A Comparison of Rigour and Intuition in Illustrations in Calculus and Analysis Textbooks

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Abstract. In this paper, illustrations in both computation-based calculus and proof-based analysis textbooks are analysed and with particular attention to the differences in their roles. Via a thematic analysis approach, we examine how visualisations for the topics continuity, differentiability and integration function within the corpus of three textbooks. Our analysis identifies the juxtaposition of rigor with intuition as a red thread. The coded illustrations reveal differences that support this distinction and are divided into the subthemes: definitions, examples and proofs. For definitions, analysis textbooks seem to use illustrations related to more formal definitions, and calculus books frequently use the ones tied to informal definitions and introductions to definitions. Examples show a similar distinction between illustrations in the analysis textbooks and calculus textbooks: counterexamples and examples that build intuition versus exercises and confirming examples. For the proofs, this study shows how illustrations are modified such that the same illustration serves a different role given the textbook: substitute for the proof or general outline of the proof.

Keywords: Illustration · Calculus · Analysis · Proof · Example · Rigour · Intuition · Definition

1 Introduction

When studying mathematics, visualisations are often employed to aid the reader's understanding of the topic. Although there is a dominance of non-visual, linguistic elements in mathematics, in light of renewed interest in visualisations made possible by computers [1] [2], it is important to consider how mathematical knowledge is represented and accessed. Because access to mathematical objects is not direct but mediated through their representations, visualisation, being one of them, plays a key role in the understanding of mathematics alongside symbolic notation and natural language descriptions.[3]

Visualisations are also relevant in that they help us translate abstract ideas into something that is mentally more 'visible'.[4] They help reveal patterns that mere symbols do not easily show and sometimes they even drive the entire solution process by enabling recognition of patterns. Thus, they form a source of mathematical knowledge, because they, for instance, help to prove a conjecture. [2] [5], [6], [7]. In fact, illustrations serve as bridges between intuition, symbolic manipulation and formal proof.[8]

Textbooks are widely used in mathematics education and most make use of illustrations to some extent. Reading mathematics textbooks is found to be more difficult than reading ordinary books, because you have to actively switch between different contexts: the symbolical notation, text and illustrations.[9] [10, p. 78] Therefore, information including illustrations must be presented in a streamlined, cognitively efficient way to improve this.[11] Even the position of illustrations within the textbook in relation to the text which references it is of the essence.[12]

With their ubiquity, illustrations serve different communicative roles in different textbooks. In computation-oriented contexts, illustrations often support intuition or demonstrate procedures [13], [14], whereas in proof-based contexts they may guide reasoning without replacing formal argumentation.[5] For this possible distinction among textbooks, we will consider calculus and analysis textbooks due to their widespread use in the hard sciences and because they mark the transition from high-school mathematics to more advanced and rigorous mathematics. The distinction is relevant, because concept definitions, which are the formal mathematical definitions, often do not agree with concept images in the minds of the students – the mental structure associated with a concept, which would include intuitive pictures and informal reasoning.[15]

This raises the central question of this study: How are illustrations used, and what communicative functions do they serve, in computation-oriented calculus versus proof-oriented analysis textbooks? To address this, the study investigates patterns in how illustrations function across a calculus textbook, an analysis textbook and a book that forms a bridge between *calculus* and *analysis* – a distinction which is to a great extent based on findings by Raman.[16] It examines whether figures primarily support computation or conceptual understanding, what communicative roles they take, and how these roles vary between more application-oriented and more formal textbooks.

Using a thematic analysis approach, the study aims to identify systematic differences both across textbooks and topics, contributing to a deeper understanding of how visual representations shape mathematical practice. We find three themes that establish the roles of illustrations for definitions, examples and proofs, which are standard elements of the mathematical exposition. We will see that these are related to the distinction between rigour and intuition as the overarching theme.

2 Related Work

An important distinction in mathematics education is between conceptual knowledge and procedural knowledge. It was firstly formalized and popularized by James Hiebert [17]. In his work, conceptual knowledge emphasises understanding relationships and underlying structures, whereas procedural knowledge focuses on the execution of algorithms and computations.

Indirectly building on this distinction, Raman [16], in a comparative study of pre-calculus, calculus, and analysis textbooks, shows that each of these groups conveys different 'epistemological messages': intuitive and graphical reasoning dominates pre-calculus, procedural manipulation characterises calculus, and definition-based, proof-oriented reasoning defines analysis. This progression highlights a shift from computation towards proof, a transition that is often challenging for students [18].

Reading mathematics textbooks is a nonlinear process, where one is more likely to succeed when actively updating one's understanding and correcting confusions [19], a task which is even hard for 'good' mathematics students [20].

It has also been argued that a multitude of exercises in calculus textbooks lend themselves to be solved by way of pattern recognition and constant application of procedures that are learnt by rote. [21]

Several studies attempt to study the role of visual elements in mathematical reasoning across calculus and analysis textbooks. In a study on high-school textbooks, researchers found that pictures had inquiry-based and explanation based pedagogical roles.[22] It was found in another study how students often rely on an archetype of a graph on which they procedurally apply pre-learned interpretations to it.[13]

In another study, the teaching of the concept of infinite series was studied in several calculus and analysis textbooks.[23] It was found that 'understanding series' in some textbooks is essentially characterized as being able to apply the correct procedures to it, whereas in a rare case of a 'conceptually rich approach', the goal was to understand the representations of series and their link to other mathematical objects.

While studies such as [23] and [16] analyze differences in mathematical reasoning across calculus and analysis textbooks, the role of illustrations within these practices remains underexplored within the context of rigor versus intuition. This study addresses that gap by studying the epistemic roles of illustrations across analysis and calculus textbooks with particular attention to the distinction between rigour and intuition.

3 Methodology

Following the papers [15] [16] [23], we create a spectrum for mathematical texts on the topics Continuity, Differentiation and Integration: from proof-based, *analysis textbooks*, that emphasize reasoning, justification and the derivation of results from definitions to the computation-oriented, *calculus textbooks*, which in contrast prioritize techniques, formulas, and problem-solving procedures.

In this research, an iterative open coding system as well as thematic analysis is used to identify structural patterns in the illustrations and in the connections between them and the surrounding text. The thematic analysis is used as opposed to content analysis due to the nature of the research question. The aim of this study is to identify the epistemic roles of the illustrations and how these might contribute to rigour or intuition. The interpretive analysis therefore better fits the qualitative research question, rather than counting the frequencies of the occurrences of certain elements.

Illustrations along with their captions are defined as the units of analysis in this study. This unit of illustration consists of a single figure, so that one subfigure of a composite figure is also referred to as a unit. In the overall analysis, the links between the surrounding text and the illustrations are taking into account to study the epistemic role of the illustrations.

3.1 Data selection

The following selection of books was made:

Introduction to Analysis by Jan van Neerven [24] – Van Neerven
Analysis : An introduction to Proof by Steven R. Lay [25] – Lay
Calculus: Early Transcendentals by James Stewart [26] – Stewart

where the labels, given after the dashes, indicate how these books will be referred to in the following. Van Neerven's book is in its essence a reader compiled by the professor of the course Analysis I. Lay's book was previously used for the same course, but the difference is that it is more standardized and also does not assume that reader knows how to prove in mathematics. Stewart's book is mainly chosen for its wide use among engineering students and standardization in calculus textbooks.[27]

The books were selected using purposively sampling [28] to capture a spectrum of books, that goes from proof-based to calculation-heavy, procedural based textbooks. Due to this, we aim to find potential differences in the roles fulfilled by the illustration in these textbooks. These books were also included in the study, because they are used by TU Delft students, where this research is carried out.

There were several other analysis textbooks considered, but preliminary inspection of several analysis textbooks indicated they are comparable in their use of illustrations. In addition, another good candidate book, used in the study by Raman [16], Principles of mathematical analysis by Rudin [29], and known for its rigour, was not selected, as it does not contain any illustrations. Since this study focuses on the role of illustrations, the book did not fit within the scope of this research paper.

In addition, the topics Continuity, Differentiation and Integration are selected from these three books. This reduction of scope keeps the entire analysis manageable and focused. The other reason for choosing these topics is motivated by Raman's paper [16], as it highlights the difficulty students usually have with understanding the concept of continuity. This is noteworthy when considered along the fact that these topics have already been introduced by the time students reach tertiary education level. These concepts are presented in both calculus and analysis, whereas advanced topics such as uniform continuity or Cauchy sequences are not selected, since their rare use in most calculus books renders them unsuitable for our comparative study. Further, they are often presented in sequential order and are strongly interconnected, which goes even further than what is presented in first-year calculus and analysis textbooks (see, e.g., [30]).

Calculus and analysis form the gateway from the *pre-rigourous* stage to the *rigorous* stage in mathematics education, coined by Terence Tao in [32]. It makes them very relevant in our study, because these are then precisely the topics where lots of students develop mismatches between intuitions and the formal definitions [15].

3.2 Data Analysis

A combination of deductive and inductive open coding was used to analyse the illustrations and relevant text. This was an iterative process, for which data was first organised in document groups in atlas.ti, important sections for the selected topic were marked and relationships between text and illustrations were established. A subset of the highlighted sections were investigated to find an initial set of descriptive and interpretive codes that touch on the distinction of rigour and intuition/procedure. Afterwards, these were reviewed and refined primarily by making them more granular and descriptive. After any potential patterns were also identified, the rest of the selected data was assigned codes and the data in its entirety was reviewed to obtain a final version of the codebook.

In Table 1, the obtained family of codes can be found. This family is extended by child codes, that reveal more specific information about the picture and in that way also categorize it.

The important codes that will be used in the results section for the themes are more concretely worked out. In the following, the code families of *Definition*, *Example* and *Proof* are given along their codes and some explanation. The three code families are partially based on the paper by M. Giaquinto [34], where a list of 'mathematical activities' is given, where discovery, justification, application, representation are mentioned. The other code families are summarized. No codes for these families will be given, as these codes are not relevant for the analysis described in the result section.

The *Definition* family has *formal definition* and *informal definition* to refer to rigorous mathematical and intuitive, less abstract definitions, respectively The other code that is included in this family is *motivation / introduction* and it aims to describe illustrations that are trying to motivate or introduce a certain topic. The codes for the *Definition* family are based on Raman's paper [16]. While the paper defines those codes only for the topic of continuity, we extend this to our other topics as well.

The *Proof* family comprises of the codes *proof by picture*, *outline of proof* and *statement*. With *proof by picture*, we refer to an illustration that serves as a substitute for the proof in the text, whereas an *outline of proof* refers to an illustration that only sketches (the main idea behind) the proof. The proof is then more rigorously formulated in the text. The code *statement* refers to only the mathematical theorem being sketched.

The *Example* family consists of *confirming example*, *counterexample*, *exercise* and *intuition-building example*. With *intuition-building example*, we refer to illustrations for examples that are trying to build intuition for a certain problem by exploring or for instance showing surprising behaviour. This is often displayed when introducing or motivating a topic. An illustration for *confirming example* attempts to show why a specific example satisfies a specific definition or theorem, whereas *counterexample* shows why a certain statement might not hold up for a specific case, showing that for instance the seemingly plausible converse version of a theorem might not hold.

Other code families were used to indicate (*Color*) if RGB colors were used, (*Emphasis*) if elements such as linewidth, shading or zooming on part of picture were used to highlight part of the picture, (*Spatial Contiguity*) the positioning of the illustrations with respect to the text, (*Representation Type*) if for example a graph or a table was used, (*Understanding*) if the caption was needed to better understand the illustration. These code families were inspired by the paper [13].

Code Family	**Purpose/Explanation**
Color	Black/White or RGB coloring and its purpose
Emphasis	Visual highlights of parts of the figures, e.g. marking
Representation Type	E.g. tables, graph, graph + symbols
Definition	Role of illustration in defining concepts
Example	Role of illustration in exemplification
Proof	Role of illustration in the mathematical justification
Spatial Contiguity	Relationship between figure and its surrounding text
Understanding	Need for caption to better convey the meaning visually

Table 1: Code families and their purposes / explanation.

The three families (*Definition*, *Example* and *Proof*) underpin the distinction of rigour and intuition with their own patterns. Due to this and the formulation of the theme, we purposefully have worked out these families specifically in detail. The three code families do not serve as an exhaustive classification of the illustrations. Rather, the aim is to answer the research question with the focus on the contrast between analysis and calculus textbooks, which the three families fit well.

4 Results

We have coded 129 illustrations across the three books and the result of the analysis thereof will be presented in this section. The main theme is: illustrations are adapted or even re-contextualized in the analysis textbook and in the calculus book to fit the focus of rigour and the focus of intuition, respectively. To exemplify this and study this in more detail, we will divide this theme up in three subthemes: definition, example and proofs. Each has its own patterns, but we will observe the red thread that connects these three subthemes on a higher level: the distinction of rigour and intuition. This has, as we will observe, all kinds of implications for the kinds and roles of illustrations that are included in a calculus textbook and analysis textbook.

4.1 Definitions

This subtheme concerns illustrations that are associated with definitions or the motivation behind definitions. The subtheme is: illustrations associated with definitions are more frequently used in calculus textbooks to motivate and introduce a concept or illustrate an informal definition, whereas analysis textbooks overwhelmingly employ illustrations as a visual representation for formal definitions. To come to this conclusion, we will look at several illustrations for definitions.

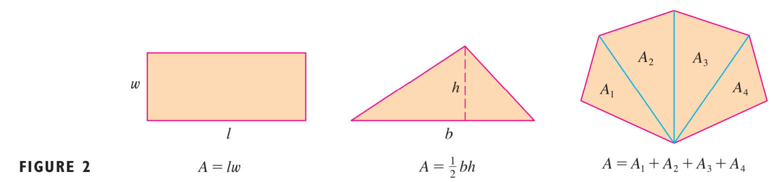


Fig. 1: Figure from Stewart's book [26, p. 355] showing how straightforward determining the area is for polygons. It is placed in a chapter about integration to highlight the question of how area can be found of a region that curves in its boundary.

With view on Figure 1 and its surrounding context, which is an illustration in the beginning of the chapter about integration in Stewart's book, we can deduce that this example is chosen to build intuition and motivate the topic of integration before stating the definition of the integral. To this end, a link is made to determining areas of polygons, which is assumed to be familiar to a reader. The third shape shown in the figure shows that the shape is divided into four smaller triangles, suggesting that the areas of these triangles is known or can be determined in a similar fashion as for the triangle shape in the middle of Figure 1. This suggests the same can be done for other shapes and it is a step towards defining the integral, where the area will be divided into many more 'known' areas. This illustration was for these reasons coded with *motivation / introduction*.

The illustration at the beginning of the chapter of integration in Van Neerven's book, as shown in figure 2, does not motivate the definition at all. A mere depiction of the definition of the Riemann sum is given, where dashed values indicate the *scattering*. The term *scattering* refers to the values along the x-axis by which the height of the shaded rectangles are determined – by using these x-coordinates as input to the function. These scattering values are not necessarily fixed uniformly for each rectangle. The illustration corresponds well with a

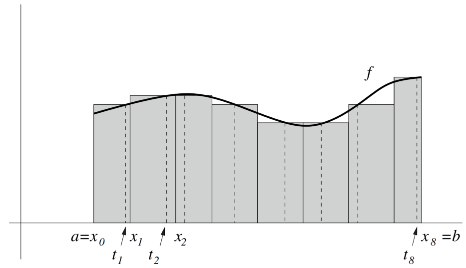


Fig. 2: Figure from Van Neerven's book [24, p. 92] showing the Riemann sum which is needed for a rigorous definition of the better known Riemann integral.

formal definition of a concept and is coded as such, because it presents a direct visualisation of the definition rather than an introduction or motivation for the Riemann sum.

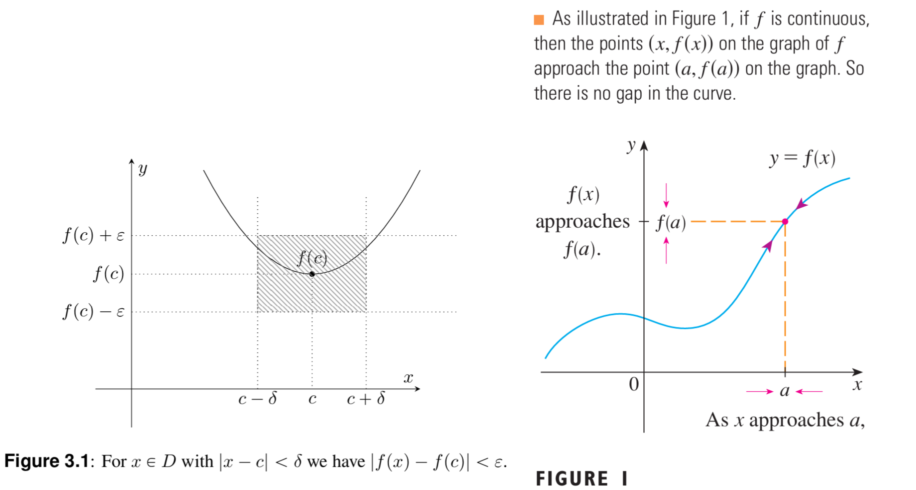
Figure 2 is also presented in Stewart's book, but this is done after several other illustrations that were coded with *introduction / motivation*. There are also two examples with specific functions to build intuition for how the area under the curve is partitioned. For this, the scattering values are fixed at the endpoints of the rectangles. Only after introducing the illustrations for the examples and the definition using the right endpoints, the illustration for the same definition as in Van Neerven's book is given.

In Figure 3, two subfigures for the definition of continuity are shown, that can be found in Van Neerven's and Stewart's book. Left Figure 3a fits the rigorous epsilon-delta definition given for continuity. This definition is stated as the following in Van Neerven's book [24, p. 47]:

Definition 3.1.1. Let D be a non-empty subset of R, and let a point c $\in$ D be given. A function f : D $\to \mathbb{C}$ is called continuous at c if for every $\epsilon > 0$ there exists a $\delta > 0$ such that for all x $\in$ D with $|x - c| < \delta$ we have $|f(x) - f(c)| < \epsilon$. The function f is called continuous if it is continuous at every point c $\in$ D. A function f is called discontinuous (at c) if it is not continuous (at c).

In Figure 3a, the area of interest is marked, which is defined by the δ and ϵ. One sees, that in similar fashion as for Figure 2, the left subfigure closely coincides with the definition by incorporating several elements of it and is consequently coded with *formal definition*.

The other illustration 3b in the Stewart's book helps to build intuition, without having to dive into the rigorous definition of 'x' approaching the value a. This could however give the wrong impression, as there is no distinction made between continuity at a point and on an interval, such that an explanation like "So there is no gap in the curve." could potentially reinforce incorrect intuition about continuity. Because the informal definition of continuity is illustrated without the use of ϵ, δ or other characteristics of the formal definition, this figure was coded as an *informal definition*.



(a) Rigorous definition

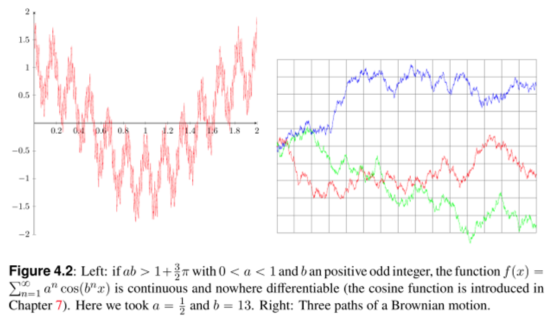
(b) Informal definition

Fig. 3: Illustrations for the definition of continuity in Van Neerven's book [24, p. 48] (left) and Stewart's book [26, p. 119] (right).

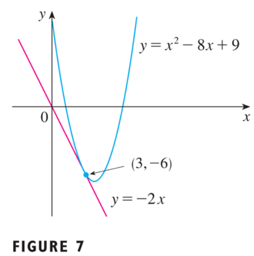
This comparison also suggests a pattern that is more frequently found when comparing illustrations from calculus to those from analysis textbooks: The calculus textbook attempts to build more intuition and give extended introduction and motivation for the topics through their illustrations. We observe that even if the same illustration for a rigorous definition is given, then that is only done after some introduction and motivation. In the analysis textbooks, on the other hand, no codings related to introductory or motivating illustrations are assigned. Also, the difference between analysis and calculus books in the order in which illustrations are presented is noteworthy. In analysis textbooks, the definition is first given in an unambiguous manner, after which an illustration may provide an graphical interpretation of the rigorous definition. Stewart's book introduces the problem via text and illustration and makes an attempt to connect it to familiar, already-existing intuition of the reader. Only after this, a rigorous definition and its potential illustration might be considered.

4.2 Examples

This subtheme concerns illustrations that are associated with examples. The subtheme is: The epistemic role of illustrations for the examples varies. Example-based learning is mainly supported in calculus textbooks. On the other hand, the validity of mathematical statements is to a greater degree challenged in analysis textbooks.



(a) Counterexample: functions that are continuous yet nowhere differentiable



(b) Finding derivative at a point on the graph

Fig. 4: Illustrations for a counterexample in the chapter about differentiation in Van Neerven's book [24, p. 74] (left) and an example of derivatives in Stewart's book [26, p. 147] (right).

Figure 4 displays an illustration from Van Neerven's book, 4a, and from Stewart's book, 4b. The left subfigure showcases a composite illustration, where the left graph in 4a illustrates a counterexample to the the statement "Continuous functions are also differentiable." and the right plot in 4a shows three paths of Brownian motion to highlight the unexpected abundance of functions that are a counterexample to that statement.

When trying to understand the validity of a statement in mathematics, two paths can usually be taken: prove the statement or disprove a statement, which is done by means of a counterexample. Because the left subfigure 4a is associated with text that comes after a proposition that mentions that "Differentiable functions are also continuous." (see Figure 6) and mentions that the converse of that proposition does not hold, this figure is coded with *counterexample*.

The right subfigure 4b shows an illustration associated with an example with the aim of determining the tangent line at a point (see also Figure 4). The example comes after a statement which mentions that the slope of the tangent line is equal to the derivative at that point. The illustration helps with visually verifying that the tangent line found using that statement is indeed correct. For this reason, this illustrations is coded with *confirming example*.

In comparison between the two Figures in 4, we note that much more information, that is not easily inferrable from the text (see also Figure 6 in the appendix) or from the picture, is given for the counterexample in the caption. At first, the information about which parameter might not seem highly important. However, this does fit in the framework used for counterexamples: to be an counterexample the function has to satisfy all the assumptions, or rather the antecedents, but violate the conclusion ('consequent') of the conditional statement. To disprove a statement rigorously, it is logical to include as much information as possible about the specifics of the counterexample to understand why a particular statement is not valid. The right figure also admittedly gives some additional information in the figure itself, but this is more so to link the different elements discussed in the text for the example (see also Figure 7 in the appendix) to the illustration of it.

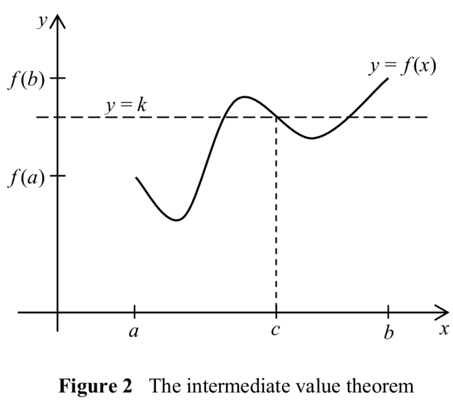
We also noticed a more general pattern: Analysis textbooks contain mostly illustrations coded with *counterexample* or *intuition-building example*. An instance of the latter is given in both analysis books; an illustration of the modified Dirichlet function, which is discontinuous at all rational points of its domain but continuous at all irrational points. It helps to illustrate the fact that continuous functions are not properly characterized by the informal definition that 'these are the functions that can be drawn without taking your pen off the paper'. The calculus books contain no illustrations classified with *counterexample*. These books include much more illustrations coded with *confirming example* and *exercise*, where the exercise refers to a graph or table for the extraction of specific information. This would indeed suggest that an example-based learning strategy is adopted in calculus textbooks, whereas in analysis textbooks the illustrations reflect that mathematical statements are contextualised by considering both possibilities and limitations within the theory.

4.3 Proofs

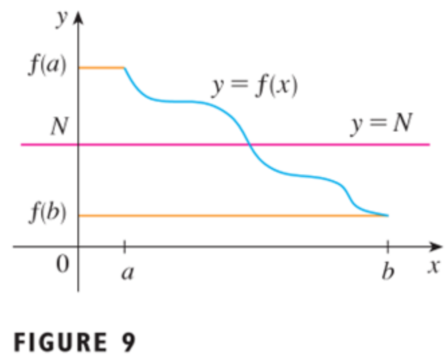
This subtheme concerns illustrations that are associated with proofs and the subtheme is: In calculus books illustrations sometimes take the role of a proof, whereas analysis books employ illustrations to demonstrate the main idea of a proof. We demonstrate this by considering an illustration in the calculus textbook and in the analysis textbook.

Figure 5 shows how essentially the same illustrations are tweaked according to the contexts they are found in to serve different roles from each other. They both concern the intermediate value theorem in mathematics that states a continuous function f on an interval $[a, b]$ achieves all values that lie in the interval $[f(a), f(b)]$. Both plots illustrate the idea and some intuition behind the proof for this statement. The manner in which this is done by the illustrations is different for the books.

Figure 5a, from Lay's book, outlines a sketch of the proof. It is a short way to build intuition for the main idea that is used in the proof and should be complemented by the proof itself in the text. Because subfigure 5a is not posed as a self-contained proof of the intermediate value theorem, it was coded with *outline of proof*. Van Neerven's book does not include 5a, but includes an illustration that is coded identically for a proposition that was used to prove the intermediate theorem.



(a) Main idea of the proof



(b) A 'proof by picture'

Fig. 5: Illustrations for the proof of the Intermediate Value Theorem in Lay's book [25, p. 228] (left) and Stewart's book [26, p. 126] (right).

Subfigure 5b in Stewart's book, however, aims to provide a stand-alone proof of the intermediate value theorem. Thus, it plays the role of convincing the reader of validity of the particular theorem. No proof in the text is further provided. Hence, the picture functions as a substitute for a proper proof. This practice is commonly referred to as 'proof by picture' in the mathematics community [35]. Hence the illustration is classified as *proof by picture*.

We encounter some more instances of *proof by picture* in Stewart's book, where a statement is not formally proved, but merely made plausible for the reader by giving a picture which paints the idea behind the proof. The potential reason behind this could be that the author has tried to cut back on some rigour to build more intuition. It seems to be difficult to reintroduce the rigour to prove a statements that are less central in the whole procedural based learning process. It is up to the reader to be convinced by this style of proving or look up the proof in an advanced analysis textbook.

Notably, Stewart's textbook will sometimes only detail a figure of the statement itself without even proving the statement in any form. These illustrations were coded with *statement*. There was also an instance in Van Neerven's book to illustrate a theorem, but in that case the statement for that theorem did include a proof in the text.

We even observe that Stewart's book does not include any illustrations coded with *outline of proof* within the selection of topics. The illustrations with that code were only found in the analysis books and illustrations classified as *proof by picture* were only found in Stewart's book. This goes on to show that from an intuition perspective 'proofs by picture' are fully allowed, whereas a figure in a rigorous analysis textbook is only allowed to sketch the idea of the proof but never replace it entirely.

5 Discussion

First, we refer back to the main theme and discuss how the three subthemes from the previous sections are connected under this umbrella theme.

In the subtheme associated with definitions, we saw that lots of 'calculus illustrations' fulfill the role of introducing a concept. This introduction is often done by connecting to notions that are more likely to be familiar to the reader. This shows that illustrations, which fit how the story around a concept is told to the reader in the text, are adapted to serve that purpose. In analysis books, the definition is given almost immediately at the start of a chapter and the illustrations follow this structuring of the books and fit the narrative of rigour.

In the subtheme regarding examples, we observed that analysis textbooks only incorporate illustrations for examples that help us gauge the strength of our theoretical framework. It is the endeavour of the authors to avoid intuitive yet speculative claims but give a more precise and rigorous indication of the possibilities our theoretical machinery. In contrast, the figures in the calculus books were mostly fitted to the framework of example- and procedural-based learning with lots of pictures for examples and, in particular, exercises and confirming examples.

In the subtheme regarding proofs, we noted the calculus books even employ illustrations as standalone proofs. Because a proof applies often to infinitely many cases and an illustration displays one specific instance of those cases, we have not rigorously proved that statement [31]. We have only given intuition how the idea used for that one case can be extrapolated to all cases. The analysis books illustrate the idea applied to one specific instance, but to satisfy the requirements in relation to rigour do provide the full formal proof in text. The comparison between the subfigures in Figure 5 highlighted another important difference: Not only are different illustrations used for distinct purposes in the two kinds of books, but also nearly identical illustrations serve different roles depending on whether their appearance is in a calculus or an analysis textbook.

In general, these three subthemes underline one general theme and distinction, which is of rigour and intuition. This contrast implies for readers of both of these textbooks that they should be aware that illustrations are in their roles tied to the pedagogical aims of the entire books. Illustrations are not merely decorations along the text but are there to help the reader build intuition or visually represent a part of a mathematical exposition without losing much rigour.

An interesting question for future research would be to what extent illustrations in analysis or even higher level analysis courses, such as real analysis or functional analysis, actually build the correct intuition and preserve rigour. A known example that comes to mind are the pictures for finite dimensional spaces that are supposed to be representative for infinite dimensional spaces.[30] This is important to research, because mathematical readers do not merely see illustrations, but they understand based on prior knowledge and experience. Because of this, illustrations can also help one misunderstand a concept.[4]

The valid point can now be made, however, that analysis textbooks also build intuition, by for example, including illustration that were coded with *intuition-building example*. Similarly, calculus textbooks do include many of the formal definition for instance. This research paper does not refute this, but it stresses the tendency of the respective books to focus more on rigour or intuition.

Nevertheless, this query of analysis having perhaps too much focus on rigour is somewhat important to look into. In the introduction, it was argued that calculus and analysis form the gateway to more advanced mathematics. Therefore, the mathematics education can actually be split in three stages. In his blog [32], Terence Tao describes the stages: the *pre-rigour phase* - *rigour phase* - *post rigorous phase*, where in the latter stage more emphasis is put on intuition and techniques, because one is able to very quickly introduce formalisation to make their argument rigorous. With this, Tao emphasises that rigour should not trump conjecturing and building intuition for problems, because that is essential to understanding problems and their underlying structures and meanings. Rigour should be seen as tool to place one's *good* intuition on more firm foundation.[33]

Some limitations of this research are that the coding is interpretive and that inter-rater reliability was not assessed to reduce the influence of personal bias. Also, figures were classified with only one code from the three main code families, but the possibility exists that multiple codes from the same code family suit a single figure due to overlapping roles. Further, Van Neerven's book and Lay's book turned out to be much more similar than anticipated. This meant that instead of looking at the spectrum of intuition and rigour, the research highlighted the divide between calculus and analysis textbooks to a larger extent.

6 Conclusion

The findings suggest that the illustrations in calculus textbooks have the primary role of building intuition and lending themselves well to support example-based learning. The analysis books are much more restricted in their use of illustrations. When they are used, they serve a the purpose of preserving rigour. We also found that with calculus and analysis having different epistemological goals, they not only utilise different illustrations in their respective textbooks, but they also use the same illustrations to serve a different function in the text. The main finding was elucidated by considering definitions, examples and proofs. In calculus textbooks, illustrations help to introduce definitions, reinforce procedural learning and act as self-contained proofs, whereas analysis textbooks included pictures for formal definitions, counterexamples that test the limits of theory and displaying the main idea for a proof.

7 Responsible Research

The corpus of analysis and calculus textbooks consists of two full published books and one reader compiled by a renowned professor. All of the three works were legitimately accessed through the library channels of Delft University of Technology (TU Delft). It must be noted that the illustrations and text used for this research remain intellectual property of their respective authors and they were used for the sole purpose of academic analysis within the research project.

One of the aims of this report is to make information more transparent and reproducible by sharing the approach that was used to analyse and qualitatively classify the data within the corpus. It was also shown how the interpretations follow from the collected coded illustrations. Although the process of data collection and classification is mentioned in the methodology section, full reproducibility is limited due to the absence of the entire codebook in the publication. The most relevant codes to the research question were included. What could have been done, but was not completed due to time constraints, is the process to improve the inter-coder reliability to mitigate the effect of personal biases in the project. Lastly, generative AI tools were used to improve language quality, including grammar correction, phrasing, and readability.

7.1 Positionality Statement

I am a Dutch bachelor's student in Computer Science and Engineering (CSE) at TU Delft with also a background in mathematics. Since this study was conducted under TU Delft as a part of the course Research Project, there was a strong motivation to choose the books that are used for Analysis I as part of the mathematics curriculum of the Bachelor's here. This choice was further enforced by the fact that I have both followed and have been a teaching assistant for the course Analysis I. The Calculus book written by James Stewart, is also used for the Calculus course given to CSE students at TU Delft and was therefore also considered in the comparative study.

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A Appendix: Figures to support theme about examples

Differentiable functions are continuous:

Proposition 4.1.8. *Let I be an interval. If $f : I \to \mathbb{R}$ is differentiable at the point $c \in I$, then f is continuous at c.*

Proof. This follows from

$$\lim_{x \to c} \big(f(x) - f(c)\big) = \lim_{x \to c} \left(\frac{f(x) - f(c)}{x - c} \cdot (x - c) \right) = f'(c) \cdot 0 = 0. \qquad \square$$

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The converse to Proposition 4.1.8 does not hold: there exist continuous functions on $\mathbb{R}$ that are *nowhere differentiable*. The discovery of such functions in the 19th century by the German mathematician KARL WEIERSTRASS (see Figure 4.2, left) came as a shock to the mathematical community. In the first half of the 20th century it was realised, however, that, within the class of continuous functions, nowhere differentiability is the rule rather than the exception. For example, it can be shown that with probability one, the path of a point particle diffusing along the real line (mathematically: the path of a Brownian motion) is continuous but nowhere differentiable (see Figure 4.2, right).

Fig. 6: Figure of text in analysis textbook.

EXAMPLE 5 Find an equation of the tangent line to the parabola $y = x^2 - 8x + 9$ at the point $(3, -6)$.

SOLUTION From Example 4 we know that the derivative of $f(x) = x^2 - 8x + 9$ at the number a is $f'(a) = 2a - 8$. Therefore the slope of the tangent line at $(3, -6)$ is $f'(3) = 2(3) - 8 = -2$. Thus an equation of the tangent line, shown in Figure 7, is

$$y - (-6) = (-2)(x - 3) \qquad \text{or} \qquad y = -2x$$

Fig. 7: Figure of text in calculus textbook.