Langemheem

FREE MOVING INTERFACES

IN

UNSTEADY COMPRESSIBLE FLOW

Impact problems in fluid dynamics

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Helmus van de Langemheen
Department of Applied Mathematics

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delft hydraulics

Voorwoord

Aan het begin van dit stage- en afstudeerverslag zou ik graag een aantal mensen willen bedanken.
Allereerst ben ik de personen die de beoordelingscommissie vormen (prof.dr.ir. P.J. Zandbergen, dr. J.G.M. Kuerten, dr.ir. B.J. Geurts, ir. G. Klopman en ir. H.A.H. Petit) dankbaar voor hun inspirerende en motiverende begeleiding en commentaar. Hiervoor ben ik naar mijn mening tevens dr.ir. M.A.J. Borsboom en dr. D. Dijkstra dank verschuldigd.
Ten slotte wil ik mijn erkentelijkheid betuigen voor de geboden mogelijkheid mijn afstudeerwerk aan het Waterloopkundig Laboratorium te Marknesse uit te voeren.

Helmus van de Langemheen
augustus 1992

Een stage aan het Waterloopkundig Laboratorium

Dit verslag is het resultaat van een aan het Waterloopkundig Laboratorium (WL, internationaal bekend onder de naam Delft Hydraulics) uitgevoerde gecombineerde stage- en afstudeeropdracht.
Een periode van zeven maanden (van augustus '91 tot maart '92) is daartoe doorgebracht op locatie 'De Voorst' (Noordoostpolder) van dit instituut. Het WL houdt zich bezig met technisch-wetenschappelijk onderzoek en advisering inzake aan water of andere vloeistoffen gerelateerde problemen.
Bijvoorbeeld kan men denken aan het bepalen van ontwerpcriteria voor waterbouwkundige constructies in verband met belastingtoestanden hierop, aan het bepalen van de gevolgen van sedimenttransport in rivieren, het voorspellen van waterhoogten, enzovoorts. Het onderzoek wordt veelal uitgevoerd in opdracht van derden (bedrijven, instanties en overheden) en is grotendeels toegepast van aard. Ter verbreding van de basis voor het opdrachtenwerk wordt echter ook vrij fundamenteel onderzoek verricht, vaak in de vorm van eigen zogenaamde 'speurwerkprojecten'. Hierdoor vervult het instituut een brugfunctie tussen wetenschap en praktijk.
De kracht van het WL ligt vooral in de combinatie van het gebruik van enerzijds wiskundige modellen en anderzijds de beschikbare experimentele faciliteiten. Het heeft een rijke traditie op het gebied van simulatie met schaalmodellen. Ondanks de toenemende betekenis van geavanceerde computer-faciliteiten en wiskundige modellen blijven de schaalmodellen van belang ter aanvulling en als verificatie van de resultaten die met wiskundige modellen zijn verkregen.
Het WL heeft in totaal (op de locaties in Delft, 'De Voorst' en Haren) zo'n 550 medewerkers, waarvan ongeveer 40% universitair geschoold is. Mijn stage heb ik uitgevoerd bij de afdeling '*Havens, Kusten en Offshore technologie*'.
Als stagiair krijg je indien mogelijk een plaats op de betreffende afdeling toebedeeld, waardoor je omringd bent door de mensen die de kennis in huis hebben die voor jouw probleem van belang kan zijn. Je krijgt in elk geval

een PC tot je beschikking, maar voor zwaar rekenwerk zijn er ook nog andere mogelijkheden. In het werken aan je opdracht wordt je een enorme vrijheid gegund; de belangstelling en de grote bereidwilligheid van anderen je te helpen, geven je verder 'automatisch' de motivatie en drang in je onderzoek tot resultaten te komen. Terugkijkend is de stage aan het WL voor mij een interessante en leerzame periode geweest.

SAMENVATTING

Doel van deze studie is de fysische en numerieke modellering van tijdsafhankelijke compressibele stromingen met een vrij bewegend scheidingsvlak tussen gebieden van water en lucht, ten einde impact problemen te kunnen simuleren.

Voor dit soort problemen zijn visceuze effecten van zowel water en lucht verwaarloosbaar. De aanwezigheid van lucht, die meestal nauwelijks van invloed is op de beweging van vrije wateroppervlakken, is hier wel van belang: singulier gedrag van de stroming op het moment van impact wordt hiermee voorkomen.

Ten eerste is onderzocht in hoeverre de beweging van water en lucht kan worden beschreven als de beweging van een hypothetische vloeistof die zich bij lage waarden van de massadichtheid gedraagt als lucht, voor hoge waarden als water (met betrekking tot de samendrukbaarheid). Naar mijn mening is het echter onmogelijk een realistische toestandsvergelijking te formuleren voor deze vloeistof (slechts gebaseerd op de massadichtheid).

In de tweede plaats zijn daarom water en lucht apart beschouwd; dat wil zeggen, behalve de massadichtheid van het mengsel wordt nu ook de fractie lucht bepaald. Met behulp van deze extra informatie kan een fysisch betere toestandsvergelijking worden gegeven. Bovendien is nu massabehoud van zowel water als lucht gegarandeerd.

Vanuit numeriek oogpunt is één van de grootste problemen het vrijwel discontinue gedrag van grootheden over het scheidingsvlak tussen water en lucht. Ruimtelijk centraal discretiseren leidt tot onechte oscillaties in de numerieke oplossing (fenomeen van Gibbs). Waar nodig worden kunstmatig dissipatieve termen toegevoegd (op een behoudende manier), die deze oscillaties 'wegfilteren'. Hoewel het scheidingsvlak zo uitsmeert over een aantal roosterpunten, werkt deze 'interface-capturing' methode redelijk en lijkt bruikbaar voor simulatie van impact problemen.

Enkele één-dimensionale berekeningen zijn uitgevoerd en vergeleken met analytisch verkregen oplossingen.

SUMMARY

The object of this study is the physical and numerical modeling of time-dependent compressible flow with a moving free interface between regions of water and air, in order to simulate impact phenomena.
For this type of problem the effect of viscosity of both water and air is negligible. The presence of air, which often can be neglected for other free-surface problems, is very important for impact problems: it prevents singular behaviour of the flow at the moment of impact.
First, it is investigated to what extent the motion of water and air can be described by the motion of a hypothetical fluid, which for small values of its density behaves like air, for large values like water (with regard to its compressibility). However, in my opinion it is impossible to formulate a realistic equation of state for this fluid (merely based on its density).
Second, water and air are considered separately; that is, the fraction air in the mixture is considered besides its density. A better equation of state can be given using this information. Furthermore conservation of mass for both water and air are guaranteed.
From a numerical point of view, one of the main problems is the almost discontinuous behaviour of quantities at the interface between water and air. Central-space discretization leads to spurious oscillations in the numerical solution (Gibbs' phenomenon). Artificial dissipation terms are added to the equations when necessary (and in a conservative way), which 'filter out' these oscillations. Although the interface smears out over some grid cells, this method of interface-capturing works well and seems appropriate for the computation of impact phenomena.
Some (one-dimensional) computations are carried out and compared with theoretically obtained results.

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Chapter one : The physical properties of water and air

§ 1 Liquids and gases

When we want to describe the motion of water and air, it is important to be familiar with some elementary properties of these fluids. Aspects which are of importance to this study, are briefly resumed in this chapter. It is strongly based on the first chapter of **[3]**.

The first remark to be made, is that under the circumstances we are interested in, water is in the liquid phase, while air is in the gaseous phase. The distance between the molecules in a gas is about 10 times the distance between the molecules in a liquid.

The molecules in a gas are very weakly attracted to each other, they move almost independently -excluding the occasional collisions- and the importance of the potential energy is negligible compared with the kinetic energy for the contribution to the force field.

In liquids, on the other hand, the arrangement of the molecules is partially ordered. The motion of molecules (or groups of molecules) is much more determined by the strong force field of the neighbors. Consequences for the macroscopic mechanical behaviour of liquids and gases are:

- the mass density of liquids is much larger than the mass density of gases; in fact this distinction between the two phases is not very fundamental: the main difference is that for a given acceleration, the change in momentum for a liquid will be much larger than this change for a gas; the momentum of a gas is often negligible compared with the momentum of a liquid;

- gases can be compressed much more readily than liquids; so, for a given change in pressure, in the case of a gas the specific volume will change much more than in the case of a liquid; compressibility of a liquid is negligible compared with the compressibility of a gas.

So the motion of a fluid will strongly depend on its compressibility and its density. Besides these properties, the motion for example also may depend on the viscosity and the temperature of the fluid. However, for the type of flow that will be considered here (impact problems), we shall in the first instance neglect the contribution of viscous forces to the motion. This assumption seems acceptable (at least until the moment of impact).

First, water and air are not very viscous fluids. In a large area the convective terms will be large compared to the viscous terms in the equations of motion.

Second, the impact phenomenon is a very fast one: the time needed for the development of boundary layers (in which viscous terms are dominating) is too large.

We shall assume that changes in temperature and entropy are small. Then we can, for example assuming the flow behaves isothermal or isentropical, find a relation (the equation of state) between the pressure and the density of the fluid. This means the motion of the fluid is determined (given external forces, initial and boundary conditions) by the continuity equation and the momentum equations, since the pressure can be replaced by some function of the fluid density.

§ 2 Equations of state for water and air

Volume changes which take place under the influence of longitudinal waves at ordinary frequencies are adiabatic, not isothermal. This can be seen by examining the properties of ordinary matter ([2], chapter 5, section 7).

Therefore the use of an equation of state which describes isentropic (reversible and adiabatic) changes might be most realistic. An accurate isentropic relation between the density of water ρ_w and the pressure p_w reads ([3], chapter 1, section 8, equation (1.8.1))

$$p_w(\rho_w) = p_{w0}\left[(1+B)(\rho_w/\rho_{w0})^n - B\right] , \qquad (1.2.a)$$

where ρ_{w0} is the density of water at atmospheric pressure p_{w0} and the parameters are chosen n=7, B=3000. At 15 °C one can take ρ_{w0}= 999.1 kg/m^3, while $p_{w0}=1.013\cdot10^5$ Pa. This relation only holds for pure water.
Assuming air behaves like an ideal gas and changes are isentropic, the density ρ_a of air and the corresponding pressure p_a are related via

$$p_a(\rho_a) = p_{a0}(\rho_a/\rho_{a0})^{\gamma} , \qquad (1.2.b)$$

where at atmospheric pressure $p_{a0}=1.013\cdot10^5$ Pa and 15 °C the density of air is about ρ_{a0}= 1.226 kg/m^3. The parameter γ represents the ratio of the specific heats c_p and c_v. For air, γ=1.4 approximately.
In the following we shall write $p_0= p_{w0}= p_{a0}$ for the atmospheric pressure. When the densities ρ_w and ρ_a do not differ very much from ρ_{w0} and ρ_{a0} respectively, the linearized versions of (*1.2.a*) and (*1.2.b*) will be good enough for calculation of the pressure. The linearized equation for the pressure in water then becomes

$$p_w - p_0 = \left(\frac{n(1+B)p_0}{\rho_{w0}}\right)(\rho_w - \rho_{w0}) \simeq 1460^2(\rho_w - \rho_{w0}) . \qquad (1.2.c)$$

For the pressure in air we find

$$p_a - p_0 = \left(\frac{\gamma\, p_0}{\rho_{a0}}\right)(\rho_a - \rho_{a0}) \simeq 340^2(\rho_a - \rho_{a0}) . \qquad (1.2.d)$$

§ 3 Compressibility and the speed of sound

The speed with which pressure disturbances are propagated in a fluid is called the speed of sound *c*, which may be defined as

$$c = \sqrt{\left(\frac{\partial p}{\partial \rho}\right)_s} \tag{1.3.a}$$

where the subscript s says the partial derivative holds under isentropic conditions. In that case c is only a function of the density ρ. Applying this definition of the speed of sound to the equations *(1.2.c)* and *(1.2.d)*, we find for the speed of sound in water and air respectively $c_w \simeq 1460$ m/s and $c_a \simeq 340$ m/s.

The isentropic compressibility K of a fluid is defined as

$$K = \frac{1}{\rho}\left(\frac{\partial \rho}{\partial p}\right)_s = \frac{1}{\rho c^2} \; . \tag{1.3.b}$$

The compressibility of water and air are approximately $K_w \simeq 4.7 \cdot 10^{-10}$ Pa^{-1} and $K_a \simeq 7.1 \cdot 10^{-6}$ Pa^{-1}. As expected, it is much easier to compress air than water.

CHAPTER TWO : AN ATTEMPT TO MODEL THE FLOW

§ 1 Introduction

We want to describe the motion of water, air and the interface between these phases in a certain domain. When this domain is divided up into grid cells for numerical computation, there will be a number of cells containing both water and air. So although investigation of two-phase flow is not our aim, we have a problem here: it will be necessary to have an equation of state that can be used in these cells.

It has already been mentioned that compressibility and density of the fluid are important for its motion. In this chapter we will investigate whether the motion of both water and air (or perhaps more important: a liquid and a damp phase) can be treated by considering a hypothetical fluid with very variable density and compressibility. If an equation of state for this fluid can be found that is acceptable from a physical point of view, we might obtain a relatively simple model for the flow, which doesn't have the problem mentioned above. This idea will be worked out, the description of this idea can be found in **[1]**.

§ 2 Proposing a simple model

Consider a fluid which has variable mass density $\rho(\mathbf{x},t)$, where $\mathbf{x}$ denotes a position in space with for example Cartesian components x, y and z. The variable t denotes time.

The velocity of the fluid is given by the vector $\mathbf{u}(\mathbf{x},t)$, which may have Cartesian components $u=u(\mathbf{x},t)$, $v=v(\mathbf{x},t)$ and $w=w(\mathbf{x},t)$. The pressure in the fluid may also depend on position in space and time and will be denoted by $p=p(\mathbf{x},t)$.

The continuity equation, which expresses conservation of the mass of the fluid reads

$$\frac{\partial \rho}{\partial t} + \nabla.(\rho \mathbf{u}) = 0 \ . \tag{2.2.a}$$

For the present we shall assume that viscous effects are negligible, and thus the equations of motion reduce to the Euler equations

$$\frac{\partial (\rho \mathbf{u})}{\partial t} + \nabla.(\rho \mathbf{u}\mathbf{u}) = -\nabla p + \rho \mathbf{g} \ . \tag{2.2.b}$$

The vector $\mathbf{g}$ may represent some external force acting on the fluid, for example gravitation.
Furthermore we assume the existence of an equation of state that relates density and pressure in the fluid, so we may write

$$f(\rho, p) = 0 \tag{2.2.c}$$

where f is some known function relating ρ and p.
This relation should be adequate for both water and air, and this may be possible because the mass-densities of these fluids differ very much. When the pressure p can be resolved from *(2.2.c)*, then it can be eliminated from the equations of motion. Given some initial state of the fluid, its motion is then determined, the only unknowns ρ and $\mathbf{u}$ can be found from equations *(2.2.a)* and *(2.2.b)*.
We shall have to specify *(2.2.c)* now in a way that is physically relevant. In the first place it should be noted that the pressure in any fluid must increase when its density increases. Therefore we may assume the existence of a function g which satisfies $f(\rho, g(\rho))=0$ and is monotonously increasing, continuously differentiable, so we may write (inverse mapping theorem)

$$p = g(\rho) \quad \text{and} \quad \rho = g^{-1}(p) \ . \tag{2.2.d}$$

This defines a unique relation between density and pressure.
However, this relation also must represent the compressibility of air for low densities of the fluid (since g is an increasing function also for

small values of the pressure). For high density the fluid should represent water and thus have its compressibility. A simple equation of state therefore at least should satisfy

$$p(\rho) - p_s = \begin{cases} c_a^2(\rho-\rho_{a0}) \text{ when } \rho<\rho_{a0} \\ c_w^2(\rho-\rho_{w0}) \text{ when } \rho\geq\rho_{w0} \end{cases} \qquad (2.2.e)$$

where p_s is some reference pressure and the other variables have the same meaning as they had in the preceding chapter. The value of p_s is not very important: the motion is determined by pressure differences (see *(2.2.b)*), not by absolute values of the pressure.
Notice that *(2.2.e)* also can be written as

$$\rho(p) = \begin{cases} \rho_{a0} + (p-p_s)/c_a^2 \quad \text{when } p<p_s \\ \rho_{w0} + (p-p_s)/c_w^2 \quad \text{when } p\geq p_s \end{cases} \qquad (2.2.f)$$

By this definition it makes sense to define the interface between water and air as the set of points for which $p=p_s$, to define the area that contains air by the set of points for which $p<p_s$ and the area that contains water by the set of points for which $p>p_s$.

Now *(2.2.f)* has a discontinuity for $p=p_s$ and thereby does not satisfy our conditions. It can however be smoothened using the blending functions as described in **appendix I**. For example, when we define

$$B(x,\sigma) = \frac{1}{2}\left(1 + \frac{x}{|x|+\sigma}\right) \quad , \qquad (2.2.g)$$

relation *(2.2.f)* can be approximated by the following bijection

$$\rho = g^{-1}(p) = \left(1-B(p-p_s,\sigma)\right)\left(\rho_{a0} + \frac{p-p_s}{c_a^2}\right) + B(p-p_s,\sigma)\left(\rho_{w0} + \frac{p-p_s}{c_w^2}\right) \qquad (2.2.h)$$

and the smaller the parameter σ, the better this approximation will be. A physical demand is, as mentioned before, that *(2.2.h)* is a monotonously increasing mapping. This is true:

$$\frac{d\rho}{dp} = \frac{1-B}{c_a^2} + \frac{B}{c_w^2} + \left\{\left(\rho_{w0} + \frac{p-p_s}{c_w^2}\right) - \left(\rho_{a0} + \frac{p-p_s}{c_a^2}\right)\right\} \frac{dB}{dp} \qquad (2.2.i)$$

which is positive for any realistic value of the pressure p, since

- $0 \le B \le 1$ by definition,
- dB/dp is positive (B is monotonously increasing) and
- the coefficient of dB/dp is positive (for realistic values of p).

The requirement that *(2.2.h)* should be an increasing function may even be too weak: since *(2.2.i)* represents the inverse of the square of the speed of sound, we only claim this square be not negative (what is the meaning of a complex speed of sound?); it might be more realistic to assume it has a a certain minimum value. This will be discussed later on.

The equations *(2.2.a)*, *(2.2.b)* and *(2.2.h)* determine the flow of some hypothetical fluid with *(2.2.h)* the equation of state of this fluid. Now we have to find out whether this model is appropriate in describing the flow of our interest.

§ 3 Qualitative interpretation for water and air

The hypothetical fluid is interpreted as water when its density is large, or, according to the preceding section when $p>p_s$, and as air when its density is small, that is, when $p<p_s$. This means we can already draw the following conclusions with respect to the model:

- *the pressure in water is always higher than the pressure in air and there will be a force field caused by differences in density: water will tend to expand in the direction of areas containing air;*
- *areas, followed in a Lagrangian way, in which* $p-p_s$ *changes sign, implicate the violation of conservation of water and air, although the total conservation of mass (of the hypothetical fluid) is guaranteed (by equation (2.2.a));*
- *although the model takes the elasticity of air into account, pressure peaks will only be found in water; this is a consequence of the monotonic pressure-density relation.*

For impact problems the air between the water and the structure plays an important role. We can distinguish two cases:

1) a volume of air is locked up between the water and the structure and is not able to stream out; in this situation the pressure in the air may become high and may determine the force on the structure; considering the conclusions up here, the proposed model is not good enough to describe this flow;
2) the air between the water and the structure can stream out easily; although its motion may not be described very well, the air prevents the singular behaviour of the impact; since the most part of the impact force comes from the water, the model may still be good enough to deal with this type of flow.

These rather qualitative remarks already show the restrictions of the model. In the following sections we shall try to obtain more results by analyzing the one-dimensional case of the model.

§ 4 The model in one dimension

The one-dimensional version of the equations *(2.2.a)* and *(2.2.b)* reads, for the case no external forces are present,

$$\frac{\partial}{\partial t}(\rho\ ) + \frac{\partial}{\partial x}(\rho u) = 0 \quad , \tag{2.4.a}$$

$$\frac{\partial}{\partial t}(\rho u) + \frac{\partial}{\partial x}\left(p+\rho u^2\right) = 0 \quad . \tag{2.4.b}$$

These equations represent the conservation of mass and momentum. In general a system of differential-equations which represent conservation laws can be written as (in one dimension)

$$\frac{\partial}{\partial t}(\mathbf{w}) + \frac{\partial}{\partial x}\left(F(\mathbf{w})\right) = 0 \ . \tag{2.4.c}$$

Every component of the vector **w** integrated over a closed volume does not depend on time, or, is conserved. $F(\mathbf{w})$ is the flux-function which determines the motion of the conserved quantity inside the volume.
Notice that $\rho(x,t)$ and $u(x,t)$ which satisfy *(2.4.a)* and *(2.4.b)* are differentiable functions, which from a physical point of view is not necessary. Therefore it is more natural to write *(2.4.a)* and *(2.4.b)* in integral form, as a matter of fact these equations are derived from integral forms.
Integration of the equations in question leads to

$$\int_{x=x_1}^{x_2} \left(\rho(x,t_2)\right)dx \ - \int_{x=x_1}^{x_2} \left(\rho(x,t_1)\right)dx \ =$$

$$= \int_{t=t_1}^{t_2} \left(\rho(x_2,t)u(x_2,t)\right)dt \ - \int_{t=t_1}^{t_2} \left(\rho(x_1,t)u(x_1,t)\right)dt \tag{2.4.d}$$

and

$$\int_{x=x_1}^{x_2} \Big(\rho(x,t_2)u(x,t_2)\Big)dx \quad - \int_{x=x_1}^{x_2} \Big(\rho(x,t_1)u(x,t_1)\Big)dx \quad = \tag{2.4.e}$$

$$\int_{t=t_1}^{t_2} \Big(p(x_2,t) + \rho(x_2,t)u^2(x_2,t)\Big)dt - \int_{t=t_1}^{t_2} \Big(p(x_1,t) + \rho(x_1,t)u^2(x_1,t)\Big)dt \quad .$$

These equations are valid for any x_1, x_2, t_1 and t_2. Furthermore $\rho(x,t)$ and $u(x,t)$ can satisfy *(2.4.d)* and *(2.4.e)* without being differentiable or even continuous.

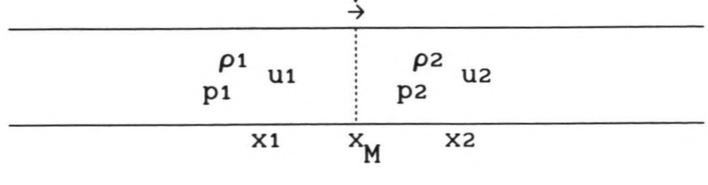
§ 5 Conditions for the existence of a discontinuity

By applying conservation laws to small volumes containing a discontinuity, conditions for the existence of such a discontinuity can be found. These conditions are the well-known Rankine-Hugoniot relations and will be derived in this section. Consider the following initial state:

$$\rho(x,0) = \rho_1 \quad \text{for} \quad x < x_M \quad ,$$
$$\rho(x,0) = \rho_2 \quad \text{for} \quad x > x_M \quad ,$$
$$u(x,0) = u_1 \quad \text{for} \quad x < x_M \quad ,$$
$$u(x,0) = u_2 \quad \text{for} \quad x > x_M$$

and let

$$p(\rho_1) = p_1 \quad \text{and} \quad p(\rho_2) = p_2 \quad .$$



Thus we have a discontinuity at $x = x_M$. Assume that this discontinuity moves with (unknown) speed v. Conditions for the existence of such a discontinuity can be found by applying the conservation laws *(2.4.d)* and *(2.4.e)*.

Therefore let $x_1 < x_M$, $x_2 > x_M$, $t_1 = 0$ and $t_2 = \Delta t$. For small Δt equation *(2.4.d)* then reduces to

$$(\rho_2 - \rho_1)\cdot v\cdot\Delta t = (\rho_2\cdot u_2 - \rho_1\cdot u_1)\cdot\Delta t$$

or, the discontinuity travels with speed

$$v = \frac{\rho_2\cdot u_2 - \rho_1\cdot u_1}{\rho_2 - \rho_1} \quad . \qquad (2.5.a)$$

Equation *(2.4.e)* yields

$$(\rho_2\cdot u_2 - \rho_1\cdot u_1)\cdot v\cdot\Delta t = \left(p_2 + \rho_2\cdot(u_2)^2 - p_1 - \rho_1\cdot(u_1)^2\right)\cdot\Delta t \quad . \qquad (2.5.b)$$

Substitution of *(2.5.a)* leads us after some rewriting to the following result:

$$(u_2 - u_1)^2 = \frac{(p_2 - p_1)(\rho_2 - \rho_1)}{\rho_1\cdot\rho_2} \quad . \qquad (2.5.c)$$

This relation will have to be satisfied crossing any discontinuity. When this discontinuity is a contact-discontinuity, for example the interface between water and air, we have $\rho_1 \neq \rho_2$, $p_1 \simeq p_2$ (neglecting surface tension) and therefore $u_1 \simeq u_2 \simeq v$.

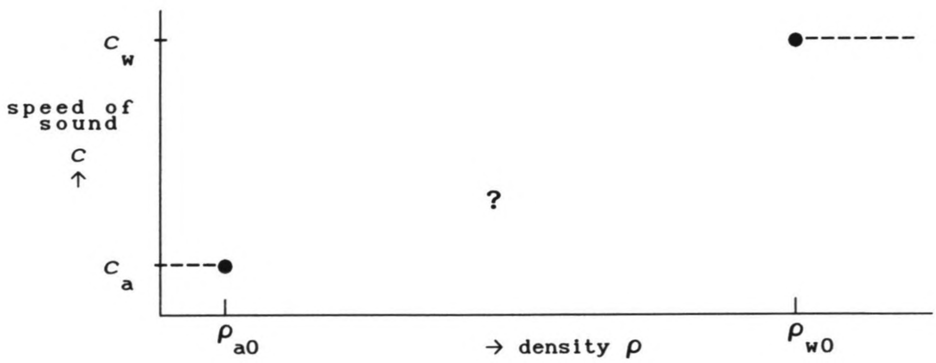
§ 6 Implications for the equation of state

The result of the preceding section has consequences for the choice of the equation of state.

Choosing a pressure-density relation is in fact nothing else than choosing the sound speed c (or the compressibility) as a function of the density. As mentioned in § 2, a realistic relation should obey

$$c(\rho) = c_a \,(\simeq\ 340\text{ m/s}) \quad \text{for } \rho<\rho_{a0}\ ,$$
$$c(\rho) = c_w \,(\simeq\ 1460\text{ m/s}) \quad \text{for } \rho>\rho_{w0}$$

and choosing a pressure-density relation is in fact the same as connecting the points (ρ_{a0}, c_a) and (ρ_{w0}, c_w) in the (ρ,c)-plane in a convenient way.



The difference in pressure between points with density ρ_{a0} and ρ_{w0} is given by

$$p(\rho_{w0}) - p(\rho_{a0}) = \int_{\rho=\rho_{a0}}^{\rho_{w0}} \frac{dp}{d\rho}\, d\rho = \int_{\rho=\rho_{a0}}^{\rho_{w0}} c^2(\rho) d\rho\ . \qquad (2.6.a)$$

We shall distinguish two characteristic cases now.

Case I:

According to the result of the preceding section the left-hand side of *(2.6.a)* should vanish, therefore it is necessary to choose the speed of sound

$$c \simeq 0 \text{ m/s} \quad \text{for} \quad \rho_{a0} < \rho < \rho_{w0} \; .$$

This means the pressure will be about constant for $\rho_{a0} < \rho < \rho_{w0}$. In that range equation *(2.4.b)* therefore can approximately be written as

$$\frac{\partial}{\partial t}(\rho u) + \frac{\partial}{\partial x}\left(\rho u^2\right) =$$

$$= \rho\frac{\partial}{\partial t}(u) + u\frac{\partial}{\partial t}(\rho) + u\frac{\partial}{\partial x}(\rho u) + (\rho u)\frac{\partial}{\partial x}(u) =$$

$$= u\Big(\frac{\partial}{\partial t}(\rho) + \frac{\partial}{\partial x}(\rho u)\Big) + \rho\Big(\frac{\partial}{\partial t}(u) + u \frac{\partial}{\partial x}(u)\Big) = 0 \; .$$

So the equations of motion in this area become

$$\frac{\partial}{\partial t}(\rho) + \frac{\partial}{\partial x}(\rho u) = 0 \quad , \qquad (2.6.b)$$

$$\frac{\partial}{\partial t}(u) + u\frac{\partial}{\partial x}(u) = 0 \quad . \qquad (2.6.c)$$

Equation *(2.6.c)* is known as Burgers' equation. When the velocity u is constant it becomes a triviality and we are merely solving the continuity equation. Note that the flow, as a consequence of the extreme low speed of sound, becomes supersonic very easily in the areas of question.

Case II:

Now consider the case in which we want to have some more realistic (?) values for the speed of sound. For example, let it be greater than a minimum-value c_{min} for all ρ with $\rho_{a0} < \rho < \rho_{w0}$. Applying equation *(2.6.a)* yields

$$p(\rho_{w0}) - p(\rho_{a0}) = \int_{\rho=\rho_{a0}}^{\rho_{w0}} \frac{dp}{d\rho}\, d\rho \;>\; c^2_{min}(\rho_{w0}-\rho_{a0}) \;. \qquad (2.6.d)$$

This means that the difference in pressure between points with density ρ_{a0} and ρ_{w0}, when for example c_{min} = 10 m/s is chosen, is already somewhere about 10^5 Pa! (In fact it is much more, because the substituted value c_{min} for c in *(2.6.d)* yields an underestimation.)
Now consider the situation of water surrounded by air. The change of momentum of the fluid is given by

$$\frac{\partial}{\partial t}(\rho u) = - \frac{\partial}{\partial x}(p) - \frac{\partial}{\partial x}\left(\rho u^2\right) =$$

$$= - \frac{\partial}{\partial x}\left(\rho c^2\right) - \frac{\partial}{\partial x}\left(\rho u^2\right) = - \frac{\partial}{\partial x}\left(\rho (c^2+u^2)\right) \qquad (2.6.e)$$

The magnitude of the terms on the right-hand side is now estimated for the case we are crossing the water-air surface:

$$- \frac{\partial}{\partial x}(p) \simeq \frac{10^5}{\Delta x}$$

according to the foregoing, while for a typical value of the fluid-velocity of 5 m/s,

$$- \frac{\partial}{\partial x}\left(\rho u^2\right) \simeq 1000\cdot 5^2/(\Delta x) = 0.25\, \frac{10^5}{\Delta x} \;,$$

where the momentum of air is neglected. It is seen that in this case the change of momentum is mostly caused by the density-gradient.
This means that the development of the flow at the surface in time is determined by a term which is physically completely irrelevant.

§ 7 Conclusions

The consequences of the preceding are far-reaching, because both above discussed cases are unacceptable (from my point of view). The implications that follow from the discussion in § 6 to me are:

a) If we want to capture the interface between water and air in a realistic way, it is necessary that the speed of sound is very small over a wide density-range. If this not is the case, a volume of water surrounded by air will 'explode'.

b) A very small speed of sound implies the flow will be supersonic. The momentum-equation is replaced by Burgers' equation in that case. 'Information' then travels with the speed of the fluid and there will be no possibility of 'a wall feeling a mass of water coming nearer'. Moreover, occurrence of shocks is possible; these shocks however do not have any physical meaning in our situation.

Chapter Three : Improving the Model

§ 1 Introduction

The problems we had in the preceding were mainly due to the impossibility of formulating a good equation of state. We merely used the density of the 'air-water mixture' to formulate this equation. A more realistic equation of state can probably be found when we have more information (except this density), for example when we know the volume-fraction or mass-fraction of air in the mixture. In this chapter a proposition is made for an improved model. This means we introduce a two-phase flow in its most simple form.

§ 2 Some elementary definitions

Consider a small volume δV which contains water and air. Assume that a volume-fraction α consists of air and consequently a fraction $1-\alpha$ that consists of water. The total mass of air in δV is denoted by M_a , the total mass of water by M_w . Now define

$$\rho_a = \frac{M_a}{\alpha \cdot \delta V} \quad \text{for } 0<\alpha\leq 1, \tag{3.2.a}$$

which is the density of air with respect to the volume $\alpha \cdot \delta V$. Similarly we write

$$\rho_w = \frac{M_w}{(1-\alpha)\delta V} \quad \text{for } 0\leq\alpha<1, \tag{3.2.b}$$

which is the density of water with respect to the volume $(1-\alpha)\delta V$. It makes sense to define an average mass-density ρ in the volume δV by

$$\rho = \begin{cases} \rho_w & , \ \alpha=0 \\ \alpha\rho_a + (1-\alpha)\rho_w & , \ 0<\alpha<1 \\ \rho_a & , \ \alpha=1 \end{cases} \tag{3.2.c}$$

since substitution of *(3.2.a)* and *(3.2.b)* shows that ρ is equal to the quotient of the total mass M_a+M_w and the volume δV.
Using equation *(3.2.c)* we find

$$\alpha = \frac{\rho - \rho_w}{\rho_a - \rho_w} \quad , \ 0\le\alpha\le1 \ . \tag{3.2.d}$$

Besides the volume-fraction air α in the mixture it also may be useful to define the mass-fraction air k in the mixture

$$k = \frac{M_a}{M_a + M_w} \quad , \ 0\le k\le1 \ . \tag{3.2.e}$$

Using the foregoing equations we can also write

$$k = \frac{\alpha\rho_a}{\rho} \quad \text{and} \quad 1-k = \frac{(1-\alpha)\rho_w}{\rho} \ . \tag{3.2.f}$$

Replacing α by the right-hand side of *(3.2.d)* and some rewriting yields

$$\frac{1}{\rho} = \frac{k}{\rho_a} + \frac{1-k}{\rho_w} \ . \tag{3.2.g}$$

§ 3 The equations of motion

The flow considered here consists of two fields (water and air) which are assumed to be inviscid. These fields communicate in two ways: they cannot occupy the same volume at the same time (i.e. $\alpha+(1-\alpha)=1$), and through the

pressure terms. One may assume both water and air have its own velocity field; this assumption is essential when we for example want to describe small (on sub-cell scale) air-bubbles rising in water. However, the modelling of this type of flow becomes very complicated; for example, it is well-known that the velocity of rising air-bubbles eventually becomes constant: one should investigate the exchange of momentum of the fields caused by viscosity to model this phenomenon. Such specific problems however are beyond the scope of this report, in which we assume the fluids are inviscid. Therefore, our model will be based on the hypothesis that there should be an equation of motion for the mixture, like we in fact did in the preceding chapter. The difference will be that we shall require conservation of mass for water and air separately. This will enable us to formulate a more realistic equation of state.
Referring to the preceding section, we notice that the density of air in the mixture is $\alpha \cdot \rho_a$ and not ρ_a, so when $\mathbf{u}$ is the velocity of the mixture, conservation of the mass of air requires

$$\frac{\partial}{\partial t}\left(\alpha\rho_a\right) + \nabla.\left(\alpha\rho_a\mathbf{u}\right) = 0 \qquad (3.3.a)$$

and in the same way the conservation of the mass of water can be written as

$$\frac{\partial}{\partial t}\left((1-\alpha)\rho_w\right) + \nabla.\left((1-\alpha)\rho_w\mathbf{u}\right) = 0 \ . \qquad (3.3.b)$$

These equations can be written in terms of k and ρ (instead of α, ρ_a and ρ_w) by means of equations *(3.2.f)*, we then have

$$\frac{\partial}{\partial t}\left(k\rho\right) + \nabla.\left(k\rho\mathbf{u}\right) = 0 \qquad (3.3.c)$$

and

$$\frac{\partial}{\partial t}\left((1-k)\rho\right) + \nabla.\left((1-k)\rho\mathbf{u}\right) = 0 \ . \qquad (3.3.d)$$

Adding these equations, it follows that

$$\frac{\partial \rho}{\partial t} + \nabla.(\rho \mathbf{u}) = 0 \,, \tag{3.3.e}$$

which states that the total mass of the mixture is conserved. It is clear that any combination of two of the three equations *(3.3.c)*, *(3.3.d)* and *(3.3.e)* guarantees the conservation of mass for both water and air.

It may be imagined that the fact that both water and air have the same velocity field, implicates that the mass fraction k of air in the mixture is constant when we move along with the fluid. This is so indeed, since k satisfies

$$\frac{Dk}{Dt} = \frac{\partial k}{\partial t} + \mathbf{u}.\nabla k =$$

$$= \frac{\partial}{\partial t}\left(\frac{\alpha\rho_a}{\rho}\right) + \mathbf{u}.\nabla\left(\frac{\alpha\rho_a}{\rho}\right) =$$

$$= \frac{1}{\rho}\frac{\partial}{\partial t}\left(\alpha\rho_a\right) - \left(\frac{\alpha\rho_a}{\rho^2}\right)\frac{\partial\rho}{\partial t} + \left(\frac{\mathbf{u}}{\rho}\right).\nabla\left(\alpha\rho_a\right) - \left(\frac{\alpha\rho_a\mathbf{u}}{\rho^2}\right).\nabla\rho =$$

$$= \frac{1}{\rho}\left(\frac{\partial}{\partial t}\left(\alpha\rho_a\right) + \mathbf{u}.\nabla\left(\alpha\rho_a\right)\right) - \left(\frac{\alpha\rho_a}{\rho^2}\right)\left(\frac{\partial\rho}{\partial t} + \mathbf{u}.\nabla\rho\right) =$$

$$= \frac{1}{\rho}\left(\frac{\partial}{\partial t}\left(\alpha\rho_a\right) + \mathbf{u}.\nabla\left(\alpha\rho_a\right) + \alpha\rho_a(\nabla.\mathbf{u})\right) - \left(\frac{\alpha\rho_a}{\rho^2}\right)\left(\frac{\partial\rho}{\partial t} + \mathbf{u}.\nabla\rho + \rho(\nabla.\mathbf{u})\right) =$$

$$= \frac{1}{\rho}\left(\frac{\partial}{\partial t}\left(\alpha\rho_a\right) + \nabla.\left(\alpha\rho_a\mathbf{u}\right)\right) - \left(\frac{\alpha\rho_a}{\rho^2}\right)\left(\frac{\partial\rho}{\partial t} + \nabla.(\rho\mathbf{u})\right) = 0 \,. \tag{3.3.f}$$

So, following the motion of the fluid, k is constant.

As mentioned up here, we assume the existence of an equation of motion for the mixture, that is, we require

$$\frac{\partial}{\partial t}(\rho \mathbf{u}) + \nabla.(\rho \mathbf{u}\mathbf{u}) = -\nabla p + \rho \mathbf{g} \qquad (3.3.g)$$

where p is the pressure in the mixture and **g** represents possible external forces. A convenient equation of state has to be found , from which p can be determined.

§ 4 An equation of state for the mixture

We now assume both ρ and k are known, so it would be desirable to have a function of these variables, which represents the pressure in the mixture. Assume we have equations of state for water and air, for example the linearized equations *(1.2.c)* and *(1.2.d)*. Resuming, we have

$$p_w(\rho_w) - p_0 = c_w^2(\rho_w - \rho_{w0}) \qquad (3.4.a)$$

and

$$p_a(\rho_a) - p_0 = c_a^2(\rho_a - \rho_{a0}) \ . \qquad (3.4.b)$$

When we consider a water-air mixture, the pressure p in a certain point in space is still unique. To determine this pressure we should first discuss what we mean by a water-air mixture.

First, we shall assume both fields (water and air) are not mixing on a molecular level, as would two gases. So the scale of the regions occupied by one field is coarser than the molecular. In a way this justifies the use of equations *(3.4.a)* and *(3.4.b)*.

Second, effects of surface tension will be neglected. So we shall assume the pressure p in the mixture is continuous across any interface between the two fields.

Therefore we should require the following 'interface condition':

$$p = p_a(\rho_a) = p_w(\rho_w) \ . \qquad (3.4.c)$$

This means ρ_a and ρ_w should satisfy

$$c_a^2(\rho_a - \rho_{a0}) = c_w^2(\rho_w - \rho_{w0}) \ . \qquad (3.4.d)$$

Furthermore we have

$$\frac{1}{\rho} = \frac{k}{\rho_a} + \frac{1-k}{\rho_w} \ . \qquad (3.4.e)$$

The result is obvious now. When ρ and k are known, ρ_a and ρ_w can be solved uniquely from equations *(3.4.d)* and *(3.4.e)*. The pressure p then is found by means of equation *(3.4.a)* or *(3.4.b)*. The resulting equation for the pressure will be derived now.

Equations *(3.4.a)* and *(3.4.b)* respectively state

$$\rho_w = \rho_{w0} + \frac{p-p_0}{c_w^2} \quad \text{and} \quad \rho_a = \rho_{a0} + \frac{p-p_0}{c_a^2} \ .$$

Substitution of these equations in *(3.4.e)* yields after some rewriting

$$(p-p_0)^2 + \beta(\rho,k)(p-p_0) + \gamma(\rho,k) = 0 \qquad (3.4.f)$$

$$\text{where} \quad \beta(\rho,k) = \rho_{a0}c_a^2 + \rho_{w0}c_w^2 - \rho\left(kc_a^2 + (1-k)c_w^2\right)$$

$$\text{and} \quad \gamma(\rho,k) = c_a^2c_w^2\left(\rho_{a0}\rho_{w0} - \rho\left(k\rho_{w0} + (1-k)\rho_{a0}\right)\right) \ .$$

This equation has two solutions for $p-p_0$. However, only one of these is physically correct.

Define the set $\mathcal{P}_0$ by

$$\mathcal{P}_0 = \left\{ (\rho,k) \in \mathbb{R}^+ x[0,1] \,\middle|\, \frac{1}{\rho} = \frac{k}{\rho_{a0}} + \frac{1-k}{\rho_{w0}} \,,\; p_a(\rho_{a0}) = p_w(\rho_{w0}) = p_0 \right\}. \qquad (3.4.g)$$

Then the physically correct solution is determined by the necessity that

$$(\rho_0,k_0) \in \mathcal{P}_0 \quad \Leftrightarrow \quad p(\rho_0,k_0) = p_0 \quad .$$

Using the equations *(3.2.f)* we can gain more insight in the behaviour of the coefficients β and γ in *(3.4.f)*. We find for $(\rho_0,k_0) \in \mathcal{P}_0$,

$$\beta(\rho_0,k_0) = (1-\alpha)\rho_{a0}c_a^2 + \alpha\rho_{w0}c_w^2 \quad \text{and} \quad \gamma(\rho_0,k_0) = 0 \;, \qquad (3.4.h)$$

reminding α is the volume fraction of air in the mixture, $0 \le \alpha \le 1$. (From now on the subscript o express the restriction $p=p_0$, so we shall write β_0 instead of $\beta(\rho_0,k_0)$ et cetera.)
Obviously the only relevant solution to equation *(3.4.f)* reads

$$p(\rho,k) - p_0 = \frac{1}{2}\left(-\beta(\rho,k) + \sqrt{\beta^2(\rho,k) - 4\gamma(\rho,k)}\right) \;. \qquad (3.4.i)$$

Chapter four : Mathematical analysis in one dimension

§ 1 The resulting system in one dimension

The system of equations which describe the motion of the mixture are given by equations *(3.3.c)*, *(3.3.e)* and *(3.3.g)* and read in the one-dimensional case

$$\frac{\partial}{\partial t}(\rho) + \frac{\partial}{\partial x}(\rho u) = 0 ,$$

$$\frac{\partial}{\partial t}(\rho k) + \frac{\partial}{\partial x}(\rho k u) = 0 \text{ and} \qquad (4.1.a)$$

$$\frac{\partial}{\partial t}(\rho u) + \frac{\partial}{\partial x}(p + \rho u^2) = \rho g ,$$

where $p = p(\rho,k)$ and $g = g(x)$ a force along the x-axis. This system can also be given by the equations

$$\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + \rho\frac{\partial u}{\partial x} = 0 ,$$

$$\frac{\partial k}{\partial t} + u\frac{\partial k}{\partial x} = 0 \text{ and} \qquad (4.1.b)$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{1}{\rho}\frac{\partial p}{\partial x} = g .$$

Now define $\mathbf{w} = \begin{pmatrix} \rho \\ k \\ u \end{pmatrix}$ and $\mathbf{g} = \begin{pmatrix} 0 \\ 0 \\ g \end{pmatrix}$. Then the equations can be written as

$$\frac{\partial \mathbf{w}}{\partial t} + \mathbf{A}\frac{\partial \mathbf{w}}{\partial x} = \mathbf{g} \qquad (4.1.c)$$

where $$\mathbf{A} = \begin{pmatrix} u & 0 & \rho \\ 0 & u & 0 \\ (\partial p/\partial \rho)/\rho & (\partial p/\partial k)/\rho & u \end{pmatrix} . \qquad (4.1.d)$$

The eigenvalues λ of matrix $\mathbf{A}$ follow from

$$\mathrm{DET}(\lambda\mathbf{I}-\mathbf{A}) = (\lambda-u)\Big((\lambda-u)^2 - \partial p/\partial\rho\Big) = 0 \ .$$

We find

$$\lambda = u, \quad \lambda = u-\sqrt{\partial p/\partial\rho} \quad \text{and} \quad \lambda = u+\sqrt{\partial p/\partial\rho} \ , \tag{4.1.e}$$

thus the system is strictly hyperbolic provided $\partial p/\partial\rho>0$.

§ 2 The speed of sound of the mixture

The behaviour of solution *(3.4.i)* for a small disturbance $\delta\rho$ in the density, resulting in a pressure disturbance $\delta p = p(\rho_0+\delta\rho,k_0) - p_0$, can be found by approximating *(3.4.i)* round (ρ_0,k_0); Taylor expansion yields

$$\delta p = \frac{1}{2}\left\{\left(-\beta_0+\sqrt{\beta_0^2-4\gamma_0}\right)+\left(-\left(\frac{\partial\beta}{\partial\rho}\right)_0 + \frac{2\beta_0\left(\frac{\partial\beta}{\partial\rho}\right)_0 - 4\left(\frac{\partial\gamma}{\partial\rho}\right)_0}{2\sqrt{\beta_0^2-4\gamma_0}}\right)\delta\rho\right\} + O(\delta\rho^2)$$

which, since $\gamma_0= 0$, reduces to

$$\delta p = -\left(\frac{(\partial\gamma/\partial\rho)_0}{\beta_0}\right)\delta\rho + O(\delta\rho^2) \ . \tag{4.2.a}$$

Defining $c_0^2 = \delta p/\delta\rho$, we can write, neglecting second and higher order terms in $\delta\rho$,

$$\frac{1}{c_0^2} = \frac{\rho_{a0}c_a^2 + \rho_{w0}c_w^2 - \rho_0\left(k_0c_a^2 + (1-k_0)c_w^2\right)}{c_a^2c_w^2\left(k_0\rho_{w0} + (1-k_0)\rho_{a0}\right)}$$

and some rewriting of this relation (using equation *(3.2.g)*) yields the nice expression

$$\frac{1}{(\rho_0c_0)^2} = \frac{k_0}{(\rho_{a0}c_a)^2} + \frac{1-k_0}{(\rho_{w0}c_w)^2} \qquad (4.2.b)$$

Some combinations of values are shown in the table down here. The variable c_0 may be interpreted as the speed of sound in the mixture for $(\rho_0, k_0) \in \mathcal{P}_0$ (which is obvious considering the eigenvalues of the system of equations found in the preceding section).

k_0	ρ_0 [kg/m^3]	c_0 [m/sec]
0	999.10	1460.00
10^{-5}	991.06	132.89
10^{-4}	924.13	45.23
10^{-3}	551.60	23.97
10^{-2}	109.64	38.14
10^{-1}	12.17	108.71
1	1.23	340.00

For the derivation of equation *(4.2.b)*, we made use of the fact that $(\rho_0, k_0) \in \mathcal{P}_0$ (since we used $\gamma_0=0$). However, equation *(4.2.b)* is valid in general, which can be seen from the fact that the choice of p_0 (in definition *(3.4.g)*) is an arbitrary one. So the subscript o in *(4.2.b)* may be omitted. Furthermore equation *(4.2.b)* can also be derived as follows.
The density ρ may be regarded as a function of p and k. Defining c^2 as the

partial derivative of p with respect to ρ (thus fixing k), we may write

$$\frac{1}{c^2} = \frac{\partial \rho}{\partial p} = -\rho^2 \frac{\partial}{\partial p}\left(\frac{1}{\rho} \right)$$

which by equation *(3.4.e)*, *(3.4.a)* and *(3.4.b)* becomes

$$-\rho^2\left(\frac{d\rho_a}{dp}\frac{\partial}{\partial \rho_a}\left(\frac{1}{\rho}\right) + \frac{d\rho_w}{dp}\frac{\partial}{\partial \rho_w}\left(\frac{1}{\rho}\right)\right) = -\rho^2\left(\frac{1}{c_a^2}\cdot\frac{-k}{\rho_a^2} - \frac{1}{c_w^2}\cdot\frac{1-k}{\rho_w^2}\right)$$

and thus leads to (compare with *(4.2.b)*)

$$\frac{1}{(\rho c)^2} = \frac{k}{(\rho_a c_a)^2} + \frac{1-k}{(\rho_w c_w)^2} \quad . \qquad (4.2.c)$$

This equation is also given in **[4]**.
Using equations *(3.2.f)* and recalling the definition of the compressibility of a fluid (see equation *(1.3.b)*) this result can be written in its most simple form:

$$K = \alpha \cdot K_a + (1-\alpha) \cdot K_w \quad . \qquad (4.2.d)$$

So the compressibility K of the mixture depends linearly on α, the volume-fraction of air, and is bounded by the compressibility K_a of air and the compressibility K_w of water, $K_a \le K \le K_w$.

A relation for the speed of sound c in terms of the volume-fraction air α, is given by van Wijngaarden in **[5]**. This relation and its equivalence with *(4.2.c)* is shown in **appendix II**.

When we take $\rho_a \simeq \rho_{a0}$ and $\rho_w \simeq \rho_{w0}$ we can give c as a function of k. In that case we find c has a minimum value for $k \simeq \rho_{ao}/\rho_{Wo}$. The speed of sound c for that value of k is about 24 m/sec . **FIGURE I** shows a plot of the speed of sound for different values of the pressure p.

From *(4.2.c)* it can be seen that $c = \sqrt{\partial p/\partial \rho}$ is strictly positive, and therefore the system of equations *(4.1.c)* is strictly hyperbolic. The non-linearity of the system is not only caused by the term $u\frac{\partial u}{\partial x}$ in the momentum equation, but also due to the strong variation of c.

§ 3 Stationary solution

The existence of a stationary solution to the equations *(4.1.b)*, where g represents gravity, is obvious from a physical point of view. The mathematical condition for the existence of this solution reads

$$\frac{dp}{dx} = \rho g \qquad (4.3.a)$$

which seems to be a simple equation at first sight. However, ρ is given by *(3.2.c)*, so we get

$$\frac{dp}{dx} = \Big(\alpha(x)\rho_a(x) + (1-\alpha(x))\rho_w(x)\Big)g$$

where $\rho_a(x)$ and $\rho_w(x)$ may be replaced using equations *(3.4.a)* and *(3.4.b)*. This leads to

$$\frac{dp}{dx} = \left\{\alpha(x)\left(\rho_{a0} + \frac{p-p_0}{c_a^2}\right) + (1-\alpha(x))\left(\rho_{w0} + \frac{p-p_0}{c_w^2}\right)\right\}g \ . \qquad (4.3.b)$$

To be able to solve this differential problem, it is necessary that the distribution $\alpha(x)$ of the air-fraction is known. When α is constant, the right-hand side of *(4.3.b)* does not depend on x, and one can easily find an exponential solution for p.
However, it would be interesting to take a blending function (**appendix I**) for α , and thus having a part containing water, a part containing air,

smoothly separated. I have not been able to solve this problem, although it seems interesting to me. Of course numerical solution of equation *(4.3.b)* is straightforward for a given distribution $\alpha(x)$.

CHAPTER FIVE : NUMERICAL TREATMENT

§ 1 Introduction

The numerical integration of the system of equations *(4.1.a)* will be based on the method that is presented in **[6]**. This means a finite-volume method will be used for space-integration, while time-integration is carried out using an explicit Runge-Kutta scheme. However, the integration here is much simpler, mainly because:

- the viscous terms are neglected here;
- no energy-equation is formulated;
- only one-dimensional problems will be considered (we don't have to calculate complex geometric quantities) and the distance between the grid-points will be uniform.

Despite these simplifications, straightforward integration of the equations will not be possible. The following points may cause difficulties and require a special treatment:

- at the interface between water and air very steep gradients of some quantities (density, fraction air) occur;
- small errors in the density of water give large errors in the pressure, since water is almost incompressible (in comparison with air);
- in a numerical computation values $k<0$ or $k>1$ may be found, which is not allowed from a physical point of view (k is the mass-fraction air in the mixture);
- boundary conditions cannot be imposed (or will lead to spurious oscillations), since the equations are hyperbolic.

It will be seen that in fact all above problems can be overcome, adding dissipation terms to the equations. For the interpretation of solutions, we have to keep in mind these terms are artificial in some sense.

Remark: in the following we shall write $w_k^n=w(x=x_k,t=t_n)$, $w_k=w(x=x_k,t)$ and $w^n=w(x,t=t_n)$ where $x_k=k\cdot\Delta x$, $t_n=n\cdot\Delta t$, Δx and Δt fixed steps in respectively space and time, and $k,n \in \mathbb{Z}$.

§ 2 Space-integration

In this section the differential-equations we consider,

$$\begin{aligned}
\frac{\partial}{\partial t}\left(\rho\right) + \frac{\partial}{\partial x}\left(\rho u\right) &= 0 \ , \\
\frac{\partial}{\partial t}\left(k\rho\right) + \frac{\partial}{\partial x}\left(k\rho u\right) &= 0 \ , \\
\frac{\partial}{\partial t}\left(\rho u\right) + \frac{\partial}{\partial x}\left(p+\rho u^2\right) &= \rho g \ .
\end{aligned} \qquad (5.2.a)$$

will be discretized. Since the flow is one-dimensional, it can be interpreted as the motion of a mixture of water and air in a (frictionless) tube, with a possible force acting along it. The physical domain is the interval

$$\mathfrak{P} = [0,L] \ .$$

The set of points that is used for numerical integration is given by

$$\mathfrak{D} = \{\ x_k=k\Delta x \mid k=-2,-1,0,1,\ldots,N-1,N,N+1,N+2;\ \Delta x=L/N\ \} \ .$$

For each point in $\mathfrak{D}$, except the virtual points x_{-2}, x_{-1}, x_{N+1} and x_{N+2} a 'control volume' is introduced now, namely

$$\Omega_k = [x_{k-1},\ x_{k+1}] \ ,\ k=0,1,\ldots,N-1,N.$$

The equations that have to be integrated have the form

$$\frac{\partial w}{\partial t} + \frac{\partial F}{\partial x} = G. \tag{5.2.b}$$

Space-integration of this equation yields

$$\int_{\Omega_k} \left(\frac{\partial}{\partial t}w(x,t) - G(x,t)\right)dx + F(x_{k+1},t) - F(x_{k-1},t) = 0 \tag{5.2.c}$$

and this can be approximated by

$$2\Delta x\left(\frac{d}{dt}w(x_k,t) - G(x_k,t)\right) + F(x_{k+1},t) - F(x_{k-1},t) - D_k = 0 \tag{5.2.d}$$

where D_k represents artificial dissipation. This dissipation is necessary because the Euler-equations do not provide any natural dissipation-mechanism and it is necessary to capture 'almost-discontinuities' (see also the remarks in the preceding section and **appendix III**) . This dissipation is added in the following form:

$$D_k = (D_x)w_k = d_{k+1/2} - d_{k-1/2} \tag{5.2.e}$$

where

$$d_{k+1/2} = S^{(2)}_{k+1/2}\varepsilon^{(2)}_{k+1/2}\left(w_{k+1} - w_k\right) -$$

$$S^{(4)}_{k+1/2}\ \varepsilon^{(4)}_{k+1/2}\left(w_{k+2} - 3w_{k+1} + 3w_k - w_{k-1}\right) . \tag{5.2.f}$$

The factors S are scaling factors, while the variables ε are 'sensors' which detect the need for dissipation. Note that *(5.2.e)* is the discretized form of

$$\Delta x \frac{\partial}{\partial x} \left\{ S^{(2)} \varepsilon^{(2)} \Delta x \frac{\partial w}{\partial x} - S^{(4)} \varepsilon^{(4)} \Delta x^3 \frac{\partial^3 w}{\partial x^3} \right\} . \qquad (5.2.g)$$

The parameter $\varepsilon^{(2)}$ is chosen in the following way:

$$\varepsilon^{(2)}_{k+1/2} = \kappa^{(2)} \max\left(\nu_k, \nu_{k+1} \right) \qquad (5.2.h)$$

where ν_k in the ISNaS-project represents a 'shock sensor'. In our case there is a discontinuity in the mass-fraction of air k in the mixture of water and air (the 'free surface') and therefore here is proposed to choose

$$\nu_k = \frac{|k_{k+1} - 2k_k + k_{k-1}|}{k_{k+1} + 2k_k + k_{k-1}} . \qquad (5.2.i)$$

(During a computation a small positive number ε will be added to the denominator, since it may be equal zero.) All terms are positive, so we may write

$$\nu_k = \frac{\big|\, |k_{k+1} + k_{k-1}| - |2k_k| \,\big|}{|k_{k+1} + k_{k-1}| + |2k_k|} \quad \text{, and thereby } 0 \le \nu_k \le 1.$$

The fourth order dissipation will be added when no discontinuity is present; more precisely, the parameter $\varepsilon^{(4)}$ is chosen

$$\varepsilon^{(4)}_{k+1/2} = \mathrm{MAX}\left(0,\ \kappa^{(4)} - \varepsilon^{(2)}_{k+1/2} \right) . \qquad (5.2.j)$$

The constants $\kappa^{(2)}$ and $\kappa^{(4)}$ and the scaling factors $S^{(2)}$ and $S^{(4)}$ have to be chosen such that time-integration is stable.

§ 3 Time-integration

After space-integration we have a system of ordinary differential equations. The equations in this system look like

$$\frac{d}{dt}(w_k) + C_k + D_k = 0 \qquad (5.3.a)$$

where C_k and D_k respectively represent the convective and the dissipative part of the flux. Time-integration of these equations will be carried out using an explicit three-stage Runge-Kutta scheme; w_k^{n+1} is found from w_k^n by walking through following scheme

$$\begin{aligned}
w_k^{(0)} &= w_k^n \ , \\
w_k^{(1)} &= w_k^{(0)} - 0.6\cdot\Delta t\left(C_k^{(0)} + D_k^{(0)}\right) \ , \\
w_k^{(2)} &= w_k^{(0)} - 0.6\cdot\Delta t\left(C_k^{(1)} + D_k^{(1)}\right) \ , \\
w_k^{(3)} &= w_k^{(0)} - \Delta t\left(C_k^{(2)} + D_k^{(2)}\right) \quad \text{and} \\
w_k^{n+1} &= w_k^{(3)} \ .
\end{aligned} \qquad (5.3.b)$$

In order to save calculation time, sometimes the dissipative fluxes are evaluated only at the first stage, that is, the following scheme is used:

$$\begin{aligned}
w_k^{(0)} &= w_k^n \ , \\
w_k^{(1)} &= w_k^{(0)} - 0.6\cdot\Delta t\left(C_k^{(0)} + D_k^{(0)}\right) \ , \\
w_k^{(2)} &= w_k^{(0)} - 0.6\cdot\Delta t\left(C_k^{(1)} + D_k^{(0)}\right) \ , \\
w_k^{(3)} &= w_k^{(0)} - \Delta t\left(C_k^{(2)} + D_k^{(0)}\right) \quad \text{and} \\
w_k^{n+1} &= w_k^{(3)} \ .
\end{aligned} \qquad (5.3.c)$$

§ 4 Stability analysis for a model equation

We consider the equation

$$\frac{\partial w}{\partial t} + a\,\frac{\partial w}{\partial x} - \mu_2 \Delta x\,\frac{\partial^2 w}{\partial x^2} + \mu_4 \Delta x^3\,\frac{\partial^4 w}{\partial x^4} = 0. \tag{5.4.a}$$

Central-space discretization yields

$$\frac{dw_k}{dt} = -a\,\frac{w_{k+1}-w_{k-1}}{2\Delta x} + \mu_2 \frac{w_{k+1}-2w_k+w_{k-1}}{\Delta x} - \mu_4 \frac{w_{k+2}-4w_{k+1}+6w_k-4w_{k-1}+w_{k-2}}{\Delta x}\,,$$

which can be rewritten as

$$\Delta t\,\frac{dw_k}{dt} = -\frac{\lambda}{2}\Big(w_{k+1}-w_{k-1}\Big) + \frac{\mu_2\lambda}{a}\Big(w_{k+1}-2w_k+w_{k-1}\Big) - \frac{\mu_4\lambda}{a}\Big(w_{k+2}-4w_{k+1}+6w_k-4w_{k-1}+w_{k-2}\Big) \tag{5.4.b}$$

when λ is defined by

$$\lambda = \frac{a\,\Delta t}{\Delta x}\,. \tag{5.4.c}$$

A Fourier mode $\hat{w}_k = e^{ik\xi}$ (where ξ=pΔx and i represents the imaginary unit) can be substituted into equation *(5.4.b)*, whereby it reads

$$\Delta t\,\frac{d\hat{w}_k}{dt} = \hat{w}_k\left[-\frac{\lambda}{2}\Big(2i\,\sin(\xi)\Big) + \frac{\mu_2\lambda}{a}\Big(2\cos(\xi)-2\Big) - \frac{\mu_4\lambda}{a}\Big(2\cos(2\xi)-8\cos(\xi)+6\Big)\right].$$

This can also be written as

$$\Delta t\,\frac{d\hat{w}_k}{dt} = z\cdot\hat{w}_k\,, \tag{5.4.d}$$

where z is defined by

$$z = \left(-\lambda \sin(\xi)\right)i + \frac{2\mu_2\lambda}{a}\left(\cos(\xi)-1\right) - \frac{4\mu_4\lambda}{a}\left(\cos(\xi)-1\right)^2 \quad . \qquad (5.4.e)$$

For time-integration a multi-stage Runge-Kutta scheme will be used. The stability region of such a scheme can be determined independently of the terms in the differential-equation. In general one time-step yields

$$w^{n+1} = A(z)\cdot w^{n} \ , \qquad (5.4.f)$$

where $A(z)$ is known as the amplification factor. The stability region $\mathcal{S}$ is then given by

$$\mathcal{S} = \left\{ z\in\mathbb{C} \;\middle|\; |A(z)| \le 1\right\} . \qquad (5.4.g)$$

Leaving undecided which scheme is used for time-integration, in general a rectangle $\mathcal{R}$,

$$\mathcal{R} = \left\{ z\in\mathbb{C} \;\middle|\; -\mathcal{R}e \le \mathrm{Re}(z) \le 0,\ |\mathrm{Im}(z)| \le \mathcal{I}m;\ \mathcal{R}e,\ \mathcal{I}m \in \mathbb{R}^+ \right\} \qquad (5.4.h)$$

can be found, which is a subset of the stability region: $\mathcal{R} \subset \mathcal{S}$. Stability is obtained, comparing *(5.4.e)* and *(5.4.h)*, when the following conditions are satisfied:

$$-\mathcal{R}e \le \frac{2\mu_2\lambda}{a}\left(\cos(\xi)-1\right) - \frac{4\mu_4\lambda}{a}\left(\cos(\xi)-1\right)^2 \le 0 \quad \text{and} \qquad (5.4.i)$$

$$\left|-\lambda \sin(\xi)\right| \le \mathcal{I}m \qquad (5.4.j)$$

for all $\xi \in [0,2\pi]$.

The coefficient a of the convective term will be chosen as the maximum of the absolute eigenvalues (see equation *(4.1.e)*) that belongs to the non-linear system we consider. Thus $\lambda>0$.
Furthermore recall that the coefficients μ_2 and μ_4 for artificial dissipation approximately have the form

$$\mu_2 = S^{(2)}\varepsilon^{(2)} = S^{(2)}\kappa_2\nu \quad \text{and}$$
$$\mu_4 = S^{(4)}\varepsilon^{(4)} = S^{(4)}\text{MAX}\Big(0,\ \kappa_4 - \varepsilon^{(2)}\Big) = S^{(4)}\text{MAX}\Big(0,\ \kappa_4 - \kappa_2\nu\Big);$$

$S^{(2)}$, $S^{(4)}$, κ_2 and κ_4 are strictly positive parameters and $0\le\nu\le1$, thereby the right inequality of *(5.4.i)* becomes a triviality. The scaling factors $S^{(2)}$and $S^{(4)}$ are chosen equal a in **[6]**; to me **appendix III** is a motivation to choose $S^{(2)}$= u, the local fluid velocity. Anyway, in both cases the following inequalities will hold:

$$\mu_2 \le a\ \kappa_2\nu \quad \text{and} \quad \mu_4 \le a\ \text{MAX}\Big(0,\ \kappa_4 - \kappa_2\nu\Big).$$

Now *(5.4.i)*, *(5.4.j)* will be satisfied when

$$4\lambda\Big\{\kappa_2\nu + 4\ \text{MAX}\Big(0,\ \kappa_4 - \kappa_2\nu\Big)\Big\} \le \mathcal{Re} \quad \text{and} \tag{5.4.k}$$

$$\lambda \le \mathcal{Im}\ . \tag{5.4.l}$$

When $\mathcal{Re}$ and $\mathcal{Im}$ are known and the parameters κ_2 and κ_4 are set, the maximum allowable time step Δt follows from *(5.4.c)*, *(5.4.k)* and *(5.4.l)*. $\mathcal{Re}$ and $\mathcal{Im}$ will be determined for three-stage Runge-Kutta schemes in the next section.

§ 5 Stability region of three-stage Runge-Kutta schemes

The model equation for a Fourier-mode has the form

$$\Delta t \frac{dw}{dt} = z \cdot w, \tag{5.5.a}$$

as shown in the preceding section. The derivation of z shows that Im(z) represents the convective term, whereas Re(z) represents the dissipative terms. For example the three-stage Runge-Kutta scheme *(5.3.b)* can be used for integration of equation *(5.5.a)*. We then have

$$\begin{aligned} w^{(0)} &= w^{n} , \\ w^{(1)} &= w^{(0)} + 0.6\ z\ w^{(0)} , \\ w^{(2)} &= w^{(0)} + 0.6\ z\ w^{(1)} , \\ w^{(3)} &= w^{(0)} + z\ w^{(2)} , \\ w^{n+1} &= w^{(3)} . \end{aligned} \tag{5.5.b}$$

Note that both convective and dissipative fluxes are evaluated at all stages. According to *(5.5.b)* we have

$$w^{n+1} = w^{n}+z\left(w^{n}+0.6z\left(w^{n}+0.6z\cdot w^{n}\right)\right) = \left(1+z+0.6z^{2}+0.36z^{3}\right)w^{n}. \tag{5.5.c}$$

For stability it is necessary that the amplification factor $A(z)$ satisfies (see *(5.4.g)*) $|A(z)| \le 1$ or

$$|1+z+0.6z^{2}+0.36z^{3}| \le 1 . \tag{5.5.d}$$

We also can use scheme *(5.3.c)* which evaluates the dissipative fluxes once. In that case we find, writing $z = x+iy$,

$$
\begin{aligned}
w^{(0)} &= w^{n} , \\
w^{(1)} &= w^{(0)} + 0.6\ z\ w^{(0)} , \\
w^{(2)} &= w^{(0)} + 0.6(x\ w^{(0)} + iy\ w^{(1)}) , \\
w^{(3)} &= w^{(0)} + x\ w^{(0)} + iy\ w^{(2)} , \\
w^{n+1} &= w^{(3)} .
\end{aligned}
\tag{5.5.e}
$$

Substitution now yields

$$
w^{n+1} = \Big((1+x-0.36xy^2-0.6y^2)+i(0.6xy+y-0.36y^3)\Big)w^{n}. \tag{5.5.f}
$$

Of course for stability it is for this scheme required that

$$
\left|(1+x-0.36xy^2-0.6y^2)+i(0.6xy+y-0.36y^3)\right| \le 1 . \tag{5.5.g}
$$

The stability regions are plotted in **FIGURE II** and **FIGURE III**.

CHAPTER SIX : PROPAGATION OF DISCONTINUITIES

§ 1 Description of the numerical experiment

A type of discontinuity we are interested in, is the interface between two fluids with vastly different densities. Some numerical experiments will be carried out here to show the behaviour of numerical solutions. To do this the following initial density distribution along the interval I=[0,L] is used:

$$\rho(x,0) = \begin{cases} \rho_l \ , & x \in [0\ ,x1[\\ \rho_h \ , & x \in [x1,x2[\\ \rho_l \ , & x \in [x2,\ L] \end{cases} \qquad (6.1.a)$$

where $\rho_l < \rho_h$ and $0 < x1 < x2 < L$.

The initial velocity is chosen

$$u(x,0) = U_0 \ , \ x \in \mathbf{I} \qquad (6.1.b)$$

and the fraction air k(x,0) such that the pressure equals

$$p(x,0) = p(\rho(x,0),\ k(x,0)) = p_0, \qquad (6.1.c)$$

the standard atmospheric pressure. The system of equations *(5.2.a)* is integrated as proposed in chapter five. The boundaries x=0 and x=L are treated as if they were fully permeable (as much as possible).
Computations are performed with and without artificial dissipation and varying the number of gridpoints. A gravitational force will be switched on during the last computation.

§ 2 Theoretically obtained results

The pressure in the fluid is initially constant. When no gravitational force is applied, there is no driving term which could change the velocity of the fluid. Thus we have at t>0

$$u(x,t) = u(x,0) \tag{6.2.a}$$

and the density-profile of the fluid is given by

$$\rho(x,t) = \rho(x-u(x,0)\cdot t) = \rho(x-U_0 t) \tag{6.2.b}$$

which means the profile is propagating with constant velocity. When a gravitational force is applied (gravitational constant g), the solution reads at t>0

$$u(x,t) = u(x,0) + gt = U_0 + gt \tag{6.2.c}$$

and

$$\rho(x,t) = \rho(x - u(x,0)t - \tfrac{1}{2}gt^2) = \rho(x-U_0 t-\tfrac{1}{2}gt^2). \tag{6.2.d}$$

§ 3 Numerical results and interpretation

We shall start with values $\rho_l = 250\ kgm^{-3}$, $\rho_h = 750\ kgm^{-3}$, $U_0 = 1.0\ msec^{-1}$ and $g=0.0\ msec^{-2}$.

FIGURE IV.I shows the first result . No dissipative terms are added here, and, as predicted (**appendix III**), the well-known Gibbs' phenomenon appears. Although the profile is transported with the right velocity, the solution is poisoned with spurious oscillations. For real water and air this computation would not have been possible at all, since already after a few timesteps negative mass densities occur.

So the result here is unacceptable. Refinement of the grid is not the remedy to this problem (**FIGURE IV.II**) , the oscillations essentially stay.

A better solution can be obtained (in fact we first should discuss the properties of a good solution) by adding dissipation as proposed in chapter five. The applied sensor is the second difference of the fraction air in the mixture (see *(5.2.i)*, a small positive number is added to the denominator, since it may become zero).

FIGURE IV.III and **FIGURE IV.IV** show the result (about the best I could find by observing the solutions qualitatively and varying the parameters $\kappa^{(2)}$ and $\kappa^{(4)}$).

During the last computation a gravitational term was added ($g = 1\ ms^{-2}$).

Despite the increasing velocity of the fluid, the dissipative term does a good job, without changing the coefficient $\kappa^{(2)}$. The table down here shows the theoretical center of the 'block with high density' at the shown points of time.

t	$x_{c,g=0.0}$	$x_{c,g=1.0}$
0.0	3.5	3.5
1.0	4.5	5.0
2.0	5.5	7.5
3.0	6.5	11.0

Comparison of this table with **FIGURE IV.IV** and **FIGURE IV.V** shows the profile is (approximately) transported with the right velocity in both cases.

CHAPTER SEVEN : A WATER HAMMER

§ 1 Description of the numerical experiment

When the flow of a fluid in a pipe is stopped at once somewhere, a pressure wave starts running from this point. This wave travels with the speed of sound and at its front density, pressure and velocity are theoretically (when we assume the fluid inviscid) discontinuous. This will be explained in the next section.
We shall try to show such a pressure wave by means of a numerical experiment. Therefore, along the interval **I**=[0,L] we initially take

$$\rho(x,0) = \rho_0 \quad , \; x \in \mathbf{I}$$

and the initial velocity is chosen

$$u(x,0) = U_0 \quad , \; x \in \mathbf{I}.$$

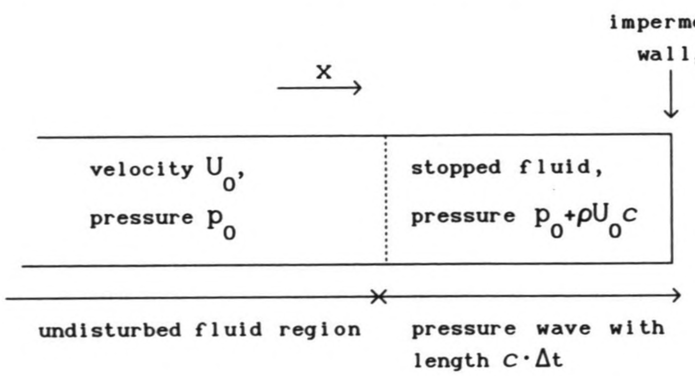
The fraction of air $k_0 = k(x,0)$ such that the pressure equals

$$p(x,0) = p(\rho_0, k_0) = p_0,$$

the standard atmospheric pressure. For t>0 the boundary condition

$$u(L,t) = 0 \text{ msec}^{-1}$$

is applied (the fluid is stopped in x=L).



§ 2 Theoretically obtained results

A pressure wave travels with speed c in negative direction. So, after some time Δt, the length Δx of the pressure wave equals $c\Delta t$. From $t=0$ till $t=\Delta t$, the velocity of the fluid in $x=L-c\Delta t$ was undisturbed and equal to U_0, so, the amount of mass per unit area that came into the interval $[L-c\Delta t,L]$ equals $\rho_0 U_0 \Delta t$.
This means the mass density in the interval has increased by an amount $\Delta\rho$, where $\Delta\rho = (\rho_0 U_0 \Delta t)/(c\Delta t) = \rho_0 U_0/c$.
This density change involves a pressure change $\Delta p = c^2\Delta\rho = \rho_0 U_0 c$.
This result was first found by T. von Kármán. The derivation can be found in many books; one of the most simple and beautiful (to my opinion) is given in **[33]**, chapter 1.1, page 4, where it is derived as a basis for the theory of sound waves.

§ 3 Numerical results and interpretation

The experiment was carried out for pure water with initial velocity equal 0.1 ms^{-1}. **FIGURE V.I** shows the result when no artificial dissipation is added. The sudden changes at the front of the pressure wave again cause spurious oscillations. The effect of some basic fourth order dissipation can be seen by looking at **FIGURE V.II** : the solution is smoothened, but a little under- and overshoot at the front of the wave is still present. When extra second order dissipation is added, with second differences of pressure as sensor for its necessity, the solution becomes like it found in **FIGURE V.III**.
The plots show that after 0.004 seconds the wave moved up about 6 meters.
This is in good agreement with the speed of sound of pure water, which is about 1460 m/sec.
The pressure change should be about $\rho_0 U_0 c = 999.1\cdot 0.1\cdot 1460 \simeq 1.5$ bar. This also can be read from the plots.

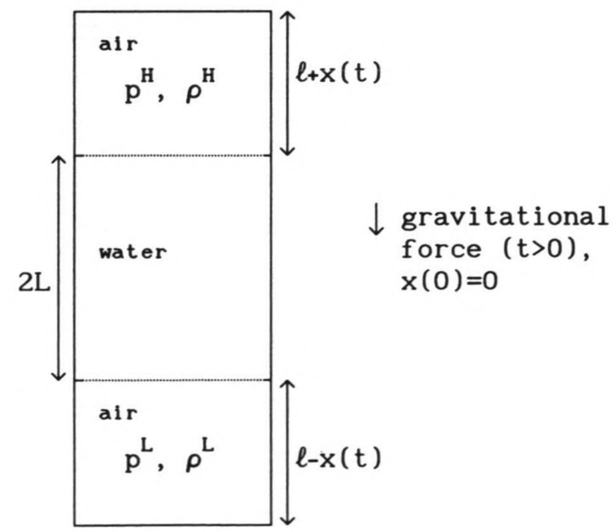
CHAPTER EIGHT : HARMONIC OSCILLATION OF A COLUMN OF WATER

§ 1 Introduction

We consider a closed tube that has length $2(L+\ell)$ and is filled with air and water. The variable x represents the displacement of the column of water. At time t=0 the tube contains (approximately)

air for $-L-\ell < x < -L$,
water for $-L < x < L$ and
air for $L < x < L+\ell$

At time t=0 gravitation is 'switched on' in positive x-direction. What we then see is that the column of water starts oscillating between the two columns of air. The equation of motion of this oscillation can be derived when we make some assumptions. This will be done in the following section.



§ 2 Analytically obtained results

The assumptions that are made are the following:

- the momentum of air is negligible compared to the momentum of water
- compressibility of water is negligible compared to the compressibility of air
- the changes in the columns of air are small, furthermore the process is quasi-static and the pressure in air does not depend on position (in the seperate columns of course);

The variables in the upper and lower column of air are respectively 'superscripted' by H and L. The pressure and density are initially such as under standard atmospheric conditions (p_0 and ρ_{ao}). The cross-sectional area of the tube is A. The mass of air in a column thus equals $A\ell\rho_{ao}$. Suppose the displacement of the column of water is x(t). So x(0)=0. This causes in the upper column of air a density change

$$d\rho_a^H = \rho_a^H(t)-\rho_{ao} = \rho_{ao}\left(\frac{\ell}{\ell+x(t)} - 1\right) = -\rho_{ao}\,\frac{x(t)}{\ell+x(t)}$$

and thus a pressure change

$$dp_a^H = c_a^2\, d\rho_a^H = -c_a^2\rho_{ao}\,\frac{x(t)}{\ell+x(t)}\ .$$

Likewise in the lower column of air we have

$$d\rho_a^L = \rho_a^L(t)-\rho_{ao} = \rho_{ao}\left(\frac{\ell}{\ell-x(t)} - 1\right) = +\rho_{ao}\,\frac{x(t)}{\ell-x(t)}$$

and a pressure change

$$dp_a^L = c_a^2\, d\rho_a^L = +c_a^2\rho_{ao}\,\frac{x(t)}{\ell-x(t)}\ .$$

The force on the column of water caused by the pressure changes in the columns of air therefore equals

$$F_p = -Ac_a^2\rho_{ao}\left(\frac{x(t)}{\ell+x(t)} + \frac{x(t)}{\ell-x(t)} \right) = -2Ac_a^2\rho_{ao}\left(\frac{\ell\, x(t)}{\ell^2-x^2(t)}\right) . \qquad (8.2.a)$$

The gravitational force acting on the water is

$$F_g = 2AL\rho_{wo}g \qquad (8.2.b)$$

where ρ_{wo} represents the density of water (assumed constant here) and g the gravitational constant (about 9.8 m/s^2). According to Newton's second law the equation of motion of the water is given by

$$2AL\rho_{wo}\frac{d^2x}{dt^2} = F_p + F_g$$

which by using equations *(8.2.a)* and *(8.2.b)* becomes

$$2AL\rho_{wo}\frac{d^2x}{dt^2} = 2AL\rho_{wo}g - 2Ac_a^2\rho_{ao}\left(\frac{\ell\, x}{\ell^2-x^2}\right)$$

or

$$\frac{d^2x}{dt^2} + f^2\left(\frac{\ell^2 x}{\ell^2-x^2}\right) - g = 0 \qquad (8.2.c)$$

where f is defined by

$$f^2 = \frac{c_a^2\rho_{ao}}{\rho_{wo}L\ell} .$$

The initial conditions for this second-order O.D.E. are given by

$$\begin{cases} \text{initial position is } x(0) = 0 \\ \text{initial velocity is } x'(0) = 0 \end{cases} . \qquad (8.2.d)$$

A solution to the problem can easily be found when the displacement of the water is small compared to the length of the column of air ($x \ll \ell$) , since

$$\frac{\ell^2 x}{\ell^2 - x^2} = \ell\left\{(x/\ell) + (x/\ell)^3 + O\Big((x/\ell)^5\Big)\right\}.$$

When $x \ll \ell$, the first term will be an accurate approximation for this expression. The equation of motion then can be written as

$$\frac{d^2x}{dt^2} + f^2x - g = 0, \quad x(0)=0, \quad x'(0)=0 \ . \qquad (8.2.e)$$

Now define

$$y = f^2x - g \ ,$$

then problem *(8.2.e)* can be transformed to

$$\frac{d^2y}{dt^2} + f^2y = 0, \quad y(0)=-g, \quad y'(0)=0$$

with simple harmonic solution

$$y(t) = -g\cos(ft)$$

and thus the solution to *(8.2.e)* becomes

$$x(t) = \frac{g}{f^2}\Big(1 - \cos(ft)\Big) \ . \qquad (8.2.f)$$

The period τ of this harmonic motion equals

$$\tau = (2\pi)/f = \frac{2\pi}{c_a}\sqrt{\frac{\rho_w L\ell}{\rho_{ao}}} \ , \qquad (8.2.g)$$

while the the amplitude x_{amp} of the oscillation satisfies

$$x_{amp} = g/(f^2) = \frac{g\rho_w L\ell}{c_a^2 \rho_{ao}} \quad . \tag{8.2.h}$$

Notice that x is in the interval $[0,2x_{amp}]$ and not in $[-x_{amp},x_{amp}]$; so the column of water oscillates around $x=x_{amp}$ and not around x=0. The maximum velocity x'_{amp} of the column of water is

$$x'_{amp} = g/f = g\sqrt{\frac{\rho_w L\ell}{c_a^2 \rho_{ao}}} \quad . \tag{8.2.i}$$

We have to keep in mind that this solution is only valid for $x \ll \ell$.

§ 3 Numerical results

The experiment described in the preceding sections has been simulated numerically. The length of the column of water in this computation was 2L = 4 meters, the length of the columns of air was chosen ℓ = 3 meters.
The interface between water and air could be captured very well during this computation and the results are in good agreement with the analytically obtained solution. At first, **FIGURE VI.I** shows the velocity of the column of water as a function of the elapsed time (this velocity can be taken anywhere in the water: since compressibility effects of water are almost negligible for this experiment, the velocity in an uninterrupted column does not depend on position, only on time).
The motion looks harmonical, which was expected. Furthermore we can check the period and the amplitude by means of equations *(8.2.g)* and *(8.2.i)*. Substitution in *(8.2.g)* yields for the period of the oscillation

$$\tau = \frac{2\pi}{340}\sqrt{\frac{999.1 \cdot 2.0 \cdot 3.0}{1.23}} \simeq 1.3 \text{ sec},$$

while the maximum velocity becomes

$$x'_{amp} = 9.8\sqrt{\frac{999.1 \cdot 0.2 \cdot 0.3}{340^2 \cdot 1.23}} \simeq 2.0 \text{ m/sec} .$$

These values are in good agreement with the values that can be read from **FIGURE VI.I**. The period of the oscillation in the numerical result is about 5% smaller than the period of the harmonic solution: this difference must not be interpreted as a numerical error, but is due to small non-linear effects. The amplitude of the oscillation is

$$x_{amp} = \frac{9.8 \cdot 999.1 \cdot 2.0 \cdot 3.0}{340^2 \cdot 1.23} \simeq 0.41 \text{ m} ,$$

so $x \in [0, 2x_{amp}] = [0.0,\ 0.82]$. Since $2x_{amp}/\ell \simeq 0.27$ and $(2x_{amp}/\ell)^3 \simeq 0.02$ the 'harmonic approximation' is applicable, but the effect of the first non-linear term in equation *(8.2.c)* may cause differences in the order of some percents in the solution.

Values of pressure, velocity, density and interface-sensor at fixed points of time are plotted in **FIGURE VI.II** and **FIGURE V1.III**. Looking at pressure values, it can be seen that in both columns of air the 'quasi-static' assumption indeed holds.
At about t=0.65 sec the pressure change Δp in the compressed column of air equals

$$\Delta p^L = c_a^2\, \Delta\rho^L = 340^2 \cdot 1.23\, \frac{0.82}{3.0-0.82} \simeq 0.5 \text{ bar}$$

while for the expanded column

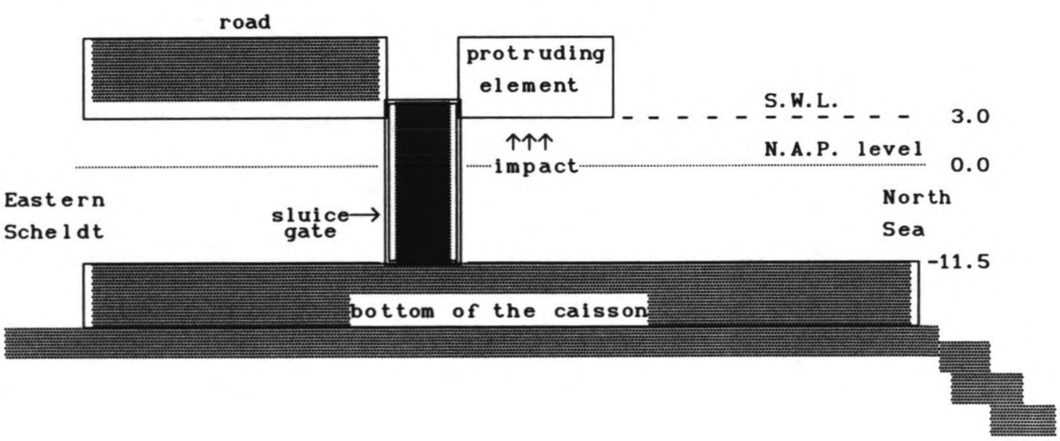
$$\Delta p^{H} = c_{a}^{2}\, \Delta\rho^{H} = -340^{2}\cdot 1.23\, \frac{0.82}{3.0+0.82} \simeq -0.3 \text{ bar}.$$

This also is in good agreement with the result in the last figure.

CHAPTER NINE : WAVE IMPACTS ON STRUCTURES

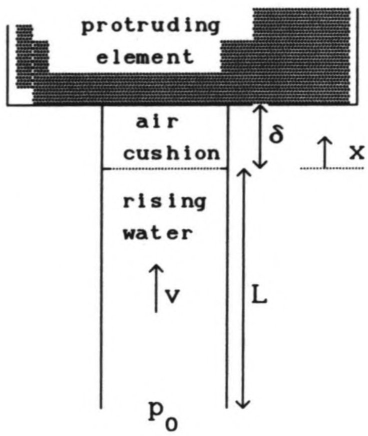
§ 1 Introduction

At the end of the seventies a study on wave impacts was carried out at Delft Hydraulics, for the design of the storm surge barrier in the Eastern Scheldt. Also some tests in a wave channel were performed, to verify the results of the study. It was concluded that the so-called Bagnold's piston model was the most appropriate to describe the wave impacts caused by standing waves against protruding elements. Due to the specific shape of the structure, a volume of air is trapped between the protrusion and the rising water level, which acts as a spring (as we already observed in the preceding chapter).
Wave impacts induced by breaking waves were not observed, because of the relatively large foreshore depth.



The rising water and the air layer between the water and the protruding element gave evidence for the choice of Bagnold's piston model to describe the impact process. This one-dimensional model considers a column of water of length L which approaches an unelastic wall (the protruding element) with (initial) velocity v. Between the water and the wall a column of air with (initial) length δ is trapped. We write c_w for the speed of sound in water, ρ_w for its density and p_0 for the atmospheric pressure.

piston model of Bagnold



It was found (see for example **[30]**) that the following dimensionless numbers characterize the impact phenomenon:

$$S = \frac{\frac{1}{2}\rho_w v^2}{p_0} \cdot \frac{L}{\delta} \qquad \text{(impact number)} \qquad (9.1.a)$$

and

$$\beta = \frac{v}{c_w} \cdot \frac{L}{\delta} \qquad \text{(water compressibility number).} \qquad (9.1.b)$$

Roughly the following subdivision can be made. For values $\beta < 0.01$ the water may be assumed incompressible, when $0.01 < \beta < 1.0$ compressibility of both water and air is important and when $\beta > 1.0$ only the elasticity of water is important. Each case will be discussed briefly in the following section.

§ 2 Analytically obtained results

In this section some results that can be found in **[28]**, **[29]**, **[30]** and **[31]** are reviewed.

$\beta < 0.01$

In this case the water is treated as a 'rigid body' and the problem looks like the one we considered in chapter eight. However, no gravitional force is added here (it was concluded that gravity does not play an important role for the impact problem) and the initial conditions are different. Again, Newton's second law states for the displacement x(t) of the water column

$$\rho_w L \frac{d^2x}{dt^2} = p_0 - p \ , \tag{9.2.a}$$

where p is the pressure in the air cushion. The initial conditions for this problem are

$$x(0) = 0 \quad \text{and} \quad \frac{dx}{dt}(0) = v. \tag{9.2.b}$$

Furthermore we assume (Boyle)

$$p_0 \delta = p(\delta - x) \ . \tag{9.2.c}$$

Defining

$$\alpha = \rho_w L / p_0 \ , \tag{9.2.d}$$

and using *(9.2.c)* the differential equation *(9.2.a)* becomes

$$\alpha \frac{d^2x}{dt^2} + \frac{x}{\delta - x} = 0 \ . \tag{9.2.e}$$

Integration of this equation is possible after multiplication by dx/dt. This yields

$$x + \delta \ln\left(\frac{\delta - x}{\delta}\right) = \frac{\alpha}{2}\left((dx/dt)^2 - v^2\right) . \qquad (9.2.f)$$

The maximum displacement x_{max} is found for dx/dt=0 and thus follows from

$$\frac{x_{max}}{\delta} + \ln\left(\frac{\delta - x_{max}}{\delta}\right) = -\frac{\alpha v^2}{2\delta} = -S \qquad (9.2.g)$$

by definition of the impact number S and equation *(9.2.d)*. For $x_{max} << \delta$ we can write

$$\ln\left(\frac{\delta - x_{max}}{\delta}\right) \simeq -\frac{x_{max}}{\delta} - \frac{1}{2}\left(\frac{x_{max}}{\delta}\right)^2$$

and we find

$$\frac{x_{max}}{\delta} \simeq \sqrt{2S} . \qquad (9.2.h)$$

The maximum pressure p_{max} then is (see equation *(9.2.c)*)

$$p_{max} \simeq p_0\left(\frac{1}{1 - \sqrt{2S}}\right) . \qquad (9.2.i)$$

<u>$0.01 < \beta < 1.0$</u>

This situation is more complicated since the compressibility of both water and air is taken into account. A Lagrangian variable a can be introduced: a is the initial x-coordinate of a fluid particle. The position of a fluid particle may be regarded as a function of its initial position and of time, so $x = x(a,t)$. The continuity equation for water now reads **[32]**

$$\rho_w \frac{\partial x}{\partial a} = \rho_{w0} , \qquad (9.2.j)$$

while the momentum equation is given by

$$\frac{\partial^2 x}{\partial t^2}\frac{\partial x}{\partial a} + \frac{1}{\rho_w}\frac{\partial p}{\partial a} = 0. \qquad (9.2.k)$$

By *(9.2.j)* this can also be written as

$$\rho_{w0}\frac{\partial^2 x}{\partial t^2} + \frac{\partial p}{\partial \rho_w}\,\frac{\partial}{\partial a}(\rho_w) = 0. \qquad (9.2.l)$$

Furthermore differentiation of *(9.2.j)* with respect to *a* yields

$$\frac{\partial}{\partial a}(\rho_w)\,\frac{\partial x}{\partial a} + \rho_w\,\frac{\partial^2 x}{\partial a^2} = 0\ . \qquad (9.2.m)$$

Now $\frac{\partial}{\partial a}(\rho_w)$ may be eliminated using *(9.2.l)* whereby we find (again using the continuity equation *(9.2.j)*)

$$\frac{\partial^2 x}{\partial t^2} - \left((\rho_w/\rho_{w0})^2\,\frac{\partial p}{\partial \rho_w}\right)\frac{\partial^2 x}{\partial a^2} = 0\ .$$

When the variation in the term between brackets is neglected (which is reasonable) and is taken to be c_w^2 we find an equation that looks familiar to the most of us:

$$\frac{\partial^2 x}{\partial t^2} - c_w^2\,\frac{\partial^2 x}{\partial a^2} = 0 \qquad \text{or}$$

$$\left(\frac{\partial}{\partial t} - c_w\,\frac{\partial}{\partial a}\right)\left(\frac{\partial}{\partial t} + c_w\,\frac{\partial}{\partial a}\right)(x) = 0\ . \qquad (9.2.n)$$

The initial conditions for this equation are

$$x(a,0) = a \quad \text{and} \quad \frac{\partial x}{\partial t}(a,0) = v. \qquad (9.2.o)$$

At the surfaces the water should have the pressure of the adjacent air, that is, we'd like to have

$$p(a{=}-L,t) = p_0 \quad \text{and} \quad p(a{=}0,t) = p_0 + c_w^2(\rho_w - \rho_{w0})$$

which also can be written as

$$\frac{\partial x}{\partial a}(a{=}-L) = 1 \quad \text{and} \quad \frac{\partial x}{\partial a}(a{=}0) = \frac{1}{1 + (p_0 x)/(\rho_{w0}c_w^2(\delta - x))}\ . \qquad (9.2.p)$$

The problem was solved numerically ([30], method of characteristics) to find the maximum pressure.

$\beta > 1.0$

Here only compressibility of water is important and we refer to the results found in chapter seven. Notice that the maximum pressure found in this case is equal to $(1+2S/\beta)p_0$.

§ 3 Numerical results

Computations were carried out for a few values of S and β. The maximum pressure in the tube as a function of time can be read from **FIGURE VII.I** . The heights of the pressure peaks (which can be read from the plots) are in the following table.

β	S	p_{max} (bar)
0.01	0.25	2.2
0.01	1.0	6.4

The shape of the plots and the values of the pressure peaks are (for these values of β and S) in good agreement with the results presented in **[30]**. For a comparison, see **FIGURE VII.II**.

Unfortunately, computations with large values for the compressibility number β (for example β=0.3, β=1.0, like in **[30]**) could not be carried out, since in those cases the following (practical) problems occurred.

When the values of β and S are given, the values of v and L/δ can be computed from equations *(9.1.a)* and *(9.1.b)*. For example, when β = 0.3 and S = 1.0, we find v $\simeq$ 0.46 and $L/\delta \simeq 946$. This large value for L/δ in combination with the uniform grid I use causes the problem. For an accurate computation it is necessary that the column of air (with length δ) at least takes a few grid-points. However, this means thousands of grid-points are necessary for the total computation, which makes it much too expensive or even impossible.

Conclusions and Recommendations

The method that is presented for the computation of time-dependent compressible flow of water and air works well for one-dimensional problems of short duration. It is quite expensive for two reasons:

- the large eigenvalues (determined by the speed of sound waves) require very small time-steps (otherwise the explicit time-integration process becomes unstable);
- to capture the interface between water and air with some accuracy a relatively fine grid has to be used. The artificial dissipation model *(Jameson)*, which is appropriate for the capturing of shock waves, is less appropriate for the capturing of linear discontinuities like interfaces.

More research and sophisticated techniques are needed to overcome these problems.
For example, adaptive grid refinement could be used to get a sharper interface. However, in that case a new grid has to be computed several times during a computation, since the interface will be moving. Furthermore it leads to even smaller time-steps.
For the computation of steady state solutions often acceleration methods are used. The system of equations then is 'preconditioned' by introducing artificial time derivatives which allow for a faster convergence **[27]**. An interesting idea is the application of this method to time-accurate computations. A 'pseudo-time' is introduced and pseudo-time derivatives are added to the equations of motion. The physical solution at an advanced time-level then is obtained as the steady state solution of the system of preconditioned equations in pseudo-time. This treatment allows for much larger time-steps **[26]**.
To my opinion it would be very interesting to investigate the possibilities of the techniques mentioned here. When this will be done perhaps one day

all types of flow can be attacked by compressible solvers and the compressibility of fluid flows will be more the result of a computation (which it should be, I think) than an assumption made beforehand.
However, the feasibility and the use of these ideas in the near future is questionable, since a lot more computer-power than available at the moment will be required.

Appendix I : Blending functions

A set of blending functions $\mathcal{B}$ may be defined as follows.
Let $B:\mathbb{R}\times\mathbb{R}^{+}\rightarrow[0,1]$ be a continuously differentiable mapping. Then $B\in\mathcal{B}$ if and only if the following conditions are fulfilled:

(i) $B(x,\sigma)$ is monotonously increasing for any fixed σ,

(ii) $\lim_{x\rightarrow-\infty} B(x,\sigma) = 0$ for any fixed σ,

(iii) $\lim_{x\rightarrow+\infty} B(x,\sigma) = 1$ for any fixed σ and

(iv) $\lim_{\sigma\downarrow 0} B(x,\sigma) = \begin{cases} 0 & \text{when } x<0 \\ 1 & \text{when } x>0 \end{cases}$.

Blending functions can be used to smear out a discontinuity. Suppose we have $f:\mathbb{R}\rightarrow\mathbb{R}$ and $g:\mathbb{R}\rightarrow\mathbb{R}$ both continuous mappings. Assume $f(\hat{x}) \neq g(\hat{x})$ and define $h:\mathbb{R}\rightarrow\mathbb{R}$ by $h(x) = \begin{cases} f(x) & \text{when } x<\hat{x} \\ g(x) & \text{when } x\geq\hat{x} \end{cases}$, so h is discontinuous for $x=\hat{x}$.

Furthermore let B_1, $B_2 \in \mathcal{B}$. Then h can be approximated by a continuous function p_σ , defined by

$$p_\sigma(x) = \left(1-B_1(x-\hat{x},\sigma)\right)f(x) + \left(B_2(x-\hat{x},\sigma)\right)g(x).$$

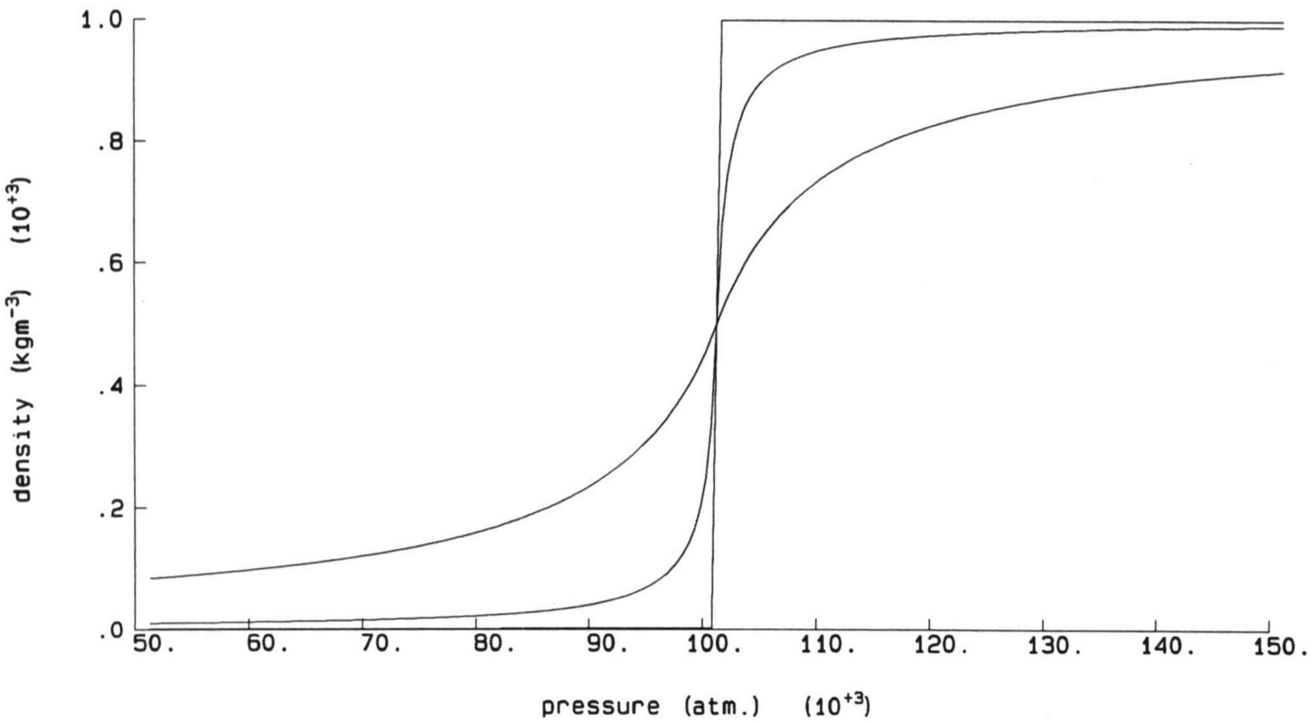
This function is continuous, since all terms are.
The smaller the positive parameter σ, the better the approximation is in the following sense,

$$\lim_{\sigma\downarrow 0} \int |p_\sigma(x)-h(x)|dx = 0.$$

Examples of blending functions are

$$B(x,\sigma) = \frac{1}{2}\left(1+\tanh(x/\sigma)\right) \quad \text{and} \quad B(x,\sigma) = \frac{1}{2}\left(1+\frac{x}{|x|+\sigma}\right) .$$

A PRESSURE-DENSITY RELATION
FOR SPREAD 10^0, 10^3 AND 10^4



Appendix II : The speed of sound in a mixture of water and air

According to the definitions given in chapter 3, section 1, the following equations hold:

(1) $k\rho = \alpha\rho_a$,

(2) $(1-k)\rho = (1-\alpha)\rho_w$ and

(3) $k/(1-k) = (\alpha\rho_a)/((1-\alpha)\rho_w)$.

Furthermore we shall use

(4) $\rho_a^2 c_a^2 \ll \rho_w^2 c_w^2$

and (following from equation *(1.2.d)*)

(5) $\rho_a c_a^2 = \gamma \cdot p$.

The equation we found for the speed of sound reads (eq. *(3.5.c)*)

$$\frac{1}{(\rho c)^2} = \frac{k}{(\rho_a c_a)^2} + \frac{1-k}{(\rho_w c_w)^2} .$$

Straightforward rewriting of this relation yields

$$\frac{1}{c^2} = \frac{k\rho^2}{(\rho_a c_a)^2} + \frac{(1-k)\rho^2}{(\rho_w c_w)^2} =$$

$$= \left(\frac{k\rho}{\rho_a}\right)^2\left(\frac{1}{kc_a^2}\right) + \left(\frac{(1-k)\rho}{\rho_w}\right)^2\left(\frac{1}{(1-k)c_w^2}\right) \overset{(1),(2)}{=}$$

$$= \alpha^2\left(\frac{1}{kc_a^2}\right) + (1-\alpha)^2\left(\frac{1}{(1-k)c_w^2}\right) =$$

$$= \left(1 + \frac{1-k}{k}\right)\left(\frac{\alpha}{c_a}\right)^2 + \left(1 + \frac{k}{1-k}\right)\left(\frac{1-\alpha}{c_w}\right)^2 \quad \overset{(3)}{=}$$

$$= \left(\frac{\alpha}{c_a}\right)^2 + \left(\frac{1-\alpha}{c_w}\right)^2 + \frac{\alpha(1-\alpha)\rho_w}{\rho_a c_a^2} + \frac{\alpha(1-\alpha)\rho_a}{\rho_w c_w^2} \quad \overset{(4)}{\simeq}$$

$$\simeq \left(\frac{\alpha}{c_a}\right)^2 + \left(\frac{1-\alpha}{c_w}\right)^2 + \frac{\alpha(1-\alpha)\rho_w}{\rho_a c_a^2} \quad \overset{(5)}{=}$$

$$= \left(\frac{\alpha}{c_a}\right)^2 + \left(\frac{1-\alpha}{c_w}\right)^2 + \frac{\alpha(1-\alpha)\rho_w}{\gamma p} \quad .$$

This expression for the speed of sound can be found in **[5]**.

Appendix III : Discontinuities, central-space discretization and artificial dissipation

The finite-volume method that is used in [6] reduces to central-space discretization in our case, a one-dimensional problem where a uniform grid is used.
It will be shown here qualitatively that central-space integration causes problems for functions which are almost discontinuous somewhere (in fact, second differences are the point), and therefore require a special treatment.
For example consider a function w(x,t) which initially is given by

$$w(x,0) = \begin{cases} w_l & \text{for} \quad x \le x_l \\ w_h & \text{for} \quad x_l < x \le x_r \\ w_l & \text{for} \quad x_r < x \end{cases} \qquad \text{where } w_h > w_l .$$

Grid-points along the x-axis are given by $x_k = k\Delta x$ ($k \in \mathbb{Z}$ and Δx fixed) such that

$$w(x_k,0) = \begin{cases} w_l & \text{for} \quad k \le k_l \\ w_h & \text{for} \quad k_l < k \le k_r \\ w_l & \text{for} \quad k_r < k \end{cases} .$$

Thus w(x,0) has a discontinuity between the points x_{k_l} and x_{k_l+1} and between the points x_{k_r} and x_{k_r+1} .

Suppose we want to transport this profile with a constant and positive velocity U. This can be described by

$$\frac{\partial w}{\partial t} + U \frac{\partial w}{\partial x} = 0 ,$$

which strictly spoken is not valid in the discontinuous points. Now, the time-derivatives in the point $x=x_k$ are estimated, after central-space integration, by

$$\frac{d}{dt}(w_k) = -U\,\frac{w_{k+1}-w_{k-1}}{2\Delta x} \quad .$$

Notice, that although this discretization is second-order, the information in $x=x_k$ is not used at all!
The time-derivative

$$\frac{d}{dt}(w_{k_l}) = -U\,\frac{w_h - w_l}{2\Delta x} < 0 \quad , \tag{1}$$

$$\frac{d}{dt}(w_{k_l+1}) = -U\,\frac{w_h - w_l}{2\Delta x} < 0 \quad , \tag{2}$$

$$\frac{d}{dt}(w_{k_r}) = -U\,\frac{w_l - w_h}{2\Delta x} > 0 \quad \text{and} \tag{3}$$

$$\frac{d}{dt}(w_{k_r+1}) = -U\,\frac{w_l - w_h}{2\Delta x} > 0 \quad , \tag{4}$$

while in all other points the time-derivatives are equal zero, as they should be. The inequalities (2) and (4) are necessary for the transport of the profile, but to conserve its shape we'd like to have

$$\frac{d}{dt}(w_{k_l}) = 0 \quad \text{and} \quad \frac{d}{dt}(w_{k_r}) = 0 \ .$$

and not (1) and (3); these inequalities cause in the first time-step respectively an under- and overshoot, and later on spurious oscillations in the numerical solution (Gibbs' phenomenon).

For this simple one-dimensional problem upwind-discretization could be the solution; however, upwind-methods are only first-order accurate, while central-methods can reach second-order accuracy (in smooth regions). Furthermore for upwind methods the 'upwind-direction' has to be determined, which requires a lot of work for two- or three-dimensional problems.

A finite-volume method in combination with artificial dissipation seems a better choice. This dissipation will only be added when necessary, that is, when large second differences of the quantity in question appear. Roughly one can say that central-space integration (second order accuracy) with a convenient amount of dissipation yields upwind integration (first order accuracy), since

$$-U \frac{w_k - w_{k-1}}{\Delta x} = -U \frac{w_{k+1} - w_{k-1}}{2\Delta x} + U \frac{w_{k+1} - 2w_k + w_{k-1}}{2\ \Delta x} \qquad \text{(upwind for } U>0\text{)}$$

and

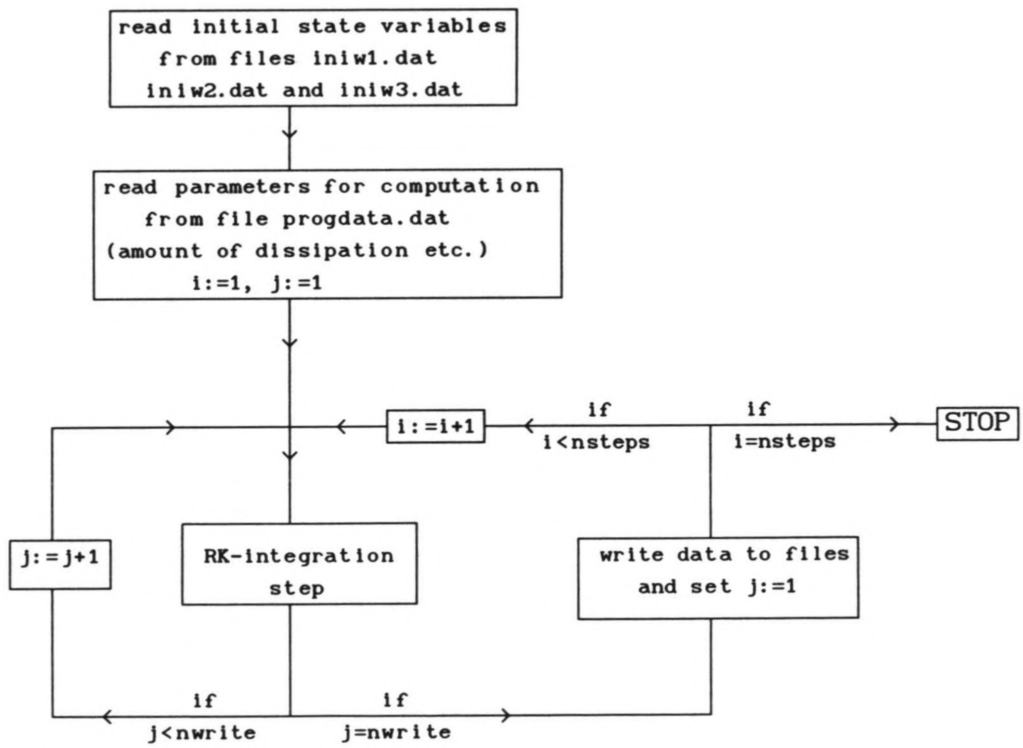
$$-U \frac{w_{k+1} - w_k}{\Delta x} = -U \frac{w_{k+1} - w_{k-1}}{2\Delta x} - U \frac{w_{k+1} - 2w_k + w_{k-1}}{2\ \Delta x} \qquad \text{(upwind for } U<0\text{).}$$

So, it is seen that in both cases we have added a dissipation term with (positive) artificial viscosity $|U| \cdot \Delta x/2$.

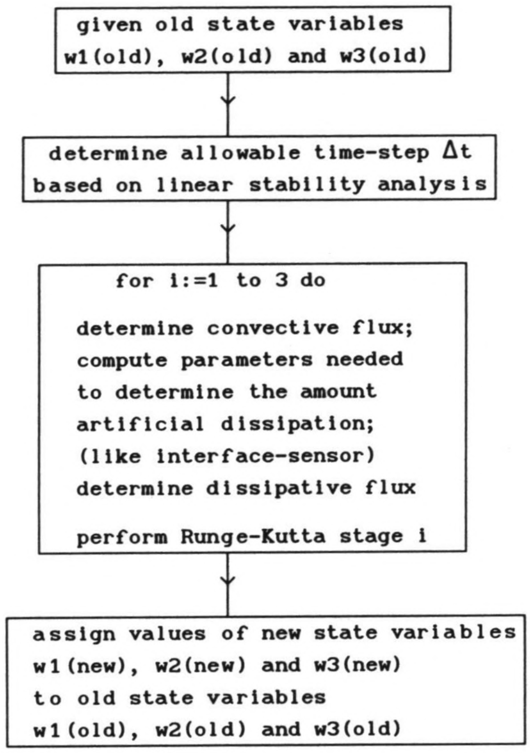
For this reason it seems convenient to me to choose the scaling factor $S^{(2)}$ of the second order dissipation term (chapter five), proportional to the velocity of the fluid.
Moreover, this means no dissipation is added when it isn't necessary , that is, when the fluid is not moving.
Since we have a hyperbolic system of equations, we always have to add some dissipation; therefore the scaling factor $S^{(4)}$ of the fourth order dissipation term will be chosen as the absolute maximum eigenvalue of the system.

Appendix IV : The computer program

The listing of the computer program which I used for the computation of one-dimensional compressible time-dependent flow of water and air is part of this appendix. The fact that my program is not an example of elegance, partly stems from my poor knowledge of FORTRAN (which is the most popular language in the computational world, I guess?). However, it does work and can roughly be schematized as is down here.



The block that performs a Runge-Kutta integration step is a bit more specified by the following scheme.



The following pages contain the FORTRAN-code.

```
C     ******************************************************************
C     This code is an attempt to compute one-dimensional, inviscid,
C     time-dependent, compressible flow of water and air;
C     Helmus van de Langemheen, version February 1992
C     ******************************************************************


      SUBROUTINE PRESSURE(DENS,FRAC,DP)
C     ******************************************************************
C     for given density DENS and mass-fraction air FRAC of water-air
C     mixture, the relative pressure DP is calculated (relative means
C     DP is absolute pressure minus standard-atmospheric pressure);
C     ******************************************************************
      DOUBLE PRECISION DENS,FRAC,DP
      DOUBLE PRECISION D1,D2,DA,DW,CA,CW
      DOUBLE PRECISION EPS
      PARAMETER (DA=1.23D0, DW=999.1D0,
     +           CA=3.4D2, CW=1460.0D0,
     +           EPS=1.0D-12)
      IF (FRAC.LE.EPS) THEN
        DP=(DENS-DW)*CW**2
      ELSE IF (FRAC.GE.(1.0D0-EPS)) THEN
        DP=(DENS-DA)*CA**2
      ELSE
        D1=DA*CA**2+DW*CW**2-DENS*(FRAC*CA**2+(1.0D0-FRAC)*CW**2)
        D2=((CA*CW)**2)*(DA*DW-DENS*(FRAC*DW+(1.0D0-FRAC)*DA))
        DP=0.5D0*(-D1+SQRT(D1*D1-4.0D0*D2))
      ENDIF
      END


      SUBROUTINE SPSOUND(DENS,FRAC,CM)
C     ******************************************************************
C     for given density DENS and mass-fraction air FRAC of water-air
C     mixture, the speed of sound CM of the mixture is calculated
C     ******************************************************************
      DOUBLE PRECISION DENS,FRAC,DP,CM
      DOUBLE PRECISION D1,D2,DA,DW,CA,CW
      DOUBLE PRECISION EPS
      PARAMETER (DA=1.23D0, DW=999.1D0,
     +           CA=3.4D2, CW=1460.0D0,
     +           EPS=1.0D-12)
      CALL PRESSURE(DENS,FRAC,DP)
      D1=(DENS*CA)/(DA*(CA**2)+DP)
      D2=(DENS*CW)/(DW*(CW**2)+DP)
      IF (FRAC.LE.EPS) THEN
        CM=CW
      ELSE IF (FRAC.GE.(1.0D0-EPS)) THEN
        CM=CA
```

```
      ELSE
        CM=SQRT(1.0D0/(FRAC*D1**2+(1.0D0-FRAC)*D2**2))
      ENDIF
      END


      SUBROUTINE AIRFRAC(DENS,FRAC)
C     ***************************************************************
C     for given density DENS of a water-air mixture, the fraction air
C     FRAC is calculated (given standard-atmospheric pressure)
C     ***************************************************************
      DOUBLE PRECISION DA,DW,DENS,FRAC
      PARAMETER (DA=1.23D0, DW=999.1D0)
      FRAC=DA*(DENS-DW)/(DENS*(DA-DW))
      END


      SUBROUTINE CONVFLUX(W1,W2,W3,FW1,FW2,FW3,GRAVCON,PRES,N,DX)
C     ***************************************************************
C     convective terms FW1, FW2 and FW3 for variables W1, W2 and W3
C     are calculated; the gravitational constant is given by GRAVCON;
C     pressure PRES is calculated here for each point, based on state
C     variables W1 and W2; FW1, FW2 and FW3 become convective fluxes
C     after multiplying by timestep
C     ***************************************************************
      INTEGER J,N
      DIMENSION W1(-2:N+2),W2(-2:N+2),W3(-2:N+2)
      DIMENSION FW1(-2:N+2),FW2(-2:N+2),FW3(-2:N+2)
      DIMENSION PRES(-2:N+2)
      DOUBLE PRECISION W1,W2,W3,FW1,FW2,FW3
      DOUBLE PRECISION PRES,DX,GRAVCON
      DO 80 J=-1,N+1
        FW1(J)=(W3(J+1 )-W3(J-1))/(2*DX)
   80 CONTINUE
      DO 90 J=-1,N+1
        FW2(J)=(W2(J+1)*W3(J+1)/W1(J+1)-
     +          W2(J-1)*W3(J-1)/W1(J-1))/(2*DX)
   90 CONTINUE
      DO 100 J=-1,N+1
        CALL PRESSURE(W1(J),W2(J)/W1(J),PRES(J))
  100 CONTINUE
      DO 110 J=-1,N+1
        FW3(J)=(PRES(J+1)+W3(J+1)*W3(J+1)/W1(J+1)-
     +          PRES(J-1)-W3(J-1)*W3(J-1)/W1(J-1))/(2*DX)+
     +          W1(J)*GRAVCON
  110 CONTINUE
C     The following statements should be added in case of
C     impermeable boundaries
      FW3(N)=FW3(N)-W1(N)*GRAVCON
      FW3(0)=FW3(0)-W1(0)*GRAVCON
      END
```

```
      SUBROUTINE DISSDATA(W1,W2,W3,K2,K4,LAM1,LAM2,NUU,EP2,EP4,N)
C     ****************************************************************
C     for given state variables W1, W2 and W3, parameters K2, K4,
C     absolute eigenvalues LAM1 and LAM2 (¦U¦ and ¦U¦+C) are
C     calculated; furthermore values of some sensor NUU are calculated
C     resulting in coefficients EP2 and EP4 for the dissipative terms
C     (respectively second and fourth order); these values are used in
C     subroutine DISSFLUX to calculate the dissipative flux in each
C     point
C     ****************************************************************
      INTEGER J,N
      DIMENSION W1(-2:N+2),W2(-2:N+2),W3(-2:N+2)
      DIMENSION LAM1(-2:N+2),LAM2(-2:N+2),NUU(-2:N+2)
      DIMENSION EP2(-2:N+2),EP4(-2:N+2)
      DOUBLE PRECISION W1,W2,W3,K2,K4,CMIX
      DOUBLE PRECISION LAM1,LAM2,NUU,EP2,EP4
      DO 140 J=-2,N+2
        CALL SPSOUND(W1(J),W2(J)/W1(J),CMIX)
        LAM1(J)=ABS(W3(J)/W1(J))
        LAM2(J)=ABS(W3(J)/W1(J))+CMIX
  140 CONTINUE
C     Values of sensor are calculated now
      DO 120 J=-1,N+1
        NUU(J)=ABS(W2(J-1)/W1(J-1)-2*W2(J)/W1(J)+W2(J+1)/W1(J+1))
     +      /(W2(J-1)/W1(J-1)+2*W2(J)/W1(J)+W2(J+1)/W1(J+1)+1.D-8)
  120 CONTINUE
      DO 130 J=-1,N
        EP2(J)=K2*MAX(NUU(J),NUU(J+1))
        EP4(J)=MAX(0.0D0,K4-EP2(J))
  130 CONTINUE
      DO 150 J=-1,N
        LAM1(J)=0.5D0*(LAM1(J)+LAM1(J+1))
        LAM2(J)=0.5D0*(LAM2(J)+LAM2(J+1))
  150 CONTINUE
      END


      SUBROUTINE DISSFLUX(W1,W2,W3,FW1,FW2,FW3,LAM1,LAM2,EP2,EP4,DX,N)
C     ****************************************************************
C     for given state variables W1, W2 and W3, the total dissipative
C     flux in each point is calculated, where LAM1, LAM2, EP2 and EP4
C     are used and already calculated in the routine DISSDATA;
C     according to the last statements, no dissipation is added at
C     boundaries
C     ****************************************************************
      INTEGER J,N
      DIMENSION W1(-2:N+2),W2(-2:N+2),W3(-2:N+2)
```

```
      DIMENSION FW1(-2:N+2),FW2(-2:N+2),FW3(-2:N+2)
      DIMENSION LAM1(-2:N+2),LAM2(-2:N+2)
      DIMENSION EP2(-2:N+2),EP4(-2:N+2)
      DOUBLE PRECISION W1,W2,W3,FW1,FW2,FW3
      DOUBLE PRECISION LAM1,LAM2,EP2,EP4,DX
      DO 160 J=0,N
        FW1(J)=LAM1(J)*(EP2(J)*(W1(J+1)-W1(J)))-
     +         LAM2(J)*(EP4(J)*(W1(J+2)-3*W1(J+1)+3*W1(J)-W1(J-1)))-
     +         LAM1(J-1)*(EP2(J-1)*(W1(J)-W1(J-1)))+
     +         LAM2(J-1)*(EP4(J-1)*(W1(J+1)-3*W1(J)+3*W1(J-1)-W1(J-2)))
 160  CONTINUE
      DO 170 J=0,N
        FW2(J)=LAM1(J)*(EP2(J)*(W2(J+1)-W2(J)))-
     +         LAM2(J)*(EP4(J)*(W2(J+2)-3*W2(J+1)+3*W2(J)-W2(J-1)))-
     +         LAM1(J-1)*(EP2(J-1)*(W2(J)-W2(J-1)))+
     +         LAM2(J-1)*(EP4(J-1)*(W2(J+1)-3*W2(J)+3*W2(J-1)-W2(J-2)))
 170  CONTINUE
      DO 180 J=0,N
        FW3(J)=LAM1(J)*(EP2(J)*(W3(J+1)-W3(J)))-
     +         LAM2(J)*(EP4(J)*(W3(J+2)-3*W3(J+1)+3*W3(J)-W3(J-1)))-
     +         LAM1(J-1)*(EP2(J-1)*(W3(J)-W3(J-1)))+
     +         LAM2(J-1)*(EP4(J-1)*(W3(J+1)-3*W3(J)+3*W3(J-1)-W3(J-2)))
 180  CONTINUE
       DO 190 J= 0,N
        FW1(J)=-FW1(J)/(2*DX)
        FW2(J)=-FW2(J)/(2*DX)
        FW3(J)=-FW3(J)/(2*DX)
 190  CONTINUE
C     No dissipation at
C     boundaries, so
      FW1(0)=0.0D0
      FW2(0)=0.0D0
      FW3(0)=0.0D0
      FW1(N)=0.0D0
      FW2(N)=0.0D0
      FW3(N)=0.0D0
      END


      SUBROUTINE BOPEN(W1,W2,W3,N)
C     ****************************************************************
C     (approximately) the boundary condition d/dx(.)=0 is applied here
C     by requiring the values of the state variables in the virtual
C     points are the same as in the boundary points; this routine can
C     be used when no changes are present at the boundaries; in that
C     case it is the same as requiring D/Dt(.)=0, but this does not
C     hold in general!
C     ****************************************************************
      INTEGER N
      DIMENSION W1(-2:N+2),W2(-2:N+2),W3(-2:N+2)
```

```
      DOUBLE PRECISION W1,W2,W3
      W1(-1)  = W1(0)
      W1(-2)  = W1(0)
      W1(N+1) = W1(N)
      W1(N+2) = W1(N)
      W2(-1)  = W2(0)
      W2(-2)  = W2(0)
      W2(N+1) = W2(N)
      W2(N+2) = W2(N)
      W3(-1)  = W3(0)
      W3(-2)  = W3(0)
      W3(N+1) = W3(N)
      W3(N+2) = W3(N)
      END


      SUBROUTINE BCLOSED(W1,W2,W3,N)
C     ****************************************************************
C     the values of the state variables W1, W2 and W3 in the virtual
C     points are chosen such that the boundaries may be assumed
C     impermeable;
C     ****************************************************************
      INTEGER N
      DIMENSION W1(-2:N+2),W2(-2:N+2),W3(-2:N+2)
      DOUBLE PRECISION W1,W2,W3
      W1(-1)  = W1(1)
      W1(-2)  = W1(2)
      W1(N+1) = W1(N-1)
      W1(N+2) = W1(N-2)
      W2(-1)  = W2(1)
      W2(-2)  = W2(2)
      W2(N+1) = W2(N-1)
      W2(N+2) = W2(N-2)
      W3(0)   = 0.0D0
      W3(-1)  = -W3(1)
      W3(-2)  = -W3(2)
      W3(N)   = 0.0D0
      W3(N+1) = -W3(N-1)
      W3(N+2) = -W3(N-2)
      END


      SUBROUTINE RKSTAGE(ALFA,W1OLD,W2OLD,W3OLD,W1NEW,W2NEW,W3NEW,
     +                     DIFL1,DIFL2,DIFL3,COFL1,COFL2,COFL3,DT,N)
C     ****************************************************************
C     one stage of the Runge-Kutta time integration scheme is carried
C     out by this routine; DIFL1, DIFL2 DIFL3, COFL1, COFL2 and COFL3
C     are respectively the known dissipative and convective fluxes;
C     DT is the actual time-step, ALFA a constant which depends on
```

```
C     the stage of the scheme (three stages: 0.6, 0.6 and 1.0);
C     *****************************************************************
      INTEGER J,N
      DIMENSION W1OLD(-2:N+2),W2OLD(-2:N+2),W3OLD(-2:N+2)
      DIMENSION W1NEW(-2:N+2),W2NEW(-2:N+2),W3NEW(-2:N+2)
      DIMENSION DIFL1(-2:N+2),DIFL2(-2:N+2),DIFL3(-2:N+2)
      DIMENSION COFL1(-2:N+2),COFL2(-2:N+2),COFL3(-2:N+2)
      DOUBLE PRECISION W1OLD,W2OLD,W3OLD,W1NEW,W2NEW,W3NEW
      DOUBLE PRECISION DIFL1,DIFL2,DIFL3,COFL1,COFL2,COFL3
      DOUBLE PRECISION ALFA,DT
      DO 200 J=0,N
        W1NEW(J)=W1OLD(J)-ALFA*DT*(DIFL1(J)+COFL1(J))
        W2NEW(J)=W2OLD(J)-ALFA*DT*(DIFL2(J)+COFL2(J))
        W3NEW(J)=W3OLD(J)-ALFA*DT*(DIFL3(J)+COFL3(J))
 200  CONTINUE
      CALL BCLOSED(W1NEW,W2NEW,W3NEW,N)
      END


      SUBROUTINE RK(INIW1,INIW2,INIW3,CONVW1,CONVW2,CONVW3,
     +              DISSW1,DISSW2,DISSW3,LAM1,LAM2,NU,EPS2,EPS4,
     +              W1,W2,W3,PR,DX,DT,KAPPA2,KAPPA4,N,GRAVCON)
C     *****************************************************************
C     performs one step of the time-integration process; the initial
C     state variables have values INIW1, INIW2 and INIW3, which after
C     this time-step become W1, W2 and W3
C     *****************************************************************
      INTEGER N
      DIMENSION INIW1(-2:N+2),INIW2(-2:N+2),INIW3(-2:N+2)
      DIMENSION CONVW1(-2:N+2),CONVW2(-2:N+2),CONVW3(-2:N+2)
      DIMENSION DISSW1(-2:N+2),DISSW2(-2:N+2),DISSW3(-2:N+2)
      DIMENSION LAM1(-2:N+2),LAM2(-2:N+2),NU(-2:N+2)
      DIMENSION EPS2(-2:N+2),EPS4(-2:N+2)
      DIMENSION W1(-2:N+2),W2(-2:N+2),W3(-2:N+2),PR(-2:N+2)
      DOUBLE PRECISION INIW1,INIW2,INIW3
      DOUBLE PRECISION CONVW1,CONVW2,CONVW3
      DOUBLE PRECISION DISSW1,DISSW2,DISSW3
      DOUBLE PRECISION LAM1,LAM2,NU,EPS2,EPS4
      DOUBLE PRECISION W1,W2,W3,PR,GRAVCON
      DOUBLE PRECISION ALPHA,DX,DT,KAPPA2,KAPPA4
      CALL CONVFLUX(INIW1,INIW2,INIW3,CONVW1,
     +              CONVW2,CONVW3,GRAVCON,PR,N,DX)
      CALL DISSDATA(INIW1,INIW2,INIW3,KAPPA2,KAPPA4,
     +                        LAM1,LAM2,NU,EPS2,EPS4,N)
      CALL DISSFLUX(INIW1,INIW2,INIW3,DISSW1,DISSW2,
     +              DISSW3,LAM1,LAM2,EPS2,EPS4,DX,N)
      ALPHA=0.6D0
      CALL RKSTAGE(ALPHA,INIW1,INIW2,INIW3,W1,W2,W3,
     +             DISSW1,DISSW2,DISSW3,CONVW1,CONVW2,CONVW3,DT,N)
      CALL CONVFLUX(W1,W2,W3,CONVW1,CONVW2,CONVW3,GRAVCON,PR,N,DX)
```

```
      CALL DISSDATA(INIW1,INIW2,INIW3,KAPPA2,KAPPA4,
     +                          LAM1,LAM2,NU,EPS2,EPS4,N)
      CALL DISSFLUX(INIW1,INIW2,INIW3,DISSW1,DISSW2,
     +              DISSW3,LAM1,LAM2,EPS2,EPS4,DX,N)
      CALL RKSTAGE(ALPHA,INIW1,INIW2,INIW3,W1,W2,W3,
     +            DISSW1,DISSW2,DISSW3,CONVW1,CONVW2,CONVW3,DT,N)
      CALL CONVFLUX(W1,W2,W3,CONVW1,CONVW2,CONVW3,GRAVCON,PR,N,DX)
      CALL DISSDATA(INIW1,INIW2,INIW3,KAPPA2,KAPPA4,
     +                          LAM1,LAM2,NU,EPS2,EPS4,N)
      CALL DISSFLUX(INIW1,INIW2,INIW3,DISSW1,DISSW2,
     +              DISSW3,LAM1,LAM2,EPS2,EPS4,DX,N)
      ALPHA=1.0D0
      CALL RKSTAGE(ALPHA,INIW1,INIW2,INIW3,W1,W2,W3,
     +            DISSW1,DISSW2,DISSW3,CONVW1,CONVW2,CONVW3,DT,N)
      END


      SUBROUTINE TIMESTEP(W1,W2,W3,K2,K4,EP2,EP4,LAM1,LAM2,NUU,XSTP,DT,N)
C     ******************************************************************
C     the maximum allowable timestep DT is calculated here, based on
C     a linear stability analysis; for both dissipative and convective
C     'direction' a restriction on the time-step is found;
C     the minimum of these is assigned to DT; the parameters REB and
C     IMB determine a subset [-REB,0]x[-IMB,IMB] of the stability
C     region as mentioned in chapter five;
C     ******************************************************************
      INTEGER J,N
      DIMENSION W1(-2:N+2),W2(-2:N+2),W3(-2:N+2)
      DIMENSION LAM1(-2:N+2),LAM2(-2:N+2),NUU(-2:N+2)
      DIMENSION EP2(-2:N+2),EP4(-2:N+2)
      DOUBLE PRECISION W1,W2,W3,K2,K4,CMIX
      DOUBLE PRECISION LAM1,LAM2,NUU,EP2,EP4,EPS
      DOUBLE PRECISION TMRE,TMIM,DT,TMSTP,XSTP,REB,IMB
      PARAMETER (REB=1.0D0,
     +           IMB=1.0D0,
     +           EPS=1.0D-10)
      DO 140 J=-2,N+2
        CALL SPSOUND(W1(J),W2(J)/W1(J),CMIX)
        LAM1(J)=ABS(W3(J)/W1(J))
        LAM2(J)=ABS(W3(J)/W1(J))+CMIX
  140 CONTINUE
      DO 120 J=-1,N+1
        NUU(J)=ABS(W2(J-1)/W1(J-1)-2*W2(J)/W1(J)+W2(J+1)/W1(J+1))
     +          /(W2(J-1)/W1(J-1)+2*W2(J)/W1(J)+W2(J+1)/W1(J+1)+1.D-12)
  120 CONTINUE
      DO 130 J=-1,N
        EP2(J)=K2*MAX(NUU(J),NUU(J+1))
        EP4(J)=MAX(0.0D0,K4-EP2(J))
  130 CONTINUE
      DO 150 J=-1,N
```

```
        LAM1(J)=0.5D0*(LAM1(J)+LAM1(J+1))
        LAM2(J)=0.5D0*(LAM2(J)+LAM2(J+1))
 150  CONTINUE
      TMIM=IMB/LAM2(-1)
      TMRE=REB/(EPS+LAM2(-1)*(4*EP2(-1)+16*EP4(-1)))
      TMSTP=TMIM
      IF (TMRE.LE.TMSTP) THEN
        TMSTP=TMRE
      ENDIF
      DO 160 J=0,N
        TMIM=IMB
        TMRE=REB/(EPS+LAM2(J)*(4*EP2(J)+16*EP4(J)))
        IF (TMIM.LE.TMSTP) THEN
          TMSTP=TMIM
        ENDIF
        IF (TMRE.LE.TMSTP) THEN
          TMSTP=TMRE
        ENDIF
160   CONTINUE
      DT=XSTP*TMSTP
      END


      PROGRAM EULERCALC
C     *****************************************************************
C     main program; performs read- and write-actions, time-integration
C     N is number of grid-points;
C     OWi, i=1,2,3 represent the 'old' state variables;
C     (before time-integration)
C     P1 is pressure;
C     CW1 is convective flux;
C     DW1 is dissipative flux;
C     LMD1 and LMD2 are absolute eigenvalues |U| and |U|+C;
C     EPS2 and EPS4 are factors used for dissipative fluxes;
C     NWi, i=1,2,3 represent the 'new' state variables;
C     (after time-integration)
C     LEN is the physical length of the interval, and DX the (uniform)
C     distance between the grid-points;
C     GRAVC is the gravitational constant (which is acting in positive
C     direction;
C     TIMFR is the fraction of the maximum allowable time-step that is
C     used during integration;
C     K2 and K4 stand for kappa2 and kappa4, the coefficients used for
C     the dissipative terms;
C     NSTEPS is the number of time-steps after which data are written
C     to files;
C     NWRITE is the number of times that is written to the data-files,
C     so the total number of time-steps is NWRITE*NSTEPS
C     *****************************************************************
      INTEGER I,J,N,NSTEPS,NWRITE
```

```
      PARAMETER (N=100)
      DIMENSION OW1(-2:N+2),OW2(-2:N+2),OW3(-2:N+2)
      DIMENSION P1(-2:N+2)
      DIMENSION CW1(-2:N+2),CW2(-2:N+2),CW3(-2:N+2)
      DIMENSION DW1(-2:N+2),DW2(-2:N+2),DW3(-2:N+2)
      DIMENSION LMD1(-2:N+2),LMD2(-2:N+2),NU(-2:N+2)
      DIMENSION EPS2(-2:N+2),EPS4(-2:N+2)
      DIMENSION NW1(-2:N+2),NW2(-2:N+2),NW3(-2:N+2)
      DOUBLE PRECISION OW1,OW2,OW3,P1,CW1,CW2,CW3
      DOUBLE PRECISION DW1,DW2,DW3,LMD1,LMD2,NU,EPS2,EPS4
      DOUBLE PRECISION NW1,NW2,NW3
      DOUBLE PRECISION LEN,DX,DT,K2,K4,ELTIME
      DOUBLE PRECISION GRAVC,TIMFR
      INTEGER*2 IHR,IMI,ISE,IHU
      PARAMETER (LEN=10.0D0,
     +           DX=LEN/N)
      CALL GETTIM(IHR,IMI,ISE,IHU)
      WRITE(*,*) ' '
      WRITE(*,*) ' '
      WRITE(*,*) '**********************************************'
      WRITE(*,*) '**  ONE-DIMENSIONAL FLOW OF WATER AND AIR  **'
      WRITE(*,*) '**********************************************'
      WRITE(*,*) ' '
      WRITE(*,*) '**********************************************'
      WRITE(*,*) 'COMPUTATION STARTED : ',IHR,':',IMI
      WRITE(*,*) '**********************************************'
      WRITE(*,*) ' '
      WRITE(*,*) '****************************'
      WRITE(*,*) '  ACTUAL /  LAST    TIMESTEP'
      WRITE(*,*) '  ------  ------'
      WRITE(*,*) ' '
      OPEN(UNIT=10,FILE='DENS.DAT')
      OPEN(UNIT=11,FILE='PRES.DAT')
      OPEN(UNIT=12,FILE='VELO.DAT')
      OPEN(UNIT=13,FILE='TIME.DAT')
      OPEN(UNIT=17,FILE='PROGDATA.DAT')
      OPEN(UNIT=18,FILE='EP24.DAT')
      OPEN(UNIT=19,FILE='FLUX.DAT')
      OPEN(UNIT=20,FILE='NUUS.DAT')
      OPEN(UNIT=21,FILE='FRAC.DAT')
      OPEN(UNIT=31,FILE='INIW1.DAT')
      OPEN(UNIT=32,FILE='INIW2.DAT')
      OPEN(UNIT=33,FILE='INIW3.DAT')
      READ(17,*) K2
      READ(17,*) K4
      READ(17,*) NSTEPS
      READ(17,*) NWRITE
      READ(17,*) TIMFR
      READ(17,*) GRAVC
      READ(17,*) DT
C     DT MAY BE A FIXED TIMESTEP
```

```
C     NOW READ INITIAL STATE VARIABLES
      DO 505 J=0,N
        READ(31,*) OW1(J)
        READ(32,*) OW2(J)
        READ(33,*) OW3(J)
  505 CONTINUE
C     AND ASSIGN VALUES TO VARIABLES
C     IN VIRTUAL POINTS
      CALL BCLOSED(OW1,OW2,OW3,N)
C     CALCULATE MAXIMUM ALLOWABLE TIME-STEP
C     WHEN YOU DON'T WANT TO USE A FIXED TIME-STEP
      CALL TIMESTEP(OW1,OW2,OW3,K2,K4,EPS2,EPS4,
     +                      LMD1,LMD2,NU,DX,DT,N)
      DT=TIMFR*DT
C     IS ACTUAL TIME-STEP
C     ELAPSED TIME IS
      ELTIME=0.0D0
      WRITE(13,*) ELTIME,OW3(N/2)/OW1(N/2)
C     WRITING OF DATA TO FILES
      DO 600 J=0,N
        WRITE(10,*) J*DX,OW1(J)
        CALL PRESSURE(OW1(J),OW2(J)/OW1(J),P1(J))
        WRITE(11,*) J*DX,P1(J)
        WRITE(12,*) J*DX,OW3(J)/OW1(J)
        WRITE(20,*) J*DX,NU(J)
  600 CONTINUE
C     NOW PERFORM THE REQUIRED NUMBER OF INTEGRATION-STEPS
      DO 400 L=1,NSTEPS
        DO 350 I=1,NWRITE
          CALL RK(OW1,OW2,OW3,CW1,CW2,CW3,
     +            DW1,DW2,DW3,LMD1,LMD2,NU,EPS2,
     +            EPS4,NW1,NW2,NW3,P1,DX,DT,K2,K4,N,GRAVC)
          DO 300 J=-2,N+2
            OW1(J)=NW1(J)
            OW2(J)=NW2(J)
            OW3(J)=NW3(J)
 300      CONTINUE
          CALL BCLOSED(OW1,OW2,OW3,N)
          ELTIME=ELTIME+DT
C     CALCULATE MAXIMUM ALLOWABLE TIME-STEP
C     WHEN YOU DON'T WANT TO USE A FIXED TIME-STEP
          CALL TIMESTEP(OW1,OW2,OW3,K2,K4,EPS2,EPS4,
     +                          LMD1,LMD2,NU,DX,DT,N)
          DT=TIMFR*DT
C     IS ACTUAL TIMESTEP
          WRITE(*,345) (L-1)*NWRITE+I,NSTEPS*NWRITE
 345      FORMAT(1H+,I8,I8)
 350    CONTINUE
C     SOME DATA WILL BE WRITTEN
C     TO FILES NOW
        WRITE(13,*) ELTIME,OW3(N/2)/OW1(N/2)
```

```
        DO 375 J=0,N
          WRITE(10,*) J*DX,NW1(J)
          CALL PRESSURE(NW1(J),NW2(J)/NW1(J),P1(J))
          WRITE(11,*) J*DX,P1(J)
          WRITE(12,*) J*DX,NW3(J)/NW1(J)
          WRITE(18,*) J*DX,EPS2(J),EPS4(J)
          WRITE(19,*) J*DX,CW1(J),DW1(J)
          WRITE(20,*) J*DX,NU(J)
          WRITE(21,*) J*DX,NW2(J)/NW1(J)
 375    CONTINUE
 400  CONTINUE
C     COMPUTATION HAS BEEN FINISHED HERE
      CALL GETTIM(IHR,IMI,ISE,IHU)
      WRITE(*,*) '******************************'
      WRITE(*,*) ' '
      WRITE(*,*) '**********************************************'
      WRITE(*,*) 'COMPUTATION FINISHED:',IHR,':',IMI
      WRITE(*,*) '**********************************************'
      CLOSE(10)
      CLOSE(11)
      CLOSE(12)
      CLOSE(13)
      CLOSE(17)
      CLOSE(18)
      CLOSE(19)
      CLOSE(20)
      CLOSE(21)
      CLOSE(31)
      CLOSE(32)
      CLOSE(33)
      END
```

Appendix V : Accuracy

Central-space discretisation yields a second-order truncation error for the numerical solution (which can be found by Taylor expansion). In areas where a lot of artificial dissipation is added, this accuracy cannot be obtained, then the error is first-order.
The three-stage Runge-Kutta time-integration scheme is, as is well-known, third-order accurate.

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FIGURE I

SPEED OF SOUND IN A MIXTURE OF WATER AND AIR FOR EQUILIBRIUM STATES P=0.5, 1.0 AND 2.0 ATMOSPHERE

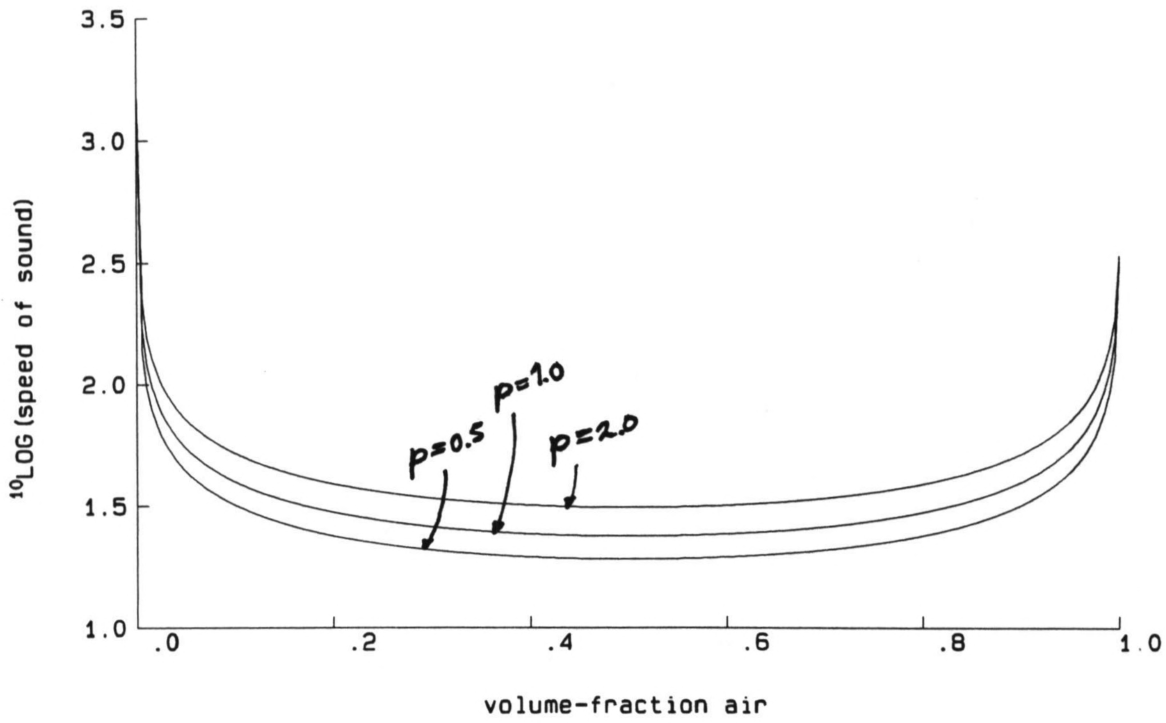

FIGURE II

Stability region of 3-stage Runge-Kutta scheme
with three evaluations of dissipative flux

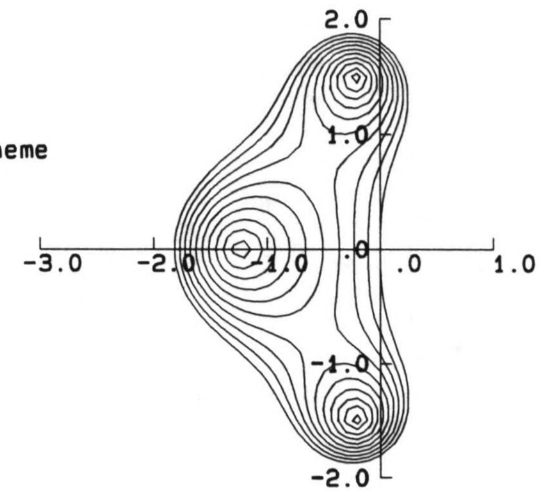

FIGURE III

Stability region of 3-stage Runge-Kutta scheme
with single evaluation of dissipative flux

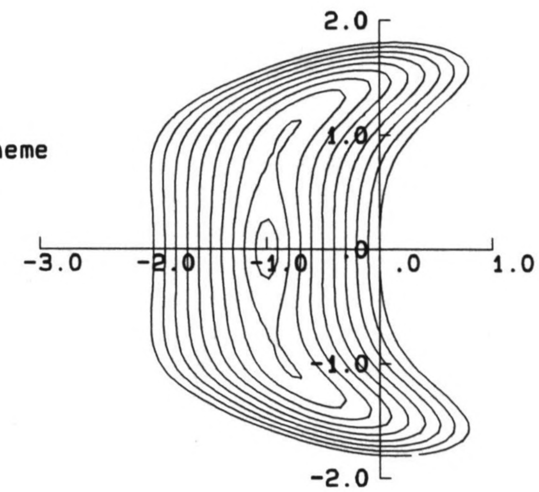

FIGURE IV.I

Mass-density at different points of time
No artificial dissipation is added
Fluid velocity U = 1 m/sec
N=50

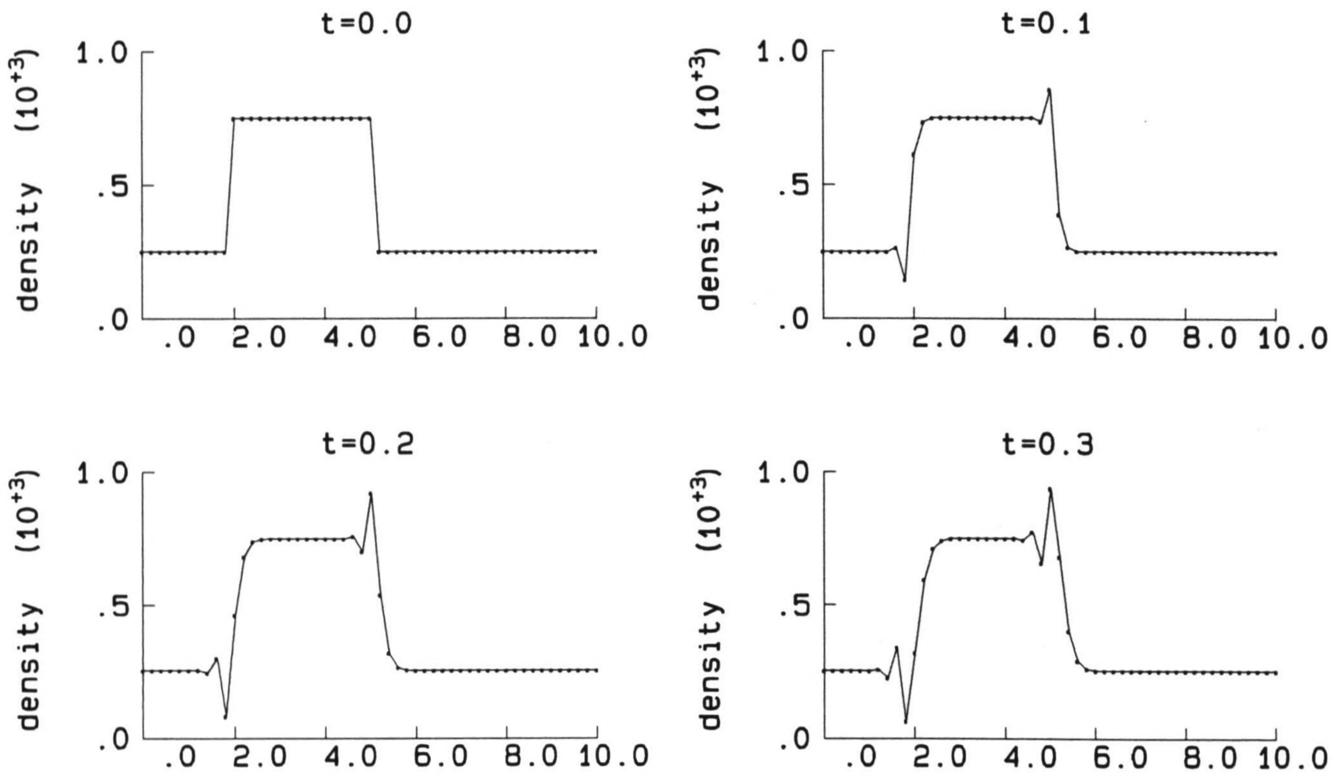

FIGURE IV.II

Mass-density at different points of time
No artificial dissipation is added
Fluid velocity U = 1 m/sec
N=200

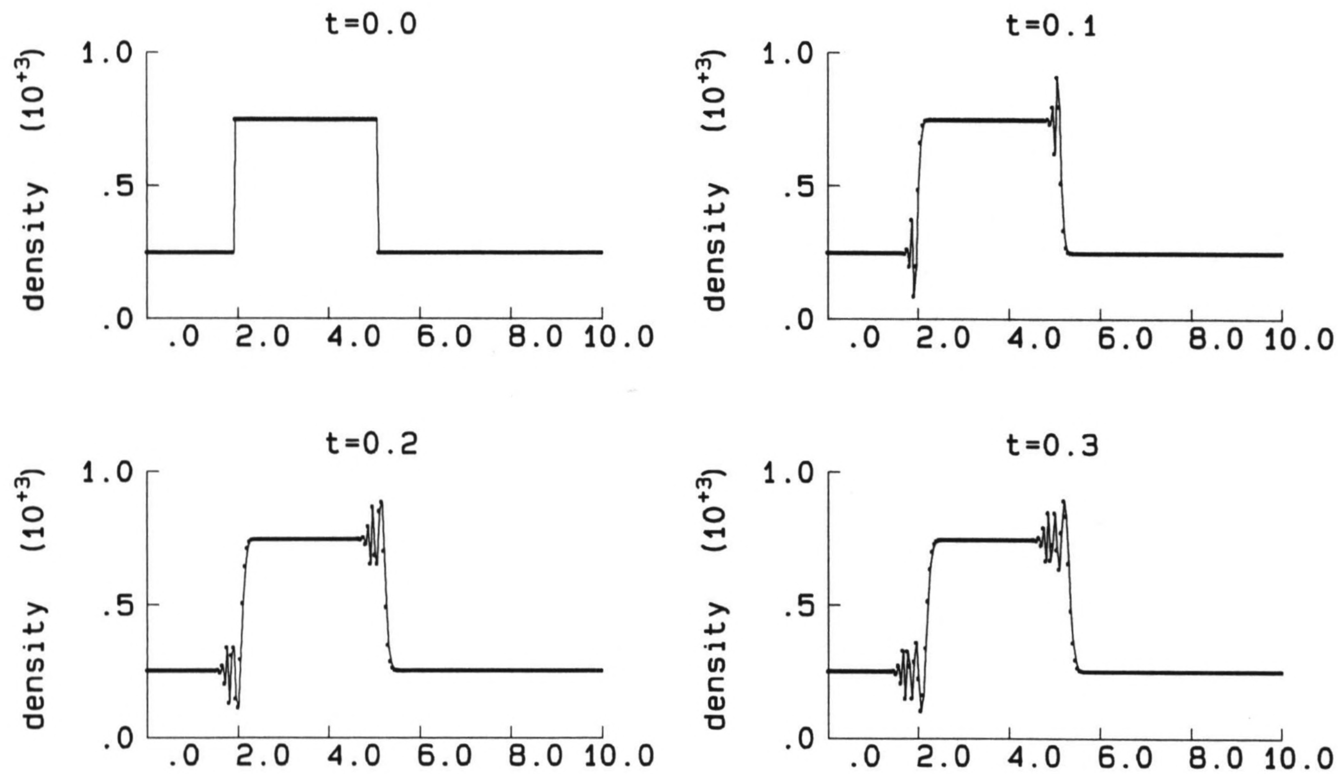

FIGURE IV.III

Mass-density at different points of time
Artificial dissipation: K2=15.0, K4=0.2
Fluid velocity: U=1m/sec
N=50

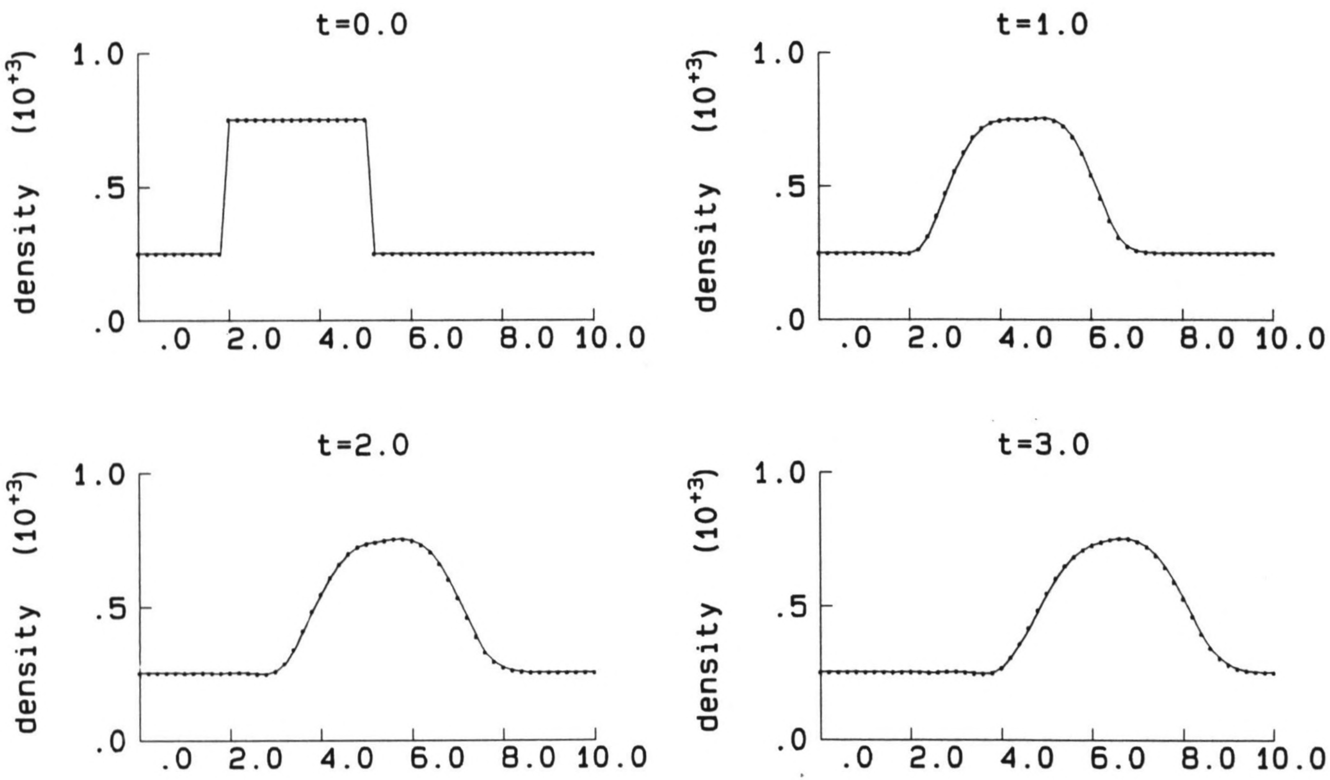

FIGURE IV.IV

Mass-density at different points of time
Artificial dissipation: K2=20.0, K4=0.4
Fluid velocity U = 1 m/sec
N=200

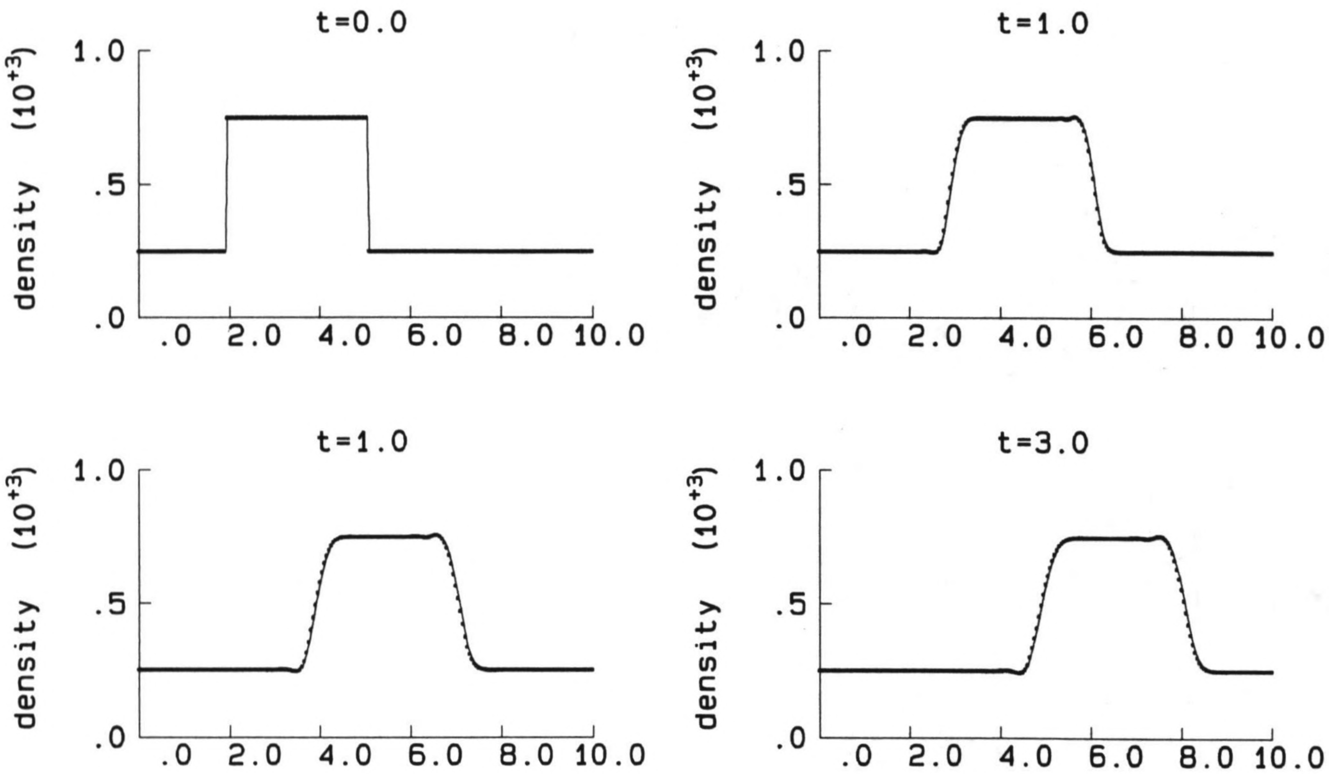

FIGURE IV.V

Mass-density at different points of time
Artificial dissipation: K2=20.0, K4=0.4
Initial fluid velocity U = 1 m/sec, g = 1 m/sec^2
N=200

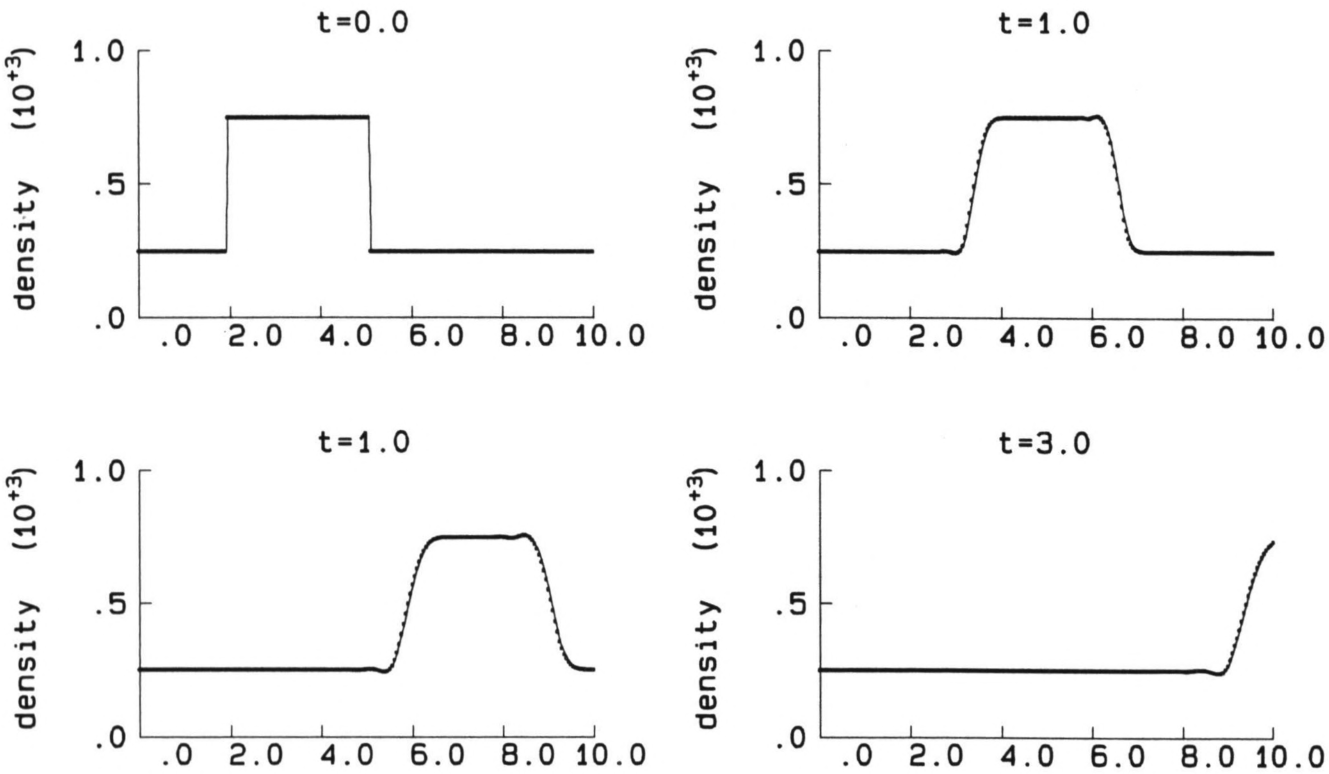

FIGURE V.I

Values of pressure in tube at different times
Sensor is normalized second difference of pressure
Initial fluid velocity U = 0.1 m/s
Boundary condition U(x=10.0) = 0.0 m/s (t>0)
kappa(2) = 0.0, kappa(4) = 0.0

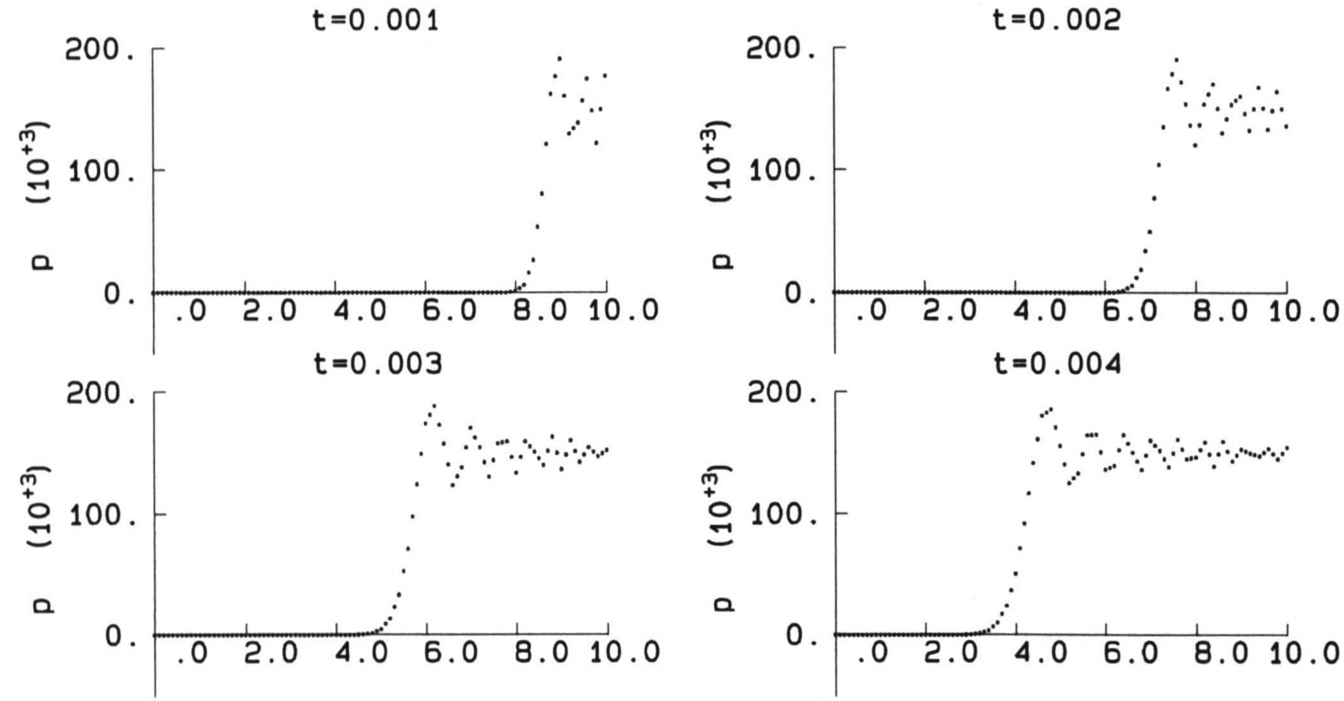

FIGURE V.II

Values of pressure in tube at different times
Sensor is normalized second difference of pressure
Initial fluid velocity U = 0.1 m/s
Boundary condition U(x=10.0) = 0.0 m/s (t>0)
kappa(2) = 0.0, kappa(4) = 0.01

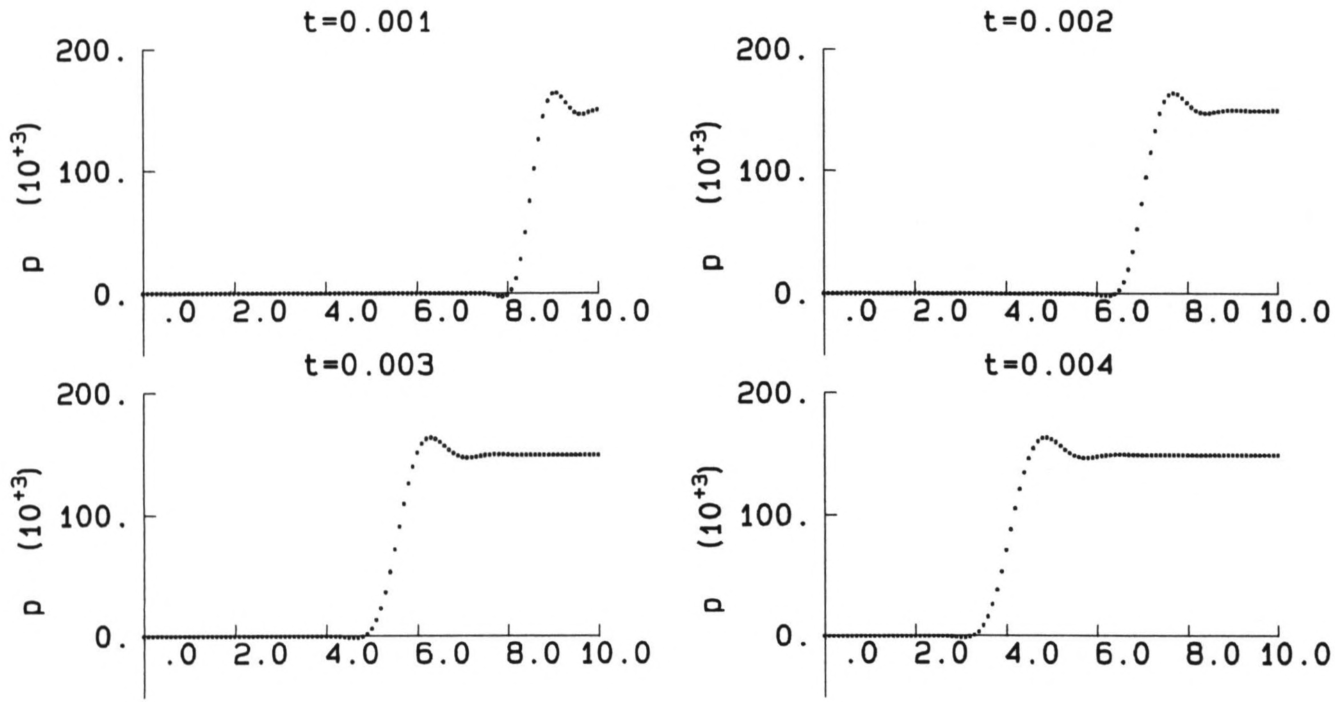

FIGURE V.III

Values of pressure in tube at different times
Sensor is normalized second difference of pressure
Initial fluid velocity U = 0.1 m/s
Boundary condition U(x=10.0) = 0.0 m/s (t>0)
kappa(2) = 1.0, kappa(4) = 0.01

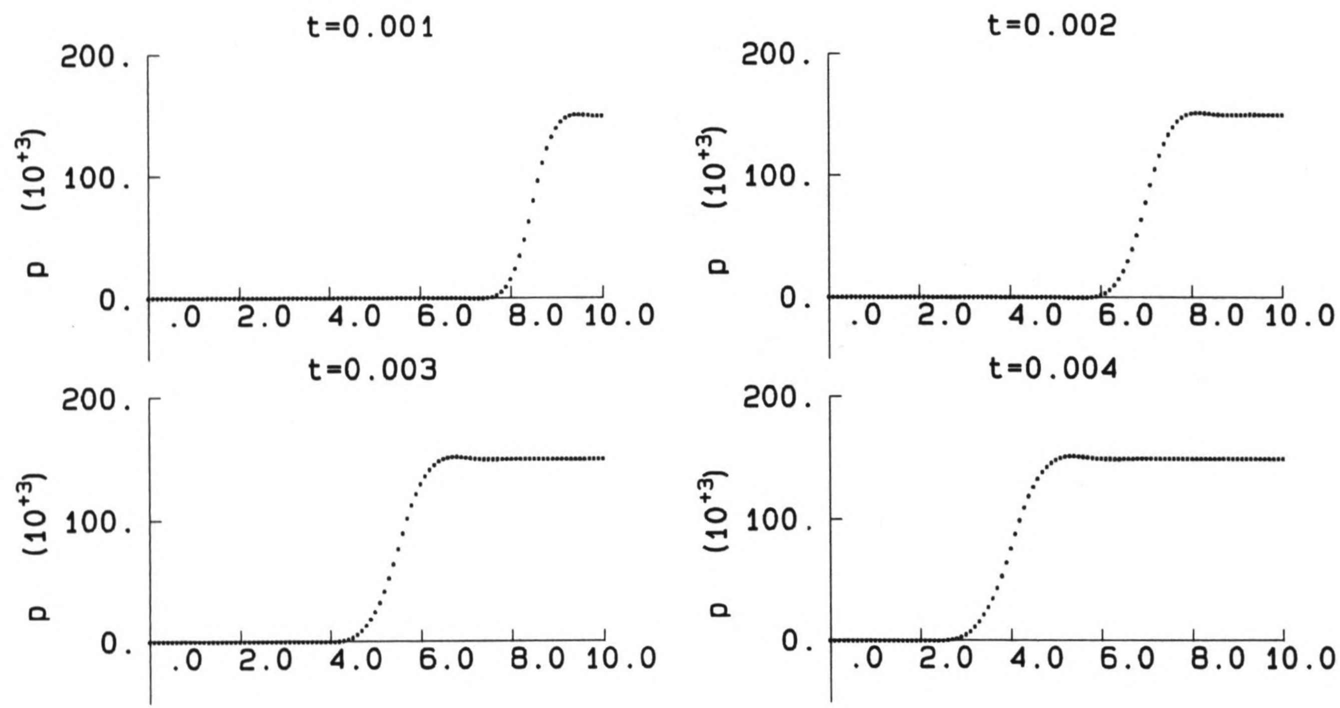

FIGURE VI.I

Harmonic oscillation of water and air

Time-velocity plot of column of water

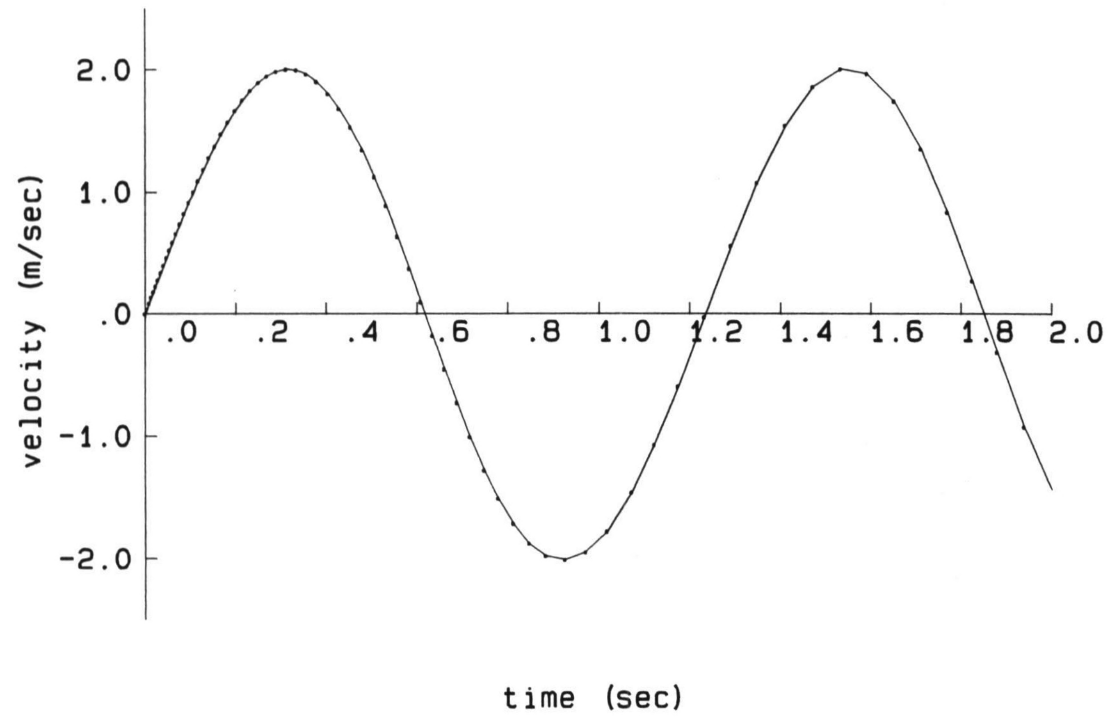

FIGURE VI.II

Pressure, velocity, density and interface-sensor along the tube at t = 0.1 sec.

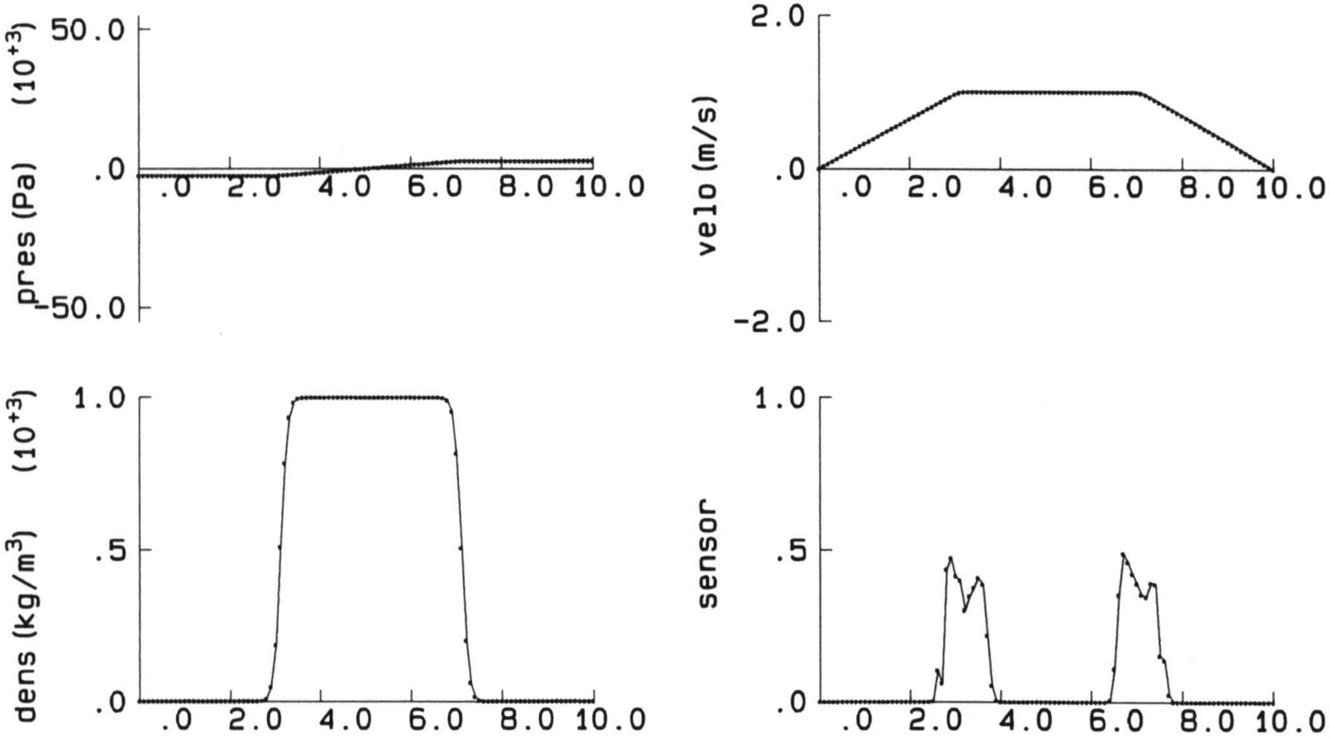

FIGURE VI.III

Pressure, velocity, density and interface-sensor along the tube at t = 0.63 sec.

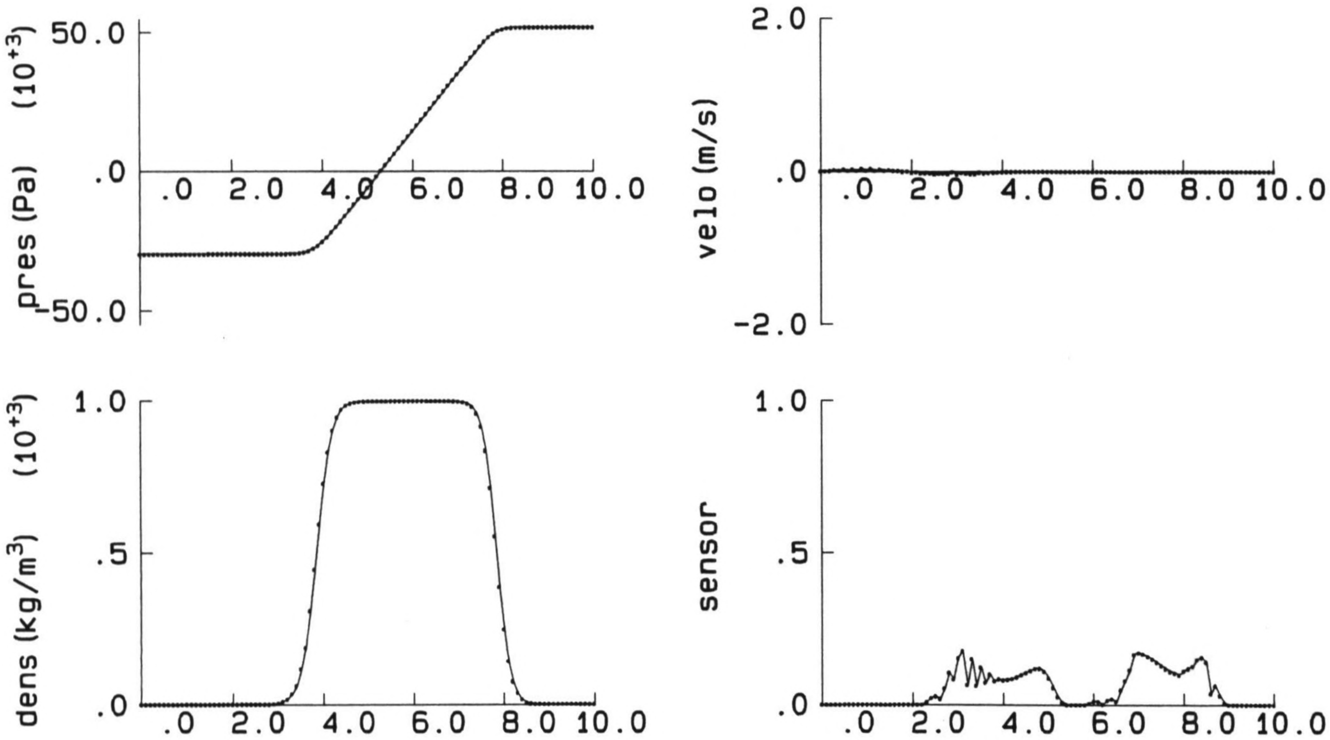

FIGURE VII.I

Impact pressure as a function of time
Compressibility number is beta=0.01
Impact numbers are S=0.25,   S=1.0

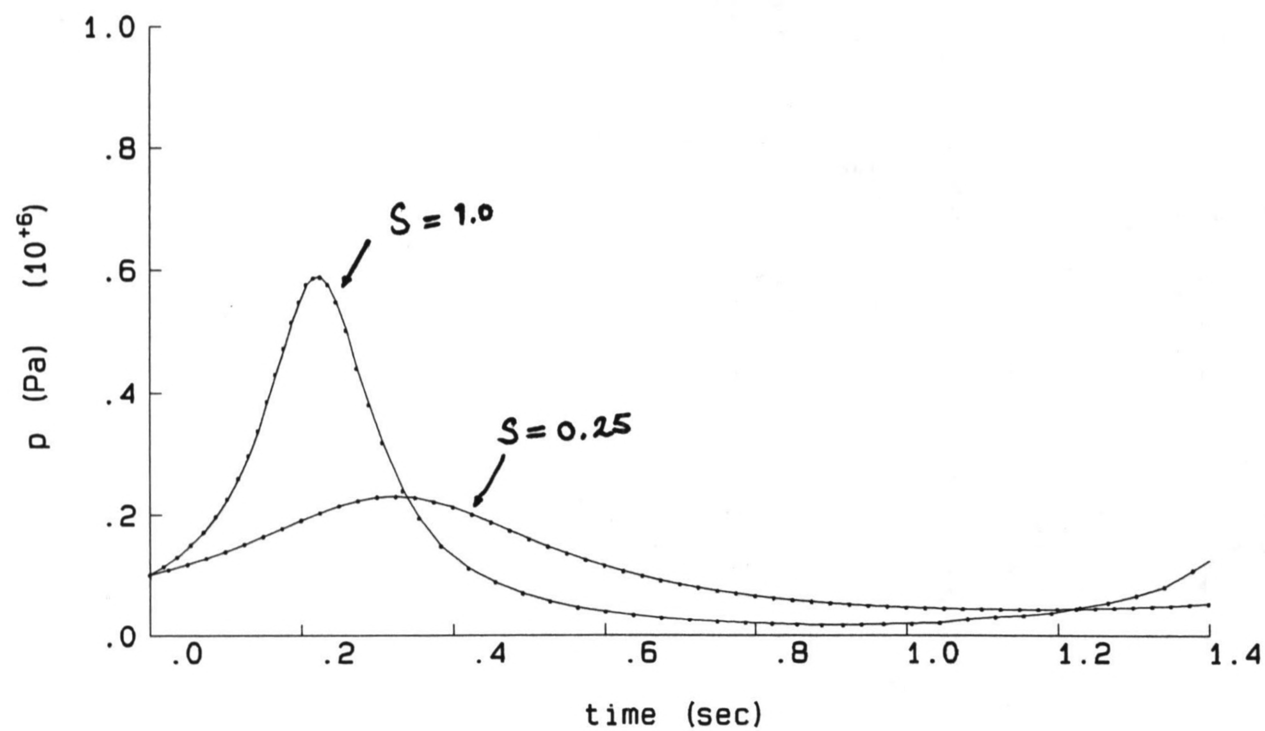

FIGURE VII.II (taken from [30])