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Lorist, Emiel

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# THE $\ell^s$-BOUNDEDNESS OF A FAMILY OF INTEGRAL OPERATORS ON UMD BANACH FUNCTION SPACES

EMIEL LORIST

*Dedicated to Ben de Pagter on the occasion of his 65th birthday.*

Abstract. We prove the $\ell^s$-boundedness of a family of integral operators with an operator-valued kernel on UMD Banach function spaces. This generalizes and simplifies the earlier work by Gallarati, Veraar and the author [12], where the $\ell^s$-boundedness of this family of integral operators was shown on Lebesgue spaces. The proof is based on a characterization of $\ell^s$-boundedness as weighted boundedness by Rubio de Francia.

## 1. Introduction

Over the past decades there has been a lot of interest in the $L^p$-maximal regularity of PDEs. Maximal $L^p$-regularity of the abstract Cauchy problem

$$\begin{cases} u'(t) + Au(t) = f(t), & t \in (0,T] \\ u(0) = x, \end{cases} \tag{1.1}$$

where $A$ is a closed operator on a Banach space $X$, means that for all $f \in L^p((0,T];X)$ the solution $u$ has "maximal regularity", i.e. both $u'$ and $Au$ are in $L^p((0,T];X)$. Maximal $L^p$-regularity can for example be used to solve quasi-linear and fully nonlinear PDEs by linearization techniques combined with the contraction mapping principle, see e.g. [1, 8, 30, 36].

In the breakthrough work of Weis [40, 41], an operator theoretic characterization of maximal $L^p$-regularity on UMD Banach spaces was found in terms of the $\mathcal{R}$-boundedness of the resolvents of $A$ on a sector. $\mathcal{R}$-boundedness is a random boundedness condition on a family of operators which is a strengthening of uniform boundedness. We refer to [7, 21] for more information on $\mathcal{R}$-boundedness.

In [13, 14] Gallarati and Veraar developed a new approach to maximal $L^p$-regularity for the case where the operator $A$ in (1.1) is time-dependent and $t \mapsto A(t)$ is merely assumed to be measurable. In this new approach $\mathcal{R}$-boundedness is once again one of the main tools. For their approach the $\mathcal{R}$-boundedness of the family of integral operators $\{I_k : k \in \mathcal{K}\}$ on $L^p(\mathbb{R};X)$

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is required. Here $I_k$ is defined for $f \in L^p(\mathbb{R};X)$ as

$$I_k f(t) := \int_{-\infty}^{t} k(t-s)T(t,r)f(r)\,\mathrm{d}r, \qquad t \in \mathbb{R},$$

where $T(t,s)$ is the two-parameter evolution family associated to $A(t)$ and $\mathcal{K}$ contains all kernels $k \in L^1(\mathbb{R})$ such that $|k| * |g| \leq Mg$ for all simple $g : \mathbb{R} \to \mathbb{C}$.

In the literature there are many $\mathcal{R}$-boundedness results for integral operators, see [21, Chapter 8] for an overview. However none of these are applicable to the operator family of $\{I_k : k \in \mathcal{K}\}$. Therefore in [12] Gallarati, Veraar and the author show a sufficient condition for the $\mathcal{R}$-boundedness of $\{I_k : k \in \mathcal{K}\}$ on $L^p(\mathbb{R};X)$ in the special case where $X = L^q$. This is done through the notion of $\ell^s$-boundedness, which states that for all finite sequences $(I_{k_j})_{j=1}^n$ in $\{I_k : k \in \mathcal{K}\}$ and $(x_j)_{j=1}^n$ in $X$ we have

$$\Big\|\Big(\sum_{j=1}^n |I_{k_j} x_j|^s\Big)^{1/s}\Big\|_X \lesssim \Big\|\Big(\sum_{j=1}^n |x_j|^s\Big)^{1/s}\Big\|_X.$$

For $s = 2$ this notion coincides with $\mathcal{R}$-boundedness as a consequence of the Kahane-Khintchine inequalities.

Our main contribution is the generalization of the main result in [12] to the setting of UMD Banach function spaces $X$. For the proof we will follow the general scheme of [12] with some simplifications. As in case $X = L^q$, for any UMD Banach function space the notions of $\ell^2$-boundedness and $\mathcal{R}$-boundedness coincide, so the following theorem in particular implies the $\mathcal{R}$-boundedness of $\{I_k : k \in \mathcal{K}\}$.

**Theorem 1.1.** *Let $X$ be a* UMD *Banach function space and $p \in (1,\infty)$. Let $T\colon \mathbb{R}\times\mathbb{R} \to \mathcal{L}(X)$ be such that the family of operators*

$$\big\{T(t,r) : t,r \in \mathbb{R}\big\}$$

*is $\ell^s$-bounded for all $s \in (1,\infty)$. Then $\{I_k : k \in \mathcal{K}\}$ is $\ell^s$-bounded on $L^p(\mathbb{R};X)$ for all $s \in (1,\infty)$.*

We will prove Theorem 1.1 in a more general setting in Section 3. In particular we allow weights in time, which in applications for example allow rather rough initial values (see e.g. [23, 26, 31, 37]).

For certain UMD Banach function spaces the $\ell^s$-boundedness assumption in Theorem 1.1 can be checked by weighted extrapolation techniques, see Corollary 3.5 and Remark 3.6.

**Notation.** For a measure space $(S,\mu)$ we denote the space of all measurable functions by $L^0(S)$. We denote the Lebesgue measure of a Borel set $E \in \mathcal{B}(\mathbb{R}^d)$ by $|E|$. For Banach spaces $X$ and $Y$ we denote the vector space of bounded linear operators from $X$ to $Y$ by $\mathcal{L}(X,Y)$ and we set $\mathcal{L}(X) := \mathcal{L}(X,X)$. For a operator family $\Gamma \subset \mathcal{L}(X,Y)$ we set $\Gamma^* := \{T^* : T \in \Gamma\}$. For $p \in [1,\infty]$ we let $p' \in [1,\infty]$ be such that $\frac{1}{p} + \frac{1}{p'} = 1$.

Throughout the paper we write $C_{a,b,\cdots}$ and $\phi_{a,b,\cdots}$ to denote a constant and a nondecreasing function on $[1,\infty)$ respectively, which only depend on the parameters $a,b,\cdots$ and the dimension $d$ and which may change from line to line.

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## 2. Preliminaries

2.1. **Banach function spaces.** Let $(S, \mu)$ be a $\sigma$-finite measure space. An order ideal $X$ of $L^0(S)$ equipped with a norm $\|\cdot\|_X$ is called a *Banach function space* if it has the following properties:

(i) *Compatibility:* If $\xi, \eta \in L^0(S)$ with $|\xi| \leq |\eta|$, then $\|\xi\|_X \leq \|\eta\|_X$
(ii) *Weak order unit:* There is a $\xi \in X$ with $\xi > 0$.
(iii) *Fatou property:* If $0 \leq \xi_n \uparrow \xi$ for $(\xi_n)_{n=1}^\infty$ in $X$, $\xi \in L^0(S)$ and $\sup_{n\in\mathbb{N}}\|\xi_n\|_X < \infty$, then $\xi \in X$ and $\|\xi\|_X = \sup_{n\in\mathbb{N}}\|\xi_n\|_X$.

A Banach function space is called *order continuous* if for any sequence $0 \leq \xi_n \uparrow \xi \in X$ we have $\|\xi_n - \xi\|_X \to 0$. Every reflexive Banach function space is order continuous. Order continuity ensures that the dual of $X$ is also a a Banach function space. For a thorough introduction to Banach function spaces we refer to [28, section 1.b] or [3, Chapter 1].

A Banach function space $X$ is said to be $p$-*convex* for $p \in [1, \infty]$ if

$$\Big\|\Big(\sum_{j=1}^n |\xi_k|^p\Big)^{1/p}\Big\|_X \leq \Big(\sum_{j=1}^n \|\xi_j\|_X^p\Big)^{1/p}$$

for all $\xi_1, \cdots, \xi_n \in X$ with the sums replaced by suprema if $p = \infty$. The defining inequality for $p$-convexity often includes a constant, but $X$ can always be renormed such that this constant equals 1. If a Banach function space is $p$-convex for some $p \in [1, \infty]$, then $X$ is also $q$-convex for all $q \in [1, q]$.

For a $p$-convex Banach function space $X$ we can define another Banach function space by

$$X^p := \big\{|\xi|^p \operatorname{sgn} \xi : \xi \in X\big\} = \big\{\xi \in L^0(S) : |\xi|^{1/p} \in X\big\}$$

equipped with the norm $\|\xi\|_{X^p} := \big\||\xi|^{1/p}\big\|_X^p$. We refer the interested reader to [28, section 1.d] for an introduction to $p$-convexity.

2.2. $\ell^s$**-boundedness.** Let $X$ and $Y$ be Banach functions spaces and let $\Gamma \subseteq \mathcal{L}(X, Y)$ be a family of operators. We say that $\Gamma$ is $\ell^s$-*bounded* if for all finite sequences $(T_j)_{j=1}^n$ in $\Gamma$ and $(x_j)_{j=1}^n$ in $X$ we have

$$\Big\|\Big(\sum_{j=1}^n |T_j x_j|^s\Big)^{1/s}\Big\|_Y \leq C\Big\|\Big(\sum_{j=1}^n |x_j|^s\Big)^{1/s}\Big\|_X.$$

with the sums replaced by suprema if $s = \infty$. The least admissible constant $C$ will be denoted by $[\Gamma]_{\ell^s}$.

Implicitly $\ell^s$-boundedness is a classical tool in harmonic analysis for operators on $L^p$-spaces (see e.g. [16, Chapter V] and [17, 18]). For Banach function spaces the notion was introduced in [40] under the name $\mathcal{R}_s$-boundedness, underlining its connection to the more well-known notion of $\mathcal{R}$-boundedness. An extensive study of $\ell^s$-boundedness can be found in [24] and for a comparison between $\ell^2$-boundedness and $\mathcal{R}$-boundedness we refer to [25].

**Lemma 2.1.** *Let $X$ and $Y$ be Banach function spaces and let $\Gamma \subseteq \mathcal{L}(X, Y)$.*

*(i)* *Let* $1 \leq s_0 < s_1 \leq \infty$ *and assume that* $X$ *and* $Y$ *are order continuous. If* $\Gamma$ *is* $\ell^{s_0}$*- and* $\ell^{s_1}$*-bounded, then* $\Gamma$ *is* $\ell^s$*-bounded for all* $s \in [s_0, s_1]$ *with* $[\Gamma]_{\ell^s} \leq \max\{[\Gamma]_{\ell^{s_0}}, [\Gamma]_{\ell^{s_1}}\}$

*(ii)* *Let* $s \in [1,\infty]$ *and assume that* $\Gamma$ *is* $\ell^s$*-bounded. Then the adjoint family* $\Gamma^*$ *is* $\ell^{s'}$*-bounded with* $[\Gamma^*]_{\ell^{s'}} = [\Gamma]_{\ell^s}$

*Proof.* Lemma 2.1(i) follows from Calderón's theory of complex interpolation of vector-valued function spaces, see [6] or [24, Proposition 2.14]. Lemma 2.1(ii) is direct from the identification $X(\ell^s_n)^* = X^*(\ell^{s'}_n)$, see [28, Section 1.d] or [24, Proposition 2.17] □

The following characterization of $\ell^s$-boundedness for $s \in [1,\infty)$ will be one of the key ingredients of our main result. This characterization relating $\ell^s$-boundedness to a certain weighted boundedness comes from the work of Rubio de Francia [16, 38, 39].

**Proposition 2.2.** *Let* $s \in [1,\infty)$ *and let* $X$ *and* $Y$ *be* $s$*-convex order continuous Banach function spaces over* $(S_X, \mu_X)$ *and* $(S_Y, \mu_Y)$ *respectively. Let* $\Gamma \subseteq \mathcal{L}(X)$ *and take* $C > 0$*. Then the following are equivalent:*

*(i)* $\Gamma$ *is* $\ell^s$*-bounded with* $[\Gamma]_{\ell^s} \leq C$.

*(ii)* *For all nonnegative* $u \in (Y^s)^*$*, there exists a nonnegative* $v \in (X^s)^*$ *with* $\|v\|_{(Y^s)^*} \leq \|u\|_{(X^s)^*}$ *and*
$$\Big(\int_{S_Y} |T(\xi)|^s u \,\mathrm{d}\mu_Y\Big)^{1/s} \leq C\,\Big(\int_{S_X} |\xi|^s v \,\mathrm{d}\mu_X\Big)^{1/s}$$
*for all* $\xi \in X$ *and* $T \in \Gamma$.

*Proof.* The statement is a combination of [39, Lemma 1, p. 217] and [16, Theorem VI.5.3], which for $X = Y$ is proven [2, Lemma 3.4]. The statement for $X \neq Y$ is can be extracted from the proof of [2, Lemma 3.4] and can in full detail be found in [29, Proposition 6.1.3] □

2.3. **Muckenhoupt weights.** A locally integrable function $w : \mathbb{R}^d \to (0,\infty)$ is called a *weight*. For $p \in (1,\infty)$ and a weight $w$ we let $L^p(w)$ be the space of all $f \in L^0(\mathbb{R}^d)$ such that
$$\|f\|_{L^p(w)} := \Big(\int_{\mathbb{R}^d} |f|^p w\Big)^{1/p} < \infty.$$
We will say that a weight $w$ lies in the *Muckenhoupt class* $A_p$ and write $w \in A_p$ if it satisfies
$$[w]_{A_p} := \sup_Q \frac{1}{|Q|}\int_Q w \cdot \Big(\frac{1}{|Q|}\int_Q w^{1-p'}\Big)^{p-1} < \infty,$$
where the supremum is taken over all cubes $Q \subseteq \mathbb{R}^d$ with sides parallel to the coordinate axes.

**Lemma 2.3.** *Let* $p \in (1,\infty)$ *and* $w \in A_p$.

*(i)* $w \in A_q$ *for all* $q \in (p,\infty)$ *with* $[w]_{A_q} \leq [w]_{A_p}$.

*(ii)* $w^{1-p'} \in A_{p'}$ *with* $[w]_{A_p}^{1/p} = [w^{1-p'}]_{A_p'}^{1/p'}$.

*(iii)* $w \in A_{p-\varepsilon}$ *for* $\varepsilon = \frac{1}{\phi_p([w]_{A_p})}$ *with* $[w]_{A_{p-\varepsilon}} \leq \phi_p([w]_{A_p})$.

The first two properties of Lemma 2.3 follow directly from the definition. The third is for example proven in [18, Exercise 9.2.4]. For a more thorough introduction to Muckenhoupt weights we refer to [18, Chapter 9].

2.4. **The UMD property.** A Banach space $X$ is said to have the UMD property if the martingale difference sequence of any finite martingale in $L^p(\Omega; X)$ is unconditional for some (equivalently all) $p \in (1, \infty)$. We will work with UMD Banach function spaces, of which standard examples include reflexive Lebesgue, Lorentz and Orlicz spaces. In this Festschrift it is shown that reflexive Musielak-Orlicz spaces, so in particular reflexive variable Lebesgue spaces, have the UMD property, see [27]. The UMD property implies reflexivity, so in particular $L^1$ and $L^\infty$ do not have the UMD property. For a thorough introduction to the theory of UMD Banach spaces we refer to [5, 20].

For an order continuous Banach function space $X$ over $(S, \mu)$ there is also a characterization of the UMD property in terms of the *lattice Hardy–Littlewood maximal operator*, which for simple functions $f\colon \mathbb{R}^d \to X$ is given by

$$\widetilde{M} f(x) := \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y)| \,\mathrm{d}y, \qquad x \in \mathbb{R}^d$$

where the supremum is taken pointwise in $S$ and over all cubes $Q \subseteq \mathbb{R}^d$ with sides parallel to the coordinate axes (see [15] or [19, Lemma 5.1] for a detailed definition of $\widetilde{M}$). It is a deep result by Bourgain [4] and Rubio de Francia [39] that $X$ has the UMD property if and only if $\widetilde{M}$ is bounded on $L^p(\mathbb{R}^d; X)$ and $L^p(\mathbb{R}^d; X^*)$ for some (equivalently all) $p \in (1, \infty)$. For weighted $L^p$-spaces we have the following proposition, which was proven in [15]. The increasing dependence on $[w]_{A_p}$ is shown in [19, Corollary 5.3].

**Proposition 2.4.** *Let $X$ be a* UMD *Banach function space, $p \in (1, \infty)$ and $w \in A_p$. Then for all $f \in L^p(w; X)$ we have*

$$\big\|\widetilde{M} f\big\|_{L^p(w;X)} \leq \phi_{X,p}\big([w]_{A_p}\big) \|f\|_{L^p(w;X)}.$$

The UMD property of a Banach function space $X$ also implies that $X^q$ has the UMD property for a $q > 1$, which is a deep result by Rubio de Francia [39, Theorem 4].

**Proposition 2.5.** *Let $X$ be a* UMD *Banach function space. Then there is a $p > 1$ such that $X$ is $p$-convex and $X^q$ is a* UMD *Banach function space for all $q \in [1, p]$.*

## 3. Integral operators with an operator-valued kernel

Before turning to our main result on the $\ell^s$-boundedness of a family of integral operators on $L^p(w; X)$ with operator-valued kernels, we will first study the $\ell^s$-boundedness of a family of convolution operators on $L^p(w; X)$ with scalar-valued kernels. For this define

$$\mathcal{K} := \{k \in L^1(\mathbb{R}^d) : |k| * |f| \leq Mf \text{ a.e. for all simple } f : \mathbb{R}^d \to \mathbb{C}\}.$$

As an example any radially decreasing $k \in L^1(\mathbb{R}^d)$ with $\|k\|_{L^1(\mathbb{R}^d)} \leq 1$ is an element of $\mathcal{K}$. For more examples see [17, Chapter 2] and [34, Proposition 4.6].

Let $X$ be a Banach function space. For a kernel $k \in \mathcal{K}$ and a simple function $f\colon \mathbb{R}^d \to X$ we define

$$T_k f := k * f = \int_{\mathbb{R}^d} k(x-y) f(y) \, \mathrm{d}y.$$

As

$$\|T_k f\|_X \le |k| * \|f\|_X \le M\big(\|f\|_X\big),$$

and since the Hardy-Littlewood maximal operator $M$ is bounded on $L^p(w)$ for all $p \in (1,\infty)$ and $w \in A_p$, $T_k$ extends to a bounded linear operator on $L^p(w;X)$ by density. This argument also shows that the family of convolution operators given by $\Gamma := \{T_k : k \in \mathcal{K}\}$ is uniformly bounded on $L^p(w;X)$.

If $X$ is a UMD Banach function space we can say more. The following lemma was first developed by van Neerven, Veraar and Weis in [33, 34] in connection to stochastic maximal regularity. As in [33, 34], the endpoint case $s = 1$ will play a major role in the proof of our main theorem in the next section.

**Proposition 3.1.** *Let $X$ be a* UMD *Banach function space, $s \in [1,\infty]$, $p \in (1,\infty)$ and $w \in A_p$. Then $\Gamma = \{T_k : k \in \mathcal{K}\}$ is $\ell^s$-bounded on $L^p(w;X)$ with*

$$[\Gamma]_{\ell^s} \le \phi_{X,p}\big([w]_{A_p}\big).$$

The proof is a weighted variant of [34, Theorem 4.7], which for the special case where $X$ is an iterated Lebesgue space is presented in [12, Proposition 3.6]. For convenience of the reader we sketch the proof in the general case.

*Proof.* As $X$ is reflexive and therefore order-continuous, $\widetilde{M}$ is well-defined on $L^p(w;X)$ and we have $T_k f \le \widetilde{M} f$ pointwise a.e. for all simple $f\colon \mathbb{R}^d \to X$.

If $s = \infty$ take simple functions $f_1, \cdots, f_n \in L^p(w;X)$ and $k_1, \cdots, k_n \in \mathcal{K}$. Using Proposition 2.4 we have

$$\begin{aligned}\Big\| \sup_{1\le j\le n} |T_{k_j} f_j| \Big\|_{L^p(w;X)} &\le \Big\| \sup_{1\le j\le n} \widetilde{M} f_j(x) \Big\|_{L^p(w;X)} \\ &\le \Big\| \widetilde{M}\Big( \sup_{1\le j\le n} |f_j| \Big)(x) \Big\|_{L^p(w;X)} \\ &\le \phi_{X,p}\big([w]_{A_p}\big) \Big\| \sup_{1\le j\le n} |f_j| \Big\|_{L^p(w;X)}.\end{aligned}$$

The result now follows by the density of simple functions in $L^p(w;X)$.

If $s = 1$ we use duality. Note that since $X$ is reflexive we have $L^p(w;X)^* = L^{p'}(w';X^*)^*$ with $w' = w^{1-p'}$ under the duality pairing

$$\langle f, g \rangle_{L^p(w;X), L^{p'}(w';X^*)} = \int_{\mathbb{R}^d} \big\langle f(x), g(x) \big\rangle_{X,X^*} \, \mathrm{d}x \tag{3.1}$$

by Lemma 2.3(ii) and [20, Corollary 1.3.22]. One can routinely check that $T_k^* = T_{\tilde{k}}$ with $\tilde{k}(x) = k(-x)$ and that $k \in \mathcal{K}$ if and only if $\tilde{k} \in \mathcal{K}$. Since $X^*$ is also a UMD Banach function space (see [20, Proposition 4.2.17]) we know from the case $s = \infty$ that the adjoint family $\Gamma^*$ is $\ell^\infty$-bounded on $L^{p'}(\mathbb{R}^d, w'; X^*)$, so the result follows by Lemma 2.1(ii). Finally if $s \in (1,\infty)$ the result follows by Lemma 2.1(i). □

With these preparations done we can now introduce the family of integral operators with operator-valued kernel that we will consider. Let $X$ and $Y$ be a Banach function space and let $\mathcal{T}$ be a family of operators $\mathbb{R}^d \times \mathbb{R}^d \to \mathcal{L}(X,Y)$ such that $(x,y) \mapsto T(x,y)\xi$ is measurable for all $T \in \mathcal{T}$ and $\xi \in X$. The integral operators that we will consider are for simple $f : \mathbb{R}^d \to X$ given by

$$I_{k,T}f(x) = \int_{\mathbb{R}^d} k(x-y)T(x,y)f(y)\,\mathrm{d}y$$

with $k \in \mathcal{K}$ and $T \in \mathcal{T}$. If $\|T(x,y)\|_{\mathcal{L}(X,Y)} \leq C$ for all $T \in \mathcal{T}$ and $x, y \in \mathbb{R}^d$, we have

$$\|I_{k,T}f\|_X \leq C\,|k| * \|f\|_X \leq C\,M\big(\|f\|_X\big).$$

So as before $I_{k,T}$ extends to a bounded linear operator from $L^p(w;X)$ to $L^p(w;Y)$ for all $p \in (1,\infty)$ and $w \in A_p$, and

$$\mathcal{I}_{\mathcal{T}} := \big\{I_{k,T} : k \in \mathcal{K}, T \in \mathcal{T}\big\}$$

is uniformly bounded. For the details see [12, Lemma 3.9].

If $X$ and $Y$ are Hilbert spaces, this implies that $\mathcal{I}_{\mathcal{T}}$ is also $\ell^2$-bounded from $L^2(\mathbb{R}^d;X)$ to $L^2(\mathbb{R}^d;Y)$, as these notions coincide on Hilbert spaces. However if $X$ and $Y$ are not Hilbert spaces, but a UMD Banach function space or if we move to weighted $L^p$-spaces, the $\ell^2$-boundedness of $\mathcal{I}_{\mathcal{T}}$ is a lot more delicate.

Our main theorem is a quantitative and more general version of Theorem 1.1 in the introduction:

**Theorem 3.2.** *Let $X$ and $Y$ be a* UMD *Banach function spaces and let $p, s \in (1,\infty)$. Let $\mathcal{T}$ be a family of operators $\mathbb{R}^d \times \mathbb{R}^d \to \mathcal{L}(X,Y)$ such that*

*(i) $(x,y) \mapsto T(x,y)\xi$ is measurable for all $T \in \mathcal{T}$ and $\xi \in X$.*
*(ii) The family of operators $\widetilde{\mathcal{T}} := \{T(x,y) : T \in \mathcal{T}, x, y \in \mathbb{R}^d\}$ is $\ell^\sigma$-bounded for all $\sigma \in (1,\infty)$.*

*Then $\mathcal{I}_{\mathcal{T}}$ is $\ell^s$-bounded from $L^p(w;X)$ to $L^p(w;Y)$ for all $w \in A_p$ with*

$$\begin{aligned}[\mathcal{I}_{\mathcal{T}}]_{\ell^s} &\leq \phi_{X,Y,p}\big([w]_{A_p}\big)\,\max\big\{\big[\widetilde{\mathcal{T}}\big]_{\ell^\sigma}, \big[\widetilde{\mathcal{T}}\big]_{\ell^{\sigma'}}\big\}, \qquad \sigma = 1 + \frac{1}{\phi_{p,s}\,[w]_{A_p}}\\ &\leq \phi_{X,Y,\mathcal{T},p,s}\big([w]_{A_p}\big).\end{aligned}$$

We will first prove a result assuming the $\ell^s$-boundedness of $\widetilde{\mathcal{T}}$ for a fixed $s \in [1,\infty)$.

**Proposition 3.3.** *Fix $1 \leq s \leq r < p < \infty$ and let $X$ and $Y$ be $s$-convex Banach function spaces such that $X^s$ has the* UMD *property. Let $\mathcal{T}$ be a family of operators $\mathbb{R}^d \times \mathbb{R}^d \to \mathcal{L}(X,Y)$ such that*

*(i) $(x,y) \mapsto T(x,y)\xi$ is measurable for all $T \in \mathcal{T}$ and $\xi \in X$.*
*(ii) The family of operators $\widetilde{\mathcal{T}} := \{T(x,y) : T \in \mathcal{T}, x, y \in \mathbb{R}^d\}$ is $\ell^s$-bounded.*

*Then $\mathcal{I}_{\mathcal{T}}$ is $\ell^s$-bounded from $L^p(w;X)$ to $L^p(w;Y)$ for all $w \in A_{p/s}$ with*

$$[\mathcal{I}_{\mathcal{T}}]_{\ell^s} \leq \phi_{X,p,r}\big([w]_{A_{p/s}}\big)\,\big[\widetilde{\mathcal{T}}\big]_{\ell^s}.$$

*Proof.* Let $(S_X, \mu_X)$ and $(S_Y, \mu_Y)$ be the measure spaces associated to $X$ and $Y$ respectively. For $j = 1, \cdots, n$ take $I_j \in \mathcal{I}_{\mathcal{T}}$ and let $k_j \in \mathcal{K}$ and $T_j \in \mathcal{T}$ be such that $I_j = I_{k_j,T_j}$. Fix simple functions $f_1, \cdots, f_n \in L^p(w; X)$ and note that

$$\Big\| \Big( \sum_{j=1}^n |I_j f_j|^s \Big)^{1/s} \Big\|_{L^p(w;Y)} = \Big\| \sum_{j=1}^n |I_j f_j|^s \Big\|_{L^{p/s}(w;Y^s)}^{1/s}. \tag{3.2}$$

Fix $x \in \mathbb{R}^d$, then by Hahn-Banach we can find a nonnegative $u_x \in (Y^s)^*$ with $\|u_x\|_{(X^s)^*} = 1$ such that

$$\Big\| \sum_{j=1}^n |I_j f_j(x)|^s \Big\|_{Y^s} = \sum_{j=1}^n \int_{S_Y} |I_j f_j(x)|^s u_x \,\mathrm{d}\mu_Y. \tag{3.3}$$

With Proposition 2.2 we can then find a nonnegative $v_x \in (X^s)^*$ with $\|v_x\|_{(X^s)^*} \leq 1$ such that

$$\int_{S_Y} |T_j(x,y)\xi|^s v_x \,\mathrm{d}\mu_Y \leq [\widetilde{\mathcal{T}}]_{\ell^s} \int_{S_X} |\xi|^s v_x \,\mathrm{d}\mu_X \tag{3.4}$$

for $j = 1, \cdots, n$, $y \in \mathbb{R}^d$ and $\xi \in X$. Since $\|k_j\|_{L^1(\mathbb{R}^d)} \leq 1$ by [34, Lemma 4.3], Holder's inequality yields

$$|I_j f_j(x)|^s \leq \int_{\mathbb{R}^d} |k_j(x-y)| |T_j(x,y) f_j(y)|^s \,\mathrm{d}y. \tag{3.5}$$

Applying (3.5) and (3.4) successively we get

$$\begin{aligned}
\sum_{j=1}^n \int_{S_Y} |I_j f_j(x)|^s u_x \,\mathrm{d}\mu_Y &\leq \sum_{j=1}^n \int_{S_Y} \int_{\mathbb{R}^d} |k_j(x-y)| |T_j(x,y) f_j(y)|^s \,\mathrm{d}y \, u_x \,\mathrm{d}\mu_Y \\
&= \sum_{j=1}^n \int_{\mathbb{R}^d} |k_j(x-y)| \int_{S_Y} |T_j(x,y) f_j(y)|^s \, u_x \,\mathrm{d}\mu_Y \,\mathrm{d}y \\
&\leq [\widetilde{\mathcal{T}}]_{\ell^s} \sum_{j=1}^n \int_{S_X} \int_{\mathbb{R}^d} |k_j(x-y)| |f_j(y)|^s \,\mathrm{d}y \, v_x \,\mathrm{d}\mu_X \\
&\leq [\widetilde{\mathcal{T}}]_{\ell^s} \Big\| \sum_{j=1}^n (|k_j| * |f_j|^s)(x) \Big\|_{X^s},
\end{aligned}$$

using duality and $\|v_x\|_{(X^s)^*} \leq 1$ in the last step. We can now use the $\ell^1$-boundedness result of Proposition 3.1, since $(X^s)^*$ has the UMD property by [21, Proposition 4.2.17]. Combined with (3.2) and (3.3) we obtain

$$\begin{aligned}
\Big\| \Big( \sum_{j=1}^n |I_j f_j|^s \Big)^{1/s} \Big\|_{L^p(w;Y)} &\leq [\widetilde{\mathcal{T}}]_{\ell^s} \Big\| \sum_{j=1}^n |k_j| * |f_j|^s \Big\|_{L^{p/s}(w;X^s)}^{\frac{1}{s}} \\
&\leq \phi_{X,p/s}\big([w]_{A_{p/s}}\big) \, [\widetilde{\mathcal{T}}]_{\ell^s} \Big\| \sum_{j=1}^n |f_j|^s \Big\|_{L^{p/s}(w;X^s)}^{1/s} \\
&\leq \phi_{X,p,r}\big([w]_{A_{p/s}}\big) \, [\widetilde{\mathcal{T}}]_{\ell^s} \Big\| \Big( \sum_{j=1}^n |f_j|^s \Big)^{\frac{1}{s}} \Big\|_{L^p(w;X)},
\end{aligned}$$

where we can pick the increasing function $\phi$ in the last step independent of $s$, since the increasing function in Proposition 3.1 depends continuously on $p$. This can for example be seen by writing out the exact dependence on $p$ in Theorem 2.4 using [19, Theorem 1.3] and [32, Theorem 3.1]. $\square$

Using this preparatory proposition, we will now prove Theorem 3.2.

*Proof of Theorem 3.2.* Let $w \in A_p$. We shall prove the theorem in three steps.

**Step 1.** First we shall prove the theorem very small $s > 1$. By Proposition 2.5 we know that there exists a $\sigma_{X,Y} \in (1,p)$ such that $X$ and $Y$ are $s$-convex and $X^s$ has the UMD property for all $s \in [1, \sigma_X]$. By Lemma 2.3(iii) we can then find a $\sigma_{p,w} \in (1, \sigma_{X,Y}]$ such that for all $s \in [1, \sigma_{p,w}]$

$$[w]_{A_{p/s}} \leq [w]_{A_{p/\sigma_{p,w}}} \leq \phi_p\big([w]_{A_p}\big)$$

Let $\sigma_1 = \min\{\sigma_{X,Y}, \sigma_{p,w}\}$, then by Proposition 3.3 we know that $\mathcal{I}_\mathcal{T}$ is $\ell^s$-bounded from $L^p(w;X)$ to $L^p(w;Y)$ for $s \in (1, \sigma_1]$ with

$$[\mathcal{I}_\mathcal{T}]_{\ell^s} \leq \phi_{X,p,\sigma_{X,Y}}([w]_{A_{p/s}})\big[\widetilde{\mathcal{T}}\big]_{\ell^s} \leq \phi_{X,Y,p}([w]_{A_p})\big[\widetilde{\mathcal{T}}\big]_{\ell^s}. \tag{3.6}$$

**Step 2.** Now we use a duality argument to prove the theorem for large $s < \infty$. As noted in the proof of Proposition 3.1, we have $L^p(w;X)^* = L^{p'}(w';X^*)$ with $w' = w^{1-p'}$ under the duality pairing as in (3.1) and similarly for $Y$. Furthermore $X^*$ and $Y^*$ have the UMD property.

It is routine to check that under this duality $I_{k,T}^* = I_{\tilde{k},\tilde{T}}$ with $\tilde{k}(x) = k(-x)$ and $\tilde{T}(x,y) = T^*(y,x)$ for any $I_{k,T} \in \mathcal{I}_\mathcal{T}$. Trivially $\tilde{k} \in \mathcal{K}$ if and only if $k \in \mathcal{K}$ and by Proposition 3.1(ii) the adjoint family $\widetilde{\mathcal{T}}^*$ is $\ell^{\sigma'}$-bounded with

$$\big[\widetilde{\mathcal{T}}^*\big]_{\ell^{\sigma'}} = \big[\widetilde{\mathcal{T}}\big]_{\ell^\sigma}$$

for all $\sigma \in (1,\infty)$. Therefore, it follows from step 1 that there is a $\sigma_2 > 1$ such that $\mathcal{I}_\mathcal{T}^*$ is $\ell^s$-bounded from $L^{p'}(w';Y^*)$ to $L^{p'}(w';X^*)$ for all $s \in (1,\sigma_2]$. Using Proposition 3.1(ii) again, we deduce that $\mathcal{I}_\mathcal{T}$ is $\ell^s$-bounded from $L^p(w;X)$ to $L^p(w;Y)$ for all $s \in [\sigma_2', \infty)$ with

$$[\mathcal{I}_\mathcal{T}]_{\ell^s} = [\mathcal{I}_\mathcal{T}^*]_{\ell^{s'}} \leq \phi_{X,Y,p}\big([w]_{A_p}\big)\big[\widetilde{\mathcal{T}}\big]_{\ell^s}. \tag{3.7}$$

**Step 3.** We can finish the prove by an interpolation argument for $s \in (\sigma_1, \sigma_2')$. By Proposition 2.2(i) we get for $s \in (\sigma_1, \sigma_2')$ that $\mathcal{I}_\mathcal{T}$ is $\ell^s$-bounded from $L^p(w;X)$ to $L^p(w;Y)$ with

$$[\mathcal{I}_\mathcal{T}]_{\ell^s} \leq \phi_{X,Y,p}([w]_{A_p}) \max\Big\{\big[\widetilde{\mathcal{T}}\big]_{\ell^{\sigma_1}}, \big[\widetilde{\mathcal{T}}\big]_{\ell^{\sigma_2'}}\Big\}. \tag{3.8}$$

Now note that by Lemma 2.3 there is a $\sigma \in (1,\infty)$ such that $\sigma < \sigma_1, \sigma_2$ and $\sigma < s < \sigma'$ and

$$\sigma = 1 + \frac{1}{\phi_{p,s}([w]_{A_p})}.$$

Thus combining (3.6), (3.7) and (3.8) we obtain

$$[\mathcal{I}_\mathcal{T}]_{\ell^s} \leq \phi_{X,Y,p}\big([w]_{A_p}\big) \max\big\{\big[\widetilde{\mathcal{T}}\big]_{\ell^\sigma}, \big[\widetilde{\mathcal{T}}\big]_{\ell^{\sigma'}}\big\} \leq \phi_{X,Y,\mathcal{T},p,s}\big([w]_{A_p}\big),$$

using the fact that $t \mapsto \max\big\{\big[\widetilde{\mathcal{T}}\big]_{\ell^t}, \big[\widetilde{\mathcal{T}}\big]_{\ell^{t'}}\big\}$ is increasing for $t \to 1$ by Proposition 2.2(i). This proves the theorem. $\square$

*Remark* 3.4.

- From Theorem 3.2 one can also conclude that $\mathcal{I}_{\mathcal{T}}$ is $\mathcal{R}$-bounded, since $\mathcal{R}$- and $\ell^2$-boundedness coincide if $X$ and $Y$ have the UMD property, see e.g. [21, Theorem 8.1.3].
- The UMD assumptions in Theorem 3.2 are necessary. Indeed already if $X = Y$, $w = 1$ and if $\widetilde{\mathcal{T}}$ only contains the identity operator, it is shown in [22] that the $\ell^2$-boundedness of $\mathcal{I}_{\mathcal{T}}$ implies the UMD property of $X$.
- The main result of [12] is Theorem 3.2 for the special case $X = Y = L^q(S)$. In applications to systems of PDEs one needs Theorem 3.2 on $L^q(S;\mathbb{C}^n)$ with $s = 2$, see e.g. [13]. This could be deduced from the proof of [12, Theorem 3.10], by replacing absolute values by norms in $\mathbb{C}^n$. In our more general statement the case $L^q(S;\mathbb{C}^n)$ is included, since $L^q(S;\mathbb{C}^n)$ is a UMD Banach function space over $S \times \{1, \cdots, n\}$

If $X = Y$ is a rearrangement invariant Banach function space on $\mathbb{R}^e$, we can check the $\ell^\sigma$-boundedness of $\widetilde{\mathcal{T}}$ for all $\sigma \in (1,\infty)$ by weighted extrapolation. Examples of such Banach function spaces are Lebesgue, Lorentz and Orlicz spaces. See [28, Section 2.a] for an introduction to rearrangement invariant Banach function spaces.

**Corollary 3.5.** *Let $X$ be a rearrangement invariant* UMD *Banach function space on $\mathbb{R}^e$ and let $p, s \in (1,\infty)$. Let $\mathcal{T}$ be a family of operators $\mathbb{R}^d \times \mathbb{R}^d \to \mathcal{L}(X)$ such that*

*(i) $(x, y) \mapsto T(x, y)\xi$ is measurable for all $T \in \mathcal{T}$ and $\xi \in X$.*
*(ii) For some $q \in (1,\infty)$ and all $v \in A_q$ we have*

$$\sup_{T\in\mathcal{T},\, x,y\in\mathbb{R}^d} \|T(x, y)\|_{\mathcal{L}(L^q(v))} \le \phi_{\mathcal{T},q}\big([v]_{A_q}\big)$$

*Then $\mathcal{I}_{\mathcal{T}}$ is $\ell^s$-bounded on $L^p(w;X)$ for all $w \in A_p$ with*

$$[\mathcal{I}_{\mathcal{T}}]_{\ell^s} \le \phi_{X,Y,\mathcal{T},p,q,s}\big([w]_{A_p}\big).$$

Note that in Corollary 3.5 we need that $T(x, y)$ is well-defined on $L^q(v)$ for all $T \in \mathcal{T}$ and $x, y \in \mathbb{R}^d$. This is indeed the case, since $X \cap L^q(v)$ is dense in $L^q(v)$.

*Proof.* Let $Y$ be the linear span of

$$\{\mathbf{1}_K\,\xi : K \subseteq \mathbb{R}^e \text{ compact},\ \xi \in X \cap L^\infty(\mathbb{R}^e)\}.$$

Then $Y \subseteq L^q(v)$ for all $v \in A_p$ and $Y$ is dense in $X$ by order continuity. Define

$$\mathcal{F} := \big\{\big(|T(x, y)\xi|, |\xi|\big) : T \in \mathcal{T},\ x, y \in \mathbb{R}^d,\ \xi \in Y\big\}.$$

Note that $X$ has upper Boyd index $q_X < \infty$ by the UMD property (see [21, Proposition 7.4.12] and [28, Section 2.a]). So we can use the extrapolation result for Banach function spaces in [11, Theorem 2.1] to conclude that for $\sigma \in (1,\infty)$

$$\Big\|\Big(\sum_{j=1}^{n}|T_j(x_j, y_j)\xi_j|^\sigma\Big)^{1/\sigma}\Big\|_X \le C_{\mathcal{T},q}\Big\|\Big(\sum_{j=1}^{n}|\xi_j|^\sigma\Big)^{1/\sigma}\Big\|_X$$

for any $T_j \in \mathcal{T}$, $x_j, y_j \in \mathbb{R}^d$ and $\xi_j \in Y$ for $j = 1, \cdots, n$. By the density this extends to $\xi_j \in X$, so

$$\{T(x, y) : x, y \in \mathbb{R}^d, T \in \mathcal{T}\}$$

is $\ell^\sigma$-bounded for all $\sigma \in (1, \infty)$. Therefore the corollary follows from Theorem 3.2. □

*Remark* 3.6.

- A sufficient condition for the weighted boundedness assumption in Corollary 3.5 is that $T(x, y)\xi \leq C\, M\xi$ for all $T \in \mathcal{T}$, $x, y \in \mathbb{R}^d$ and $\xi \in L^q(\mathbb{R}^e)$, which follows directly from [18, Theorem 9.1.9].
- Corollary 3.5 holds more generally for UMD Banach function spaces $X$ such that the Hardy-Littlewood maximal operator is bounded on both $X$ and $X^*$ (see [10, Theorem 4.6]). For example the variable Lebesgue spaces $L^{p(\cdot)}$ satisfy this assumption if $p_+, p_- \in (1, \infty)$ and $p(\cdot)$ satisfies a certain continuity condition, see [9, 35].
- The conclusion of Corollary 3.5 also holds for $X(v)$ for all $v \in A_{p_X}$ where $p_X$ is the lower Boyd index of $X$ and $X(v)$ is a weighted version of $X$, see [11, Theorem 2.1].

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Delft Institute of Applied Mathematics, Delft University of Technology, P.O. Box 5031, 2600 GA Delft, The Netherlands

*E-mail address*: e.lorist@tudelft.nl