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AAK MODEL REDUCTION FOR TIME-VARYING SYSTEMS

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The paper describes a generalization of the Hankel norm model reduction theory to time-varying systems. For time-invariant systems, this problem has a solution which goes back to the work of Adamyan, Arov and Krein on the Schur-Takagi problem. Our approach extends that theory to cover general, not only Toeplitz, upper operators as well. We derive the usual parametrization of all possible Hankel norm approximants of a given upper operator with respect to a given approximation tolerance.

1. INTRODUCTION

In classical model reduction theory, one is given a transfer function $T(z)$ belonging to H_∞ of the unit circle, $T(z) = t_0 + t_1 z + t_2 z^2 + \cdots$. Associated to $T(z)$ are its transfer operator T and Hankel operator H_T:

$$T = \begin{bmatrix} \ddots & \vdots & & \vdots & \\ & t_0 & t_1 & t_2 & \cdots \\ & & t_0 & t_1 & \\ & & & t_0 & \cdots \\ & & & & \ddots \end{bmatrix}, \qquad H_T = \begin{bmatrix} t_1 & t_2 & t_3 & \cdots \\ t_2 & t_3 & & \\ t_3 & & \ddots & \\ \vdots & & & \end{bmatrix}.$$

The model order of $T(z)$ is equal to the rank of H_T and finite if and only if the system has a rational transfer function. Adamyan, Arov and Krein [1] showed that H_T can be approximated by a Hankel matrix of low rank, n say, such that the Euclidian norm difference between the original Hankel operator and the approximant is equal to the $(n+1)$-st singular value of H_T. This approximation leads to the *optimal reduced system* in Hankel norm. It was soon recognized that this can be utilized to solve the problem of optimal model-order reduction of a dynamical system [2, 3, 4, 5]. Calculations can be done in a recursive fashion [6] based on interpolation theory of Schur-Takagi type. The state space theory for this interpolation problem was extensively studied in the book [7].

In the present paper, the aim is to extend the model reduction theory to the time-varying context, by considering bounded upper ℓ_2-operators with matrix representation

$$T = \begin{bmatrix} \ddots & & & \vdots & \\ & \boxed{T_{00}} & T_{01} & T_{02} & \cdots \\ & & T_{11} & T_{12} & \\ & \mathbf{0} & & T_{22} & \cdots \\ & & & & \ddots \end{bmatrix} \tag{1.1}$$

which are now no longer taken to be Toeplitz. The 00-entry in the matrix representation is distinguished by a surrounding square. T maps ℓ_2-sequences $u = [\cdots\ \boxed{u_0}\ u_1\ u_2\ \cdots]$ into ℓ_2-sequences y via $y = uT$, and is thus seen to be a *causal* operator: an entry y_i only depends on entries u_k for $k \le i$. The rows of T can be viewed as the impulse responses of the system. We will be interested in systems T that admit a realization in the form of the recursion

$$\begin{aligned} x_{k+1} &= x_k A_k + u_k B_k \\ y_k &= x_k C_k + u_k D_k \end{aligned} \qquad \mathbf{T}_k = \begin{bmatrix} A_k & C_k \\ B_k & D_k \end{bmatrix} \tag{1.2}$$

in which the matrices $\{A_k, B_k, C_k, D_k\}$ all have finite (but possibly time-varying) dimensions. We call such systems *locally finite*. Let A_k be of size $d_k \times d_{k+1}$, then the size of x_k, *i.e.*, the *system order* at point k, is equal to d_k. Define a sequence of operators $\{H_k\}_{-\infty}^{\infty}$ with matrix representations

$$H_k = \begin{bmatrix} T_{k-1,k} & T_{k-1,k+1} & T_{k-1,k+2} & \cdots \\ T_{k-2,k} & T_{k-2,k+1} & & \\ T_{k-3,k} & & \ddots & \\ \vdots & & & \end{bmatrix}. \tag{1.3}$$

We will call the H_k time-varying Hankel matrices of T, although they have no Hankel structure unless T is a Toeplitz operator. Their matrix representations are mirrored submatrices of T. Although we have lost the traditional anti-diagonal Hankel structure, a number of important properties are retained, for example, if $\{A_k, B_k, C_k, D_k\}$ is a realization of T, then H_k has a factorization into

$$H_k = \begin{bmatrix} B_{k-1} \\ B_{k-2}A_{k-1} \\ B_{k-3}A_{k-2}A_{k-1} \\ \vdots \end{bmatrix} [C_k \quad A_k C_{k+1} \quad A_k A_{k+1} C_{k+2} \quad \cdots] =: \mathcal{C}_k \mathcal{O}_k. \tag{1.4}$$

$\mathcal{C}_k$ and $\mathcal{O}_k$ can be regarded as time-varying controllability and observability operators. If the realization is minimal, then one can show that *the rank of H_k is equal to the system order of any minimal realization of T at point k.*

The Hankel norm of an operator T can be defined at present as

$$\| T \|_H = \sup_k \| H_k \|. \tag{1.5}$$

This definition is a generalization of the time-invariant Hankel norm and reduces to it if all H_k are the same. We will prove the following theorem:

Theorem 1.1. *Let T be a bounded operator which is strictly upper, strictly stable and locally finite, and let Γ be an invertible Hermitian diagonal operator. Let H_k be the Hankel matrix of $\Gamma^{-1}T$ at time instant k. Suppose that the singular values of each H_k decompose into two sets $\sigma_{-,k}$ and $\sigma_{+,k}$, with all $\sigma_{-,k}$ larger than 1, uniformly over k, and all $\sigma_{+,k}$ uniformly smaller than 1. Let N_k be equal to the number of singular values of H_k which are larger than 1.*

Then there exists a strictly upper locally finite operator T_a of system order at most N_k at point k, such that

$$\| \Gamma^{-1}(T - T_a) \|_H \le 1. \tag{1.6}$$

Operators T_a satisfying (1.6) are called *Hankel norm approximants* of T, parameterized by the error tolerance Γ. We are interested in Hankel norm approximants of minimal system order. There is a collection of such T_a. Theorem 4.3 gives a parametrization of all solutions. A consequence is that no Hankel norm approximants of order lower than N_k exist. This paper is a summary of [8] in which full proofs appear.

2. PRELIMINARIES

Spaces

Starting with a realization (1.2), we can assemble the matrices $\{A_k\}$, $\{B_k\}$ etc. as operators on spaces of sequences of appropriate dimensions, by

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defining $A = \mathrm{diag}(A_k)$, $B = \mathrm{diag}(B_k)$, $C = \mathrm{diag}(C_k)$ and $D = \mathrm{diag}(D_k)$. Together these operators define a realization $\mathbf{T}$ of T:

$$\begin{aligned} xZ^{-1} &= xA + uB \\ y &= xC + uD \end{aligned} \qquad \mathbf{T} = \begin{bmatrix} A & C \\ B & D \end{bmatrix}. \tag{2.1}$$

The diagonal operators act on sequences $u = [\cdots, \boxed{u_0}\ u_1\ u_2\ \cdots]$, $x = [\cdots\ \boxed{x_0}\ x_1\ x_2\ \cdots]$, and the causal shift operator Z on these sequences is defined by $xZ^{-1} = [\cdots\ \boxed{x_1}\ x_2\ x_3\ \cdots]$. The realization in (2.1) is equivalent to (1.2), but more convenient to handle in equations because the time-index has been suppressed. Shifted diagonal operators are $A^{(1)} = Z^{-1}AZ = \mathrm{diag}(A_{k-1})$ and $A^{(-1)} = ZAZ^{-1} = \mathrm{diag}(A_{k+1})$. An important aspect of these sequences is that the dimensions of their components can vary in time. Suppose that $x_k \in \mathcal{B}_k$, with $\mathcal{B}_k = \mathbf{C}^{N_k}$ an Euclidean space of dimension N_k, then we define $\mathcal{B} = \cdots \times \mathcal{B}_0 \times \mathcal{B}_1 \times \cdots$ to be the space of sequences x with entries in $\mathcal{B}_k$, and hence $x \in \mathcal{B}$ and $A : \mathcal{B} \to \mathcal{B}^{(-1)}$, where $\mathcal{B}^{(-1)}$ is a shifted space sequence corresponding to xZ^{-1}. We write $N = \#\mathcal{B}$ for the dimension sequence of $\mathcal{B}$. Even input- and output sequences can have varying dimensions. We will typically use $\mathcal{M}$ for input sequences and $\mathcal{N}$ for output sequences, and hence $T : \mathcal{M} \to \mathcal{N}$. Let $\ell_A = \lim \| [AZ]^n \|^{1/n}$ be the spectral radius of (AZ). If $\ell_A < 1$, then $(I - AZ)$ has a bounded inverse that is again upper, and the realization $\mathbf{T}$ is such that $T = D + BZ(I - AZ)^{-1}C$. We call such realizations *strictly stable.*

The space of non-uniform sequences $\mathcal{N}$ with index sequence N and with finite 2-norm is denoted by $\ell_2^{\mathcal{N}}$. It is a Hilbert space. Let $\mathcal{M}$ and $\mathcal{N}$ be space sequences corresponding to index sequences M, N. We denote by $\mathcal{X}(\mathcal{M},\mathcal{N})$ the space of *bounded* linear operators $\ell_2^{\mathcal{M}} \to \ell_2^{\mathcal{N}}$: an operator T is in $\mathcal{X}(\mathcal{M},\mathcal{N})$ if and only if for each $u \in \ell_2^{\mathcal{M}}$, the result $y = uT$ is in $\ell_2^{\mathcal{N}}$, in which case the induced operator norm of T is bounded. Such operators have a block matrix representation $[T_{ij}]$, much as in (1.1). They have an upper part and a lower part (which taken on themselves are not necessarily bounded): all entries T_{ij} above the main (0-th) diagonal and including this diagonal form the upper part, while all entries below the diagonal, including the diagonal, form the lower part. We define $\mathcal{U}(\mathcal{M},\mathcal{N})$, $\mathcal{L}(\mathcal{M},\mathcal{N})$ and $\mathcal{D}(\mathcal{M},\mathcal{N})$ to be, respectively, the space of bounded upper, lower and diagonal operators $\ell_2^{\mathcal{M}} \to \ell_2^{\mathcal{N}}$.

Besides the spaces $\mathcal{X}, \mathcal{U}, \mathcal{L}, \mathcal{D}$ in which the operator norm reigns, we shall need Hilbert-Schmidt spaces $\mathcal{X}_2, \mathcal{U}_2, \mathcal{L}_2, \mathcal{D}_2$ which consist of elements of $\mathcal{X}, \mathcal{U}, \mathcal{L}, \mathcal{D}$ respectively, and for whom the norms of the entries are square summable. These spaces are Hilbert spaces for the usual Hilbert-Schmidt inner product. They will often be considered to be input or output spaces for our system operators. Indeed, if T is a bounded operator $\ell_2^{\mathcal{M}} \to \ell_2^{\mathcal{N}}$, then it may be extended as a bounded operator $\mathcal{X}_2^{\mathcal{M}} \to \mathcal{X}_2^{\mathcal{N}}$ by stacking an infinite collection of sequences in ℓ_2 to form elements of $\mathcal{X}_2$. This leads for example to the expression $y = uT$, where $u \in \mathcal{X}_2^{\mathcal{M}} = \mathcal{X}_2(\mathbf{C}^{\mathbf{Z}}, \mathcal{M})$ and $y \in \mathcal{X}_2^{\mathcal{N}} = \mathcal{X}_2(\mathbf{C}^{\mathbf{Z}}, \mathcal{N})$ [9].

We define $\mathbf{P}$ as the projection operator of $\mathcal{X}_2$ on $\mathcal{U}_2$, $\mathbf{P}_0$ as the projection operator of $\mathcal{X}_2$ on $\mathcal{D}_2$, and $\mathbf{P}_{\mathcal{L}_2Z^{-1}}$ as the projection operator of $\mathcal{X}_2$ on $\mathcal{L}_2Z^{-1}$. The domain of $\mathbf{P}_0$ can be extended to $\mathcal{X}$. $T \in \mathcal{X}$ has a formal decomposition into a sum of shifted diagonal operators as in $T = \sum_{k=-\infty}^{\infty} Z^{[k]}T_{[k]}$, where $T_{[k]} = \mathbf{P}_0(Z^{-k}T) \in \mathcal{D}(\mathcal{M}^{(k)},\mathcal{N})$ is the k-th diagonal above the main (0-th) diagonal.

Left D-invariant subspaces

$\mathcal{X}_2$, as a Hilbert space, has subspaces in the usual way. We say that a subspace $\mathcal{H} \subset \mathcal{X}_2$ is *left D-invariant* if $A \in \mathcal{H} \Rightarrow DA \in \mathcal{H}$ for all $D \in \mathcal{D}$.

Let $\Delta_k = \mathrm{diag}[\cdots\ 0\ 0\ I\ 0\ 0\ \cdots]$, where the unit operator appears at the k-th position, and let $\mathcal{H}$ be a left D-invariant subspace. Define $\mathcal{H}_k = \Delta_k\mathcal{H}$, then $\mathcal{H}_k$ is also left D-invariant, and $\mathcal{H}_k \subset \mathcal{H}$. It follows that $\mathcal{H} = \oplus_k\mathcal{H}_k$. A left D-invariant subspace is said to be *locally finite* if $\dim \mathcal{H}_k$ is uniformly bounded by some finite number. In that case, there exists a *local basis* for $\mathcal{H}$, where each basisvector is itself a basisvector of some $\mathcal{H}_k$. The conjunction of the basisvectors of all $\mathcal{H}_k$ span $\mathcal{H}$. With $d_k = \dim \mathcal{H}_k$, we

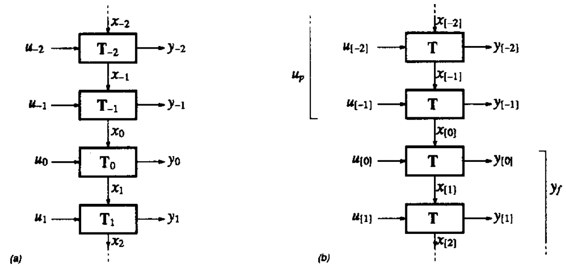

Fig. 1. Realization $\mathbf{T}$ (*a*) on ℓ_2-sequences, (*b*) on $\mathcal{X}_2$ sequences of diagonals.

will call the sequence $[\cdots\ d_0\ d_1\ d_2\ \cdots]$ the *sequence of dimensions of* $\mathcal{H}$, in notation s-dim $\mathcal{H}$.

Hankel operators and state spaces

Let $T \in \mathcal{X}$ be a bounded operator. An abstract version of the Hankel operator maps "inputs" in $\mathcal{L}_2Z^{-1}$ to outputs restricted to $\mathcal{U}_2$: the Hankel operator H_T connected to T is the map $u \in \mathcal{L}_2Z^{-1} \mapsto \mathbf{P}(uT)$. Note that only the strictly upper part of T plays a role in this definition. The operators H_k of equation (1.3) are "snapshots" of it: H_k can be obtained from H_T by considering a further restriction to inputs $\Delta_k u$ of which only the k-th row is non-zero: the operator $(\Delta_k \cdot)H_T$ is isomorphic to H_k. Realization theory is based on distinguishing characteristic spaces in $\mathcal{L}_2Z^{-1}$ and $\mathcal{U}_2$,

– the input state space $\mathcal{H}(T) = \mathrm{ran}\ (H_T^*) = \{\mathbf{P}_{\mathcal{L}_2Z^{-1}}(yT^*) : y \in \mathcal{U}_2\} \subset \mathcal{L}_2Z^{-1}$,

– the output state space $\mathcal{H}_0(T) = \mathrm{ran}\ (H_T) = \{\mathbf{P}(uT) : u \in \mathcal{L}_2Z^{-1}\} \subset \mathcal{U}_2$.

These spaces are left D-invariant: $D\mathcal{H} \subset \mathcal{H}$, $D\mathcal{H}_0 \subset \mathcal{H}_0$. $\mathcal{H}$ and $\mathcal{H}_0$ are not necessarily closed; their closures $\overline{\mathcal{H}}$ and $\overline{\mathcal{H}}_0$ are left D-invariant subspaces, Throughout the paper, it will be assumed that T is such that $\overline{\mathcal{H}}(T)$ and $\overline{\mathcal{H}}_0(T)$ are locally finite subspaces. Such T will be called *locally finite* operators. On a local level, it holds that $\dim \mathcal{H}_k = \dim \mathrm{ran}\ (H_k^*) = \mathrm{rank}(H_k^*)$ and $\dim (\mathcal{H}_0)_k = \dim \mathrm{ran}\ (H_k) = \mathrm{rank}(H_k) = \dim \mathcal{H}_k$. Hence s-dim $\mathcal{H}$ = s-dim $\mathcal{H}_0 = [\mathrm{rank}\ H_k]_{-\infty}^{\infty}$ is equal to the minimal state dimension sequence of T.

Let the Hankel norm of T be defined as the operator norm of its Hankel operator: $\|T\|_H = \|H_T\|$. This definition is equivalent to (1.5). It is straightforward to show that the Hankel norm is weaker than the operator norm: $\|T\|_H \le \|T\|$.

Realizations

The realization (2.1) can be generalized further, by considering inputs in $\mathcal{X}_2^{\mathcal{M}}$, outputs in $\mathcal{X}_2^{\mathcal{N}}$, and states in $\mathcal{X}_2^{\mathcal{B}}$, for which again the same relations hold. By projecting onto the k-th diagonal, and using the fact that A, B, C, D are diagonal operators, a generalization of the recursive realization (1.2) is obtained as

$$\begin{aligned} x_{[k+1]}^{(-1)} &= x_{[k]}A + u_{[k]}B \\ y_{[k]} &= x_{[k]}C + u_{[k]}D \end{aligned} \tag{2.2}$$

(see figure 1). Note the diagonal shift in $x_{[k+1]}^{(-1)}$.

The Hankel operator H_T has a factorization: if $u_p \in \mathcal{L}_2Z^{-1}$ then $y_f = \mathbf{P}(y) = u_pH_T$ can be written as a map $u_p \mapsto x_{[0]}$ followed by a map $x_{[0]} \mapsto y_f$:

$$\begin{aligned} x_{[0]} &= \mathbf{P}_0(x) = \mathbf{P}_0(\, uBZ(I - AZ)^{-1}\,) = \left[u_{[-1]}^{(1)}\ \ u_{[-2]}^{(2)}\ \ u_{[-3]}^{(3)} \cdots\right] \mathcal{C} \\ y_f &= x_{[0]}(I - AZ)^{-1}C\,. \end{aligned}$$

or in expanded form $[\,y_{[0]}\ \ y_{[1]}^{(-1)}\ \ y_{[2]}^{(-2)}\ \cdots\,] = x_{[0]}\mathcal{O}$, where

$$\mathcal{C} := \begin{bmatrix} B^{(1)} \\ B^{(2)}A^{(1)} \\ B^{(3)}A^{(2)}A^{(1)} \\ \vdots \end{bmatrix} \qquad \mathcal{O} := \begin{bmatrix} C & AC^{(-1)} & AA^{(-1)}C^{(-2)} & \cdots \end{bmatrix}$$

$\mathcal{C}$ and $\mathcal{O}$ are the controllability and observability operators of the realization. $\mathcal{C}_k$ and $\mathcal{O}_k$ in (1.4) are obtained by taking the k-th entry along each diagonal of $\mathcal{C}$ and $\mathcal{O}$. The realization is said to be controllable if $\mathcal{C}^*\mathcal{C} > 0$ and observable if $\mathcal{O}\mathcal{O}^* > 0$, and uniformly controllable/observable if the expressions are uniformly positive. If a realization is uniformly controllable then $\mathcal{H}_0(T) = \mathcal{D}_2^{\mathcal{B}}(I - AZ)^{-1}C$, if it is uniformly observable then $\mathcal{H}(T) = \mathcal{D}_2^{\mathcal{B}}[BZ(I-AZ)^{-1}]^*$. This shows, again, that s-dim $\mathcal{H}$ = s-dim $\mathcal{H}_0 = \#\mathcal{B}$, the state dimension sequence. It also shows that the input state space is determined by the pair (A, B).

Lyapunov Equations

A state transformation on a given realization $\mathbf{T}$ has the form

$$\mathbf{T}' = \begin{bmatrix} R & \\ & I \end{bmatrix} \begin{bmatrix} A & C \\ B & D \end{bmatrix} \begin{bmatrix} (R^{(-1)})^{-1} & \\ & I \end{bmatrix}.$$

where R is a boundedly invertible diagonal operator. State transformations are often used to bring a transition operator into some desirable form. For example, an input normal form $(A^*A + B^*B = I)$ is obtained by putting $M = R^*R$ and solving the Lyapunov equation

$$M^{(-1)} = A^*MA + B^*B, \qquad M \in \mathcal{D}(\mathcal{B}, \mathcal{B}) \tag{2.3}$$

Equation (2.3) will have a unique solution provided $\ell_A < 1$. By taking the k-th entry of each diagonal which appears in (2.3), this equation leads to $M_{k+1} = A_k^*M_kA_k + B_k^*B_k$, which can be solved recursively if an initial value for some M_k is known. If $\mathcal{C}$ is the controllability operator of the given realization, then $M = \mathcal{C}^*\mathcal{C}$ is the solution of (2.3), which shows that M is boundedly invertible if the realization is uniformly controllable. Likewise, if the realization is strictly stable and uniformly observable ($\mathcal{O}$ is such that $Q = \mathcal{O}\mathcal{O}^*$ is boundedly invertible), then Q is the unique bounded solution of the Lyapunov equation

$$Q = AQ^{(-1)}A^* + CC^*$$

and with the factoring of $Q = R^{-1}R^{-*}$ this yields a state transformation R such that $A'A'^* + C'C'^* = I$. The resulting realization then forms an output normal realization for the operator. In section 3 we shall assume that the operator to be approximated is indeed specified by a realization in output normal form. This is always possible to achieve.

J-unitary operators and J-unitary realizations

If an operator is at the same time unitary and upper, we shall call it an inner operator. A J-unitary operator Θ is an operator with 2×2 block decomposition so that the input and output spaces of Θ are split into sequences $\mathcal{M}_1 \oplus \mathcal{N}_1$ and $\mathcal{M}_2 \oplus \mathcal{N}_2$, and has corresponding signature operators:

$$\Theta = \begin{bmatrix} \Theta_{11} & \Theta_{12} \\ \Theta_{21} & \Theta_{22} \end{bmatrix}, \quad J_1 = \begin{bmatrix} I_{\mathcal{M}_1} & \\ & -I_{\mathcal{N}_1} \end{bmatrix}, \quad J_2 = \begin{bmatrix} I_{\mathcal{M}_2} & \\ & -I_{\mathcal{N}_2} \end{bmatrix} \tag{2.4}$$

such that $\Theta^*J_1\Theta = J_2$, $\Theta^*J_2\Theta = J_1$. Let be given a state operator $\mathbf{\Theta}$, and let $\mathcal{B}$ be the space sequence of the state of Θ. Suppose that $\mathcal{B} = \mathcal{B}_+ \oplus \mathcal{B}_-$ is a certain decomposition of $\mathcal{B}$ into two space sequences. Let

$$J_{\mathcal{B}} = \begin{bmatrix} I_{\mathcal{B}_+} & \\ & -I_{\mathcal{B}_-} \end{bmatrix} \tag{2.5}$$

be a corresponding signature matrix, which we call the state signature sequence of Θ.

Theorem 2.1. *If a state realization operator $\mathbf{\Theta}$ is strictly stable and satisfies*

$$\mathbf{\Theta}^* \begin{bmatrix} J_{\mathcal{B}} & \\ & J_1 \end{bmatrix} \mathbf{\Theta} = \begin{bmatrix} J_{\mathcal{B}}^{(-1)} & \\ & J_2 \end{bmatrix} \tag{2.6}$$

$$\mathbf{\Theta} \begin{bmatrix} J_{\mathcal{B}}^{(-1)} & \\ & J_2 \end{bmatrix} \mathbf{\Theta}^* = \begin{bmatrix} J_{\mathcal{B}} & \\ & J_1 \end{bmatrix} \tag{2.7}$$

then the corresponding transfer operator Θ will be J-unitary in the sense that

$$\Theta^*J_1\Theta = J_2, \qquad \Theta J_2\Theta^* = J_1. \tag{2.8}$$

With '#' indicating the sequence of dimensions of a space sequence, the dimensions of the signatures satisfy the inertia relations

$$\begin{aligned} \#\mathcal{B}_+ + \#\mathcal{M}_1 &= \#\mathcal{B}_+^{(-1)} + \#\mathcal{M}_2 \\ \#\mathcal{B}_- + \#\mathcal{N}_1 &= \#\mathcal{B}_-^{(-1)} + \#\mathcal{N}_2. \end{aligned} \tag{2.9}$$

A J-unitary upper operator has an interpolation-type property: it maps its input state space (in $[\mathcal{L}_2Z^{-1} \;\; \mathcal{L}_2Z^{-1}]$) to $[\mathcal{U}_2 \;\; \mathcal{U}_2]$. This general property, formulated for a J-unitary state realization of Θ, reads

Lemma 2.2. *If $\{\alpha, \beta, \gamma, \delta\}$ is a J-unitary state realization for a J-unitary block-upper operator Θ, then $Z^*(I - \alpha^*Z^*)^{-1}\beta^*J_1\Theta \in [\mathcal{U} \;\; \mathcal{U}]$.*

Scattering operators

Associated to Θ is an operator Σ such that $[a_1 \;\; b_2]\Sigma = [a_2 \;\; b_1] \Leftrightarrow [a_1 \;\; b_1]\Theta = [a_2 \;\; b_2]$. Σ can be evaluated in terms of the block-entries of Θ as

$$\Sigma = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix} = \begin{bmatrix} \Theta_{11} - \Theta_{12}\Theta_{22}^{-1}\Theta_{21} & -\Theta_{12}\Theta_{22}^{-1} \\ \Theta_{22}^{-1}\Theta_{21} & \Theta_{22}^{-1} \end{bmatrix} \tag{2.10}$$

where J-unitarity of Θ ensures that Θ_{22}^{-1} is bounded and that Σ is unitary. One fact which will be essential for the approximation theory in the later sections is that, although Θ has block-entries which are upper, Θ_{22}^{-1} need not be upper but can be of mixed causality, so that the block-entries of Σ are in general not upper.

Partition the state x of the realization $\mathbf{\Theta}$ according to the signature $J_{\mathcal{B}}$ into $x = [x_+ \;\; x_-]$, and partition $\mathbf{\Theta}$ likewise, then a corresponding scattering operator $\mathbf{\Sigma}$ can be defined by the relation

$$\begin{aligned} & [x_+ \;\; x_- \;\; a_1 \;\; b_1]\mathbf{\Theta} &= [x_+Z^{-1} \;\; x_-Z^{-1} \;\; a_2 \;\; b_2] \\ \Leftrightarrow \;\; & [x_+ \;\; x_-Z^{-1} \;\; a_1 \;\; b_2]\mathbf{\Sigma} &= [x_+Z^{-1} \;\; x_- \;\; a_2 \;\; b_1] \end{aligned} \tag{2.11}$$

(inputs of $\mathbf{\Sigma}$ have positive signature). $\mathbf{\Sigma}$ can be computed independently for each time instant from $\mathbf{\Theta}$. It is a kind of generalized or implicit realization for Σ, which can be obtained after elimination of x_- and x_+. $\mathbf{\Sigma}$ is unitary: $\mathbf{\Sigma}\mathbf{\Sigma}^* = I$; $\mathbf{\Sigma}^*\mathbf{\Sigma} = I$, which is easily derived from the J-unitarity of $\mathbf{\Theta}$.

3. APPROXIMATION PROCEDURE

The problem that we shall solve in this section is the model reduction problem for a strictly upper operator described by a strictly stable "higher order model". Let Γ be a diagonal and hermitian operator. We shall use Γ as a measure for the local accuracy of the reduced order model. It will also parametrize the solutions. We will look for a contractive operator E such that $E = (T^* - T'^*)\Gamma^{-1}$ where T' is an operator which is not necessarily upper triangular, but whose strictly causal part will assumed to be bounded and have state space dimensions of low order — as low as possible for a given Γ. Once we have such a contractive E, it is immediately verified that it satisfies $\| \Gamma^{-1}(T - T') \| = \| E \| \le 1$. Let T_a be the strictly causal part of T'. Then

$$\| \Gamma^{-1}(T - T_a) \|_H \le \| \Gamma^{-1}(T - T') \| \le 1,$$

and T_a is a Hankel-norm approximant when T' is an operator-norm approximant. The construction of a suitable T' consists of three steps. We start by determining a (minimal) factorization of T in the form $T = \Delta^*U$ where Δ and U are upper operators which have finite state space dimensions of the same size as that of T, and U is inner: $UU^* = I$; $U^*U = I$. Next, we look for a locally finite J-unitary operator Θ with upper block entries chosen such that

$$\begin{bmatrix} U^* & -T^*\Gamma^{-1} \end{bmatrix} \Theta = [A' \;\; -B'] \tag{3.1}$$

consists of two upper operators. Θ will again be locally finite. Then, because Θ is J-unitary, we have that Θ_{22}^{-1} will exist (but not necessarily be upper) and $\Sigma_{12} = -\Theta_{12}\Theta_{22}^{-1}$ will be contractive. From (3.1) we have $B' = -U^*\Theta_{12} + T^*\Gamma^{-1}\Theta_{22}$. Define the approximating operator T' as

$$T'^* = B'\Theta_{22}^{-1}\Gamma\,, \tag{3.2}$$

then $E = (T^* - T'^*)\Gamma^{-1} = -U^*\Sigma_{12}$ has $\|E\| \le 1$, so that $T'^* = B'\Theta_{22}^{-1}\Gamma$ is indeed an approximant with an admissible modeling error. In view of the target theorem 1.1, we have to show that the strictly causal part T_a of T' has the stated number of states and to verify the relation with the Hankel singular values of $\Gamma^{-1}T$. This will done at the end of this section.

Factorization of T

Theorem 3.1. *Let T be an upper operator which has a strictly stable locally finite and uniformly observable state space realization $\{A, B, C, D\}$. Then there exists a factorization of T as $T = \Delta^* U$, where Δ and U are upper operators, again locally finite and strictly stable, and U is inner, i.e., upper and unitary.*

PROOF We start from a realization of T in output normal form, *i.e.*, such that $AA^* + CC^* = I$. For each time instant k, augment the state transition matrices $[A_k \ \ C_k]$ of T with as many extra rows as needed to yield a unitary (hence square) matrix $\mathbf{U}_k$:

$$\mathbf{U}_k \quad = \quad \begin{bmatrix} A_k & C_k \\ (B_U)_k & (D_U)_k \end{bmatrix}. \tag{3.3}$$

Assemble the individual matrices $\{A_k, (B_U)_k, C_k, (D_U)_k\}$ in diagonal operators $\{A, B_U, C, D_U\}$, then the corresponding operator $\mathbf{U}$ is a state space realization for U; $U = D_U + B_U Z(I - AZ)^{-1}C$. U is well-defined and upper, and it is unitary because it has a unitary realization (as in theorem 2.1). It is straightforward to verify that $\Delta = UT^*$ is indeed upper. □

Note that the number of rows added to $[A_k \ \ C_k]$ is time-varying, so that U (and hence also Δ) has a time-varying number of inputs. The varying number of inputs of U will of course be matched by a varying number of outputs of Δ^*.

Construction of Θ

The next step is to construct a locally finite and block-upper J-unitary Θ that satisfies equation (3.1). Let $\mathcal{B}$ be the space in which the state sequences of the realization $\mathbf{\Theta}$ of Θ live. Θ will be J-unitary in the sense of (2.4) if $\mathbf{\Theta}$ satisfies (2.6) with some state signature matrix $J_\mathcal{B}$ to be determined yet. Let $\{A, B, C, 0\}$ be the realization for T used in the previous section (it is in output normal form), and let $\{A, B_U, C, D_U\}$ be the realization for the inner factor U of T. We submit that Θ satisfying (3.1) has a realization $\mathbf{\Theta}$ of the form

$$\mathbf{\Theta} = \left[\begin{array}{c|c} X & \\ \hline & I \end{array}\right] \left[\begin{array}{c|cc} A & C_1 & C_2 \\ \hline B_U & D_{11} & D_{12} \\ \Gamma^{-1}B & D_{21} & D_{22} \end{array}\right] \left[\begin{array}{c|c} (X^{(-1)})^{-1} & \\ \hline & I \end{array}\right] =: \begin{bmatrix} \alpha & \gamma \\ \beta & \delta \end{bmatrix} \tag{3.4}$$

which is a square matrix at each time instant k, and where X and C_i, D_{ij} are yet to be determined. Note that the state sequence space $\mathcal{B}$ is the same for Θ and T. X is a boundedly invertible diagonal state transformation operator which is such that $\mathbf{\Theta}$ is J-unitary as in (2.6). Writing $\Lambda = X^* J_\mathcal{B} X$, the signature of Λ will determine $J_\mathcal{B}$, and it is straightforward to derive that $\Lambda = I - M$, where M is given by

$$M^{(-1)} = A^*MA + B^*\Gamma^{-1}\Gamma^{-1}B\,. \tag{3.5}$$

M is the solution of one of the Lyapunov equations associated to $\Gamma^{-1}T$, and can be determined recursively from the given realization of T via $M_{k+1} = A_k^* M_k A_k + B_k^* \Gamma_k^{-1} \Gamma_k^{-1} B_k$.

Theorem 3.2. *Let T be a strictly upper locally finite operator mapping $\ell_2^\mathcal{M}$ to $\ell_2^\mathcal{N}$, with output normal realization $\{A, B, C, 0\}$ such that $\ell_A < 1$, and let Γ be a Hermitian diagonal operator. Also let U be the inner factor of a coprime factorization of T. If the solution M of the Lyapunov equation (3.5) is such that $\Lambda = I - M$ is boundedly invertible, then there exists a J-unitary block upper operator Θ such that $[U^* \ \ -T^*\Gamma^{-1}]\Theta$ is block upper.*

PROOF The condition insures that there exists a state transformation X such that $\alpha = XA(X^{(-1)})^{-1}$, $\beta = \begin{bmatrix} B_U(X^{(-1)})^{-1} \\ \Gamma^{-1}B(X^{(-1)})^{-1} \end{bmatrix}$ satisfy

$$\alpha^* J_\mathcal{B} \alpha + \beta^* J_1 \beta = J_\mathcal{B}^{(-1)}\,. \tag{3.6}$$

X is obtained by solving the Lyapunov equation (3.5) for M, putting $\Lambda = I - M$, and factoring Λ into $\Lambda = X^* J_\mathcal{B} X$. This also determines the signature operator $J_\mathcal{B}$ and thus the space sequence decomposition $\mathcal{B} = \mathcal{B}_+ \oplus \mathcal{B}_-$. The next step is the construction of a realization $\mathbf{\Theta}$ of the form (3.4) which is a square matrix at each point k, and where the X and C_i, D_{ij} are yet to be determined. $\mathbf{\Theta}$ is to satisfy (2.6) for

$$J_1 = \begin{bmatrix} I_{\mathcal{M}_U} & \\ & -I_\mathcal{M} \end{bmatrix}, \qquad J_2 := \begin{bmatrix} I_{\mathcal{M}_2} & \\ & -I_{\mathcal{N}_2} \end{bmatrix} \tag{3.7}$$

where the dimensionality of the output space sequences $\mathcal{M}_2$ and $\mathcal{N}_2$ follow from theorem 2.1, equation (2.9) as

$$\begin{aligned} \#\mathcal{M}_2 &= \#\mathcal{B}_+ - \#\mathcal{B}_+^{(-1)} + \#\mathcal{M}_U \ge 0 \\ \#\mathcal{N}_2 &= \#\mathcal{B}_- - \#\mathcal{B}_-^{(-1)} + \#\mathcal{M} \ \ge 0\,. \end{aligned}$$

Finally, to obtain $\mathbf{\Theta}$, it remains to show that $\begin{bmatrix} \alpha \\ \beta \end{bmatrix}$ can be completed to form $\mathbf{\Theta}$ in (3.4), in such a way that the whole operator is now J-unitary in the sense of (2.6). It can be shown that this completion exists under the present conditions and can be achieved at the local level: it is for each time instant k an independent problem of matrix algebra. To conclude the proof, we have to show that $[U^* \ \ -T^*\Gamma^{-1}]\Theta$ is block upper. We have

$$[U^* \ \ -T^*\Gamma^{-1}] = [D_U^* \ \ -D^*\Gamma^{-1}] + C^*Z^*(I - A^*Z^*)^{-1}[B_U^* \ \ -B^*\Gamma^{-1}]$$

and it will be enough to show that

$$\mathcal{D}_2 Z^*(I - A^*Z^*)^{-1}[B_U^* \ \ -B^*\Gamma^{-1}]\,\Theta$$

is block upper. With entries as in equation (3.4), and using the state equivalence transformation defined by X, this is equivalent to showing that $\mathcal{D}_2 X Z^*(I - \alpha^* Z^*)^{-1} \beta^* J_1 \Theta$ is block-upper. That this is indeed the case follows directly from lemma 2.2. □

We conclude this section by establishing the link between the Lyapunov equation and the Hankel operator connected with $\Gamma^{-1}T$.

Theorem 3.3. *Under the hypothesis on the singular values of the Hankel operators H_k of $\Gamma^{-1}T$ in theorem 1.1, the solution M of the Lyapunov equation (3.5) is such that $\Lambda = I - M$ is boundedly invertible and has signature $J_\mathcal{B}$ having N_k negative entries at point k, where N_k is the number of singular values of H_k that are larger than 1.*

PROOF The solutions of the two Lyapunov equations associated to $\Gamma^{-1}T$,

$$\begin{aligned} M^{(-1)} &= A^*MA + B^*\Gamma^{-2}B \\ Q &= AQ^{(-1)}A^* + CC^* \end{aligned}$$

may be expressed in terms of the controllability and observability operators of $\Gamma^{-1}T$ as $M = \mathcal{C}^*\mathcal{C}$, $Q = \mathcal{O}\mathcal{O}^*$. The Hankel operator H_k of $\Gamma^{-1}T$ at time instant k satisfies the decomposition $H_k = \mathcal{C}_k\mathcal{O}_k$. Hence $H_k H_k^* = \mathcal{C}_k\mathcal{O}_k\mathcal{O}_k^*\mathcal{C}_k^*$. We have started from a state realization in output normal form: $Q = \mathcal{O}\mathcal{O}^* = I$. With the current finiteness assumption, the non-zero eigenvalues of $H_k H_k^* = \mathcal{C}_k\mathcal{C}_k^*$ will be the same as those of $\mathcal{C}_k^*\mathcal{C}_k = M_k$. In particular, the number of singular values of H_k that are larger than 1 is equal to the number of eigenvalues of M_k that are larger than 1. Writing $\Lambda_k = I - M_k$, this is in turn equal to the number of negative eigenvalues of Λ_k. □

Figure 2 shows a simple instance of the application of the theory developed in this section, especially with regard to the dimensions of the input, output and state sequence spaces related to the Θ-matrix. While the signal flow of Θ runs strictly from top to bottom and from left to right, the directions of the arrows in the figure correspond to the signal flow of the unitary state space operator $\mathbf{\Sigma}$ that can be associated to $\mathbf{\Theta}$, and that will play an important role in the next section. Upward arrows in the state of $\mathbf{\Sigma}$ are caused by the negative entries in the state signature $J_\mathcal{B}$ of $\mathbf{\Theta}$.

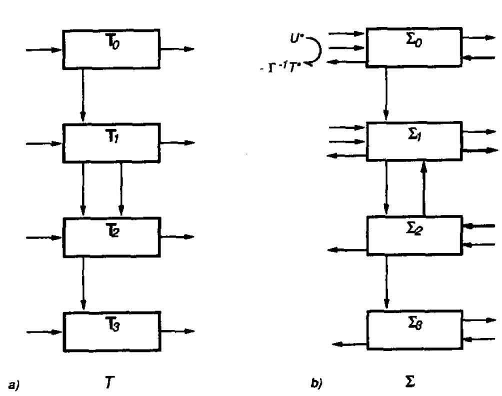

Fig. 2. (*a*) Computational scheme for T, (*b*) Computational scheme for a possible Σ, where it is assumed that one singular value of the Hankel operator of $\Gamma^{-1}T$ at time 1 is larger than 1.

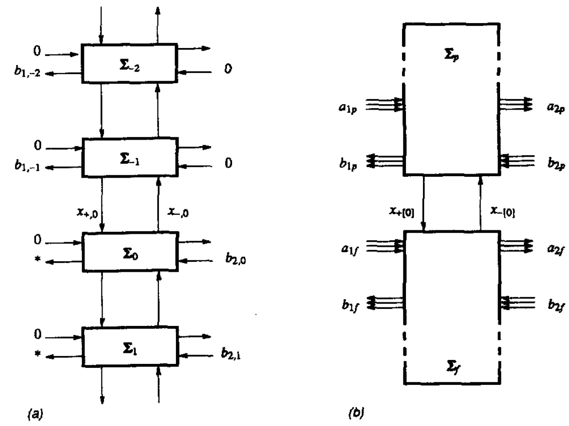

Fig. 3. (*a*) The state transition scheme for Σ, with ℓ_2-sequences as inputs. (*b*) The decomposition of Σ into a past operator Σ_p and a future operator Σ_f linked by the state $[x_{+[0]} \;\; x_{-[0]}]$. This summarizes the figure on the left for all time.

Complexity of the approximant

At this point we have proven the first part of theorem 1.1: we have constructed a J-unitary operator Θ and from it an operator T_a which is a Hankel-norm approximant of T. It remains to verify the complexity assertion, which stated that the dimension of the state space of T_a is at most equal to N: the number of Hankel singular values of $\Gamma^{-1}T$ that are larger than one. In view of theorems 3.2 and 3.3, N is equal to the number of negative entries in the state signature J_B of Θ. Suppose that the conditions of theorem 3.2 are fulfilled so that Θ satisfies

$$[U^* \;\; -T^*\Gamma^{-1}]\,\Theta = [A' \;\; -B']$$

with $A', B' \in \mathcal{U}$. Let $T'^*\Gamma^{-1} = B'\Theta_{22}^{-1}$. The approximating transfer function T_a is given by the strictly upper part of T'. It might not be a bounded operator but its Hankel map $H_{T_a} = H_{T'}$ is well-defined and bounded. We have

Lemma 3.4. *Under the conditions of theorem 3.2, the input state space of* $\Gamma^{-1}T_a$ *satisfies* $\mathcal{H}(\Gamma^{-1}T_a) \subset \mathcal{H}(\Theta_{22}^{-*})$.

PROOF From the definition of $\mathcal{H}$ and the operators we have

$$\begin{aligned}
\mathcal{H}(\Gamma^{-1}T_a) &= \mathbf{P}_{\mathcal{L}_2Z^{-1}}(\mathcal{U}_2 T_a^*\Gamma^{-1}) \\
&= \mathbf{P}_{\mathcal{L}_2Z^{-1}}(\mathcal{U}_2 T'^*\Gamma^{-1}) \\
&= \mathbf{P}_{\mathcal{L}_2Z^{-1}}(\mathcal{U}_2 B'\Theta_{22}^{-1}) \\
&\subset \mathbf{P}_{\mathcal{L}_2Z^{-1}}(\mathcal{U}_2 \Theta_{22}^{-1}) \\
&= \mathcal{H}(\Theta_{22}^{-*}).
\end{aligned}$$

□

Hence the dimension sequence of $\mathcal{H}(\Theta_{22}^{-*})$ is of interest. Define the "conjugate-Hankel" operator $H' := H'_{\Theta_{22}^{-1}} = \mathbf{P}_{\mathcal{L}_2Z^{-1}}(\,\cdot\,\Theta_{22}^{-1})|_{\mathcal{U}_2}$ Then $\mathcal{H}(\Theta_{22}^{-*}) = \mathrm{ran}\,(H')$.

Let the signals a_1, b_1, a_2, b_2 and the state sequences x_+, x_- be in $\mathcal{X}_2$ and be related by Θ as in (2.11). Define decompositions into past and future parts of signals in $\mathcal{X}_2$: $a_1 = a_{1p} + a_{1f}$ with $a_{1p} = \mathbf{P}_{\mathcal{L}_2Z^{-1}}(a_1)$ and $a_{1f} = \mathbf{P}(a_1)$. Because $\Theta_{22}^{-1} = \Sigma_{22}$, the conjugate-Hankel operator H' is a restriction of the partial map $\Sigma_{22} : b_2 \mapsto b_1$, that is, $H' : b_{2f} \mapsto b_{1p}$ is such that b_{2p} and b_{1p} satisfy the input-output relations defined by Σ under the conditions $a_1 = 0$ and $b_{2p} = 0$. Inspection of figure 3 shows that H' can be factored as $H' = \sigma\tau$, where the operators

$$\sigma:\; b_{2f} \mapsto x_{-[0]} \qquad \tau:\; x_{-[0]} \mapsto b_{1p}$$

can be derived from Σ by elimination of $x_{+[0]}$, again taking $a_1 = 0$ and $b_{2p} = 0$. It can be shown [8] that the operator σ is 'onto' while τ is 'one-to-one', so that the factorization of H' into these operators is minimal. It follows, in lemma 3.5, that the dimension of $x_{-[0]}$ at each point in time determines the local dimension of the subspace $\mathcal{H}(\Theta_{22}^{-*})$ at that point.

Lemma 3.5. *The s-dimension of* $\mathcal{H}(\Theta_{22}^{-*})$ *is equal to* $N = \#(\mathcal{B}_-)$, *i.e., the number of negative entries in the state signature sequence of* Θ.

Lemma 3.5 completes the proof of theorem 1.1. It is possible to derive explicit formulas for a realization of the approximant T_a [8]. This realization is given in terms of four recursions: two that run forward in time, the other two run backward in time and depend on the first two recursions. One implication of this is that it is not possible to compute part of an optimal approximant of T if the model of T is known only partly, say up to time instant k.

4. PARAMETRIZATION OF ALL APPROXIMANTS

The present section is devoted to the description of all possible solutions to the Hankel norm approximation problem of order smaller than or equal to N, where $N = \text{s-dim}\ \mathcal{H}(\Theta_{22}^{-*})$ is the sequence of dimensions of the input state space of Θ_{22}^{-*}. We shall determine all possible bounded operators $T' \in \mathcal{X}$ for which it is true that

(1) $\|\Gamma^{-1}(T - T')\| \le 1$,
and (2) the state dimension sequence of T_a = (upper part of T') is at most equal to N.

It turns out that there are no Hankel norm approximants with state dimension sequence lower than N. The result is that all solutions are obtained by a linear fractional transform (chain scattering transformation) of Θ with an upper and contractive parameter S_L. That this procedure effectively generates all approximants of with state dimensions at most equal to N can be seen from the fact that if $\|\Gamma^{-1}(T - T_a)\|_H \le 1$, then an extension T' of T_a must exist such that $\|\Gamma^{-1}(T - T')\| \le 1$ (Nehari's theorem).

We will use the following preliminary fact.

Theorem 4.1. ([8]) *Let* $A \in \mathcal{U}$, $B = I - X$ *where* $X \in \mathcal{X}$, $\|X\| < 1$, *and let* A *be invertible in* $\mathcal{X}$. *Suppose that* s-dim $\mathbf{P}_{\mathcal{L}_2Z^{-1}}(\mathcal{U}_2A^{-1}) = N$. *Then* s-dim $\mathbf{P}_{\mathcal{L}_2Z^{-1}}(\mathcal{U}_2B^{-1}A^{-1}) = N + p \;\Rightarrow\;$ s-dim $\mathbf{P}_{\mathcal{L}_2Z^{-1}}(\mathcal{U}_2AB) = p$.

Generating new solutions of the interpolation problem

Theorem 4.2. *Let* T, Γ *and* U *be as in theorem 3.2, and let* N *be the sequence of number of singular values of the Hankel operator of* $\Gamma^{-1}T$ *that are larger than 1. Let* Θ *be a* J*-unitary block-upper operator such that* $[U^* \;\; -T^*\Gamma^{-1}]\,\Theta = [A' \;\; -B'] \in [\mathcal{U}\;\; \mathcal{U}]$, *which exists by theorem 3.2.*

Let $S_L \in \mathcal{U}$ *be a contractive operator.* (1) $\Theta_{22} - \Theta_{21}S_L$ *is boundedly invertible, and* $S = (\Theta_{11}S_L - \Theta_{12})(\Theta_{22} - \Theta_{21}S_L)^{-1}$ *is contractive.*

(2) *Let, furthermore,* $T' = T + \Gamma S^* U$. *Then*

(*a*) $\| \Gamma^{-1}(T - T') \| = \| S^* U \| \le 1$,

(*b*) *the state dimension sequence of* T_a = *(upper part of* T'*) is precisely equal to* N.

That is, T_a is a Hankel norm approximant of T. The Hankel norm approximant of the previous section is obtained for $S_L = 0$.

PROOF (1) is true by J-unitarity of Θ and contractiveness of S_L. (2*a*) follows immediately since $\Gamma^{-1}(T - T') = S^* U$ and U is unitary. To prove (2*b*), use the following equality:

$$\begin{aligned} T'^{*}\Gamma^{-1} &= [U^* \;\; -T^*\Gamma^{-1}] \begin{bmatrix} S \\ -I \end{bmatrix} \\ &= [U^* \;\; -T^*\Gamma^{-1}] \begin{bmatrix} \Theta_{11} & \Theta_{12} \\ \Theta_{21} & \Theta_{22} \end{bmatrix} \begin{bmatrix} S_L \\ -I \end{bmatrix} (\Theta_{22} - \Theta_{21} S_L)^{-1} \\ &= [A' \;\; -B'] \begin{bmatrix} S_L \\ -I \end{bmatrix} (\Theta_{22} - \Theta_{21} S_L)^{-1} \\ &= (A' S_L + B')(\Theta_{22} - \Theta_{21} S_L)^{-1}. \end{aligned}$$

Since $(A' S_L + B') \in \mathcal{U}$, the state dimension sequence of T' is *at most* equal to the s-dim $\mathbf{P}_{\mathcal{L}_2 Z^{-1}}[\mathcal{U}_2(\Theta_{22} - \Theta_{21} S_L)^{-1}]$. Because the latter operant is equal to $(I - \Theta_{22}^{-1}\Theta_{21} S_L)^{-1}\Theta_{22}^{-1}$, and $\|\Theta_{22}^{-1}\Theta_{21} S_L\| < 1$, application of theorem 4.1 with $A = \Theta_{22}$ and $B = I - \Theta_{22}^{-1}\Theta_{21} S_L$ shows that s-dim $\mathbf{P}_{\mathcal{L}_2 Z^{-1}}[\mathcal{U}_2(\Theta_{22} - \Theta_{21} S_L)^{-1}]$ = s-dim $\mathbf{P}_{\mathcal{L}_2 Z^{-1}}(\mathcal{U}_2 \Theta_{22}^{-1})$ = s-dim $\mathcal{H}(\Theta_{22}^{-*})$, *i.e.*, equal to N. Hence s-dim $\mathcal{H}(T') \le N$ (pointwise).

The proof terminates by showing that also s-dim $\mathcal{H}(T') \ge N$, so that in fact s-dim $\mathcal{H}(T') = N$. We omit this part here. □

So all S of the form $S = (\Theta_{11} S_L - \Theta_{12})(\Theta_{22} - \Theta_{21} S_L)^{-1}$ with $S_L \in \mathcal{U}$, $\| S_L \| \le 1$ give rise to Hankel norm approximants of T. It is well known that this type of expression for S is a chain scattering transformation of S_L by Θ. The reverse question is: are all Hankel norm approximants obtained this way? That is, given some T' whose strictly upper part is a Hankel norm approximant of T, is there a corresponding upper and contractive S_L such that T' is given by $T' = T + \Gamma S^* U$, with S as above. This problem is addressed in the next theorem. The main issue is to prove that S_L as defined by the equations is upper.

Generating all approximants

Theorem 4.3. *Let* T, Γ, U *and* Θ *be as in theorem 4.2, and let* N *be the number of Hankel singular values of* $\Gamma^{-1}T$ *that are larger than 1. Let be given a bounded operator* $T' \in \mathcal{X}$ *such that*

(1) $\| \Gamma^{-1}(T - T') \| \le 1$,

(2) *the state dimension sequence of* T_a = *(upper part of* T'*) is at most equal to* N.

Define $S = U(T'^{*} - T^*)\Gamma^{-1}$. *Then there is an operator* S_L *with* $(S_L \in \mathcal{U}, \| S_L \| \le 1)$ *such that* $S = (\Theta_{11} S_L - \Theta_{12})(\Theta_{22} - \Theta_{21} S_L)^{-1}$ *(i.e.,* Θ *generates all Hankel norm approximants). The state dimension of* T_a *is in fact precisely equal to* N.

PROOF The main line of the proof runs in parallel with [7], but differs in detail. In particular, the 'winding number' argument to determine state dimensions is replaced by theorem 4.1.

1. From the definition of S, and using the factorization $T = \Delta^* U$, we know that

$$\| S \| = \| U(T'^{*} - T^*)\Gamma^{-1} \| = \| \Gamma^{-1}(T' - T) \| \le 1$$

so S is contractive. Since $S = -\Delta\Gamma^{-1} + UT'^{*}\Gamma^{-1}$, where Δ and U are upper, s-dim $\mathcal{H}(S^*) \le$ s-dim $\mathcal{H}(T') \le N$, *i.e.*, the state dimension sequence of a minimal realization of the causal part of S^* is at most equal to N.

2. Define $[G_1^* \;\; G_2^*] := [S^* \;\; I]\Theta$. Then $\mathcal{H}(G_1^*) \subset \mathcal{H}(T')$ and $\mathcal{H}(G_2^*) \subset \mathcal{H}(T')$. (Proof omitted.)

3. The definition of G_1 and G_2 can be rewritten using $\Theta^{-1} = J\Theta^* J$ as

$$\begin{bmatrix} S \\ -I \end{bmatrix} = \Theta \begin{bmatrix} G_1 \\ -G_2 \end{bmatrix} \tag{4.1}$$

G_2 is boundedly invertible, and S_L defined by $S_L = G_1 G_2^{-1}$ is well defined and contractive: $\| S_L \| \le 1$. In addition, S satisfies $S = (\Theta_{11} S_L - \Theta_{12})(\Theta_{22} - \Theta_{21} S_L)^{-1}$ as required. (Proof omitted.)

4. $G_2^{-1} \in \mathcal{U}$, the space $\mathcal{H}(T')$ has the same s-dimension as $\mathcal{H}(\Theta_{22}^{-*})$, and $\mathcal{H}(G_1^*) \subset \mathcal{H}(G_2^*)$.

PROOF According to equation (4.1), G_2^{-1} satisfies

$$\begin{aligned} G_2^{-1} &= \Theta_{22}(I - \Theta_{22}^{-1}\Theta_{21} S_L) \\ G_2 &= (I - \Theta_{22}^{-1}\Theta_{21} S_L)^{-1}\Theta_{22}^{-1}. \end{aligned}$$

Let p = s-dim $\mathcal{H}(G_2^{-*})$ = s-dim $\mathbf{P}_{\mathcal{L}_2 Z^{-1}}(\mathcal{U}_2 G_2^{-1})$, N_2 = s-dim $\mathcal{H}(G_2^*)$ = s-dim $\mathbf{P}_{\mathcal{L}_2 Z^{-1}}(\mathcal{U}_2 G_2)$, and N = s-dim $\mathcal{H}(\Theta_{22}^{-*})$ = s-dim $\mathbf{P}_{\mathcal{L}_2 Z^{-1}}(\mathcal{U}_2 \Theta_{22}^{-1})$. Then $N_2 \le N$. Application of theorem 4.1 with $A = \Theta_{22}$ and $B = (I - \Theta_{22}^{-1}\Theta_{21} S_L)$ shows that $N_2 = N + p$, and hence $N_2 = N$ and $p = 0$: $G_2^{-1} \in \mathcal{U}$, and $\mathcal{H}(G_2^*)$ has s-dimension N. Step 2 claimed $\mathcal{H}(G_2^*) \subset \mathcal{H}(T')$, and because the latter space has at most s-dimension N, we must have that in fact $\mathcal{H}(G_2^*) = \mathcal{H}(T')$, and hence $\mathcal{H}(G_1^*) \subset \mathcal{H}(G_2^*)$.

5. $S_L \in \mathcal{U}$.

PROOF This can be inferred from $G_2^{-1} \in \mathcal{U}$, and $\mathcal{H}(G_1^*) \subset \mathcal{H}(G_2^*)$, as follows. $S_L \in \mathcal{U}$ is equivalent to $\mathbf{P}_{\mathcal{L}_2 Z^{-1}}(\mathcal{U}_2 S_L) = 0$, and

$$\begin{aligned} \mathbf{P}_{\mathcal{L}_2 Z^{-1}}(\mathcal{U}_2 S_L) &= \mathbf{P}_{\mathcal{L}_2 Z^{-1}}(\mathcal{U}_2 G_1 G_2^{-1}) \\ &= \mathbf{P}_{\mathcal{L}_2 Z^{-1}}(\mathbf{P}_{\mathcal{L}_2 Z^{-1}}(\mathcal{U}_2 G_1) G_2^{-1}) \end{aligned}$$

since $G_2^{-1} \in \mathcal{U}$. Using $\mathcal{H}(G_1^*) \subset \mathcal{H}(G_2^*)$, or $\mathbf{P}_{\mathcal{L}_2 Z^{-1}}(\mathcal{U}_2 G_1) \subset \mathbf{P}_{\mathcal{L}_2 Z^{-1}}(\mathcal{U}_2 G_2)$ we obtain that

$$\begin{aligned} \mathbf{P}_{\mathcal{L}_2 Z^{-1}}(\mathcal{U}_2 S_L) &\subset \mathbf{P}_{\mathcal{L}_2 Z^{-1}}(\mathbf{P}_{\mathcal{L}_2 Z^{-1}}(\mathcal{U}_2 G_2) G_2^{-1}) \\ &= \mathbf{P}_{\mathcal{L}_2 Z^{-1}}(\mathcal{U}_2 G_2 G_2^{-1}) \qquad (\text{since } G_2^{-1} \in \mathcal{U}) \\ &= 0. \end{aligned}$$

□

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