ADT, MDADT, MMDADT

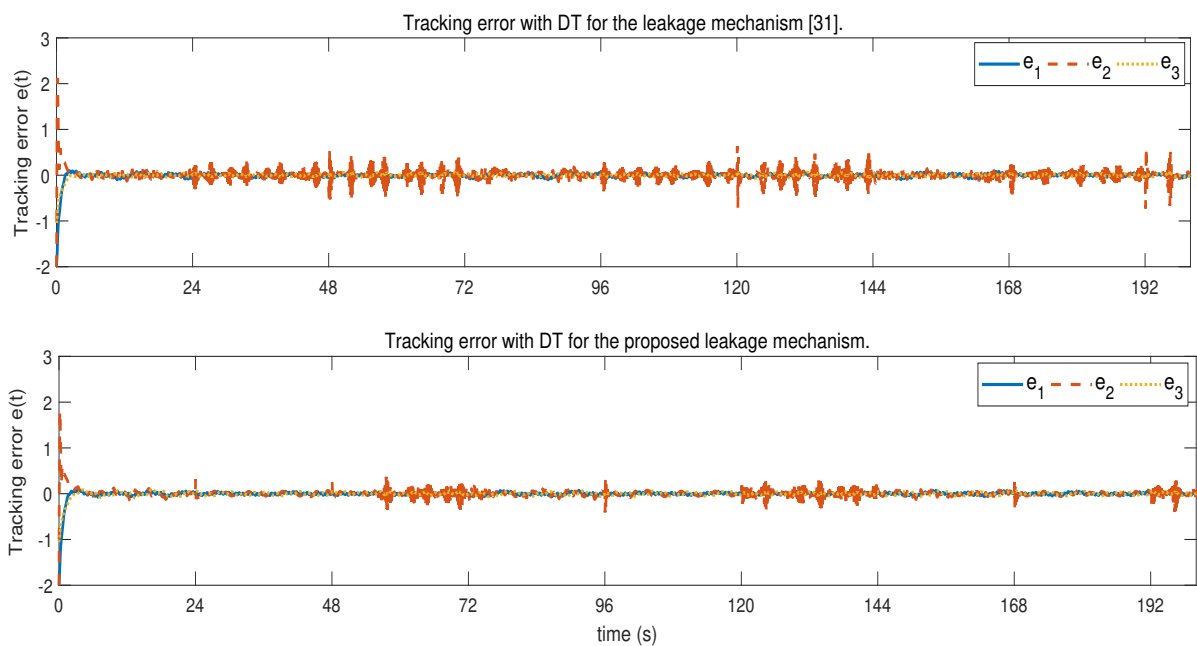


Figure 3: Tracking error for DT switching law

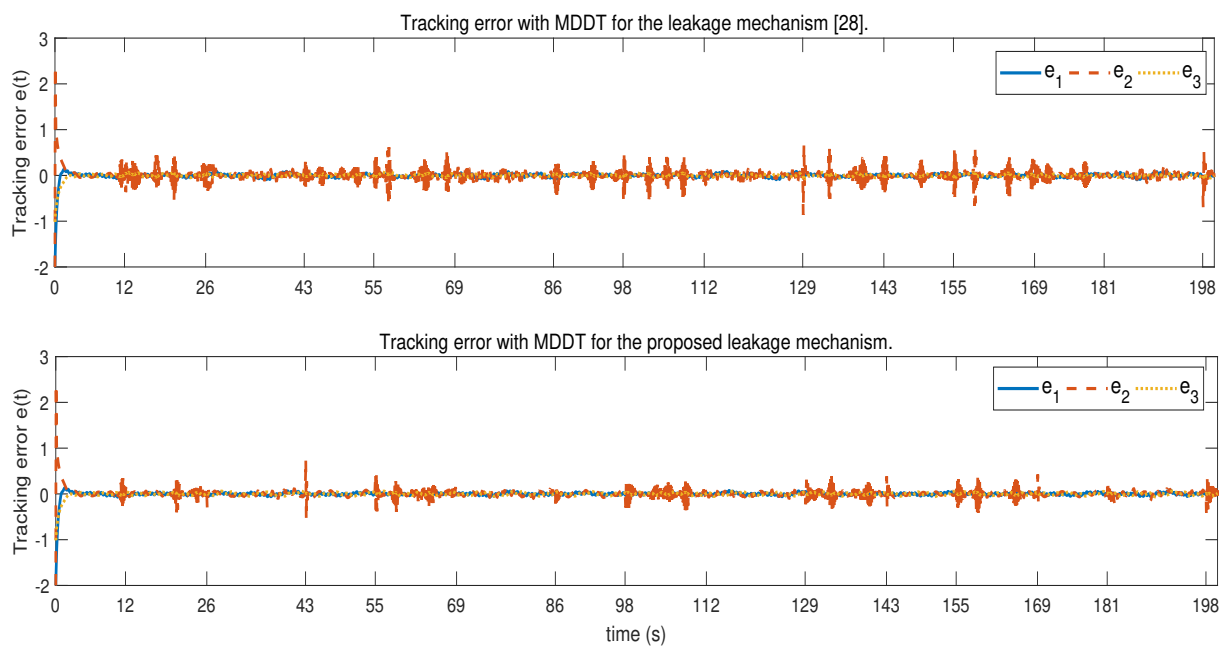


Figure 4: Tracking error for MDDT switching law

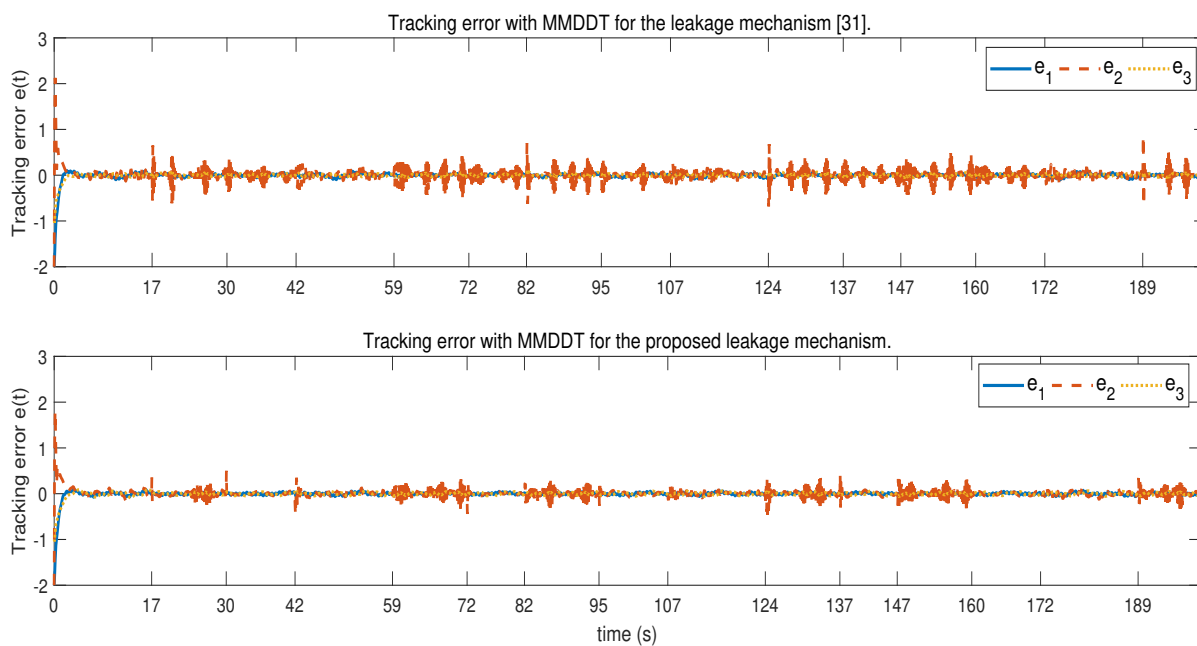


Figure 5: Tracking error for MDDT switching law

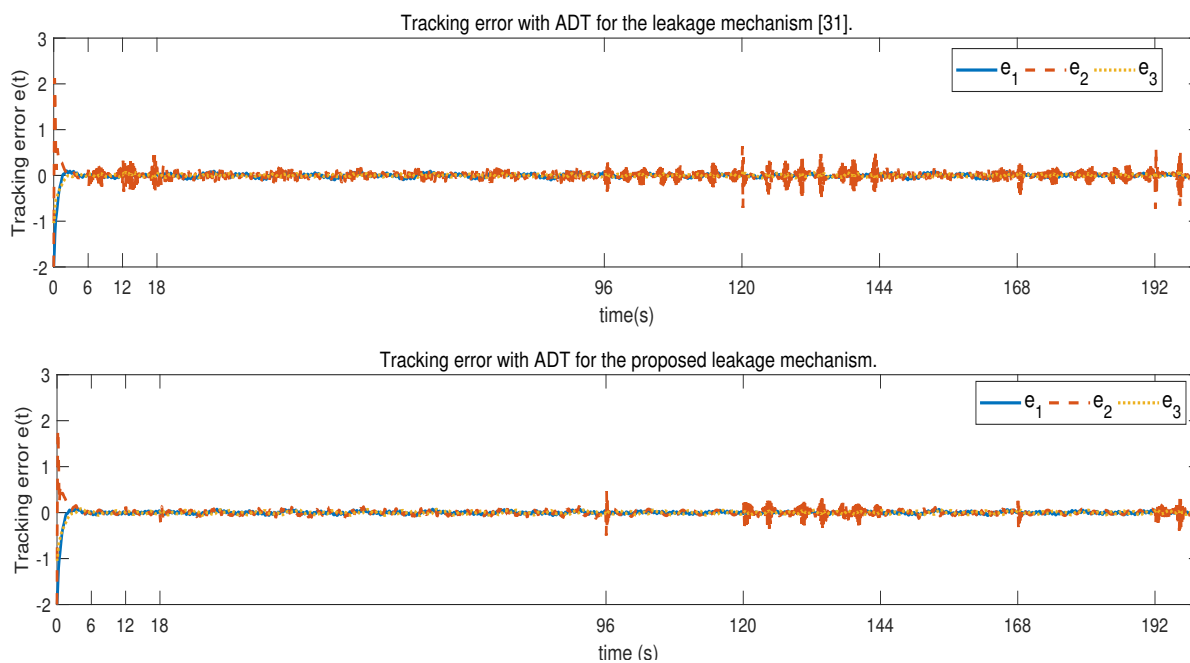


Figure 6: Tracking error for ADT switching law

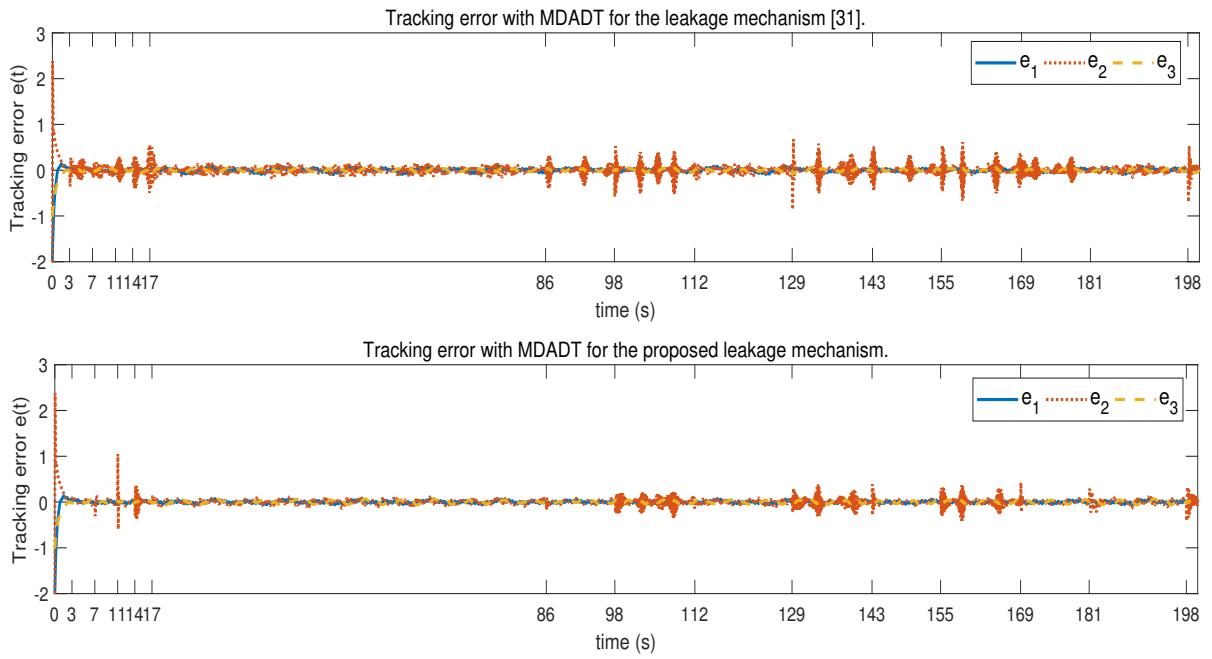


Figure 7: Tracking error for MDADT switching law

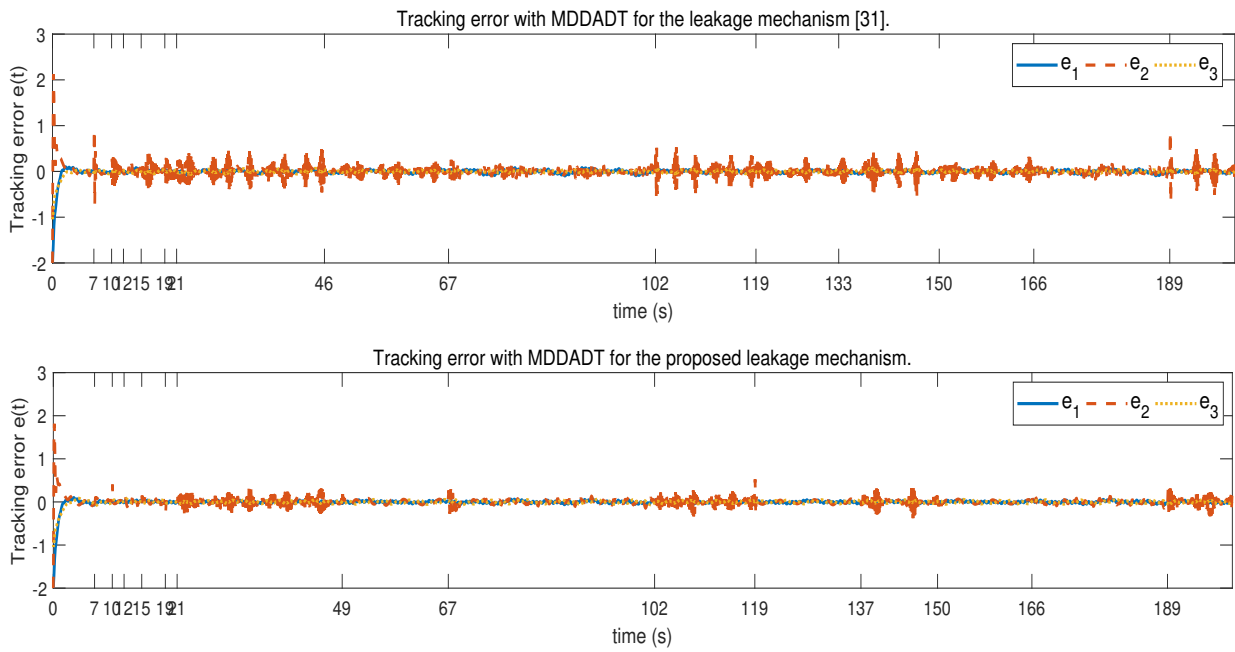


Figure 8: Tracking error for MMDADT switching law

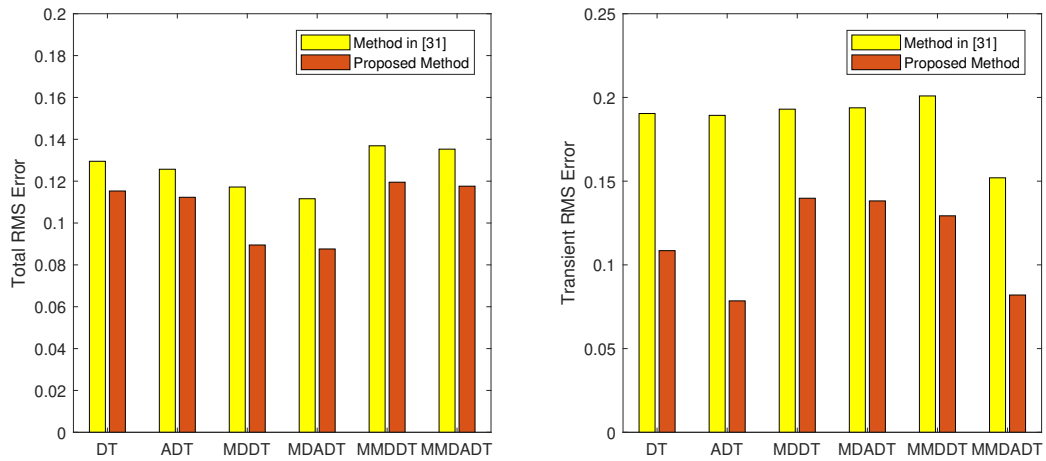


Figure 9: Visualization of the comparative results in Table 2.

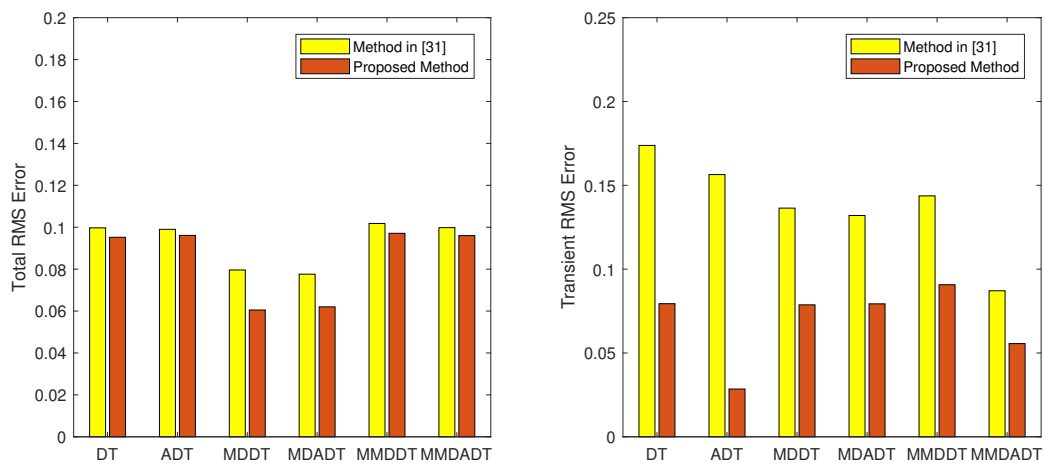


Figure 10: Visualization of the comparative results in Table 3.

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