Distributed power consensus algorithms in DC microgrids

C.W. Zevenbergen

Master of Science Thesis

Delft University of Technology
Delft Center for Systems and Control

Distributed power consensus algorithms in DC microgrids

Master of Science Thesis

For the degree of Master of Science in Systems and Control at Delft University of Technology

C.W. Zevenbergen

June 20, 2019

Faculty of Mechanical, Maritime and Materials Engineering (3mE) · Delft University of Technology

Copyright © Delft Center for Systems and Control (DCSC)
All rights reserved.

Abstract

Energy distribution systems are undergoing rapid change in terms of energy generation techniques and network topology. The shift towards sustainable energy generation leads to distributed energy networks, thereby replacing the old-fashioned centralized networks. A promising type of distributed network is the microgrid. A distributed topology has many benefits, but also introduces many challenges. One such a challenge is to attain a fair distribution of energy generation among the generation units in the network. In the thesis, the goal is to develop protocols which fairly distributed power generation in a DC microgrid. In the process, the theory of nonlinear consensus protocols is generalized to aid the implementation of power consensus algorithms. Furthermore, the implementation of adaptive gain consensus protocols is investigated and new protocols of this type are proposed.

Table of Contents

List of Figures

List of Tables

Preface

The future of energy distribution is an interesting topic due to the shift of energy generation methods from fossil fuels to sustainable energy forms, such as wind and solar. The underlying distribution networks are undergoing change and new control strategies are required to implement sustainable energy generation units in a distributed manner. With my master thesis, I hope to contribute to the theory by investigating one particular part, namely the fair sharing of power generation between the generation units. Specifically, the networks of interest are those of DC microgrids.

I would to thank my thesis supervisor Dr. Sergio Grammatico for his guidance throughout the process. Without his help I would not have been successful in performing relevant research on the topic of power consensus in DC microgrids.

Chapter 1

Introduction

The structure of electrical networks is undergoing rapid change due to the ongoing energy transition. The energy transition is a term used to describe the transformation of energy systems from old-fashioned fossil fuel based energy generation methods to sustainable energy generation. Sustainable energy generation takes advantages of solar power, wind energy and many more phenomena. A key difference in the implementation of the two is the structure of the network that results from the strategy for energy generation. Fossil fuel based methods generally consist of a central unit, such as a coal-fired plant, from which the energy is dissipated throughout a region. One coal-fired plant is capable of delivering huge amounts of energy, and can be located in most geographical areas.

Sustainable energy generation units tend to perform better in specific areas. For example, a solar cell produces energy in areas where there is a lot a of sun, and wind turbines convert more energy in windy regions and require large open spaces. Furthermore, larger numbers of energy generation units are required to meet the demands of our modern society. These factors result in a network topology in which the energy generation units are distributed across the region.

1-1 Microgrid

These distributed electrical networks and their implementation are a popular research topic with a variety of solutions being proposed. One of these is the concept of the microgrid. Microgrids have great potential to become the electrical network of the future. A microgrid provides a limited region with electricity, and is capable of handling distributed generation units.

A microgrid is defined as "a group of interconnected loads and distributed energy resources within clearly defined electrical boundaries that acts as a single controllable entity with respect to the grid. A microgrid can connect and disconnect from the grid to enable it to operate in both grid-connected or islanded-mode" [1]. The main components of a microgrid are loads, generators, storage devices and power converters, which connect the units to the microgrid.

The network may consist of AC and DC loads and generators. The transmission lines may be dominantly resistive, dominantly inductive or a combination of the two. As a result, some devices may need to be connected to the microgrid using power electronic interfaces [2].

The control objectives in microgrids are broad, and challenging to implement simultaneously. The objectives differ depending on the type (AC, DC, resistive, inductive, etc.) of microgrid and whether it is in islanded or grid mode. The individual components must achieve frequency synchronization with the microgrid, as should the microgrid be synchronized with the utility grid. The voltage throughout the microgrid should be uniform and constant. Power generation among the generators should be proportionally shared. Power losses within the system should be minimized. It is required to allow for plug-and-play capability, that is, the desired behaviour across the microgrid must be maintained without readjusting of the control system [3]. Furthermore, it is required that the microgrid functions in such a way that the controllers do not require knowledge of the topology of the entire system, that is, a decentralized or distributed approach is preferred [4].

Microgrids are highly reliable, due to their plug-and-play capabilities and distributed nature. The failing of a single link will rarely lead to the failing of the entire network. The transmission of power can be done with a lower voltage and the transmission lines are shorter than in classic energy networks, resulting in a higher power efficiency across the network. Furthermore, since the implementation of renewable energy sources is made possible, carbon emissions are reduced greatly [5].

1-2 Research objective

The thesis focuses on one particular control objective of a DC microgrid. The main objective is to propose control algorithms that achieve Proportional power consensus (PPC) among the generation units of a DC microgrid, while stabilizing the voltage. As a starting point, a Power consensus algorithm (PCA) from the literature is studied and its properties analyzed to gain insights into the workings of the algorithm. With these insights, the connection of the algorithm to nonlinear consensus theory from the literature is investigated. This investigation leads to the further development of nonlinear consensus theory, such that a bridge from the current theory to the PCA is constructed. In the process, Agreement function (AF)s of Nonlinear consensus protocol (NCP)s are studied and additional types of consensus are proposed.

The PCAs that are proposed in the thesis have the property of implementing adaptive gains. In order to strategically construct these algorithms, a study of the connection between adaptive gain algorithms from the literature and the existing and developed theory on NCPs is executed. The study will result in a variety of general adaptive gain consensus protocols.

The proposed adaptive gain consensus protocols are applied to the problem of PPC in DC microgrids. Throughout the thesis, results will be proved analytically and/or verified my means of numerical simulation.

Finally, the problem of PPC and voltage regulation, instead of just voltage stability, in DC microgrids is considered. Here, the goal is to steer the voltages to some (not necessarily identical) reference values.

1-3 Report organization

The report is organized as follows. In Chapter 2, some theory on NCPs [6, 7], adaptive gain protocols [8, 9] and DC microgrid modelling [10] is introduced. The PCA [11] is also introduced in this chapter. In Chapter 3, a new type of AF, namely the Weighted geometric mean (WGM), is introduced, and alternative types of consensus reaching are introduced. These include proportional consensus and general consensus on some function. In Chapter 4, four nonlinear adaptive gain consensus protocols are proposed and their properties analyzed and verified. In Chapter 5, the connection between the PCA and the nonlinear consensus theory from Chapter 3 is investigated. Then, the adaptive gain protocols from Chapter 4 are applied to achieve the objective of proportional power sharing and voltage stabilization. Furthermore, a result on voltage regulation is given. In Chapter 6, the proposed theory and the developed algorithms are evaluated and discussed. In Chapter 7, conclusions on the research are drawn and future research topics are proposed.

1-4 Summary of literature review

The master thesis began with an extensive review of current microgrid control technology. Although the thesis would focus on power consensus in microgrids, knowledge of other microgrid control objectives was vital for the understanding of the topic as a whole. Strategies to achieve a certain objective may be of interest for power consensus. A short summary of the literature review is given here. Overviews of the current state of microgrid technology are given in [2, 3, 5]. Topics related to power consensus and (reactive) power compensation are considered in [11, 12, 13, 14, 15, 16]. The hierarchical control structure, often applied to microgrids, is discussed in [4, 17, 18]. Frequency control and synchronization algorithms are proposed in [13, 14, 4, 19]. The topic of voltage control is discussed in [20, 15, 16, 21, 22, 19]. NCPs are presented in [6, 7]. Consensus protocols that implement an adaptive gain are proposed in [8, 9]. Finally, microgrid modelling strategies are discussed in [10, 23].

Chapter 2

Mathematical background

A review of the current state of nonlinear consensus theory, adaptive gain consensus algorithms and DC microgrid modelling strategies is performed in this chapter. The results presented in this chapter act as a summary of the relevant topics which are used as a basis for this thesis. Some results are implemented directly in later chapters, while others are expanded upon. Specifically, the DC microgrid modelling strategies from the literature are applied directly, and the theory on Nonlinear consensus protocol (NCP)s and adaptive gain consensus protocols is extended to accommodate the implementation of Power consensus algorithm (PCA)s. Finally, a PCA from the literature is presented.

2-1 DC Microgrid modelling

A microgrid consists of interconnected sources and loads. In the literature, microgrids are generally modelled as a network graph by applying a graph theoretic approach. The results in this section follow from the extensive review on microgrid modelling in [10] and the paper on PCAs [11].

2-1-1 Microgrid and communication network

In the thesis, the considered networks are modelled as connected and undirected graphs $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, which are described by a set of nodes $\mathcal{V}$ and a set of edges $\mathcal{E}$. The cardinality of the set of edges is denoted by $m \in \mathbb{R}$, i.e., the network consists of m edges. The set of nodes is partitioned into a set of sources $\mathcal{V}_s$ and a set loads $\mathcal{V}_l$, such that their union is equal to the set of nodes: $\mathcal{V} = \mathcal{V}_s \cup \mathcal{V}_l$. The set of edges consists of pairs of interconnected nodes (i, j) for $i, j \in \mathcal{V}$. If two nodes i and j are connected, then $(i, j) \in \mathcal{E}$. The sources of the DC microgrid are capable of communicating with each other. Their interconnections are, too, described by a connected and undirected graph. The graph is denoted by $\mathcal{G}_c = (\mathcal{V}_s, \mathcal{E}_c)$, where the definition of $\mathcal{E}_c$ is equivalent to its counterparts in the microgrid network.

Set of neighbouring nodes

Associated to each node i is a set of neighbouring nodes $\mathcal{N}_i$, defined as $\mathcal{N}_i := \{j \in \mathcal{V} : (i,j) \in \mathcal{E}\}$. Equivalently, the set of neighbouring nodes of the communication network are described by $\mathcal{N}_{ci} := \{j \in \mathcal{V}_s : (i,j) \in \mathcal{E}_c\}$. The neighbouring states of some node i are denoted by $x^{(i)}$. To ensure that the controlled system is distributed, each node must communicate only with its neighbouring nodes.

Adjacency and incidence matrices

The interconnections of a connected and undirected graph are described by the symmetric adjacency matrix $\mathcal{A} \in \mathbb{R}^{n \times n}$ with elements $a_{ij} \in \mathcal{A}$, which is defined as follows.

$$a_{ij} = \begin{cases} 1 & \text{if } (i,j) \in \mathcal{E} \\ 0 & \text{otherwise} \end{cases} \tag{2-1}$$

In words, this means that if $a_{ij} = 1$, the nodes i and j are connected. Otherwise, they are not connected. It is important to note that the sources are numbered first. That is, $i < j$ for all $i \in \mathcal{V}_s$ and $j \in \mathcal{V}_l$.

The edges of the graph are numbered and the interconnections of the nodes and edges are described by the incidence matrix $\mathcal{B} \in \mathbb{R}^{n \times m}$ with elements $b_{ij} \in \mathcal{B}$. The edges are numbered such that for each $(i,j) \in \mathcal{E}$ there exists a unique $k \in \{1, \ldots, m\}$. The incidence matrix is defined as

$$b_{ij} = \begin{cases} 1 & \text{if } i \in \mathcal{V}_s \text{ and } (i,j) \in \mathcal{E} \\ -1 & \text{if } i \in \mathcal{V}_l \text{ and } (i,j) \in \mathcal{E} \\ 0 & \text{otherwise} \end{cases} \tag{2-2}$$

The differentiation between sources and loads in the incidence matrix describes direction of the flow of current in the microgrid. A positive value b_{ij} indicates that current flows from node i to node j, and a negative value indicates the opposite.

Laplacian and conductance matrices

Associated to each edge k is a conductance $\gamma_k = 1/r_k$, which are summarized in the matrix of conductances $\Gamma = \text{diag}(\gamma_1, \ldots, \gamma_m)$. Then, relation between the voltage and currents can be described by the weighted Laplacian matrix

$$Y = B\Gamma B^\top \tag{2-3}$$

such that

$$\begin{bmatrix} I_s \\ I_l \end{bmatrix} = \begin{bmatrix} Y_{ss} & Y_{sl} \\ Y_{ls} & Y_{ll} \end{bmatrix} \begin{bmatrix} V_s \\ V_l \end{bmatrix}. \tag{2-4}$$

2-2 Nonlinear consensus

Within multi-agent systems of distributed networks, an objective may be to let the values of some function of the agent's achieve consensus. That is, the value of the function associated to each agent should converge to some value on which all agents agree. A protocol that achieves such an objective is called a consensus protocol. In [6, 7], nonlinear protocols are proposed. In particular, these protocols steer the states of the distributed system to a consensus value. In this section, the protocols are introduced. Furthermore, their properties are discussed and conditions to achieve consensus given. The proposed consensus protocols have the property that the value of some function of the states if preserved. This function is termed the Agreement function (AF), and shall be elaborated next. All results in this section follow from [6, 7].

2-2-1 Agreement function

The AF is a function of the states whose value is to be preserved throughout the evolution of the system. For state consensus protocols, this is the value to which the states converge. Knowledge of the AF results in some insight into the trajectories of the states of the system. An AF $\chi : \mathbb{R}^n \to \mathbb{R}$ is defined as follows. χ is a continuous and differentiable function such that $\chi(x_1, \ldots, x_n) = \chi(x_{\sigma_1}, \ldots, x_{\sigma_n})$ for any one-to-one permutation $\sigma : \mathcal{V} \to \mathcal{V}$. The AF must satisfy the following condition.

$$\forall x \in \mathbb{R}^n : \quad \min_{i\in\mathcal{V}} x_i \leq \chi(x) \leq \max_{i\in\mathcal{V}} x_i \tag{2-5}$$

That is, the value of the AF must be bounded above and below by the maximum and the minimum state value. Examples of an AF are the arithmetic mean, geometric mean, harmonic mean and mean of order p.

The AF can be decomposed into two functions $g, h : \mathbb{R} \to \mathbb{R}$ with $\vartheta'(x_i) \neq 0$ such that $\chi(x) = h(\sum_{i\in\mathcal{V}} g(x_i))$. This decomposition is necessary to allow for the implementation of the AF in the NCP. To illustrate this, let us present a protocol which preserves the value of the AF. The protocol is distributed and consists of a state-dependent 'gain'.

$$\forall i \in \mathcal{V} : \quad \dot{x}_i = (g'(x_i))^{-1} \sum_{j\in\mathcal{N}_i} \phi(x_j, x_i) \tag{2-6}$$

The state-dependent 'gain' is a function of $g(x_i)$ of the decomposed AF. Interestingly, the gain ensures that the value of the AF is preserved, given the correct conditions on $\phi(\cdot)$. If $\phi(x_j, x_i)$ is anti-symmetric, that is $\phi(x_j, x_i) = -\phi(x_i, x_j)$, then the value of the AF is preserved. Note that the s, implying that it is distributed.

2-2-2 State consensus protocol

Given the correct conditions on $\phi(\cdot)$, the time-invariant protocol (Eq. (2-6)) can achieve state consensus. To this end, the difference function $\phi(\cdot)$ is expanded, resulting in the following protocol.

$$\forall i \in \mathcal{V}: \quad \dot{x} = (g'(x_i))^{-1} \sum_{j \in \mathcal{N}_i} \hat{\phi}(\vartheta(x_j) - \vartheta(x_i)) \tag{2-7}$$

Protocol Eq. (2-7) achieves state consensus of all agents if $g(\cdot)$ is strictly increasing, $\hat{\phi}(\cdot)$ is continuous, locally Lipschitz, odd and strictly increasing, $\vartheta(\cdot)$ is differentiable and $\vartheta'(x_i)$ is locally Lipschitz and strictly positive. These conditions will be referred to repeatedly in the thesis and are summarized in Table 2-1 for readability.

Table 2-1: Conditions for consensus

$g(\cdot)$	$\hat{\phi}(\cdot)$	$\vartheta(\cdot)$	$\vartheta'(x_i)$
Strictly increasing	Continuous, locally Lipschitz, odd, strictly increasing	Differentiable	Locally Lipschitz, strictly positive

2-3 Adaptive gain

In [8, 9], a variety of consensus protocols which make use of adaptive gains are proposed. The goal of the adaptive gain is to increase the response of agents whose states are the furthest from the consensus value. Thus, the response time of the normally 'slow' agents is increased, and, as a consequence, the response of the entire system is improved. In [9], adaptive gains are implemented in two ways. The first is done by associating an adaptive gain to each edge of the network. That is, each pair of connected agents has an adaptive gain, which increases as a function of the difference of the states. The second method is to associate a gain to each agent, which increases as a function of the sum of the difference of the local and the neighbouring states. Two edge-based adaptive gain protocols and one node-based adaptive gain protocol are given in this section.

2-3-1 Edge-based adaptive gain

Two edge-based adaptive gain consensus protocols from the literature are presented in this section. The summary of the protocols will serve as a basis for the derivation of the adaptive gain consensus protocols proposed in Chapter 4.

Edge-based protocol - Example 1

The first edge-based protocol is proposed in [8]. It considers the control of a network of agents with linear dynamics.

$$\forall i \in \mathcal{V}: \quad \dot{x}_i = Ax_i + Bu_i \tag{2-8}$$

The input u_i is designed in such a way that the states are driven towards some consensus value. The input is a function of the states and of a gain c. The gain dynamics quadratic and given by a function of the states of the system. The quadratic nature of the gain dynamics ensures that the gain is monotonically increasing.

$$\forall i \in \mathcal{V}: \quad u_i = F \sum_{j\in\mathcal{N}_i} c_{ij}(x_i - x_j) \tag{2-9a}$$

$$\dot{c}_{ij} = (x_i - x_j)^\top \Gamma (x_i - x_j) \tag{2-9b}$$

It immediately becomes evident that the structure of the input (Eq. (2-9a)) resembles the consensus protocol Eq. (2-7). The matrices F and Γ are defined as $F = -B^\top P^{-1}$ and $\Gamma = P^{-1}BB^\top P^{-1}$, where $P > 0$ is a solution to the linear matrix inequality

$$AP + PA^\top - 2BB^\top < 0. \tag{2-10}$$

In [8, Theorem 1], it is proved that the protocol Eq. (2-9) achieves state consensus.

2-3-2 Edge-base protocol - Example 2

The following edge-based protocol from [9] introduces a auxiliary state v to the system. The input u_i becomes a function of the auxiliary state v_i. The agent dynamics are equivalent to Eq. (2-8), with the addition of a virtual output $y_i = Cx_i$. The gain dynamics are again given by a quadratic function to ensure a monotonic increase in gain. However, now they are a function of both the virtual output and the auxiliary state.

$$\forall i \in \mathcal{V}: \quad u_i = Fv_i \tag{2-11a}$$

$$\dot{v}_i = (A + BF)v_i + L \sum_{j\in\mathcal{N}_i} c_{ij}[C(v_i - v_j) - (y_i - y_j)] \tag{2-11b}$$

$$\dot{c}_{ij} = a_{ij} \begin{bmatrix} y_i - y_j \\ C(v_i - v_j) \end{bmatrix}^\top \Gamma \begin{bmatrix} y_i - y_j \\ C(v_i - v_j) \end{bmatrix} \tag{2-11c}$$

In [9, Theorem 3], it is stated and proven that consensus is reached under this protocol for a connected and undirected graph, assuming that $A + BF$ is Hurwitz and

$$\Gamma = \begin{bmatrix} I_q & -I_q \\ -I_q & I_q \end{bmatrix} \tag{2-12}$$

$$L = -Q^{-1}C^\top \tag{2-13}$$

where $Q > 0$ is a solution to the linear matrix inequality

$$A^\top Q + QA - 2C^\top C < 0. \tag{2-14}$$

2-3-3 Node-based adaptive gain

In [9], an adaptive gain consensus protocol is proposed. The protocol is similar to Eq. (2-11) and differs only in the implementation of the adaptive gains. The protocol implements a node-based gain. This results in a minor change in the dynamics of the auxiliary state v and a considerable change in the gain dynamics d. As mentioned earlier, a node-based gain is a function of the states of all neighbouring gains, by summing the difference of the neighbouring and local states.

$$\forall i \in \mathcal{V}: \quad \dot{x}_i = Fv_i \tag{2-15a}$$

$$\dot{v}_i = (A + BF)v_i + d_i L \sum_{j \in \mathcal{N}_i} [C(v_i - v_j) - (y_i - y_j)] \tag{2-15b}$$

$$\dot{d}_i = \tau_i \sum_{j \in \mathcal{N}_i} \begin{bmatrix} y_i - y_j \\ C(v_i - v_j) \end{bmatrix}^\top \Gamma \sum_{j \in \mathcal{N}_i} \begin{bmatrix} y_i - y_j \\ C(v_i - v_j) \end{bmatrix} \tag{2-15c}$$

The matrices F, L and Γ are constructed equivalently to those in the previous section. Then, in [9, Theorem 3], it is proved that consensus is reached under this protocol for a connected and undirected graph.

2-4 Power consensus

The PCAs that are proposed in this paper are based on one from the literature [11]. The objective of the PCA is to proportionally share the source power whilst stabilizing the voltages. The state of the system is the voltage V, whose dynamics are a function of the local voltage and the local and neighbouring power. Power measurements are made at each node, whose values are then shared among the neighbouring source nodes. These dependencies imply that the algorithm is distributed.

$$\forall i \in \mathcal{V}: \quad \dot{V}_i = C_i^{-1} V_i \sum_{j \in \mathcal{N}_{ci}} (C_j^{-1} P_j - C_i^{-1} P_i) \tag{2-16}$$

The term C_i represents the power sharing coefficient at node i. It is this value that determines how the power should be shared. The reader is invited to compare the structure of the PCA to that of the NCP (Eq. (2-7)) to notice the similarity. Of course, there are some differences, which shall be discussed in later chapters. Finally, it is important to mention that the PCA preserves the Weighted geometric mean (WGM) of the voltages. It is this observation that prompted the authors of [11] to suggest an investigation of the connection between the PCA and the NCPs form the literature.

Chapter 3

Nonlinear consensus

The goal of this chapter is to build a bridge between the theory on Nonlinear consensus protocol (NCP)s [6, 7] and the Power consensus algorithm (PCA) [11]. By comparing the NCP (Eq. (2-7)) and the PCA (Eq. (2-16)), a slight resemblance can be immediately observed. The similarities and differences between the two are discussed in this chapter. To this end, some new concepts regarding Agreement function (AF)s and consensus protocols are introduced.

3-1 Weighted geometric mean preservation

The PCA has been shown to preserve the Weighted geometric mean (WGM) of the voltages. However, in [6], the WGM is not proposed as a valid AF. In this section, the WGM is analyzed to determine its validity as an AF. The WGM $\bar{x}^{\mathrm{w}}$ is defined as

$$\bar{x}^{\mathrm{w}} = (\prod_{i\in\mathcal{V}} x_i^{w_i})^{(\sum_{i\in\mathcal{V}} w_i)^{-1}} \tag{3-1}$$

where $x_i, w_i > 0$ for all $i \in \mathcal{V}$. If $\bar{x}^{\mathrm{w}}$ can be decomposed as described in Section 2-2-1, then it can be implemented as an AF for the time invariant protocol Eq. (2-6). That is, we require that $\bar{x}^{\mathrm{w}} = h(\sum_{i\in\mathcal{V}} g_i(x_i))$ for some $g_i, h : \mathbb{R} \to \mathbb{R}$. It is found that this is achieved by letting

$$h(y) = e^{(\sum_{i\in\mathcal{V}} w_i)^{-1} y} \tag{3-2a}$$
$$g_i(x_i) = \log x_i^{w_i}. \tag{3-2b}$$

By implementing Eq. (3-2b) in the time-invariant protocol Eq. (2-6), a protocol that preserves the WGM is constructed. The derivative $g_i'(x_i) = w_i x_i^{-1}$ is substituted to obtain

$$\forall i \in \mathcal{V}: \quad \dot{x}_i = w_i^{-1} x_i \sum_{j \in \mathcal{N}_i} \phi(x_j, x_i). \tag{3-3a}$$

The dynamics is a function of the local and neighbouring states and consists of a state-dependent 'gain' and the sum of some anti-symmetric function. Time-invariance of the WGM is proved in [6, theorem 1] by considering Eq. (3-2).

3-2 Generalized consensus

The theory presented in [6, 7] considers the problem of state consensus. This concept is generalized to consider a wider variety of consensus problems. Specifically, conditions to ensure consensus on some function $f(\cdot)$ are investigated. To this end, the following protocol is proposed.

$$\forall i \in \mathcal{V}: \quad \dot{x}_i = (g_i'(x_i))^{-1} \sum_{j \in \mathcal{N}_i} \hat{\phi}(\vartheta(f(x_j, x^{(j)})) - \vartheta(f(x_i, x^{(i)}))) \tag{3-4}$$

The conditions summarized in Table 2-1 are valid. Recall that the term $x^{(i)}$ represents the set of neighbouring states of node i, implying that $f(\cdot)$ is a function of not only the local, but also the neighbouring states. The attentive reader will recognize a problem with this. The protocol is no longer distributed since the dynamics of some node i are now a function of the states of the neighbours of neighbouring nodes.

Remark 1. *The presence of $f(x_j, x^{(j)})$ for $j \in \mathcal{V}$ implies that knowledge of not only the neighbouring states are required, but also those of the neighbours of the neighbouring agents. Thus, the system is no longer distributed. This issue can be resolved by measuring the function $f(x_j, x^{(j)})$ at each agent and sharing these measurements among neighbours.*

A quick analysis of the equilibria shows that consensus on $f(\cdot)$ is achieved at equilibrium. Equilibria are found where $\dot{x}_i = 0$ for all $i \in \mathcal{V}$. Because $(g_i'(x_i))^{-1} \neq 0$, it follows that

$$\sum_{j \in \mathcal{N}_i} \hat{\phi}(\vartheta(f(x_j, x^{(j)})) - \vartheta(f(x_i, x^{(i)}))) = 0 \qquad \forall i \in \mathcal{V} \tag{3-5a}$$

$$\Rightarrow \hat{\phi}(\vartheta(f(x_j, x^{(j)})) - \vartheta(f(x_i, x^{(i)}))) = 0 \tag{3-5b}$$

$$\Rightarrow \vartheta(f(x_j, x^{(j)})) - \vartheta(f(x_i, x^{(i)})) = 0 \tag{3-5c}$$

$$\Rightarrow f(x_j, x^{(j)}) - f(x_i, x^{(i)}) = 0 \qquad \forall i, j \in \mathcal{V}. \tag{3-5d}$$

Eq. (3-5b) follows from the connected and undirected properties of the network. Eq. (3-5c) and Eq. (3-5d) follow from the strictly increasing properties of $\hat{\phi}(\cdot)$ and $\vartheta(\cdot)$.

Finding conditions on $f(\cdot)$ that guarantee asymptotic stability of the equilibria is a challenging task and is left as an open problem. Nonetheless, some properties are investigated in the following section.

3-2-1 Convergence properties

Convergence properties of the general consensus protocol Eq. (3-4) are investigated in this section. The reader is warned that these results are empirical, and no necessary and sufficient conditions for general consensus are found in this thesis. The results may, however, serve as a basis for future research on the topic. Some functions $f(\cdot)$ with relevant properties are simulated in the simple network described in Appendix C-1.

The AF is chosen such that the arithmetic mean of the states is preserved, i.e., $g_i(x_i) = x_i$ and, subsequently, $g_i'(x_i) = 1$. Furthermore, $g(\ \vartheta(x) = x$ and $\hat{\phi}(x) = x$. We begin by looking at the function $f(x) = -x$. Intuitively, this should make the functions diverge. Figure 3-1 shows that this is indeed the case.

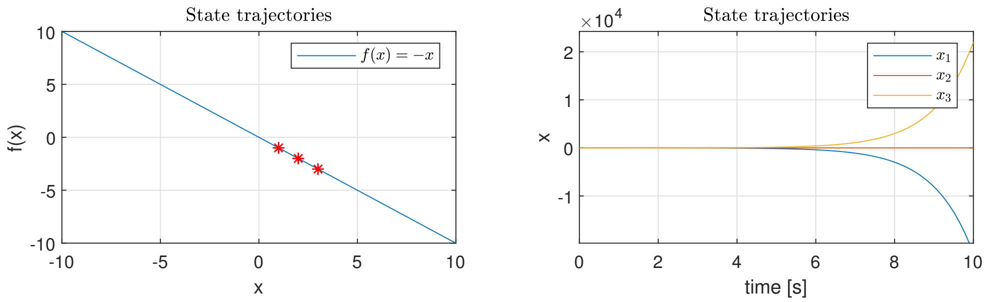

Figure 3-1: Monotonic consensus function - divergence

The case where $f(x) = x$ is simulated next, with initial conditions $x(0) = [1, 2, 3]$. As expected, the states converge to the arithmetic mean (see Figure 3-2). This specific case meets all conditions for state convergence as stated in [6] and summarized in Table 2-1. Next, we look at three functions which do not satisfy these conditions. Specifically, functions where the derivative $d\vartheta(f(x_i))/dx_i$ is not strictly positive are considered.

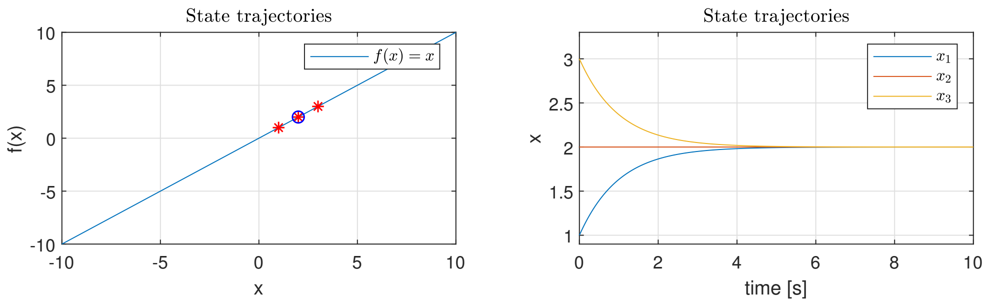

Figure 3-2: Monotonic consensus function - convergence

Consider the function $f(x)$ in Figure 3-3, with initial conditions $x(0) = [1, 2, 3]$. The function is similar to $f(x) = -x$ for $x \in \{x : |x| \leq 2.5\}$. Nonetheless, the states to converge to finite values. This is due to the increase of $f(x)$ for $x > 3$. Since the arithmetic mean of the states is preserved, two states evolve to the right while the other evolves to the left. Then, a point where consensus on $f(x)$ is reached and where the arithmetic mean is preserved is found.

Next, the function $f(x)$ is flipped. With initial conditions $x(0) = [1, 2, 3]$, the states converge (see Figure 3-4). It is interesting that the states converge for this case and for the previous

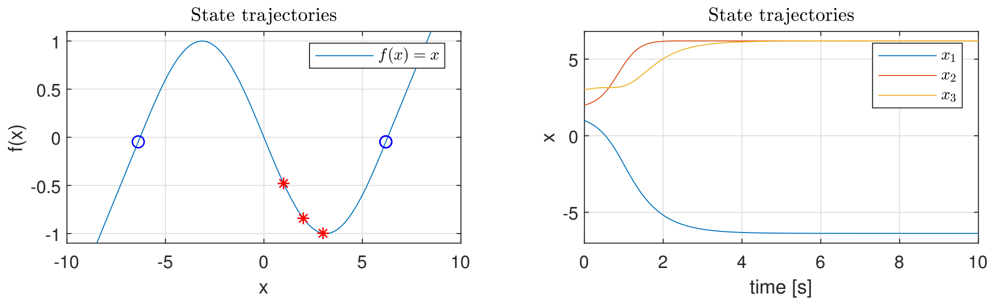

Figure 3-3: Non-monotonic consensus function - convergence

case, since the functions have the opposite increasing/decreasing properties due to them being flipped.

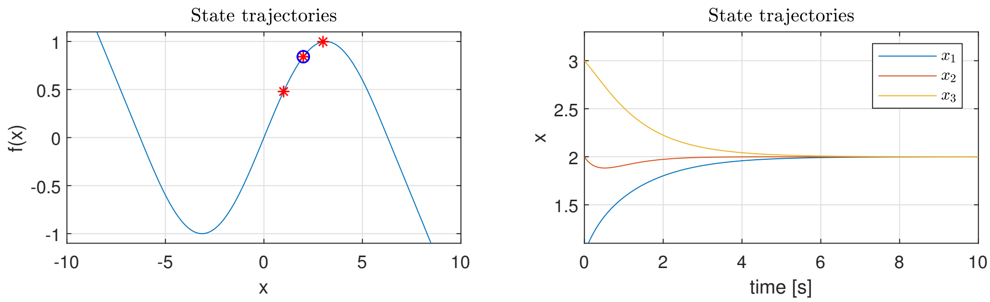

Figure 3-4: Non-monotonic consensus function - local convergence

The same function $f(x)$ is used, but now the initial conditions are $x(0) = [1, 2, 5]$. It is interesting to observe that the states now do not converge, implying that the function is not globally asymptotically stable.

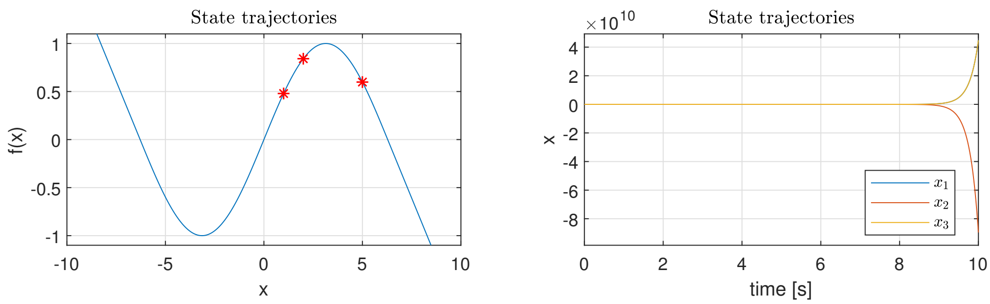

Figure 3-5: Initial conditions - divergence

These insights lead us to the final two related scenarios. In Figure 3-6 and the states are initialized at $x(0) = \begin{bmatrix} 0.3 & 0 & 2\pi - 0.3 \end{bmatrix}$, which has an equilibrium at $x(0) = \begin{bmatrix} 0 & 0 & 2\pi & -0 \end{bmatrix}$. The states converge to the equilibrium. Figure 3-7 shows the case where the states are initialized at $x(0) = \begin{bmatrix} -0.3 & 0 & 2\pi + 0.3 \end{bmatrix}$. The arithmetic mean remains the same, and so does the equilibrium point. However, the states do not converge to the equilibrium.

The following conclusions are drawn based on the simulations above. First of all, it seems that the system is globally asymptotically stable if the function is strictly increasing, excluding

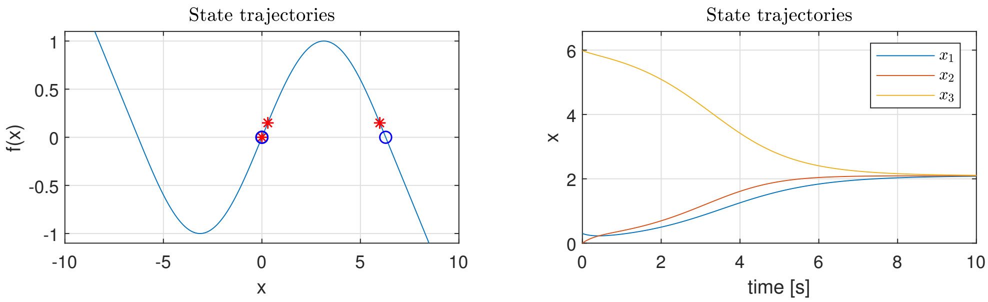

Figure 3-6: Initial conditions - convergence

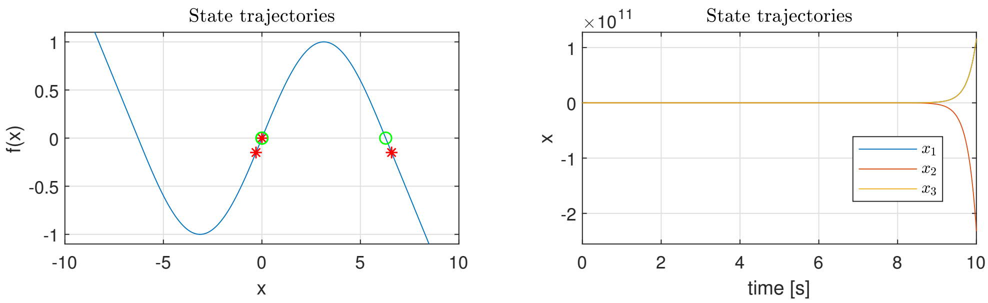

Figure 3-7: Initial conditions - divergence

some connected region. For example, in Figure 3-3, we see that $f(x)$ is strictly increasing outside the region $\{x : |x| \leq \pi\}$.

Furthermore, it appears that local asymptotic stability can be established if the function is not strictly decreasing over the entire domain, which becomes evident from Figure 3-5, Figure 3-6 and Figure 3-7.

3-3 Proportional consensus

In some applications, it may be required that consensus is reached on some proportional values. Let us define the notion of proportional consensus. We say that proportional consensus is reached when

$$w_i^{-1} f(x_i, x^{(i)}) = w_j^{-1} f(x_j, x^{(j)}) \qquad \forall i, j \in \mathcal{V}. \tag{3-6}$$

A protocol that achieves proportional consensus is proposed next. The protocol extends the NCP Eq. (2-7) to the more general case with $f(x_i, x^{(i)})$ and its proportional counterparts.

$$\forall i \in \mathcal{V}: \quad \dot{x}_i = (g_i'(x_i))^{-1} \sum_{j \in \mathcal{N}_i} \hat{\phi}(\vartheta(w_j^{-1} f(x_j, x^{(j)})) - \vartheta(w_i^{-1} f(x_i, x^{(i)}))) \tag{3-7a}$$

The protocol has the following components. The decomposed AF $g(\cdot)$ which ensures preservation of the AF given the right conditions. The consensus function $f(\cdot)$, which determines the value at which consensus to be reached. Two functions $\hat{\phi}(\cdot)$ and $\vartheta(\cdot)$, which may be chosen freely as long as they satisfy the conditions of Table 2-1.

An investigation of the equilibria, similar to that of Section 3-2, leads to the conclusion that $w_i^{-1} f(x_i, x^{(i)}) = w_j^{-1} f x_j, x^{(j)})$ at equilibrium. Asymptotic stability of the equilibria is a standing assumption which is verified by means of simulation in Section 3-4.

3-3-1 Proportional state consensus

The special case of proportional state consensus is further investigated. Proportional state consensus is achieved when

$$w_i^{-1} x_i = w_j^{-1} x_j \qquad \forall i, j \in \mathcal{V}. \tag{3-8}$$

This is equivalent to $z_i = z_j$ for all $i, j \in \mathcal{V}$, where z follows from a linear transformation of the states:

$$z = \mathcal{W}^{-1} x. \tag{3-9}$$

The matrix $\mathcal{W} = \operatorname{diag}(w_1, \ldots, w_n)$ denotes the diagonal matrix of weights. It now becomes clear that proportional consensus is reached when the transformed states reach state consensus. Thus, the proposed proportional consensus algorithm in constructed as

$$\dot{x}_i = (g_i'(x_i))^{-1} \sum_{j \in \mathcal{N}_i} \hat{\phi}(\vartheta(z_j) - \vartheta(z_i)) \quad \forall i \in \mathcal{V}. \tag{3-10a}$$

The conditions in Table 2-1 are imposed on the protocol to guarantee proportional state consensus. This result is summarized in the following theorem.

Theorem 1 (Proportional state consensus)**.** *Assume that $g(\cdot)$ is strictly increasing, $g_i'(cx_i) = cg_i'(x_i)$ for some $c \in \mathbb{R}$, $\hat{\phi}(\cdot)$ is continuous, locally Lipschitz, odd and strictly increasing, $\vartheta(\cdot)$ is differentiable and $\vartheta'(x_i)$ is locally Lipschitz and strictly positive. Then, a network of agents under protocol Eq. (3-10) achieves proportional state consensus according to weighted $w_i > 0$ for $i \in \mathcal{V}$.*

Proof. See Appendix A-1 □

3-4 Numerical simulations

Numerical simulations are performed to verify the properties of the NCPs proposed in this chapter. The properties of interest are (proportional) state consensus and (proportional) general consensus on some function $f(\cdot)$. The simulations are performed in the simple network described in Appendix C-1.

The structure and conditions of the protocols leave the designer with unlimited possibilities for the functions $\hat{\phi}(\cdot)$ and $\vartheta(\cdot)$. In this section, three cases are considered that satisfy the conditions in Table 2-1. The first case, referred to as case 1, represents the standard case where $\hat{\phi}(x) = x$ and $\vartheta(x) = x$. To investigate the effect of the function $\hat{\phi}(\cdot)$, the second case (case 2) implements the functions $\hat{\phi}(x) = x + x^3$ and $\vartheta(x) = x$, where the third case (case 3), implements $\hat{\phi}(x) = x$ and $\vartheta(x) = x + x^3$. Intuitively, case 2 and case 3 should increase the response time, since the value of the state dynamics $\dot{x}_i$ become larger. The three cases are summarized in Table C-1 in Appendix C, and will be considered for simulations in later chapters.

3-4-1 State consensus

To compare the responses of the three cases, each is implemented in the state consensus protocol (Eq. (2-7)). To verify that the AF is preserved and that the WGM is implemented appropriately, an additional objective is to preserve the WGM. To achieve this, the decomposed WGM Eq. (3-2b) is implemented accordingly, and its evolution plotted. The results are shown in Figure 3-8 (case 1), Figure 3-9 (case 2) and Figure 3-10 (case 3).

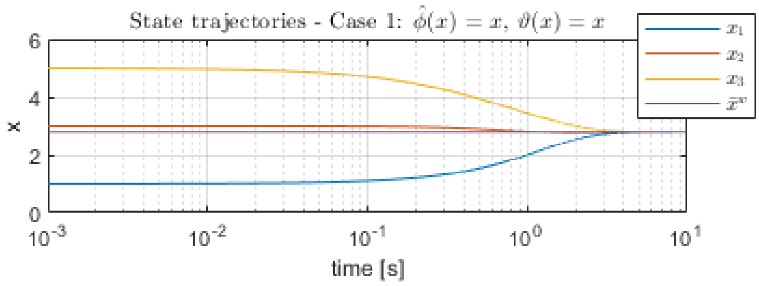

Figure 3-8: State trajectories - state consensus (case 1)

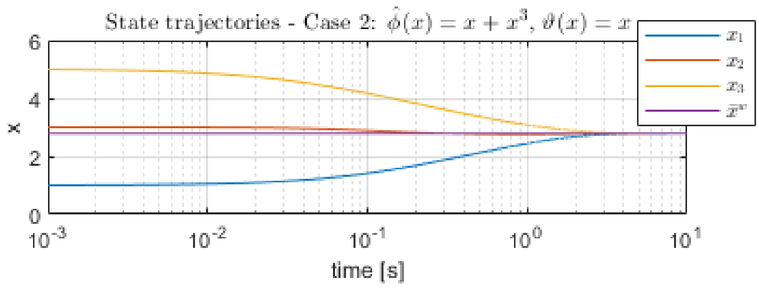

Figure 3-9: State trajectories - state consensus (case 2)

Observe that the WGM is constant in each case. Also, the states reach consensus on the consensus value, the WGM. An improvement in response time with respect to the linear case, case 1, is observed in both cases. The states in case 1 converge in about $4s$, in case 2 in about $3s$ and in case 3 in roughly $0.2s$.

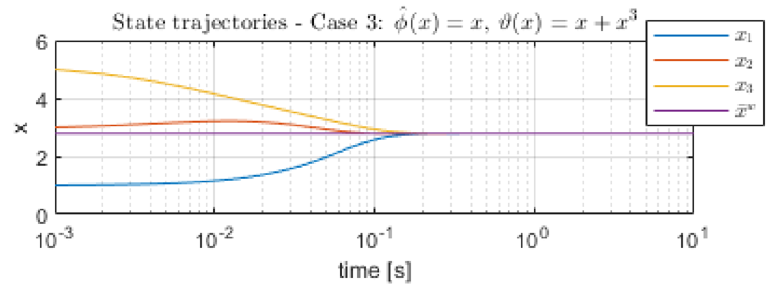

Figure 3-10: State trajectories - state consensus (case 3)

3-4-2 General consensus

To verify that the general consensus protocol (Eq. (3-4)) achieves its objective of reaching general consensus on some function, we let $f(x) = \sin(x)$. Recall that state consensus is no longer guaranteed because the condition that $d\vartheta(f(x))/dx$ is strictly increasing, is no longer satisfied. The trajectories of $f(x) = \sin(x)$ are plotted in Figure 3-11 and the state trajectories are plotted in Figure 3-12. Consensus on $f(x) = \sin(x)$ is reached and the states converge to finite values. Note that these values differ, i.e. state consensus is not reached. Figure 3-12 also shows that the WGM is preserved.

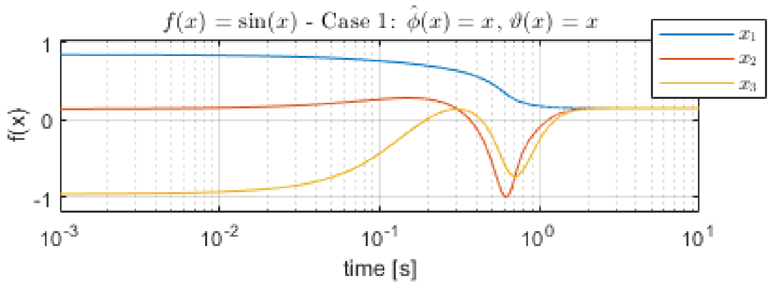

Figure 3-11: Trajectories of $\sin(x)$ - general consensus

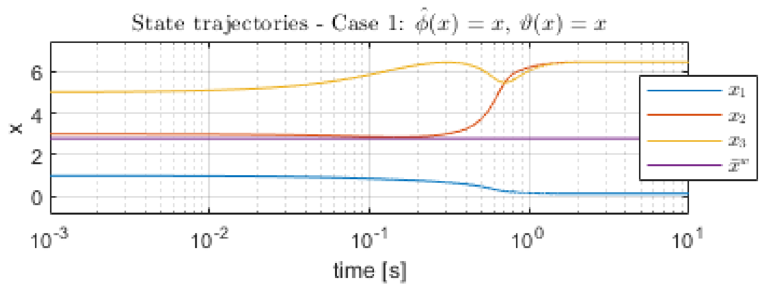

Figure 3-12: State trajectories - general consensus

3-4-3 Proportional consensus

We begin with a numerical simulation of the proportional state consensus protocol for case 1 to verify that proportional consensus on some function $f(\cdot)$ is reached and the AF, the WGM, is preserved. We let $f(x) = \sin(x)$, which is implemented in Eq. (3-7). Figure 3-13 depicts the proportional values $w^{-1}\sin(x)$, which evidently reach consensus. Figure 3-14 shows that the states converge to finite values. Again, state consensus is not reached, which is to be expected. The WGM is preserved.

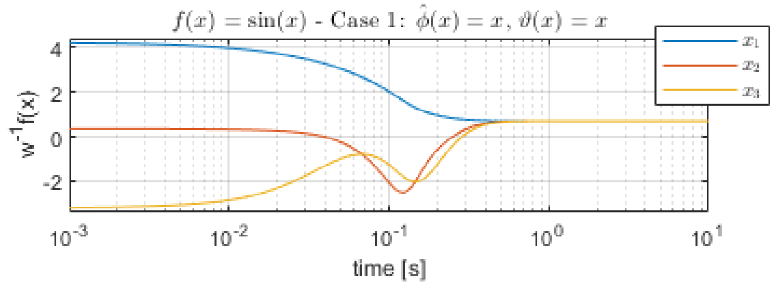

Figure 3-13: Trajectories of $w^{-1}\sin(x)$ - general proportional consensus

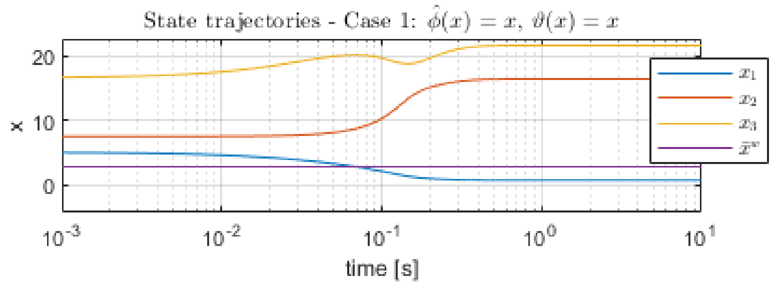

Figure 3-14: State trajectories - general proportional consensus

Chapter 4

Adaptive gain consensus

In this chapter, the theory on Nonlinear consensus protocol (NCP)s is extended to the case with adaptive gains. Four protocols are proposed, two of which implement an edge-based gain and two implement a node-based gain. Before the protocols are introduced, the notions of node-based and edge-based gains are elaborated. An edge-based gain is assigned to each edge of the network. The gain increases whilst the two nodes, connected by the edge, have not reached consensus. The gains converge to a finite value, which is reached once consensus between the two nodes is reached. The node-based case assigns a gain to each node of the network. Now, the gain increases as long as the node and all its neighbours have not yet reached consensus. Once consensus is reached, the gain will have simultaneously reached steady-state. The adaptive gain dynamics are quadratic to ensure the gains are non-decreasing.

4-1 Edge-based protocol I

The first nonlinear adaptive gain consensus protocol is inspired by the edge-based adaptive protocol from [8]. For readability, the protocol is repeated here.

$$\forall i \in \mathcal{V}: \quad \dot{x}_i = Ax_i + Bu_i \tag{4-1a}$$

$$u_i = F \sum_{j \in \mathcal{N}_i} c_{ij}(x_i - x_j) \tag{4-1b}$$

$$\dot{c}_{ij} = (x_i - x_j)^\top \Gamma (x_i - x_j), \quad \forall i \in \mathcal{V} \tag{4-1c}$$

In the following section, the derivation of the new edge-based consensus protocol is given. The implementation of the agent dynamics is discussed, and alternative difference functions for reaching state or generalized consensus are proposed. Furthermore, the consensus protocol is constructed in such a way that any Agreement function (AF) can be preserved.

4-1-1 Derivation

First of all, the agent dynamics should be determined solely by the input to the input u_i. To this end, the matrices of Eq. (4-1a) are set to $A = 0$ and $B = 1$. Recall from Section 2-3 that the matrices F and Γ follow from the solution $P > 0$ of the linear matrix inequality Eq. (2-10). Upon inspection of the inequality, it becomes clear that the designer is free to choose the matrix P, and hence F, freely, since the state matrix $A = 0$. The matrix F is utilized to make sure that the value of the AF is preserved. By observing the structure of Eq. (2-6), a logical choice of F is the following.

$$F = (g_i'(x_i))^{-1}. \tag{4-2}$$

The freedom in choosing F also goes for the matrix Γ, by following the same reasoning of the absence of A in the linear matrix inequality. For now, Γ remains in the protocol to serve as a tuning parameter. Finally, the difference functions are generalized to include nonlinear functions. A structure similar to the NCP (Eq. (2-7)) is implemented in the state dynamics. This same structure is then adopted in the gain dynamics.

$$\forall i \in \mathcal{V}: \quad \dot{x}_i = (g_i'(x_i))^{-1} \sum_{j \in \mathcal{N}_i} c_{ij} \hat{\phi}(\vartheta(x_j) - \vartheta(x_i)), \tag{4-3a}$$

$$\dot{c}_{ij} = \hat{\phi}(\vartheta(x_j) - \vartheta(x_i))^\top \Gamma \hat{\phi}(\vartheta(x_j) - \vartheta(x_i)) \tag{4-3b}$$

In the next sections, the properties of time-invariance and consensus reaching of Edge-based protocol I (EB1) are investigated.

4-1-2 Agreement function

EB1 is constructed such that the dynamics of the agent's states are similar to the NCP of Section 2-2-1. The only difference is that the state dynamics contain an adaptive gain term. The similarity in structure implies that the time-invariant AF property may also be valid for EB1. In this section, an investigation on the inclusion of the adaptive gain and it's effect on the preservation of the AF of EB1 is performed. First, a result on the anti-symmetry of the function $c_{ij}\hat{\phi}(\vartheta(x_j) - \vartheta(x_i))$ is proposed.

Lemma 1. *Assume that $\hat{\phi}(\vartheta(x_j) - \vartheta(x_i))$ is odd and $c_{ij}(t) = c_{ji}(t)$ for all $i, j \in \mathcal{V}$ and all $t > 0$. Then, the function $c_{ij}\hat{\phi}(\vartheta(x_j) - \vartheta(x_i))$ is anti-symmetric. Furthermore, $c_{ij}(t) = c_{ji}(t)$ for all $t > 0$ if $\dot{c}_{ij}$ is defined as in* (4-3) *and $c_{ij}(0) = c_{ji}(0)$ for all $i, j \in \mathcal{V}$.*

Proof. See Appendix A-2 □

By setting $\phi(x_j, x_i) = c_{ij}\hat{\phi}(\vartheta(x_j) - \vartheta(x_i))$, which has been shown to be anti-symmetric in the preceding Lemma, in Eq. (2-6), it follows from [6, Theorem 1] that the AF of EB1 is time-invariant.

Corollary 1. *(Time-invariance of EB1). Assume that $g_i(\cdot)$ is strictly increasing, $\hat{\phi}(\cdot)$ is continuous, locally Lipschitz, odd and strictly increasing, $\vartheta(\cdot)$ is differentiable and $\vartheta'(x_i)$ is locally Lipschitz and strictly positive. From [6, Theorem 1] and Lemma 1, it follows that EB1 ((4-3)) preserves the value of the AF $\chi = h(\sum_{i=1}^{n} g_i(x_i))$.*

4-1-3 State consensus

Under the right conditions, EB1 can steer the agents to a consensus value. The consensus value is determined by the AF. The conditions of Table 2-1 must be be satisfied and the adaptive gains must be initialized such that $c_{ij}(0) = c_{ji}(0)$ for all $i, j \in \mathcal{V}$.

Theorem 2. *Assume that $g_i(\cdot)$ is strictly increasing, $\hat{\phi}(\cdot)$ is continuous, locally Lipschitz, odd and strictly increasing, $\vartheta(\cdot)$ is differentiable and $\vartheta'(x_i)$ is locally Lipschitz and strictly positive. Then, a network of agents under protocol Eq. (4-3) achieves state consensus if the adaptive gains are initialized such that $c_{ij}(0) > 0$ for all $i, j \in \mathcal{V}$.*

Proof. See Appendix A-3 □

The adaptive gains are given by a function of the difference of the states. Once state consensus is reached, the gain dynamics equals zero. Intuitively, the states converge to finite values. Proving this fact, however, is challenging. Implementing a standard Lyapunov approach resulted in restrictive conditions on the functions $\hat{\phi}(\cdot)$ and $\vartheta(\cdot)$. These restrictions would impose the following constraint:

$$\forall i \in \mathcal{V}: \quad g_i(x_i) = \hat{\phi}(\vartheta(x_i)). \tag{4-4}$$

This restriction limits the use of the protocol considerably, which is undesirable. Numerical simulations show that this restriction is not necessary. Therefore, a standing assumption is that the adaptive gains converge to finite values.

The results are verified numerically. The protocol is implemented in the simple network described in Appendix C-1 with $\hat{\phi}(x) = x$, $\vartheta(x) = x + x^3$ and $g_i(x_i) = \log(x_i^{w_i})$. First, the objective is to reach state consensus, and the state trajectories are depicted in Figure 4-1. Then, the proportional state consensus protocol is implemented according to Theorem 1. These results are depicted in Figure 4-2. Note that the AF is set as the Weighted geometric mean (WGM) $\bar{x}^{\mathrm{w}}$. The evolution of $\bar{x}^{\mathrm{w}}$ is also shown in the figures.

The figures verify that the objectives of (proportional) state consensus are achieved under EB1, and that the value of the AF is preserved. Figure 4-3 shows that the adaptive gains c converge to finite values.

4-1-4 Generalized consensus

The generalized consensus protocol from Section 3-2 is implemented in EB1, such that consensus can be reached on some function $f(\cdot)$ of the states. This results in the following protocol.

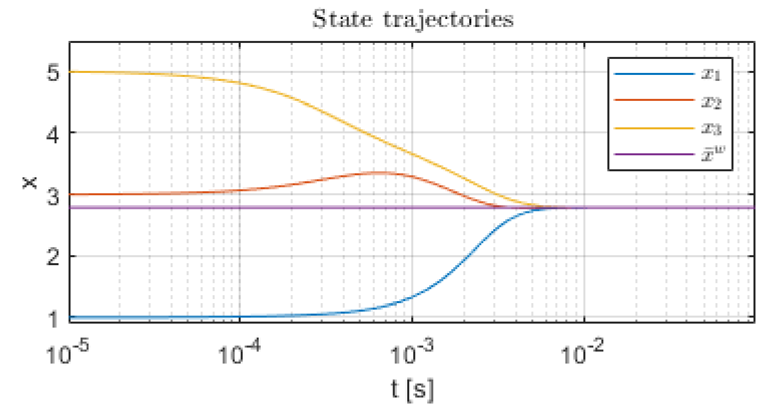

Figure 4-1: Edge-based I - State consensus

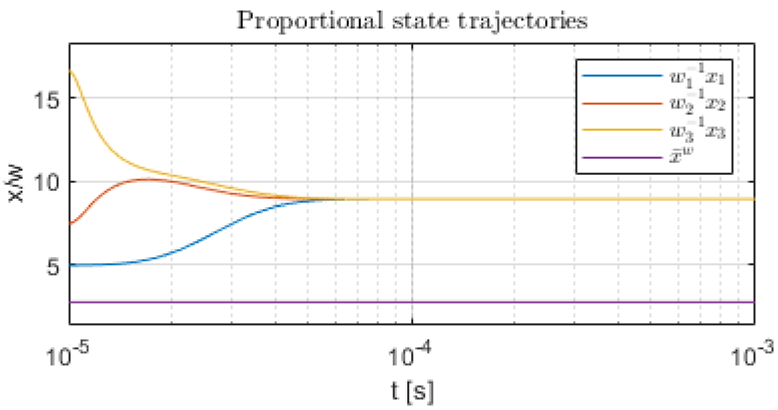

Figure 4-2: Edge-based I - Proportional state consensus

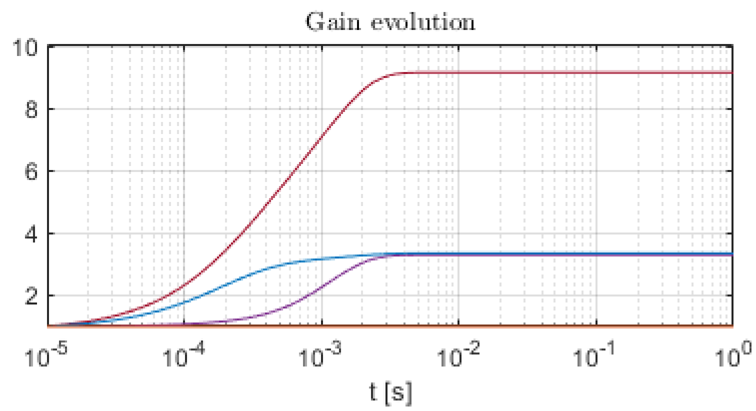

Figure 4-3: Edge-based I - Gain evolution

$$\forall i \in \mathcal{V}: \quad \dot{x}_i = (g_i'(x_i))^{-1} \sum_{j \in \mathcal{N}_i} c_{ij} \hat{\phi}(\vartheta(f(x_j, x^{(j)})) - \vartheta(f(x_i, x^{(i)}))) \tag{4-5a}$$

$$\dot{c}_{ij} = \hat{\phi}(\vartheta(f(x_j, x^{(j)})) - \vartheta(f(x_i, x^{(i)})))^\top \Gamma \hat{\phi}(\vartheta(f(x_j, x^{(j)})) - \vartheta(f(x_i, x^{(i)}))) \tag{4-5b}$$

Determining conditions on $f(\cdot)$ which guarantee consensus is currently an open problem. Some results are given in Section 3-2. Nonetheless, an investigation of the equilibrium points is performed.

Equilibria

Equilibria are found where the dynamics equals zero. For Eq. (4-5a), the following can be concluded.

$$\dot{x}_i = (g_i'(x_i))^{-1} \sum_{j \in \mathcal{N}_i} c_{ij} \hat{\phi}(\vartheta(f(x_j, x^{(j)})) - \vartheta(f(x_i, x^{(i)}))) = 0 \qquad \forall i \in \mathcal{V} \tag{4-6a}$$

$$\Rightarrow \hat{\phi}(\vartheta(f(x_j, x^{(j)})) - \vartheta(f(x_i, x^{(i)}))) = 0 \tag{4-6b}$$

$$\Rightarrow \vartheta(f(x_j, x^{(j)})) - \vartheta(f(x_i, x^{(i)})) = 0 \tag{4-6c}$$

$$\Rightarrow f(x_j, x^{(j)}) - f(x_i, x^{(i)}) = 0 \qquad \forall i, j \in \mathcal{V} \tag{4-6d}$$

Step Eq. (4-6a) to Eq. (4-6b) follows from the undirected and connected properties of the network. The two following steps are a result of the strictly increasing property of $\hat{\phi}(\cdot)$ and $\vartheta(\cdot)$. Eq. (4-6d) shows that consensus on $f(\cdot)$ is reached at equilibrium.

The claim that (proportional) consensus is reached is verified by means of numerical simulation. The system is simulated with $f(x) = \sin(x)$, and the results are given in Figure 4-4 for the generalized consensus case and in Figure 4-5. Clearly, (proportional) generalized consensus on $f(\cdot)$ is reached.

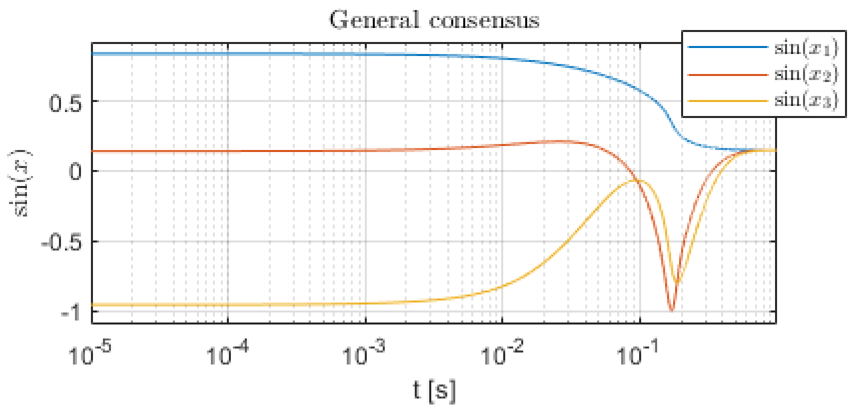

Figure 4-4: Edge-based I - generalized consensus

4-2 Edge-based protocol II

Inspired by the second edge-based protocol presented in Section 2-3, an alternative nonlinear edge-based adaptive gain consensus protocol is proposed. For readability, the original protocol from [9] is repeated here. However, a distinction is made between the gain matrices F_1 and F_2,

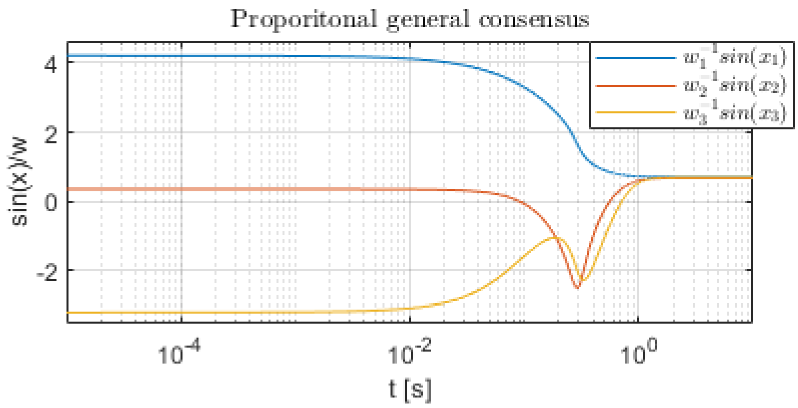

Figure 4-5: Edge-based I - Proportional generalized consensus

which, in [9], are equal. The reason for the distinction will become apparent in the following section.

$$\forall i \in \mathcal{V}: \quad \dot{v}_i = (A + BF_2)v_i + L \sum_{j \in \mathcal{N}_i} c_{ij}[C(v_i - v_j) - (y_i - y_j)] \tag{4-7a}$$

$$\dot{c}_{ij} = a_{ij} \begin{bmatrix} y_i - y_j \\ C(v_i - v_j) \end{bmatrix}^\top \Gamma \begin{bmatrix} y_i - y_j \\ C(v_i - v_j) \end{bmatrix} \tag{4-7b}$$

$$u_i = F_1 v_i \tag{4-7c}$$

The linear agent dynamics considered in [9] are given by the following state-space system.

$$\forall i \in \mathcal{V}: \quad \dot{x}_i = Ax_i + Bu_i \tag{4-8a}$$

$$y_i = Cx_i \tag{4-8b}$$

4-2-1 Derivation

The protocol Eq. (4-7) is generalized following a similar strategy as in Section 4-1-1. First of all, the case where the agents themselves are stationary, such that the dynamics solely follow from the input u_i, is investigated. That is, $A = 0$ and $B = 1$. The linear output Eq. (4-8b) is replaced by a more general, nonlinear function of the states.

$$\forall i \in \mathcal{V}: \quad \dot{x}_i = u_i \tag{4-9a}$$

$$y_i = \vartheta(x_i) \tag{4-9b}$$

Now, since $A = 0$, the designer of the controller is free to choose the matrix Q in the linear matrix inequality Eq. (2-14). By implementing the matrix appropriately, preservation of the

AF can be guaranteed, given the correct conditions. It turns out that this is achieved by selecting

$$F_1 = (g_i'(x_i))^{-1}. \tag{4-10}$$

To let the AF be time-invariant, the auxiliary state v needs to preserve the value of the arithmetic mean. This is elaborated in Section 4-2-2. In Appendix B-1-1 and Appendix B-2, it is stated that this is achieved by choosing $F_2 = -dg_v(x_i)/dx_i$ with $g_v(x_i) = x$, from which it follows that $F_2 = -1$. The matrices Γ and L may also be chosen freely, and are left as tuning parameters.

Remark 2. *The resulting dynamics of the auxiliary state (Eq. (4-13b)) preserves the arithmetic mean of the initial states. However, the form of the protocol does not agree with the form proposed in Eq. (2-6) due to the subtraction of the state v_i. In Appendix B-2, it is proved that this form preserves the value of the arithmetic mean of the initial states, given that the mean is initially zero.*

The difference functions of the dynamics of the auxiliary state v and the adaptive gains c are generalized to include classes of nonlinear functions, similarly to Section 4-3-1. The resulting nonlinear edge-based adaptive gain consensus protocol, from now on referred to as Edge-based protocol II (EB2), takes the following form.

$$\forall i \in \mathcal{V}: \quad \dot{x}_i = -(g_i'(x_i))^{-1} v_i \tag{4-11a}$$

$$\dot{v}_i = -v_i - \sum_{j \in \mathcal{N}_i} c_{ij} \left[\hat{\phi}(\vartheta(x_j) - \vartheta(x_i)) - \hat{\phi}(\vartheta(v_j) - \vartheta(v_i)) \right] \tag{4-11b}$$

$$\dot{c}_{ij} = \begin{bmatrix} \hat{\phi}(\vartheta(x_j) - \vartheta(x_i)) \\ \hat{\phi}(\vartheta(v_j) - \vartheta(v_i)) \end{bmatrix}^\top \Gamma \begin{bmatrix} \hat{\phi}(\vartheta(x_j) - \vartheta(x_i)) \\ \hat{\phi}(\vartheta(v_j) - \vartheta(v_i)) \end{bmatrix} \tag{4-11c}$$

By defining

$$z_i = (x_i, v_i) \tag{4-12a}$$

$$\hat{\phi}_v(z_j, z_i) = \left[\hat{\phi}(\vartheta(x_j) - \vartheta(x_i)) - \hat{\phi}(\vartheta(v_j) - \vartheta(v_i)) \right] \tag{4-12b}$$

$$\hat{\phi}_c(z_j, z_i) = \begin{bmatrix} \hat{\phi}(\vartheta(x_j) - \vartheta(x_i)) \\ \hat{\phi}(\vartheta(v_j) - \vartheta(v_i)) \end{bmatrix} \tag{4-12c}$$

Eq. (4-11) can be written in compact form as

$$\forall i \in \mathcal{V}: \quad \dot{x}_i = -(g_i'(x_i))^{-1} v_i \tag{4-13a}$$

$$\dot{v}_i = -v_i - \sum_{j \in \mathcal{N}_i} c_{ij} \hat{\phi}_v(z_j, z_i) \tag{4-13b}$$

$$\dot{c}_{ij} = \hat{\phi}_c(z_j, z_i)^\top \Gamma \hat{\phi}_c(z_j, z_i). \tag{4-13c}$$

4-2-2 Agreement function

The time-invariance property of the AF of EB2 is considered next. At first glance, it becomes clear that the structure of the state dynamics differs from NCP Eq. (2-6) due to the absence of the sum. Thus, a different approach to determine time-invariance of the protocol is required. As will be proved below, protocol Eq. (4-11) does indeed preserve the value of the AF over time. To show this, the time-derivative of the AF is taken, which is then shown to equal zero, implying that it is time-invariant.

Theorem 3 gives conditions on EB2 (Eq. (4-13)) such that the AF $\chi(x)$ is time-invariant.

Theorem 3 (Time-invariance of EB2)**.** *Assume that $\phi_c(z_j z_i)$ and $\phi_v(z_j z_i)$ are anti-symmetric, $g_i(x_i)$ is strictly increasing, the auxiliary states v are initialized such that $\sum_{i\in\mathcal{V}} v_i(0) = 0$ and the adaptive gains c are initialized such that $c_{ij}(0) = c_{ji}(0)$ for all $i, j \in \mathcal{V}$. Then, the value of the AF $\chi = h(\sum_{i\in\mathcal{V}} g_i(x_i))$ is preserved.*

Proof. See Appendix A-4. □

4-2-3 State consensus

State consensus under protocol Eq. (4-11) is reached if the conditions in Table 2-1 are satisfied. The result is summarized in the following theorem.

Conjecture 1. *Assume that $g_i(\cdot)$ is strictly increasing, $\hat{\phi}(\cdot)$ is continuous, locally Lipschitz, odd and strictly increasing, $\vartheta(\cdot)$ is differentiable and $\vartheta'(x_i)$ is locally Lipschitz and strictly positive. Then, EB2 (Eq. (4-11)) achieves state consensus. Furthermore, the auxiliary states v converge to zero and the adaptive gains c converge to finite values.*

A proof of state consensus of EB2 is not given in the thesis. The proof is currently in progress and will be given included in the research paper (see Appendix D) once completed. For now, it is assumed that the conditions are sufficient for reaching consensus, and that the problem is in determining a suitable Lyapunov function for the given system. The complexity arises in the implementation of the auxiliary state v. Some intuitive quadratic Lyapunov functions which do not do the trick are summarized below. A similar strategy to the proofs of 2 and 4 is implemented, with error functions $e_{xi} = g(x_i) - g(\chi(x(0)))$ and $e_{vi} = v_i$.

$$V = \frac{1}{2}\sum_{i\in\mathcal{V}} e_{xi}^2 + \frac{1}{2}\sum_{i\in\mathcal{V}} e_{vi}^2$$

$$V = \frac{1}{2}\sum_{i\in\mathcal{V}} (e_{xi} + e_{vi})^2$$

$$V = \frac{1}{2} e_x^\top \mathcal{L} e_x + \frac{1}{2} e_v^\top \mathcal{L} e_v$$

$$V = \frac{1}{2}\begin{bmatrix} e_x \\ e_v \end{bmatrix}^\top (\mathcal{L} \otimes \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}) \begin{bmatrix} e_x \\ e_v \end{bmatrix}$$

Unfortunately, the time-derivatives of the Lyapunov functions cannot be shown to be non-positive for all $x, v \in \mathbb{R}$.

The result is verified by means of numerical simulation. The simple network with three agents, described in Appendix C-1, is controlled under EB2 with $\hat{\phi}(x) = x$, $\vartheta(x) = x + x^3$ and $g_i(x_i) = \log(x_i^{w_i})$. The WGM is implemented as AF. First, the objective is to reach state consensus (Figure 4-6). Then, the objective is to reach proportional state consensus (Figure 4-7). The figures show that the objectives are satisfied and the AF is time-invariant. The evolution of the auxiliary states v is presented in Figure 4-8, and shows that the arithmetic mean $\bar{x}$ is preserved and the states converge to the origin.

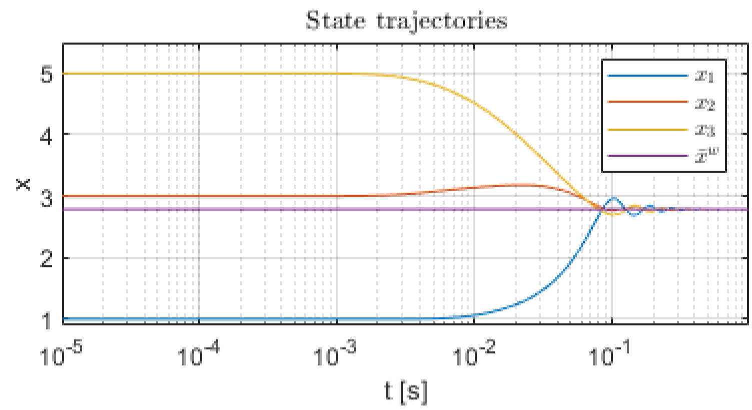

Figure 4-6: Edge-based II - State consensus

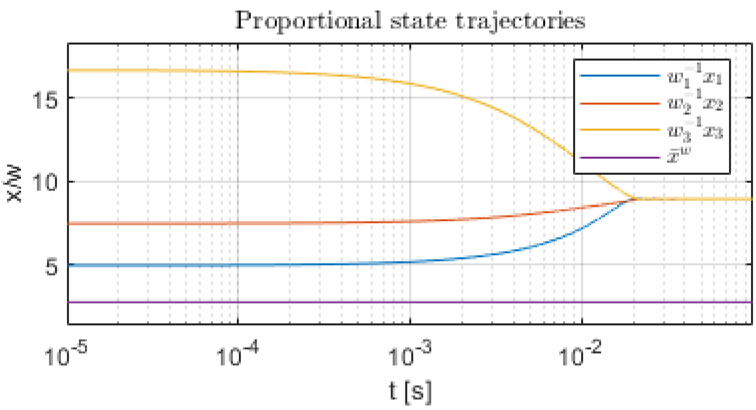

Figure 4-7: Edge-based II - Proportional state consensus

4-2-4 Generalized consensus

To reach generalized consensus on some function $f(\cdot)$, the following protocol is proposed.

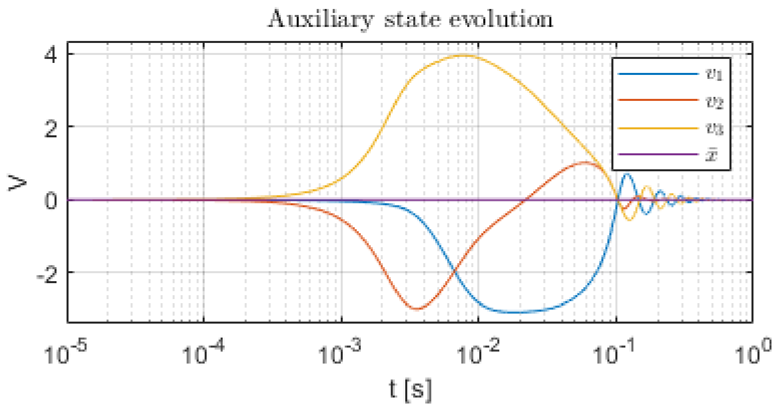

Figure 4-8: Edge-based II - Auxiliary state trajectories

$$\forall i \in \mathcal{V}: \quad \dot{x}_i = -(g'(x_i))^{-1} v_i \tag{4-14a}$$

$$\dot{v}_i = -v_i - \sum_{j \in \mathcal{N}_i} c_{ij} \left[\hat{\phi}(\vartheta(f(x_j, x^{(j)})) - \vartheta(f(x_i, x^{(i)}))) - \hat{\phi}(\vartheta(v_j) - \vartheta(v_i)) \right] \tag{4-14b}$$

$$\dot{c}_{ij} = \begin{bmatrix} \hat{\phi}(\vartheta(f(x_j, x^{(j)})) - \vartheta(f(x_i, x^{(i)}))) \\ \hat{\phi}(\vartheta(v_j) - \vartheta(v_i)) \end{bmatrix}^\top \Gamma \begin{bmatrix} \hat{\phi}(\vartheta(f(x_j, x^{(j)})) - \vartheta(f(x_i, x^{(i)}))) \\ \hat{\phi}(\vartheta(v_j) - \vartheta(v_i)) \end{bmatrix} \tag{4-14c}$$

Finding conditions that guarantee generalized consensus is an open problem. An investigation of the equilibria is performed, and the claim that consensus on $f(\cdot)$ is reached is verified by means of numerical simulation. Figure 4-9 shows the generalized consensus case where $f(x) = \sin(x)$, and Figure 4-10 shows the case with the objective of reaching proportional consensus of $f(x)$. In both cases, the objectives are satisfied.

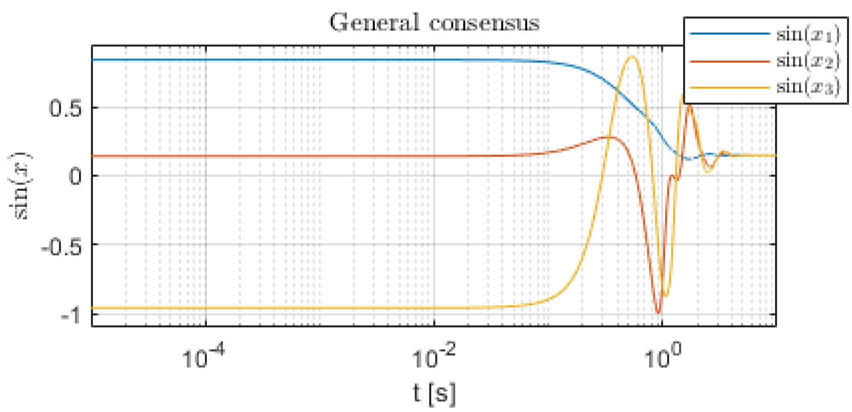

Figure 4-9: Edge-based II - generalized consensus

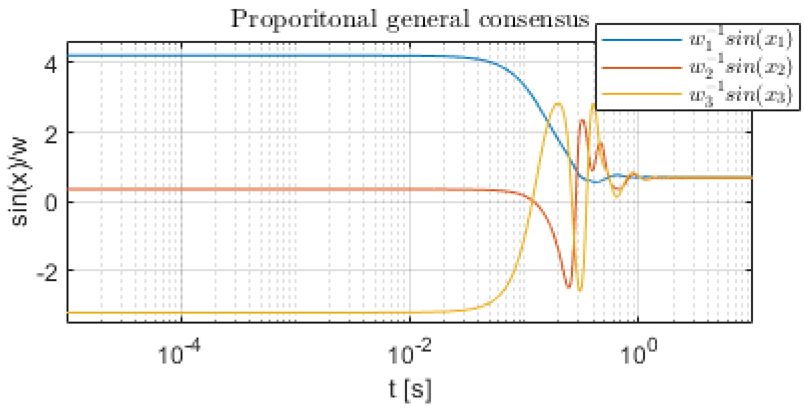

Figure 4-10: Edge-based II - Proportional generalized consensus

Equilibria

An investigation of the equilibria of the system is performed next. Recall that $dg_i(x_i)/dx_i$ is strictly positive, and, as a consequence, so is $(g'(x_i))^{-1}$. Thus, from Eq. (4-14a), it follows that $v_i = 0$ for all $i \in \mathcal{V}$ at steady-state. Then, Eq. (4-14b) becomes

$$\dot{v}_i = -\sum_{j\in\mathcal{N}_i} c_{ij}\left[\hat{\phi}(\vartheta(f(x_j, x^{(j)})) - \vartheta(f(x_i, x^{(i)})))\right] = 0. \tag{4-15}$$

Due to the undirectedness and connectivity of the graph, it follows that

$$\vartheta(f(x_j, x^{(j)})) - \vartheta(fx_i, x^{(i)})) = 0 \tag{4-16}$$

$$f(x_j, x^{(j)}) - f(x_i, x^{(i)}) = 0 \quad \forall i, j \in \mathcal{V} \tag{4-17}$$

Hence, consensus on $f(\cdot)$ is reached at steady-state. The existence of the equilibria is an open problem and is left for future research.

4-3 Node-based protocol I

In this section, the first node-based adaptive gain consensus protocol is proposed. The protocol is similar to EB1 (Eq. (4-3)), and implements a node-based adaptive gain.

4-3-1 Derivation

The node-based adaptive gain in [9] is implemented in a consensus protocol similar to the protocol proposed in Section 4-1. The adaptive gain is implemented in a similar fashion to the consensus protocol proposed in Section 4-2. That is, an adaptive gain is assigned to each node. The gain dynamics are described by a quadratic function of the sum of the difference of some function of local and neighbouring states. The Node-based protocol I (NB1)s then described by

$$\forall i \in \mathcal{V}: \quad \dot{x}_i = (g_i'(x_i))^{-1} d_i \sum_{j \in \mathcal{N}_i} \hat{\phi}(\vartheta(x_j) - \vartheta(x_i)) \tag{4-18a}$$

$$\dot{d}_i = \sum_{j \in \mathcal{V}} \hat{\phi}(\vartheta(x_j) - \vartheta(x_i))^\top \Gamma \sum_{j \in \mathcal{V}} \hat{\phi}(\vartheta(x_j) - \vartheta(x_i)). \tag{4-18b}$$

The protocol is referred to as NB1 for the remainder of the thesis.

The reader is invited to compare the state dynamics of NB1 (Eq. (4-18a)) to those of EB1 (Eq. (4-3a)). Note that the equations differ only in the implementation of the adaptive gain. This allows for an investigation of the effect of the type of gain (edge-based vs. node-based) on the properties of the protocol. In particular, the effect on the AF is investigated.

4-3-2 Agreement function

In Section 4-1-2, the preservation property of the AF is determined by showing that the function $\phi(x_j, x_i) = c_{ij}\hat{\phi}(\vartheta(x_j) - \vartheta(x_i))$ is anti-symmetric and then applying [6, Theorem 1]. However, since [6, Theorem 1] only states a sufficient condition for time-invariance and $\phi(x_j, x_i) = d_i\hat{\phi}(\vartheta(x_j) - \vartheta(x_i))$ is symmetric if and only if $d_i = d_j$ for all $i, j \in \mathcal{V}$, time-invariance cannot be concluded using this approach.

Thus, another approach is needed. In Section 4-2-2, it is stated that time-invariance of the AF is equivalent to time-invariance of it's argument, $\sum_{i\in\mathcal{V}} g_i(x_i)$. We have

$$\sum_{i \in \mathcal{V}} \dot{g}_i(x_i) = \sum_{i=1}^{n} g_i'(x_i)\dot{x}_i = \sum_{i \in \mathcal{V}} d_i \sum_{j \in \mathcal{N}_i} \hat{\phi}(\vartheta(x_j) - \vartheta(x_i)) = 0 \tag{4-19a}$$

$$\iff$$

$$d_i = d_j \quad \forall i, j \in \mathcal{V}, \tag{4-19b}$$

which follows from the connectivity and undirectedness of the network. Therefore, NB1 is time-invariant if and only if all gains d are equal. Of course, if all gains are equal, the implementation of an adaptive gain at each node becomes pointless.

4-3-3 State consensus

NB1 reaches state consensus if the conditions of Table 2-1 are satisfied. This is summarized in Theorem 4.

Theorem 4. *Assume that $g_i(\cdot)$ is strictly increasing, $\hat{\phi}(\cdot)$ is continuous, locally Lipschitz, odd and strictly increasing, $\vartheta(\cdot)$ is differentiable and $\vartheta'(x_i)$ is locally Lipschitz and strictly positive. Then, EB2 (Eq. (4-18)) achieves state consensus.*

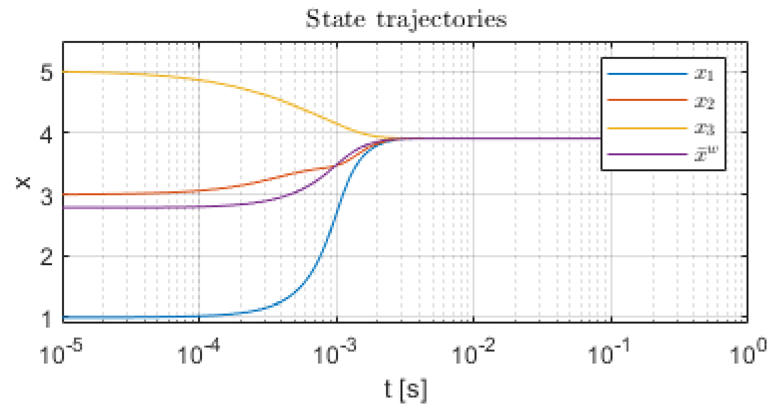

Figure 4-11: Node-based I - State consensus

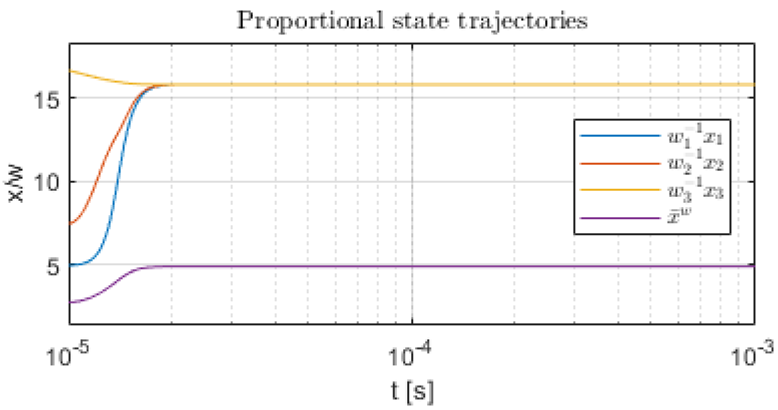

Figure 4-12: Node-based I - Proportional state consensus

Proof. See Appendix A-5 □

An investigation on convergence of the adaptive gains shows that they should converge to finite values. The adaptive gain dynamics is a function of the sum of the difference of the states. Thus, once state consensus is reached, the gain dynamics $\dot{d}_i = 0$ for all $i \in \mathcal{V}$. The proof is left for future research, and the claim is verified numerically.

The results are verified for the state consensus (Figure 4-11) and the proportional state consensus (Figure 4-12) case with $\hat{\phi}(x) = x$, $\vartheta(x) = x + x^3$ and $g_i(x_i) = \log(x_i^{w_i})$. To illustrate that the AF is no longer guaranteed to be time-invariant, the WGM is implemented as the AF. The figures show that the objective of (proportional) state consensus is achieved. Furthermore, it is evident from Figure 4-13 that the adaptive gains d converge to finite values, as previously assumed.

4-3-4 Generalized consensus

The protocol is extended such that consensus on some function $f(\cdot)$ can be achieved. To this end, the following protocol is proposed:

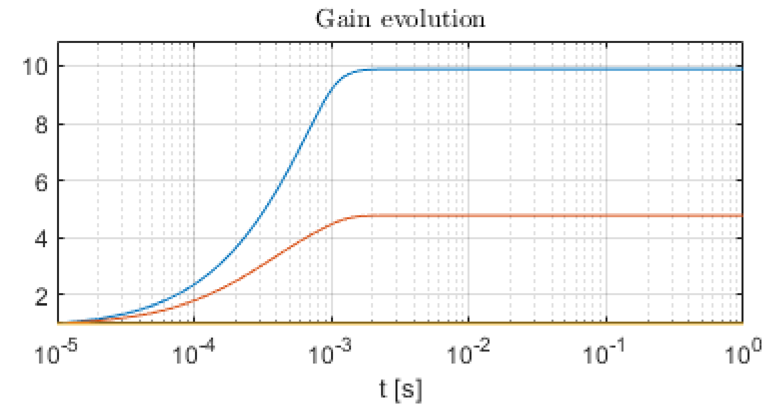

Figure 4-13: Node-based I - Gain evolution

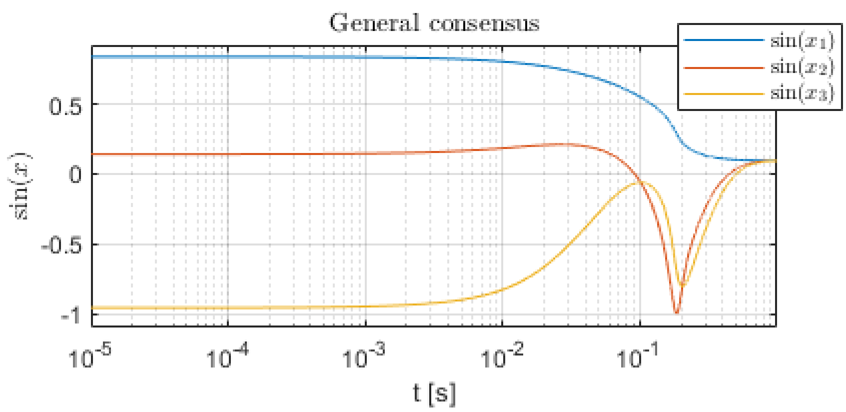

Figure 4-14: Node-based I - generalized consensus

$$\forall i \in \mathcal{V}: \quad \dot{x}_i = (g_i'(x_i))^{-1} d_i \sum_{j \in \mathcal{N}_i} \hat{\phi}(\vartheta(f(x_j, x^{(j)})) - \vartheta(f(x_i, x^{(i)}))) \tag{4-20a}$$

$$\dot{d}_i = \sum_{j \in \mathcal{N}_i} \hat{\phi}(\vartheta(f(x_j, x^{(j)})) - \vartheta(f(x_i, x^{(i)})))^\top \Gamma \sum_{j \in \mathcal{N}_i} \hat{\phi}(\vartheta(f(x_j, x^{(j)})) - \vartheta(f(x_i, x^{(i)}))) \tag{4-20b}$$

To verify that the protocol is successful in reaching consensus on some function $f(\cdot)$, we let $f(x) = \sin(x)$. Numerical simulations are performed in the simple network (see Appendix C-1). Figure 4-14 shows convergence for the generalized consensus case and Figure 4-15 for the proportional generalized consensus case.

Determining conditions on $f(\cdot)$ that guarantee consensus is left for future research. For now, an investigation of the equilibria is given.

Equilibria

Similarly to Section 4-1-4, it is found that

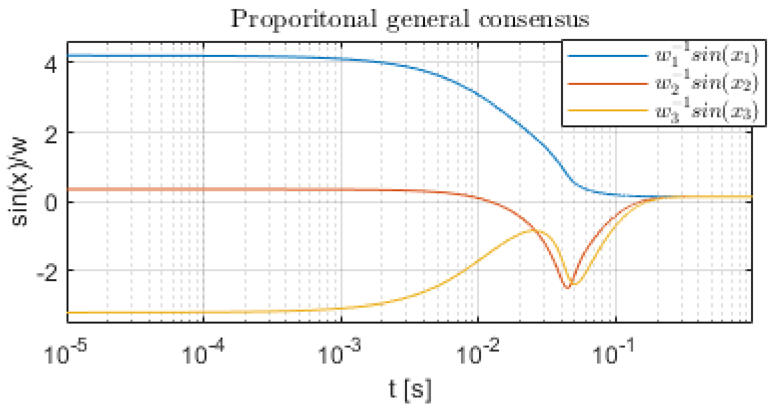

Figure 4-15: Node-based I - Proportional generalized consensus

$$f(x_j, x^{(j)}) = f(x_i, x^{(i)}), \quad \forall i, j \in \mathcal{V} \tag{4-21}$$

at equilibrium. Hence, generalized consensus on $f(\cdot)$ is reached. The determination of necessary and sufficient conditions that guarantee the existence of the equilibria is an open problem.

4-4 Node-based protocol II

Inspired by [9], a second node-based consensus protocol is proposed. The protocol is similar to EB2, and differs only in the implementation of the adaptive gains.

4-4-1 Derivation

The derivation of the second NB1s equivalent to EB2. The steps are the same as those described in Section 4-2-1. The resulting protocol is given next.

$$\forall i \in \mathcal{V}: \quad \dot{x}_i = -(g_i'(x_i))^{-1} v_i \tag{4-22a}$$

$$\dot{v}_i = -v_i - d_i \sum_{j \in \mathcal{N}_i} \left[\hat{\phi}(\vartheta(x_j) - \vartheta(x_i)) - \hat{\phi}(\vartheta(v_j) - \vartheta(v_i)) \right] \tag{4-22b}$$

$$\dot{d}_i = \sum_{j \in \mathcal{V}} \begin{bmatrix} \hat{\phi}(\vartheta(x_j) - \vartheta(x_i)) \\ \hat{\phi}(\vartheta(v_j) - \vartheta(v_i)) \end{bmatrix}^\top \Gamma \sum_{j \in \mathcal{V}} \begin{bmatrix} \hat{\phi}(\vartheta(x_j) - \vartheta(x_i)) \\ \hat{\phi}(\vartheta(v_j) - \vartheta(v_i)) \end{bmatrix} \tag{4-22c}$$

The protocol can be written in compact form, according to Eq. (4-12).

$$\begin{aligned}\forall i \in \mathcal{V}: \quad \dot{x}_i &= -\frac{1}{dg_i(x_i)/dx_i} v_i && \text{(4-23a)}\\ \dot{v}_i &= -v_i - d_i \sum_{j \in \mathcal{N}_i} \hat{\phi}_v(z_j, z_i) && \text{(4-23b)}\\ \dot{d}_i &= \sum_{j \in \mathcal{V}} \hat{\phi}_c(z_j, z_i)^\top \Gamma \sum_{j \in \mathcal{V}} \hat{\phi}_c(z_j, z_i). && \text{(4-23c)}\end{aligned}$$

In the remainder of the thesis, the protocol is referred to as Node-based protocol II (NB2).

4-4-2 Agreement function

Similarly to Section 4-2-2, it is shown that NB2 preserves the value of the AF if and only if all adaptive gains are equal. Consider the time-derivative of the argument $\sum_{i \in \mathcal{V}} g_i(x_i)$ of the AF.

$$\sum_{i \in \mathcal{V}} \dot{g}_i(x_i) = \sum_{i \in \mathcal{V}} g_i'(x_i)\dot{x}_i = \sum_{i \in \mathcal{V}} v_i = 0 \tag{4-24}$$

Since $n > 0$, this is equivalent to stating that the arithmetic mean of the auxiliary states is preserved at zero. In Appendix B-2, a protocol that preserves the arithmetic mean given that the initial arithmetic mean of the states equals zero is proposed, assuming that $\phi(x_j, x_i)$ is anti-symmetric.

$$\dot{x}_i = -x_i - \sum_{i \in \mathcal{V}} \phi(x_j, x_i), \quad \forall i \in \mathcal{V} \tag{4-25}$$

Comparing this protocol to Eq. (4-13b), it becomes clear that $d_i \phi(x_j, x_i)$ must be anti-symmetric for the AF to be preserved. Clearly, this is only the case when $d_i = d_j$ for all $i, j \in \mathcal{V}$. Thus, preservation of the AF cannot be guaranteed.

4-4-3 State consensus

NB2 reaches state consensus if the conditions of Table 2-1 are satisfied. This is summarized in Theorem 2.

Conjecture 2. *Assume that $g_i(\cdot)$ is strictly increasing, $\hat{\phi}(\cdot)$ is continuous, locally Lipschitz, odd and strictly increasing, $\vartheta(\cdot)$ is differentiable and $\vartheta'(x_i)$ is locally Lipschitz and strictly positive. Then, NB2 (Eq. (4-22)) achieves state consensus. Furthermore, the auxiliary states v converge to zero and the adaptive gains d converge to finite values.*

See Conjecture 1 for a discussion on the shortcomings of the state consensus proof of NB2. A similar approach has been implemented here, however without success.

The results are tested by means of simulation with $\hat{\phi}(x) = x$, $\vartheta(x) = x + x^3$ and $g_i(x_i) = \log(x_i^{w_i})$. First, the objective is to reach state consensus (see Figure 4-16). Then, the objective is to reach proportional state consensus (see Figure 4-12). The WGM is included to show that it is not time-invariant.

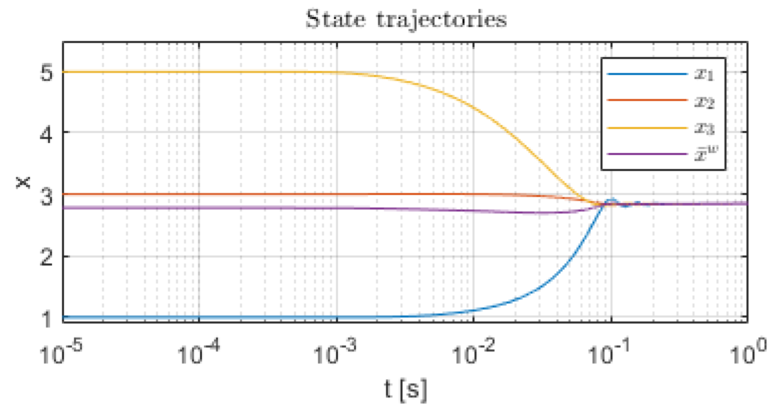

Figure 4-16: Node-based II - State consensus

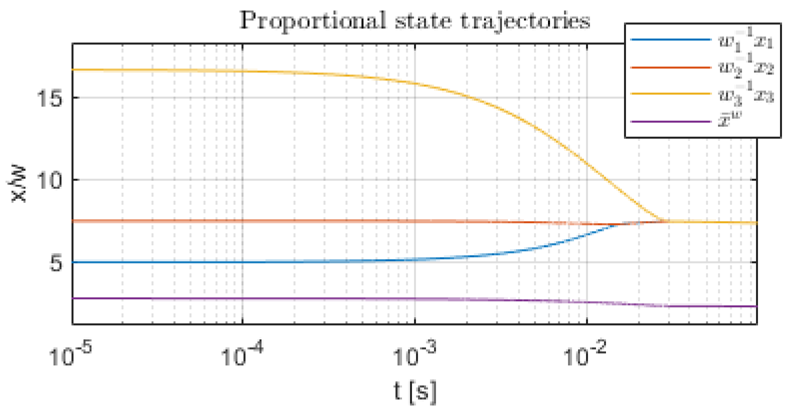

Figure 4-17: Node-based II - Proportional state consensus

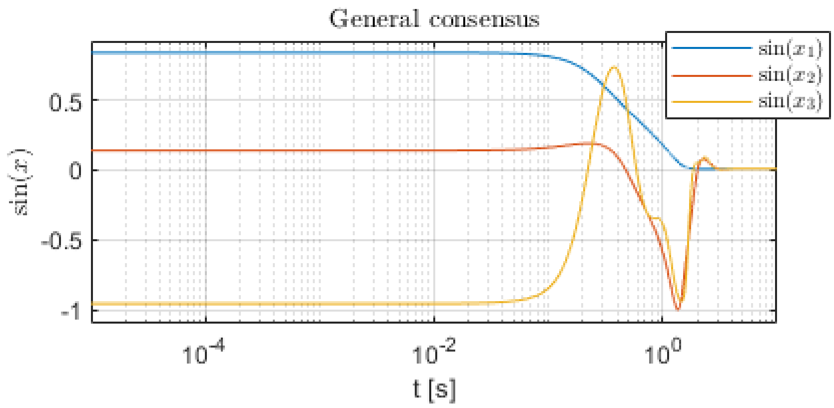

Figure 4-18: Node-based II - generalized consensus

4-4-4 Generalized consensus

Similarly to 4-2-4, the NB2 can be further generalized to allow for consensus reaching of some function of the local and neighbouring states $f(x_i, x^{(i)})$. The resulting protocol is given below.

$$\forall i \in \mathcal{V}: \quad \dot{x}_i = -(g_i'(x_i))^{-1} v_i \tag{4-26a}$$

$$\dot{v}_i = -v_i - d_i \sum_{j \in \mathcal{N}_i} \left[\hat{\phi}(\vartheta(f(x_j, x^{(j)})) - \vartheta(f(x_i, x^{(i)}))) - \hat{\phi}(\vartheta(v_j) - \vartheta(v_i)) \right] \tag{4-26b}$$

$$\dot{c}_{ij} = \sum_{j \in \mathcal{V}} \begin{bmatrix} \hat{\phi}(\vartheta(f(x_j, x^{(j)})) - \vartheta(f(x_i, x^{(i)}))) \\ \hat{\phi}(\vartheta(v_j) - \vartheta(v_i)) \end{bmatrix}^\top \times \Gamma \sum_{j \in \mathcal{V}} \begin{bmatrix} \hat{\phi}(\vartheta(f(x_j, x^{(j)})) - \vartheta(f(x_i, x^{(i)}))) \\ \hat{\phi}(\vartheta(v_j) - \vartheta(v_i)) \end{bmatrix} \tag{4-26c}$$

To verify the claims numerically, the system is simulated for $f(x) = \sin(x)$. First, the objective is to reach generalized consensus. The results are depicted in Figure 4-18. Then, the objective is proportional consensus, and the results are shown in Figure 4-19.

Equilibria

Applying an equivalent approach as in Section 4-3-4, it is found that

$$f(x_j, x^{(j)}) = f(x_i, x^{(i)}), \quad \forall i, j \in \mathcal{V} \tag{4-27}$$

at steady-state.

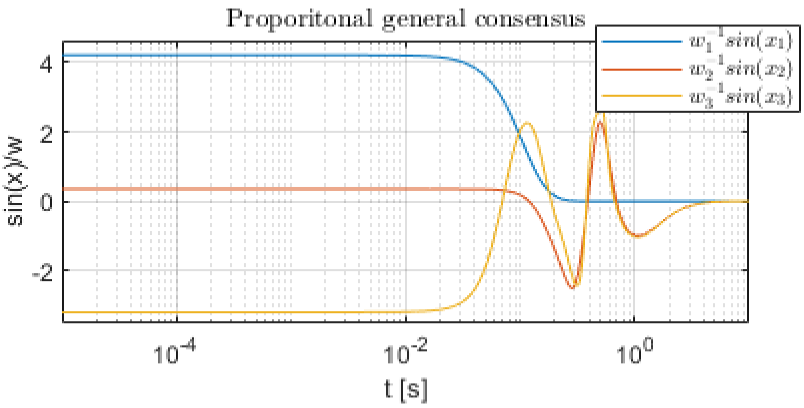

Figure 4-19: Node-based II - Proportional generalized consensus

Chapter 5

Power consensus

Chapters 2 and 3 provide a solid theoretical foundation on Nonlinear consensus protocol (NCP)s. Protocols that allow networked systems to reach several types of consensus have been proposed. In this chapter, the theory is applied to the Proportional power consensus (PPC) algorithm [11] presented in Section 2-4. The research in Section 5-1 has the objective to determine whether the Power consensus algorithm (PCA) is a special case of the NCPs discussed in Section 2-2 and in Chapter 3. In Section 5-2, the PCA [11] is extended to the general nonlinear consensus case, such that additional difference functions can be implemented. Then, in Section 5-3, the adaptive gain protocols proposed in the previous chapter are utilized to achieve the objectives of PPC in DC microgrids, whilst stabilizing the voltage.

5-1 Power consensus algorithm as nonlinear consensus protocol

The NCP from Section 2-2 consists of two parts. The first is the decomposed function $g(x_i)$ of the Agreement function (AF). The second is the sum of the anti-symmetric function $\phi(x_j, x_i) = \hat{\phi}(\vartheta(f(x_j, x^{(j)})) - \vartheta(f(x_i, x^{(i)})))$. Each part, and its connection to the PCA [11], will be analyzed in this chapter. The PCA and the NCP are repeated here to aid the comparison of the two. The state of the NCP is set as the voltage $V \in \mathbb{R}$, and the weight at node i is denoted by C_i.

Power consensus algorithm

$$\forall i \in \mathcal{V} : \quad \dot{V}_i = C_i^{-1} V_i \sum_{j \in \mathcal{N}_{ci}} (C_j^{-1} P_j - C_i^{-1} P_i) \tag{5-1}$$

Nonlinear consensus protocol

$$\forall i \in \mathcal{V} : \quad \dot{V}_i = (g_i'(V_i))^{-1} \sum_{j \in \mathcal{N}_i} \hat{\phi}(\vartheta(w_j^{-1} f(V_j, V^{(j)})) - \vartheta(w_i^{-1} f(V_i, V^{(i)}))) \tag{5-2}$$

Let us consider the properties of the PCA. The Weighted geometric mean (WGM) of the voltages is preserved. The voltage is stabilized at some value, and consensus among the voltages is not guaranteed. Proportional consensus is reached among the source powers. Each of these properties is discussed next.

5-1-1 Agreement function

By observing the structure of Eq. (5-1) and Eq. (5-2), it quickly becomes evident that

$$(g_i'(V_i))^{-1} = C_i^{-1} V_i. \tag{5-3}$$

Eq. (5-3) is solved for $g(V_i)$ by taking the inverse and the integral:

$$\begin{aligned}
(g_i'(V_i))^{-1} &= C_i^{-1} V_i && \text{(5-4a)}\\
&\Rightarrow \\
g_i'(V_i)^{-1} &= C_i V_i^{-1} && \text{(5-4b)}\\
&\Rightarrow \\
g_i(V_i) &= C_i \log V_i = \log V_i^{C_i} \qquad \forall V_i > 0 && \text{(5-4c)}
\end{aligned}$$

This is in agreement with the decomposed function $g(\cdot)$ Eq. (3-2b) of the WGM, discussed in Section 3-1. Clearly, Eq. (5-3) is strictly positive for $C_i, V_i > 0$. Thus, the conditions on $g(x_i)$ for preservation of the AF are satisfied.

Now, if $\hat{\phi}(\vartheta(w_j^{-1} f(V_j, V^{(j)})) - \vartheta(w_i^{-1} f(V_i, V^{(i)}))) = C_j^{-1} P_j - C_i^{-1} P_i$ is anti-symmetric, then the conditions for time-invariance of the WGM of the states are satisfied. This is clearly the case, since $C_j^{-1} P_j - C_i^{-1} P_i = -(C_i^{-1} P_i - C_j^{-1} P_j)$. Thus, it has been verified that the PCA satisfies the conditions for time-invariance of the AF.

5-1-2 Power function

The properties of the power function $f(V_i, V^{(i)}) = P_i(V_i, V^{(i)})$ and the functions $\hat{\phi}(\cdot)$ and $\vartheta(\cdot)$ are analyzed to conclude on the type of consensus that is reached. Recall that (proportional) state consensus is reached if $\hat{\phi}$ is continuous, locally Lipschitz, odd and strictly increasing, and ϑ is differentiable with $\vartheta(f(V_i, V^{(i)}))/dV_i$ locally Lipschitz and strictly positive.

Clearly, $\hat{\phi}(x) = x$ satisfies the conditions. However, determining the properties of $\vartheta(f(V_i, V^{(i)}))$ is not so straightforward. The power at the sources is given by [11]

$$P_s = [V_s] I_s \tag{5-5}$$
$$I_s = Y_{ss} V_s + Y_{sl} V_l \tag{5-6}$$

where the source voltage V_s is measured and the load voltage V_l follows from the algebraic equation

$$V_l B_l \Gamma B^\top B V - V_l I_l(V_l) = 0. \tag{5-7}$$

Equivalently, the source power can be represented as

$$P_i = V_i \sum_{j \in \mathcal{N}_i} y_{ij} V_j \tag{5-8a}$$

$$= V_i(\sum_{j \in \mathcal{N}_i \cap \mathcal{V}_s} y_{ij} V_j + \sum_{j \in \mathcal{N}_i \cap \mathcal{V}_l} y_{ij} V_j(V_i), \quad \forall i \in \mathcal{V}_s \tag{5-8b}$$

with $y_{ij} \in Y$. Note that the differentiation between neighbouring source and load nodes is made, to indicate the functional dependence of the load voltages on the source voltages. Then, the derivative with respect to the state is

$$\frac{d\vartheta(f(V_i, V^{(i)}))}{dV_i} = C_i^{-1}(\sum_{j \in \mathcal{N}_i \cap \mathcal{V}_s} y_{ij} V_j + \sum_{j \in \mathcal{N}_i \cap \mathcal{V}_l} y_{ij} (V_j(V_i) + V_i \frac{dV_j(V_i)}{dV_i})). \tag{5-9a}$$

From the construction of the conductance matrix (Eq. (2-3)) and the definition of the incidence matrix (Eq. (2-2)), it becomes evident the values $y_{ij} \in Y$ are negative if $j \in \mathcal{V}_l$. Thus, it cannot be guaranteed that $\vartheta(f(V_i, V^{(i)}))/dV_i$ is strictly positive. Therefore, (proportional) consensus of the state voltages cannot be guaranteed.

The above verifies that the PCA is a special case of the generalized consensus algorithm proposed in Section 3-2 with implementation of the proportional consensus strategy proposed in Section 3-3. The equilibria are found where the proportional power values at the source nodes are equal. Consensus reaching and asymptotic stability are proved in [11].

5-2 Nonlinear consensus implementation

The WGM as AF and the source power as generalized consensus are implemented in the NCP Eq. (3-7) to obtain the extended power consensus protocol.

$$\forall i \in \mathcal{V}: \quad \dot{V}_i = C_i^{-1} V_i \sum_{j \in \mathcal{N}_{ci}} \hat{\phi}(\vartheta(C_j^{-1} P_j) - \vartheta(C_i^{-1} P_i)) \tag{5-10}$$

The voltage dynamics are a function of the local voltage and the local and neighbouring power measurements.

Conjecture 3 (Proportional power consensus - NCP)**.** *Assume that $\hat{\phi}(\cdot)$ is continuous, locally Lipschitz, odd and strictly increasing, $\vartheta(\cdot)$ is differentiable and $\vartheta(x_i)/dx_i$ is locally Lipschitz and strictly positive. The nonlinear PCA (Eq. (5-10)) achieves PPC among the sources of a DC microgrid according to power sharing coefficients $C_i > 0$ for all $i \in \mathcal{V}_s$. Furthermore, the voltages are stabilized and the WGM of the source voltages is preserved.*

The case where $\hat{\phi}(x) = x$ and $\vartheta(x) = x$ is proved in [11]. The proof of the general case is an open problem.

An investigation of the effect of the functions $\hat{\phi}(\cdot)$ and $\vartheta(\cdot)$ on the response of the protocol Eq. (5-10) is performed. To this end, the three cases of Table C-1 are considered in the

DC microgrid network described in Appendix C-2. An interesting result for the case where $\vartheta(x) = x^3 + x$, follows directly. The system response is far too aggressive. The system stabilizes the voltage within $10^{-7}s$. Such a response is unrealistic and not applicable to an actual microgrid. Thus, case 2 is not considered for the simulation of protocol Eq. (5-10). The reason for the aggressive behaviour is that the values of the source power can be large, and cubing these values in $\vartheta(C_i^{-1}P_i) = C_i^{-1}P_i + (C_i^{-1}P_i)^3$ results in huge inside the difference function. The reader is referred to Chapter 6 for an in-depth discussion on this behaviour.

The voltage evolution of cases 1 and 3 and depicted in Figure 5-1. To avoid cluttered plots, only the voltage of source 3 is given. The original PCA [11] (case 1) stabilizes the voltage in about 10^{-2} s and exhibits an overshoot. Case 3, however, shows no overshoot and reacts quicker in the first 10^{-4} s, after which it converges to its steady-state value. Figure 5-2 shows the WGM of the two cases. Clearly, the WGM is preserved.

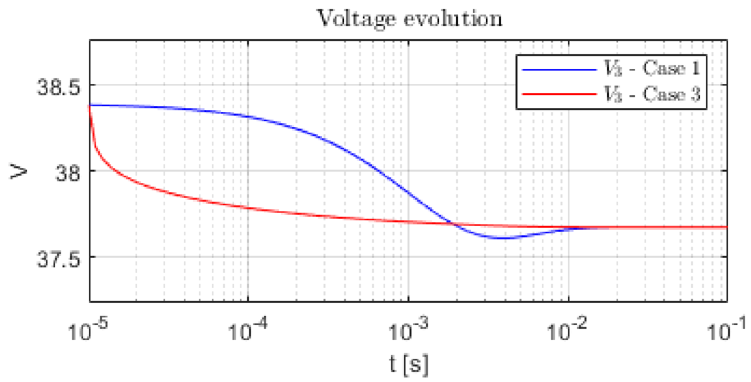

Figure 5-1: Voltage evolution - No adaptive gain

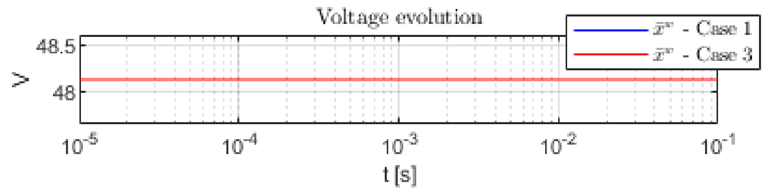

Figure 5-2: WGM - No adaptive gain

Convergence of the proportional power values to the consensus value is depicted in Figure 5-3. PPC is reached in both cases. The first case converges in about 10^{-2} s, whereas case 3 converges in about 10^{-3} s, indicating an improvement in response time of a factor 10.

5-3 Adaptive gain implementation

The four protocols from Section 4 are utilized to achieve the objectives of proportional power sharing and voltage stability. An additional objective for the edge-based cases is to preserve the value of the WGM of the voltages. The three cases of Table C-1 are considered, and simulation results are presented to verify the properties and compare the different cases.

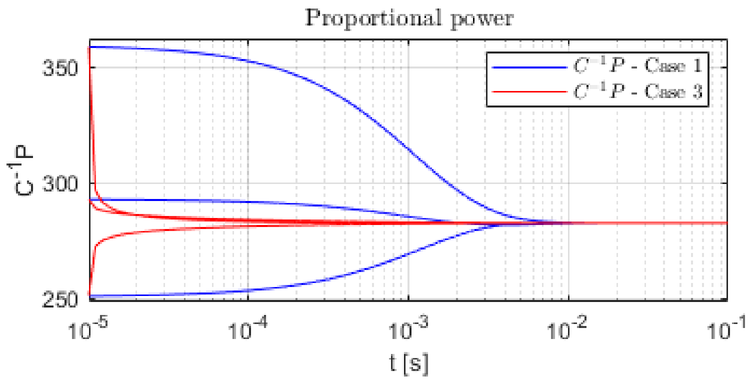

Figure 5-3: Proportional power - No adaptive gain

5-3-1 Edge-based protocol I

In this section, the general edge-based consensus protocol Eq. (4-5) is considered. The WGM and the source power function are implemented appropriately and proportional consensus is applied, to obtain the following adaptive gain power consensus protocol.

$$\forall i \in \mathcal{V}: \quad \dot{V}_i = C_i^{-1} V_i \sum_{j \in \mathcal{N}_{ci}} c_{ij} \hat{\phi}(\vartheta(C_j^{-1} P_j) - \vartheta(C_i^{-1} P_i)) \tag{5-11a}$$

$$\dot{c}_{ij} = \hat{\phi}(\vartheta(C_j^{-1} P_j) - \vartheta(C_i^{-1} P_i)) \Gamma \hat{\phi}(\vartheta(C_j^{-1} P_j) - \vartheta(C_i^{-1} P_i)) \tag{5-11b}$$

The voltage dynamics is a function of the local voltage and the local and neighbouring power measurements. The gain dynamics is quadratic and increases as a function of the difference in proportional power between connected nodes.

Conjecture 4 (Proportional power consensus - Edge-based protocol I)**.** *Assume that $\hat{\phi}(\cdot)$ is continuous, locally Lipschitz, odd and strictly increasing, $\vartheta(\cdot)$ is differentiable, $\vartheta(x_i)/dx_i$ is locally Lipschitz and strictly positive and $\Gamma > 0$. The edge-based PCA (Eq. (5-11)) achieves PPC among the sources of a DC microgrid according to power sharing coefficients $C_i > 0$ for all $i \in \mathcal{V}_s$, and the voltages are stabilized. Furthermore, the WGM of the source voltages is preserved if the adaptive gains are initialized such that $c_{ij}(0) = c_{ji}(0)$ for all $i, j \in \mathcal{V}_s$.*

The results are verified numerically. As in Section 5-2, the response for case 2 is too aggressive and is not considered. Figure 5-4 shows the evolution of the voltage of source node 3. Clearly, the voltage is stabilized. Again, the case where $\vartheta(x) = x^3 + x$ converges faster than the linear case. Figure 5-5 shows that the WGM is constant over time.

The evolution of the proportional power at each source node is depicted in Figure 5-6. Clearly, consensus on the proportional power is reached. It should be noted that the response of case 3 is very fast, and may be considered too aggressive to be implemented into a microgrid. The overly-aggressive response of the system is due to the dynamics of the adaptive gains c. The dynamics are quadratic and $\hat{\phi}(x) = x^3 + x$ is contains a cubic function. Thus, the cubic difference function $\hat{\phi}(x)$ is also squared, resulting in an huge increase of the adaptive gains. Figure 5-7 shows the increase of adaptive gains for case 1 and Figure 5-8 for case 3. For

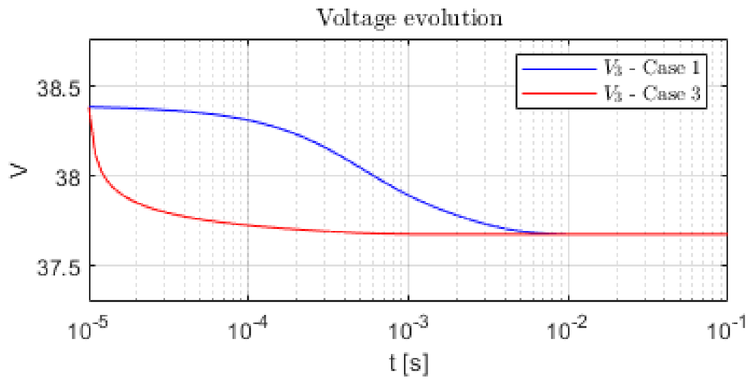

Figure 5-4: Voltage evolution - Edge-based I

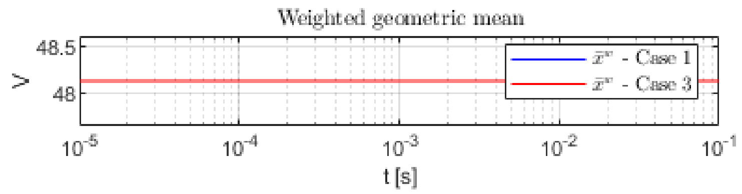

Figure 5-5: WGM - Edge-based I

case 1, the gains increase to a reasonable value of 4. However, for case 3, the gains increase almost to 300 within 10^{-7} s. This explains the aggressive response. Solutions to this issue are discussed in Chapter 6.

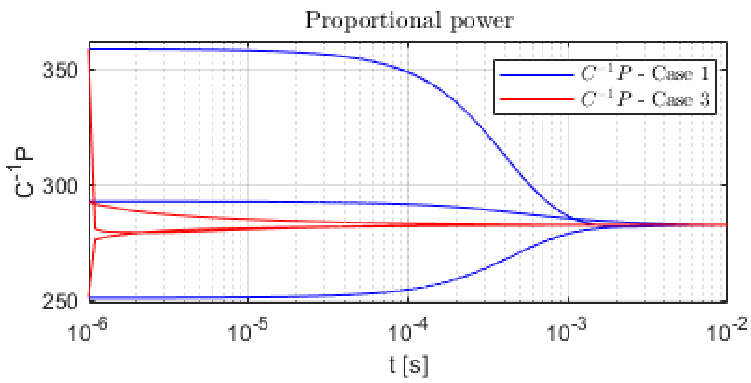

Figure 5-6: Proportional power - Edge-based I

5-3-2 Edge-based protocol II

The WGM and power function are implemented in Eq. (4-14). The three cases are considered and simulations are performed in the DC microgrid network. The consensus protocol is given below.

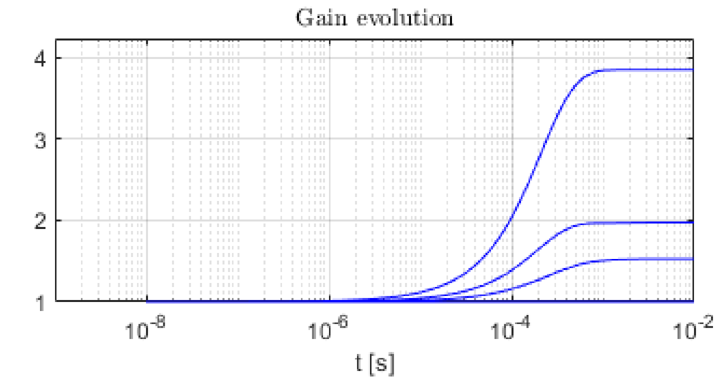

Figure 5-7: Adaptive gain - Edge-based I, case 1

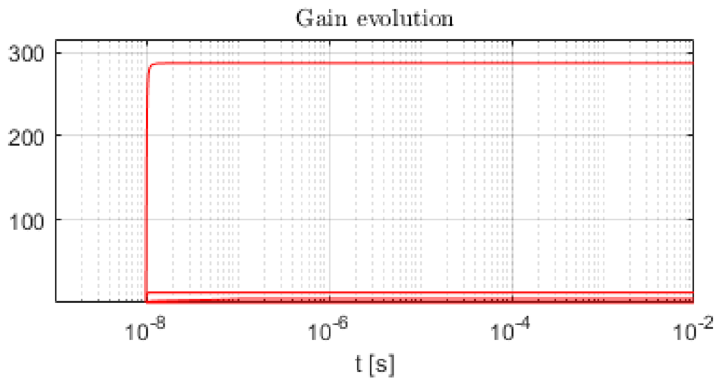

Figure 5-8: Adaptive gain - Edge-based I, case 3

$$\forall i \in \mathcal{V}: \quad \dot{V}_i = -C_i^{-1} P_i v_i \tag{5-12a}$$

$$\dot{v}_i = -v_i - \sum_{j \in \mathcal{N}_{ci}} c_{ij} \left[\hat{\phi}(\vartheta(C_j^{-1} P_j) - \vartheta(C_i^{-1} P_i)) - \hat{\phi}(\vartheta(v_j) - \vartheta(v_i)) \right] \tag{5-12b}$$

$$\dot{c}_{ij} = a_{ij} \begin{bmatrix} \hat{\phi}(\vartheta(C_j^{-1} P_j) - \vartheta(C_i^{-1} P_i)) \\ \hat{\phi}(\vartheta(v_j) - \vartheta(v_i)) \end{bmatrix}^\top \Gamma \begin{bmatrix} \hat{\phi}(\vartheta(C_j^{-1} P_j) - \vartheta(C_i^{-1} P_i)) \\ \hat{\phi}(\vartheta(v_j) - \vartheta(v_i)) \end{bmatrix}. \tag{5-12c}$$

The voltage dynamics is now a function of the auxiliary state v. The auxiliary state dynamics consist of a difference function of the auxiliary states and the proportional power. As stated in Section 4-2, the auxiliary states must be initialized such that the arithmetic mean $1/n \sum_{i \in \mathcal{V}} v_i(0) = 0$. An obvious choice is to set the initial values to zero. The gain dynamics is represented by a quadratic function of the difference function of auxiliary states and of the proportional power.

Conjecture 5 (Proportional power consensus - Edge-based protocol II)**.** *Assume that $\hat{\phi}(\cdot)$ is continuous, locally Lipschitz, odd and strictly increasing, $\vartheta(\cdot)$ is differentiable, $\vartheta(x_i)/dx_i$ is locally Lipschitz and strictly positive and $\Gamma > 0$. The edge-based PCA (Eq. (5-12)) achieves PPC among the sources of a DC microgrid according to power sharing coefficients $C_i > 0$ for all $i \in \mathcal{V}_s$, and the voltages are stabilized. Furthermore, the WGM of the source voltages is* *preserved if the adaptive gains are initialized such that* $c_{ij}(0) = c_{ji}(0)$ *and the auxiliary states are initialized such that the arithmetic mean* $\bar{v}(0) = 0$ *for all* $i, j \in \mathcal{V}_s$.

At first, all three cases of Table C-1 are considered. However, it immediately becomes evident that cases 2 and 3 are not suited for the application of PPC using Edge-based protocol II (EB2). The reason for this will become clear shortly. Protocol Eq. (5-12) is simulated for case 1 only.

The voltage evolution of source node 3 is depicted in Figure 5-9. The objective of voltage stabilization is achieved. However, the response oscillates around the steady-state value, an undesirable property for a DC microgrid. The WGM is preserved (Figure 5-10). Figure 5-11 shows that PPC is reached in about 2×10^{-1} s. Again, the proportional power oscillates before consensus is reached. Figure 5-13 depicts the evolution of the auxiliary states v and Figure 5-14 verifies that the arithmetic mean is preserved.

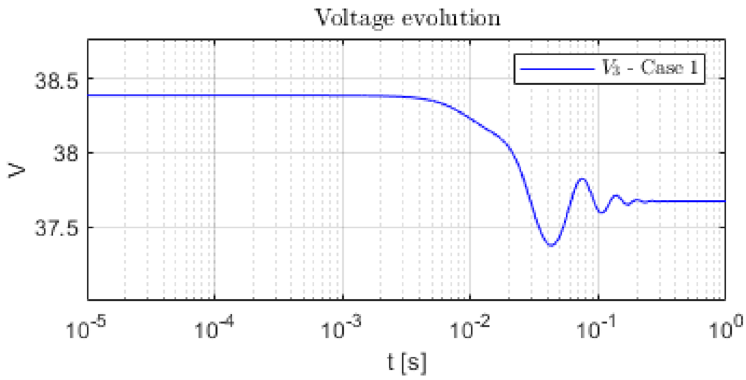

Figure 5-9: Voltage evolution - Edge-based II

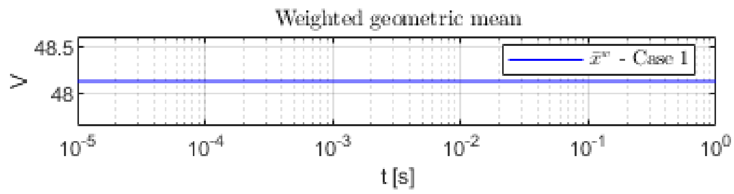

Figure 5-10: WGM - Edge-based II

The adaptive gain c increases to almost 120 (see Figure 5-12). This is a result of the inclusion of the auxiliary state v in the protocol. The voltage dynamics is a function of the auxiliary state. This results in an extra step in the protocol, which increases the response time. The protocol must first let the auxiliary state v increase from its initial value. When this is happening, the voltage dynamics slowly begins to react. This process takes some time, and all the while the adaptive gain is increasing. The protocol is generally slower than the protocols without auxiliary state, and this gives the adaptive gain more time to increase.

5-3-3 Node-based protocol I

Node-based protocol I (NB1) (Eq. (4-20)) is adjusted such that the objectives of PPC and voltage stability are achieved. In Section 4-3 it is shown that the AF is not preserved.

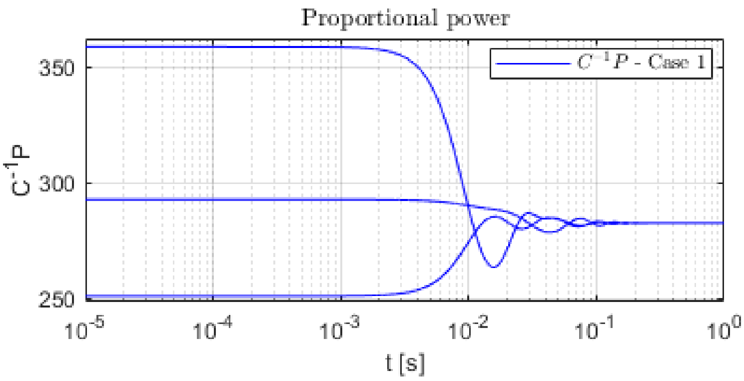

Figure 5-11: Proportional power - Edge-based II

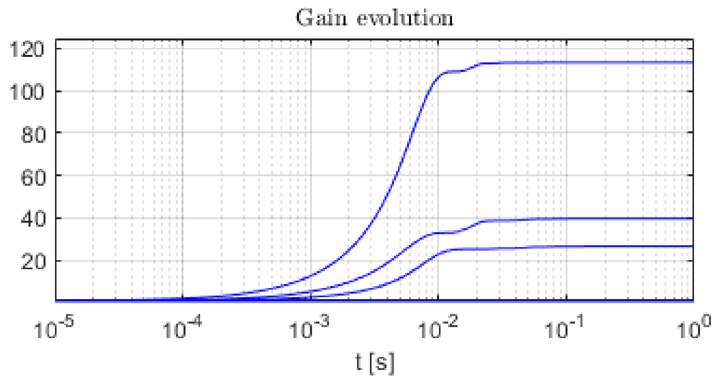

Figure 5-12: Adaptive gain - Edge-based II

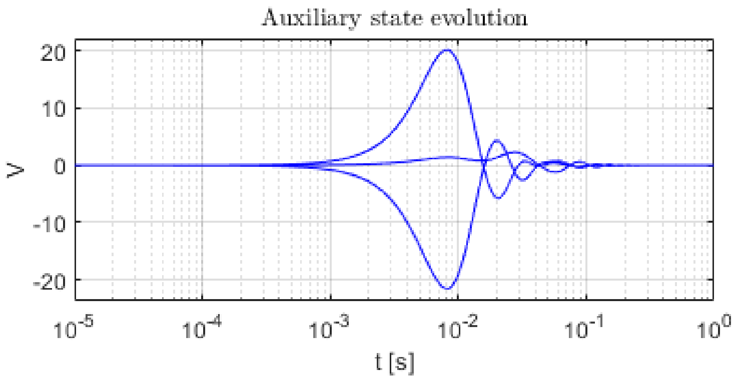

Figure 5-13: Auxiliary state evolution - Edge-based II

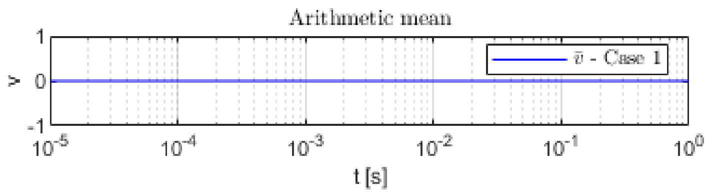

Figure 5-14: Auxiliary state arithmetic mean - Edge-based I

Nonetheless, the function $g_i = \log(V_i^{C_i})$ is implemented to allow for a fair comparison of the PPC protocols. The protocol is similar to Eq. (5-11) and differs in the implementation of the adaptive gains.

$$\forall i \in \mathcal{V}: \quad \dot{x}_i = C_i^{-1} P_i d_i \sum_{j \in \mathcal{N}_{ci}} \hat{\phi}(\vartheta(C_j^{-1} P_j) - \vartheta(C_i^{-1} P_i)) \tag{5-13a}$$

$$\dot{d}_i = (\sum_{j \in \mathcal{N}_i} \hat{\phi}(\vartheta(C_j^{-1} P_j) - \vartheta(C_i^{-1} P_i)))^2 \tag{5-13b}$$

The voltage dynamics is a function of the local voltage and the local and neighbouring power measurements. The gain dynamics is quadratic and increases as a function of sum of the difference in proportional power between neighbouring nodes.

Conjecture 6 (Proportional power consensus - Node-based protocol I)**.** *Assume that $\hat{\phi}(\cdot)$ is continuous, locally Lipschitz, odd and strictly increasing, $\vartheta(\cdot)$ is differentiable, $\vartheta(x_i)/dx_i$ is locally Lipschitz and strictly positive and $\Gamma > 0$. The node-based PCA (Eq. (5-13)) achieves PPC among the sources of a DC microgrid according to power sharing coefficients $C_i > 0$ for all $i \in \mathcal{V}_s$, and the voltages are stabilized.*

The simulations are performed for case 1 and case 3 of Table C-1. The response of case 2 is too aggressive and is not considered. Figure 5-15 shows that the voltage is stabilized. As expected, the WGM is not preserved. Figure 5-16 shows the evolution of the WGM of the two cases. Figure 5-17 shows the proportional source power over time. PPC is reached in about 10^{-3} s in case 1 and about 10^{-4} s in case 3. The consensus value is different for each case, in contrast to the edge-based cases, where the consensus value is equal for each case. The aggressive response of case 2 is due to the fast increase in adaptive gain, similarly to that of Edge-based protocol I (EB1). The presence of the cubic function $\vartheta(\cdot)$ in the gain dynamics causes the gains to increase to over 300 within 10^{-7} s.

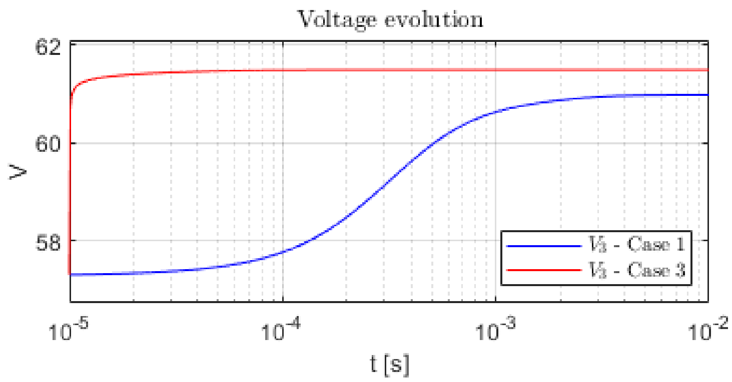

Figure 5-15: Voltage evolution - Node-based I

5-3-4 Node-based protocol II

Node-based protocol II (NB2) is utilized to achieve proportional power sharing and voltage stability. As before, the decomposed WGM $g_i = \log(V_i^{C_i})$ is implemented, although it is not

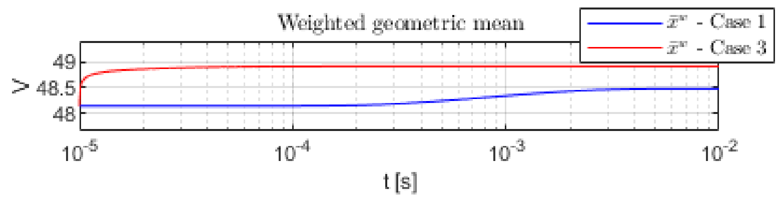

Figure 5-16: WGM - Node-based I

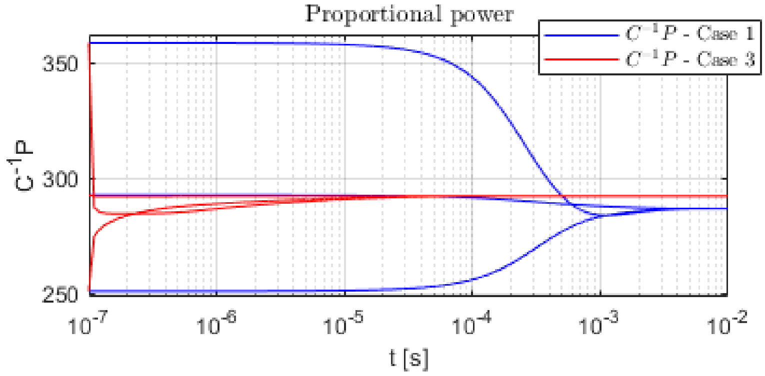

Figure 5-17: Proportional power - Node-based I

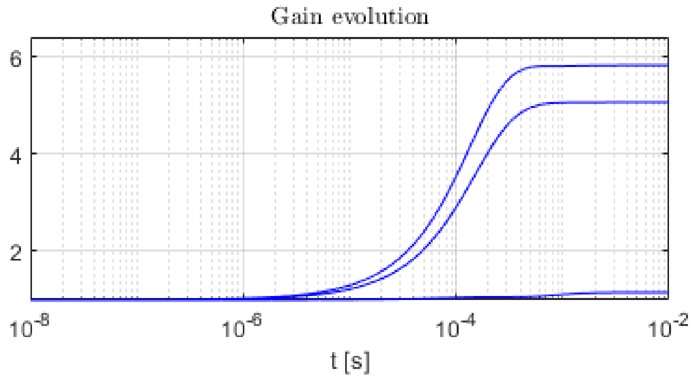

Figure 5-18: Adaptive gain - Node-based I, case 1

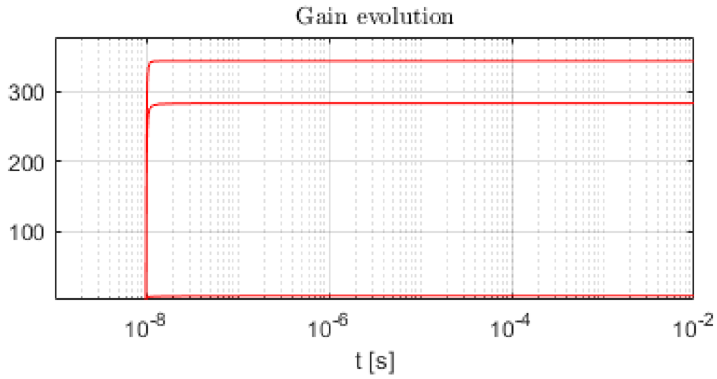

Figure 5-19: Adaptive gain - Node-based I, case 3

expected to be preserved. This is done so that a fair comparison of the protocols can be performed in Chapter 6.

$$\forall i \in \mathcal{V}: \quad \dot{x}_i = -C_i^{-1} P_i v_i \tag{5-14a}$$

$$\dot{v}_i = -v_i - d_i \sum_{j \in \mathcal{N}_{ci}} \left[\hat{\phi}(\vartheta(C_j^{-1} P_j) - \vartheta(C_i^{-1} P_i)) - \hat{\phi}(\vartheta(v_j) - \vartheta(v_i)) \right] \tag{5-14b}$$

$$\dot{d}_{ij} = \sum_{j \in \mathcal{V}} a_{ij} \begin{bmatrix} \hat{\phi}(\vartheta(C_j^{-1} P_j) - \vartheta(C_i^{-1} P_i)) \\ \hat{\phi}(\vartheta(v_j) - \vartheta(v_i)) \end{bmatrix}^\top \Gamma \sum_{j \in \mathcal{V}} a_{ij} \begin{bmatrix} \hat{\phi}(\vartheta(C_j^{-1} P_j) - \vartheta(C_i^{-1} P_i)) \\ \hat{\phi}(\vartheta(v_j) - \vartheta(v_i)) \end{bmatrix} \tag{5-14c}$$

The voltage dynamics is now a function of the auxiliary state v. The auxiliary state dynamics consist of a difference function of the auxiliary states and the proportional power. The gain dynamics are a quadratic function of the sum of proportional power difference, and a sum of the difference in auxiliary state.

Conjecture 7 (Proportional power consensus - Node-based protocol II)**.** *Assume that $\hat{\phi}(\cdot)$ is continuous, locally Lipschitz, odd and strictly increasing, $\vartheta(\cdot)$ is differentiable, $\vartheta(x_i)/dx_i$ is locally Lipschitz and strictly positive and $\Gamma > 0$. The edge-based PCA (Eq. (5-14)) achieves PPC among the sources of a DC microgrid according to power sharing coefficients $C_i > 0$ for all $i \in \mathcal{V}_s$, and the voltages are stabilized.*

As previously discussed in Section 5-3-2, the presence of the auxiliary state makes the adaptive gains increase to extremely high values when cubic functions $\hat{\phi}(\cdot)$ and $\vartheta(\cdot)$ are considered. Thus, only case 1 is considered. Simulations are performed in the DC microgrid network. The state trajectory of source node 1 is plotted in Figure 5-20. The protocol exhibits the slowest response of the four protocols, taking about 10 s to reach steady-state. The proportional power reaches consensus after about 10^{-1} s. However, the proportional power is not stabilized until about 5 s. This is because the auxiliary states v require more time to converge to zero. Since the voltage is a function of the auxiliary state, the system does not reach steady-state until the auxiliary states have converged to zero.

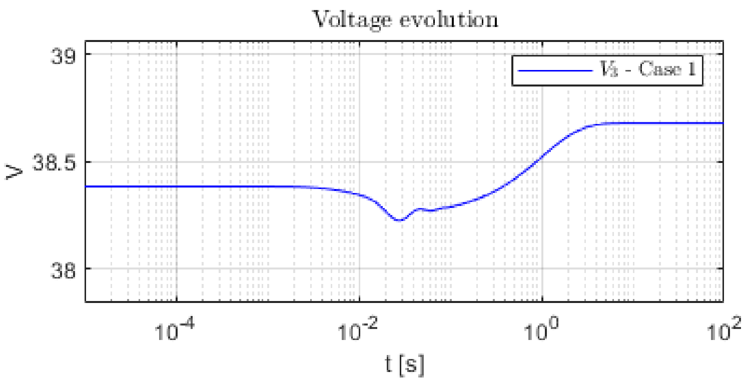

Figure 5-20: Voltage evolution - Node-based II

The evolution of the adaptive gain is shown in Figure 5-23. By the same reasoning as in Section 5-3-2, the adaptive gains increase to such high values because the protocol is slow,

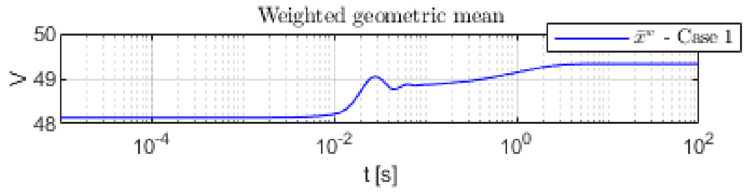

Figure 5-21: WGM - Node-based II

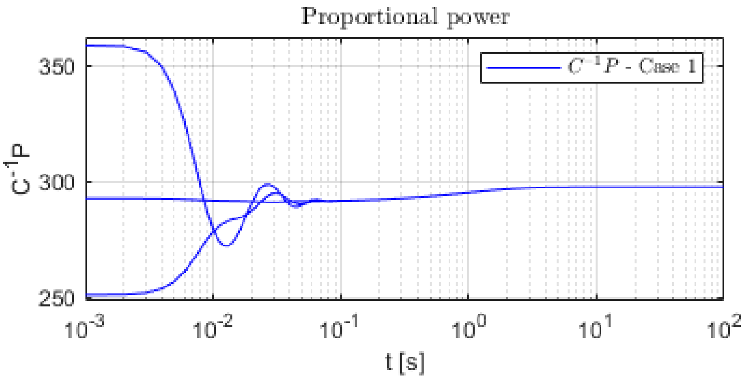

Figure 5-22: Proportional power - Node-based II

giving the adaptive gain more time to grow. The WGM of the voltages and the arithmetic mean of the protocol states are plotted in Figure 5-21 and Figure 5-25, respectively. As expected, the means are not preserved.

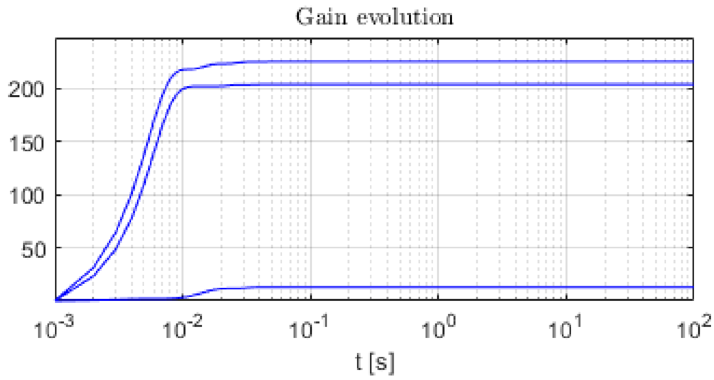

Figure 5-23: Adaptive gain - Node-based II

5-4 Voltage regulation

A minor addition to the thesis is the extension of the PCA to voltage regulation. An alternative PCA that achieves PPC and voltage regulation is proposed. The protocol introduces a source current dynamics term, $\hat{I}_s$, which ensures that PPC is reached. The controllable current is added to the source current term:

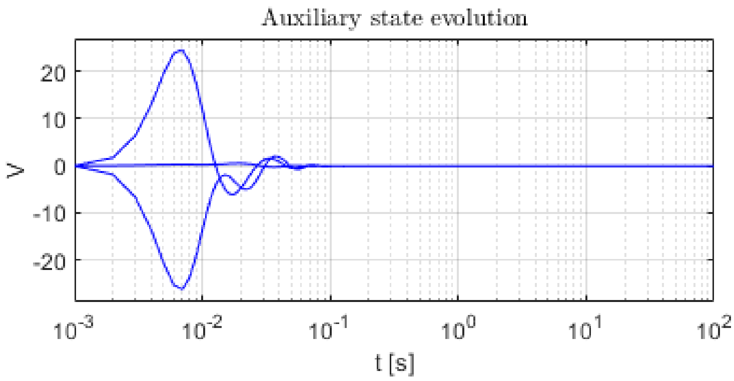

Figure 5-24: Auxiliary state evolution - Node-based II

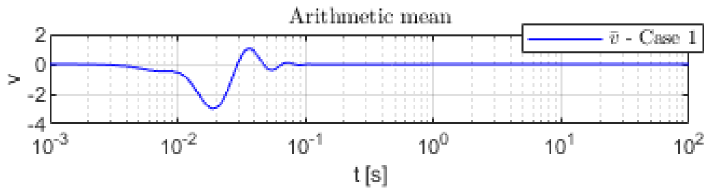

Figure 5-25: Auxiliary state arithmetic mean - Node-based I

$$\begin{bmatrix} I_s \\ I_l \end{bmatrix} = \begin{bmatrix} Y_{ss} & Y_{sl} \\ Y_{ls} & Y_{ll} \end{bmatrix} \begin{bmatrix} V_s \\ V_l \end{bmatrix} + \begin{bmatrix} \hat{I}_s \\ 0 \end{bmatrix} \tag{5-15}$$

The voltage dynamics steer the voltage to the reference value. The protocol is constructed as follows.

$$\dot{\hat{I}}_i = C_i V_i \sum_{j \in \mathcal{N}_{ci}} \hat{\phi}(\vartheta(C_j^{-1} P_j) - \vartheta(C_i^{-1} P_i)) \tag{5-16a}$$

$$\dot{V}_i = \tau(V_i^{\text{ref}} - V_i) \tag{5-16b}$$

The results are simulated in the DC microgrid network described in Appendix C-2. The reference voltages are set at $V^{\text{ref}} = \begin{bmatrix} 40 & 45 & 50 \end{bmatrix}^\top$. The case where $\hat{\phi}(x) = x$ and $\vartheta(x) = x$ is considered. The voltage gain τ is set to $\tau = 100$, because the voltage dynamics are much slower than the current dynamics.

Figure 5-26 shows that the voltages converge to their reference values. Figure 5-27 depicts the trajectories of the controllable current term $\hat{I}_s$, and Figure 5-28 shows the evolution of the source currents. Clearly, both are stable. Finally, Figure 5-29 shows that the source power is shared proportionally. The given voltage regulation and power sharing protocol serves as an introduction into voltage regulation. The further development and investigation of implementability is left for future research. Generally, controlling both the current and the voltage is problematic.

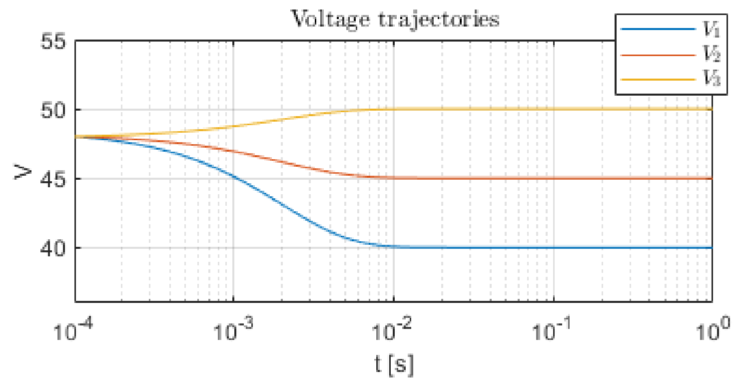

Figure 5-26: Voltage trajectories

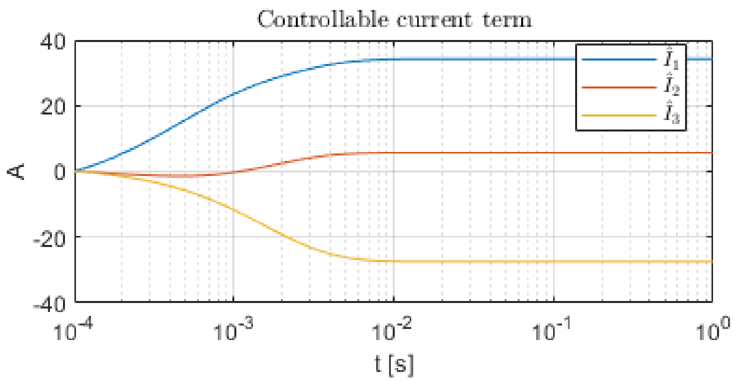

Figure 5-27: Controllable current term

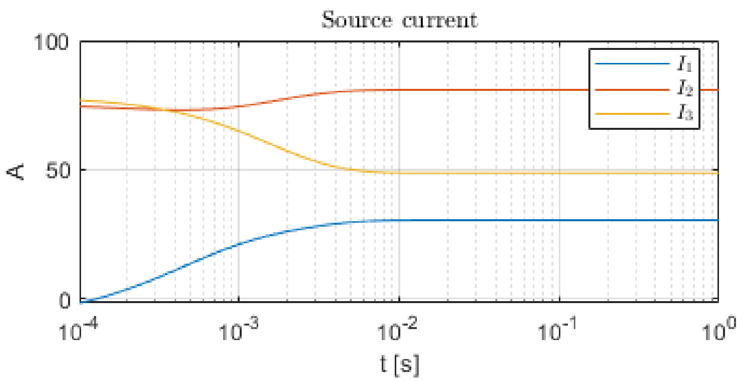

Figure 5-28: Current trajectories

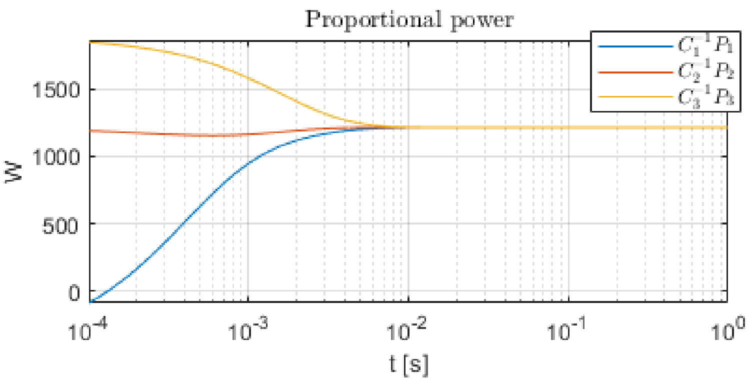

Figure 5-29: Proportional power

Chapter 6

Discussion and evaluation

6-1 Power consensus algorithms evaluation

Five alternative Power consensus algorithm (PCA)s have been proposed in this chapter. The protocols are presented in such a way that there is plenty of room for the implementation of nonlinear difference functions. To compare the five protocols, the case where $\hat{\phi}(\cdot)$ and $\vartheta(\cdot)$ are represented by the identity function are considered. The response time, overshoot and time-invariance of the Weighted geometric mean (WGM) shall be taken as measures of the effectiveness of each protocol.

The response time is defined as the time it takes to reach Proportional power consensus (PPC) and voltage stability. The results follow from the simulation in Section 5-2 and Section 5-3 and are summarized in Table 6-1. Node-based protocol I (NB1) has the fastest response time, with Edge-based protocol I (EB1) coming in second place. Comparing Figure 5-18 to Figure 5-7, we see that the adaptive gains increase to values up to 6 for the node-based case, whereas with the edge-based case this is only 4. This explains why the node-based case responds faster. This observation allows us to conclude that a node-based adaptive gain tends to increase faster. This makes sense, because the gain dynamics sum the difference between all neighbouring nodes, instead of just of one as is the case for the edge-based case, potentially resulting in a greater value. However, the node-based case does not preserve the value of the WGM.

Edge-based protocol II (EB2) and Node-based protocol II (NB2), the cases which implement an auxiliary state v, have the worst properties. First of all, they are slower than the original PCA that does not implement an adaptive gain. Furthermore, they exhibit an overshoot.

6-1-1 Adaptive gain response

In electrical networks, the value of the source power can become large. The implementation of gain dynamics as a quadratic function of the difference in power has the risk of resulting in very fast gain increases, and, as a result, aggressive system responses. Especially when considering

	Response time (s)	Overshoot	Time-invariance
No adaptive gain	10^{-2}	Yes	Yes
EB1	$5 \cdot 10^{-3}$	No	Yes
EB2	$5 \cdot 10^{-1}$	Yes	Yes
NB1	$4 \cdot 10^{-3}$	No	No
NB2	5	Yes	No

Table 6-1: Response times of PCAs

cubic terms in the difference function the effect becomes amplified. This is visualized in Figure 5-12 and Figure 5-23. Therefore, combining adaptive gains with cubic difference functions with the application of power consensus in DC microgrids is not advisable. However, choosing one of the two shows fast and smooth results without overshoot.

6-1-2 Auxiliary state implementation

The implementation of auxiliary states in EB2 and NB2 results in a slow response with oscillations around the equilibrium. This is because the auxiliary state v needs to grow from its initial value 0 before the state responds. This causes a delay in the system response. Nonetheless, this may be a desirable property for applications where a sudden, fast change in state should be avoided.

Chapter 7

Conclusion and recommendation

7-1 Conclusion

The objective of this thesis is to build a bridge between nonlinear consensus theory and the Power consensus algorithm (PCA) from [11] and to implement adaptive gain into Nonlinear consensus protocol (NCP)s. These results are then implemented to achieve Proportional power consensus (PPC) in microgrids. In this section, a discussion on the extent to which the objectives have been achieved is performed.

7-1-1 Nonlinear consensus theory

The theory on NCPs has been extended to meet the properties of the PCA [11]. The theory from the literature is limited to protocol which reach state consensus. The PCA, however, does not reach consensus on the states, but on a function of the states, namely the power. Furthermore, the objective is to share the power proportionally. Thus, the first step in the extension of nonlinear consensus theory is to find conditions which allow for proportional consensus to be reached. In the thesis, conditions for proportional state consensus are found and proved analytically. This is then extended to proportional consensus on some function $f(\cdot)$ (generalized consensus). Necessary and sufficient conditions for generalized consensus are not found. Nonetheless, an investigation on the equilibrium points is performed and numerical simulations show that the general consensus protocol is capable of achieving its objective.

The PCA has the property that the Weighted geometric mean (WGM) is preserved. The theory on Agreement function (AF) is limited to the arithmetic mean, geometric mean, harmonic mean and mean of order p. The WGM is introduced and conditions for WGM preservation are given. With these results, the bridge from the existing nonlinear consensus theory to the PCA is successfully built, although some results are left as conjecture.

7-1-2 Adaptive gain consensus protocols

The adaptive gain consensus protocols from the literature are limited to linear difference functions, and no results on AF preservation are given. In the thesis, three protocols from the literature are extended to include nonlinear difference functions, and an additional node-based consensus protocol is proposed. Furthermore, conditions to preserve an AF are given. An interesting result is that a node-based adaptive gain protocol is not capable of guaranteeing preservation of the of the AF, whereas this is possible with an edge-based adaptive gain. Not only the structure of the protocols, but also the initial conditions of the adaptive gain and auxiliary state play a roll in the preservation of the AF.

7-1-3 Proportional power consensus

The general NCP, without adaptive gain, and the four adaptive gain consensus protocols are applied to the problem of PPC in DC microgrids. By allowing for nonlinear terms in the difference function, a great improvement in response time is observed. This is further improved by the implementation of adaptive gains for the protocols without auxiliary state. The protocols with auxiliary state, however, are much slower than the original power consensus protocol [11], and therefore less suited for the considered application. Furthermore, it is found that combining adaptive gains with cubic difference functions results in overly aggressive response times, due to the large values of power in DC microgrids. Edge-based protocol I (EB1) and Node-based protocol I (NB1) are capable of quickly regulating the power, and are promising protocols to consider for the regulation of power consensus in DC microgrids.

7-2 Recommendation

Performing a theoretical thesis constantly opens the door to new interesting research topics. Once one result is achieved, the next question presents itself. In this section, the open problems and future research topics encountered throughout the thesis are discussed.

Finding conditions for general consensus of the NCP and the four adaptive gain protocols is an open problem. The results in Section 3-2-1 serve as a basis for this investigation. If successful, these results can be used to prove power consensus of the adaptive gain power consensus protocols. Alternatively, one can prove consensus and stability of the applied power consensus case directly by considering a Bregman storage function [13, 11] approach.

The proposed adaptive gain consensus protocols implement the same difference function for the (auxiliary) state dynamics and the gain dynamics. A separation of these functions will lead to more freedom in the design of the protocols. As observed in Section 5-3, the adaptive gains increase too fast in some cases. By separating the difference function of the state and the gain dynamics, the gain dynamics can be constructed in such a way that they grow more slowly, or even contain an upper bound. To give an example, EB1 can be extended to the following protocol.

$$\forall i \in \mathcal{V}: \quad \dot{x}_i = (g_i'(x_i))^{-1} \sum_{j \in \mathcal{N}_i} c_{ij} \hat{\phi}_x(\vartheta_x(x_j) - \vartheta_x(x_i)) \tag{7-1}$$

$$\dot{c}_{ij} = \hat{\phi}_c(\vartheta_c(x_j) - \vartheta_c(x_i))^\top \Gamma \hat{\phi}_c(\vartheta_c(x_j) - \vartheta_c(x_i)) \tag{7-2}$$

To reduce the rate of increase of the gain, one could let $\hat{\phi}_c(x) = x$ and $\vartheta_c(x) = x$ and $\hat{\phi}_x(x) = x^3 + x$ and $\vartheta_x(x) = x^3 + x$. This way, the state dynamics respond quickly, while the adaptive gain is kept at a reasonable level.

Finally, the algorithms developed in Chapter 4 lend themselves to all kinds of applications. The control engineer is left with a lot of freedom to design the protocols to meet the demands of their specific case, thus allowing the protocols to be implemented in a variety of engineering problems.

Appendix A

Proofs

A-1 Proof of Theorem 1

Proof. The proof follows a similar logic to [6, Theorem 1, 2]. For proportional state consensus, we require that (3-8) holds. That is, the equilibrium $x^* = \lambda w$ for some $\lambda \in \mathbb{R}$. This is equivalent to $z^* = \lambda \mathbf{1}$ with $z = \mathcal{W}^{-1}x$, where $\mathcal{W} = \text{diag}(w_1, \dots, w_n)$. Assume $z_i = \lambda \mathbf{1}$ for all $i \in \mathcal{V}$. Then, we have $\hat{\phi}(z_j - z_i) = \hat{\phi}(\lambda - \lambda) = 0$. Since $\hat{\phi}$ is continuous and odd, it follows that $\dot{x}_i = 0$. Thus, $z^* = \lambda \mathbf{1}$ is an equilibrium point. Next, we show that the equilibrium point is unique.

Assume that there exists an equilibrium point $z^* \neq \lambda \mathbf{1}$. Let $\mathcal{I}$ represent the set of nodes i whose proportional equilibrium states z_i^* are greater than or equal to those of all nodes $j \in \mathcal{V}$. That is, $\mathcal{I} = \{i \in \mathcal{V} : z_i^* \geq z_j^*,\ \forall j \in \mathcal{V}\}$. We note that $\mathcal{I} \subset \mathcal{V}$. In fact, if $\mathcal{I} = \mathcal{V}$, then $z_i \geq z_j$ for all $i, j \in \mathcal{V}$, from which it follows that $z_i = z_j$ for all $i, j \in \mathcal{V}$. Thus, $z^* = \lambda \mathbf{1}$, which is a contradiction. This allows us to select $i \in \mathcal{I}$ such that $z_i^* > z_j^*$ for some $j \in \mathcal{V}$ and $z_i^* \geq z_k^*$ for all $k \in \mathcal{V}$. The resulting sum $\sum_{j \in \mathcal{N}_i} \hat{\phi}(z_j - z_i) < 0$. Since, $z_i > 0$, it follows that $\dot{x}_i < 0$. Therefore, $z^* \neq \lambda \mathbf{1}$ is not an equilibrium point, and we have shown that $z^* = \lambda \mathbf{1}$ is a unique equilibrium point.

Next, we show that the states reach proportional consensus. This is achieved when $w_i^{-1} x_i = z_i = \chi(z_0)$ for all $i \in \mathcal{V}$. Then, since $g_i(x_i)$ is strictly increasing, we define $e := g_i(z_i) - g_i(\chi(z_0)) = 0$ as the consensus error function. Then, proportional consensus is equivalent to proving asymptotic stability of the point $e = 0$ with

$$\dot{e}_i = \dot{g}_i(z_i) = \dot{g}_i(w_i^{-1} x_i) = g_i'(w_i^{-1} x_i)\dot{x}_i = w_i^{-1} g_i'(x_i)\dot{x}_i. \quad \text{(A-1a)}$$

The assumption that $g_i'(cx_i) = c g_i'(x_i)$ is used above.

Consider the candidate Lyapunov function $V(e) = \frac{1}{2}\sum_{i\in\mathcal{V}} w_i e_i^2$. Clearly, $V(0) = 0$ and $V(e) > 0$ for all $e \neq 0$. The time-derivative of the Lyapunov function is given by

$$\begin{aligned}
\dot{V} &= \sum_{i\in\mathcal{V}} w_i e_i \dot{e}_i && \text{(A-2a)}\\
&= \sum_{i\in\mathcal{V}} w_i e_i w_i^{-1} g_i'(x_i)\dot{x}_i && \text{(A-2b)}\\
&= \sum_{i\in\mathcal{V}} w_i e_i w_i^{-1} g_i'(x_i)(g_i'(x_i))^{-1} \sum_{j\in\mathcal{N}_i} \hat{\phi}(\vartheta(z_j) - \vartheta(z_i)) && \text{(A-2c)}\\
&= \sum_{i\in\mathcal{V}} e_i \sum_{j\in\mathcal{N}_i} \hat{\phi}(\vartheta(z_j) - \vartheta(z_i)) && \text{(A-2d)}\\
&= \sum_{i\in\mathcal{V}} g_i(z_i) - g_i(\chi(z(0))) \sum_{j\in\mathcal{N}_i} \hat{\phi}(\vartheta(z_j) - \vartheta(z_i)) && \text{(A-2e)}\\
& && \text{(A-2f)}
\end{aligned}$$

The term $g_i(\chi(z(0))$ cancels because it is constant and the network in connected and undirected.

$$\begin{aligned}
\dot{V} &= \sum_{i\in\mathcal{V}} g_i(z_i) \sum_{j\in\mathcal{N}_i} \hat{\phi}(\vartheta(z_j) - \vartheta(z_i)) && \text{(A-3a)}\\
&= -\sum_{(i,j)\in\mathcal{E}} (g_i(z_j) - g_i(z_i))\hat{\phi}(\vartheta(z_j) - \vartheta(z_i)) && \text{(A-3b)}
\end{aligned}$$

The final step uses the fact that $i \in \mathcal{N}_j$ if and only if $j \in \mathcal{N}_i$. This allows Eq. (A-3a) to be written as Eq. (A-3b). Since $g_i(\cdot)$, $\hat{\phi}(\cdot)$ and $\vartheta(\cdot)$ are strictly increasing, we find that $\dot{V} \leq 0$, and $\dot{V} = 0$ if and only if $z_i = z_j$ for all $i, j \in \mathcal{V}$, implying that $e_i = 0$ for all $i \in \mathcal{V}$. □

A-2 Proof of Lemma 1

Proof. Clearly,

$$\begin{aligned}
c_{ij}\hat{\phi}(\vartheta(x_j) - \vartheta(x_i)) &= c_{ji}\hat{\phi}(\vartheta(x_j) - \vartheta(x_i))\\
&= -c_{ji}\hat{\phi}(\vartheta(x_i) - \vartheta(x_j)),
\end{aligned}$$

where the final equality follows from anti-symmetry of $\hat{\phi}(\vartheta(x_j) - \vartheta(x_i))$. This shows that $c_{ij}\hat{\phi}(\vartheta(x_j) - \vartheta(x_i))$ is anti-symmetric. Furthermore,

$$\begin{aligned}
\dot{c}_{ij} &= \hat{\phi}(\vartheta(x_j) - \vartheta(x_i))^\top \Gamma \hat{\phi}(\vartheta(x_j) - \vartheta(x_i))\\
&= (-\hat{\phi}(\vartheta(x_i) - \vartheta(x_j))^\top \Gamma(-\hat{\phi}(\vartheta(x_i) - \vartheta(x_j))\\
&= \hat{\phi}(\vartheta(x_i) - \vartheta(x_j))^\top \Gamma \hat{\phi}(\vartheta(x_i) - \vartheta(x_j))\\
&= \dot{c}_{ji}
\end{aligned}$$

□

A-3 Proof of Theorem 2

Proof. State consensus is reached when $\tilde{e}_i = x_i - \chi(x(0)) = 0$ for all $i \in \mathcal{V}$, where $\chi(x(0))$ represents the consensus value. Since $g_i(\cdot)$ is strictly increasing, we may say that consensus is reached when $e_i = g_i(x_i) - g_i(\chi(x(0))) = 0$. Thus, proving consensus of the states is equivalent to proving asymptotic stability of the $e = 0$. We have

$$\dot{e}_i = g_i'(x_i)\dot{x}_i = \sum_{j \in \mathcal{N}_i} c_{ij}\hat{\phi}(\vartheta(x_j) - \vartheta(x_i))$$

Consider the candidate Lyapunov function $V = \frac{1}{2}\sum_{i \in \mathcal{V}} e_i^2$. Clearly, $V(0) = 0$ and $V(e) > 0$ if $e \neq 0$. The time-derivative is determined next.

$$\dot{V} = \sum_{i \in \mathcal{V}} e_i \dot{e}_i \tag{A-4}$$

$$= \sum_{i \in \mathcal{V}} e_i \sum_{j \in \mathcal{N}_i} c_{ij}\hat{\phi}(\vartheta(x_j) - \vartheta(x_i)) \tag{A-5}$$

$$= \sum_{i \in \mathcal{V}} (g_i(x_i) - g_i(\chi(x(0)))) \sum_{j \in \mathcal{N}_i} c_{ij}\hat{\phi}(\vartheta(x_j) - \vartheta(x_i)) \tag{A-6}$$

Since $g_i(\chi(x(0)))$ is constant and the network is connected and undirected, the term cancels.

$$\dot{V} = \sum_{i \in \mathcal{V}} g_i(x_i) \sum_{j \in \mathcal{N}_i} c_{ij}\hat{\phi}(\vartheta(x_j) - \vartheta(x_i)) \tag{A-7}$$

$$= -\sum_{(i,j) \in \mathcal{E}} c_{ij}(g_i(x_j) - g_i(x_i))\hat{\phi}(\vartheta(x_j) - \vartheta(x_i)) \tag{A-8}$$

The final step uses the fact that $i \in \mathcal{N}_j$ if and only if $j \in \mathcal{N}_i$. This allows Eq. (A-7) to be written as Eq. (A-8). The adaptive gains c_{ij} initially have positive values, which remain positive due to the quadratic form of the gain dynamics. Then, since $g_i(\cdot)$, $\hat{\phi}(\cdot)$ and $\vartheta(\cdot)$ are odd and strictly increasing, it follows that each product is positive. Hence, $\dot{V} \leq 0$. Finally, we have that $\dot{V} = 0$ implies that $x_i = x_j$ for all $i, j \in \mathcal{V}$. Thus, $e_i = 0$ for all $i \in \mathcal{V}$, and we have shown that the system reaches consensus. □

A-4 Proof of Theorem 3

Proof. It suffices to show that the argument $\sum_{i \in \mathcal{V}} g_i(x_i)$ of the Agreement function (AF) is constant.

$$\sum_{i \in \mathcal{V}} \dot{g}_i(x_i) = \sum_{i \in \mathcal{V}} g_i'(x_i)\dot{x}_i = -\sum_{i \in \mathcal{V}} v_i \tag{A-9}$$

Thus, the AF is time-invariant when $\sum_{i\in\mathcal{V}} v_i(t) = 0$ for all $t \geq 0$. Clearly, the protocol state must be initialized such that $\sum_{i\in\mathcal{V}} v_i(0) = 0$. Next, we show that $\sum_{i\in\mathcal{V}} v_i(t) = 0$ for all $t > 0$.

$$\begin{aligned}\frac{d\sum_{i\in\mathcal{V}} v_i}{dt} &= \sum_{i\in\mathcal{V}} \frac{dv_i}{dt} \\ &= \sum_{i\in\mathcal{V}} \dot{v}_i = \sum_{i\in\mathcal{V}} (v_i - \sum_{i\in\mathcal{N}_i} c_{ij}\phi) \\ &= \sum_{i\in\mathcal{V}} v_i - \sum_{i\in\mathcal{V}} \sum_{i\in\mathcal{N}_i} c_{ij}\phi = \sum_{i\in\mathcal{V}} v_i\end{aligned}$$

The final equality results from anti-symmetry of ϕ and connectedness of $\mathcal{G}$. Define $\hat{v} := \sum_{i\in\mathcal{V}} v_i$. The solution of $\hat{v}$ then follows from solving the differential equation $\dot{\hat{v}} - \hat{v} = 0$ with initial value $\hat{v} = 0$. This results in $\hat{v} = \sum_{i\in\mathcal{V}} v_i = 0$, showing that the argument of the AF is constant and subsequently that the value of the AF is preserved. □

A-5 Proof of Theorem 4

Proof. Define $\tilde{e}_i := x_i - \bar{x}$, with $\bar{x}$ the arithmetic mean of the states, as the error function for reaching consensus such that $\tilde{e}_i = 0$ implies that agent i equals the consensus value. Since $g_i(x_i)$ is strictly increasing, an alternative error function is $e_i := g_i(x_i) - g_i(\bar{x})$. Asymptotic stability of e is equivalent to consensus reaching of the states x. The Lyapunov direct method is applied to prove asymptotic stability of

$$\dot{e}_i = g_i'(x_i)\dot{x}_i = d_i \sum_{j\in\mathcal{N}_i} \hat{\phi}(\vartheta(x_j) - \vartheta(x_i)) \tag{A-10a}$$

Consider the Lyapunov candidate function $V = \frac{1}{2}e^\top \mathcal{L} e$ where $\mathcal{L}$ is the symmetric Laplacian matrix of the connected and undirected graph. The Lyapunov function satisfies $V(0) = 0$ and $V(e) > 0$ for all $(e) \neq (0)$. The time-derivative is then given by

$$\dot{V} = e^\top \mathcal{L}\dot{e} \tag{A-11a}$$

$$= \begin{bmatrix} g_i(x_1) - g_i(\bar{x}) \\ \vdots \\ g_i(x_n) - g_i(\bar{x}) \end{bmatrix}^\top \mathcal{L} \begin{bmatrix} d_1 \sum_{j\in\mathcal{N}_1} \hat{\phi}(\vartheta(x_j) - \vartheta(x_1)) \\ \vdots \\ d_n \sum_{j\in\mathcal{N}_n} \hat{\phi}(\vartheta(x_j) - \vartheta(x_n)) \end{bmatrix} \tag{A-11b}$$

$$= \begin{bmatrix} \sum_{j\in\mathcal{N}_1} g_i(x_1) - g_i(x_j) \\ \vdots \\ \sum_{j\in\mathcal{N}_n} g_i(x_n) - g_i(x_j) \end{bmatrix}^\top \begin{bmatrix} d_1 \sum_{j\in\mathcal{N}_1} \hat{\phi}(\vartheta(x_j) - \vartheta(x_1)) \\ \vdots \\ d_n \sum_{j\in\mathcal{N}_n} \hat{\phi}(\vartheta(x_j) - \vartheta(x_n)) \end{bmatrix} \tag{A-11c}$$

$$= - \begin{bmatrix} \sum_{j\in\mathcal{N}_1} g_i(x_j) - g_i(x_1) \\ \vdots \\ \sum_{j\in\mathcal{N}_n} g_i(x_j) - g_i(x_n) \end{bmatrix}^\top \begin{bmatrix} d_1 \sum_{j\in\mathcal{N}_1} \hat{\phi}(\vartheta(x_j) - \vartheta(x_1)) \\ \vdots \\ d_n \sum_{j\in\mathcal{N}_n} \hat{\phi}(\vartheta(x_j) - \vartheta(x_n)) \end{bmatrix} \tag{A-11d}$$

The adaptive gain is initialized such that $d_i > 0$ for all $i, j \in \mathcal{V}$. The gain dynamics $\dot{d}_i$ are quadratic, implying that $d_i(t) > 0$ for all $t > 0$. The functions $g_i(\cdot)$, $\hat{\phi}(\cdot)$ and $\vartheta(\cdot)$ are strictly increasing and odd. Thus, each product resulting from A-11d results in a positive value, from which it follows that $\dot{V} \leq 0$. From A-11d it is clear that $\dot{V} = 0$ implies that $x_i = x_j$ for all $i, j \in \mathcal{V}$, which in turn implies that $e_i = 0$ for all $i \in \mathcal{V}$. □

A-6 Proof of Theorem 5

Proof. The arithmetic mean $\bar{x}$ is defined as $\bar{x} = \frac{1}{n}\sum_{i\in\mathcal{V}} x_i$. The time-derivative of $\bar{x}$ along the state trajectories x_i of agents i is given by

$$\begin{aligned}\frac{d\bar{x}}{dt} &= \sum_{i\in\mathcal{V}} \frac{d\bar{x}}{dx_i}\dot{x}_i = \sum_{i\in\mathcal{V}} \frac{d\frac{1}{n}\sum_{j\in\mathcal{V}} x_j}{dx_i}\dot{x}_i = \sum_{i\in\mathcal{V}} n\frac{1}{n}\dot{x}_i \\ &= \sum_{i\in\mathcal{V}} \dot{x}_i = \sum_{i\in\mathcal{V}}(-x_i + \sum_{j\in\mathcal{N}_i} \phi(x_j, x_i)) \\ &= -\sum_{i\in\mathcal{V}} x_i + \sum_{i\in\mathcal{V}}\sum_{j\in\mathcal{N}_i} \phi(x_j, x_i)\end{aligned}$$

Due to connectivity and undirectedness of the graph, the second term equals zero. Then, since the states are initialized such that the arithmetic mean equals zero, it follows that

$$\frac{d\bar{x}(0)}{dt} = -\sum_{i\in\mathcal{V}} x_i(0) = 0 \tag{A-13}$$

from which it follows that the time-derivative of $\bar{x}$ will remain at zero. Hence, the arithmetic mean $\bar{x}$ remains at zero. □

Appendix B

Agreement functions

For completeness, the Agreement function (AF) from [6] and their decomposition are given. The considered AFs are the arithmetic mean, geometric mean, harmonic mean and mean of order p.

B-1 Alternative agreement functions

B-1-1 Arithmetic mean

$$\chi(x) = \sum_{i \in \mathcal{V}} \frac{1}{n} x_i \tag{B-1a}$$

$$= f(\sum_{i=1}^{n} g(x_i)) \tag{B-1b}$$

$$f(y) = \frac{1}{n} y \tag{B-1c}$$

$$g(x_i) = x_i \tag{B-1d}$$

$$g'(x_i) = 1 \tag{B-1e}$$

B-1-2 Geometric mean

$$\chi(x) = (\prod_{i\in\mathcal{V}} x_i)^{1/n} \quad \text{(B-2a)}$$
$$= f(\sum_{i=1}^{n} g(x_i)) \quad \text{(B-2b)}$$
$$f(y) = e^{1/ny} \quad \text{(B-2c)}$$
$$g(x_i) = \log x_i \quad \text{(B-2d)}$$
$$g'(x_i) = \frac{1}{x_i} \quad \text{(B-2e)}$$

B-1-3 Harmonic mean

$$\chi(x) = \frac{1}{\sum_{i\in\mathcal{V}} \frac{n}{x_i}} \quad \text{(B-3a)}$$
$$= f(\sum_{i=1}^{n} g(x_i)) \quad \text{(B-3b)}$$
$$f(y) = \frac{n}{y} \quad \text{(B-3c)}$$
$$g(x_i) = \frac{1}{z} \quad \text{(B-3d)}$$
$$g'(x_i) = -\frac{1}{z^2} \quad \text{(B-3e)}$$

B-1-4 Mean of order p

$$\chi(x) = (\prod_{i\in\mathcal{V}} \frac{1}{n} x_i)^{1/p} \quad \text{(B-4a)}$$
$$= f(\sum_{i=1}^{n} g(x_i)) \quad \text{(B-4b)}$$
$$f(y) = (\frac{1}{n} y)^{1/p} \quad \text{(B-4c)}$$
$$g(x_i) = z^p \quad \text{(B-4d)}$$
$$g'(x_i) = pz^{(p-1)} \quad \text{(B-4e)}$$

B-2 Alternative time-invariant protocol

An alternative consensus protocol which preserves the value of the arithmetic mean is proposed in the following theorems.

Theorem 5. *Consider a network of agents described by a connected and undirected graph. Assuming that the arithmetic mean of the initial states equals zero, the following consensus protocol preserves the arithmetic mean of the initial states if ϕ is anti-symmetric.*

$$\dot{x}_i = -x_i + \sum_{j \in \mathcal{N}_i} \phi(x_j, x_i) \tag{B-5}$$

Proof. See Appendix A-6. □

Appendix C

Simulation environment

Three cases are considered in the simulations (see Table C-1). The first consists of identity functions $\hat{\phi}(\cdot)$ and $\vartheta(\cdot)$. In case 2, $\hat{\phi}(\cdot)$ is chosen such that $\vartheta(x) = x^3 + x \geq x$. In case 3, $\hat{\phi}(\cdot)$ is chosen such that $\hat{\phi}(x) = x^3 + x \geq x$. With this property, it is expected that the response of the system becomes faster, because the values in the state dynamics will be greater or equal to those in case 1. Following the same logic, for case 3 we let $\vartheta(x) = x^3 + x \geq x$. The gains are set at $\Gamma = 1$ for Edge-based protocol I (EB1) and Node-based protocol I (NB1), and $\Gamma = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ for Edge-based protocol II (EB2) and Node-based protocol II (NB2).

	Case 1	Case 2	Case 3
$\hat{\phi}(x) =$	x	x	$x + x^3$
$\vartheta(x) =$	x	$x + x^3$	x

Table C-1: Simulation cases

C-1 Simple network

The simple simulation network consists of three agents connected as in Figure C-1.

Figure C-1: Simple network

The initial values x_0 and weights w for are set at

$$x_0 = \begin{bmatrix} 1 & 3 & 5 \end{bmatrix}^\top, w = \begin{bmatrix} 0.2 & 0.4 & 0.3 \end{bmatrix}^\top. \tag{C-1}$$

C-2 DC microgrid topology

The DC microgrid considered for the simulations consists of three sources (blue) and six loads (white), which are described by a connected and undirected graph. The network graph is shown in Figure C-2. The loads are modeled as ZIP loads, which is elaborated in Section C-2-2.

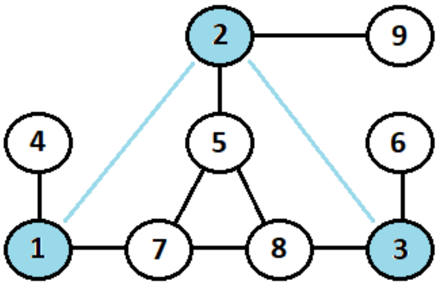

Figure C-2: Microgrid network

C-2-1 System parameters

The system parameters are summarized in Table C-2, Table C-3 and Table C-4.

Line	1	2	3	4	5	6	7	8	9
Conductance γ (Ω^{-1})	0.165	0.141	0.135	0.195	0.105	0.171	0.138	0.171	0.258

Table C-2: Line conductances

Source node	1	2	3
Power sharing coefficient C	1.0	3.0	2.0

Table C-3: Power sharing coefficients

Load	4	5	6	7	8	9
Type	Z	I	P	Z	I	P
Value	0.09 (Ω^{-1})	-1.4 (A)	-80 (W)	0.04 (Ω^{-1})	-1.1 (A)	-80 (W)

Table C-4: Load models

C-2-2 Load models

Initially, loads 6, 7, 8 and 9 are switched on. After one time step, the loads 4 and 5 are switched on instantly and load 6 is switched off. This is done to simulate an extreme case of change in the microgrid.

ZIP loads

The loads are modelled as constant impedance (Eq. (C-2a)), constant current (Eq. (C-2b)) and constant power loads (Eq. (C-2c)) as described in [11], abbreviated by ZIP.

$$I_l = -Y_l^* V_l, \qquad Y_l^* > 0 \tag{C-2a}$$
$$I_l = I_l^* \qquad I_l^* < 0 \tag{C-2b}$$
$$I_l = V_l^{-1} P_l^* \qquad P_l^* < 0 \tag{C-2c}$$

C-2-3 Kron reduction

The network is reduced by means of Kron reduction [23, 11] to obtain a relation between the source voltages, source currents and load currents. This is done by eliminating the load voltages from Eq. (2-4) to obtain

$$I_s - Y_{ll}^{-1} Y_{ls} I_l = Y_{red} V_s \tag{C-3a}$$
$$Y_{red} = Y_{ss} - Y_{sl} Y_{ll}^{-1} Y_{ls} \tag{C-3b}$$

C-3 Python

The simulations are performed in Python. The differential equations of the voltage, gain and protocol state dynamics are calculated using the ordinary differential equation solver using the open source Python library SciPy [24]. Furthermore, the open source Python library NumPy [25] has been used to perform calculations.

Appendix D

Research paper

On Distributed Power Consensus Algorithms in DC Microgrids

Chris Zevenbergen and Sergio Grammatico

***Abstract*—In this paper, we investigate nonlinear consensus algorithms and their application to proportional power sharing in DC microgrids. Specifically, the connection to existing nonlinear consensus protocols is investigated with the objective of proving that the power sharing algorithm is a special case of the nonlinear consensus protocols. This being the case, a thorough analysis of the power sharing algorithm is performed based on existing nonlinear consensus theory in the literature and that developed in this paper. The reason for this investigation is the presence of an agreement function in the power sharing algorithm, which is evident due to the preservation of the weighted geometric mean. The performance of the algorithm is improved by applying the theory of consensus protocols. Then, the implementation of adaptive gains is investigated and performance analyzed.**

I. Introduction

The way that energy is produced has been undergoing rapid change over the past decades and will continue to do so in the future. The implementation of sustainable energy resources, such as wind and solar, will change the topology of power systems, by introducing a distributed network of energy resources. Furthermore, sustainable energy resources have an unpredictable nature compared to the old-fashioned fossil fuel-based methods of energy production, as they are reliant on, among others, weather conditions.

To allow for implementation of these technologies, the traditional power systems will require alteration. The process of implementing sustainable energy resources is a slow and complicated one. There are several ideas on how to tackle this problem, one of which is the implementation of microgrids. A microgrid is a power system, consisting of several loads and generation units, and is seen in remote villages, hospitals and university campuses, to name a few examples. Microgrids can consist of AC power, DC power, or a combination of the two. In this paper, DC microgrids are considered.

The control objectives within a microgrid are diverse. For DC microgrids, the main control objectives are the following. The voltage should be stabilized quickly when changes in the grid occur. Power generation should be shared among the generators proportionally, according to some weight. Power losses should be minimized, and the microgrid should have plug-and-play capabilities. That is, loads and generators should be able to be connected and disconnected from the system whilst maintaining the desired behaviour across the microgrid. In this paper, the problems of proportional power sharing and voltage regulation with plug-and-play capabilities are considered.

The authors are with the Delft Center for Systems and Control, TU Delft, The Netherlands. E-mail addresses: c.w.zevenbergen@student.tudelft.nl, s.grammatico@tudelft.nl.

Overviews of the current state of microgrid technology are given in [13], [7], [8]. Topics related to power consensus and (reactive) power compensation are considered in [5], [10], [15], [14], [12], [16]. The hierarchical control structure, often applied to microgrids, is discussed in [9], [17], [18]. Frequency control and synchronization algorithms are proposed in [15], [14], [9], [19]. The topic of voltage control is discussed in [20], [12], [16], [21], [22], [19]. Nonlinear consensus protocols are presented in [1], [2]. Consensus protocols that implement an adaptive gain are proposed in [3], [4]. Finally, microgrid modelling strategies are discussed in [23], [6].

The main contribution of this paper is threefold. The first part considers nonlinear consensus theory [1]. An additional agreement function, the weighted geometric mean, whose value is to be preserved throughout the evolution of a dynamical system, is proposed. Two new types of consensus, namely proportional state consensus and general consensus of some function of the states, are introduced. In the second part, four general adaptive gain consensus protocols are developed by combining the insights gained in the first part with the existing theory of adaptive gain consensus protocols [3], [4]. The third part concerns proportional power consensus in DC microgrids. The properties of a power consensus algorithm proposed in [1] are analyzed by looking at the connection to nonlinear consensus theory, followed by the implementation of proportional power consensus in the the adaptive gain consensus protocols proposed before.

The paper is organized as follows. In Section II, the modelling approach of a DC microgrid is discussed. In Section III, a summary of nonlinear consensus protocol theory from the literature and some new results is given. In Section IV, several adaptive gain consensus protocols are proposed and their properties discussed. In Section V, the developed theory is applied to address the problem of proportional power consensus in DC microgrids. In Section VI, numerical simulations of the power consensus protocols are performed and discussed. Finally, Section VII concludes the paper and suggests some interesting future research topics.

II. DC microgrid

A. *Microgrid and communication network*

The modelling approach from [5] is adopted here. The DC microgrid is modelled as a connected and undirected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V} = \{1, \ldots, n\}$ represents the set of nodes and $\mathcal{E}$ the set of edges, connecting the nodes. The cardinality of the set $\mathcal{E}$ is denoted by $|\mathcal{E}| = m$. The set of nodes consists of sources $\mathcal{V}_s$ and loads $\mathcal{V}_l$ such that $\mathcal{V} = \mathcal{V}_s \cup \mathcal{V}_l$. The set of edges consists of pairs of interconnected nodes (i, j) where $i, j \in \mathcal{V}$.

The sources are capable of communicating with each other. Their interconnections are described by a connected and undirected graph $\mathcal{G}_c = (\mathcal{V}_c, \mathcal{E}_c)$, where the definitions $\mathcal{V}_c$ and $\mathcal{E}_c$ are equivalent to those of the microgrid network. The set of neighbouring nodes $\mathcal{N}_i$ of node i is defined as $\mathcal{N}_i := \{j \in \mathcal{V} : (i, j) \in \mathcal{E}\}$. Equivalently, the set of neighbouring source node N_{ci} is defined as $\mathcal{N}_{ci} := \{j \in \mathcal{V}_s : (i, j) \in \mathcal{E}_c\}$.

The interconnections of a connected and undirected graph $\mathcal{G}$ are described by the symmetric adjacency matrix $\mathcal{A} \in \mathbb{R}^{n\times n}$ with elements $a_{ij} \in \mathcal{A}$, which is defined as follows.

$$a_{ij} = \begin{cases} 1 & \text{if } (i, j) \in \mathcal{E} \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

The edges of the graph, and hence the interconnections of the nodes and edges, are described by the incidence matrix $\mathcal{B} \in \mathbb{R}^{n\times m}$ with elements $b_{ij} \in \mathcal{B}$. The edges are numbered such that for each $(i, j) \in \mathcal{E}$ there exists a unique $k \in \{1, \ldots, m\}$. The incidence matrix is defined as

$$b_{ij} = \begin{cases} 1 & \text{if } i \in \mathcal{V}_s \text{ and } (i, j) \in \mathcal{E} \\ -1 & \text{if } i \in \mathcal{V}_l \text{ and } (i, j) \in \mathcal{E} \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

The differentiation between sources and loads in the incidence matrix describes direction of the flow of current in the microgrid.

Associated to each edge k is a conductance $\gamma_k = \frac{1}{r_k}$ summarized in the matrix of conductances $\Gamma = \mathrm{diag}(\gamma_1, \ldots, \gamma_k)$ where γ is the vector of conductances. Then, relation between the voltage and currents can be described by the weighted Laplacian matrix

$$Y = B\Gamma B^\top \tag{3}$$

such that

$$\begin{bmatrix} I_s \\ I_l \end{bmatrix} = \begin{bmatrix} Y_{ss} & Y_{sl} \\ Y_{ls} & Y_{ll} \end{bmatrix} \begin{bmatrix} V_s \\ V_l \end{bmatrix}. \tag{4}$$

Since power consensus is the objective, a study of consensus protocols from the literature is performed. In the following section, some nonlinear consensus theory from the literature is presented and additional results are proposed.

III. Mathematical background

The objective of the consensus protocol from [1] is to steer the states of a dynamical system to the same value in such a way that the nodes communicate only with neighbouring nodes. The former is referred to as state consensus for the remainder of the paper. The latter implies that the protocol must be distributed. The states converge to some value which is determined by the agreement function, introduced next.

A. Agreement function

An agreement function $\chi(x) : \mathbb{R}^n \to \mathbb{R}$ is a continuous and differentiable function of the states of the system and is permutation invariant [1]. Agreement functions that have the property

$$\forall x \in \mathcal{V} : \min_{i\in\mathcal{V}} x_i \leq \chi(x) \leq \max_{i\in\mathcal{V}} x_i \tag{5}$$

are of interest. For example, agreement functions that satisfy condition (5) are the arithmetic mean, geometric mean, harmonic mean and mean of order p.

The consensus protocols from the literature and those proposed in this paper have the objective of preserving the value of an agreement function. Then, the agreement function is said to be time-invariant. A protocol that achieves this objective is referred to as a time-invariant protocol.

To ensure time-invariance of the agreement function it must be implemented into the consensus protocol appropriately. To this end, the agreement function is decomposed into two functions $f, g : \mathbb{R} \to \mathbb{R}$ with $g'(x_i) \neq 0$ such that $\chi(x) = h(\sum_{i\in\mathcal{V}} g_i(x_i))$. The reason for this becomes evident next.

B. Time-invariant protocol

The structure of a time-invariant protocol is proposed in [1], and is repeated here for completeness. The dynamics at node i are a function of the local state x_i and the neighbouring states $x^{(i)}$. The dynamics consist of the product of the sum of a anti-symmetric difference function of the states and a time-dependent gain that depends on the agreement function. The protocol is constructed as follows.

$$\forall i \in \mathcal{V} : \quad \dot{x}_i = (g_i'(x_i))^{-1} \sum_{j\in\mathcal{N}_i} \phi(x_j, x_i) \tag{6}$$

The "gain" is given by the inverse of the derivative w.r.t. to the state of the function $g(\cdot)$ of the decomposed agreement function. In [1], it is proved that protocol (6) is time-invariant if $g(\cdot)$ is increasing and $\phi(x_j, x_i)$ is anti-symmetric. Before the function $\phi(\cdot)$ is elaborated, the notion of the weighted geometric mean is introduced.

C. Weighted geometric mean as agreement function

The particular agreement function of interest in this paper is the weighted geometric mean $\bar{x}^{\mathrm{w}}$, which is defined as.

$$\bar{x}^w = \left(\prod_{i\in\mathcal{V}} x_i^{w_i}\right)^{(\sum_{i\in\mathcal{V}} w_i)^{-1}} \tag{7}$$

The power consensus algorithm proposed in [1] is such that the weighted geometric mean of the states is preserved. Thus, a logical first step in the investigation of the connection between the power consensus algorithm and nonlinear consensus protocols is to determine whether the weighted geometric mean is an appropriate agreement function, capable of being time-invariant under protocol (6). The weighted geometric mean can conveniently be decomposed as $\chi(x) = h(\sum_{i\in\mathcal{V}} g_i(x_i))$ with

$$h(y) = e^{(\sum_{i\in\mathcal{V}} w_i)^{-1} y} \tag{8a}$$

$$g_i(x_i) = \log x_i^{w_i}. \tag{8b}$$

Then, a nonlinear consensus protocol which lets the weighted geometric mean of the states be time-invariant is found by substituting (8b) in (6):

$$\forall i \in \mathcal{V}: \quad \dot{x}_i = w_i^{-1} x_i \sum_{j\in\mathcal{N}_i} \phi(x_j, x_i) \tag{9a}$$

D. Proportional consensus

In [1], conditions on $\phi(x_j, x_i)$ to achieve state consensus while preserving the value of the agreement function are proposed. The results are summarized next. Consider the time-invariant protocol (6) with $\phi(x_j, x_i)$ expanded as follows.

$$\phi(x_j, x_i) = \hat{\phi}(\vartheta(x_j) - \vartheta(x_i)) \tag{10}$$

$\phi(x_j, x_i)$ represents a difference function of some function of the states. If $\hat{\phi} : \mathbb{R} \to \mathbb{R}$ is continuous, locally Lipschitz, odd and strictly increasing, $\vartheta : \mathbb{R} \to \mathbb{R}$ is differentiable and $\vartheta'(x_i)$ is locally Lipschitz and strictly positive, then protocol

$$\dot{x}_i = (g_i'(x_i))^{-1} \sum_{j\in\mathcal{N}_i} \hat{\phi}(\vartheta(x_j) - \vartheta(x_i)), \quad \forall i \in \mathcal{V} \tag{11}$$

achieves state consensus. The conditions are relevant to following sections.

In some cases, it may be needed to let the states reach values proportional to each other. In this section, the theory on nonlinear consensus protocols is extended to include the notion of proportional state consensus.

Definition 1 (Proportional state consensus): We say that proportional state consensus according to weights $w \in \mathbb{R}^n$ is achieved when

$$w_i^{-1} x_i = w_j^{-1} x_j \qquad \forall i, j \in \mathcal{V}. \tag{12}$$

This is equivalent to $z_i = z_j$ for all $i, j \in \mathcal{V}$, where z follows from a linear transformation of the states:

$$z = W^{-1} x \tag{13}$$

Note that $W = \text{diag}(w_1, \ldots, w_n)$ is the $n \times n$ matrix with weights w on the diagonal. It now becomes clear that proportional consensus is reached when the transformed states reach state consensus. Thus, the proposed proportional consensus algorithm is constructed as

$$\dot{x}_i = (g_i'(x_i))^{-1} \sum_{j\in\mathcal{N}_i} \hat{\phi}(\vartheta(z_j) - \vartheta(z_i)), \quad \forall i \in \mathcal{V} \tag{14a}$$

The conditions for proportional state consensus are summarized in Theorem 1.

Theorem 1 (Proportional state consensus): Assume that $g(\cdot)$ is strictly increasing and differentiable, $g_i'(cx_i) = cg_i'(x_i)$ for some $c \in \mathbb{R}$, $\hat{\phi}(\cdot)$ is continuous, locally Lipschitz, odd and strictly increasing, $\vartheta(\cdot)$ is differentiable and $\vartheta'(x_i)$ is locally Lipschitz and strictly positive. Then, a network of agents under protocol (14) achieves proportional state consensus , whilst preserving the value of the agreement function $\chi(x)$.

Proof: The proof follows a similar logic to [1, Theorem 1, 2]. For proportional state consensus, we require that (12) holds. That is, the equilibrium $x^* = \lambda w$ for some $\lambda \in \mathbb{R}$. This is equivalent to $z^* = \lambda\mathbf{1}$ with $z = \mathcal{W}^{-1}x$, where $\mathcal{W} = \text{diag}(w_1, \ldots, w_n)$. Assume $z_i = \lambda\mathbf{1}$ for all $i \in \mathcal{V}$. Then, we have $\hat{\phi}(z_j - z_i) = \hat{\phi}(\lambda - \lambda) = 0$. Since $\hat{\phi}$ is continuous and odd, it follows that $\dot{x}_i = 0$. Thus, $z^* = \lambda\mathbf{1}$ is an equilibrium point. Next, we show that the equilibrium point is unique.

Assume that there exists an equilibrium point $z^* \neq \lambda\mathbf{1}$. Let $\mathcal{I}$ represent the set of nodes i whose proportional equilibrium states z_i^* are greater than or equal to those of all nodes $j \in \mathcal{V}$. That is, $\mathcal{I} = \{i \in \mathcal{V} : z_i^* \geq z_j^*, \ \forall j \in \mathcal{V}\}$. We note that $\mathcal{I} \subset \mathcal{V}$. In fact, if $\mathcal{I} = \mathcal{V}$, then $z_i \geq z_j$ for all $i, j \in \mathcal{V}$, from which it follows that $z_i = z_j$ for all $i, j \in \mathcal{V}$. Thus, $z^* = \lambda\mathbf{1}$, which is a contradiction. This allows us to select $i \in \mathcal{I}$ such that $z_i^* > z_j^*$ for some $j \in \mathcal{V}$ and $z_i^* \geq z_k^*$ for all $k \in \mathcal{V}$. The resulting sum $\sum_{j\in\mathcal{N}_i} \hat{\phi}(z_j - z_i) < 0$. Since, $z_i > 0$, it follows that $\dot{x}_i < 0$. Therefore, $z^* \neq \lambda\mathbf{1}$ is not an equilibrium point, and we have shown that $z^* = \lambda\mathbf{1}$ is a unique equilibrium point.

Next, we show that the states reach proportional consensus. This is achieved when $w_i^{-1}x_i = z_i = \chi(z_0)$ for all $i \in \mathcal{V}$. Then, since $g_i(x_i)$ is strictly increasing, we define $e := g_i(z_i) - g_i(\chi(z_0)) = 0$ as the consensus error function. Then, proportional consensus is equivalent to proving asymptotic stability of the point $e = 0$ with

$$\dot{e}_i = \dot{g}_i(z_i) = \dot{g}_i(w_i^{-1}x_i) = g_i'(w_i^{-1}x_i)\dot{x}_i = w_i^{-1}g_i'(x_i)\dot{x}_i. \tag{15a}$$

The assumption that $g_i'(cx_i) = cg_i'(x_i)$ is used above.

Consider the candidate Lyapunov function $V(e) = \frac{1}{2}\sum_{i\in\mathcal{V}} w_i e_i^2$. Clearly, $V(0) = 0$ and $V(e) > 0$ for all $e \neq 0$. The time-derivative of the Lyapunov function is given by

$$\dot{V} = \sum_{i\in\mathcal{V}} w_i e_i \dot{e}_i \tag{16a}$$

$$= \sum_{i\in\mathcal{V}} w_i e_i w_i^{-1} g_i'(x_i)\dot{x}_i \tag{16b}$$

$$= \sum_{i\in\mathcal{V}} w_i e_i w_i^{-1} g_i'(x_i)(g_i'(x_i))^{-1} \sum_{j\in\mathcal{N}_i} \hat{\phi}(\vartheta(z_j) - \vartheta(z_i)) \tag{16c}$$

$$= \sum_{i\in\mathcal{V}} e_i \sum_{j\in\mathcal{N}_i} \hat{\phi}(\vartheta(z_j) - \vartheta(z_i)) \tag{16d}$$

$$= \sum_{i\in\mathcal{V}} g_i(z_i) - g_i(\chi(z(0))) \sum_{j\in\mathcal{N}_i} \hat{\phi}(\vartheta(z_j) - \vartheta(z_i)) \tag{16e}$$

(16f)

The term $g_i(\chi(z(0))$ cancels because it is constant and the network in connected and undirected.

$$\dot{V} = \sum_{i\in\mathcal{V}} g_i(z_i) \sum_{j\in\mathcal{N}_i} \hat{\phi}(\vartheta(z_j) - \vartheta(z_i)) \tag{17a}$$

$$= -\sum_{(i,j)\in\mathcal{E}} (g_i(z_j) - g_i(z_i))\hat{\phi}(\vartheta(z_j) - \vartheta(z_i)) \tag{17b}$$

The final step uses the fact that $i \in \mathcal{N}_j$ if and only if $j \in \mathcal{N}_i$. This allows (17a) to be written as (17b). Since $g_i(\cdot)$, $\hat{\phi}(\cdot)$ and

$\vartheta(\cdot)$ are strictly increasing, we find that $\dot{V} \leq 0$, and $\dot{V} = 0$ if and only if $z_i = z_j$ for all $i, j \in \mathcal{V}$, implying that $e_i = 0$ for all $i \in \mathcal{V}$. ■

E. Generalized consensus

The previous section focused on achieving (proportional) consensus of the states of a system. In some cases consensus of the states is not the objective. For example, power should be shared among the generation units of an electrical network. That is, consensus on the power is required, not consensus on the states. In this section, a protocol that achieves consensus on some function is suggested. This function is denoted by $f(\cdot)$ and may be a function of local and neighbouring states. In fact, as we will see soon, the state consensus protocol (11) is the special case of the general consensus protocol, where $f(x_i) = x_i$.

The proposed protocol is similar to (11).

$$\dot{x}_i = (g_i'(x_i))^{-1} \sum_{j \in \mathcal{N}_i} \hat{\phi}(\vartheta(f(x_j, x^{(j)})) - \vartheta(f(x_i, x^{(i)}))) \tag{18}$$

Remark 1: The function $f(x_j, x^{(j)})$ is not only dependent on the local state x_i and neighbouring states $x^{(i)}$, but also on the neighbouring states $x^{(j)}$ of node j, the protocol is not fully distributed. In fact, each node requires information from nodes which are neighbours of neighbouring nodes. This issue can be resolved if each agent reads the values of the function $f(\cdot)$, and shares them to its neighbours.

Finding conditions on $f(\cdot)$ that guarantee consensus is a challenging task and currently an open problem. However, a quick investigation of the equilibria of the system is performed to show that consensus of $f(\cdot)$ is achieved at equilibrium. We assume that the conditions in Theorem 1 hold. At equilibrium, we have $\dot{x}_i = 0$. Since $g_i'(x_i)$ is strictly positive, it follows that

$$\sum_{j \in \mathcal{N}_i} \hat{\phi}(\vartheta(f(x_j, x^{(j)})) - \vartheta(f(x_i, x^{(i)}))) = 0. \tag{19}$$

Due to connectivity and undirectedness of the graph, we have

$$\vartheta(f(x_j, x^{(j)})) - \vartheta(f(x_i, x^{(i)})) = 0 \qquad \forall i, j \in \mathcal{V}. \tag{20}$$

Then, since $\hat{\phi}$ and ϑ are strictly increasing, it follows that $f(x_j, x^{(j)}) - f(x_i, x^{(i)}) = 0$. It should be noted that, as in Section III-D, proportional consensus on $f(\cdot)$ is achieved by replacing $f(\cdot)$ by $w_i^{-1} f(\cdot)$. Clearly, the equilibria are then found where $w_j^{-1} f(x_j, x^{(j)}) = w_i^{-1} f(x_i, x^{(i)})$.

IV. Nonlinear consensus protocols with adaptive gains

In this section, the theory on nonlinear consensus protocols is extended to the case with adaptive gains. Inspired by [3], [4], we propose four protocols, two of which implement an edge-based gain and two implement a node-based gain. Before the protocols are introduced, the notions of node-based and edge-based gains are elaborated.

An edge-based gain is assigned to each edge of the network. The gain increases while the two nodes, connected by the edge, have not reached consensus. The gains converge to a finite value, which is reached once consensus between the two nodes is reached. The node-based case assigns a gain to each node of the network. Now, the gain increases as long as the node and all its neighbours have not yet reached consensus. Once consensus is reached, the gain will have simultaneously reached steady-state. The adaptive gain dynamics are given by some quadratic function to ensure the gains are non-decreasing.

An interesting difference between the two is that, if constructed correctly, an edge-based adaptive gain protocol will preserve the value of the agreement function. This cannot be guaranteed for a node-based adaptive gain protocol. In this section, the evolution of the agreement function is investigated and conditions for state consensus are given. Furthermore, the protocols are extended to handle general consensus on some function $f(\cdot)$.

A. Edge-based protocol I

The first adaptive gain protocol is inspired by the edge-based protocol proposed in [3]. An adaptive gain is assigned to each edge. Each node communicates its state information to the neighbouring nodes, and the dynamics are a function of the local and neighbouring nodes.

$$\dot{x}_i = (g_i'(x_i))^{-1} \sum_{j \in \mathcal{N}_i} c_{ij} \hat{\phi}(\vartheta(x_j) - \vartheta(x_i)) \tag{21a}$$

$$\dot{c}_{ij} = \hat{\phi}(\vartheta(x_j) - \vartheta(x_i))^\top \Gamma \hat{\phi}(\vartheta(x_j) - \vartheta(x_i)) \tag{21b}$$

The state dynamics are similar to the nonlinear consensus protocol (11), with the difference that an adaptive gain term c_{ij} is now involved. The gain dynamics is quadratic and depends on some difference function of the states. Let us begin by investigating the evolution of the agreement function. We recall from Section III-B that the protocol is time-invariant if $g_i(\cdot)$ follows from the decomposition of the agreement function, and $\phi(x_j, x_i) = c_{ij}\hat{\phi}(\vartheta(x_j) - \vartheta(x_i))$ is anti-symmetric. To this end, we show that $c_{ij}\hat{\phi}(\vartheta(x_j) - \vartheta(x_i)) = -c_{ji}\hat{\phi}(\vartheta(x_i) - \vartheta(x_j))$. This result is presented the following Lemma.

Lemma 1: Assume that $\hat{\phi}(\vartheta(x_j) - \vartheta(x_i))$ is odd and $c_{ij}(t) = c_{ji}(t)$ for all $i, j \in \mathcal{V}$ and all $t > 0$. Then, the function $c_{ij}\hat{\phi}(\vartheta(x_j) - \vartheta(x_i))$ is anti-symmetric. Furthermore, $c_{ij}(t) = c_{ji}(t)$ for all $t > 0$ if $\dot{c}_{ij}$ is defined as in (21) and $c_{ij}(0) = c_{ji}(0)$ for all $i, j \in \mathcal{V}$.

Proof: Clearly,

$$\begin{aligned} c_{ij}\hat{\phi}(\vartheta(x_j) - \vartheta(x_i)) &= c_{ji}\hat{\phi}(\vartheta(x_j) - \vartheta(x_i)) \\ &= -c_{ji}\hat{\phi}(\vartheta(x_i) - \vartheta(x_j)), \end{aligned}$$

where the final equality follows from anti-symmetry of $\hat{\phi}(\vartheta(x_j) - \vartheta(x_i))$. This shows that $c_{ij}\hat{\phi}(\vartheta(x_j) - \vartheta(x_i))$ is anti-symmetric. Furthermore,

$$\begin{aligned} \dot{c}_{ij} &= \hat{\phi}(\vartheta(x_j) - \vartheta(x_i))^\top \Gamma \hat{\phi}(\vartheta(x_j) - \vartheta(x_i)) \\ &= (-\hat{\phi}(\vartheta(x_i) - \vartheta(x_j))^\top \Gamma (-\hat{\phi}(\vartheta(x_i) - \vartheta(x_j)) \\ &= \hat{\phi}(\vartheta(x_i) - \vartheta(x_j))^\top \Gamma \hat{\phi}(\vartheta(x_i) - \vartheta(x_j)) \\ &= \dot{c}_{ji} \end{aligned}$$

∎

This result, combined with [1, Theorem 1], proves that (21) is a time-invariant protocol.

Given the right conditions, the protocol reaches state consensus. These are summarized in the following protocol.

Theorem 2 (State consensus of edge-based protocol I): Assume that $g_i(\cdot)$ is strictly increasing, $\hat{\phi}(\cdot)$ is continuous, locally Lipschitz, odd and strictly increasing, $\vartheta(\cdot)$ is differentiable and $\vartheta'(x_i)$ is locally Lipschitz and strictly positive. Furthermore, the gains are initialized such that $c_{ij}(0) > 0$ for all $i, j \in \mathcal{V}$. Then, a network of agents under protocol (21) achieves state consensus.

Proof: State consensus is reached when $\tilde{e}_i = x_i - \chi(x(0)) = 0$ for all $i \in \mathcal{V}$, where $\chi(x(0))$ represents the consensus value. Since $g_i(\cdot)$ is strictly increasing, we may say that consensus is reached when $e_i = g_i(x_i) - g_i(\chi(x(0))) = 0$. Thus, proving consensus of the states is equivalent to proving asymptotic stability of the $e = 0$. We have

$$\dot{e}_i = g_i'(x_i)\dot{x}_i = \sum_{j\in\mathcal{N}_i} c_{ij}\hat{\phi}(\vartheta(x_j) - \vartheta(x_i))$$

Consider the candidate Lyapunov function $V = \frac{1}{2}\sum_{i\in\mathcal{V}} e_i^2$. Clearly, $V(0) = 0$ and $V(e) > 0$ if $e \neq 0$. The time-derivative is determined next.

$$\begin{aligned}
\dot{V} &= \sum_{i\in\mathcal{V}} e_i \dot{e}_i && (22)\\
&= \sum_{i\in\mathcal{V}} e_i \sum_{j\in\mathcal{N}_i} c_{ij}\hat{\phi}(\vartheta(x_j) - \vartheta(x_i)) && (23)\\
&= \sum_{i\in\mathcal{V}} (g_i(x_i) - g_i(\chi(x(0)))) \sum_{j\in\mathcal{N}_i} c_{ij}\hat{\phi}(\vartheta(x_j) - \vartheta(x_i)) && (24)
\end{aligned}$$

Since $g_i(\chi(x(0)))$ is constant and the network is connected and undirected, the term cancels.

$$\begin{aligned}
\dot{V} &= \sum_{i\in\mathcal{V}} g_i(x_i) \sum_{j\in\mathcal{N}_i} c_{ij}\hat{\phi}(\vartheta(x_j) - \vartheta(x_i)) && (25)\\
&= -\sum_{(i,j)\in\mathcal{E}} c_{ij}(g_i(x_j) - g_i(x_i))\hat{\phi}(\vartheta(x_j) - \vartheta(x_i)) && (26)
\end{aligned}$$

The final step uses the fact that $i \in \mathcal{N}_j$ if and only if $j \in \mathcal{N}_i$. This allows (25) to be written as (26). The adaptive gains c_{ij} initially have positive values, which remain positive due to the quadratic form of the gain dynamics. Then, since $g_i(\cdot)$, $\hat{\phi}(\cdot)$ and $\vartheta(\cdot)$ are odd and strictly increasing, it follows that each product is positive. Hence, $\dot{V} \leq 0$. Finally, we have that $V = 0$ implies that $x_i = x_j$ for all $i, j \in \mathcal{V}$. Thus, $e_i = 0$ for all $i \in \mathcal{V}$, and we have shown that the system reaches consensus. ∎

The edge-based protocol is extended to the general consensus case. The state dynamics are extended as in Section III-E, by generalizing the function to include $f(x_i, x^{(i)})$. The same process must be applied the the gain dynamics, so that they increase with respect to the difference function of $f(x_i, x^{(i)})$, not that of the states. In doing so, the adaptive gains speed up the "slower" edges, and allow the gain dynamics to converge to finite values at steady-state. The resulting adaptive gain general consensus protocol has the following form.

$$\dot{x}_i = (g'(x_i))^{-1} \sum_{j\in\mathcal{N}_i} c_{ij}\hat{\phi}(\vartheta(f(x_j, x^{(j)})) - \vartheta(f(x_i, x^{(i)}))) \quad (27a)$$

$$\begin{aligned}
\dot{c}_{ij} &= \hat{\phi}(\vartheta(f(x_j, x^{(j)})) - \vartheta(f(x_i, x^{(i)})))^\top \\
&\quad \cdot \Gamma\hat{\phi}(\vartheta(f(x_j, x^{(j)})) - \vartheta(f(x_i, x^{(i)}))) \quad (27b)
\end{aligned}$$

An investigation of the equilibria, following an equivalent approach to that of Section III-E, shows that $f(x_j, x^{(j)}) = f(x_i, x^{(i)})$ for all $i, j \in \mathcal{V}$ at any equilibrium.

B. Edge-based protocol II

Next, we propose a second edge-based adaptive gain consensus protocol which is inspired by the edge-based protocol in [4]. The protocol is different to the preceding edge-based protocol in that it introduces an auxiliary protocol state, denoted by v:

$$\dot{x}_i = -(g_i'(x_i))^{-1} v_i \quad (28a)$$

$$\begin{aligned}
\dot{v}_i &= -v_i \\
&\quad - \sum_{j\in\mathcal{N}_i} c_{ij}\left[\hat{\phi}(\vartheta(x_j) - \vartheta(x_i)) - \hat{\phi}(\vartheta(v_j) - \vartheta(v_i))\right] \quad (28b)
\end{aligned}$$

$$\dot{c}_{ij} = \begin{bmatrix}\hat{\phi}(\vartheta(x_j) - \vartheta(x_i)) \\ \hat{\phi}(\vartheta(v_j) - \vartheta(v_i))\end{bmatrix}^\top \Gamma \begin{bmatrix}\hat{\phi}(\vartheta(x_j) - \vartheta(x_i)) \\ \hat{\phi}(\vartheta(v_j) - \vartheta(v_i))\end{bmatrix}. \quad (28c)$$

In contrast to edge-based protocol I, the state dynamics do not have the form of the nonlinear consensus protocol (11). The gain dynamics are now a function of both the state and the auxiliary protocol state. Letting z_i represent the pair (x_i, v_i), the protocol is written in compact form as

$$\begin{aligned}
\dot{x}_i &= -(g_i'(x_i))^{-1} v_i && (29a)\\
\dot{v}_i &= -v_i - \sum_{j\in\mathcal{N}_i} c_{ij}\hat{\phi}_v(z_j, z_i) && (29b)\\
\dot{c}_{ij} &= \hat{\phi}_c(z_j, z_i)^\top \Gamma \hat{\phi}_c(z_j, z_i). && (29c)
\end{aligned}$$

Because the protocol is not a special case of the nonlinear consensus protocol (11), [1, Theorem 1] can no longer be applied to prove time-invariance. Thus, we show that the agreement function is constant along the trajectories under some conditions.

Theorem 3 (Time-invariance of edge-based protocol II): Assume that $\phi_c(z_j z_i)$ and $\phi_v(z_j z_i)$ are anti-symmetric, $g_i(x_i)$ is strictly increasing, the auxiliary states v are initialized such that $\sum_{i\in\mathcal{V}} v_i(0) = 0$ and the adaptive gains c are initialized such that $c_{ij}(0) = c_{ji}(0)$ for all $i, j \in \mathcal{V}$. Then, the value of the agreement function $\chi = h(\sum_{i\in\mathcal{V}} g_i(x_i))$ is preserved.

Proof: It suffices to show that the argument $\sum_{i\in\mathcal{V}} g_i(x_i)$ of the agreement function is constant.

$$\sum_{i\in\mathcal{V}} \dot{g}_i(x_i) = \sum_{i\in\mathcal{V}} g_i'(x_i)\dot{x}_i = -\sum_{i\in\mathcal{V}} v_i \quad (30)$$

Thus, the agreement function is time-invariant when $\sum_{i\in\mathcal{V}} v_i(t) = 0$ for all $t \geq 0$. Clearly, the protocol state must be initialized such that $\sum_{i\in\mathcal{V}} v_i(0) = 0$. Next, we show that $\sum_{i\in\mathcal{V}} v_i(t) = 0$ for all $t > 0$.

$$\begin{aligned}\frac{d\sum_{i\in\mathcal{V}} v_i}{dt} &= \sum_{i\in\mathcal{V}} \frac{dv_i}{dt} \\ &= \sum_{i\in\mathcal{V}} \dot{v}_i = \sum_{i\in\mathcal{V}} (v_i - \sum_{i\in\mathcal{N}_i} c_{ij}\phi) \\ &= \sum_{i\in\mathcal{V}} v_i - \sum_{i\in\mathcal{V}}\sum_{i\in\mathcal{N}_i} c_{ij}\phi = \sum_{i\in\mathcal{V}} v_i\end{aligned}$$

The final equality results from anti-symmetry of ϕ and connectedness of $\mathcal{G}$. Define $\hat{v} := \sum_{i\in\mathcal{V}} v_i$. The solution of $\hat{v}$ then follows from solving the differential equation $\dot{\hat{v}} - \hat{v} = 0$ with initial value $\hat{v} = 0$. This results in $\hat{v} = \sum_{i\in\mathcal{V}} v_i = 0$, showing that the argument of the agreement function is constant and subsequently that the value of the agreement function is preserved. ■

We claim that the protocol reaches state consensus under the correct conditions, which are summarized in Conjecture 1.

Conjecture 1 (State consensus of edge-based protocol II): Assume that $g_i(\cdot)$ is strictly increasing, $\hat{\phi}(\cdot)$ is continuous, locally Lipschitz, odd and strictly increasing, $\vartheta(\cdot)$ is differentiable and $\vartheta'(x_i)$ is locally Lipschitz and strictly positive. Then, edge-based protocol II (28) achieves state consensus. Furthermore, the auxiliary states v converge to zero.

The protocol is now extended so that general consensus on some function $f(\cdot)$ can be reached. This is done by following a similar approach as in Section IV-A.

$$\dot{x}_i = -\,(g_i'(x_i))^{-1}v_i \tag{31a}$$

$$\begin{aligned}\dot{v}_i = &-v_i - \sum_{j\in\mathcal{N}_i} c_{ij}[\hat{\phi}(\vartheta(f(x_j,x^{(j)})) - \vartheta(f(x_i,x^{(i)}))) \\ &- \hat{\phi}(\vartheta(v_j) - \vartheta(v_i))]\end{aligned} \tag{31b}$$

$$\dot{c}_{ij} = \begin{bmatrix}\hat{\phi}(\vartheta(f(x_j,x^{(j)})) - \vartheta(f(x_i,x^{(i)}))) \\ \hat{\phi}(\vartheta(v_j) - \vartheta(v_i))\end{bmatrix}^\top \tag{31c}$$

$$\Gamma\begin{bmatrix}\hat{\phi}(\vartheta(f(x_j,x^{(j)})) - \vartheta(f(x_i,x^{(i)}))) \\ \hat{\phi}(\vartheta(v_j) - \vartheta(v_i))\end{bmatrix} \tag{31d}$$

As in Section III-E, it can be shown that the equilibria of the system are found where $f(x_j, x^{(j)}) = f(x_i, x^{(i)})$ for all $i, j \in \mathcal{V}$.

C. Node-based protocol I

The first node-based protocol resembles edge-based protocol I, with the distinction that the adaptive gains are implemented differently. As will become evident soon, this difference in gain implementation, although minor, has a crucial effect on the evolution of the agreement function. The node-based gains are implemented, and the resulting consensus protocol is given below.

$$\dot{x}_i = (g'(x_i))^{-1}d_i \sum_{j\in\mathcal{N}_i} \hat{\phi}(\vartheta(x_j) - \vartheta(x_i)) \tag{32a}$$

$$\dot{d}_i = \sum_{j\in\mathcal{V}} \hat{\phi}(\vartheta(x_j) - \vartheta(x_i))^\top \Gamma \sum_{j\in\mathcal{V}} \hat{\phi}(\vartheta(x_j) - \vartheta(x_i)) \tag{32b}$$

Note that the the gain dynamics are now governed by a quadratic function of the sum of the difference function, and that the gains are now represented by a vector $d \in \mathbb{R}^n$, whilst the edge-based cases had a gain matrix $c \in \mathbb{R}^{n\times n}$. The effect of this on the evolution of the agreement function is discussed next, as we shall see that node-based protocol I is not guaranteed to be time-invariant.

To illustrate this, conditions that guarantee time-invariance of the agreement function are determined. By now we know that the agreement function is preserved if its argument is constant over time. Consider the following implication.

$$\begin{aligned}\sum_{i\in\mathcal{V}} \dot{g}_i(x_i) &= \sum_{i=1}^{n} (g'(x_i))\dot{x}_i && (33a)\\ &= \sum_{i\in\mathcal{V}} d_i \sum_{j\in\mathcal{N}_i} \hat{\phi}(\vartheta(x_j) - \vartheta(x_i)) = 0 && (33b)\\ &\iff \\ d_i &= d_j \quad \forall i \in \mathcal{V}, && (33c)\end{aligned}$$

It shows that the argument is constant if and only if all adaptive gains $d_i(t)$ for $i \in \mathcal{V}$ are equal for all $t > 0$. Then, the implementation of an adaptive gain at each node becomes pointless. Furthermore, equality of the gains for all $t > 0$ cannot be guaranteed by (32).

Node-based protocol I, however, does achieve state consensus. The conditions for state consensus are summarized in Theorem 4.

Theorem 4 (State consensus of node-based protocol I): Assume that $g_i(\cdot)$ is strictly increasing, $\hat{\phi}(\cdot)$ is continuous, locally Lipschitz, odd and strictly increasing, $\vartheta(\cdot)$ is differentiable and $\vartheta'(x_i)$ is locally Lipschitz and strictly positive. Furthermore, the gains are initialized such that $d_i(0) > 0$ for all $i \in \mathcal{V}$. Then, edge-based protocol II (32) achieves state consensus.

Proof: Define $\tilde{e}_i := x_i - \bar{x}$, with $\bar{x}$ the arithmetic mean of the states, as the error function for reaching consensus such that $\tilde{e}_i = 0$ implies that agent i equals the consensus value. Since $g_i(x_i)$ is strictly increasing, an alternative error function is $e_i := g_i(x_i) - g_i(\bar{x})$. Asymptotic stability of e is equivalent to consensus reaching of the states x. The Lyapunov direct method is applied to prove asymptotic stability of

$$\dot{e}_i = g_i'(x_i)\dot{x}_i = d_i \sum_{j\in\mathcal{N}_i} \hat{\phi}(\vartheta(x_j) - \vartheta(x_i)) \tag{34a}$$

Consider the Lyapunov candidate function $V = \frac{1}{2}e^\top \mathcal{L} e$ where $\mathcal{L}$ is the symmetric Laplacian matrix of the connected and undirected graph. The Lyapunov function satisfies $V(0) =$

0 and $V(e) > 0$ for all $(e) \neq (0)$. The time-derivative is then given by

$$\dot{V} = e^\top \mathcal{L} \dot{e} \tag{35a}$$

$$= \begin{bmatrix} g_i(x_1) - g_i(\bar{x}) \\ \vdots \\ g_i(x_n) - g_i(\bar{x}) \end{bmatrix}^\top \mathcal{L} \begin{bmatrix} d_1 \sum_{j\in\mathcal{N}_1} \hat{\phi}(\vartheta(x_j) - \vartheta(x_1)) \\ \vdots \\ d_n \sum_{j\in\mathcal{N}_n} \hat{\phi}(\vartheta(x_j) - \vartheta(x_n)) \end{bmatrix} \tag{35b}$$

$$= \begin{bmatrix} \sum_{j\in\mathcal{N}_1} g_i(x_1) - g_i(x_j) \\ \vdots \\ \sum_{j\in\mathcal{N}_n} g_i(x_n) - g_i(x_j) \end{bmatrix}^\top \cdot \begin{bmatrix} d_1 \sum_{j\in\mathcal{N}_1} \hat{\phi}(\vartheta(x_j) - \vartheta(x_1)) \\ \vdots \\ d_n \sum_{j\in\mathcal{N}_n} \hat{\phi}(\vartheta(x_j) - \vartheta(x_n)) \end{bmatrix} \tag{35c}$$

$$= - \begin{bmatrix} \sum_{j\in\mathcal{N}_1} g_i(x_j) - g_i(x_1) \\ \vdots \\ \sum_{j\in\mathcal{N}_n} g_i(x_j) - g_i(x_n) \end{bmatrix}^\top \cdot \begin{bmatrix} d_1 \sum_{j\in\mathcal{N}_1} \hat{\phi}(\vartheta(x_j) - \vartheta(x_1)) \\ \vdots \\ d_n \sum_{j\in\mathcal{N}_n} \hat{\phi}(\vartheta(x_j) - \vartheta(x_n)) \end{bmatrix} \tag{35d}$$

The adaptive gain is initialized such that $d_i > 0$ for all $i, j \in \mathcal{V}$. The gain dynamics $\dot{d}_i$ are quadratic, implying that $d_i(t) > 0$ for all $t > 0$. The functions $g_i(\cdot)$, $\hat{\phi}(\cdot)$ and $\vartheta(\cdot)$ are strictly increasing and odd. Thus, each product resulting from 35d results in a positive value, from which it follows that $\dot{V} \leq 0$. From 35d it is clear that $\dot{V} = 0$ implies that $x_i = x_j$ for all $i, j \in \mathcal{V}$, which in turn implies that $e_i = 0$ for all $i \in \mathcal{V}$. ■

To allow for general consensus on some function $f(\cdot)$, the following protocol is proposed.

$$\dot{x}_i = (g'(x_i))^{-1} d_i \sum_{j\in\mathcal{N}_i} \hat{\phi}(\vartheta(f(x_j, x^{(j)})) - \vartheta(f(x_i, x^{(i)}))) \tag{36a}$$

$$\dot{d}_i = (\sum_{j\in\mathcal{N}_i} \hat{\phi}(\vartheta(f(x_j, x^{(j)})) - \vartheta(f(x_i, x^{(i)}))))^2 \tag{36b}$$

It can be shown that the equilibria of the system are found where $f(x_j, x^{(j)}) = f(x_i, x^{(i)})$ for all $i, j \in \mathcal{V}$.

D. Node-based protocol II

The final consensus protocol proposed in this paper assigns an adaptive gain to each node of the in the network as previously discussed. The protocol is inspired by the node-based protocol of [4]. A protocol state v is introduced, as before, and the structure is similar to that of the edge-based protocol in Section IV-B.

$$\dot{x}_i = -(g'(x_i))^{-1} v_i \tag{37a}$$

$$\dot{v}_i = -v_i - d_i \sum_{j\in\mathcal{N}_i} \left[\hat{\phi}(\vartheta(x_j) - \vartheta(x_i)) - \hat{\phi}(\vartheta(v_j) - \vartheta(v_i))\right] \tag{37b}$$

$$\dot{d}_i = \sum_{j\in\mathcal{V}} \begin{bmatrix} \hat{\phi}(\vartheta(x_j) - \vartheta(x_i)) \\ \hat{\phi}(\vartheta(v_j) - \vartheta(v_i)) \end{bmatrix}^\top \Gamma \sum_{j\in\mathcal{V}} \begin{bmatrix} \hat{\phi}(\vartheta(x_j) - \vartheta(x_i)) \\ \hat{\phi}(\vartheta(v_j) - \vartheta(v_i)) \end{bmatrix} \tag{37c}$$

A similar reasoning to the evaluation of the agreement function, as in the previous section on node-based protocol I, results in the observation that the protocol cannot be guaranteed to preserve the value of the agreement function.

However, the algorithm is capable of reaching state consensus under the right conditions. These conditions are summarized in Theorem 2

Conjecture 2 (State consensus of node-based protocol II): Assume that $g_i(\cdot)$ is strictly increasing, $\hat{\phi}(\cdot)$ is continuous, locally Lipschitz, odd and strictly increasing, $\vartheta(\cdot)$ is differentiable and $\vartheta'(x_i)$ is locally Lipschitz and strictly positive. Then, node-based protocol II (37) achieves state consensus. Furthermore, the auxiliary states v converge to zero.

The protocol is extended so that general consensus on some function $f(\cdot)$ can be reached. A protocol that achieves this has the following from.

$$\dot{x}_i = -\, (g'(x_i))^{-1} v_i \tag{38a}$$

$$\dot{v}_i = -\, v_i - d_i \sum_{j\in\mathcal{N}_i} [\hat{\phi}(\vartheta(f(x_j, x^{(j)})) - \vartheta(f(x_i, x^{(i)})))- \tag{38b}$$

$$\hat{\phi}(\vartheta(v_j) - \vartheta(v_i))] \tag{38c}$$

$$\dot{c}_{ij} = \sum_{j\in\mathcal{V}} \begin{bmatrix} \hat{\phi}(\vartheta(f(x_j, x^{(j)})) - \vartheta(f(x_i, x^{(i)}))) \\ \hat{\phi}(\vartheta(v_j) - \vartheta(v_i)) \end{bmatrix}^\top \tag{38d}$$

$$\Gamma \sum_{j\in\mathcal{V}} \begin{bmatrix} \hat{\phi}(\vartheta(f(x_j, x^{(j)})) - \vartheta(f(x_i, x^{(i)}))) \\ \hat{\phi}(\vartheta(v_j) - \vartheta(v_i)) \end{bmatrix} \tag{38e}$$

As in Section III-E, it can be shown that the equilibria of the system are found where $f(x_j, x^{(j)}) = f(x_i, x^{(i)})$ for all $i, j \in \mathcal{V}$.

V. Proportional power consensus in DC microgrids

In this section, we apply the theory developed in the previous sections to the problem of proportional power consensus in DC microgrids. To begin, a power consensus algorithm [5] from the literature is introduced. The algorithm achieves proportional power consensus while stabilizing the voltage and preserving the value of the weighted geometric mean. The voltage dynamics are a function of the local voltage and the local and neighbouring power measurements, implying that the algorithm is distributed.

$$\dot{V}_i = C_i^{-1} V_i \sum_{j\in\mathcal{N}_{ci}} (C_j^{-1} P_j - C_i^{-1} P_i), \quad \forall i \in \mathcal{V} \tag{39}$$

The constant term C_i represents the power sharing coefficient, V_i the voltage and P_i the power at node i. Note that the term $C_i^{-1}V_i$ follows from the decomposition of the weighted geometric mean with states V and weights C as discussed in Section III-C. The power is calculated by

$$P_i = V_i \sum_{j\in\mathcal{N}_i} y_{ij}V_j \tag{40a}$$

$$= V_i(\sum_{j\in\mathcal{N}_i\cap\mathcal{V}_s} y_{ij}V_j + \sum_{j\in\mathcal{N}_i\cap\mathcal{V}_l} y_{ij}V_j(V_i) \qquad \forall i \in \mathcal{V}_s \tag{40b}$$

Let us take a look at the power function (40b). The derivative with respect to the state is not strictly positive, which is a condition for state consensus. This is evident because the terms $y_{ij} \in Y$ are negative if $j \in \mathcal{V}_l$.

$$\frac{d\vartheta(f(V_i, V^{(i)}))}{dV_i} = C_i^{-1}(\sum_{j\in\mathcal{N}_i\cap\mathcal{V}_s} y_{ij}V_j \tag{41a}$$

$$+ \sum_{j\in\mathcal{N}_i\cap\mathcal{V}_l} y_{ij}(V_j(V_i) + V_i\frac{dV_j(V_i)}{dV_i})). \tag{41b}$$

Thus, we conclude that the power consensus algorithm from [5] is a special case of the general consensus algorithm proposed in Section III-E, where the state is the voltage V, $g_i(V_i) = \log(V_i^{C_i})$, $f(V_i, V^{(i)}) = C_i^{-1}P_i(V_i, V^{(i)})$, $\hat{\phi}(x) = x$ and $\vartheta(x) = x$. Note that proportional consensus is applied as described in Section III-D. The power consensus algorithm is expressed in terms of the general consensus protocol (18). In doing so, alternative functions $\hat{\phi}(\cdot)$ and $\vartheta(\cdot)$ can be implemented whilst still achieving the objectives.

$$\dot{V}_i = C_i^{-1}V_i \sum_{j\in\mathcal{N}_{ci}} \hat{\phi}(\vartheta(C_j^{-1}P_j) - \vartheta(C_i^{-1}P_i)) \tag{42}$$

The dependence of the power on the voltages is omitted since power measurements are made. Some alternative functions are tested and simulated in Section VI.

Conjecture 3 (Nonlinear consensus protocol): Assume that $\hat{\phi}(\cdot)$ is continuous, locally Lipschitz, odd and strictly increasing, $\vartheta(\cdot)$ is differentiable and $\vartheta'(x_i)$ is locally Lipschitz and strictly positive. The nonlinear power consensus algorithm (**??**) achieves proportional power consensus among the sources of a DC microgrid according to power sharing coefficients $C_i > 0$ for all $i \in \mathcal{V}_s$. Furthermore, the voltages are stabilized and the weighted geometric mean of the source voltages is preserved.

The gained insights on the power consensus algorithm allow us to alter the adaptive gain protocols from Section IV in such a way that power consensus is achieved. This process consists of two steps. The first is the substitution of $g(V_i)$ by $\log(V_i^{C_i})$ such that the weighted geometric mean is preserved. This is also done for the node-based case, although the weighted geometric mean will not be preserved. This way a fair comparison of the adaptive gain protocols can be performed. The second step is the substitution of $f(V_i, V^{(i)})$ by $P_i(V_i, V^{(i)})$.

Edge-based protocol I with power consensus implementation has the following form.

$$\dot{V}_i = C_i^{-1}V_i \sum_{j\in\mathcal{N}_{ci}} c_{ij}\hat{\phi}(\vartheta(C_j^{-1}P_j) - \vartheta(C_i^{-1}P_i)) \tag{43a}$$

$$\dot{c}_{ij} = \Gamma\hat{\phi}(\vartheta(C_j^{-1}P_j) - \vartheta(C_i^{-1}P_i))^2 \tag{43b}$$

Conjecture 4 (Power consensus - edge-based protocol I): Assume that $\hat{\phi}(\cdot)$ is continuous, locally Lipschitz, odd and strictly increasing, $\vartheta(\cdot)$ is differentiable, $\vartheta'(x_i)$ is locally Lipschitz and strictly positive and $\Gamma > 0$. The edge-based power consensus algorithm (43) achieves proportional power consensus among the sources of a DC microgrid according to power sharing coefficients $C_i > 0$ for all $i \in \mathcal{V}_s$, and the voltages are stabilized. Furthermore, the weighted geometric mean of the source voltages is preserved if the adaptive gains are initialized such that $c_{ij}(0) = c_{ji}(0)$ for all $i, j \in \mathcal{V}_s$.

Edge-based protocol II is constructed as follows.

$$\dot{V}_i = -C_i^{-1}V_iv_i \tag{44a}$$

$$\dot{v}_i = -v_i - \sum_{j\in\mathcal{N}_{ci}} c_{ij}[\hat{\phi}(\vartheta(C_j^{-1}P_j) - \vartheta(C_i^{-1}P_i)) - \hat{\phi}(\vartheta(v_j) - \vartheta(v_i))] \tag{44b}$$

$$\dot{c}_{ij} = \begin{bmatrix} \hat{\phi}(\vartheta(C_j^{-1}P_j) - \vartheta(C_i^{-1}P_i)) \\ \hat{\phi}(\vartheta(v_j) - \vartheta(v_i)) \end{bmatrix}^\top \Gamma \begin{bmatrix} \hat{\phi}(\vartheta(C_j^{-1}P_j) - \vartheta(C_i^{-1}P_i)) \\ \hat{\phi}(\vartheta(v_j) - \vartheta(v_i)) \end{bmatrix} \tag{44c}$$

Conjecture 5 (Power consensus - Edge-based protocol II): Assume that $\hat{\phi}(\cdot)$ is continuous, locally Lipschitz, odd and strictly increasing, $\vartheta(\cdot)$ is differentiable, $\vartheta'(x_i)$ is locally Lipschitz and strictly positive and $\Gamma > 0$. The edge-based power consensus algorithm (44) achieves proportional power consensus among the sources of a DC microgrid according to power sharing coefficients $C_i > 0$ for all $i \in \mathcal{V}_s$, and the voltages are stabilized. Furthermore, the weighted geometric mean of the source voltages is preserved if the adaptive gains are initialized such that $c_{ij}(0) = c_{ji}(0)$ and the auxiliary states are initialized such that the arithmetic mean $\bar{v}(0) = 0$ for all $i, j \in \mathcal{V}_s$.

Node-based protocol I is adjusted accordingly, resulting in the following protocol.

$$\dot{x}_i = C_i^{-1}V_id_i \sum_{j\in\mathcal{N}_{ci}} \hat{\phi}(\vartheta(C_j^{-1}P_j) - \vartheta(C_i^{-1}P_i)) \tag{45a}$$

$$\dot{d}_i = \Gamma(\sum_{j\in\mathcal{N}_{ci}} \hat{\phi}(\vartheta(C_j^{-1}P_j) - \vartheta(C_i^{-1}P_i)))^2 \tag{45b}$$

Conjecture 6 (Power consensus - Node-based protocol I): Assume that $\hat{\phi}(\cdot)$ is continuous, locally Lipschitz, odd and strictly increasing, $\vartheta(\cdot)$ is differentiable, $\vartheta'(x_i)$ is locally Lipschitz and strictly positive and $\Gamma > 0$. The node-based power consensus algorithm (45) achieves proportional power consensus among the sources of a DC microgrid according to power sharing coefficients $C_i > 0$ for all $i \in \mathcal{V}_s$, and the voltages are stabilized.

Finally, power consensus is implemented in node-based protocol II.

$$\dot{x}_i = -\, C_i^{-1} V_i v_i \tag{46a}$$

$$\dot{v}_i = -\, v_i - d_i \sum_{j \in \mathcal{N}_{ci}} [\hat{\phi}(\vartheta(C_j^{-1} P_j) - \vartheta(C_i^{-1} P_i)) - \hat{\phi}(\vartheta(v_j) - \vartheta(v_i))] \tag{46b}$$

$$\dot{d}_i = \sum_{j \in \mathcal{N}_{ci}} \begin{bmatrix} \hat{\phi}(\vartheta(C_j^{-1} P_j) - \vartheta(C_i^{-1} P_i)) \\ \hat{\phi}(\vartheta(v_j) - \vartheta(v_i)) \end{bmatrix}^\top \Gamma \sum_{j \in \mathcal{N}_{ci}} \begin{bmatrix} \hat{\phi}(\vartheta(C_j^{-1} P_j) - \vartheta(C_i^{-1} P_i)) \\ \hat{\phi}(\vartheta(v_j) - \vartheta(v_i)) \end{bmatrix} \tag{46c}$$

Conjecture 7 (Power consensus - Node-based protocol II): Assume that $\hat{\phi}(\cdot)$ is continuous, locally Lipschitz, odd and strictly increasing, $\vartheta(\cdot)$ is differentiable, $\vartheta'(x_i)$ is locally Lipschitz and strictly positive and $\Gamma > 0$. The edge-based power consensus algorithm (46) achieves proportional power consensus among the sources of a DC microgrid according to power sharing coefficients $C_i > 0$ for all $i \in \mathcal{V}_s$, and the voltages are stabilized.

VI. Numerical simulations

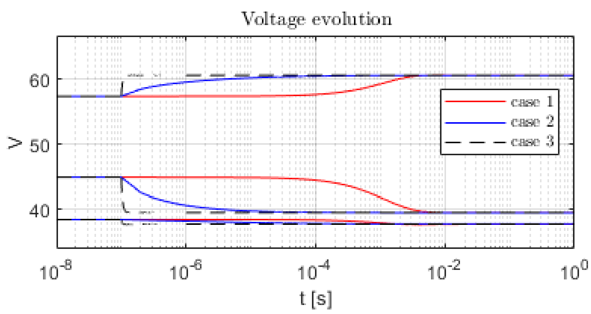

Fig. 1. State trajectories with alternative functions

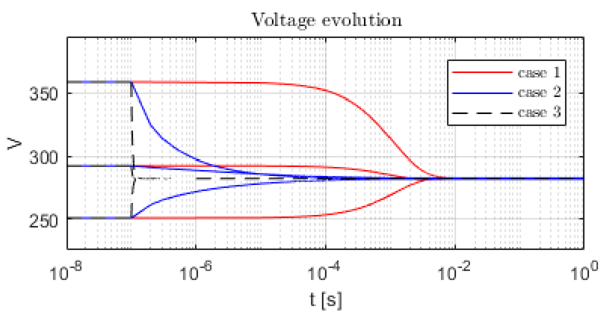

Fig. 2. Proportional power with alternative functions

The DC microgrid considered for the sake of simulation consists of three sources (blue) and six loads (white), which are described by a connected and undirected graph. The network graph is shown in Figure 3. The loads are modeled as ZIP loads [5]. Initially, loads 6, 7, 8 and 9 are turned on. After one time step, loads 6 is turned off and loads 4 and 5 are turned on. The microgrid parameters are summarized in Table I. The blue lines represent the communication network.

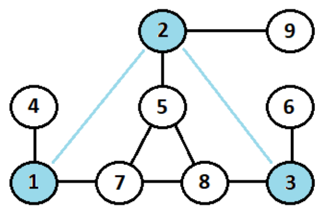

Fig. 3. Microgrid network

TABLE I
MICROGRID PARAMETERS

Line	γ (Ω^{-1})	Node	Type	C	
1	0.165	1	Source	1	-
2	0.141	2	Source	2	-
3	0.135	3	Source	3	-
4	0.195	4	Z-load		0.09 Ω^{-1}
5	0.105	5	I-load		-1.4 A
6	0.171	6	P-load		-80 W
7	0.138	7	Z-load		0.04 Ω^{-1}
8	0.171	8	I-load		-1.1 A
9	0.258	9	P-load		-80 W

The general structure of the proposed protocols allows for a variety of functions $\hat{\phi}(\cdot)$ and $\vartheta(\cdot)$ to be implemented. To illustrate the effect of each function, three cases are simulated in the general power consensus case without adaptive gain, see equation (42). The first case lets $\hat{\phi}(x) = x$ and $\vartheta(x) = x$. The second case lets $\hat{\phi}(x) = x + x^3$, and the third case lets $\vartheta(x) = x + x^3$. A substantial improvement is observed for each case. However, case 2 and 3 may be too aggressive for the power consensus algorithm. Figure 2 shows that each case achieves proportional power sharing.

The edge-based protocols are simulated for the case where $\hat{\phi}(x) = x$ and $\vartheta(x) = x$. Figure 4 shows the state response, and Figure 5.

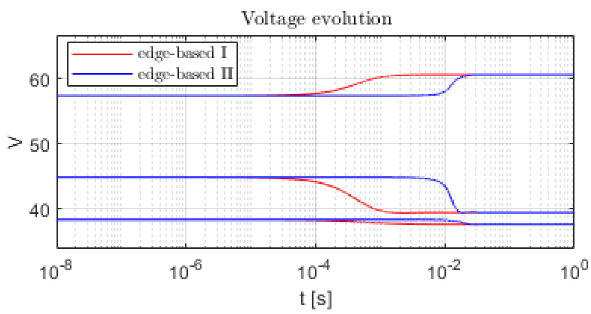

Fig. 4. State trajectories of edge-based protocols

Similarly, the simulations are performed for the node-based protocols. The results are shown in Figure 4 and Figure 5.

Edge-based protocol I and node-based protocol I show an improvement w.r.t. to the case without adaptive gain, however, edge-based protocol II and node-based protocol are

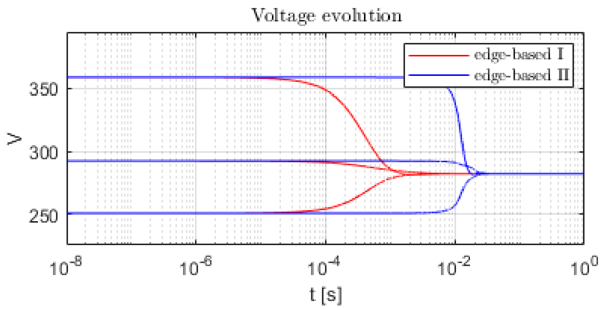

Fig. 5. Proportional power of edge-based protocols

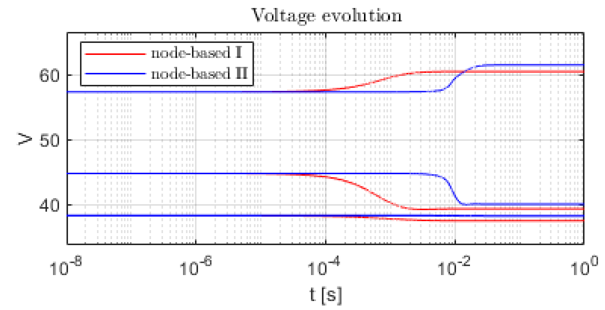

Fig. 6. State trajectories of node-based protocols

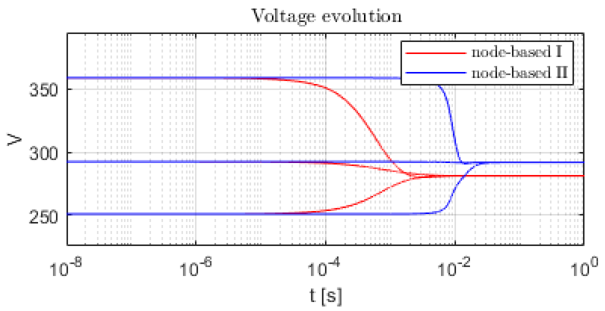

Fig. 7. Proportional power of node-based protocols

slower. Nonetheless, all protocols achieve proportional power consensus and stabilize the voltage.

VII. Conclusion

The power consensus algorithm from [5] has been analyzed and general guidelines have been presented to extend the algorithm to include nonlinear difference functions, while guaranteeing that the objectives are met. Four alternative consensus adaptive gain protocols are proposed. Their general structure allows for the implementation of nonlinear difference functions. Conditions for time-invariance of the adaptive gain consensus protocols are given, and it is shown that time-invariance cannot be guaranteed for the node-based case.

Several results of the paper are left to conjecture. All results on general consensus are numerically shown to converge, however, the paper lacks the necessary and sufficient conditions. Finding these conditions is an interesting open future research topic. Furthermore, the functions $\hat{\phi}(\cdot)$ and $\vartheta(\cdot)$ are assumed to be the same for the (protocol) state dynamics and the gain dynamics. Another future research topic is to differentiate between these functions and finding conditions that guarantee consensus, therefore allowing for more freedom in the design of the adaptive gain consensus protocols.

References

[1] D. Bauso, L. Giarré, R. Pesenti, "Non-linear protocols for optimal distributed consensus in networks of dynamic agents," Systems & Control Letters, 55(11), pp.918-928, 2006.

[2] J. Cortés, "Distributed algorithms for reaching consensus on general functions," Automatica, 44, no. 3, pp.726-737, 2008.

[3] Z. Li, W. Ren, X. Liu, M. Fu, "Consensus of multi-agent systems with general linear and Lipschitz nonlinear dynamics using distributed adaptive protocols," IEEE Transactions on Automatic Control, 58(7), pp.1786-1791, 2013.

[4] Z. Li, W. Ren, X. Liu, L. Xie, "Distributed consensus of linear multi-agent systems with adaptive dynamic protocols," Automatica, 49, no. 7, pp.1986-1995, 2013.

[5] C. De Persis, E.R.A. Weitenberg, F. Dörfler, "A power consensus algorithm for DC microgrids," Automatica, 89, pp.364-375, 2018.

[6] F. Dörfler, F. Bullo, "Kron Reduction of Graphs With Applications to Electrical Networks," IEEE Trans. on Circuits and Systems, 60(1), pp.150-163, 2013.

[7] R.H. Lasseter, P. Paolo. "Microgrid: A conceptual solution." In IEEE Power Electronics Specialists Conference, vol. 6, pp. 4285-4291. 2004.

[8] S. Parhizi, H. Lofti, A. Khodaei, S. Bahramirad. "State of the are in research on microgrids: A review." IEEE Access 3: 890-926. 2015.

[9] F. Dörfler, J.W. Simpson-Porco', F. Bullo. "Breaking the hierarchy: Distributed control and economic optimality in microgrids."IEEE Transactions on Control of Network Systems, 3(3), pp.241-253", 2016.

[10] S. Bolognaia, S. Zampieri. "A distributed control strategy for reactive power compensation in smart microgrids,"IEEE Transactions on Automatic Control, 58(11), pp.2818-2833", 2013.

[11] C. De Persis, M. Monshizadeh. "Bregman storage functions for microgrid control," IEEE Transactions on Automatic Control, 63(1), pp.53-68, 2018.

[12] J.W. Simpson-Porco, F. Dörfler, F. Bullo, Q. Shafiee, J.M. Guerrero. "Stability, power sharing & distributed secondary control in droop-controller microgrids," In Smart Grid Communications (SmartGridComm), 2013 IEEE International Conference on (pp. 672-677). IEEE, 2013.

[13] H. Jiayi, J. Chuanwen, X. Rong. "A review on distributed energy resources and MicroGrid," Renewable and Sustainable Energy Reviews 12, no. 9: 2472-2483, 2008.

[14] J.W. Simpson-Porco, F. Dörfler, F. Bullo. "Synchronization and power sharing for droop-controlled inverters in islanded microgrids," Automatica, 49(9), pp.2603-2611, 2013.

[15] C. De Persis, N. Monshizadeh. "Bregman storage functions for microgrid control," IEEE Transactions on Automatic Control, 63(1), pp.53-68, 2018.

[16] J. Schiffer, T. Seel, J. Raisch, T. Sezi. "Voltage Stability and Reactive Power Sharing in Inverter-Based Microgrids With Consensus-Based Distributed Voltage Control," IEEE Trans. Contr. Sys. Techn., 24(1), pp.96-109, 2016.

[17] A. Bidram, A. Davoudi. "Hierarchical structure of microgrids control system," IEEE Transactions on Smart Grid, 3(4), pp.1963-1976, 2012.

[18] N. Li, C. Zhao, L. Chen. "Connecting automatic generation control and economic dispatch from an optimization view," IEEE Transactions on Control of Network Systems, 3(3), pp.254-264, 2016.

[19] J.W. Simpsons-Porco, F. Dörfler, F. Bullo. "Voltage stabilization in microgrids via quadratic droop control," In 52nd IEEE Conference on Decision and Control, 2013.

[20] J. Zhao, F. Dörfler. "Distributed control and optimization in DC microgrids," Automatica, 61, pp.18-26, 2015.

[21] J. Rocabert, A. Luna, F. Blaabjerg, P. Rodriguez. "Control of power converters in AC microgrids," IEEE transactions on power electronics, 27(11), pp.4734-4749, 2012.

[22] V. Nasirian, S. Moayedi, A. Davoudi, F.L. Lewis. "Distributed cooperative control of dc microgrids," IEEE Transactions on Power Electronics, 30(4), pp.2288-2303, 2015.

[23] F. Dörfler, J.W. Simpson-Porco, F. Bullo. "Electrical Networks and Algebraic Graph Theory: Models, Properties, and Applications," Proceedings of the IEEE, 106(5), pp.977-1005, 2018.

Bibliography

[1] D. T. Ton and M. A. Smith, *The US department of energy's microgrid initiative.* The Electricity Journal 25, no. 8: 84-94, 2012.

[2] H. Jiayi, J. Chuanwen, and X. Rong, *A review on distributed energy resources and MicroGrid.* Renewable and Sustainable Energy Reviews 12.9: 2472-2483, 2008.

[3] R. Lasseter and P. Paigi, *Microgrid: A conceptual solution.* In Power Electronics Specialists Conference, 2004. PESC 04. 2004 IEEE 35th Annual (Vol. 6, pp. 4285-4290). IEEE, 2004.

[4] F. Dörfler, J. W. Simpson-Porco, and F. Bullo, *Breaking the hierarchy: Distributed control and economic optimality in microgrids.* IEEE Transactions on Control of Network Systems, 3(3), pp.241-253, 2016.

[5] S. Parhizi, L. Hossein, A. K., and S. B., *State of the Art in Research on Microgrids: A Review.* Ieee Access 3, no. 1: 890-925, 2015.

[6] D. Bauso, L. Giarré, and R. Pesenti, *Non-linear protocols for optimal distributed consensus in networks of dynamic agents.* Systems & Control Letters, 55(11), pp.918-928, 2006.

[7] J. Cortés, *Distributed algorithms for reaching consensus on general functions.* Automatica, 44, no. 3, pp.726-737, 2008.

[8] Z. Li, W. Ren, X. Liu, and M. Fu, *Consensus of multi-agent systems with general linear and Lipschitz nonlinear dynamics using distributed adaptive protocols.* IEEE Transactions on Automatic Control, 58(7), pp.1786-1791, 2013.

[9] Z. Li, W. Ren, X. Liu, and L. Xie, *Distributed consensus of linear multi-agent systems with adaptive dynamic protocols.* Automatica, 49, no. 7, pp.1986-1995, 2013.

[10] F. Dörfler, J. W. Simpson-Porco, and F. Bullo, *Electrical Networks and Algebraic Graph Theory: Models, Properties, and Applications.* Proceedings of the IEEE, 106(5), pp.977-1005, 2018.

[11] C. De Persis, E. Weitenberg, and F. Dörfler, *A power consensus algorithm for DC microgrids.* Automatica, 89, pp.364-375, 2018.

[12] S. Bolognani and S. Zampieri, *A distributed control strategy for reactive power compensation in smart microgrids.* IEEE Transactions on Automatic Control, 58(11), pp.2818-2833, 2013.

[13] C. De Persis and N. Monshizadeh, *Bregman storage functions for microgrid control.* IEEE Transactions on Automatic Control, 63(1), pp.53-68, 2018.

[14] J. W. Simpson-Porco, F. Dörfler, and F. Bullo, *Synchronization and power sharing for droop-controlled inverters in islanded microgrids.* Automatica, 49(9), pp.2603-2611, 2013.

[15] J. W. Simpson-Porco, F. Dorfler, F. Bullo, Q. Shafiee, and J. M. Guerrero, *Stability, power sharing, & distributed secondary control in droop-controlled microgrids.* In Smart Grid Communications (SmartGridComm), 2013 IEEE International Conference on (pp. 672-677). IEEE, 2013.

[16] J. Schiffer, T. Seel, J. Raisch, and T. Sezi, *Voltage Stability and Reactive Power Sharing in Inverter-Based Microgrids With Consensus-Based Distributed Voltage Control.* IEEE Trans. Contr. Sys. Techn., 24(1), pp.96-109, 2016.

[17] A. Bidram and A. Davoudi, *Hierarchical structure of microgrids control system.* IEEE Transactions on Smart Grid, 3(4), pp.1963-1976, 2012.

[18] N. Li, C. Zhao, and L. Chen, *Connecting automatic generation control and economic dispatch from an optimization view.* IEEE Transactions on Control of Network Systems, 3(3), pp.254-264, 2016.

[19] J. W. Simpson-Porco, F. Dörfler, and F. Bullo, *Voltage stabilization in microgrids via quadratic droop control.* In 52nd IEEE Conference on Decision and Control, 2013.

[20] J. Zhao and F. Dörfler, *Distributed control and optimization in DC microgrids.* Automatica, 61, pp.18-26, 2015.

[21] J. Rocabert, A. Luna, F. Blaabjerg, and P. Rodriguez, *Control of power converters in AC microgrids.* IEEE transactions on power electronics, 27(11), pp.4734-4749, 2012.

[22] V. Nasirian, S. Moayedi, A. Davoudi, and F. Lewis, *Distributed cooperative control of dc microgrids.* IEEE Transactions on Power Electronics, 30(4), pp.2288-2303, 2015.

[23] F. Dörfler and F. Bullo, *Kron Reduction of Graphs With Applications to Electrical Networks.* IEEE Trans. on Circuits and Systems, 60(1), pp.150-163, 2013.

[24] E. Jones, T. Oliphant, P. Peterson, *et al.*, *SciPy: Open source scientific tools for Python.* http://www.scipy.org/, 2001.

[25] T. Oliphant, *NumPy: A guide to NumPy.* USA: Trelgol Publishing, http://www.numpy.org/, 2006.

Glossary

List of Acronyms

WGM	Weighted geometric mean
PPC	Proportional power consensus
PCA	Power consensus algorithm
EB1	Edge-based protocol I
EB2	Edge-based protocol II
NB1	Node-based protocol I
NB2	Node-based protocol II
AF	Agreement function
NCP	Nonlinear consensus protocol

List of Symbols

χ	AF
$_c$	Communication graph
$_l$	Set of loads
$_s$	Set of sources
$\bar{x}$	Arithmetic mean
$\bar{x}^{\mathrm{w}}$	Weighted geometric mean
$\mathcal{A}$	Adjacency metrix
$\mathcal{B}$	Incidence matrix
$\mathcal{E}$	Set of edges

$\mathcal{G}$	Connected and undirected graph
$\mathcal{N}_i$	Set of neighbouring nodes
$\mathcal{V}$	Set of nodes
$\mathcal{W}$	Diagonal matrix of weights
m	Number of edges
n	Number of nodes
$x^{(i)}$	Set of neighbouring states
Γ	Conductance matrix
γ	Conductance
C	Power sharing coefficient
I	Current
P	Power
r	Resistance
V	Voltage