DELFT UNIVERSITY OF TECHNOLOGY
MSC THESIS

Gaussian Hardy spaces

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Introduction

This thesis is based on a preprint by Pierre Portal [Por12].

This thesis is about Hardy spaces and in particular the Gaussian Hardy spaces. Here we replace the Lebesgue measure with the Gaussian measure i.e.,

$$\mathrm{d}\gamma(x) = \pi^{-\frac{d}{2}} e^{-|x|^2}\,\mathrm{d}x \text{ for } x \in \mathbf{R}^d.$$

The ultimate goal is to build an equally rich theory as in the Lebesgue measure case.

There is an abundance of equivalent definitions for the Hardy spaces on $(\mathbf{R}^d, |\cdot|)$. We will only name the few of them that are relevant for this thesis. The first one is the atomic Hardy space $H^1_{\mathrm{at}}(\mathbf{R}^d)$. Here an atom is a complex-valued function a defined on $\mathbf{R}^d$ which is supported on a cube Q and is such that

$$\int_Q a(x)\,\mathrm{d}x = 0 \text{ and } \|a\|_{L^\infty(\mathbf{R}^d)} \leqslant \frac{1}{|Q|}.$$

The space atomic $H^1(\mathbf{R}^d)$ denoted by $H^1_{\mathrm{at}}(\mathbf{R}^d)$ is defined by

$$H^1_{\mathrm{at}}(\mathbf{R}^d) := \left\{ \sum_j \lambda_j a_j \colon a_j \text{ atoms}, \lambda_j \in \mathbf{C}, \sum_j |\lambda_j| < \infty \right\}$$

with norm

$$\|f\|_{H^1_{\mathrm{at}}(\mathbf{R}^d)} := \inf \left\{ \sum_j |\lambda_j| \colon f = \sum_j \lambda_j a_j \right\}.$$

This definition is equivalent to some maximal function Hardy spaces and some conical square function Hardy spaces but more about those later on.

There are now important questions one can ask. What is the dual space? It is know that this is $\mathrm{BMO}(\mathbf{R}^d)$. Furthermore the Calderón-Zygmund operators are not bounded on $L^1(\mathbf{R}^d)$. We do have that the operator is bounded on weak L^1. This is sometimes enough but a downside to this space is that it is not a Banach space. Luckily we do have boundedness on $H^1(\mathbf{R}^d)$! It can be shown that the atomic space $H^1_{\mathrm{at}}(\mathbf{R}^d)$ is a proper subspace of $L^1(\mathbf{R}^d)$.

We could now try to replace the Lebesgue measure with the Gaussian measure and try to mimic all arguments. However, this quickly fails. Many of the covering arguments in harmonic analysis rely on the doubling property of the measure. That is: if μ is a doubling measure then we have for all $r > 0$ and x in $\mathbf{R}^d$ that there exists $C > 0$ independent on r and x such that

$$\mu(B(x, 2r)) \leqslant C\mu(B(x, r)).$$

The Gaussian measure is unfortunately not doubling.

Mauceri and Meda have tried to answer these questions at least partially in [MM07] in the case of *a* Gaussian Hardy space. They have taken the route of an atomic Hardy spaces as in done in the Lebesgue measure case. For this space they have proven that the dual is BMO(γ). Unfortunately they have proven in the follow-up paper [MMS10] that some Riesz transforms are only bounded on this space in dimension one. This is surely unsatisfactory. One possible reason that this happens is that their definitions of the atoms do not relate well enough to the nature of the Ornstein-Uhlenbeck operator. They have the same cancellation condition $\int a = 0$ which appears to be unnatural. One could try to drop this condition but it can be shown that in the Euclidean case we then just get the space $L^1(\mathbf{R}^d)$. This will be subject of future research.

The Mauceri and Meda paper did develop a potentially useful technique in Gaussian harmonic analysis. This is the tool of the so called *admissible balls*. Here we are averaging only over balls where the radius is at maximum a fixed parameter a times $m(x) = \min(1, |x|^{-1})$ where x is the center of the ball. The key observation here is that on these balls the Gaussian measure is doubling. Using this observation we could again try to adapt the usual arguments. We can quickly see that this fails. Admissible balls are small when their centre is far away from the origin. Tools like the Whitney decompositions of open set at least require that the size of the balls is comparable to their distance to the boundary of that open set. This way these balls would have to be very large.

So we first try a different route. Pierre Portal in [Por12] has taken the approach of a maximal function and a conical square function Hardy space. In the Lebesgue measure case these are defined using

$$Mu(x) := \sup_{(y,t)\in\Gamma_x} |e^{t^2\Delta}u(y)|,$$

$$Su(x) := \left(\iint_{\Gamma_x \frac{1}{|B(y,t)|}} |t\nabla e^{t^2\Delta}u(y)|^2 \, \mathrm{d}y \, \frac{\mathrm{d}t}{t} \right)^{\frac{1}{2}},$$

where,

$$\Gamma_x := \left\{(y,t) \in \mathbf{R}^d \times (0,\infty) \colon |y-x| < t \right\}.$$

and $\Delta = \sum_j \partial_j^2$ is the Laplacian. Now the Hardy spaces can be defined as the completion of the space of compactly supported functions $C_c(\mathbf{R}^d)$ with respect to the norm

$$\|f\|_{H^1} := \|Tf\|_{L^1(\mathbf{R}^d)} + \|f\|_{L^1(\mathbf{R}^d)}$$

where this gives equivalent norms if we pick T to be either M or S.

In this thesis we are interested in the Gaussian versions of the Hardy spaces on $\mathbf{R}^d$. That is, we replace the Lebesgue measure with the Gaussian measure, i.e.,

$$\mathrm{d}\gamma(x) = \pi^{-\frac{d}{2}} e^{-|x|^2} \mathrm{d}x.$$

Our main question is if the Hardy spaces $h^1_{\max,a}(\gamma)$ and $h^1_{\mathrm{quad},a'}(\gamma)$ are the same for some $a, a' > 0$. These spaces are defined as follows. First let L be the Ornstein-Uhlenbeck operator

$$L := -\frac{1}{2}\sum_j \partial_j^* \partial_j$$

and let

$$T_a^* u(x) := \sup_{(y,t)\in\Gamma_x^a(\gamma)} |e^{t^2 L}u(y)|,$$

$$S_a u(x) := \left(\iint_{\Gamma_x^a(\gamma)} \frac{1}{\gamma(B(y,t))} |t\nabla e^{t^2 L} u(y)|^2 \, \mathrm{d}\gamma(y) \, \frac{\mathrm{d}t}{t} \right)^{\frac{1}{2}},$$

where,

$$\Gamma_x^a(\gamma) := \left\{ (y,t) \in \mathbf{R}^d \times (0,\infty) \colon |y-x| < t < a \min\left\{1, \frac{1}{|x|}\right\} \right\}.$$

Now $h^1_{\max,a}(\gamma)$ and $h^1_{\mathrm{quad},a'}(\gamma)$ are the completions of the smooth compactly supported functions $C_c^\infty(\mathbf{R}^d)$ with respect to the norms

$$\|u\|_{h^1_{\max,a}(\gamma)} := \|T_a^* u\|_{L^1(\gamma)} + \|u\|_{L^1(\gamma)},$$

and

$$\|u\|_{h^1_{\mathrm{quad},a'}(\gamma)} := \|S_{a'} u\|_{L^1(\gamma)} + \|u\|_{L^1(\gamma)},$$

respectively.

One direction of the equivalence of the norms is already proven in [MvNP10a], there is proven that

$$\|S_{a'} u\|_{L^1(\gamma)} \lesssim \|T_a^* u\|_{L^1(\gamma)}.$$

We are then, of course, interested in the other direction. In particular we will prove that

$$\|T_a^* u\|_{L^1(\gamma)} \lesssim \|S_{a'} u\|_{L^1(\gamma)} + \|u\|_{L^1(\gamma)}.$$

Setup of the proof

In the first chapter the required definitions are given. Also, a few preliminary lemmas, propositions and theorems are given which turn out to be very useful in the sequel. The proof is based around a Calderón reproducing formula which is also proven in this chapter.

The second chapter treats the kernel estimates of the kernels of the operators in the Calderón reproducing formula. A few technical lemmas are given to prove these estimates. Another useful lemma is the lemma that gives the off-diagonal estimates for an operator in the Calderón reproducing formula. There will be situations where these off-diagonal estimates will not work, so another lemma is given which can be used when those estimates fail.

In the third chapter, the notion of molecules is introduced. Here there is proven that a certain operator is a molecule and that the $h^1_{\max,a}$-norm of a certain class of molecules is always bounded by a constant under the right assumptions. This proof fills the remainder of this chapter.

The fourth chapters handles the remainder terms, the required estimates are given.

Finally, in the last chapter the equivalence is proven with the results from the previous chapters. This ends the proof that the Hardy spaces $h^1_{\max,a}(\gamma)$ and $h^1_{\mathrm{quad},a'}(\gamma)$ are the same for certain $a, a' > 0$.

Prerequisites

The material in this report should be understandable after following a basic course in functional analysis and measure theory. Knowing the notions of Bochner integrals and interpolation might come in handy.

Acknowledgments

I want to finish this chapter with some acknowledgments. I'm very grateful for having been given this nice topic by my supervisor Jan van Neerven. It surely was a lot of fun trying to figure out how all these arguments work. Thanks to his supervision I am slightly proud of the work I have done in this thesis, something which I never had in the past.

When I didn't ask Jan van Neerven, I did ask Sonja Cox. She has given me tips for some arguments and suggestions on how to improve the text. Thanks Sonja!

Finally, I would like to thank my office mates during the writing of this thesis, Mikko Kemppainen and Jeroen van Leeuwen. You have been responsible for a nice working environment and I hope to see you guys again in the future.

Contents

1 Preliminaries

In this chapter we introduce the preliminaries which will be used later on in this thesis. In the first two sections we introduce techniques which will be useful in the later sections of this chapter. The study of the *Ornstein-Uhlenbeck* operator is partly what this thesis is about so we introduce and some of its properties in the next section. The necessary tools to prove our main result are introduced in the next three sections. Here we give a Calderón reproducing formula, a doubling property for the Gaussian measure and we introduce the local and global regions of for example the Mehler kernel. The next section introduces the spaces this thesis is about. Finally, the last three sections give technical results which will be useful in the sequel.

1.1 Interchanging integrals and derivatives

In this section we present some results that will be used to rigorously justify interchanging integrals and derivatives.

We will use Hille's theorem which states that under some conditions closed operators and Bochner integrals commute to interchange integrals and weak derivatives.

The following theorem is a theorem taken from [DU77].

1.1 Theorem (Hille)**.** *Let (A, μ) be a σ-finite measure space and let $u : A \to E$ be μ-Bochner integrable and let T be a closed linear operator with domain $\mathscr{D}(T)$ in E taking values in a Banach space F. Assume that f takes its values in $\mathscr{D}(T)$ μ-almost everywhere and the μ-almost everywhere defined function $Tu : A \to F$ is μ-Bochner integrable. Then $\int_A u \,\mathrm{d}\mu$ in $\mathscr{D}(T)$ and*

$$T \int_A u \,\mathrm{d}\mu = \int_A Tu \,\mathrm{d}\mu.$$

Remark. In [DU77] this theorem is proven for finite measure spaces but the proof extends to σ-finite measure spaces.

The following lemma will turn out to be very useful together with theorem 1.1.

1.2 Lemma. *The weak derivative ∂_{x_i} for $i = 1, \ldots, d$ with domain $W^{1,2}(\mathbf{R}^d)$ is closed on $L^2(\mathbf{R}^d)$.*

Proof. Let u_n in $W^{1,2}(\mathbf{R}^d)$ and u be such that $u_n \to u$ in $L^2(\mathbf{R}^d)$. Furthermore assume that $\partial_{x_i} u_n \to v$. We will show that u in $W^{1,2}(\mathbf{R}^d)$ and $\partial_{x_i} u = v$. Let ϕ be a test function. Then

$$\left| \int_{\mathbf{R}^d} (u_n - u) \partial_{x_i} \phi \,\mathrm{d}\lambda \right| \leqslant \|u_n - u\|_{L^2(\mathbf{R}^d)} \|\partial_{x_i} \phi\|_{L^2(\mathbf{R}^d)} \to 0 \text{ as } n \to \infty.$$

And similarly

$$\left|\int_{\mathbf{R}^d}(\partial_{x_i}u_n - v)\phi\,\mathrm{d}\lambda\right| \leqslant \|\partial_{x_i}u_n - v\|_{L^2(\mathbf{R}^d)}\|\phi\|_{L^2(\mathbf{R}^d)} \to 0 \text{ as } n \to \infty.$$

So u in $W^{1,2}(\mathbf{R}^d)$ and $\partial_{x_i}u = v$. ■

Using this lemma we can apply theorem 1.1 to interchange the integral and derivative.

1.2 Interpolation

In what follows we will need to use some interpolation results. We will recall the Riesz-Thorin and the Marcinkiewicz interpolation theorems from [Gra08].

1.3 Theorem (Riesz-Thorin interpolation theorem)**.** *Let (X,μ) and (Y,ν) be two measure spaces. Let T be a linear operator defined on the set of all simple functions on X taking values in the set of measurable functions on Y. Let $1 \leqslant p_0, p_1, q_0, q_1 \leqslant \infty$ and assume that*

$$\|Tu\|_{L^{q_0}} \leqslant M_0\|u\|_{L^{p_0}},$$
$$\|Tu\|_{L^{q_1}} \leqslant M_1\|u\|_{L^{p_1}},$$

for all simple functions u on X. Then for all $0 < \theta < 1$ we have

$$\|Tu\|_{L^q} \leqslant M_0^{1-\theta}M_1^{\theta}\|u\|_{L^p}$$

for all simple functions u on X, where

[1.1] $$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1} \text{ and } \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}.$$

By density, T has a unique extension as a bounded operator from $L^p(X,\mu)$ to $L^q(Y,\nu)$ for all p and q as in [1.1]*.*

Before we recall the Marcinkiewicz interpolation we define the weak L^p spaces.

1.4 Definition. *For u a measurable function on X, the distribution function of u is the function $d_u : [0,\infty) \to [0,\infty]$ defined as follows:*

$$d_u(\alpha) := \mu(\{|u| > \alpha\}).$$

For $0 < p < \infty$ the space weak $L^p(X,\mu)$ is defined as the set of all μ-measurable functions u such that

$$\|u\|_{L^{p,\infty}} := \sup\left\{\gamma d_u(\gamma)^{\frac{1}{p}} : \gamma > 0\right\}$$

is finite.

An operator is said to be of *weak type* (p, q) if it maps L^p to weak L^q.

1.5 Theorem (Marcinkiewicz interpolation theorem)**.** *Let* (X, μ) *and* (Y, ν) *be measure space and let* $0 < p_0 < p_1 \leqslant \infty$. *Let* T *be a sublinear mapping defined on the space* $L^{p_0}(X) + L^{p_1}(X)$ *and taking values in the space of the measurable functions on* Y. *Assume that there exist two positive constants* A_0 *and* A_1 *such that*

$$\|Tu\|_{L^{p_0,\infty}(Y)} \leqslant A_0 \|u\|_{L^{p_0}(X)}, \text{ for all } u \text{ in } L^{p_0}(X),$$
$$\|Tu\|_{L^{p_1,\infty}(Y)} \leqslant A_1 \|u\|_{L^{p_1}(X)}, \text{ for all } u \text{ in } L^{p_1}(X).$$

Then for all $p_0 < p < p_1$ *and for all* u *in* $L^p(X)$ *we have the estimate*

$$\|Tu\|_{L^p(Y)} \leqslant A\|u\|_{L^p(X)},$$

where the constant A *only depends on* p, p_0, p_1, A_0 *and* A_1.

1.3 The Ornstein-Uhlenbeck operator

We will be primarily concerned with the Ornstein-Uhlenbeck operator on $\mathbf{R}^d$ and in particular its semigroup. The Ornstein-Uhlenbeck operator is the correct replacement in the Gaussian case for the Laplacian as the latter one is not symmetric in $L^2(\gamma)$ where

$$\mathrm{d}\gamma(x) = \pi^{-\frac{d}{2}} e^{-|x|^2}\,\mathrm{d}x$$

is the Gaussian measure. The Ornstein-Uhlenbeck operator is given by

$$L := -\frac{1}{2}\sum_{i=1}^{d} \partial_i^* \partial_i = \frac{1}{2}\Delta - x \cdot \nabla,$$

where

$$\partial_i^* = -\partial_i + 2x_i$$

is the formal adjoint of ∂_i in $L^2(\gamma)$.

The survey of [Sjö97] gives some results about the Ornstein-Uhlenbeck operator which we will now briefly summarize. On $L^2(\gamma)$, the closure[1] of the Ornstein-Uhlenbeck operator L generates a semigroup e^{tL} and for this semigroup the normalized Hermite polynomials $(H_\alpha)_{\alpha \in \mathbf{Z}_+^n}$ form an orthonormal basis of eigenfunctions. In particular we have the action of e^{tL}

$$e^{tL}\left(\sum_{\beta \in \mathbf{Z}_+^n} c_\beta H_\beta\right) = \sum_{\beta \in \mathbf{Z}_+^n} e^{-t|\beta|} c_\beta H_\beta \tag{1.2}$$

[1] For the definition of closure see [RS72, page 250]. This book is also useful for the other functional analysis needed in this thesis.

where

$$H_\alpha = \bigotimes_{i=1}^{d} h_{\alpha_i} \text{ and } h_{\alpha_i}(x) = \frac{2^{-\frac{\alpha_i}{2}}}{\sqrt{\alpha_i!}}(-1)^{\alpha_i}\frac{\partial^{\alpha_i}}{\partial x^{\alpha_i}}e^{-x^2}$$

are the normalized Hermite polynomials, $|\beta| = \beta_1 + \ldots + \beta_d$ and $\mathbf{Z}_+ = \{0, 1, 2, \ldots\}$. Different formulas for these polynomials are given in [Sjö97]. We also have the identity which follows immediately from the properties of the semigroup

[1.3] $$Le^{tL}u = \partial_t e^{tL}u \text{ for } t > 0 \text{ and } u \text{ in } L^2(\gamma).$$

Another property of the Ornstein-Uhlenbeck operator that will be often used is the expression for the action of the semigroup as integration against the Mehler kernel, that is,

[1.4] $$e^{tL}u(x) = \int_{\mathbf{R}^d} M_t(x, y)u(y)\,\mathrm{d}y,$$

where the Mehler kernel M_t is given by

$$M_t(x, y) = \pi^{-\frac{d}{2}}\left(1 - e^{-2t}\right)^{-\frac{d}{2}} \exp\left(-\frac{|e^{-t}x - y|^2}{1 - e^{-2t}}\right).$$

One important thing to note is that

$$M_t(x, y) = M_t(y, x)e^{|x|^2 - |y|^2}.$$

From the Mehler kernel expression the pointwise estimate $|e^{tL}f| \leqslant e^{tL}|f|$ of the Ornstein-Uhlenbeck semigroup can be easily deduced as follows

$$|e^{tL}u|(x) = \left|\int_{\mathbf{R}^d} M_t(x, y)u(y)\,\mathrm{d}y\right| \leqslant \int_{\mathbf{R}^d} M_t(x, y)|u(y)|\,\mathrm{d}y = e^{tL}|u|(x).$$

Another thing we can deduce from [1.4] is that e^{tL} is self-adjoint for all $t \geqslant 0$. To see this let u and v be in $L^2(\gamma)$ and note

$$\begin{aligned}
\left\langle e^{tL}u, v\right\rangle &= \int_{\mathbf{R}^d}\int_{\mathbf{R}^d} M_t(x, y)u(y)v(x)\,\mathrm{d}y\,\mathrm{d}\gamma(x) \\
&= \int_{\mathbf{R}^d}\int_{\mathbf{R}^d} M_t(x, y)u(y)v(x)\mathrm{d}\gamma(x)\,\mathrm{d}y \\
&= \int_{\mathbf{R}^d}\int_{\mathbf{R}^d} M_t(y, x)e^{|x|^2 - |y|^2}u(y)v(x)\mathrm{d}\gamma(x)\,\mathrm{d}y \\
&= \int_{\mathbf{R}^d}\int_{\mathbf{R}^d} M_t(y, x)u(y)v(x)\mathrm{d}x\,\mathrm{d}\gamma(y) \\
&= \int_{\mathbf{R}^d} u(x)e^{tL}v(x)\,\mathrm{d}x \\
&= \left\langle u, e^{tL}v\right\rangle.
\end{aligned}$$

We can also show that the Ornstein-Uhlenbeck semigroup is bounded on $L^p(\gamma)$ for $1 \leqslant p \leqslant \infty$ using Riesz-Thorin and duality. We proceed by using the theorem 1.3 from section 1.2. First remark that e^{tL} is bounded on $L^2(\gamma)$ because it is a strongly continuous semigroup on $L^2(\gamma)$. Furthermore remark that e^{tL} is bounded on $L^\infty(\gamma)$. To see this note that

$$\|e^{tL}u\|_{L^\infty(\gamma)} \leqslant \|u\|_{L^\infty(\gamma)} \int_{\mathbf{R}^d} M_t(x,y) 1 \,\mathrm{d}y = \|u\|_{L^\infty(\gamma)}$$

where we have used that $e^{tL}1 = 1$. By the preceding theorem we can now conclude that e^{tL} is bounded on $L^p(\gamma)$ for $2 \leqslant p \leqslant \infty$.

We can prove that e^{tL} is bounded on $L^p(\gamma)$ for $1 \leqslant p \leqslant 2$ by duality. Let f in $L^p(\gamma)$ for $1 \leqslant p < 2$. Let q be the conjugate exponent of p. Then,

$$\begin{aligned}
\|e^{tL}u\|_{L^p(\gamma)} &= \sup_{\|v\|_{L^q(\gamma)} \leqslant 1} |\langle e^{tL}u, v\rangle| \\
&= \sup_{\|v\|_{L^q(\gamma)} \leqslant 1} |\langle u, e^{tL}v\rangle| \\
&\leqslant \sup_{\|v\|_{L^q(\gamma)} \leqslant 1} \|u\|_{L^p(\gamma)} \|e^{tL}v\|_{L^q(\gamma)} \\
&\lesssim \|u\|_{L^p(\gamma)}.
\end{aligned}$$

So we have

$$\|e^{tL}u\|_{L^p(\gamma)} \lesssim \|u\|_{L^p(\gamma)}$$

for all u in $L^p(\gamma)$ and $1 \leqslant p \leqslant \infty$. Another method is to use Jensen's inequality and $e^{tL}1 = 1$. Note that for u in $L^p(\gamma)$ and $1 \leqslant p < \infty$ we have by Jensen's inequality and the convexity of $x \mapsto x^p$ that

$$(e^{tL}|u(x)|)^p = \left(\int_{\mathbf{R}^d} M_t(x,y)|u(y)| \,\mathrm{d}y\right)^p \leqslant \int_{\mathbf{R}^d} M_t(x,y)|u(y)|^p \,\mathrm{d}y = e^{tL}|u(x)|^p.$$

A good reference about the properties of strongly continuous semigroups is [EN06].

1.4 A Calderón reproducing formula

In this section we will prove a Calderón reproducing formula (the name goes back to [Cal64]) which will be central in the proof of the equivalence of the norms on $h^1_{\max,a}$ and $h^1_{\mathrm{quad},a}$.

1.6 Lemma. *For all N in $\mathbf{Z}_+$ and $A > 0$ and for all u in $L^2(\gamma)$ we have that*

$$u = C \int_0^\infty (t^2L)^{N+1} e^{At^2L} u \,\frac{\mathrm{d}t}{t} + \int_{\mathbf{R}^d} u \,\mathrm{d}\gamma, \tag{1.5}$$

where $C = \frac{2}{N!} A^{N+1}$.

Proof. We first prove the result for the Hermite polynomial $u := H_\beta$ for some multi-index β and then we use [1.2]. First let $\beta = \mathbf{0}$. Then $H_{\mathbf{0}} = 1$, so we can calculate the right-hand side

$$C\int_0^\infty (t^2L)^{N+1}e^{At^2L}1\,\frac{\mathrm{d}t}{t} + \int_{\mathbf{R}^d} 1\,\mathrm{d}\gamma = C\cdot 0 + 1 = H_{\mathbf{0}}.$$

So now assume that $\beta \neq \mathbf{0}$. For these H_β the last integral in [1.5] will evaluate to zero (by integration by parts) so we compute the first one. Using $L^{N+1}H_\beta = |\beta|^{N+1}H_\beta$ we obtain

$$\begin{aligned}\int_0^\infty (t^2L)^{N+1}e^{At^2L}H_\beta\,\frac{\mathrm{d}t}{t} &= \int_0^\infty (t^2L)^{N+1}e^{-At^2|\beta|}H_\beta\,\frac{\mathrm{d}t}{t}\\ &= |\beta|^{N+1}H_\beta\int_0^\infty t^{2(N+1)}e^{-At^2|\beta|}\,\frac{\mathrm{d}t}{t}\\ &= \frac{N!}{2}\left(\frac{|\beta|}{A|\beta|}\right)^{N+1}H_\beta\\ &= \frac{N!}{2}A^{-(N+1)}H_\beta.\end{aligned}$$

Hence we see that $C = \frac{2}{N!}A^{N+1}$ is the right constant. To finish the proof we apply [1.2] to this result. [1.5] holds for all Hermite polynomials whose span is a dense subset of $L^2(\gamma)$ and the LHS of [1.5] depends continuously in $L^2(\gamma)$ on u. So by continuity the result now follows for general u in $L^2(\gamma)$. ■

1.5 A doubling property for the Gaussian measure

The Gaussian measure is non-doubling. This means that there does not exist a constant $C > 0$ such that

$$\gamma(B(x,2r)) \leqslant C\gamma(B(x,r))$$

for all x in $\mathbf{R}^d$ and $r > 0$. The Lebesgue measure does have the doubling property, and that is what for example makes the Whitney type decompositions work. To work around this problem we first define classes of so called admissible balls on which the Gaussian measure is doubling. These admissible balls $\mathcal{B}_a$ with admissibility parameter a are introduced in [MM07] as follows. Set

$$\mathcal{B}_a := \left\{B(x,r) : x \in \mathbf{R}^d, 0 < r \leqslant am(x)\right\}$$

where

$$m(x) := \min\left\{1, \frac{1}{|x|}\right\}.$$

We recall the result from [MM07] which will act as a substitute for the doubling property of the Lebesgue measure on admissible balls when working with the Gaussian measure.

1.7 Lemma. *There exists a constant $C_d > 0$ only depending on the dimension d such that for all $a, b \geqslant 1$ and all $B(x,r)$ in $\mathcal{B}_a$ we have that*

[1.6] $$\gamma(B(x,br)) \leqslant C_d e^{2a^2(2b+1)^2} \gamma(B(x,r)).$$

1.6 The local and global regions

The technique in Gaussian harmonic analysis of splitting the kernels such as the Mehler kernel in a global and local part is well known and goes back to [Muc69]. The idea behind this is that the local part behaves like some kind of Calderón-Zygmund operator (for more about these operators see [Gra09]) and the global part has nice decay properties.

We will split the Mehler kernel into a local and a global part. For all $a > 0$, the *local region* is defined as

$$N_a := \left\{(x,y) \in \mathbf{R}^{2d} : |x-y| \leqslant am(x)\right\},$$

The *global region* is then the complement of N_a. A typical result that can be obtained using this splitting technique is the weak type $(1,1)$ of the local part of the Hardy-Littlewood maximal operator and the $L^1(\gamma)$ boundedness of its global part as proven in [HTV00, theorem 2.7]. Furthermore, we also define the local region $N_a(B)$

$$N_a(B) := \{y \in \mathbf{R}^d : |c_B - y| \leqslant am(c_B)\}.$$

1.7 The Hardy spaces

This is a preliminary section on Hardy spaces with the Gaussian measure. The Hardy spaces in the Euclidean case are interesting because for example certain singular integral operators map H^1 (which is a closed subspace of L^1) to L^1 while they do not necessarily map L^1 to L^1. An example of this phenomenon is the Riesz transform. In the Euclidean cases there exists an abundance of equivalent characterizations for the Hardy space $h^1(\gamma)$. However, one has to go through a considerable amount of work to obtain these equivalences.

In this report we consider two possible characterizations of the Hardy spaces with respect to the Gaussian measure. The eventual goal is to prove the equivalence of these.

These spaces are defined using the non-tangential maximal function T_a^* and the conical square function S_a^*

$$T_a^* u(x) := \sup_{(y,t)\in\Gamma_x^a(\gamma)} |e^{t^2 L} u(y)|,$$

$$S_a u(x) := \left(\int_{\Gamma_x^a(\gamma)} \frac{1}{\gamma(B(y,t))} |t\nabla e^{t^2 L} u(y)|^2 \, \mathrm{d}\gamma(y) \, \frac{\mathrm{d}t}{t} \right)^{\frac{1}{2}},$$

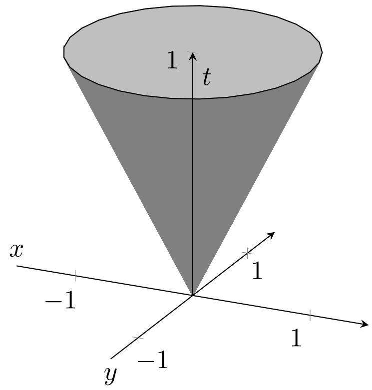

Figure 1.1: *The cone* Γ_0 *in* $\mathbf{R}^2 \times (0, \infty)$.

where

$$\Gamma_x^a(\gamma) := \left\{(y,t) \in \mathbf{R}^d \times (0,\infty) : |y-x| < t < am(x)\right\}.$$

Now $h^1_{\max,a}(\gamma)$ and $h^1_{\text{quad},a}(\gamma)$ are defined as the completions of the space of smooth compactly supported functions $C_c^\infty(\mathbf{R}^d)$ with respect to the norms

$$\|u\|_{h^1_{\max,a}(\gamma)} := \|T_a^* u\|_{L^1(\gamma)} + \|u\|_{L^1(\gamma)},$$

and

$$\|u\|_{h^1_{\text{quad},a}(\gamma)} := \|S_a u\|_{L^1(\gamma)} + \|u\|_{L^1(\gamma)},$$

respectively.

Given $A, a > 0$ we define the *admissible cone* $\Gamma_x^{(A,a)}(\gamma)$ *with aperture* A *and admissibility parameter* a *based at the point* x as

$$\Gamma_x^{(A,a)}(\gamma) := \left\{(y,t) \in \mathbf{R}^d \times (0,\infty) : |y-x| < At \text{ and } t \leqslant am(x)\right\}$$

For simplicity we will also write $\Gamma_x(\gamma) := \Gamma_x^{(1,1)}(\gamma)$ and $\Gamma_x^a(\gamma) := \Gamma_x^{(1,a)}(\gamma)$.

For an example of such a cone see figure 1.1.

The following lemma about the cones states a fact that we will often use.

1.8 Lemma. *If* (y,s) *in* $\Gamma_x^a(\gamma)$ *and* z *in* $B(y,s)$ *then* (z,s) *in* $\Gamma_x^{(2,a)}(\gamma)$.

The proof is straightforward, so we skip it.

1.8 A useful lemma

In this section we give a lemma that will be useful throughout the text.

We recall the lemma from [MvNP10b, lemma 2.3].

1.9 Lemma. *Let $a > 0$ and x, y in $\mathbf{R}^d$. If $|x - y| < am(x)$ then $m(x) \leqslant (1 + a)m(y)$ and $m(y) \leqslant 2(1 + a)m(x)$.*

This lemma wil turn out to be extremely useful for example when changing domains.

1.9 The boundedness of some non-tangential maximal operators

In this section we will prove the boundedness of some non-tangential maximal operators which will turn out to be very useful in the sequel. The proof of the first part depends heavily on the global/local dichotomy. For the second part we also use interpolation.

1.10 Proposition. *Let $A, a > 0$ and let $\tau := \frac{(1+aA)(1+2aA)}{2}$. Then, for u in $C_c^\infty(\mathbf{R}^d)$,*

(i) $$\left\| x \mapsto \sup_{(y,t)\in\Gamma_x^{(A,a)}(\gamma)} \int_{\mathbf{R}^d} M_{t^2}(y,z) 1_{\complement N_\tau}(y,z)|u(z)|\,\mathrm{d}z \right\|_{L^1(\gamma)} \lesssim \|u\|_{L^1(\gamma)};$$

(ii) $$\left\| x \mapsto \sup_{(y,t)\in\Gamma_x^{(A,a)}(\gamma)} \int_{\mathbf{R}^d} M_{t^2}(y,z)|u(z)|\,\mathrm{d}z \right\|_{L^p(\gamma)} \lesssim \|u\|_{L^p(\gamma)} \text{ for all } 1 < p \leqslant \infty.$$

Additionally, the sublinear operator

$$u \mapsto \sup_{(y,t)\in\Gamma_x^{(A,a)}(\gamma)} \int_{\mathbf{R}^d} M_{t^2}(y,z)|u(z)|\,\mathrm{d}z$$

is of weak type $(1,1)$.

Before we continue with the proof we state a theorem which will give us an lemma that will be useful. The following lemma is a small modification of [PUR08, lemma 1.1].

1.11 Lemma. *Let $A, a > 0$. For all x in $\mathbf{R}^d$ and all u in $L^2(\gamma)$ we have*

$$\sup_{(y,t)\in\Gamma_x^{(A,a)}(\gamma)} |e^{t^2 L}u(y)| \lesssim \sup_{r>0} \frac{1}{\gamma(B(x,r))} \int_{B(x,r)} |u(z)|\,\mathrm{d}\gamma(z).$$

Furthermore, from [HTV00, theorem 2.7] we get the following theorem

1.12 Theorem. *Consider the maximal function $M_\gamma u$ defined by*

$$M_\gamma u(x) := \sup_{r>0} \frac{1}{\gamma(B(x,r))} \int_{B(x,r)} 1_{\complement N_{\frac{1}{2}}}(x,y)|u(y)|\,\mathrm{d}\gamma(y).$$

Then the operator M_γ is bounded on $L^1(\gamma)$.

We will also need the following weak type $(1,1)$ estimate on an "admissible maximal function" which is proven in [MvNP10a, lemma 3.2].

1.13 Lemma. *Let $a > 0$. For u in $L^1_{loc}(\mathbf{R}^d)$ put*

$$M_a^* u(x) := \sup_{B(x,r) \in \mathcal{B}_a} \frac{1}{\gamma(B(x,r))} \int_{B(x,r)} |u(y)| \, \mathrm{d}\gamma(y).$$

Then for all $\tau > 0$,

$$\tau \gamma(\{M_a^* u > \tau\}) \lesssim \|u\|_{L^1(\gamma)}$$

with the implied constant only depending on a and d.

Proof of proposition 1.10. We begin with the proof of (i). Let x in $\mathbf{R}^d$, (y,z) in $\complement N_\tau$ and (y,t) in $\Gamma_x^{(A,a)}(\gamma)$. We claim that

$$|x - z| > \frac{1}{2} m(x).$$

To see this first note that by definition of N_τ we have that $|y - z| > \tau m(y)$ and by definition of $\Gamma_x^{(A,a)}(\gamma)$ that $|y - x| < At \leqslant Aam(x)$. Furthermore we have by the reverse triangle inequality that

$$\begin{aligned} |x - z| &\geqslant |z - y| - |x - y| \\ &> \tau m(y) - Aam(x). \end{aligned}$$

Now, by lemma 1.9 we have that $m(y)(1 + aA) \geqslant m(x)$ so we get

$$|x - z| > \tau m(y) - aAm(x) \geqslant \left(\frac{\tau}{1 + aA} - aA \right) m(x) = \frac{1}{2} m(x),$$

where the last equality follows from the definition of τ. So,

$$\begin{aligned} &\left\| x \mapsto \sup_{(y,t) \in \Gamma_x^{(A,a)}(\gamma)} \int_{\mathbf{R}^d} M_{t^2}(y,z) 1_{\complement N_\tau}(y,z) |u(z)| \, \mathrm{d}z \right\|_{L^1(\gamma)} \\ &\overset{\text{(i)}}{\leqslant} \int_{\mathbf{R}^d} \left| \sup_{(y,t) \in \Gamma_x^{(A,a)}(\gamma)} \int_{\mathbf{R}^d} M_{t^2}(y,z) 1_{\complement N_{\frac{1}{2}}}(x,z) |u(z)| \, \mathrm{d}z \right| \mathrm{d}\gamma(x) \\ &\overset{\text{(ii)}}{\lesssim} \int_{\mathbf{R}^d} \sup_{r > 0} \frac{1}{\gamma(B(x,r)} \int_{B(x,r)} 1_{\complement N_{\frac{1}{2}}}(x,z) |u(z)| \, \mathrm{d}\gamma(z) \, \mathrm{d}\gamma(x). \end{aligned}$$

Where we have used in (i) that by previous inequality and the definition of $N_{\frac{1}{2}}$ we have that $1_{\complement N_\tau}(y,z) \leqslant 1_{\complement N_{\frac{1}{2}}}(x,z)$. Furthermore in (ii) we have used [1.4] and lemma 1.11. Theorem 1.12 gives us that this is smaller than a constant times $\|u\|_{L^1(\gamma)}$. This concludes the proof of (i).

To prove (ii) we first apply lemma 1.9 to $|y - z| \leqslant m(y)$, $|y - x| \leqslant Aam(x)$ to find a τ' such that $|z - x| \leqslant \tau' m(x)$. So,

$$\sup_{(y,t) \in \Gamma_x^{(A,a)}(\gamma)} \int_{\mathbf{R}^d} M_{t^2}(y,z) 1_{N_\tau}(y,z) |u(z)| \, \mathrm{d}z$$

$$
\begin{aligned}
&= \sup_{(y,t)\in\Gamma_x^{(A,a)}(\gamma)} \int_{\mathbf{R}^d} M_{t^2}(y,z) 1_{N_\tau}(y,z) 1_{B(x,\tau' m(x))}(z)|u(z)|\,\mathrm{d}z \\
&\overset{(i)}{\lesssim} \sup_{r>0} \frac{1}{\gamma(B(x,r))} \int_{B(x,r)} 1_{B(x,\tau' m(x))}(z)|u(z)|\,\mathrm{d}\gamma(z) \\
&= \sup_{r\in(0,\tau' m(x))} \frac{1}{\gamma(B(x,r))} \int_{B(x,r)} 1_{N'_\tau}(x,z)|u(z)|\,\mathrm{d}\gamma(z) \qquad [1.7]
\end{aligned}
$$

where we have used lemma 1.11 in (i).

We can now finish the proof using the Marcinkiewicz interpolation theorem. We will first show that

$$
x \mapsto \sup_{(y,t)\in\Gamma_x^{(A,a)}(\gamma)} \int_{\mathbf{R}^d} M_{t^2}(y,z)|u(z)|\,\mathrm{d}z \qquad [1.8]
$$

is of weak type $(1,1)$ and bounded on $L^\infty(\gamma)$. By lemma 1.13 the RHS of [1.7] is of weak type $(1,1)$. Combining this with part (i) we see that [1.8] is of weak type $(1,1)$. The $L^\infty(\gamma)$ boundedness result for [1.8] follows from lemma 1.11 as follows. Let u in $\|u\|_{L^\infty(\gamma)}$. Then

$$
\sup_{(y,t)\in\Gamma_x^{(A,a)}(\gamma)} \int_{\mathbf{R}^d} M_{t^2}(y,z)|u(z)|\,\mathrm{d}z \lesssim \sup_{r>0} \frac{1}{\gamma(B(x,r))} \int_{B(x,r)} |u(z)|\,\mathrm{d}\gamma(z) \leqslant \|u\|_{L^\infty(\gamma)}.
$$

Hence (ii) now follows from the Marcinkiewicz interpolation theorem (theorem 1.5). ■

Remark. Many results in this thesis are given for functions in $C_c^\infty(\mathbf{R}^d)$. Since the Gaussian measure is a finite regular measure we have that $C_c^\infty(\mathbf{R}^d)$ is dense in $L^p(\gamma)$ for $1 \leqslant p < \infty$. This way we can extend all these results on $C_c^\infty(\mathbf{R}^d)$ to the appropriate $L^p(\gamma)$-spaces by density.

We will also need the $L^2(\gamma)$ boundedness of T_a^*. We now state this as a lemma.

1.14 Lemma. *The operator T_a^* is bounded on $L^2(\gamma)$.*

Proof.

$$
\|T_a^* u\|_{L^2(\gamma)}^2 = \int_{\mathbf{R}^d} \left| \sup_{(y,t)\in\Gamma_x^a(\gamma)} e^{t^2 L}|u(y)| \right|^2 \gamma(x) \overset{(i)}{\leqslant} \|u\|_{L^2(\gamma)}^2.
$$

Where (i) follows from proposition 1.10(ii). ■

1.15 Definition. *Let $A, a > 0$. We define the global part of $T_a^{*(A,a)}$ by*

$$
T_{glob}^{*(A,a)} u(x) := \sup_{(y,t)\in\Gamma_x^{(A,a)}(\gamma)} \left| \int_{\mathbf{R}^d} 1_{\complement N_\tau}(z,w) M_{t^2}(z,w) u(w)\,\mathrm{d}w \right|,
$$

where $\tau := \frac{(1+aA)(1+2aA)}{2}$.

Remark. We will also write $T^{*(1,a)}_{\text{glob}} = T^{*a}_{\text{glob}}$.

Proposition 1.10(i) gives the boundedness of the global part of T^*_a.

1.16 Corollary. *Let u in $L^2(\gamma)$, $A, a > 0$. Then we have*

$$\|T^{*(A,a)}_{glob} u\|_{L^1(\gamma)} \lesssim \|u\|_{L^1(\gamma)}.$$

1.10 Gaussian tent spaces

In [MvNP10b] the Gaussian tent spaces are introduced as follows. Let

$$D := \left\{(x,t) \in \mathbf{R}^d \times (0,\infty) : t < m(x)\right\}.$$

The Gaussian tent space $t^{1,2}(\gamma)$ is defined as the completion of $C_c(D)$ with respect to the norm

$$\|A\|_{t^{1,2}(\gamma)} := \int_{\mathbf{R}^d} \left(\iint_{\Gamma_x(\gamma)} \frac{1}{\gamma(B(y,t))} |A(y,t)|^2 \,\mathrm{d}\gamma(y) \frac{\mathrm{d}t}{t} \right)^{\frac{1}{2}} \mathrm{d}\gamma(x).$$

Compared to [MvNP10b] we will use the notation $t^{1,2}(\gamma)$ rather than $T^{1,2}(\gamma)$ to emphasise the local nature of this space (and we do the sam
e with the Hardy spaces).

In the same article ([MvNP10b]) theorem 3.4 gives an atomic decomposition for $t^{1,2}(\gamma)$. As in the Euclidean case, this atomic decomposition will turn out to be very useful. Using an atomic decomposition we will only have to check results for atoms and then the rest follows reasonably easy. We first define what an atom is in the Gaussian context.

1.17 Definition. *Given $a > 0$ a function $A\colon D \mapsto \mathbf{C}$ is called a $t^{1,2}(\gamma)$ a-atom if there exists a ball B in $\mathcal{B}_a$ such that*

(i) $supp(A) \subset \left\{(y,t) \in D\colon t \leqslant d(y, \complement B)\right\}$ and,

(ii) $\displaystyle\int_{\mathbf{R}^d} \int_0^\infty |A(y,t)|^2 \frac{\mathrm{d}\gamma(y)\mathrm{d}t}{t} \leqslant \frac{1}{\gamma(B)}$.

Now the atomic decomposition is as follows.

1.18 Theorem. *For all u in $t^{1,2}(\gamma)$ and $a > 1$, there exists a sequence $(\lambda_n)_{n\geqslant 1} \in \ell^1$ and a sequence of $t^{1,2}(\gamma)$ a-atoms $(A_n)_{n\geqslant 1}$ such that*

(i) $\displaystyle u = \sum_{n=1}^\infty \lambda_n A_n$ and,

(ii) $\displaystyle\sum_{n=1}^\infty |\lambda_n| \lesssim \|f\|_{t^{1,2}(\gamma)}$.

Using the Calderón reproducing formula [1.5] and the atomic decomposition we can prove the following corollary. The proof follows quite directly from those results. This corollary will be the actual underlying identity when proving the equivalence of the "non-tangential maximal function Hardy space" and the "conical square function Hardy space" in the last chapter.

1.19 Corollary. *For all N in $\mathbf{Z}_+$, $a > 1$, $b \geqslant \frac{1}{2}$ and $\alpha > a^2$ there exists $C_1, C_2, C_3, C_4 > 0$ and d sequences of atoms $(A_{n,j})_{n\in\mathbf{Z}_+}$ and numbers $(\lambda_{n,j})_{n\in\mathbf{Z}_+}$ such that for all u in $C_c^\infty(\mathbf{R}^d)$ and x in $\mathbf{R}^d$:*

[1.9]

$$\begin{aligned} u(x) = \int_{\mathbf{R}^d} u \,\mathrm{d}\gamma - C_1 \sum_{j=1}^{d}\sum_{n=1}^{\infty} \lambda_{n,j} \int_0^2 (t^2L)^N e^{\frac{t^2}{\alpha}L} t\partial_{x_j}^* A_{n,j}(x,t)\frac{\mathrm{d}t}{t} \\ + C_2 \sum_{j=1}^{d}\sum_{n=1}^{\infty} \lambda_{n,j} \int_0^2 1_{[\frac{m(x)}{b},2]}(t)(t^2L)^N e^{\frac{t^2}{\alpha}L} t\partial_{x_j}^* A_{n,j}(x,t)\frac{\mathrm{d}t}{t} \\ - C_3 \sum_{j=1}^{d} \int_0^{\frac{m(x)}{b}} (t^2L)^N e^{\frac{t^2}{\alpha}L} t\partial_{x_j}^* (1_{\complement D}(t,x) t\partial_{x_j} e^{\frac{a^2t^2}{\alpha}L} u(x))\frac{\mathrm{d}t}{t} \\ + C_4 \int_{\frac{m(x)}{b}}^{\infty} (t^2L)^{N+1} e^{\frac{(1+a)^2t^2}{\alpha}L} u(x)\frac{\mathrm{d}t}{t}, \end{aligned}$$

and

$$\sum_{j=1}^{d}\sum_{n=1}^{\infty} |\lambda_{n,j}| \lesssim \|u\|_{h^1_{quad,a}}.$$

Where $\partial_{x_j}^* = -\partial_{x_j} + 2x_j$ denotes the adjoint of ∂_{x_j} in $L^2(\gamma)$.

Proof. First remark that $L = -\frac{1}{2}\sum_j \partial_j^*\partial_j$ and hence

[1.10]

$$\begin{aligned} (t^2L)^{N+1} e^{\frac{(1+a^2)t^2}{\alpha}L} u(x) &= -\sum_{i=1}^{d} (t^2L)^N \frac{1}{2} t^2 \partial_{x_j}^*\partial_{x_j} e^{\frac{t^2}{\alpha}L} e^{\frac{a^2}{\alpha}t^2L} u(x) \\ &= -\frac{1}{2}\sum_{i=1}^{d} (t^2L)^N e^{\frac{t^2}{\alpha}L} t\partial_{x_j}^* [1_D(x,t) + 1_{\complement D}(x,t)] t\partial_{x_j} e^{\frac{a^2t^2}{\alpha}L} u(x). \end{aligned}$$

The first line follows from the definition of L and the second follows from the fact that L and its semigroup commute.

We would like to have an atomic decomposition for $x \mapsto 1_D(x,t)t\partial_{x_j} e^{\frac{a^2t^2}{\alpha}L}u$, $j = 1,\dots,d$. To show that this exists, we show that this term lies in $t^{1,2}(\gamma)$. First let $\tilde{\Gamma}_x^{a'}(\gamma) = \{(y,t) \in \mathbf{R}^d \times (0,\infty) \colon |x-y| < t < a'm(y)\}$. Furthermore let $\tilde{S}_{a'}$ be $S_{a'}$ with $\Gamma_x^{a'}(\gamma)$ replaced by $\tilde{\Gamma}_x^{a'}(\gamma)$. Then by [MvNP10a, remark 4.2]

$$\left\| (x,t) \mapsto 1_D(x,t) t\partial_{x_j} e^{\frac{a^2}{\alpha}t^2L} u(x) \right\|_{t^{1,2}(\gamma)}$$

$$= \int_{\mathbf{R}^d} \left(\iint_{\Gamma_x(\gamma)} \frac{1_D(y,t)}{\gamma(B(y,t))} |t\partial_{y_j} e^{\frac{a^2}{\alpha}t^2 L} u(y)|^2 \,\mathrm{d}\gamma(y) \frac{\mathrm{d}t}{t} \right)^{\frac{1}{2}} \mathrm{d}\gamma(x)$$
$$\leqslant \int_{\mathbf{R}^d} \left(\int_0^\infty \int_{B(x,t)} \frac{1_D(y,t)}{\gamma(B(y,t))} |t\partial_{y_j} e^{\frac{a^2}{\alpha}t^2 L} u(y)|^2 \,\mathrm{d}\gamma(y) \frac{\mathrm{d}t}{t} \right)^{\frac{1}{2}} \mathrm{d}\gamma(x).$$

We can now substitute $t \to \sqrt{\alpha} t$ to get that the RHS is smaller than a constant times

$$\int_{\mathbf{R}^d} \left(\int_0^\infty \int_{B(x,\sqrt{\alpha}t)} \frac{1_D(y,\sqrt{\alpha}t)}{\gamma(B(y,\sqrt{\alpha}t))} |t\partial_{y_j} e^{a^2t^2 L} u(y)|^2 \,\mathrm{d}\gamma(y) \frac{\mathrm{d}t}{t} \right)^{\frac{1}{2}} \mathrm{d}\gamma(x)$$
$$[*] \qquad \leqslant \int_{\mathbf{R}^d} \left(\int_0^\infty \int_{B(x,\sqrt{\alpha}t)} \frac{1_D(y,\sqrt{\alpha}t)}{\gamma(B(y,\sqrt{\alpha}t))} |t\nabla e^{a^2t^2 L} u(y)|^2 \,\mathrm{d}\gamma(y) \frac{\mathrm{d}t}{t} \right)^{\frac{1}{2}} \mathrm{d}\gamma(x).$$

Because $\gamma(B(y,\sqrt{\alpha}t)) \geqslant \gamma(B(y,t))$ we get that the RHS in [*] is smaller than a constant times

$$\int_{\mathbf{R}^d} \left(\int_0^\infty \int_{B(x,\sqrt{\alpha}t)} \frac{1_D(y,\sqrt{\alpha}t)}{\gamma(B(y,t))} |t\nabla e^{a^2t^2 L} u(y)|^2 \,\mathrm{d}\gamma(y) \frac{\mathrm{d}t}{t} \right)^{\frac{1}{2}} \mathrm{d}\gamma(x).$$

By the change of aperture formula [MvNP10b, theorem 3.8] we have

$$\begin{aligned}
&\left\|(x,t) \mapsto 1_D(x,t) t\partial_{x_j} e^{\frac{a^2}{\alpha}t^2 L} u(x)\right\|_{t^{1,2}(\gamma)} \\
&\lesssim \int_{\mathbf{R}^d} \left(\int_0^\infty \int_{B(x,a^2t)} \frac{1_D(y,a^2t)}{\gamma(B(y,t))} |t\nabla e^{a^2t^2 L} u(y)|^2 \,\mathrm{d}\gamma(y) \frac{\mathrm{d}t}{t} \right)^{\frac{1}{2}} \mathrm{d}\gamma(x) \\
&\lesssim \int_{\mathbf{R}^d} \left(\int_0^\infty \int_{B(x,at)} \frac{1_D(y,at)}{\gamma(B(y,t))} |t\nabla e^{t^2 L} u(y)|^2 \,\mathrm{d}\gamma(y) \frac{\mathrm{d}t}{t} \right)^{\frac{1}{2}} \mathrm{d}\gamma(x) \\
&\leqslant \|\tilde{S}_a u\|_{L^1(\gamma)} \\
&\leqslant \|u\|_{h^1_{\mathrm{quad},a}} < \infty,
\end{aligned}$$

where the second inequality follows from the substitution $at \to t$. By theorem 1.18 we can now conclude that $h : (x,t) \mapsto 1_D(x,t)t\partial_{x_j} e^{\frac{a^2}{\alpha}t^2 L} u$ has an atomic decomposition for $j = 1, \ldots, d$. I.e.,

$$1_D(x,t)t\partial_{x_j} e^{\frac{a^2}{\alpha}t^2 L} u = \sum_{n=1}^\infty \lambda_{n,j} A_{n,j}(x,t) \text{ with } \sum_{n=1}^\infty |\lambda_{n,j}| \lesssim \|h\|_{t^{1,2}(\gamma)} < \infty$$

for $j = 1, \ldots, d$.

Using lemma 1.6 and [1.10] we get after setting $C' := \frac{1}{2}C$ that

$$u(x) = \int_{\mathbf{R}^d} u \,\mathrm{d}\gamma + C \int_0^\infty (t^2 L)^{N+1} e^{\frac{(1+a^2)}{\alpha}t^2 L} u(x) \,\frac{\mathrm{d}t}{t}$$

$$
\begin{aligned}
&= \int_{\mathbf{R}^d} u \mathrm{d}\gamma + C \int_{\frac{m(x)}{b}}^{\infty} (t^2 L)^{N+1} e^{\frac{1+a^2}{\alpha} t^2 L} u(x) \frac{\mathrm{d}t}{t} \\
&\quad - C' \sum_{j=1}^{d} \int_0^{\frac{m(x)}{b}} (t^2 L)^N e^{\frac{t^2}{\alpha} L} t \partial_{x_j}^* [1_D(x,t) + 1_{\complement D}(x,t)] t \partial_{x_j} e^{\frac{a^2}{\alpha} t^2 L} u(x) \frac{\mathrm{d}t}{t} \\
&= \int_{\mathbf{R}^d} u \mathrm{d}\gamma + C \int_{\frac{m(x)}{b}}^{\infty} (t^2 L)^{N+1} e^{\frac{1+a^2}{\alpha} t^2 L} u(x) \frac{\mathrm{d}t}{t} \\
&\quad - C' \sum_{j=1}^{d} \int_0^{\frac{m(x)}{b}} (t^2 L)^N e^{\frac{t^2}{\alpha} L} t \partial_{x_j}^* 1_D(x,t) t \partial_{x_j} e^{\frac{a^2}{\alpha} t^2 L} u(x) \frac{\mathrm{d}t}{t} \\
&\quad - C[\sum_{j=1}^{d} \int_0^{\frac{m(x)}{b}} (t^2 L)^N e^{\frac{t^2}{\alpha} L} t \partial_{x_j}^* 1_{\complement D}(x,t) t \partial_{x_j} e^{\frac{a^2}{\alpha} t^2 L} u(x) \frac{\mathrm{d}t}{t} \\
&= \int_{\mathbf{R}^d} u \mathrm{d}\gamma + C \int_{\frac{m(x)}{b}}^{\infty} (t^2 L)^{N+1} e^{\frac{1+a^2}{\alpha} t^2 L} u(x) \frac{\mathrm{d}t}{t} \\
&\quad - C' \sum_{j=1}^{d} \sum_{n=1}^{\infty} \lambda_{n,j} \int_0^{\frac{m(x)}{b}} (t^2 L)^N e^{\frac{t^2}{\alpha} L} t \partial_{x_j}^* A_{n,j}(x,t) \frac{\mathrm{d}t}{t} \qquad [1.11] \\
&\quad - C' \sum_{j=1}^{d} \int_0^{\frac{m(x)}{b}} (t^2 L)^N e^{\frac{t^2}{\alpha} L} t \partial_{x_j}^* 1_{\complement D}(x,t) t \partial_{x_j} e^{\frac{a^2}{\alpha} t^2 L} u(x) \frac{\mathrm{d}t}{t}.
\end{aligned}
$$

We have switched the (Bochner) integrals and the sum. To see that this is allowed first note that if the series $\sum_n g_n$ converges in $t^{1,2}(\gamma)$ to f then the series converges in $L^2(\mathrm{d}\gamma \frac{\mathrm{d}t}{t})$ as well. From this we can deduce that for almost all x we have that

$$
\int_0^{\infty} \sum_{n=1}^{N} g_n(x,t) \frac{\mathrm{d}t}{t} \to \int_0^{\infty} f(x,t) \frac{\mathrm{d}t}{t}.
$$

We can now switch integration and summation to yield the desired result.

We can split [1.11] to obtain

$$
\begin{aligned}
&\sum_{j=1}^{d} \sum_{n=1}^{\infty} \lambda_{n,j} \int_0^{\frac{m(x)}{b}} (t^2 L)^N e^{\frac{t^2}{\alpha} L} t \partial_{x_j}^* A_{n,j}(x,t) \frac{\mathrm{d}t}{t} \\
&\overset{\text{(i)}}{=} \sum_{j=1}^{d} \sum_{n=1}^{\infty} \lambda_{n,j} \int_0^{2} (t^2 L)^N e^{\frac{t^2}{\alpha} L} t \partial_{x_j}^* A_{n,j}(x,t) \frac{\mathrm{d}t}{t} \\
&\quad - \sum_{j=1}^{d} \sum_{n=1}^{\infty} \lambda_{n,j} \int_0^{2} 1_{[\frac{m(x)}{b},2]} (t^2 L)^N e^{\frac{t^2}{\alpha} L} t \partial_{x_j}^* A_{n,j}(x,t) \frac{\mathrm{d}t}{t}
\end{aligned}
$$

where (i) uses that $\frac{m(x)}{b} \leqslant 2$. This gives [1.9]. We have shown above that

$$
\left\| (x,t) \mapsto 1_D(x,t) t \partial_{x_j} e^{\frac{a^2}{\alpha} t^2 L} u(x) \right\|_{t^{1,2}(\gamma)} \lesssim \|u\|_{h^1_{\mathrm{quad},a}}
$$

so,

$$\sum_{j=1}^{d}\sum_{n=1}^{\infty}|\lambda_{n,j}| \lesssim \|u\|_{h^1_{\mathrm{quad},a}}$$

follows. This concludes the proof. ■

2 Kernel estimates

In this chapter we will find explicit expressions for some of the kernels of the operators that occur in the reproducing formula [1.9]. For these kernels we will find the needed estimates for the next two chapters. We will also give some appropriate off-diagonal estimates and give an inequality that can be useful when those off-diagonal estimates fail.

2.1 Some useful integral kernels

Having integral kernels for operators can make the analysis of those operators much easier as one can see in the section on the Ornstein-Uhlenbeck operator. In this section we will compute the kernels of the operators that occur in the corollary to the Calderón reproducing formula [1.9].

2.1 Definition. *Given $t, \alpha > 0, j = 1, \ldots, d$ and N in $\mathbf{Z}_+$ we denote by $K_{t^2,N,\alpha}$ and $\tilde{K}_{t^2,N,\alpha,j}$ the kernels defined, given u in $L^2(\gamma)$ by*

[2.1]
$$\begin{aligned} K_{t^2,N,\alpha}(x,y) &= t^{2N}\left[\partial_s^N M_s(x,y)\right]_{s=\frac{t^2}{\alpha}} \\ \tilde{K}_{t^2,N,\alpha,j}(x,y) &= t^{2N+1}\partial_{y_j}\left[\partial_s^N M_s(y,x)\right]_{s=\frac{t^2}{\alpha}} \exp(|x|^2-|y|^2). \end{aligned}$$

We can easily find expressions for these kernels by using the Mehler kernel as kernel to the semigroup e^{tL}.

2.2 Proposition. *The kernels $K_{t^2,N,\alpha}$ and $\tilde{K}_{t^2,N,\alpha,j}$ are given by*

[2.2]
$$\int_{\mathbf{R}^d} K_{t^2,N,\alpha}(x,y)u(y)\,\mathrm{d}y = (t^2L)^N e^{\frac{t^2}{\alpha}L}u(x),$$

[2.3]
$$\int_{\mathbf{R}^d} \tilde{K}_{t^2,N,\alpha,j}(x,y)u(y)\,\mathrm{d}y = (t^2L)^N e^{\frac{t^2}{\alpha}L}t\partial_{x_j}^* u(x).$$

Proof. The proof is based around the identity [1.4]. By [1.3] we have

$$\begin{aligned} (t^2L)^N e^{\frac{t^2}{\alpha}L}u(x) &= t^{2N}\left[\partial_s^N \int_{\mathbf{R}^d} M_s(x,y)u(y)\,\mathrm{d}y\right]_{s=\frac{t^2}{\alpha}} \\ &= \int_{\mathbf{R}^d} t^{2N}\left[\partial_s^N M_s(x,y)\right]_{s=\frac{t^2}{\alpha}} u(y)\,\mathrm{d}y, \end{aligned}$$

where the second equality follows from theorem 1.1 together with lemma 1.2. So we can conclude that [2.2] holds. We can compute [2.3] using duality. Let u, v in $C_c^\infty(\mathbf{R}^d)$ then

$$\begin{aligned}
&\left\langle (t^2L)^N e^{\frac{t^2}{\alpha}L} t\partial_{x_j}^* u, v\right\rangle \\
&= \left\langle u, \partial_{x_j} e^{\frac{t^2}{\alpha}L} t(t^2L)^N v\right\rangle \\
&= \left\langle u, \partial_{x_j} t(t^2L)^N e^{\frac{t^2}{\alpha}L} v\right\rangle \\
&= \int_{\mathbf{R}^d} u(y)\partial_{y_j}\left[t(t^2L)^N e^{\frac{t^2}{\alpha}L} v(y)\right] \,\mathrm{d}\gamma(y) \\
&= \pi^{-\frac{d}{2}} \int_{\mathbf{R}^d} u(y)\partial_{y_j}\left[t(t^2L)^N e^{\frac{t^2}{\alpha}L} v(y)\right] e^{-|y|^2} \,\mathrm{d}y \\
&= -\pi^{-\frac{d}{2}} \int_{\mathbf{R}^d} \partial_{y_j}\left[u(y)e^{-|y|^2}\right] t(t^2L)^N e^{\frac{t^2}{\alpha}L} v(y) \,\mathrm{d}y \\
&= -\pi^{-\frac{d}{2}} \int_{\mathbf{R}^d} \partial_{y_j}\left[u(y)e^{-|y|^2}\right] \left(\int_{\mathbf{R}^d} t^{2N+1}\left[\partial_s^N M_s(y,x)\right]_{s=\frac{t^2}{\alpha}} v(x)\,\mathrm{d}x\right) \mathrm{d}y \\
&= -\int_{\mathbf{R}^d}\int_{\mathbf{R}^d} \partial_{y_j}\left[u(y)e^{-|y|^2}\right] t^{2N+1}\left[\partial_s^N M_s(y,x)\right]_{s=\frac{t^2}{\alpha}} v(x)e^{|x|^2}\,\mathrm{d}\gamma(x)\,\mathrm{d}y \\
&= -\int_{\mathbf{R}^d}\int_{\mathbf{R}^d} \partial_{y_j}\left[u(y)e^{|x|^2-|y|^2}\right] t^{2N+1}\left[\partial_s^N M_s(y,x)\right]_{s=\frac{t^2}{\alpha}} v(x)\,\mathrm{d}\gamma(x)\,\mathrm{d}y \\
&= \int_{\mathbf{R}^d}\int_{\mathbf{R}^d} u(y)e^{|x|^2-|y|^2} t^{2N+1}\partial_{y_j}\left[\partial_s^N M_s(y,x)\right]_{s=\frac{t^2}{\alpha}} v(x)\,\mathrm{d}\gamma(x)\,\mathrm{d}y \\
&= \int_{\mathbf{R}^d}\left[\int_{\mathbf{R}^d} u(y)e^{|x|^2-|y|^2} t^{2N+1}\partial_{y_j}\left[\partial_s^N M_s(y,x)\right]_{s=\frac{t^2}{\alpha}} \,\mathrm{d}y\right] v(x)\,\mathrm{d}\gamma(x).
\end{aligned}$$

We have applied Fubini in the last line. This is allowed because u, v are in $C_c^\infty(\mathbf{R}^d)$. So we can conclude [2.3] holds. ■

2.2 Three technical results

In the sequel we need to know how certain derivatives of the Mehler kernel behave, so one goal of this section is to find quantitative information about these. Useful estimates on the exponential terms that occur in Mehler kernel will also be given.

2.3 Lemma. *Let N in $\mathbf{Z}_+$. There exists a polynomial of $2d+1$ variables P_N such that for all x, y in $\mathbf{R}^d$*

$$\begin{aligned}
\partial_s^N M_s(x,y) &= (1-e^{-2s})^{-N} \\
&\quad \times P_N\left(e^{-s}, \left(\frac{e^{-s}x_j - y_j}{\sqrt{1-e^{-2s}}}\right)_{j=1,\dots,d}, \left(\sqrt{1-e^{-2s}}\,x_j\right)_{j=1,\dots,d}\right) M_s(x,y).
\end{aligned}$$

Proof. Let $j = 1, \dots, d$, $s > 0$ and x, y in $\mathbf{R}^d$. We have, for $N = 1$,

$$\partial_s M_s(x,y) = -(1-e^{-2s})^{-1} \times de^{-2s} M_s(x,y) + \partial_s \left(\frac{|e^{-s}x - y|^2}{1-e^{-2s}} \right) M_s(x,y),$$

which is of the asserted form and,

[*] $$\partial_s \left(\frac{e^{-s}x_j - y_j}{\sqrt{1-e^{-2s}}} \right) = -(1-e^{-2s})^{-1} \times \left(e^{-s}x_j\sqrt{1-e^{-2s}} + e^{-2s}\frac{e^{-s}x_j - y_j}{\sqrt{1-e^{-2s}}} \right),$$

[*] $$\partial_s \left(\sqrt{1-e^{-2s}x_j} \right) = \left(1-e^{-2s}\right)^{-1} \left(e^{-s}\sqrt{1-e^{-2s}} \right),$$

[*] $$\partial_s \left(\frac{[e^{-s}x_j - y_j]^2}{1-e^{-s}} \right) = -\left(1-e^{-2s}\right)^{-1} \left[\left(2e^{-s}\sqrt{1-e^{-2s}}x_j\right) \times \left(\frac{e^{-s}x_j - y_j}{\sqrt{1-e^{-2s}}} \right) + \left(\frac{e^{-s}x_j - y_j}{\sqrt{1-e^{-2s}}} \right)^2 2e^{-2s} \right].$$

For $N \geqslant 2$ the proof now follows by induction using [*]. ■

2.4 Corollary. *Let N in $\mathbf{Z}_+$ and $j = 1, \dots, d$. There exists a polynomial of $2d+1$ variables Q_N such that for all x, y in $\mathbf{R}^d$ and $s > 0$ we have that*

$$\partial_{x_j}\partial_s^N M_s(x,y) = (1-e^{-2s})^{-(N+\frac{1}{2})} \times Q_N \left(e^{-s}, \left(\frac{e^{-s}x_j - y_j}{\sqrt{1-e^{-2s}}} \right)_{j=1,\dots,d}, \left(\sqrt{1-e^{-2s}}x_j \right)_{j=1,\dots,d} \right) M_s(x,y).$$

Proof. We skip this proof since it is similar to the proof above. ■

The following lemma will be useful when transfering estimates from $M_{\frac{t^2}{\alpha}}$ to M_{t^2}. It follows quite directly after applying the mean value theorem to the function $\xi \mapsto \xi^\alpha$.

2.5 Lemma. *For $C, T > 0, \alpha > 1, t$ in $(0,T]$ and all x, y in $\mathbf{R}^d$ we have that*
[2.4]
$$\exp\left(-C\frac{|e^{-\frac{t^2}{\alpha}}x - y|^2}{1-e^{-2\frac{t^2}{\alpha}}} \right) \leqslant \exp\left(-C\frac{\alpha}{2e^{2a^2}}\frac{|e^{-t^2}x - y|^2}{1-e^{-2t^2}} \right) \exp\left(C\frac{t^4 \min(|x|^2, |y|^2)}{1-e^{-2\frac{t^2}{\alpha}}} \right)$$

Proof. Let t in $(0,T]$ and $\alpha > 1$. Applying the mean value theorem to the function $f(\xi) = \xi^\alpha$ gives, for $0 < \xi < \xi'$

$$f(\xi) - f(\xi') = \alpha\hat{\xi}^{\alpha-1}(\xi - \xi') \text{ for some } \hat{\xi} \text{ in } [\xi, \xi'].$$

Picking $\xi = 1$ and $\xi' = e^{-2\frac{t^2}{\alpha}}$ gives

[*]
$$\frac{1-e^{-2t^2}}{1-e^{-2\frac{t^2}{\alpha}}} = \alpha\hat{\xi}^{\alpha-1} \text{ for some } \hat{\xi} \text{ in } \left[e^{-2\frac{t^2}{\alpha}}, 1\right].$$

Using that $\frac{\alpha-1}{\alpha}t^2 \leqslant T^2$ we have that

[2.5]
$$\alpha e^{-2T^2} \leqslant \alpha e^{-2t^2\frac{\alpha-1}{\alpha}} \overset{\text{(i)}}{\leqslant} \frac{1-e^{-2t^2}}{1-e^{-2\frac{t^2}{\alpha}}} \overset{\text{(ii)}}{\leqslant} \lim_{t\downarrow 0}\frac{1-e^{-2t^2}}{1-e^{-2\frac{t^2}{\alpha}}} = \lim_{t\downarrow 0}\frac{\alpha e^{-2t^2}}{e^{-2\frac{t^2}{\alpha}}} = \alpha.$$

Where (i) uses [*] and the monotonicity of $\xi \mapsto \alpha\xi^{\alpha-1}$. (ii) uses basic calculus since the function in question is monotone. To prove [2.4] note that

$$\begin{aligned}
|e^{-\frac{t^2}{\alpha}}x - y| &\geqslant |e^{-t^2}x - y| - |e^{-t^2} - e^{-\frac{t^2}{\alpha}}||x| \\
&\geqslant |e^{-t^2}x - y| - |e^{-t^2} - 1||x| \\
&\geqslant |e^{-t^2}x - y| - t^2|x|.
\end{aligned}$$

By $2ab \leqslant a^2 + b^2$ we have

$$\begin{aligned}
2(|e^{-\frac{t^2}{\alpha}}x - y|^2 + t^4|x|^2) &\geqslant (|e^{-\frac{t^2}{\alpha}}x - y| + t^2|x|)^2 \\
&\geqslant |e^{-t^2}x - y|^2.
\end{aligned}$$

Hence,

$$|e^{-\frac{t^2}{\alpha}}x - y|^2 \geqslant \frac{|e^{-t^2}x - y|^2}{2} - t^4|x|^2.$$

Using this we get,

$$\begin{aligned}
\exp\left(-C\frac{|e^{-\frac{t^2}{\alpha}}x - y|^2}{1-e^{-2\frac{t^2}{\alpha}}}\right) &\leqslant \exp\left(-\frac{C}{2}\left[\frac{1-e^{-2t^2}}{1-e^{-2\frac{t^2}{\alpha}}}\right]\frac{|e^{-t^2}x - y|^2}{1-e^{-2t^2}}\right)\exp\left(C\frac{t^4|x|^2}{1-e^{-2\frac{t^2}{\alpha}}}\right) \\
&\leqslant \exp\left(-C\frac{\alpha}{2e^{2a^2}}\frac{|e^{-t^2}x - y|^2}{1-e^{-2t^2}}\right)\exp\left(C\frac{t^4|x|^2}{1-e^{-2\frac{t^2}{\alpha}}}\right),
\end{aligned}$$

where the last line follows from [2.5]. We can prove the other part the same way, noting

$$\begin{aligned}
\left|e^{-\frac{t^2}{\alpha}}x - y\right| &= \left|e^{-(\frac{\alpha}{\alpha}-\frac{\alpha-1}{\alpha})t^2}x - y\right| \\
&= e^{\frac{\alpha-1}{\alpha}t^2}\left|e^{-t^2}x - e^{-\frac{\alpha-1}{\alpha}t^2}y\right| \\
&\geqslant \left|e^{-t^2}x - e^{-\frac{\alpha-1}{\alpha}t^2}y\right| \\
&\geqslant \left|e^{-t^2}x - y\right| - \left|1 - e^{-\frac{\alpha-1}{\alpha}t^2}\right||y|
\end{aligned}$$

$$\geqslant \left|e^{-t^2}x - y\right| - t^2|y|.$$

And as before we conclude that

$$|e^{-\frac{t^2}{\alpha}}x - y|^2 \geqslant \frac{|e^{-t^2}x - y|^2}{2} - t^4|y|^2,$$

and

$$\exp\left(-C\frac{|e^{-\frac{t^2}{\alpha}}x - y|^2}{1 - e^{-2\frac{t^2}{\alpha}}}\right) \leqslant \exp\left(-\frac{C}{2}\left[\frac{1 - e^{-2t^2}}{1 - e^{-2\frac{t^2}{\alpha}}}\right]\frac{|e^{-t^2}x - y|^2}{1 - e^{-2t^2}}\right)\exp\left(C\frac{t^4|y|^2}{1 - e^{-2\frac{t^2}{\alpha}}}\right)$$
$$\leqslant \exp\left(-C\frac{\alpha}{2e^{2a^2}}\frac{|e^{-t^2}x - y|^2}{1 - e^{-2t^2}}\right)\exp\left(C\frac{t^4|y|^2}{1 - e^{-2\frac{t^2}{\alpha}}}\right).$$

This concludes the proof of the lemma. ■

2.3 Kernel estimates

Here we will compute useful estimates for the integral kernels that occur in the Calderón reproducing formula [1.9]. They will turn out to be useful when estimating the terms in that formula. The estimates will be proven using the results from the previous section.

2.6 Lemma. *Let N in $\mathbf{Z}_+$, $j = 1, \dots, d$, $a > 0$ and $\alpha \geqslant 4e^{2a^2}$. Let t in $(0, T]$ for some $T > 0$ and let x, y in $\mathbf{R}^d$. Furthermore let $C > 0$ be a positive constant independent on t, x and y not necessarily the same at all instances. Then we have that*

(i) If $t\min(|x|, |y|) \leqslant C$ then $M_{\frac{t^2}{\alpha}}(x, y) \lesssim \exp\left(-\frac{\alpha}{4e^{2a^2}}\frac{|e^{-t^2}x - y|^2}{1 - e^{-2t^2}}\right)M_{t^2}(x, y),$

(ii) If $t|x| \leqslant C$ then $|K_{t^2,N,\alpha}(x, y)| \lesssim \exp\left(-\frac{\alpha}{4e^{2a^2}}\frac{|e^{-t^2}x - y|^2}{1 - e^{-2t^2}}\right)M_{t^2}(x, y),$

(iii) If $t|y| \leqslant C$ then $|\tilde{K}_{t^2,N,\alpha,j}(x, y)| \lesssim \exp\left(-\frac{\alpha}{4e^{2a^2}}\frac{|e^{-t^2}y - x|^2}{1 - e^{-2t^2}}\right)M_{t^2}(x, y).$

The implicit constants in the inequalities are only depending on the dimension d, a and α.

Proof. We will first show (i) using [2.4]. C is a general constant, not necessarily the same at all instances.

$$M_{\frac{t^2}{\alpha}}(x, y)$$
$$= C_d\left(1 - e^{-2\frac{t^2}{\alpha}}\right)^{-\frac{d}{2}}\exp\left(-\frac{|e^{-\frac{t^2}{\alpha}}x - y|^2}{1 - e^{-2\frac{t^2}{\alpha}}}\right)$$

$$
\begin{aligned}
&\leqslant C_d\left(1-e^{-2\frac{t^2}{\alpha}}\right)^{-\frac{d}{2}}\exp\left(-\frac{\alpha}{2e^{2a^2}}\frac{|e^{-t^2}x-y|^2}{1-e^{-2t^2}}\right)\exp\left(\frac{t^4\max(|x|^2,|y|^2)}{1-e^{-2\frac{t^2}{\alpha}}}\right)\\
&=C_d\left(\frac{1-e^{-2\frac{t^2}{\alpha}}}{1-e^{-2t^2}}\right)^{-\frac{d}{2}}\exp\left(\left[1-\frac{\alpha}{4e^{2a^2}}\right]\frac{|e^{-t^2}x-y|^2}{1-e^{-2t^2}}\right)\exp\left(-\frac{\alpha}{4e^{2a^2}}\frac{|e^{-t^2}x-y|^2}{1-e^{-2t^2}}\right)\\
&\quad\times\exp\left(\frac{t^4\max(|x|^2,|y|^2)}{1-e^{-2\frac{t^2}{\alpha}}}\right)M_{t^2}(x,y)\\
&\leqslant C_{d,T,\alpha}\exp\left(\left[1-\frac{\alpha}{4e^{2a^2}}\right]\frac{|e^{-t^2}x-y|^2}{1-e^{-2t^2}}\right)\exp\left(-\frac{\alpha}{4e^{2a^2}}\frac{|e^{-t^2}x-y|^2}{1-e^{-2t^2}}\right)\\
&\quad\times\exp\left(\frac{t^4\max(|x|^2,|y|^2)}{1-e^{-2\frac{t^2}{\alpha}}}\right)M_{t^2}(x,y)\\
&\leqslant C_{d,T,\alpha}\exp\left(-\frac{\alpha}{4e^{2a^2}}\frac{|e^{-t^2}x-y|^2}{1-e^{-2t^2}}\right)M_{t^2}(x,y).
\end{aligned}
$$

Where we have used [2.5] in the fourth line and in the last line that $\alpha\geqslant 4e^{2a^2}$ and

$$
\exp\left(\frac{t^4\max(|x|^2,|y|^2)}{1-e^{-2\frac{t^2}{\alpha}}}\right)\leqslant\exp\left(C\frac{t^2}{1-e^{-2\frac{t^2}{\alpha}}}\right)\leqslant\exp\left(C\frac{T^2}{1-e^{-2\frac{T^2}{\alpha}}}\right).
$$

Next we will show (ii) in a similar way using [2.4].

$$
\begin{aligned}
&|K_{t^2,N,\alpha}(x,y)|\\
&=C_d\left(1-e^{-2\frac{t^2}{\alpha}}\right)^{-N}|P_N|t^{2N}\left(1-e^{-2\frac{t^2}{\alpha}}\right)^{-\frac{d}{2}}\exp\left(-\frac{|e^{-\frac{t^2}{\alpha}}x-y|^2}{1-e^{-2\frac{t^2}{\alpha}}}\right)\\
&=C_d\left(1-e^{-2\frac{t^2}{\alpha}}\right)^{-N}|P_N|t^{2N}\left(\frac{1-e^{-2\frac{t^2}{\alpha}}}{1-e^{-2t^2}}\right)^{-\frac{d}{2}}\exp\left(-\frac{|e^{-\frac{t^2}{\alpha}}x-y|^2}{1-e^{-2\frac{t^2}{\alpha}}}\right)\\
&\quad\times\exp\left(\frac{|e^{-t^2}x-y|^2}{1-e^{-2t^2}}\right)M_{t^2}(x,y)\\
[*]\quad&\leqslant C_d|P_N|t^{2N}\left(1-e^{-2\frac{t^2}{\alpha}}\right)^{-N}\exp\left(\left[1-\frac{\alpha}{4e^{2a^2}}\right]\frac{|e^{-t^2}x-y|^2}{1-e^{-2t^2}}\right)\\
&\quad\times\exp\left(-\frac{\alpha}{4e^{2a^2}}\frac{|e^{-t^2}x-y|^2}{1-e^{-2t^2}}\right)\exp\left(\frac{t^4\max(|x|^2,|y|^2)}{1-e^{-2\frac{t^2}{\alpha}}}\right)M_{t^2}(x,y)
\end{aligned}
$$

The only problem for the function $t\mapsto t^{2N}\left(1-e^{-2\frac{t^2}{\alpha}}\right)^{-N}$ on $(0,T]$ lies in the point 0. However, the limit $t\downarrow 0$ is finite. This means that this function is bounded. Another issue that can arise is that

$$
|P_N|\exp\left(\left[1-\frac{\alpha}{4e^{2a^2}}\right]\frac{|e^{-t^2}x-y|^2}{1-e^{-2t^2}}\right)
$$

is unbounded in x, y or t. However, we can quickly see that

$$\frac{|e^{-t^2}x - y|^M}{\sqrt{1-e^{-2t^2}}^M} \exp\left(\left[1 - \frac{\alpha}{4e^{2a^2}}\right] \frac{|e^{-t^2}x-y|^2}{1-e^{-2t^2}}\right)$$

is uniformly bounded in t, x and y for all integers M and for α sufficiently large. The same holds for

$$\sqrt{1-e^{-2t^2}}^M |x|^M \exp\left(\left[1 - \frac{\alpha}{4e^{2a^2}}\right] \frac{|e^{-t^2}x-y|^2}{1-e^{-2t^2}}\right)$$

because we have that $|x| \leqslant Ct^{-1}$. Hence the RHS in [*] is smaller than

$$C_{d,N,T,\alpha} \exp\left(-\frac{\alpha}{4e^{2a^2}} \frac{|e^{-t^2}x-y|^2}{1-e^{-2t^2}}\right) M_{t^2}(x,y).$$

We end with the proof of (iii) using [2.4].

$$\begin{aligned}
|\tilde{K}_{t^2,N,\alpha,j}(x,y)| &= t^{2N+1} \left|\partial_{y_j}\partial_s^N M_s(y,x)\right|\Big|_{s=\frac{t^2}{\alpha}} \exp(|x|^2 - |y|^2) \\
&= t^{2N+1}(1-e^{-2\frac{t^2}{\alpha}})^{-\left(N+\frac{1}{2}\right)} |Q_N| M_{\frac{t^2}{\alpha}}(y,x) \\
&\quad \times \exp(|x|^2-|y|^2) \\
&= t^{2N+1}(1-e^{-2\frac{t^2}{\alpha}})^{-\left(N+\frac{1}{2}\right)} |Q_N| \left(1-e^{-2\frac{t^2}{\alpha}}\right)^{-\frac{d}{2}} \exp\left(-\frac{|e^{-\frac{t^2}{\alpha}}y-x|^2}{1-e^{-2\frac{t^2}{\alpha}}}\right) \\
&\quad \times \exp(|x|^2-|y|^2) \\
&\lesssim t^{2N+1}(1-e^{-2\frac{t^2}{\alpha}})^{-\left(N+\frac{1}{2}\right)} |Q_N| \exp\left(-\frac{\alpha}{2e^{2a^2}} \frac{|e^{-t^2}y-x|^2}{1-e^{-2\frac{t^2}{\alpha}}}\right) \\
&\quad \times \exp\left(\frac{t^4\max(|x|^2,|y|^2)}{1-e^{-2\frac{t^2}{\alpha}}}\right) \exp(|x|^2-|y|^2) \\
&\leqslant C_{d,\alpha} t^{2N+1}(1-e^{-2\frac{t^2}{\alpha}})^{-\left(N+\frac{1}{2}\right)} |Q_N| \left(\frac{1-e^{-2\frac{t^2}{\alpha}}}{1-e^{-2t^2}}\right)^{-\frac{d}{2}} \\
&\quad \times \exp\left(\left[1-\frac{\alpha}{4e^{2a^2}}\right] \frac{|e^{-t^2}y-x|^2}{1-e^{-2\frac{t^2}{\alpha}}}\right) \exp\left(-\frac{\alpha}{4e^{2a^2}} \frac{|e^{-t^2}y-x|^2}{1-e^{-2\frac{t^2}{\alpha}}}\right) M_{t^2}(x,y) \\
&\leqslant C_{d,N,T,\alpha} \exp\left(-\frac{\alpha}{4e^{2a^2}} \frac{|e^{-t^2}x-y|^2}{1-e^{-2\frac{t^2}{\alpha}}}\right) M_{t^2}(x,y).
\end{aligned}$$

where we have estimated $t \mapsto t^{2N+1}(1-e^{-2\frac{t^2}{\alpha}})^{-\left(N+\frac{1}{2}\right)}$ and $(x,y,t) \mapsto |Q_N| \exp\left(\left[1-\frac{\alpha}{4e^{2a^2}}\right] \frac{|e^{-t^2}y-x|^2}{1-e^{-2\frac{t^2}{\alpha}}}\right)$ in the same was as before. ■

2.4 Off-diagonal estimates

Decomposition into annuli

We will often need to estimate integrals which are integrated over the whole of $\mathbf{R}^d$. This will often not be possible to do directly, so we decompose $\mathbf{R}^d$ into annuli. Then we can estimate the integrals over those sets and then sum to obtain an estimate for the integral over the whole of $\mathbf{R}^d$.

So, given $a > 0$, $B = B(c_B, r_B)$ in $\mathcal{B}_a$ and k in $\mathbf{Z}_+$ we consider the following sets

$$C_k(B) := \begin{cases} B(c_B, 2r_B) & \text{if } k = 0, \\ B(c_B, 2^{k+1})r_B) \setminus B(c_B, 2^k r_B) & \text{if } k \geqslant 1. \end{cases}$$

The estimates

The following lemma will play an important role in the next chapter.

2.7 Lemma (Off-diagonal estimates)**.** *Let N in $\mathbf{Z}_+$, $a > 0$, $j = 1, \ldots, d$, B in $\mathcal{B}_a$, $\alpha > 8e^{2a^2}$ and k in $\mathbf{N}$. Then we have for all u in $L^2(\gamma)$ that*

$$\left\| 1_{C_k(B)} 1_{(0,r_B)}(t)(t^{2N+1} L^N e^{\frac{t^2}{\alpha} L} \partial_{x_j}^*) 1_B u \right\|_{L^2(\gamma)} \lesssim \exp\left(-\frac{\alpha}{128 e^{2a^2}} \left(\frac{r_B}{t}\right)^2 4^k \right) \|u\|_{L^2(\gamma)}.$$

Where the implied constant only depends on α, a, d and N.

Proof. For $t \leqslant r_B \leqslant a m(c_B)$ and y in B we have by lemma 1.9 that $t \leqslant a(1+a)m(y)$. Given x in $\mathbf{R}^d$ we have by the triangle inequality that

$$|y - x|^2 \leqslant 2(|e^{-t^2} y - x|^2 + (1 - e^{-t^2})^2 |y|^2). \tag{2.6}$$

So we have that

$$\frac{1}{2}|y - x|^2 - (1 - e^{-t^2})^2 |y|^2 \leqslant |e^{-t^2} y - x|^2.$$

Furthermore note that

$$1 - e^{-2t^2} \leqslant 2t^2,$$

so

$$\begin{aligned} \exp\left(-\frac{\alpha}{4e^{2a^2}} \frac{|e^{-t^2} y - x|^2}{1 - e^{-2t^2}} \right) &\leqslant \exp\left(-\frac{\alpha}{8e^{2a^2}} \frac{|y - x|^2}{1 - e^{-2t^2}} \right) \exp\left(\frac{\alpha}{4e^{2a^2}} (t|y|)^2 \right) \\ &\lesssim \exp\left(-\frac{\alpha}{16 e^{2a^2}} \frac{|y - x|^2}{t^2} \right). \end{aligned} \tag{2.7}$$

Where the last line follows from $t \leqslant a(1+a)m(y)$. So using lemma 2.6, definition 2.1 and proposition 2.2 we get

$$\begin{aligned}
&\int_{C_k(B)} \left(\int_B |\tilde{K}_{t^2,N,\alpha,j}(x,y)| 1_{(0,r_B)}(t) |u(y)| \, \mathrm{d}y \right)^2 \mathrm{d}\gamma(x) \\
&\lesssim \int_{C_k(B)} \left(\int_B \exp\left(-\frac{\alpha}{4e^{2a^2}} \frac{|e^{-t^2} y - x|^2}{1 - e^{-2t^2}} \right) M_{t^2}(x,y) 1_{(0,r_B)}(t) |u(y)| \, \mathrm{d}y \right)^2 \mathrm{d}\gamma(x) \\
&\lesssim \int_{C_k(B)} \left(\int_B \exp\left(-\frac{\alpha}{16e^{2a^2}} \frac{|y-x|^2}{t^2} \right) M_{t^2}(x,y) 1_{(0,r_B)}(t) |u(y)| \, \mathrm{d}y \right)^2 \mathrm{d}\gamma(x).
\end{aligned}$$

Now we have for x in $C_k(B)$ and y in B that $|x-y| \geqslant 2^{k-1} r_B$ for k in $\mathbf{N}$ from the definition of $C_k(B)$, so

$$\begin{aligned}
\int_{C_k(B)} &\left(\int_B |\tilde{K}_{t^2,N,\alpha,j}(x,y)| 1_{(0,r_B)}(t) |u(y)| \, \mathrm{d}y \right)^2 \mathrm{d}\gamma(x) \\
&\leqslant \exp\left(-\frac{\alpha}{16e^{2a^2}} \left(\frac{r_B}{t} \right)^2 4^{k-1} \right) \int_{C_k(B)} \left(\int_B M_{t^2}(x,y) |u(y)| \, \mathrm{d}y \right)^2 \mathrm{d}\gamma(x) \\
&\leqslant \exp\left(-\frac{\alpha}{16e^{2a^2}} \left(\frac{r_B}{t} \right)^2 4^{k-1} \right) \| e^{t^2 L} |u| \|_{L^2(\gamma)}^2 \\
&\lesssim \exp\left(-\frac{\alpha}{16e^{2a^2}} \left(\frac{r_B}{t} \right)^2 4^{k-1} \right) \|u\|_{L^2(\gamma)}^2.
\end{aligned}$$

Which concludes the proof of this lemma. ■

2.5 When the off-diagonal estimates fail

We conclude with a property of the sets $C_k(B)$ in the local region $N_\tau(B)$. We recall

$$N_a := \left\{ (x,y) \in \mathbf{R}^{2d} : |x-y| \leqslant a m(x) \right\},$$

It will be helpful when the off-diagonal estimates fail.

Before we state the main lemma of this section we first give two auxillary results.

2.8 Lemma. *If* $|x-y| \leqslant \tau m(y) \leqslant \tau(1+\tau)m(x)$ *then we have* $e^{-|x|^2} \simeq e^{-|y|^2}$.

Proof. By the reverse triangle inequality we have

$$|y| - |x| \leqslant \tau(1+\tau)m(x)$$

and,

$$|x| - |y| \leqslant \tau m(y).$$

Hence we have that,

$$|x|^2 \leqslant |y|^2 + 2\tau m(y)y| + \tau^2 m(y)^2$$

and,

$$|y|^2 \leqslant |x|^2 + 2\tau(1+\tau)m(x)|x| + \tau^2(1+\tau)^2 m(x)^2.$$

So we have

$$e^{-|x|^2} \simeq e^{-|y|^2}.$$

This concludes the proof of this lemma. ∎

2.9 Lemma. *If we have* $e^{|x|^2} \simeq e^{|y|^2}$ *then* $\mathrm{d}x\,\mathrm{d}\gamma(y) \simeq \mathrm{d}\gamma(x)\,\mathrm{d}y$.

As the proof of this lemma is obvious we omit it.

2.10 Lemma. *Let* $a, \tau > 0$ *and* $B = B(c_B, r_B)$ *in* $\mathcal{B}_a$. *Then for all* k *in* $\mathbf{Z}_+$

$$\gamma(C_k(B) \cap N_\tau(B)) \lesssim 2^{kd}\gamma(B).$$

Where the implied constant is independent on k.

Proof. Let k in $\mathbf{Z}_+$ and x in $C_k(B) \cap N_\tau(B)$. By lemma 1.9 we have $|x - c_B| \leqslant \tau m(c_B) \leqslant \tau(1+\tau)m(x)$, hence by lemma 2.8 we have that $e^{-|x|^2} \simeq e^{-|c_B|^2}$ for all x in $N_\tau(B)$ where the implicit constants are independent on τ, k, B and x. Furthermore we have

$$\gamma(B) = \int_B \mathrm{d}\gamma \simeq e^{-|c_B|^2} \int_B \mathrm{d}x \simeq r_B^d e^{-|c_B|^2}.$$

For k in $\mathbf{Z}_+$ we have

$$\begin{aligned}
\gamma(C_k(B) \cap N_\tau(B)) &= \int_{C_k(B)\cap N_\tau(B)} \mathrm{d}\gamma \\
&\simeq e^{-|c_B|^2} \int_{C_k(B)\cap N_\tau(B)} \mathrm{d}\lambda \\
&\lesssim e^{-|c_B|^2} \int_{2^{k+1}B} \mathrm{d}\lambda \\
&\lesssim (2^k r_B)^d e^{-|c_B|^2} \\
&\simeq 2^{kd}\gamma(B).
\end{aligned}$$

This concludes the proof of the lemma. ∎

3 Molecules

In this chapter we will introduce molecules. We will prove that the $h^1_{\max,a}$ norm of a certain class of molecules is always bounded by a constant. Furthermore, we prove that the function [3.1] on the molecules from the Calderón reproducing formula [1.9] is such a molecule.

3.1 Molecules

We first define molecules.

3.1 Definition. *Let N in $\mathbf{N}$, $a > 0$ and $C > 0$. A function u in $L^2(\gamma)$ is said to be a (β, N, C)-molecule if there exists $B = B(c_B, r_B)$ in $\mathcal{B}_\beta$ and $\tilde{u}$ in the domain $\mathcal{D}(L^N)$ of L^N such that $u = L^N \tilde{u}$ and*

(i) $\|1_{C_k(B)} u\|_{L^2(\gamma)} \leqslant e^{-C4^k} \dfrac{1}{\sqrt{\gamma(B)}}$ *for all k in $\mathbf{Z}_+$,*

(ii) $\|1_{C_k(B)} \tilde{u}\|_{L^2(\gamma)} \leqslant r_B^{2N} e^{-C4^k} \dfrac{1}{\sqrt{\gamma(B)}}$ *for all k in $\mathbf{Z}_+$.*

The next proposition will be useful with the theorem of the next section.

3.2 Proposition. *Let N in $\mathbf{N}$, $j = 1, \ldots, d$ and $\alpha > 8e^{2a^2}$. Furthermore, let $B = B(c_B, r_B)$ in $\mathcal{B}_2$ and A a $t^{1,2}(\gamma)$ atom associated with B. The function*

$$x \mapsto \int_0^{r_B} (t^2 L)^N e^{\frac{t^2}{\alpha} L} t \partial^*_{x_j} A(x,t) \,\frac{\mathrm{d}t}{t} \qquad [3.1]$$

is a $(2, N, 2^{-29}\alpha)$-molecule.

Proof. We first treat the case $k = 0$ separately. For this it suffices to bound

$$\left\| x \mapsto \int_0^{r_B} (t^2 L)^N e^{\frac{t^2}{\alpha} L} t \partial^*_{x_j} A(x,t) \,\frac{\mathrm{d}t}{t} \right\|_{L^2(\gamma)}$$

from above by a constant times $\sqrt{\gamma(B)}^{-1}$. We first claim that the Riesz transforms $R_j = \partial_{x_j} L^{-\frac{1}{2}}$ for $j = 1, \ldots, d$ are bounded on $L^2(\gamma)$. To see this we will use Hermite polynomials. Let $u = \sum c_\alpha H_\alpha$ be a function in $L^2(\gamma)$. First note that $\partial^*_{x_j} \partial_{x_j} H_\alpha(x) = 2\alpha_j H_\alpha(x)$. Then

$$\left\langle u, L^{-\frac{1}{2}} \partial^*_{x_j} \partial_{x_j} L^{-\frac{1}{2}} u \right\rangle = \left\langle u, L^{-\frac{1}{2}} \partial^*_{x_j} \partial_{x_j} L^{-\frac{1}{2}} \sum_{\alpha \in \mathbf{Z}^d_+} c_\alpha H_\alpha \right\rangle$$

$$
\begin{aligned}
&= \sum_{\alpha\in\mathbf{Z}_+^d} c_\alpha \left\langle u, L^{-\frac{1}{2}}\partial_{x_j}^*\partial_{x_j} L^{-\frac{1}{2}} H_\alpha \right\rangle \\
&= \sum_{\alpha\in\mathbf{Z}_+^d} c_\alpha \left\langle u, |\alpha|^{-\frac{1}{2}} 2\alpha_j |\alpha|^{-\frac{1}{2}} H_\alpha \right\rangle \\
&= \sum_{\alpha\in\mathbf{Z}_+^d} c_\alpha 2\alpha_j |\alpha|^{-1} \left\langle \sum_{\beta\in\mathbf{Z}_+^d} c_\beta H_\beta, H_\alpha \right\rangle \\
&= 2\sum_{\alpha\in\mathbf{Z}_+^d}\sum_{\beta\in\mathbf{Z}_+^d} c_\alpha \overline{c_\beta} \alpha_j |\alpha|^{-1} \langle H_\beta, H_\alpha\rangle \\
&= 2\sum_{\alpha\in\mathbf{Z}_+^d} |c_\alpha|^2 \alpha_j |\alpha|^{-1} \\
&\leqslant 2\sum_{\alpha\in\mathbf{Z}_+^d} |c_\alpha|^2 \\
&= 2\|u\|_{L^2(\gamma)}^2.
\end{aligned}
$$

To finish the case $k = 0$ it suffices to bound

[3.2] $$\left| \int_0^{r_B} \int_{\mathbf{R}^d} (t^2L)^N e^{\frac{t^2}{\alpha}L} t\partial_{x_j}^* A(x,t) b(x)\,\mathrm{d}\gamma(x)\,\frac{\mathrm{d}t}{t} \right|$$

for all b with $b = \sum c_\beta H_\beta$ and $\sum |c_\beta|^2 \leqslant 1$. We can view this as an inner product on $L^2(\gamma)$ and apply duality to obtain that [3.2] is equal to

[3.3] $$\left| \int_0^{r_B} \int_{\mathbf{R}^d} A(x,t) R_j (t^2L)^{N+\frac{1}{2}} e^{\frac{t^2}{\alpha}L} b(x)\,\mathrm{d}\gamma(x)\,\frac{\mathrm{d}t}{t} \right|$$

We can now apply the boundedness of the Riesz transforms and Cauchy-Schwarz to obtain that [3.3] is smaller than or equal to a constant times

$$
\begin{aligned}
&\left(\int_0^\infty \int_{\mathbf{R}^d} |A(x,t)|^2\,\mathrm{d}\gamma(x)\,\frac{\mathrm{d}t}{t} \right)^{\frac{1}{2}} \left(\int_0^{r_B} \int_{\mathbf{R}^d} |(t^2L)^{N+\frac{1}{2}} e^{\frac{t^2}{\alpha}L} b(x)|^2\,\mathrm{d}\gamma(x)\,\frac{\mathrm{d}t}{t} \right)^{\frac{1}{2}} \\
&\lesssim \frac{1}{\sqrt{\gamma(B)}} \left(\int_0^{r_B} \int_{\mathbf{R}^d} |(t^2L)^{N+\frac{1}{2}} e^{\frac{t^2}{\alpha}L} b(x)|^2\,\mathrm{d}\gamma(x)\,\frac{\mathrm{d}t}{t} \right)^{\frac{1}{2}}.
\end{aligned}
$$

This reduces to problem to proving that

$$\int_0^{r_B} \int_{\mathbf{R}^d} |(t^2L)^{N+\frac{1}{2}} e^{\frac{t^2}{\alpha}L} b(x)|^2\,\mathrm{d}\gamma(x)\frac{\mathrm{d}t}{t}$$

is bounded. Now,

$$\int_0^{r_B} \int_{\mathbf{R}^d} |(t^2L)^{N+\frac{1}{2}} e^{\frac{t^2}{\alpha}L} b(x)|^2\,\mathrm{d}\gamma(x)\frac{\mathrm{d}t}{t}$$

$$
\begin{aligned}
&\leqslant \int_0^{r_B} \int_{\mathbf{R}^d} \Big| \sum_{\beta \in \mathbf{Z}_+^d} c_\beta (t^2 L)^{N+\frac{1}{2}} e^{\frac{t^2}{\alpha} L} H_\beta(x) \Big|^2 \, \mathrm{d}\gamma(x) \, \frac{\mathrm{d}t}{t} \\
&\leqslant \sum_{\beta \in \mathbf{Z}_+^d} |c_\beta|^2 \int_0^\infty (t^2 |\beta|)^{2N+1} e^{-2\frac{t^2}{\alpha}|\beta|} \, \frac{\mathrm{d}t}{t} \\
&\lesssim \sum_{\beta \in \mathbf{Z}_+^d} |c_\beta|^2 \\
&\leqslant 1.
\end{aligned}
$$

As required. Furthermore we have

$$
\int_0^{r_B} (t^2 L)^N e^{\frac{t^2}{\alpha} L} t \partial_{x_j}^* A(x,t) \frac{\mathrm{d}t}{t} = L^N \tilde{u} \text{ for } \tilde{u}(x) := \int_0^{r_B} t^{2N} e^{\frac{t^2}{\alpha} L} t \partial_{x_j}^* A(x,t) \frac{\mathrm{d}t}{t}.
$$

The same argument now gives

$$
\|\tilde{u}\|_{L^2(\gamma)} \lesssim r_B^{2N} \frac{1}{\sqrt{\gamma(B)}}.
$$

We can now prove the result for k in $\mathbf{N}$. So let k in $\mathbf{N}$. By lemma 2.7 we have

$$
\begin{aligned}
&\int_0^{r_B} \Big\| x \mapsto 1_{C_k(B)}(x) (t^2 L)^N e^{\frac{t^2}{\alpha} L} t \partial_{x_j}^* A(x,t) \Big\|_{L^2(\gamma)} \frac{\mathrm{d}t}{t} \\
&\quad \lesssim \int_0^{r_B} \exp\left(-\frac{\alpha}{128 e^8} 4^k \left(\frac{r_B}{t} \right)^2 \right) \|x \mapsto A(x,t)\|_{L^2(\gamma)} \frac{\mathrm{d}t}{t} \\
&\quad = \int_0^{r_B} \left[\exp\left(-\frac{\alpha}{256 e^8} 4^k \left(\frac{r_B}{t} \right)^2 \right) \right]^2 \|x \mapsto A(x,t)\|_{L^2(\gamma)} \frac{\mathrm{d}t}{t} \\
&\quad \leqslant \exp\left(-\frac{\alpha}{256 e^8} 4^k \right) \int_0^{r_B} \exp\left(-\frac{\alpha}{256 e^{2a^2}} 4^k \left(\frac{r_B}{t} \right)^2 \right) \|x \mapsto A(x,t)\|_{L^2(\gamma)} \frac{\mathrm{d}t}{t} \\
&\quad \leqslant \exp\left(-\frac{\alpha}{256 e^8} 4^k \right) \left[\int_0^{r_B} \exp\left(-\frac{\alpha}{128 e^{2a^2}} 4^k \left(\frac{r_B}{t} \right)^2 \right) \frac{\mathrm{d}t}{t} \right]^{\frac{1}{2}} \\
&\qquad \times \left(\int_0^{r_B} \|x \mapsto A(x,t)\|_{L^2(\gamma)}^2 \frac{\mathrm{d}t}{t} \right)^{\frac{1}{2}} \\
&\quad \leqslant \exp\left(-\frac{\alpha}{256 e^8} 4^k \right) \left[\int_0^1 \exp\left(-\frac{\alpha}{128 e^8} \frac{1}{t^2} \right) \frac{\mathrm{d}t}{t} \right]^{\frac{1}{2}} \\
&\qquad \times \left(\int_0^{r_B} \|x \mapsto A(x,t)\|_{L^2(\gamma)}^2 \frac{\mathrm{d}t}{t} \right)^{\frac{1}{2}} \\
&\quad \lesssim \exp\left(-\frac{\alpha}{2^{29}} 4^k \right) \frac{1}{\sqrt{\gamma(B)}}.
\end{aligned}
$$

Furthermore we have

$$
\int_0^{r_B} (t^2 L)^N e^{\frac{t^2}{\alpha} L} t \partial_{x_j}^* A(x,t) \frac{\mathrm{d}t}{t} = L^N \tilde{u} \text{ for } \tilde{u}(x) := \int_0^{r_B} t^{2N} e^{\frac{t^2}{\alpha} L} t \partial_{x_j}^* A(x,t) \frac{\mathrm{d}t}{t}.
$$

And finally note that as before, when we replace N by 0 we get by using the same argument as before that

$$\begin{aligned}\|1_{C_k(B)}\tilde{u}\|_{L^2(\gamma)} &\leqslant r_B^{2N} \int_0^{r_B} \left\| x \mapsto 1_{C_k(B)}(x) 1_{(0,r_B)}(t) e^{\frac{t^2}{\alpha}L} t \partial_{x_j}^* A(x,t) \right\|_{L^2(\gamma)}^2 \frac{\mathrm{d}t}{t} \\ &\lesssim r_B^{2N} \exp\left(-\frac{\alpha}{2^{29}} 4^k\right) \frac{1}{\sqrt{\gamma(B)}}.\end{aligned}$$

This conclude the proof. ■

3.2 The $h^1_{\max,a}$ norm of a molecule

Here we prove that the $h^1_{\max,a}$ norm of any molecule is bounded. We do this by splitting the integral that defines the norm into pieces and then we estimate them separately.

3.3 Theorem. *Let $a > 0$ and let u be a $(2, N, C)$-molecule with $\alpha \geqslant 2^{35}$, $N > \frac{d}{4}$ and $C > 2^{12}$. Then u in $h^1_{max,a}$ and $\|u\|_{h^1_{max,a}} \lesssim 1$.*

Proof. Let

$$\|u\|_{h^1_{\max,a}} \leqslant I + \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} I'_{k,l} + \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} I''_{k,l}.$$

where

$$\begin{aligned} I &:= \int_{\mathbf{R}^d} \sup_{(y,s)\in\Gamma_x^a(\gamma),\, s\leqslant \frac{r_B}{\sqrt{\alpha}}} |e^{s^2 L} u(y)| \,\mathrm{d}\gamma(x), \\ I'_{k,l} &:= \int_{C_k(B)} \sup_{(y,s)\in\Gamma_x^a(\gamma),\, s\geqslant \frac{r_B}{\sqrt{\alpha}}} |e^{s^2 L} 1_{C_l(B)}(y) u(y)| 1_{\left(0, \frac{2^l r_B}{C_a}\right)}(m(x)) \,\mathrm{d}\gamma(x), \\ I''_{k,l} &:= \int_{C_k(B)} \sup_{(y,s)\in\Gamma_x^a(\gamma),\, s\geqslant \frac{r_B}{\sqrt{\alpha}}} |L^N e^{s^2 L} 1_{C_l(B)}(y) \tilde{u}(y)| 1_{\left[\frac{2^l r_B}{C_a}, 1\right]}(m(x)) \,\mathrm{d}\gamma(x). \end{aligned}$$

where the appropriate C_a will be chosen later on. $B(c_B, r_B)$ (which is a ball from $\mathcal{B}_2$) contains the support of u. Note that using lemma 1.8 and with the τ from proposition 1.10 (with $A = 2$) we get

$$\begin{aligned} I &= \int_{\mathbf{R}^d} \sup_{(y,s)\in\Gamma_x^a(\gamma),\, s\leqslant \frac{r_B}{\sqrt{\alpha}}} \left| \int_{\mathbf{R}^d} M_{s^2}(z,w) u(w) \,\mathrm{d}w \right| \mathrm{d}\gamma(x) \\ &\leqslant \int_{\mathbf{R}^d} \sup_{(z,s)\in\Gamma_x^a(\gamma),\, s\leqslant \frac{r_B}{\sqrt{\alpha}}} \int_{\mathbf{R}^d} M_{s^2}(z,w) [1_{N_\tau}(z,w) + 1_{\complement N_\tau}(z,w)] |u(w)| \,\mathrm{d}w \,\mathrm{d}\gamma(x) \\ &\leqslant \int_{\mathbf{R}^d} \sup_{(z,s)\in\Gamma_x^a(\gamma),\, s\leqslant \frac{r_B}{\sqrt{\alpha}}} \int_{\mathbf{R}^d} M_{s^2}(z,w) 1_{\complement N_\tau}(z,w) |u(w)| \,\mathrm{d}w \,\mathrm{d}\gamma(x) \\ &\quad + \int_{\mathbf{R}^d} \sup_{(z,s)\in\Gamma_x^a(\gamma),\, s\leqslant \frac{r_B}{\sqrt{\alpha}}} \int_{\mathbf{R}^d} M_{s^2}(z,w) 1_{N_\tau}(z,w) |u(w)| \,\mathrm{d}w \,\mathrm{d}\gamma(x) \end{aligned}$$

$$\lesssim \|u\|_{L^1(\gamma)} + \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} I^{\text{loc}}_{k,l},$$

where the last line follows from proposition 1.10 and where

$$I^{\text{loc}}_{k,l} := \int_{C_k(B)} \sup_{(z,s)\in\Gamma^a_x(\gamma),s\leqslant \frac{r_B}{\sqrt{\alpha}}} \int_{C_l(B)} M_{s^2}(z,w)1_{N_\tau}(z,w)|u(w)|\,\mathrm{d}w\,\mathrm{d}\gamma(x).$$

We also have using lemma 1.7 and C large enough that

$$\begin{aligned}
\|u\|_{L^1(\gamma)} &\leqslant \sum_{k=0}^{\infty} \|1_{C_k(B)}u\|_{L^1(\gamma)} \\
&\leqslant \sum_{k=0}^{\infty} \|1_{C_k(B)}\|_{L^2(\gamma)}\|1_{C_k(B)}u\|_{L^2(\gamma)} \\
&\leqslant \sum_{k=0}^{\infty} \sqrt{\frac{\gamma(2^{k+1}B)}{\gamma(B)}}e^{-C4^k} \\
&\lesssim \sum_{k=0}^{\infty} \sqrt{e^{8(2^{k+2}+1)^2}}e^{-C4^k} \\
&\lesssim 1.
\end{aligned}$$

So now we still have to estimate $I^{\text{loc}}_{k,l}$. We first estimate $I^{\text{loc}}_{k,l}$ for $k \leqslant l+2$. Using proposition 1.10(ii) we get

$$\begin{aligned}
I^{\text{loc}}_{k,l} &\leqslant \sqrt{\gamma(2^{k+1}B)} \\
&\quad\times \left[\int_{\mathbf{R}^d}\left(\sup_{(z,s)\in\Gamma^a_x(\gamma)}\int_{C_l(B)} M_{s^2}(z,w)1_{N_\tau}(z,w)|u(w)|\,\mathrm{d}w\right)^2\mathrm{d}\gamma(x)\right]^{\frac{1}{2}} \\
&\leqslant \sqrt{\gamma(2^{k+1}B)} \\
&\quad\times \left[\int_{\mathbf{R}^d}\left(\sup_{(z,s)\in\Gamma^a_x(\gamma)}\int_{\mathbf{R}^d} M_{s^2}(z,w)1_{C_l(B)}(w)|u(w)|\,\mathrm{d}w\right)^2\mathrm{d}\gamma(x)\right]^{\frac{1}{2}} \\
&\lesssim \sqrt{e^{8(2^{k+2}+1)^2}}\sqrt{\gamma(B)}\|1_{C_l(B)}u\|_{L^2(\gamma)} \\
&\lesssim \sqrt{e^{8(2^{k+2}+1)^2}}e^{-C4^l}.
\end{aligned}$$

Thus we have

$$\sum_{l=0}^{\infty}\sum_{k=0}^{l+2} I^{\text{loc}}_{k,l} \lesssim 1. \tag{3.4}$$

Finally we estimate $I_{k,l}^{\mathrm{loc}}$ for $k > l+2$. We will use lemma 2.6(i). To be able to use this lemma we should verify that $t \lesssim m(w)$ for all w in $C_l(B)$. First note that $|z-x| < 2am(x)$ hence by lemma 1.9 we get that $m(x) \leqslant (1+2a)m(z)$. Furthermore note that $|z-w| \leqslant \tau m(z)$ hence again by lemma 1.9 we get $m(z) \leqslant (1+\tau)m(w)$. Finally

$$\begin{aligned} t &\leqslant a\sqrt{\alpha} m(x) \\ &\leqslant a\sqrt{\alpha}(1+2a)m(z) \\ &\leqslant a\sqrt{\alpha}(1+2a)(1+\tau)m(w). \end{aligned}$$

This proves the claim. Substituting $s = \frac{t}{\sqrt{\alpha}}$, with lemma 2.6(i) we obtain

$$\begin{aligned} I_{k,l}^{\mathrm{loc}} &= \int_{C_k(B)} \sup_{(z,t)\in\Gamma_x^{(1/\sqrt{\alpha},a\sqrt{\alpha})}(\gamma), t\leqslant r_B} \int_{C_l(B)} M_{\frac{t^2}{\alpha}}(z,w) 1_{N_\tau}(z,w)|u(w)| \,\mathrm{d}w \,\mathrm{d}\gamma(x) \\ &\lesssim \int_{C_k(B)} \sup_{(z,t)\in\Gamma_x^{(1/\sqrt{\alpha},a\sqrt{\alpha})}(\gamma), t\leqslant r_B} \int_{C_l(B)} M_{t^2}(z,w) \exp\left(-\frac{\alpha}{4e^8}\frac{|e^{-t^2}z-w|^2}{1-e^{-2t^2}}\right) \\ &\quad \times 1_{N_\tau}(z,w)|u(w)| \,\mathrm{d}w \,\mathrm{d}\gamma(x). \end{aligned}$$

Note that this estimate holds for all k, l in $\mathbf{Z}_+$. For x in $C_k(B)$, w in $C_l(B)$, $t \leqslant \min\{r_B, a\sqrt{\alpha}(1+2a)m(z)\}$ and $|z-x| < \frac{2t}{\sqrt{\alpha}}$ we have

$$\begin{aligned} |e^{-t^2}z-w| &= |(e^{-t^2}-1)z-(x-z)-(x-w)| \\ &\geqslant |x-w|-|x-z|-(1-e^{-t^2})|z|. \end{aligned}$$

Furthermore we have $|x-w| \geqslant [(2^k+1)-(2^{l+1}+1)]r_B \geqslant 2^{k-1}r_B$ and $|x-z| \leqslant \frac{2r_B}{\sqrt{\alpha}}$. Also $(1-e^{-t^2})|z| \leqslant t^2|z|$ together with $t \leqslant a\sqrt{\alpha}(1+2a)\frac{1}{|z|}$ gives $(1-e^{-t^2})|z| \leqslant a\sqrt{\alpha}(1+2a)r_B$, hence

$$|e^{-t^2}z-w| \geqslant \left(2^{k-1}-\frac{2}{\sqrt{\alpha}}-a\sqrt{\alpha}(1+2a)\right)r_B.$$

Let $M_{a,\alpha}$ in $\mathbf{N}$ be such that $\frac{2}{\sqrt{\alpha}}+a\sqrt{\alpha}(1+2a) \leqslant 2^{M_{a,\alpha}}$. Then for $l+2 \leqslant k \leqslant M_{a,\alpha}+2$ we get that $k-2 \leqslant M_{a,\alpha}$ and hence

$$\begin{aligned} 2^{k-1}-\frac{2}{\sqrt{\alpha}}-a\sqrt{\alpha}(1+2a) &\geqslant 2^{k-1}-2^{M_{a,\alpha}} \\ &\geqslant 2^{M_{a,\alpha}+1}-2^{M_{a,\alpha}} \\ &= 2^{M_{a,\alpha}}. \end{aligned}$$

Therefore,

$$\exp\left(-\frac{\alpha}{4e^8}\frac{|e^{-t^2}z-w|^2}{1-e^{-2t^2}}\right) \leqslant \exp\left(-\frac{\alpha}{4e^8}\left(2^{k-1}-\frac{2}{\sqrt{\alpha}}-a\sqrt{\alpha}(1+2a)\right)^2\frac{r_B^2}{1-e^{-2r_B^2}}\right)$$

$$\leqslant \exp\left(-\frac{\alpha}{8e^8}\left(2^{k-1}-\frac{2}{\sqrt{\alpha}}-a\sqrt{\alpha}(1+2a)\right)^2\right)$$
$$\leqslant \exp\left(-\frac{\alpha}{8e^8}(2^{M_{a,\alpha}})^2\right).$$

So, for $l+2 \leqslant k \leqslant M_{a,\alpha}$ we get using proposition 1.10(ii)

$$I^{\mathrm{loc}}_{k,l} \lesssim \exp\left(-\frac{\alpha}{8e^8}(2^{M_{a,\alpha}})^2\right)\int_{C_k(B)} \sup_{(z,t)\in\Gamma_x^{(1/\sqrt{\alpha},a\sqrt{\alpha})}(\gamma),t\leqslant r_B} \int_{C_l(B)} M_{t^2}(z,w)$$
$$\times 1_{N_\tau}(z,w)|u(w)|\,\mathrm{d}w\,\mathrm{d}\gamma(x)$$
$$\overset{(i)}{\lesssim} \exp\left(-\frac{\alpha}{8e^8}(2^{M_{a,\alpha}})^2\right)\sqrt{\gamma(2^{k+1}B)}\|1_{C_l(B)}u\|_{L^2(\gamma)}$$
$$\lesssim \exp\left(-\frac{\alpha}{8e^8}(2^{M_{a,\alpha}})^2\right)\sqrt{\frac{\gamma(2^{k+1}B)}{\gamma(B)}}e^{-C4^l}$$
$$\overset{(ii)}{\leqslant} \exp\left(-\frac{\alpha}{8e^8}(2^{M_{a,\alpha}})^2\right)e^{8(2^{k+2}+1)^2}e^{-C4^l}.$$

Where we have used proposition 1.10 and Cauchy-Schwarz in (i) and lemma 1.7 in (ii). Hence,

[3.5] $$\sum_{l=0}^{M_{a,\alpha}}\sum_{k=l+2}^{M_{a,\alpha}} I^{\mathrm{loc}}_{k,l} \lesssim 1.$$

We still need an estimate for $k \geqslant \max\{l, M_{a,\alpha}\}+2$. For such k we have $k \geqslant M_{a,\alpha}+2$ hence $2^{k-2} \geqslant 2^{M_{a,\alpha}}$. Then

$$2^{k-1}-\frac{2}{\sqrt{\alpha}}-a\sqrt{\alpha}(1+2a) \geqslant 2^{k-1}-2^{M_{a,\alpha}}$$
$$\geqslant 2^{k-1}-2^{k-2}$$
$$= 2^{k-2}.$$

So,

$$\exp\left(-\frac{\alpha}{4e^8}\frac{|e^{-t^2}z-w|^2}{1-e^{-2t^2}}\right) \leqslant \exp\left(-\frac{\alpha}{8e^8}(2^{k-4})^2\right).$$

Hence,

$$I^{\mathrm{loc}}_{k,l} \lesssim \exp\left(-\frac{\alpha}{8e^8}(2^{k-2})^2\right)\int_{C_k(B)} \sup_{(z,t)\in\Gamma_x^{(1/\sqrt{\alpha},a\sqrt{\alpha})}(\gamma),t\leqslant r_B} \int_{C_l(B)} M_{t^2}(z,w)$$
$$\times 1_{N_\tau}(z,w)|u(w)|\,\mathrm{d}w\,\mathrm{d}\gamma(x)$$
$$\overset{(i)}{\lesssim} \exp\left(-\frac{\alpha}{8e^8}(2^{k-2})^2\right)\sqrt{\gamma(2^{k+1}B)}\|1_{C_l(B)}u\|_{L^2(\gamma)}$$

$$\lesssim \exp\left(-\frac{\alpha}{8e^8}(2^{k-2})^2\right)\sqrt{\frac{\gamma(2^{k+1}B)}{\gamma(B)}}e^{-C4^l}.$$

$$\overset{\text{(ii)}}{\leqslant} \exp\left(-\frac{\alpha}{8e^8}(2^{k-2})^2\right)e^{8(2^{k+2}+1)^2}e^{-C4^l}.$$

Where we have used proposition 1.10 and Cauchy-Schwarz in (i) and lemma 1.7 in (ii).
Finally, we have for α large enough, for example $\alpha > 2^{35}$ suffices, that

[3.6] $$\sum_{l=0}^{\infty}\sum_{k=\max(M_{a,\alpha},l)+2}^{\infty} I_{k,l}^{\text{loc}} \lesssim 1.$$

Combining [3.4], [3.5] and [3.6] this concludes our estimate for I.
Next, we will estimate $I'_{k,l}$ for $k \leqslant l$. Note that

$$\begin{aligned}
I'_{k,l} &:= \int_{C_k(B)} \sup_{(y,s)\in\Gamma_x^a(\gamma),s\geqslant\frac{r_B}{\sqrt{\alpha}}} |e^{s^2L}1_{C_l(B)}(y)u(y)|1_{\left(0,\frac{2^l r_B}{4C_a}\right)}(m(x))\,\mathrm{d}\gamma(x)\\
&= \int_{C_k(B)} \sup_{(z,s)\in\Gamma_x^a(\gamma),s\geqslant\frac{r_B}{\sqrt{\alpha}}} \int_{\mathbf{R}^d} M_{s^2}(z,w)|1_{C_l(B)}(w)u(w)|\,\mathrm{d}w\\
&\quad\times 1_{\left(0,\frac{2^l r_B}{C_a}\right)}(m(x))\,\mathrm{d}\gamma(x)\\
&\leqslant \sqrt{\gamma(2^{k+1}B)}\left\|x\mapsto \sup_{(z,s)\in\Gamma_x^a(\gamma)}\int_{\mathbf{R}^d} M_{s^2}(z,w)|1_{C_l(B)}(w)u(w)|\,\mathrm{d}w\right\|_{L^2(\gamma)}\\
&\overset{\text{(i)}}{\lesssim} \sqrt{\gamma(2^{k+1}B)}\|1_{C_l(B)}u\|_{L^2(\gamma)}\\
&\lesssim e^{8(2^{k+2}+1)^2}e^{-C4^l}.
\end{aligned}$$

Where (i) follows from proposition 1.10(ii). So,

$$\sum_{l=0}^{\infty}\sum_{k=0}^{l} I'_{k,l} \lesssim \sum_{l=0}^{\infty}\sum_{k=0}^{l} e^{8(2^{k+2}+1)^2-C4^l} \lesssim 1.$$

What is left is the estimate of $I'_{l,k}$ for $k > l$. For this let x in $C_k(B)$ such that $m(x) < \frac{2^l r_B}{C_a}$. Let (y,s) in $\Gamma_x^a(\gamma)$, then $s \leqslant am(x)$ and y in $B(x,s)$. Furthermore let z in $B(y,s)$ and w in $C_l(B)$. Note that we have

$$|x-w| \geqslant (2^{k+1}-2^l)r_B \geqslant (2^{l+1}-2^l)r_B = 2^l r_B.$$

And

$$|x-z| \leqslant |z-y| + |y-x| < 2s \leqslant 2am(x).$$

So

$$
\begin{aligned}
|z-w| &\geqslant |x-w|-|x-z| \\
&\geqslant 2^l r_B - 2am(x) \\
&\geqslant (C_a-2a)\,m(x) \\
[*] \qquad &\geqslant \frac{1}{2+4a}\,(C_a-2a)\,m(z) \\
&= \tau m(z).
\end{aligned}
$$

This shows that (z,w) in $\complement N_\tau$. Where we have taken $C_a = (2+4a)\tau + 2a$ and $\tau := \frac{(1+2a)(1+4a)}{2}$ as in corollary 1.16 (with $A=1$). We have also used that $|z-x|<2a$ implies $m(z) \leqslant (2+4a)m(x)$ by lemma 1.9. Hence by corollary 1.16 we have

$$
\begin{aligned}
&\sum_{l=0}^{\infty}\sum_{k=l}^{\infty} I'_{k,l} \\
&\leqslant \sum_{l=0}^{\infty}\int_{\mathbf{R}^d} \sup_{(y,s)\in\Gamma_x^a(\gamma),\, s\geqslant \frac{r_B}{\sqrt{\alpha}}} |e^{s^2L}(1_{C_l(B)}(y)u(y))|1_{\left(0,\frac{2^l r_B}{C_a}\right)}(m(x))\,\mathrm{d}\gamma(x) \\
&\leqslant \sum_{l=0}^{\infty}\int_{\mathbf{R}^d} \sup_{(y,s)\in\Gamma_x^a(\gamma),\, s\geqslant \frac{r_B}{\sqrt{\alpha}}} \left|\int_{\mathbf{R}^d} M_{s^2}(z,w)1_{C_l(B)}(w)u(w)\,\mathrm{d}w\right| 1_{\left(0,\frac{2^l r_B}{C_a}\right)}(m(x))\,\mathrm{d}\gamma(x) \\
&\overset{\text{(i)}}{\leqslant} \sum_{l=0}^{\infty} \|T^{*a}_{\text{glob}}|1_{C_l(B)}f|\|_{L^1(\gamma)} \\
&\overset{\text{(ii)}}{\lesssim} \|u\|_{L^1(\gamma)} \\
&\lesssim 1.
\end{aligned}
$$

Where (i) follows from definition 1.15 and [*]. (ii) follows from corollary 1.16.

Next we estimate $I''_{k,l}$. Let x in $\mathbf{R}^d$, (y,t) in $\Gamma_x^{(2/\sqrt{\alpha},a/\sqrt{\alpha})}(\gamma)$ and z in $B\left(y,\frac{t}{\sqrt{\alpha}}\right)$. We have $|x-y|<\frac{t}{\sqrt{\alpha}}$ hence by lemma 1.9 we have

$$
t \leqslant a\sqrt{\alpha}m(x) \leqslant a\sqrt{\alpha}\left(1+\frac{t}{\sqrt{\alpha}}\right)m(z) \lesssim m(z).
$$

Hence by definition 2.1, [*] and lemma 2.6(ii).

$$
\begin{aligned}
|L^N e^{\frac{t^2}{\alpha}L}(1_{C_l(B)}\tilde{f})(y)| &\lesssim t^{-2N}\int_{C_l(B)} |K_{t^2,N,\alpha}(z,w)\tilde{u}(w)|\,\mathrm{d}w \\
&\lesssim t^{-2N}\int_{C_l(B)} M_{t^2}(z,w)\exp\left(-\frac{\alpha}{4e^{2a^2}}\frac{|e^{-t^2}z-w|^2}{1-e^{-2t^2}}\right)|\tilde{u}(w)|\,\mathrm{d}w \\
&\lesssim t^{-2N}\int_{C_l(B)} M_{t^2}(z,w)|\tilde{u}(w)|\,\mathrm{d}w
\end{aligned}
$$

$$= t^{-2N} e^{t^2 L} |1_{C_l(B)} \tilde{f}|(y).$$

So,

$$\begin{aligned}
I''_{k,l} &= \int_{C_k(B)} \sup_{(y,s)\in\Gamma_x^a(\gamma), s\geqslant \frac{r_B}{\sqrt{\alpha}}} |L^N e^{s^2 L} 1_{C_l(B)}(y)\tilde{u}(y)| 1_{\left[\frac{2^l r_B}{C_a},1\right]}(m(x))\,\mathrm{d}\gamma(x) \\
&\leqslant \int_{C_k(B)} \sup_{(z,t)\in\Gamma_x^{(1/\sqrt{\alpha},a\sqrt{\alpha})}(\gamma), t\geqslant r_B} t^{-2N} e^{t^2 L} |1_{C_l(B)}(z)\tilde{u}(z)| 1_{\left[\frac{2^l r_B}{C_a},1\right]}(m(x))\,\mathrm{d}\gamma(x) \\
&\lesssim r_B^{-2N} J_{k,l}^{\mathrm{glob}} + J_{k,l}^{\mathrm{loc}}.
\end{aligned}$$

Where

$$\begin{aligned}
J_{k,l}^{\mathrm{glob}} &:= \int_{C_k(B)} \sup_{(z,t)\in\Gamma_x^{(1/\sqrt{\alpha},a\sqrt{\alpha})}(\gamma)} \int_{C_l(B)} M_{t^2}(z,w) 1_{\complement N_\tau}(z,w)|\tilde{u}(w)|\,\mathrm{d}w\,\mathrm{d}\gamma(x) \\
J_{k,l}^{\mathrm{loc}} &:= \int_{C_k(B)} \sup_{(z,t)\in\Gamma_x^{(1/\sqrt{\alpha},a\sqrt{\alpha})}(\gamma), t\geqslant r_B} t^{-2N} \int_{C_l(B)} M_{t^2}(z,w) 1_{N_\tau}(z,w)|\tilde{u}(w)|\,\mathrm{d}w \\
&\quad \times 1_{\left[\frac{2^l r_B}{C_a},1\right]}(m(x))\,\mathrm{d}\gamma(x).
\end{aligned}$$

Here τ is as in proposition 1.10 but with the A, a there equal to $1/\sqrt{\alpha}$ and $a/\sqrt{\alpha}$ here respectively. Proposition 1.10 gives that

$$\begin{aligned}
\sum_{l=0}^{\infty}\sum_{k=0}^{\infty} J_{k,l}^{\mathrm{glob}} &\lesssim \sum_{l=0}^{\infty} \|1_{C_l(B)}\tilde{u}\|_{L^1(\gamma)} \\
&\lesssim \sum_{l=0}^{\infty} \sqrt{\gamma(2^l B)} \|1_{C_l(B)}\tilde{u}\|_{L^2(\gamma)} \\
&\lesssim r_B^{2N} \sum_{l=0}^{\infty} e^{8(2^{l+2}+1)^2} e^{-C4^l} \\
&\lesssim r_B^{2N}.
\end{aligned}$$

Next we estimate $J_{k,l}^{\mathrm{loc}}$. For x in $C_k(B)$ and $m(x) \geqslant \frac{2^l r_B}{C_a}$ we have when $k \leqslant l+1$

$$\begin{aligned}
|x - c_B| &\leqslant (2^{k+1}+1) r_B \\
&\leqslant 2^{k+2} r_B \\
&\leqslant 2^{l+3} r_B \\
&\leqslant 2^3 C_a m(x) \\
&\leqslant 2^3 C_a (1 + 2^3 C_a) m(c_B) \\
&=: \tau' m(c_B).
\end{aligned}$$

Where the second to last line follows from lemma 1.9. So by definition

$$J_{k,l}^{\mathrm{loc}} \leqslant \int_{C_k(B)\cap N_{\tau'}(B)} \sup_{(y,t)\in\Gamma_x^{(1/\sqrt{\alpha},a\sqrt{\alpha})}(\gamma),t\geqslant r_B} t^{-2N} \times \int_{C_l(B)} M_{t^2}(z,w)1_{N_\tau}(z,w)|\tilde{u}(w)|\,\mathrm{d}w\,\mathrm{d}\gamma(x).$$

For $k \leqslant l+1$ we now have that

$$\begin{aligned}\sum_{l=0}^{\infty}\sum_{k=0}^{l+1} J_{k,l}^{\mathrm{loc}} &\leqslant r_B^{-2N}\sum_{l=0}^{\infty}\sum_{k=0}^{l+1}\sqrt{\gamma(C_k(B)\cap N_{\tau'}(B))} \\ &\quad \cdot \left\| x\mapsto \sup_{(y,t)\in\Gamma_x^{(1/\sqrt{\alpha},a\sqrt{\alpha})}(\gamma)} \int_{C_l(B)} M_{t^2}(z,w)1_{N_\tau}(z,w)|\tilde{u}(y)|\,\mathrm{d}w \right\|_{L^2(\gamma)} \\ &\overset{\text{(i)}}{\lesssim} r_B^{-2N}\sum_{l=0}^{\infty}\sum_{k=0}^{l+1}\sqrt{2^{kd}\gamma(B)}\|1_{C_l(B)}\tilde{u}\|_{L^2(\gamma)} \\ &\lesssim \sum_{l=0}^{\infty}\sum_{k=0}^{l+1} 2^{k\frac{d}{2}}e^{-C4^l} \\ &\lesssim 1.\end{aligned}$$

Where (i) follows from proposition 1.10 and lemma 2.10.

Finally we estimate $J_{k,l}^{\mathrm{loc}}$ for $k > l+1$ to complete the proof. We first use the substitution $t = \frac{s}{\sqrt{\alpha}}$ and lemma 2.6(i) to get

$$\begin{aligned}J_{k,l}^{\mathrm{loc}} &\overset{\text{(i)}}{\lesssim} \int_{C_k(B)\cap N_{\tau'}(B)} \sup_{(z,s)\in\Gamma_x^{(1/\alpha,a\alpha)}(\gamma),s\geqslant r_B} s^{-2N} \\ &\quad\times\int_{C_l(B)} M_{\frac{s^2}{\alpha}}(z,w)1_{N_\tau}(z,w)|\tilde{u}(w)|\,\mathrm{d}w\,\mathrm{d}\gamma(x) \\ &\lesssim \int_{C_k(B)\cap N_{\tau'}(B)} \sup s^{-2N}\int_{C_l(B)} M_{s^2}(z,w) \\ &\quad\times\exp\left(-\frac{\alpha}{4e^{2a^2}}\frac{|e^{-s^2}z-w|^2}{1-e^{-2s^2}}\right)|\tilde{u}(w)|\,\mathrm{d}w\,\mathrm{d}\gamma(x).\end{aligned}$$

Where (i) follows from lemma 2.6(i). For this we still need to verify that $s \lesssim m(w)$. We have x in $\mathbf{R}^d$, $s \leqslant a\alpha m(x)$ and $|x-z| < \frac{2}{\alpha}s$, so $|x-z| < \frac{2}{\alpha}a\alpha m(x) = 2am(x)$. From (z,w) in N_τ we obtain $|z-w| \leqslant \tau m(z)$ and hence by lemma 1.9 we have

$$s \leqslant a\alpha m(x) \leqslant a\alpha(1+2a)m(z) \leqslant a\alpha(1+2a)(1+\tau)m(w) \lesssim m(w).$$

This proves the claim. Furthermore we are in the situation that x in $C_k(B)$, w in $C_l(B)$, $s \leqslant \alpha a m(x)$ and $|x-z| < \frac{2}{\alpha}s$ so

$$|e^{-s^2}z-w| \geqslant |z-w| - (1-e^{-s^2})|z| \geqslant (2^k-2^{l+1})r_B - \alpha(a+2a^2)s,$$

where
$$(1-e^{-s^2})|z| \leqslant s^2|z| \leqslant \alpha a\left(1+\frac{2}{\alpha}a\right)m(z)|z|s \leqslant \alpha(a+2a^2)s,$$
by lemma 1.9 for large α. Let $M = M_{a,\alpha}$ be such that $a\alpha^2(a+2a^2) \leqslant 2^M$. Now we pick the region for k,l where $2^M \leqslant \frac{1}{2}(2^k - 2^{l+1})r_B \leqslant 2^k - 2^{l+1}$, then we have
$$\begin{aligned}(2^k-2^{l+1})r_B - \alpha(a+2a^2)s &\geqslant (2^k - 2^{l+1} - a\alpha^2(a+2a^2)\\ &\geqslant (2^k-2^{l+1})r_B - 2^M\\ &\geqslant \frac{1}{2}(2^k-2^{l+1})r_B.\end{aligned}$$
So, note that because $\exp(-x) \leqslant x^{-N}$
$$[*] \qquad s^{-2N}\exp\left(-C^2\left(\frac{r_B}{s}\right)^2\right) \leqslant s^{-2N}\left(-C^{2N}\left(\frac{r_B}{s}\right)^{2N}\right) = C^{-2N}r_B^{-2N}.$$
Therefore,
$$\begin{aligned}&\int_{C_k(B)\cap N_{\tau'}(B)} \sup s^{-2N}\int_{C_l(B)} M_{s^2}(z,w)\exp\left(-\frac{\alpha}{4e^{2a^2}}\frac{|e^{-s^2}z-w|^2}{1-e^{-2s^2}}\right)|\tilde{u}(w)|\,\mathrm{d}w\,\mathrm{d}\gamma(x)\\
&\leqslant \int_{C_k(B)\cap N_{\tau'}(B)} \sup s^{-2N}\exp\left(-\frac{\alpha}{16e^{2a^2}}\frac{(2^k-2^{l+1})^2r_B^2}{1-e^{-2s^2}}\right)\\
&\quad\times\int_{C_l(B)} M_{s^2}(z,w)|\tilde{u}(w)|\,\mathrm{d}w\,\mathrm{d}\gamma(x)\\
&\leqslant \int_{C_k(B)\cap N_{\tau'}(B)} \sup s^{-2N}\exp\left(-\frac{\alpha}{32e^{2a^2}}(2^k-2^{l+1})^2\left(\frac{r_B}{s}\right)^2\right)\\
&\quad\times\int_{C_l(B)} M_{s^2}(z,w)|\tilde{u}(w)|\,\mathrm{d}w\,\mathrm{d}\gamma(x)\\
&\overset{\text{(i)}}{\lesssim} [(2^k-2^{l+1})r_B]^{-2N}\int_{C_k(B)\cap N_{\tau'}(B)}\int_{C_l(B)} M_{s^2}(z,w)|\tilde{u}(w)|\,\mathrm{d}w\,\mathrm{d}\gamma(x)\\
&\overset{\text{(ii)}}{\leqslant} [(2^k-2^{l+1})r_B]^{-2N}\sqrt{\gamma(C_k(B)\cap N_{\tau'}(B))}\|1_{C_l(B)}\tilde{u}\|_{L^2(\gamma)}\\
&\overset{\text{(iii)}}{\lesssim} (2^k-2^{l+1})^{-2N}e^{-C4^l}2^{k\frac{d}{2}},\end{aligned}$$
where we have used [*] in (i), proposition 1.10 in (ii) and lemma 2.10 in (iii). We have
$$\sum\sum J_{k,l}^{\mathrm{loc}} \lesssim 1.$$
where the sum ranges over all (k,l) such that $2^M \leqslant \frac{1}{2}(2^k - 2^{l+1})r_B$. This holds for C large enough and an N such that $2N > \frac{d}{2}$. We still need an estimate for $2^M > \frac{1}{2}(2^k - 2^{l+1})$. Note that
$$J_{k,l}^{\mathrm{loc}} \overset{\text{(i)}}{\leqslant} \sqrt{\gamma(2^{k+1}B)}r_B^{-2N}\|e^{tL}1_{C_l(B)}\tilde{u}\|_{L^2(\gamma)}$$

$$
\begin{aligned}
&\leqslant \sqrt{\gamma(2^{k+1}B)} r_B^{-2N} \|1_{C_l(B)} \tilde{u}\|_{L^2(\gamma)} \\
&\lesssim e^{8(2^{k+2}+1)^2} e^{-C4^l}
\end{aligned}
$$

where (i) follows from proposition 1.10. Over a finite sum, this is certainly finite. This completes the proof. ■

4 The remainder terms

In this chapter we will handle the remainder terms

(i) $x \mapsto \displaystyle\int_0^2 1_{\left[\frac{m(x)}{b},2\right]}(t) t^{2N+1} L^N e^{\frac{t^2}{\alpha}L} \partial_{x_j}^* A(x,t) \,\frac{\mathrm{d}t}{t}$,

(ii) $x \mapsto \displaystyle\int_0^{\frac{m(x)}{b}} t^{2N+1} L^N e^{\frac{(1+a^2)t^2}{\alpha}L} \partial_{x_j}^* (1_{\complement D}(x,t) t \partial_{x_j} e^{\frac{a^2t^2}{\alpha}L}) u(x) \frac{\mathrm{d}t}{t}$,

(iii) $x \mapsto \displaystyle\int_{\frac{m(x)}{b}}^{\infty} (t^2L)^{N+1} e^{\frac{(1+a^2)t^2}{\alpha}L} u(x) \frac{\mathrm{d}t}{t}$,

where u in $L^1(\gamma)$ and A is a $t^{1,2}(\gamma)$ 2-atom. When we compare the first displayed equation with corollary 1.19 we note that we can replace the 2 in the upper bound of the integral by r_B because A is supported in $B(c_B, r_B)$. This is what we will do in the next lemma.

4.1 The estimates

4.1 Lemma. *Let N in $\mathbf{Z}_+$, $j = 1, \dots, d$, $b > 0$ and $\alpha > 2^{37}$. Furthermore, let A be a $t^{1,2}(\gamma)$ 2-atom associated with the ball $B = B(c_B, r_B)$ in $\mathcal{B}_2$. Then we have*

$$\left\| x \mapsto \int_0^{r_B} 1_{\left[\frac{m(x)}{b},2\right]}(t) t^{2N+1} L^N e^{\frac{t^2}{\alpha}L} \partial_{x_j}^* A(x,t) \,\frac{\mathrm{d}t}{t} \right\|_{L^1(\gamma)} \lesssim 1.$$

Proof. For y in B we have by lemma 1.9 that $m(y) \simeq m(c_B)$. Moreover we have

$$\begin{aligned}
&\left\| x \mapsto \int_0^{r_B} 1_{\left[\frac{m(x)}{b},2\right]}(t) t^{2N+1} L^N e^{\frac{t^2}{\alpha}L} \partial_{x_j}^* A(x,t) \,\frac{\mathrm{d}t}{t} \right\|_{L^1(\gamma)} \\
&\quad = \int_{\mathbf{R}^d} \left| \int_0^{r_B} 1_{\left[\frac{m(x)}{b},2\right]}(t) t^{2N+1} L^N e^{\frac{t^2}{\alpha}L} \partial_{x_j}^* A(x,t) \,\frac{\mathrm{d}t}{t} \right| \mathrm{d}\gamma(x) \\
&\quad \leqslant \int_{\mathbf{R}^d} \int_0^{r_B} \int_{\mathbf{R}^d} |\tilde{K}_{t^2,N,\alpha,j}(x,y) A(y,t)| \,\mathrm{d}y \,\frac{\mathrm{d}t}{t} \,\mathrm{d}\gamma(x) \\
&\quad = \sum_{k=0}^{\infty} \int_{C_k(B)} \int_0^{r_B} \int_B |\tilde{K}_{t^2,N,\alpha,j}(x,y) A(y,t)| \,\mathrm{d}y \,\frac{\mathrm{d}t}{t} \,\mathrm{d}\gamma(x).
\end{aligned}$$

Where we have used the decomposition of $\mathbf{R}^d$ into annuli and that the support of A lies in B in (i). For $t \leqslant r_B \leqslant 2m(c_B)$ and y in B we get $t \lesssim m(y)$. So we can apply lemma 2.6(iii) to obtain that the RHS of the previous estimate is smaller than

$$[*] \quad \sum_{k=0}^{\infty} \int_{C_k(B)} \int_0^{r_B} \int_B \exp\left(-\frac{\alpha}{4e^8} \frac{|e^{-t^2} y - x|^2}{1 - e^{-2t^2}} \right) M_{t^2}(x,y)|A(y,t)| \,\mathrm{d}y \,\frac{\mathrm{d}t}{t} \,\mathrm{d}\gamma(x).$$

Now we can use the same argument as in lemma 2.7, that is [2.6-2.7]. We recall that

$$\begin{aligned} \exp\left(-\frac{\alpha}{4e^8} \frac{|e^{-t^2} y - x|^2}{1 - e^{-2t^2}} \right) &\leqslant \exp\left(-\frac{\alpha}{8e^8} \frac{|y-x|^2}{t^2} \right) \exp\left(\frac{\alpha}{4e^8} (t|y|)^2 \right) \\ &\lesssim \exp\left(-\frac{\alpha}{8e^8} \frac{|y-x|^2}{t^2} \right). \end{aligned}$$

We can use this to obtain that [*] is smaller than a constant times

$$\sum_{k=0}^{\infty} \int_{C_k(B)} \int_0^{r_B} \int_B \exp\left(-\frac{\alpha}{4e^8} \frac{|y-x|^2}{t^2} \right) M_{t^2}(x,y)|A(y,t)| \,\mathrm{d}y \,\frac{\mathrm{d}t}{t} \,\mathrm{d}\gamma(x).$$

For x in $C_k(B)$ and y in B we have $|x-y| \geqslant 2^{k-1} r_B$ for $k \geqslant 1$ so we get

$$\begin{aligned} &\left\| x \mapsto \int_0^{r_B} 1_{[\frac{m(x)}{b}, 2]}(t) t^{2N+1} L^N e^{\frac{t^2}{\alpha} L} \partial_{x_j}^* A(x,t) \,\frac{\mathrm{d}t}{t} \right\|_{L^1(\gamma)} \\ &\lesssim \sum_{k=0}^{\infty} \int_{C_k(B)} \int_0^{r_B} \exp\left(-\frac{\alpha}{8e^8} \left(\frac{r_B}{t}\right)^2 4^{k-1} \right) \int_B M_{t^2}(x,y)|A(y,t)| \,\mathrm{d}y \,\frac{\mathrm{d}t}{t} \,\mathrm{d}\gamma(x) \\ &= \sum_{k=0}^{\infty} \int_0^{r_B} \exp\left(-\frac{\alpha}{8e^8} \left(\frac{r_B}{t}\right)^2 4^{k-1} \right) \int_{C_k(B)} \int_B M_{t^2}(x,y)|A(y,t)| \,\mathrm{d}y \,\mathrm{d}\gamma(x) \,\frac{\mathrm{d}t}{t} \\ &\leqslant \sum_{k=0}^{\infty} \int_0^{r_B} \exp\left(-\frac{\alpha}{8e^8} \left(\frac{r_B}{t}\right)^2 4^{k-1} \right) \sqrt{\gamma(C_k(B))} \| x \mapsto e^{t^2 L} |A(x,t)| \|_{L^2(\gamma)} \,\frac{\mathrm{d}t}{t} \\ &\overset{\text{(i)}}{\lesssim} \sum_{k=0}^{\infty} e^{8(2^{k+2}+1)^2} \sqrt{\gamma(B)} \exp\left(-\frac{\alpha}{4e^8} 4^{k-1} \right) \int_0^{r_B} \| x \mapsto e^{t^2 L} |A(x,t)| \|_{L^2(\gamma)} \,\frac{\mathrm{d}t}{t} \\ &\overset{\text{(ii)}}{\lesssim} \sum_{k=0}^{\infty} e^{8(2^{k+2}+1)^2} \exp\left(-\frac{\alpha}{4e^8} 4^{k-1} \right) \\ &\lesssim 1. \end{aligned}$$

Where (i) follows from the doubling property [1.6]. (ii) in its turn follows for a similar argument as in proposition 3.2, the boundedness of the semigroup and the fact that A is an 2-atom. This completes the proof. ■

From corollary 1.16 we immediately obtain

4.2 Corollary. *Let $a, b > 0$, N in $\mathbf{Z}_+$, $j = 1, \ldots, d$ and $\alpha > 2^{37}$. Furthermore, let A be a $t^{1,2}(\gamma)$ 2-atom associated with the ball $B = B(c_B, r_B)$ in $\mathcal{B}_2$. Then we have*

$$\left\| T^{*a}_{glob} \left(x \mapsto \int_0^{r_B} 1_{\left[\frac{m(x)}{b},2\right]}(t) t^{2N+1} L^N e^{\frac{t^2}{\alpha}L} \partial^*_{x_j} A(x,t) \, \frac{\mathrm{d}t}{t} \right) \right\|_{L^1(\gamma)} \lesssim 1.$$

We will use this corollary (which follows from lemma 4.1) to prove the required estimate for the $h^1_{\max,a}$.

4.3 Proposition. *Let $a > 0$, N in $\mathbf{Z}_+$, $j = 1, \ldots, d$ and $\alpha > 2^{40}$. Furthermore, let A be a $t^{1,2}(\gamma)$ 2-atom associated with the ball $B = B(c_B, r_B)$ in $\mathcal{B}_2$. Then we have*

$$\left\| x \mapsto \int_0^{r_B} 1_{\left[\frac{m(x)}{b},2\right]}(t) t^{2N+1} L^N e^{\frac{t^2}{\alpha}L} \partial^*_{x_j} A(x,t) \, \frac{\mathrm{d}t}{t} \right\|_{h^1_{max,a}} \lesssim 1.$$

Proof. We have

$$\begin{aligned}
&\left\| x \mapsto \int_0^{r_B} 1_{\left[\frac{m(x)}{b},2\right]}(t) t^{2N+1} L^N e^{\frac{t^2}{\alpha}L} \partial^*_{x_j} A(x,t) \, \frac{\mathrm{d}t}{t} \right\|_{h^1_{\max,a}} \\
&= \left\| T^*_a \left(x \mapsto \int_0^{r_B} 1_{\left[\frac{m(x)}{b},2\right]}(t) t^{2N+1} L^N e^{\frac{t^2}{\alpha}L} \partial^*_{x_j} A(x,t) \, \frac{\mathrm{d}t}{t} \right) \right\|_{L^1(\gamma)} \\
&= \left\| x \mapsto \sup_{(y,t)\in\Gamma^a_x(\gamma)} \left| e^{t^2 L} \int_0^{r_B} 1_{\left[\frac{m(z)}{b},2\right]}(t) t^{2N+1} L^N e^{\frac{t^2}{\alpha}L} \partial^*_{z_j} A(z,t) \, \frac{\mathrm{d}t}{t} \right| \right\|_{L^1(\gamma)} \\
&= \int_{\mathbf{R}^d} \sup_{(y,t)\in\Gamma^a_x(\gamma)} \left| \iint_{\mathbf{R}^{2d}} M_{t^2}(z,w) \right. \\
&\quad \left. \times \left(\int_0^{r_B} 1_{\left[\frac{m(w)}{b},2\right]}(s) \tilde{K}_{s^2,N,\alpha,j}(w,v) A(v,s) \, \frac{\mathrm{d}s}{s} \right) \mathrm{d}v \, \mathrm{d}w \right| \mathrm{d}\gamma(x) \\
&= \int_{\mathbf{R}^d} \sup_{(y,t)\in\Gamma^a_x(\gamma)} \left| \iint_{\mathbf{R}^{2d}} [1_{N_\tau}(z,w) + 1_{N_{\complement\tau}}(z,w)] M_{t^2}(z,w) \right. \\
&\quad \left. \times \left(\int_0^{r_B} 1_{\left[\frac{m(w)}{b},2\right]}(s) \tilde{K}_{s^2,N,\alpha,j}(w,v) A(v,s) \, \frac{\mathrm{d}s}{s} \right) \mathrm{d}v \, \mathrm{d}w \right| \mathrm{d}\gamma(x).
\end{aligned}$$

Given corollary 4.2 we only have to estimate for the τ as in corollary 1.16

$$\begin{aligned}
&\int_{\mathbf{R}^d} \sup_{(y,t)\in\Gamma^a_x(\gamma)} \left| \iint_{\mathbf{R}^{2d}} M_{t^2}(z,w) 1_{N_\tau}(z,w) \right. \\
&\quad \left. \times \int_0^{r_B} 1_{\left[\frac{m(w)}{b},2\right]}(s) \tilde{K}_{s^2,N,\alpha,j}(w,v) A(v,s) \, \frac{\mathrm{d}s}{s} \, \mathrm{d}v \, \mathrm{d}w \right| \mathrm{d}\gamma(x).
\end{aligned}$$

So, it is sufficient to estimate

$$\begin{aligned}
I := &\int_{\mathbf{R}^d} \sup_{(z,t)\in\Gamma^a_x(\gamma)} \iint_{\mathbf{R}^{2d}} M_{t^2}(z,w) 1_{N_\tau}(z,w) \\
&\times \int_0^{r_B} 1_{\left[\frac{m(w)}{b},2\right]}(s) |\tilde{K}_{s^2,N,\alpha,j}(w,v) A(v,s)| \, \frac{\mathrm{d}s}{s} \, \mathrm{d}v \, \mathrm{d}w \, \mathrm{d}\gamma(x).
\end{aligned}$$

Where we have used lemma 1.8.

Note that for v in B we have $|v - c_B| < r_B < 2m(c_B)$ and hence $r_B < 2m(c_B) \lesssim m(v)$ by lemma 1.9. So by lemma 2.6(iii) we have that

$$I \lesssim \int_{\mathbf{R}^d} \sup_{(z,t)\in\Gamma_x^a(\gamma)} \iint_{\mathbf{R}^{2d}} M_{t^2}(z,w) 1_{N_\tau}(z,w) \int_0^{r_B} 1_{[\frac{m(w)}{b},2]}(s) \times \exp\left(-\frac{\alpha}{4e^8}\frac{|e^{-s^2}v - w|^2}{1-e^{-2s^2}}\right) M_{s^2}(w,v)|A(v,s)| \frac{\mathrm{d}s}{s}\,\mathrm{d}v\,\mathrm{d}w\,\mathrm{d}\gamma(x).$$

We write

$$I \lesssim I^{\mathrm{loc}} + \sum_{k=0}^{\infty} I_k^{\mathrm{glob}},$$

where

$$I^{\mathrm{loc}} = \int_{\mathbf{R}^d} \sup_{(y,s)\in\Gamma_x^a(\gamma)} \iint_{\mathbf{R}^{2d}} M_{s^2}(y,z) 1_{N_\tau}(y,z) \int_0^{r_B} 1_{[\frac{m(z)}{b},2]}(t) \times \exp\left(-\frac{\alpha}{4e^8}\frac{|e^{-t^2}w - z|^2}{1-e^{-2t^2}}\right) M_{t^2}(z,w) 1_{N_1}(z,w)|A(w,t)| \frac{\mathrm{d}t}{t}\,\mathrm{d}w\,\mathrm{d}z\,\mathrm{d}\gamma(x)$$

$$I_k^{\mathrm{glob}} = \int_{C_k(B)} \sup_{(y,s)\in\Gamma_x^a(\gamma)} \iint_{\mathbf{R}^{2d}} M_{s^2}(y,z) 1_{N_\tau}(y,z) \int_0^{r_B} 1_{[\frac{m(z)}{b},2]}(t) \times \exp\left(-\frac{\alpha}{4e^8}\frac{|e^{-t^2}w - z|^2}{1-e^{-2t^2}}\right) M_{t^2}(z,w) 1_{\complement N_1}(z,w)|A(w,t)| \frac{\mathrm{d}t}{t}\,\mathrm{d}w\,\mathrm{d}z\,\mathrm{d}\gamma(x).$$

We will first estimate I_k^{glob}. For w in B (recall that A is supported in B), x in $C_k(B)$, $|x-y| < am(x)$, $|y-z| < \tau m(y)$, $t \leqslant r_B$ and $m(z) \leqslant br_B$, we get, using lemma 1.9

$$\begin{aligned} |w - c_B| &< r_B \\ &< 2m(c_B) \\ &\leqslant 2(1+2)m(w), \end{aligned}$$

and hence by $t \leqslant r_B$ we have $t \lesssim m(w)$. Furthermore we have

$$\begin{aligned} |x - z| &\leqslant |x-y| + |y-z| \\ &\leqslant am(x) + \tau m(y). \end{aligned}$$

We also have $m(y) \leqslant 2(1+a)m(x)$. Hence we get

$$|x - z| \leqslant [a + 2\tau(1+a)]m(x).$$

Furthermore we also have

$$m(x) \leqslant [1 + a + 2\tau(1+a)]m(z)$$

$$\leqslant [1 + a + 2\tau(1 + a)]br_B.$$

Let $C_{a,b,\tau} := b[1 + a + 2\tau(1 + a)][a + 2\tau(1 + a)]$. Then

$$[*] \quad |e^{-t^2}w - z| \geqslant |w - x| - |x - z| - (1 - e^{-t^2})|w| \geqslant 2^{k-12}r_B - C_{a,b,\tau}r_B - t^2|w|.$$

Note that $t \leqslant r_B \lesssim m(w)$ implies that $t^2|w| \leqslant C'r_B$ for some $C' > 0$. Let $M = M_{a,b,\alpha}$ in $\mathbf{N}$ be such that $C_{a,b,\tau} + C' \leqslant 2^{M_{a,b,\alpha}}$.

Note that for $k \leqslant M_{a,b,\alpha} + 2$, x in $C_k(B)$ and $|x - z| \leqslant [a + 2\tau(1 + a)]m(x)$ hence by lemma 1.9 we have $m(z) \simeq m(x) \simeq m(c_B)$. In particular we have $m(z) \geqslant \kappa m(c_B)$ for some $\kappa > 0$. We first apply Cauchy-Schwarz and $\exp(-x) \leqslant 1$ to I_k^{glob}. To complete the estimate we need to estimate the term

$$\begin{aligned}
&I_{k,l}^{\mathrm{glob}} \\
&\overset{(i)}{\leqslant} \Bigg\| x \mapsto \sup_{(y,s)\in\Gamma_x^a(\gamma)} \iint_{\mathbf{R}^{2d}} M_{s^2}(y,z) 1_{N_\tau}(y,z) \int_0^{r_B} 1_{[\frac{m(z)}{b},2]}(t) \\
&\quad \times M_{t^2}(z,w) 1_{\complement N_1}(z,w) |A(w,t)| \frac{\mathrm{d}t}{t} \,\mathrm{d}w\,\mathrm{d}z \Bigg\|_{L^2(\gamma)} \sqrt{\gamma(C_k(B))} \\
&\leqslant \Bigg\| x \mapsto \sup_{(y,s)\in\Gamma_x^a(\gamma)} \int_{\mathbf{R}^d} M_{s^2}(y,z) 1_{N_\tau}(y,z) \int_0^{r_B} 1_{[\frac{m(z)}{b},2]}(t) e^{t^2L} |A(z,t)| \frac{\mathrm{d}t}{t}\,\mathrm{d}z \Bigg\|_{L^2(\gamma)} \\
&\quad \times \sqrt{\gamma(C_k(B))} \\
&\leqslant \int_0^{r_B} 1_{[\frac{\kappa}{b}m(c_B),2]}(t) \Bigg\| x \mapsto \sup_{(y,s)\in\Gamma_x^a(\gamma)} \int_{\mathbf{R}^d} M_{s^2}(y,z) 1_{N_\tau}(y,z) e^{t^2L} |A(z,t)|\,\mathrm{d}z \Bigg\|_{L^2(\gamma)} \frac{\mathrm{d}t}{t} \\
&\quad \times \sqrt{\gamma(C_k(B))} \\
&\overset{(ii)}{\leqslant} \int_0^{r_B} 1_{[\frac{\kappa}{b}m(c_B),2]}(t) \left\| x \mapsto e^{t^2L} |A(x,t)| \right\|_{L^2(\gamma)} \frac{\mathrm{d}t}{t} \sqrt{\gamma(C_k(B))} \\
&\leqslant \left(\int_{\frac{\kappa}{b}m(c_B)}^{2m(c_B)} \frac{\mathrm{d}t}{t} \right)^{\frac{1}{2}} \left(\int_0^{r_B} \|x \mapsto A(x,t)\|_{L^2(\gamma)}^2 \frac{\mathrm{d}t}{t} \right)^{\frac{1}{2}} \sqrt{\gamma(C_k(B))} \\
&\lesssim \frac{1}{\sqrt{\gamma(B)}} \sqrt{\gamma(C_k(B))}.
\end{aligned}$$

Where (i) follows by Cauchy-Schwarz and (ii) follows by proposition 1.10(ii). So we get by Cauchy-Schwarz that

$$\begin{aligned}
\sum_{k=0}^{M_{a,b,\alpha}+2} I_k^{\mathrm{glob}} &\lesssim \sum_{k=0}^{M_{a,b,\alpha}+2} \sqrt{\frac{\gamma(2^{k+1}B)}{\gamma(B)}} \\
&\lesssim \sum_{k=0}^{M_{a,b,\alpha}+2} e^{8(2^{k+2}+1)^2}
\end{aligned}$$

$$\lesssim 1.$$

For $k > M_{a,b,\alpha} + 2$ we may estimate the RHS of [*]

$$\begin{aligned} 2^{k-1} r_B - C_{a,b,\tau} r_B - t^2|w| &\geqslant 2^{k-1} r_B - C_{a,b,\tau} r_B - C' r_B \\ &\geqslant (2^{k-1} - 2^M) r_B \\ &\geqslant 2^{k-2} r_B. \end{aligned}$$

So, similarly to the case $k \leqslant M_{a,b,\alpha} + 2$ we have to estimate after applying Cauchy-Schwarz and proposition 1.10(ii)

$$\begin{aligned} I_k^{\text{glob}} &\leqslant \sqrt{\gamma(2^{k+1}B)} \Bigg\| x \mapsto \sup_{(y,s)\in\Gamma_x^a(\gamma)} \int_{\mathbf{R}^{2d}} M_{s^2}(y,z) \\ &\times \int_0^{r_B} \exp\left(-\frac{\alpha}{4e^8} \frac{|e^{-t^2} w - z|^2}{1 - e^{-2t^2}}\right) M_{t^2}(z,w) |A(w,t)| \,\frac{\mathrm{d}t}{t} \,\mathrm{d}w\,\mathrm{d}z \Bigg\|_{L^2(\gamma)} \\ &\leqslant \sqrt{\gamma(2^{k+1}B)} \Bigg\| x \mapsto \sup_{(y,s)\in\Gamma_x^a(\gamma)} \int_{\mathbf{R}^{2d}} M_{s^2}(y,z) \\ &\times \int_0^{r_B} \exp\left(-\frac{\alpha}{4e^8} \left(\frac{2^{k-2} r_B}{t}\right)^2\right) M_{t^2}(z,w) |A(w,t)| \,\frac{\mathrm{d}t}{t} \,\mathrm{d}w\,\mathrm{d}z \Bigg\|_{L^2(\gamma)} \\ &\leqslant \sqrt{\gamma(2^{k+1}B)} \int_0^{r_B} \exp\left(-\frac{\alpha}{4e^8} \left(\frac{2^{k-2} r_B}{t}\right)^2\right) \\ &\times \Bigg\| x \mapsto \sup_{(y,s)\in\Gamma_x^a(\gamma)} \int_{\mathbf{R}^{d}} M_{s^2}(y,z) e^{t^2 L} |A(z,t)| \,\mathrm{d}z \Bigg\|_{L^2(\gamma)} \frac{\mathrm{d}t}{t} \\ &\leqslant \sqrt{\gamma(2^{k+1}B)} \int_0^{r_B} \exp\left(-\frac{\alpha}{4e^8} \left(\frac{2^{k-2} r_B}{t}\right)^2\right) \|x \mapsto e^{t^2 L} |A(x,t)|\|_{L^2(\gamma)} \frac{\mathrm{d}t}{t} \\ &\leqslant \sqrt{\gamma(2^{k+1}B)} \exp\left(-\frac{\alpha}{8e^8} \left(2^{k-2}\right)^2\right) \left(\int_0^{r_B} \|x \mapsto A(x,t)\|_{L^2(\gamma)}^2 \frac{\mathrm{d}t}{t}\right)^{\frac{1}{2}} \\ &\times \left(\int_0^{r_B} \exp\left(-\frac{\alpha}{4e^8} \left(\frac{2^{k-2} r_B}{t}\right)^2\right) \frac{\mathrm{d}t}{t}\right)^{\frac{1}{2}} \\ &\leqslant e^{8(2^{k+2}+1)^2} \exp\left(-\frac{\alpha}{8e^8} \left(2^{k-2}\right)^2\right). \end{aligned}$$

Where in the last term we have used that the integral in the penultimate line is finite. Hence,

$$\sum_{k=M_{a,b,\alpha}+2}^{\infty} I_k^{\text{glob}} \lesssim 1$$

for α large enough.

If we bound I^{loc} by a constant our proof is done, so that is what we will now do. First remark that for t in $\left[\frac{m(z)}{b}, 2\right]$ we have

$$\begin{aligned} M_{t^2}(z,w)\exp\left(-\frac{\alpha}{4e^8}\frac{|e^{-t^2}w-z|^2}{1-e^{-2t^2}}\right) &\leqslant (1-e^{-2t^2})^{-\frac{d}{2}} \\ &\leqslant (1-e^{-2\frac{m(z)^2}{b^2}})^{-\frac{d}{2}} \\ &\leqslant \left(\frac{2b^2}{1+e^{-8}}\right)^d \frac{1}{m(z)^d} \\ &\lesssim \frac{1}{m(z)^d} \end{aligned} \tag{4.1}$$

by calculus. For w in B (as A is supported in B), (z,w) in N_1, (y,z) in N_τ and (y,s) in $\Gamma_x^a(\gamma)$ we have

$$\begin{aligned} |x-c_B| &\leqslant |x-y|+|y-z|+|z-w|+|w-c_B| \\ &\leqslant am(x)+\tau m(y)+m(z)+2m(c_B) \\ &\lesssim m(c_B) \end{aligned}$$

by lemma 1.9. Similarly

$$\begin{aligned} |x-w| &\leqslant |x-y|+|y-z|+|z-w| \\ &\leqslant am(x)+\tau m(y)+m(z) \\ &\lesssim m(w). \end{aligned}$$

We have $e^{-|x|^2}\simeq e^{-|w|^2}$ which can be shown using a similar argument as in lemma 2.10. Therefore we have $\mathrm{d}w\,\mathrm{d}\gamma(x)\simeq \mathrm{d}\gamma(w)\,\mathrm{d}x$ by lemma 2.9. Hence,

$$\begin{aligned} I^{\text{loc}} &\leqslant \int_{\mathbf{R}^d}\sup_{(y,s)\in\Gamma_x^a(\gamma)}\iint_{\mathbf{R}^{2d}} M_{s^2}(y,z)1_{N_\tau}(y,z)\int_0^{r_B}1_{\left[\frac{m(z)}{b},2\right]}(t) \\ &\quad\times\frac{1}{m(z)^d}1_{N_1}(z,w)|A(w,t)|\,\frac{\mathrm{d}t}{t}\,\mathrm{d}w\,\mathrm{d}z\,\mathrm{d}\gamma(x) \\ &\lesssim \int_0^{r_B}\frac{1}{m(c_B)^d}\int_{\mathbf{R}^d}\sup_{(y,s)\in\Gamma_x^a(\gamma)}\int_{\mathbf{R}^d} M_{s^2}(y,z)1_{N_\tau}(y,z)1_{\left[\frac{m(z)}{b},2\right]}(t) \\ &\quad\times\int_{\mathbf{R}^d}1_{N_1}(z,w)|A(w,t)|\,\mathrm{d}\gamma(w)\,\mathrm{d}z\,\mathrm{d}x\,\frac{\mathrm{d}t}{t} \\ &\overset{\text{(i)}}{\leqslant} \int_{\frac{\kappa}{b}m(c_B)}^{r_B}\frac{1}{m(c_B)^d}\int_{B(c_B,\lambda m(c_B))}\sup_{(y,s)\in\Gamma_x^a(\gamma)}\int_{\mathbf{R}^d} M_{s^2}(y,z)\,\mathrm{d}z \\ &\quad\times\int_{\mathbf{R}^d}|A(w,t)|\,\mathrm{d}\gamma(w)\,\mathrm{d}x\,\frac{\mathrm{d}t}{t} \end{aligned}$$

$$\leqslant \int_{\frac{\kappa}{b}m(c_B)}^{r_B} \frac{1}{m(c_B)^d} \int_{B(c_B,\lambda m(c_B))} \|w \mapsto A(w,t)\|_{L^1(\gamma)} \,\mathrm{d}x \,\frac{\mathrm{d}t}{t}$$

where we have used in (i) that there exist $\kappa, \lambda > 0$ such that $m(z) \geqslant \kappa m(c_B)$ and $|x - c_B| \leqslant \lambda m(c_B)$. Note that by Cauchy-Schwarz and the fact that A is supported in B we have

$$\|x \mapsto A(x,t)\|_{L^1(\gamma)} \leqslant \sqrt{\gamma(B)}\|x \mapsto A(x,t)\|_{L^2(\gamma)}.$$

Furthermore, we have

$$\int_{B(c_B,\lambda m(c_B))} \mathrm{d}x \simeq m(c_B)^d.$$

Hence,

$$I^{\mathrm{loc}} \lesssim \sqrt{\gamma(B)} \left(\int_{\frac{\kappa}{b}m(c_B)}^{m(c_B)} \frac{\mathrm{d}t}{t}\right)^{\frac{1}{2}} \left(\int_0^\infty \|x \mapsto A(x,t)\|^2_{L^2(\gamma)} \frac{\mathrm{d}t}{t}\right)^{\frac{1}{2}} \lesssim 1.$$

This completes the proof of the present proposition. ■

Next we will estimate the term (ii) on page 41.

4.4 Proposition. *Let* $a, a' > 0$*,* N *in* $\mathbf{Z}_+$*,* $j = 1, \ldots, d$ *and* $\alpha > 8e^{2a^2+1}$*. Let* $b \geqslant 2e$*. Then*

$$\left\| x \mapsto \int_0^{\frac{m(x)}{b}} t^{2N+1} L^N e^{\frac{t^2}{\alpha}L} \partial^*_{x_j} [1_{\complement D}(x,t) t \partial_{x_j} e^{\frac{a^2 t^2}{\alpha^2}L}] u(x) \,\frac{\mathrm{d}t}{t} \right\|_{h^1_{max,a'}} \lesssim \|u\|_{L^1(\gamma)}.$$

Proof. First note that $\|T^*_{a'}u\|_{L^1(\gamma)} \leqslant \|u\|_{L^\infty(\gamma)}$. To see this recall that the semigroup generated by L is positive. This means that $|e^{sL}u| \leqslant e^{sL}|u|$. Furthermore note that

$$\begin{aligned} e^{sL}|u(x)| &= \int_{\mathbf{R}^d} M_s(x,y)|u(y)| \,\mathrm{d}y \\ &\leqslant \|u\|_{L^\infty(\gamma)} \int_{\mathbf{R}^d} M_s(x,y) 1 \,\mathrm{d}y \\ &= \|u\|_{L^\infty(\gamma)} \end{aligned}$$

where we have used that $e^{sL}1 = 1$. So,

$$\begin{aligned} \|T^*_{a'}u\|_{L^1(\gamma)} &= \int_{\mathbf{R}^d} \sup_{(y,t)\in\Gamma^{a'}_x(\gamma)} |e^{t^2L}u(y)| \,\mathrm{d}\gamma(x) \\ &\leqslant \int_{\mathbf{R}^d} \sup_{(y,t)\in\Gamma^{a'}_x(\gamma)} e^{t^2L}|u(y)| \,\mathrm{d}\gamma(x) \\ &\leqslant \int_{\mathbf{R}^d} \|u\|_{L^\infty(\gamma)} \,\mathrm{d}\gamma(x) \\ &= \|u\|_{L^\infty(\gamma)}. \end{aligned}$$

Thus it is sufficient to show that

$$\left\| x \mapsto \int_0^{\frac{m(x)}{b}} t^{2N+1} L^N e^{\frac{t^2}{\alpha}L} \partial_{x_j}^* [1_{\complement D}(x,t) t \partial_{x_j} e^{\frac{a^2 t^2}{\alpha^2}L}] u(x) \frac{\mathrm{d}t}{t} \right\|_{L^\infty(\gamma)} \lesssim \|u\|_{L^1(\gamma)}.$$

To prove this claim, first fix x in $\mathbf{R}^d$ and consider $t \geqslant 0$ and y in $\mathbf{R}^d$ such that $m(y) < t \leqslant \frac{m(x)}{b}$. So, $m(y) < 1$ hence $|y| \geqslant 1$. We have $2e|x| \leqslant b|x|$. Furthermore we have $|y|^{-1} \leqslant (b|x|)^{-1}$ hence $2e|x| \leqslant b|x| \leqslant |y|$. So

$$\begin{aligned} |e^{-t^2} y - x| &\geqslant e^{-t^2}|y| - |x| \\ &\geqslant \frac{|y|}{e} - |x| \\ &= \frac{|y|}{2e} + \frac{|y|}{2e} - |x| \\ &\geqslant \frac{|y|}{2e}. \end{aligned}$$

This gives, using lemma 2.6(iii) and $t|y| \leqslant b^{-1}$.

$$\begin{aligned} t^{-1}|\tilde{K}_{t^2,N,\alpha,j}(x,y)| &\overset{\text{(i)}}{\lesssim} |y|^d \exp\left(-\frac{\alpha}{4e^{2a^2}} \frac{|e^{-t^2}y - x|^2}{1 - e^{-2t^2}}\right) M_{t^2}(x,y) \\ &\leqslant |y|^d \exp\left(-\frac{\alpha}{4e^{2a^2}} |e^{-t^2}y - x|^2\right) M_{t^2}(x,y) \\ &\leqslant |y|^d \exp\left(-\frac{\alpha}{16e^{2a^2+1}} |y|^2\right) M_{t^2}(x,y) \qquad [4.2] \\ &\overset{\text{(ii)}}{\leqslant} |y|^d \exp\left(-\frac{\alpha}{16e^{2a^2+1}} |y|^2\right) \frac{1}{m(x)^d} \\ &\overset{\text{(iii)}}{\leqslant} \exp\left(-\frac{\alpha}{16e^{2a^2+1}} |y|^2\right). \end{aligned}$$

Where (i) follows from $|y| \leqslant t^{-1}$ and $|y| \geqslant 1$. (ii) follows from the argument of [4.1] that shows that $M_{t^2}(x,y) \lesssim m(x)^{-d}$. (iii) follows from $|y| \leqslant \frac{m(x)}{b}$. Now,

$$\begin{aligned} &\left| \int_0^{\frac{m(x)}{b}} t^{2N+1} L^N e^{\frac{t^2}{\alpha}L} \partial_{x_j}^* \left[1_{\complement D}(x,t) t \partial_{x_j} e^{\frac{a^2 t^2}{\alpha}L}\right] u(x) \frac{\mathrm{d}t}{t} \right| \\ &\leqslant \left| \int_0^{\frac{m(x)}{b}} \int_{\mathbf{R}^d} \tilde{K}_{t^2,N,\alpha,j}(x,y) \left[1_{\complement D}(y,t) t \partial_{y_j} e^{\frac{a^2 t^2}{\alpha}L}\right] u(y) \,\mathrm{d}y \frac{\mathrm{d}t}{t} \right| \\ &\leqslant \left| \int_0^{\frac{m(x)}{b}} \int_{\mathbf{R}^d} \int_{\mathbf{R}^d} \tilde{K}_{t^2,N,\alpha,j}(x,y) 1_{\complement D}(y,t) \partial_{y_j} M_{\frac{a^2t^2}{\alpha}}(y,z) u(z) \,\mathrm{d}z \,\mathrm{d}y \,\mathrm{d}t \right|. \end{aligned}$$

To continue this estimate we need to estimate $|\partial_{y_j} M_{\frac{a^2t^2}{\alpha}}(y,z)|$.

$$\left|\partial_{y_j} M_{\frac{a^2t^2}{\alpha}}(y,z)\right| = \left| M_{\frac{a^2t^2}{\alpha}}(y,z) \frac{2e^{-\frac{a^2t^2}{\alpha}}(e^{-\frac{a^2t^2}{\alpha}} y_j - z_j)}{1 - e^{-2\frac{a^2t^2}{\alpha}}} \right|$$

$$\lesssim M_{\frac{a^2t^2}{\alpha}}(y,z)\frac{|e^{-\frac{a^2t^2}{\alpha}}y-z|}{1-e^{-2\frac{a^2t^2}{\alpha}}}.$$

So,

$$\left|\int_0^{\frac{m(x)}{b}} t^{2N+1}L^N e^{\frac{t^2}{\alpha}L}\partial_{x_j}^*\left[1_{\complement D}(y,t)t\partial_{x_j}e^{\frac{a^2t^2}{\alpha}L}\right]u(x)\,\frac{\mathrm{d}t}{t}\right|$$
$$\leqslant \int_0^{\frac{m(x)}{b}}\int_{\mathbf{R}^d}\int_{\mathbf{R}^d}\tilde{K}_{t^2,N,\alpha,j}(x,y)1_{\complement D}(y,t)t\left|\partial_{y_j}M_{\frac{a^2t^2}{\alpha}}(y,z)u(z)\right|\,\mathrm{d}z\,\mathrm{d}y\,\frac{\mathrm{d}t}{t}$$
$$\lesssim \int_0^{\frac{m(x)}{b}}\int_{\mathbf{R}^d}\int_{\mathbf{R}^d}\tilde{K}_{t^2,N,\alpha,j}(x,y)1_{\complement D}(y,t)tM_{\frac{a^2t^2}{\alpha}}(y,z)\frac{|e^{-\frac{a^2t^2}{\alpha}}y-z|}{1-e^{-2\frac{a^2t^2}{\alpha}}}|u(z)|\,\mathrm{d}z\,\mathrm{d}y\,\frac{\mathrm{d}t}{t}$$

[*]

$$\overset{\text{(i)}}{\lesssim}\int_0^{\frac{m(x)}{b}}\int_{\mathbf{R}^d}\int_{\mathbf{R}^d}\tilde{K}_{t^2,N,\alpha,j}(x,y)1_{\complement D}(y,t)tM_{t^2}(y,z)\exp\left(-\frac{\alpha}{4e^{2a^2}}\frac{\left|e^{-\frac{a^2t^2}{\alpha}}y-z\right|^2}{1-e^{-2\frac{a^2t^2}{\alpha}}}\right)$$
$$\times\frac{|e^{-\frac{a^2t^2}{\alpha}}y-z|}{1-e^{-2\frac{a^2t^2}{\alpha}}}|u(z)|\,\mathrm{d}z\,\mathrm{d}y\,\frac{\mathrm{d}t}{t}.$$

Where (i) holds because $t|y|\leqslant b^{-1}$ so we can use lemma 2.6 with t replaced by at. Furthermore we have used that $m(y)\leqslant t$ which follows from (y,t) in $\complement D$. Note that

$$\exp\left(-c_1\frac{\xi^2}{d}\right)\frac{c_2\xi}{d}\lesssim 1$$

for all constants $c_1,c_2>0$ and for $\xi\geqslant 0$. The implied constant is independent on ξ and d.

In combination with [4.2] we now conclude that the RHS in the above [*] is smaller than

$$\int_0^1\int_{\mathbf{R}^d}\exp\left(-\frac{\alpha}{16e^{2a^2+1}}|y|^2\right)e^{t^2L}|u(y)|\,\mathrm{d}y\,\mathrm{d}t$$
$$\lesssim\int_0^1\int_{\mathbf{R}^d}\exp\left(\left[1-\frac{\alpha}{16e^{2a^2+1}}\right]|y|^2\right)e^{t^2L}|u(y)|\,\mathrm{d}\gamma(y)\,\mathrm{d}t.$$

So, for α large enough we have

$$\int_0^1\int_{\mathbf{R}^d}\exp\left(\left[1-\frac{\alpha}{8e^{2a^2+1}}\right]|y|^2\right)e^{t^2L}|u(y)|\,\mathrm{d}\gamma(y)\,\mathrm{d}t$$
$$\leqslant\|e^{t^2L}u\|_{L^1(\gamma)}$$
$$\lesssim\|u\|_{L^1(\gamma)}.$$

This estimate holds for all x in $\mathbf{R}^d$ so we get

$$\left\| x \mapsto \int_0^{\frac{m(x)}{b}} t^{2N+1} L^N e^{\frac{t^2}{\alpha}L} \partial_{x_j}^* [1_{\complement D}(x,t) t \partial_{x_j} e^{\frac{a^2 t^2}{\alpha^2}L}] u(x) \,\frac{\mathrm{d}t}{t} \right\|_{L^\infty(\gamma)} \lesssim \|u\|_{L^1(\gamma)}.$$

This concludes the proof. ■

4.5 Proposition. *Let N in $\mathbf{Z}_+$, $\alpha > 0$ and $a, a', b > 0$. For all u in $C_c^\infty(\mathbf{R}^d)$ we have that*

$$\left\| x \mapsto \int_{\frac{m(x)}{b}}^\infty (t^2 L)^{N+1} e^{\frac{(1+a^2)t^2}{\alpha}L} u(x) \,\frac{\mathrm{d}t}{t} \right\|_{h^1_{max,a'}} \lesssim \|u\|_{L^1(\gamma)}.$$

Proof. Let $M > 1$ and x in $\mathbf{R}^d$. Using the substitution $\frac{1+a^2}{\alpha} t^2 = s$ and [1.3] we get

$$\left| \int_{\frac{m(x)}{b}}^M (t^2 L)^{N+1} e^{\frac{(1+a^2)t^2}{\alpha}L} u(x) \,\frac{\mathrm{d}t}{t} \right| \simeq \left| \int_{\frac{(1+a^2)m(x)^2}{b^2\alpha}}^{\frac{(1+a^2)M^2}{\alpha}} s^{N+1} \partial_s^{N+1} e^{sL} u(x) \,\frac{\mathrm{d}s}{s} \right|.$$

We can integrate the last integral N times by parts and use the triangle inequality to obtain

$$\begin{aligned} \left| \int_{\frac{m(x)}{b}}^M (t^2 L)^{N+1} e^{\frac{(1+a^2)t^2}{\alpha}L} u(x) \,\frac{\mathrm{d}t}{t} \right| &\lesssim \sum_{k=0}^N \left| \left(\frac{(1+a^2)M^2}{\alpha} \right)^k L^k e^{\frac{(1+a^2)M^2}{\alpha}L} u(x) \right| \\ &\quad + \sum_{k=0}^N \left| \left(\frac{(1+a^2)m(x)^2}{b^2\alpha} \right)^k L^k e^{\frac{(1+a^2)m(x)^2}{b^2\alpha}L} u(x) \right| \\ &=: [A] + [B]. \end{aligned}$$

We estimate both sums separately. We begin with [A]. Note that using the chaos decomposition [1.2] and k in $\mathbf{N}$ we have

$$\begin{aligned} \|L^k e^{tL} u\|_{L^2(\gamma)}^2 = \|e^{tL} L^k u\|_{L^2(\gamma)}^2 &= \left\| \sum_{\beta \in \mathbf{Z}_+^n} e^{-t|\beta|} c_\beta |\beta|^k H_\beta \right\|_{L^2(\gamma)}^2 \\ [*] \qquad &= \sum_{\beta \in \mathbf{Z}_+^n} e^{-2t|\beta|} |\beta|^{2k} |c_\beta|^2 \leqslant e^{-2k} \left(\frac{k}{t} \right)^{2k} \|u\|_{L^2(\gamma)}^2 \end{aligned}$$

where we have used $e^{-2t|\beta|}|\beta|^{2k} \leqslant e^{-2k} \left(\frac{k}{t}\right)^{2k}$.

Hence, given $k = 1, \ldots, N$ we have

$$\left\| (M^2 L)^k e^{\frac{(1+a^2)M^2}{\alpha}L} u \right\|_{h^1_{\max,a'}} = \left\| T_{a'}^* (M^2 L)^k e^{\frac{(1+a^2)M^2}{\alpha}L} u \right\|_{L^1(\gamma)}$$

$$\begin{aligned}
&\leqslant \left\| T^*_{a'} (M^2 L)^k e^{\frac{(1+a^2)M^2}{\alpha} L} u \right\|_{L^2(\gamma)} \\
&\overset{\text{(i)}}{\leqslant} \left\| (M^2 L)^k e^{\frac{(1+a^2)M^2}{\alpha} L} u \right\|_{L^2(\gamma)} \\
&\leqslant M^{2k} \left\| L^k e^{\frac{(1+a^2)M^2}{\alpha} L} u \right\|_{L^2(\gamma)} \\
&\overset{\text{(ii)}}{\leqslant} M^{2k} e^{-2k} \left(\frac{k\alpha}{(1+a^2)M^2} \right)^{2k} \|u\|_{L^2(\gamma)} \\
&\to 0 \text{ as } M \to \infty
\end{aligned}$$

where (i) follows from lemma 1.14 and (ii) follows from [*].

Next we estimate the sum [B]. Using [2.1] where we set $t^2 = (1+a^2)m(x)^2 b^{-2}$ we get

$$\begin{aligned}
&\sum_{k=0}^{N} \left| \left(\frac{(1+a^2)m(x)^2}{b^2\alpha} \right)^k L^k e^{\frac{(1+a^2)m(x)^2}{b^2\alpha} L} u(x) \right| \\
&\qquad \lesssim \sum_{k=0}^{N} \left| \left(\frac{(1+a^2)m(x)^2}{b^2} \right)^k L^k e^{\frac{(1+a^2)m(x)^2}{b^2\alpha} L} u(x) \right| \\
&\qquad \leqslant \sum_{k=0}^{N} \int_{\mathbf{R}^d} |K_{(1+a^2)m(x)^2 b^{-2},k,\alpha)}(x,y)||u(y)| \,\mathrm{d}y.
\end{aligned}$$

It remains to prove that

$$\left\| T^*_{a'} \left(x \mapsto \int_{\mathbf{R}^d} |K_{(1+a^2)b^{-2}m(x)^2,k,\alpha}(x,y)||u(y)| \,\mathrm{d}y \right) \right\|_{L^1(\gamma)} \lesssim \|u\|_{L^1(\gamma)}.$$

By lemma 2.6(ii) it is sufficient to prove that

$$\text{[4.3]} \qquad \left\| T^*_{a'} \left(x \mapsto \int_{\mathbf{R}^d} M_{(1+a^2)b^{-2}m(x)^2}(x,y)|u(y)| \,\mathrm{d}y \right) \right\|_{L^1(\gamma)} \lesssim \|u\|_{L^1(\gamma)}.$$

We split the equation in [4.3] in a global and a local part. Using proposition 1.10(i) we estimate the global part as follows

$$\begin{aligned}
&\left\| T^{*a'}_{\text{glob}} \left(x \mapsto \int_{\mathbf{R}^d} M_{(1+a^2)b^{-2}m(x)^2}(x,y)|u(y)| \,\mathrm{d}y \right) \right\|_{L^1(\gamma)} \\
&\quad = \int_{\mathbf{R}^d} \sup_{(y,t)\in\Gamma^a_x(\gamma)} \Bigg| \int_{\mathbf{R}^d} 1_{\complement N_\tau}(z,w) M_{t^2}(z,w) \\
&\qquad \times \int_{\mathbf{R}^d} M_{(1+a^2)b^{-2}m(x)^2}(w,v)|u(v)| \,\mathrm{d}v \Bigg|^2 \mathrm{d}w \,\mathrm{d}\gamma(x)
\end{aligned}$$

$$\leqslant \int_{\mathbf{R}^d} \sup_{(z,t)\in\Gamma_x^a(\gamma)} \bigg| \int_{\mathbf{R}^d} 1_{\complement N_\tau}(z,w) M_{t^2}(z,w) \times \int_{\mathbf{R}^d} M_{(1+a^2)b^{-2}m(x)^2}(w,v)|u(v)|\,\mathrm{d}v \bigg| \,\mathrm{d}w\,\mathrm{d}\gamma(x)$$

$$\lesssim \int_{\mathbf{R}^d}\int_{\mathbf{R}^d} M_{(1+a^2)b^{-2}m(x)^2}(x,y)|u(y)|\,\mathrm{d}y\,\mathrm{d}\gamma(x).$$

We decompose the right hand side in a local and a global part. Let $\tau := \frac{1}{2}(1 + b^{-1}\sqrt{1+a^2})(1+2b^{-1}\sqrt{1+a^2})$. By proposition 1.10(i), for the global part we have

$$\int_{\mathbf{R}^d}\int_{\mathbf{R}^d} M_{(1+a^2)b^{-2}m(x)^2}(x,y) 1_{\complement N_\tau}(x,y)|u(y)|\,\mathrm{d}y\,\mathrm{d}\gamma(x) \lesssim \|u\|_{L^1(\gamma)}.$$

For (x,y) in N_τ we have $m(x) \simeq m(y)$ by lemma 1.9, hence

$$\int_{\mathbf{R}^d}\int_{\mathbf{R}^d} M_{(1+a^2)b^{-2}m(x)^2}(x,y) 1_{N_\tau}(x,y)|u(y)|\,\mathrm{d}y\,\mathrm{d}\gamma(x) \leqslant \int_{\mathbf{R}^d}\int_{B(x,\tau m(x))} \frac{1}{m(x)^d}|u(y)|\,\mathrm{d}y\,\mathrm{d}\gamma(x).$$

Where the RHS of the previous inequality follows from the argument of [4.1] that shows that $M_{t^2}(x,y) \lesssim m(x)^{-d}$. For (x,y) in N_τ we also have $e^{-|x|^2} \simeq e^{-|y|^2}$ by lemma 2.8, therefore, using lemma 1.9

$$\begin{aligned}
\int_{\mathbf{R}^d}\int_{B(x,\tau m(x))} \frac{1}{m(x)^d}|u(y)|\,\mathrm{d}y\,\mathrm{d}\gamma(x) &\lesssim \int_{\mathbf{R}^d}\int_{B(x,\tau m(x))} \frac{1}{m(y)^d}|u(y)|\,\mathrm{d}y\,\mathrm{d}\gamma(x) \\
&= \int_{\mathbf{R}^d}\int_{\mathbf{R}^d} 1_{B(x,\tau m(x))}(y)\frac{1}{m(y)^d}|u(y)|\,\mathrm{d}y\,\mathrm{d}\gamma(x) \\
&\lesssim \int_{\mathbf{R}^d}\int_{\mathbf{R}^d} 1_{B(y,\tau(1+\tau)m(y))}(x)\frac{1}{m(y)^d}|u(y)|\,\mathrm{d}y\,\mathrm{d}\gamma(x) \\
&\lesssim \int_{\mathbf{R}^d}\int_{B(y,(1+\tau)m(y))} \frac{1}{m(y)^d}|u(y)|\,dx\,\mathrm{d}\gamma(y) \\
&\lesssim \int_{\mathbf{R}^d} |u(y)|\,\mathrm{d}\gamma(y) \\
&= \|u\|_{L^1(\gamma)}.
\end{aligned}$$

Next we estimate the local part [4.3], that is we need an estimate for

$$\int_{\mathbf{R}^d} \sup_{(y,t)\in\Gamma_x^a(\gamma)} \bigg| \int_{\mathbf{R}^d} 1_{N_{\tau'}}(z,w) M_{t^2}(z,w) \times \int_{\mathbf{R}^d} |M_{(1+a^2)b^{-2}m(x)^2}(w,v)||u(v)|\,\mathrm{d}v \bigg| \,\mathrm{d}w\,\mathrm{d}\gamma(x)$$

Clearly the RHS can be estimated by

$$\int_{\mathbf{R}^d} \sup_{(y,t)\in\Gamma_x^a(\gamma)} \int_{\mathbf{R}^d} 1_{N_{\tau'}}(y,z) M_{t^2}(y,z) \\ \times \int_{\mathbf{R}^d} M_{(1+a^2)b^{-2}m(x)^2}(z,w)|u(w)| \,\mathrm{d}w \,\mathrm{d}z \,\mathrm{d}\gamma(x).$$

Once we have estimated the following terms, the proof will be complete.

$$J_{\text{glob}} := \int_{\mathbf{R}^d} \sup_{(y,t)\in\Gamma_x^a(\gamma)} \int_{\mathbf{R}^d} M_{t^2}(y,z) 1_{N_{\tau'}}(y,z) \\ \times \int_{\mathbf{R}^d} M_{(1+a^2)b^{-2}m(z)^2}(z,w) 1_{\complement N_{\tau''}}(z,w)|u(w)| \,\mathrm{d}w \,\mathrm{d}z \,\mathrm{d}\gamma(x),$$

$$J_{\text{loc}} := \int_{\mathbf{R}^d} \sup_{(y,t)\in\Gamma_x^a(\gamma)} \int_{\mathbf{R}^d} M_{t^2}(y,z) 1_{N_{\tau'}}(y,z) \\ \times \int_{\mathbf{R}^d} M_{(1+a^2)b^{-2}m(z)^2}(z,w) 1_{N_{\tau''}}(z,w)|u(w)| \,\mathrm{d}w \,\mathrm{d}z \,\mathrm{d}\gamma(x),$$

where τ' is defined for the parameters $(A,a) = (1,a')$ as in proposition 1.10 and τ'' is defined as follows. For (x,y) in N_τ and (y,z) in $N_{\tau'}$ we have by lemma 1.9 that $m(x) \simeq m(y) \simeq m(z)$. Fix τ'' as in proposition 1.10 for the parameters $(1,\tilde{a}) = (1, \sqrt{1+a^2}b^{-1})$ Using proposition 1.10(i) we have

$$\begin{aligned} J_{\text{glob}} &\lesssim \int_{\mathbf{R}^d} \sup_{(y,t)\in\Gamma_x^a(\gamma)} \int_{\mathbf{R}^d} M_{t^2}(y,z) 1_{N_{\tau'}}(y,z) \\ &\quad \times \sup_{(\eta,s)\in\Gamma_z^{\tilde{a}}} \int_{\mathbf{R}^d} M_{s^2}(\eta,w) 1_{\complement N_{\tau''}}(\eta,w)|u(w)| \,\mathrm{d}w \,\mathrm{d}z \,\mathrm{d}\gamma(x) \\ &\lesssim \int_{\mathbf{R}^d} \sup_{(y,t)\in\Gamma_x^a(\gamma)} \int_{\mathbf{R}^d} M_{t^2}(y,z) 1_{N_{\tau'}}(y,z) \|u\|_{L^1(\gamma)} \,\mathrm{d}z \,\mathrm{d}\gamma(x) \\ &\lesssim \|u\|_{L^1(\gamma)}. \end{aligned}$$

For (x,y) in N_a, (y,z) in $N_{\tau'}$ and (z,w) in $N_{\tau''}$ we have $m(x) \simeq m(y) \simeq m(z) \simeq m(w)$ by lemma 1.9. We also have

$$\begin{aligned} |x-w| &\leqslant |z-w| + |x-y| + |y-z| \\ &\leqslant \tau'' m(z) + 2am(x) + \tau' m(y) \\ &\leqslant \lambda m(x). \end{aligned}$$

for some $\lambda > 0$. Let $\kappa > 0$ be such that $m(x) \leqslant \kappa m(w)$. We also the argument of [4.1] that shows that $M_{t^2}(z,w) \lesssim m(z)^{-d} \lesssim m(x)^{-d}$.

$$J_{\text{loc}} \lesssim \int_{\mathbf{R}^d} \sup_{(y,t)\in\Gamma_x^a(\gamma)} \int_{\mathbf{R}^d} M_{t^2}(y,z) 1_{N_{\tau'}}(y,z) \frac{1}{m(x)^d} \int_{B(z,\tau'' m(z))} |u(w)| \,\mathrm{d}w \,\mathrm{d}z \,\mathrm{d}\gamma(x)$$

$$
\begin{aligned}
&\leqslant \int_{\mathbf{R}^d} \sup_{(y,t)\in\Gamma_x^a(\gamma)} \int_{\mathbf{R}^d} M_{t^2}(y,z) 1_{N_{\tau'}}(y,z) \frac{1}{m(x)^d} \int_{B(x,\lambda m(x))} |u(w)| \,\mathrm{d}w \,\mathrm{d}z \,\mathrm{d}\gamma(x) \\
&\leqslant \int_{\mathbf{R}^d} \frac{1}{m(x)^d} \int_{B(x,\lambda m(x))} |u(w)| \,\mathrm{d}w \,\mathrm{d}\gamma(x) \\
&\overset{\text{(i)}}{\lesssim} \int_{\mathbf{R}^d} \frac{1}{m(w)^d} |u(w)| \int_{B(x,\lambda\kappa m(w))} \mathrm{d}x \,\mathrm{d}\gamma(w) \\
&\lesssim \|u\|_{L^1(\gamma)}.
\end{aligned}
$$

Where we have used lemma 2.9 in (i). This completes the proof of the present proposition. ■

5 The equivalence

In this final chapter we will prove that the Hardy spaces as defined in the introduction are actually the same. To do this all the results from the previous chapters together with the Calderón reproducing formula are put together.

We first define another Hardy space using a non-tangential maximal function $T_a^{\text{avg}*}$

$$T_a^{\text{avg}*} := \sup_{(y,t)\in\Gamma_x^{(\frac{1}{2},a)}(\gamma)} \left(\frac{1}{\gamma(B(y,t))} \int_{B(y,t)} |e^{t^2 L} u(z)|^2 \, \mathrm{d}\gamma(z) \right)^{\frac{1}{2}}.$$

We can quickly see that

$$\|T_a^{\text{avg}*}\| \leqslant \|T_a^*\|.$$

5.1 Theorem. *Given $a > 0$, there exists $a' > 0$ such that $h^1_{quad,a}(\gamma) = h^1_{max,a'}(\gamma)$ with equivalent norms.*

Proof. For $a > 0$ we have by [MvNP10a, theorem 1.1] that there exists $a' > 0$ such that $\|S_a^* u\|_{L^1(\gamma)} \lesssim \|T_a^* u\|_{L^1(\gamma)}$ for all $u \in C_c^\infty(\mathbf{R}^d)$. Fix the a' from $h^1_{\text{max},a'} \subset h^1_{\text{quad},a}$ and choose α and b big enough. Let $u \in C_c^\infty(\mathbf{R}^d)$ and apply corollary 1.19. We have

$$\begin{aligned}
\|u\|_{h^1_{\text{max},a'}} &= \left\| T_{a'}^* \int_{\mathbf{R}^d} u \, \mathrm{d}\gamma \right\|_{L^1(\gamma)} \\
&+ C \sum_{j=1}^d \sum_{n=1}^\infty |\lambda_{n,j}| \left\| x \mapsto \int_0^2 (t^2 L)^N e^{\frac{t^2}{\alpha} L} t \partial_{x_j}^* A_{n,j}(x,t) \frac{\mathrm{d}t}{t} \right\|_{h^1_{\text{max},a'}} \\
&+ C \sum_{j=1}^d \sum_{n=1}^\infty |\lambda_{n,j}| \left\| x \mapsto \int_0^2 1_{[\frac{m(x)}{b},2]}(t) (t^2 L)^N e^{\frac{t^2}{\alpha} L} t \partial_{x_j}^* A_{n,j}(x,t) \frac{\mathrm{d}t}{t} \right\|_{h^1_{\text{max},a'}} \\
&+ C \sum_{j=1}^d \left\| x \mapsto \int_0^{\frac{m(x)}{b}} (t^2 L)^N e^{\frac{t^2}{\alpha} L} t \partial_{x_j}^* (1_{\complement D}(t,x) t \partial_{x_j} e^{\frac{a^2 t^2}{\alpha} L}) u(x) \frac{\mathrm{d}t}{t} \right\|_{h^1_{\text{max},a'}} \\
&+ C \left\| x \mapsto \int_{\frac{m(x)}{b}}^\infty (t^2 L)^{N+1} e^{\frac{(1+a)^2 t^2}{\alpha} L} u(x) \frac{\mathrm{d}t}{t} \right\|_{h^1_{\text{max},a'}}.
\end{aligned}$$

We have

$$\left\| T_{a'}^* \int_{\mathbf{R}^d} u \, \mathrm{d}\gamma \right\|_{L^1(\gamma)}$$

$$\leqslant \|u\|_{L^1(\gamma)} \int_{\mathbf{R}^d} \sup_{(y,t)\Gamma_x^{a'}(\gamma)} |e^{t^2 L} 1| \, \mathrm{d}\gamma(x)$$
$$= \|u\|_{L^1(\gamma)}.$$

Moreover, proposition 3.2 together with theorem 3.3 gives us that for any 2-atom A,

$$\left\| x \mapsto \int_0^2 (t^2 L)^N e^{\frac{t^2}{\alpha} L} t \partial_{x_j}^* A_{n,j}(x,t) \frac{\mathrm{d}t}{t} \right\|_{h^1_{\max,a'}} \lesssim 1.$$

By proposition 4.3 we get

$$\left\| x \mapsto \int_0^2 1_{\left[\frac{m(x)}{b},2\right]}(t) (t^2 L)^N e^{\frac{t^2}{\alpha} L} t \partial_{x_j}^* A_{n,j}(x,t) \frac{\mathrm{d}t}{t} \right\|_{h^1_{\max,a'}} \lesssim 1.$$

Hence by corollary 1.19 (final estimate),

$$\sum_{j=0}^{\infty} \sum_{n=0}^{\infty} |\lambda_{n,j}| \lesssim \|u\|_{h^1_{\mathrm{quad},a}}.$$

For $j = 1, \ldots, d$ we have by, proposition 4.4,

$$\left\| x \mapsto \int_0^{\frac{m(x)}{b}} (t^2 L)^N e^{\frac{t^2}{\alpha} L} t \partial_{x_j}^* (1_{\complement D}(t,x) t \partial_{x_j} e^{\frac{a^2 t^2}{\alpha} L}) u(x) \, \frac{\mathrm{d}t}{t} \right\|_{h^1_{\max,a'}} \lesssim \|u\|_{L^1(\gamma)}.$$

Finally, by proposition 4.5 we have

$$\left\| x \mapsto \int_{\frac{m(x)}{b}}^{\infty} (t^2 L)^{N+1} e^{\frac{(1+a)^2 t^2}{\alpha} L} u(x) \, \frac{\mathrm{d}t}{t} \right\|_{h^1_{\max,a'}} \lesssim \|u\|_{L^1(\gamma)}.$$

So

$$\|u\|_{h^1_{\max,a'}} \lesssim \|u\|_{h^1_{\mathrm{quad},a}}.$$

This completes the proof of the equivalence. ■

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